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BLACK HOLE MICROSTATES AND HOLOGRAPHY IN THE D1D5 CFT

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This thesis is dedicated to Umberto, Antonio, Silvia and Anna.

La maestra non ritenne di fare altre domande

— Offlaga Disco Pax

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Declaration

I, Emanuele Moscato, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications:

This thesis describes research carried out with my supervisor Rodolfo Russo which was published in [1, 2]. It also contains some unpublished material. We collaborated with Stefano Giusto on [1, 2] and also with Andrea Galliani on [2]. We also collaborated with Alessandro Bombini. Where other sources have been used, they are cited in the bibliography.

Abstract

In this thesis we exploit the setup of AdS_3/CFT_2 holography, and in particular the D1D5 two-dimensional CFT, to describe states dual to geometries relevant for the "fuzzball" proposal for the description of six-dimensional black hole microstates. Precise holographic dualities between CFT and bulk geometric objects are established and checked, both for 2 and 3-charge states. In particular, VEVs of CFT operators of small conformal dimension are checked to encode deviations from AdS_3 geometry near the spacetime boundary. 4-point functions of the "heavy-heavy-light-light" type are also considered and matching is found between CFT and bulk computations via the usual AdS/CFT prescription, with the heavy states being dual to (simple) microstate geometries. In this context, the issue of the presence of spurious singularities at leading order in the large N limit is assessed and cancellations are found even without considering sub-leading corrections, at the cost of considering the full detail of the D1D5 CFT (i.e. including the Virasoro blocks of operators of small dimension charged under the internal $SU(2)_L \times SU(2)_R$ R-symmetry group). Finally, more complicated 4-point functions, involving operators in the twisted sector of the CFT, are computed and the results are checked against known results in the literature with the aim of verifying the robustness of the (new) techniques used. Supersymmetric Ward identities are also derived, and checked for some cases, between correlators written in terms of bosons and in terms of fermions.

Contents

Chapter 1

Introduction

1.1 Black hole entropy

When looking for a quantum theory of gravity, among the typical systems that we should consider there are certainly black holes. These systems arise as solutions of general relativity (and supergravity theories) and present physical singularities (i.e. divergences of one of the curvature scalars that it's possible to define) covered by horizons. This feature signals that the theory is unable to describe regions in which the curvature is strong, and the fact that this is a typical short-distance effect is thought to be an indication that deviations from General Relativity should be taken into account.

This discussion already contains a problematic point, upon which there is no clear consensus: naively we would think that deviations from GR appear only in regions in which the curvature is of order $\sim l_p^{-2}$, where $l_p \equiv \left(\hbar G/c^3\right)^{1/2}$ is the Planck length This happens close to a black hole singularity, rather than near the horizon, where, especially for large black holes, the curvature remains small. Indeed, for e.g. 4-dimensional Schwarzschild black holes the horizon radius is

$$r_s = \frac{2GM}{c^2}, \tag{1.1}$$

while curvature scales like $\sim r^{-2}$, so as the black hole mass increases the curvature at the horizon gets weaker and weaker. In spite of this, many approaches to black hole physics provide indications that deviations from GR also affect the region close to the horizon, where the theory would suggest that nothing happens to a classical observer, besides being affected by the peculiar causal structure of the system (which remains the main feature of the horizon of a black hole).

Work by Bekenstein [3] also led to another piece of evidence in this direction: a black hole carries an entropy proportional to the area of its horizon, the *Bekenstein-Hawking* *entropy* S_{BH}, given in standard units by

$$S_{BH} = \frac{c^3 A}{4G\hbar}. \tag{1.2}$$

The appearance of the Bekenstein-Hawking entropy suggested many different ideas, perhaps the most important of which is the first hint to *holography*: due to the causal structure of the system, the information contained in a black hole (which is inaccessible to a classical observer outside the horizon) is encoded on the horizon. Moreover, information is not proportional to the volume of the region surrounded by the horizon, as extensivity suggests, but to the area of the horizon itself. This fact has been called *holographic principle* and led to the idea that a quantum theory of gravity should show some holographic features.

1.2 The black hole entropy paradox

The presence of an entropy associated to a black hole might already be seen as a puzzle, and for various reasons. Results by Hawking [4], derived using quantum field theory in curved spacetime, state that quantum fields in a black hole background geometry originate a process of pair production which leads to the emission of thermal radiation with a temperature given, e.g. for a Schwarzschild black hole of mass M, by

$$T_H = \frac{\hbar c^3}{8\pi G M k_B}. \tag{1.3}$$

More generally, this *Hawking temperature* is given by

$$T_H = \frac{\kappa}{2\pi}\,\frac{\hbar}{k_B c}, \tag{1.4}$$

where κ is the surface gravity of the black hole, which in standard units has the dimension of an acceleration and is defined in terms of the null Killing vector normal to the horizon χ^μ in natural units $\hbar = G = c = 1$ as[1]

$$\kappa^2_{\hbar=G=c=1} \equiv -\frac{1}{2}\left(\nabla^\mu \chi^\nu\right)\left(\nabla_\mu \chi_\nu\right) \tag{1.5}$$

and interpreted as the force that must be exerted at infinity to keep a unit mass at rest on the horizon (locally on the horizon this force would be infinite). Notice that black hole horizons are usually Killing horizons, i.e. hypersurfaces invariant under an

[1] The conversion of the definition below into standard units is

$$\kappa = \frac{c^4}{G}\,\kappa_{\hbar=G=c=1}.$$

isometry of the metric and on which the corresponding Killing vector has zero norm. Moreover they are also null surfaces: the vector field normal to them is null (and thus, because of Lorentzian signature, it lies in their tangent space).

The first part of the entropy paradox is the following: if we consider black holes that carry more charges than just their mass (for example electro-mangetically charged and/or rotating black holes), there is a precise bound between the charges and the mass that must be satisfied in order for the horizon to exist and not to have a naked singularity (which is considered unphysical), the *extremality bound*. The inequality is saturated when the mass squared equals the sum of the squares of the other charges (the black hole has a the minimal mass given the other charges) and the black hole is said to be extremal, in which case $\kappa = 0$ and thus $T_H = 0$. The horizon area is still nonzero, so we are presented with a system that, strangely enough, has a Bekenstein-Hawking entropy but no temperature. Another part of the entropy paradox regards *where* the information falling beyond the horizon is stored. Due to the uncertainty principle, if we want to maximize the accuracy of the measurement of the state of a quantum system we can't consider cells of volume less than $\Delta x \, \Delta p \sim \hbar$ in (one-dimensional) phase space. This suggests that information in phase space is stored in discrete cells, and we cannot store as much information as we want in an arbitrarily small volume, because its localization energy (roughly speaking $\sim \Delta p^2/2m$) would be infinite. If we stick to the picture from General Relativity, though, matter falling into a black hole eventually reaches the singularity, in contradiction with the above (qualitative) statement. This can be seen as another piece of evidence suggesting that quantum gravity effects should be there to modify the classical GR picture.

One idea is that information is not stored only in the high-curvature region close to the singularity, but in the whole region inside the horizon. This picture also naively justifies the second law of black hole thermodynamics [5] that states that for all physical processes

$$\frac{dA}{dt} \geq 0, \tag{1.6}$$

where A is the horizon area: matter falling beyond the horizon causes it to expand. Indeed this must be the case, if we think of information as occupying discrete cells on the horizon: when matter passes the horizon, more cells are added, and thus the horizon (or some notion of surface related to it) must get bigger.

From statistical mechanics, we are used to associating an entropy to macroscopic states given by a coarse-graining of a number N of microstates, which are all compatible with the macroscopic state in the sense that the macroscopic observables cannot distinguish between them. If the statistical ensemble of the microstates is microcanonical (each microscopic state compatible with the macroscopic one is equally probable), entropy is then given by Boltzmann's law,

$$S = k_B \log(N). \tag{1.7}$$

In this view, black holes could be seen as macroscopic systems obtained by coarse-graining over a microcanonical ensemble of states compatible with the black hole's charges. We may also go further: if we take Hawking radiation into account black holes can be seen as thermodynamic systems at the Hawking temperature T_h and should be described by microstates in a canonical ensemble (in which each microstate has probability $p \sim e^{-E/k_B T}$) with Gibbs entropy

$$S = -k_B \sum_i p_i \log p_i, \tag{1.8}$$

where the sum runs on every microstate in the ensemble. A third part of the black hole entropy paradox regards whether inaccessible microstates in fact exist and whether these they could be classical smooth (non-singular) geometries or some different kind of degrees of freedom.

1.3 The black hole information paradox

Directly connected with the possibility of the existence of microstates for black hole systems we also have the *black hole information paradox* [6–8], which come from the fact that black holes are believed to emit Hawking radiation and therefore lose mass. Historically, the starting point was the computation, in a semiclassical approximation with quantum fields on a classical background curved spacetime, of black hole radiation by Hawking [9, 10]. The relevant property of Hawking radiation is not merely its existence, but rather the fact that its spectrum is thermal, and thus typical of a statistically mixed state. In the hypothesis that black holes can radiate away all of their mass, this seems to be at odds with the unitary time evolution that is fundamentally assumed to hold at the microscopic (quantum) level, as it looks that black holes formed from matter in a pure statistical state can indeed evolve into a fully mixed state of thermal radiation. This also assumes the validity of the *no-hair conjecture* [11], according to which black hole solutions (in four dimensions) are unique upon fixing their mass and charges (including angular momentum): once these parameters are fixed there are no additional features in which information can be encoded. In $d > 4$ supergravity theories black hole solutions possessing hair do exist, but it's not clear how information may be encoded in the additional parameters needed to specify the solution, and the information paradox still holds.

Non-unitary evolution from pure states to mixed states is indeed what we call information loss: systems in states that are in principle distinguishable evolve into states that are not distinguishable any more, not even in principle. This is completely different from what we see in thermodynamic processes in our everyday life: if we burn a piece of paper we indeed obtain approximately[2] thermal radiation, but we also obtain vapour, ashes and all that, which in principle allow to reconstruct the piece of paper we started with. In the hypothesis black holes evaporate completely into Hawking radiation, nothing is left to reconstruct the initial state of the matter that underwent gravitational collapse.

Various proposals have been made to solve one of the most long-standing contradictions between two successful physical theories, General Relativity and Quantum Mechanics. Among these maybe the most natural ideas regarded the possibility of information actually being carried by Hawking radiation (i.e. the spectrum not being exactly thermal) and the possibility of a final state given by radiation plus a black hole "remnant", in which information would be encoded. The first idea is in fact now disfavoured, as it seems to be impossible to encode enough information in the Hawking radiation without its spectrum deviating too much from the thermal one [7,12]. As for black hole remnants, see e.g. [13], the hypothesis is still debated.

More recent approaches include the *fuzzball proposal* [14], *firewall* idea [15], the ER=EPR [16] model and the idea that black hole may have "soft hair" [17,18]. Despite the richness of ideas developed in the decades, no consensus exists in the community.

Hawking's computation [9,10] was performed in a semiclassical regime in which no quantum gravity effects could be taken into account. It is commonly believed that a complete theory of quantum gravity should be able to shed light upon the information paradox, and indeed black hole physics is taken to be one of the main indications that a quantum theory of gravity must exist.

Notice that the information paradox doesn't apply to extremal black holes because their Hawking temperature is zero. On the other hand the entropy paradox still applies, and indeed due to the connection between supersymmetry and extremal black holes much effort has been devoted to the description of the microstates of such systems. This thesis falls in this category. Extending the results to the nonextremal cases has historically been a difficult task, as many of the results and techniques granted by the presence of supersymmetry become unavailable. Nevertheless, progress has been made in the case of nearly-extremal black holes, treated as small deformations of the extremal ones.

[2]The deviation from perfectly thermal, i.e. black body, spectrum comes from the fact that everyday materials aren't usually black bodies.

1.4 Black holes and string theory

Black hole systems can be realized within the framework of string theory starting from D-brane configurations. In particular, if we start from a spacetime with with a product structure $\mathcal{M}\times\mathcal{K}$, where $\mathcal{M}$ is a noncompact manifold to be identified with the extended spacetime dimensions and $\mathcal{K}$ is a compact manifold, we can consider configurations consisting in D-branes wrapping (some cycles within) the compact space, while being localized in the extended dimensions. Schematically, we can think of having N D-branes wrapped around $\mathcal{K}$, with the parameter g_sN, where g_s is the string coupling constant, setting the typical scale of the system. For $g_sN \ll 1$ the backreaction of the branes on the geometry is small and the system is described in terms of string theory, while for $g_sN \gg 1$ the backreaction is large and the system is described in terms of a curved metric. It is possible to connect the two descriptions looking at the emission of closed strings by the D-branes: the backreacted metric can be reconstructed from the closed string amplitudes, which provides a link between the microscopic description in terms of strings and D-branes and the macroscopic one in terms of spacetime geometry [19–22]. This provides also a connections between string theory and the black branes solutions known to appear in supergravity theories, which are another way with which black hole solutions can be derived starting from configurations of branes.

The *Strominger-Vafa black hole* is an example of such systems obtained using D-branes. It is a solution of 10-dimensional type IIB supergravity compactified on $S^1\times T^4$ (or $S^1\times K3$) and corresponding to a 3-charge black hole in 5 noncompact asymptotically flat dimensions: it's essentially a generalization of the Reissner-Nordström black hole.. It can be obtained starting from a D-brane configuration in which n_1 $D1$-branes are wrapped around the S^1, n_5 $D5$-branes are wrapped around both the S^1 and the T^4 and n_p unit of momentum travel along the $D1$-branes. In order for some supersymmetry to remain unbroken, there must be only one kind of momentum excitations, either left or right. The resulting 5-dimensional metric in the string frame is

$$\begin{aligned} ds^2_{SV} = &-\frac{dt^2}{\left(1+\frac{Q_1}{r^2}\right)^{2/3}\left(1+\frac{Q_5}{r^2}\right)^{2/3}\left(1+\frac{Q_p}{r^2}\right)^{2/3}}+ \\ &+\left(1+\frac{Q_1}{r^2}\right)^{1/3}\left(1+\frac{Q_5}{r^2}\right)^{1/3}\left(1+\frac{Q_p}{r^2}\right)^{1/3}\left[dr^2+r^2d\Omega_3^2\right], \end{aligned} \tag{1.9}$$

where Q_1, Q_5 and Q_p are the charges corresponding to the $D1$-branes, $D5$-branes and momentum excitations, respectively (and are connected to n_1, n_5 and n_p). In this coordinates the horizon is located at $r = 0$; moreover, the area of the horizon is nonzero, which allows one to compare a "microscopic" value of the entropy, obtained studying the D-brane configuration, with the Bekenstein-Hawking entropy. This is also called the *naive 3-charge geometry*.

A 2-charge geometry (again called *naive*) can be obtained from the above by setting $Q_p = 0$. This geometry is somehow singular, in the sense that the horizon area is zero, and therefore there is no Bekenstein-Hawking entropy, but it's still possible to compute a (nonzero) entropy microscopically. If higher-derivative corrections to the Einstein-Hilbert action are considered it's possible to use the Wald formula [23] to compute the entropy, essentially a generalization of the Bekenstein-Hawking one, finding agreement with the microscopic result.

Starting from the Strominger-Vafa geometry it is possible to define a near-horizon limit, here called *decoupling limit*, which consists in taking r^2 small compared to Q_1 and Q_5 but large compared to Q_p, allowing us to approximate

$$1 + \frac{Q_i}{r^2} \sim \frac{Q_i}{r^2}, \qquad i = 1, 5. \tag{1.10}$$

In this limit the metric reduces to

$$ds^2 = R_{AdS_3} \left[\frac{du^2}{u^2} + u^2 \left(-dt^2 + dy^2 \right) \right] + R_{S^3}\, d\Omega_3^2, \tag{1.11}$$

with $u \alpha r$, which is the metric for $AdS_3 \times S^3$ spacetime with the AdS_3 factor written in Poincaré patch (see section 3.1.2 for more details). The fact that the decoupling limit allows us to obtain a geometry with such a structure is crucial, as having an AdS factor is the first indication that a description in terms of the AdS/CFT correspondence might be possible.

1.5 The AdS/CFT correspondence

In the decoupling limit the factors previously asymptotically $\mathbb{R}^{1,4} \times S^1$ become asymptotically $AdS_3 \times S^3$. Thanks to this the possibility of using the AdS/CFT correspondence [24–26] opens up, and in particular we can use AdS_3 holography to characterise the bulk fields (or, more in general, string theory in the bulk) in terms of states in a 2-dimensional CFT living on the AdS_3 boundary known as D1D5 CFT and already conjectured in [24]. More precisely, on one side of the correspondence we have type IIB string theory defined on an asymptotically $AdS_3 \times S^3$ background, while on the other side we have a 2-dimensional $\mathcal{N} = (4, 4)$ conformal field theory. In the supergravity limit we have that supergravity bulk fields are dual to states (or operators) in the (strongly coupled) CFT (see Chapter 4 for more details). We will consider two classes of CFT operators, one dual to black hole microstates and the other dual to probe fields: the next section will give more details about the hypotheses under which we will be working.

The D1D5 CFT is a $(1+1)$-dimensional sigma-model with target space $\left(T^4\right)^N / S_N$, i.e. is the tensor product of $N = n_1 n_5$ copies of the same CFT symmetric under the exchange of any two copies. It enjoys an $\mathcal{N} = (4,4)$ supersymmetry with holomorphic and antiholomorphic supercharges transforming under an $SU(2)_L \times SU(2)_R$ R-symmetry group. A twisted sector of the theory exists in which more copies are glued together in a sense that will be explained in Chapter 2. Throughout this work we will look at a specific point in the moduli space of the theory, the *free orbifold point*, in which all the couplings are zero and the CFT reduces to a collection of free bosons and free fermions. Thanks to known non-renormalization theorems [27], in all the cases considered this will be enough to make contact with the gravity side of the AdS/CFT correspondence.

Finally, we would like to emphasize how the dual CFT description can be used to give a more precise definition to the expected description of black holes as statistical ensembles. Indeed, once we know how to write black hole microstates in terms of quantum states in the dual CFT, we also have a natural way of defining statistical ensembles: it's sufficient to consider a density matrix over the Hilbert space of the CFT states.

1.6 Black hole microstates and the fuzzball proposal

One of the successes of string theory as a theory of quantum gravity was the computation of the entropy of the Strominger-Vafa solution from microscopic principles and the verification of its agreement with the corresponding Bekenstein-Hawking result [28,29]. The entropy was computed counting the number of different D-brane configurations compatible with fixed values for the charges. This result alone doesn't fully solve the black hole entropy entropy paradox, but suggests that a description in terms of microstates may be possible indeed. This thesis addresses the problem of describing black hole microstates of the Strominger-Vafa black hole in the framework of the *fuzzball proposal*. This asserts that black hole solutions arise by coarse-graining over an ensemble of states[3], some of which are given by smooth, regular geometries. An introduction to the subject is given in [14] and [30]. Each of these smooth solutions looks very different from the traditional picture of a black hole: indeed no horizon is present, and the gravitational *throat* usually associated to black holes doesn't end in a singularity, but rather in a regular *cap*, whose geometry depends on the particular microstate solution considered. The estimation for the typical size of microstates geometries (or *fuzzball geometries*) gives roughly the radius of the horizon of a black hole: these systems could give a description of gravitationally collapsed object which is completely different from

[3]The operation of coarse graining over an ensemble of solution of GR or supergravity isn't really defined, even though some notion of it should be there, if the interpretation of black holes as thermodynamical objects is true. In the dual CFT description of the microstates, however, ensemble and density matrices can be defined, which in turns can be taken as a definition of what coarse-graining could be in the bulk.

the black hole solutions of general relativity and supergravity, directly suggesting that deviations from the GR description are not at all confined to a small region within an horizon, but rather start very close to the horizon itself.

The fuzzball proposal aims at solving both the entropy paradox, giving a description of the black hole's microstates, and the information paradox [30]. In order to do this, one of the hypotheses leading to information loss is relaxed: we no longer assume that deviations from GR are negligible at the horizon, even if the black hole is big enough not to give strong curvature at the horizon radius. Indeed, the horizon scale is exactly where microstate geometries start to differ from black holes. Hawking radiation may be then seen as the thermal radiation of a highly degenerate object in which the macroscopically indistinguishable degrees of freedom are encoded in the density matrix giving the ensemble of microstates.

As for the Strominger-Vafa black hole, fuzzball geometries are solutions of 10-dimensional supergravity with a specific configuration of $D1$ and $D5$-branes, compactified to 5 dimensions, resulting in asymptotically flat geometries (in particular, the asymptotic behaviour presents $\mathbb{R}^{1,4} \times S^1$ factors, with the other 4 compact dimensions corresponding to a T^4 of a Calabi-Yau $K3$ compact manifold). These reproduce the same behaviour as the Strominger-Vafa solution at infinity (the charges of the microstates are the same as those of the black hole), but differ from it in the interior of spacetime. The D1D5 CFT is obtained starting from the stacks of $D1$ and $D5$ branes, taking the decoupling limit and separating the branes. If we then consider open strings extending between the branes we get that the low-energy description is indeed a conformal field theory [31].

Thanks to the above construction, fuzzball microstates can be seen as bound states of D-branes in an asymptotically flat bulk theory (before taking the decoupling limit). More precisely, these are *threshold bound states* in the sense that the gravitational attraction is compensated by the charge repulsion resulting in an equilibrium configuration with flat potential energy, as if we had a system of effectively non-interacting objects. Such a precise construction is crucial to preserve some supersymmetry and acts as a mechanism that decouples the dynamics of the objects from their kinematics, in the sense defined below, which is exactly what is needed to more easily address the problem of black hole microstates: indeed the entropy paradox is related to the kinematics of the system, while Hawking radiation and the information paradox are inherently dynamical processes. The fact that supersymmetry is preserved implies that fuzzball microstates are BPS, which in turns implies that their charges saturate the analogue of the extremality bound. This is a purely gravitational sign of the dynamics (Hawking radiation) being decoupled from the kinematics (entropy) of the system: fuzzballs are the microstates of extremal black holes, which have zero temperature (and therefore emit no Hawking radiation) but nonzero entropy.

The technique to obtain fuzzball geometries that differ from the naive ones but start from the same D-brane configuration [32–35] is perhaps clearer if we switch to another duality frame. Through a series of S- and T-dualities, a 2-charge D1-D5 system in type IIB supergravity can be connected to a 2-charge NS1-P system in type IIA given by a fundamental string wrapped around the S^1 and carrying momentum excitations [14]. We can then look at the oscillations of the fundamental string in the transverse (noncompact) space and parametrize it in terms of vibration profiles $F_i(t-i)$, $i = 1, \ldots, 4$, where y is the coordinate along the S^1 and we are considering only left-moving momentum excitations. These oscillations correspond to a supergravity solution which is different from the naive one and, for $n_1 = 1$ is given by

$$\begin{aligned} ds^2_{\text{String}} &= H\left(-dv\,du + K\,dv^2 + A_i\,dx_i\,dv\right) + dx_i\,dx_i + dz_a\,dz_a, \\ B_{uv} &= -\frac{1}{2}\Big(H-1\Big), \qquad B_{vi} = H\,A_i, \\ e^{2\phi} &= H, \\ H &= H(x_i,y,t) = \left(1 + \frac{Q_1}{(x-F(t-y))^2}\right)^{-1}, \\ K &= K(x_i,y,t) = Q_1\,\frac{\dot{F}^2(t-y)}{(x-F(t-y))^2}, \\ A_i &= A_i(x_i,y,t) = -Q_1\,\frac{\dot{F}_i(t-y)}{(x-F(t-y))^2}, \end{aligned} \tag{1.12}$$

where $x_1, \ldots, x_4$ are the coordinates on the $\mathbb{R}^4$, $z_1, \ldots, z_4$ are the ones on the T^4 and we defined the lightcone coordinates $v \equiv t + y$ and $u \equiv t - y$. In the more general case of $n_1 > 1$, with or without multiwound strands of the NS1, generalizations of the above constructions exist [14]. Through the opposite chain of duality transformation we can get back to the D1-D5 duality frame: the resulting geometry is asymptotically flat, smooth and horizonless and corresponds to the 2-charge solutions considered in Chapter 4. 3-charge fuzzball geometries are more difficult to find, as no construction in terms of a vibration profile is possible. Much work has been dedicated to the search for 2 and 3-charge fuzzball solutions in supergravity [32, 36–48, 48–59].

As mentioned before, in the dual CFT description there is a precise way to define statistical ensembles in terms of density matrices.In the context of the fuzzball proposal, it is believed that not all of the entropy of the black hole is accounted for by microstates that are smooth, horizonless geometries in supergravity: other states could be present that have a precise description in the CFT but are not dual to any supergravity solution (e.g. string states).

The bulk fuzzball geometries are generically dual to *heavy* CFT states, defined as states with conformal dimension scaling as $h_H \sim c$. Intuitively, these are the states that generate a strong backreaction on the geometry. As mentioned above, fuzzball microstates that have a dual geometric description are only a subset of the full ensemble: they are the semiclassical ones, understood as coherent states in the dual CFT description [60]. The D1D5 CFT is an orbifold theory consisting in the symmetrized tensor product of N CFT copies, and coherent states correspond to having the same state on many (order $\sim N$) of the copies.

Working with the CFT at the free orbifold point, we have that N copies of the Neveu-Schwarz vacuum of the fermionic sector correspond to global $AdS_3 \times S^3$ in the bulk. The simplest 2-charge geometries can be schematically obtained taking tensor products of Ramond vacua. Obtaining a CFT description for 3-charge microstates proved to be more complicated, but is now known for a class of them, called *superstrata* [58]: these are obtained starting from tensor products of excitations or Ramond vacua, where the excitations are given by the action of modes of CFT operators of small conformal dimension. CFT states dual to fuzzball geometries were studied in [32–35, 37, 38, 48, 58, 61–63].

Another line of research consisted in obtaining microstate geometries perturbatively from string amplitudes in settings in which strings were used to probe the D-brane configurations [19–22].

1.7 Main results and outline of the thesis

This thesis is the result of work focused on holographic computations: we looked for observables that can be computed both on the CFT side and in the bulk and we computed them on both sides of the duality, verifying their agreement on CFT states and bulk geometries dual to each other. This operation is crucial because if any microscopic description of black holes is possible, then there should be observables that are able to distinguish among different microstates. Moreover, the agreement between the CFT and the gravity computations is a strong signal of the duality of the specific CFT states and bulk geometries selected.

The most relevant part of the work involved the computation of correlators among operators in the D1D5 CFT. The correlators considered involve two heavy operators acting as asymptotic states, and one or two light operators acting as probes, with conformal dimension H_L such that $\lim_{c\to\infty} h_L/c = 0$. New CFT technology was developed to carry out the computations, and check were made against results known in the literature. On the bulk side we used various techniques corresponding to the different observables considered. The starting point was always the identification of the correct bulk geometry dual to a CFT state (or vice versa), and of the correct supergravity field dual to the light operator(s) in the correlators. We refer to the specific chapters for a case-by-case explanation.

The main results of the present work move in two directions, corresponding to the kinds of different fundamental objects considered: 3-point functions and 4-point functions.

3-point functions have been used to perform precision holography computations establishing the duality between microstate geometries and CFT states in many more examples than previously known in the literature. Moreover, for the first time the exact duality between 3-charge fuzzball geometries and more involved CFT states has been established: the first examples of CFT states dual to superstrata were found.

The second direction of work involved 4-point functions of the HH-LL class, where two operators are heavy and two are light. This is relevant for black hole physics [64–69], as the light operators can be seen as bulk probes in the background generated by the heavy states: issues like the typicality of the microstates, their distinguishability and the comparison between having pure (heavy) states rather then a thermal ensemble of them can be studied. In this context, the first example is given in which spurious singularities within Virasoro blocks cancel out between different blocks without including $1/c$ corrections to the $c \to \infty$ limit. Moreover, a class of 4-point correlators is discovered which is non-renormalized due to the presence of an affine symmetry: these correspond to having simple heavy states exactly given by the tensor product of the same CFT state on each of the N copies.

This thesis is organised as follows. In Chapter 2 a description as complete as possible of the D1D5 CFT is given, to lay the foundations for all the subsequent results. In particular, we work at the *free orbifold point* of the theory, a zero-coupling limit in which the D1D5 CFT reduces to a collection of free bosons and free fermions. Most operators we'll be interested in are realised in terms of these fundamental fields. Moreover, all the CFT techniques used will be presented, so that the CFT technology can have a self-contained exposition.

Chapter 3 contains a presentation of the bulk side: the Strominger-Vafa solution is presented along with its 2-charge analogue. Then the 2- and 3-charge fuzzball geometries are presented and analysed. In particular the introduction of all the objects in terms of which fuzzball geometries are defined allow for a general treatment of the microstate metrics: specifying a particular forms for those objects is equivalent to specify a fuzzball solution.

Chapter 4 contains the holographic study of classes of 2- and 3-charge microstates through the use of 1-point functions and Entanglement Entropy (EE) as observables. 1-point functions (VEVs) of light operators of small conformal dimension are checked [32, 62, 63] against the (asymptotic expansion of) corresponding objects defining the microstate metrics, where the VEVs are computed between heavy states dual to the bulk metrics. A small interval in the spatial circle of the CFT is then selected as the defining region for the computation of EE, which is carried out on the CFT side using the *replica trick* [70] in terms of the light operators' VEVs. This is checked to agree with the bulk result obtained via the Ryu-Takayanagi holographic prescription [71, 72].

Chapter 5 is devoted to the study of another class of observables, namely CFT 4-point functions of the HH-LL type, where two heavy operators are dual to a microstate bulk geometry and two light operators are dual to probes moving in that background. This time we selected a class of dual geometries simpler than in the previous chapter in that all the geometries can be locally reduced to $AdS_3 \times S^3$ via diffeomorphisms. The heavy CFT states are also simpler, as in the usual basis we use they are not superpositions. This simplicity allows on the one hand to compute the CFT 4-point functions and on the other to analytically solve the wave equation holographically dual to the CFT correlators. The 4-point functions are decomposed into Virasoro blocks and affine blocks, one of the main results being that even at leading order in the $c \to \infty$ expansion spurious singularities of Virasoro blocks [66, 68] cancel out between blocks belonging to different primaries. Finally, an a priori unexpected matching between the CFT and bulk computations is explained in terms of the affine blocks decomposition of the correlators.

Chapter 6 contains the analysis of 4-point functions in the twisted sector of the D1D5 CFT at the free orbifold point. Light operators written in terms of the free bosons are considered as well. Correlators with bosons and fermions are then successfully connected using the Ward identities derived from the $\mathcal{N} = (4, 4)$ supersymmetry structure of the theory.

A conclusion to this work is contained in Chapter 7, while Appendices A, B, C, D, E present various technical results used in the course of the main text.

Chapter 2

The $D1D5$ CFT

2.1 Field content

The CFT relevant for the fuzzball microstates is a 2-dimensional conformal field theory with $\mathcal{N} = (4,4)$ supersymmetry and central charge $c = 6n_1n_5$. Here we follow the conventions of Section 7 of [58] (see the references therein for more details) and visualize the CFT at the free orbifold point as a collection of $N \equiv n_1n_5$ strings (or "strands"), each one with four bosons and four doublets of fermions

$$\Big(X^{\dot{A}A}_{(r)}(\tau,\sigma)\,,\psi^{\alpha\dot{A}}_{(r)}(\tau+\sigma)\,,\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\tau-\sigma)\Big)\,, \tag{2.1}$$

where $r = 1, \ldots, N$ runs over the different strings and (τ, σ) are the timelike and the spacelike directions in the CFT, which in our conventions will correspond to the directions t and y on the bulk side. Here $\alpha, \dot{\alpha} = \pm$ are spinorial indices[4] for the R-symmetry group $SU(2)_L \times SU(2)_R$ which is identified with the rotations in the S^3 factor of the bulk metric (which acts as an internal space from the point of view of the CFT), while $A, \dot{A} = 1, 2$ are indices for the $SU(2)_1 \times SU(2)_2 = SO(4)_I$ rotations acting on the tangent space in the compact manifold T^4. Here τ identifies the CFT time coordinate in Lorentzian signature. We can Wick rotate to Euclidean space by taking

$$\tau \to -\mathrm{i}\tau_E$$

so that now holomorphic and antiholomorphic fermions can be written as functions

$$\psi^{\alpha\dot{A}}_{(r)}(\tau_E + \mathrm{i}\sigma), \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\tau_E - \mathrm{i}\sigma).$$

[4]We sometimes use the notation $\alpha, \dot{\alpha} = 1, 2$ with the identifications $1 \equiv +$, $2 \equiv -$. In terms of the eigenvalues of the J^3 and $\tilde{J}^3$ operators, defined below, these of course just correspond to $\pm 1/2$.

Since the boundary of AdS_3 is a cylinder, σ will naturally be a periodic coordinate, with periodicity

$$\sigma \sim \sigma + 2\pi. \tag{2.2}$$

We can then change coordinates from the (τ_E, σ) cylinder to the complex plane by defining

$$z = e^{\tau_E + \mathrm{i}\sigma}, \qquad \bar{z} = e^{\tau_E - \mathrm{i}\sigma}. \tag{2.3}$$

To avoid confusion, it's useful to define z and $\bar{z}$ only in Euclidean signature: we can always go back to the coordinates on the cylinder and Wick rotate back to Lorentzian signature when needed (e.g. to compare CFT and gravity results). The fields become

$$\psi^{\alpha\dot{A}}_{(r)}(z), \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z}).$$

Writing the bosons in the complex plane coordinates we have that their derivatives within respect to z and $\bar{z}$ are respectively holomorphic and antiholomorphic,

$$\partial X^{A\dot{A}}_{(r)}(z), \qquad \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}),$$

where $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. The $N = n_1 n_5$ CFT copies are all independent from each other, and we see that each of them brings a contribution $c_{1\text{ strand}} = \tilde{c}_{1\text{ strand}} = 6$ to the total holomorphic and antiholomorphic central charges (corresponding to 4 bosons and 4 fermions). Each fermion can also take another name according to the dictionary

$$\chi^1 = -i\psi^{1\dot{1}}, \qquad \overline{\chi}^1 = -i\psi^{2\dot{2}}, \tag{2.4a}$$
$$\chi^2 = \psi^{1\dot{2}}, \qquad \overline{\chi}^2 = \psi^{2\dot{1}}, \tag{2.4b}$$
$$\tilde{\chi}^1 = -i\tilde{\psi}^{\dot{1}\dot{1}}, \qquad \overline{\tilde{\chi}}^1 = -i\tilde{\psi}^{\dot{2}\dot{2}}, \tag{2.4c}$$
$$\tilde{\chi}^2 = \tilde{\psi}^{\dot{1}\dot{2}}, \qquad \overline{\tilde{\chi}}^2 = \tilde{\psi}^{\dot{2}\dot{1}}. \tag{2.4d}$$

Fermions and bosons are all chiral primaries operators (CPO).

The orbifold nature of the theory comes from the fact that all the states (or, dually, the operators) must be invariant under the action of an S_N group acting on the CFT copies: everything must be invariant under permutations of the N strands.

2.2 The untwisted ($k = 1$) sector

2.2.1 Boundary conditions, mode expansions and mode algebras

Let's consider N completely independent strands: this is called the **untwisted sector** of the theory and we can think of it as a collection of N strands of length $k = 1$

(or "singly wound"), in contrast with the twisted sector, in which multiple strands of length 1 are glued together to form longer ($k > 1$) ones ("multiply wound"). Having strands of length 1 simply means that the boundary (monodromy) conditions for the fields have to be imposed upon taking $\sigma \to \sigma + 2\pi$ (on the cylinder) or $z \to e^{2\pi \mathrm{i}} z$ (on the complex plane). As we will see, something different happens with $k > 1$.

The OPE of the fermions and of the bosons are, respectively,

$$\psi^{1\dot{A}}_{(r)}(z)\,\psi^{2\dot{B}}_{(s)}(w) = \frac{\epsilon^{\dot{A}\dot{B}}\,\delta_{rs}}{z-w} + [\text{reg.}]\,, \tag{2.5a}$$

$$\tilde{\psi}^{\dot{1}\dot{A}}_{(r)}(\bar{z})\,\tilde{\psi}^{\dot{2}\dot{B}}_{(s)}(\bar{w}) = \frac{\epsilon^{\dot{A}\dot{B}}\,\delta_{rs}}{\bar{z}-\bar{w}} + [\text{reg.}]\,, \tag{2.5b}$$

$$\partial X^{A\dot{A}}_{(r)}(z)\,\partial X^{B\dot{B}}_{(s)}(w) = \frac{\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\,\delta_{rs}}{(z-w)^2} + [\text{reg.}]\,, \tag{2.5c}$$

$$\bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z})\,\bar{\partial} X^{B\dot{B}}_{(s)}(\bar{w}) = \frac{\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\,\delta_{rs}}{(\bar{z}-\bar{w})^2} + [\text{reg.}]\,. \tag{2.5d}$$

where $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{A}\dot{B}}$ are the totally antisymmetric symbols and we use the convention $\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = -\epsilon^{12} = -\epsilon^{\dot{1}\dot{2}} = 1$ (and $\epsilon_{AB}\,\epsilon^{BC} = \delta^C_A$). Using the χ fermions, the OPEs are

$$\chi^i_{(r)}(z)\,\bar{\chi}^j_{(s)}(w) = \frac{\delta^{ij}\,\delta_{rs}}{z-w}, \qquad \tilde{\chi}^i_{(r)}(\bar{z})\,\bar{\tilde{\chi}}^j_{(s)}(\bar{w}) = \frac{\delta^{ij}\,\delta_{rs}}{\bar{z}-\bar{w}}. \tag{2.6}$$

The boundary conditions for the bosons are periodic,

$$X^{A\dot{A}}_{(r)}(\tau_E, \sigma + 2\pi) = X^{A\dot{A}}_{(r)}(\tau_E, \sigma), \tag{2.7}$$

or, working with chiral bosons on the complex plane,

$$\partial X^{A\dot{A}}_{(r)}\left(e^{2\pi \mathrm{i}} z\right) = \partial X^{A\dot{A}}_{(r)}(z), \qquad \bar{\partial} X^{A\dot{A}}_{(r)}\left(e^{-2\pi \mathrm{i}} \bar{z}\right) = \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}). \tag{2.8}$$

Fermions on the other hand can have either Ramond (R) or Neveu-Schwarz (NS) boundary conditions, corresponding to the following scheme,

	Cylinder	$\mathbb{C}$ plane
R sector	periodic	antiperiodic
NS sector	antiperiodic	periodic

which on the cylinder correspond to

- Ramond: $\psi^{\alpha\dot{A}}_{(r)}(\tau, \sigma + 2\pi) = \psi^{\alpha\dot{A}}_{(r)}(\tau, \sigma)$,
- Neveu-Schwarz: $\psi^{\alpha\dot{A}}_{(r)}(\tau, \sigma + 2\pi) = -\psi^{\alpha\dot{A}}_{(r)}(\tau, \sigma)$,

and on the complex plane correspond to

- Ramond: $\psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) = -\psi^{\alpha\dot{A}}_{(r)}(z)$,
- Neveu-Schwarz: $\psi^{\alpha\dot{A}}_{(r)}(e^{2\pi i}z) = \psi^{\alpha\dot{A}}_{(r)}(z)$.

The boundary conditions are reflected in the mode expansions of the fields. For the bosons we have

$$\partial X^{A\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \alpha^{A\dot{A}}_{(r)\,n}\, z^{-n-1}, \qquad \bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z}) = \sum_{n\in\mathbb{Z}} \tilde{\alpha}^{A\dot{A}}_{(r)\,n}\, \bar{z}^{-n-1}, \tag{2.9}$$

while for the fermions in the **R sector** we have

$$\psi^{\alpha\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \psi^{\alpha\dot{A}}_{(r)\,n}\, z^{-n-\frac{1}{2}}, \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z}) = \sum_{n\in\mathbb{Z}} \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)\,n}\, \bar{z}^{-n-\frac{1}{2}}, \tag{2.10}$$

and in the **NS sector** we have

$$\psi^{\alpha\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{\alpha\dot{A}}_{(r)\,n}\, z^{-n-\frac{1}{2}}, \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z}) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)\,n}\, \bar{z}^{-n-\frac{1}{2}}. \tag{2.11}$$

The mode expansions for the χ fermions in the two sectors are the same as the above, i.e. in the **R sector** we have

$$\begin{aligned} \chi^{i}_{(r)}(z) &= \sum_{n\in\mathbb{Z}} \chi^{i}_{(r)\,n} z^{-n-\frac{1}{2}}, & \bar{\chi}^{i}_{(r)}(z) &= \sum_{n\in\mathbb{Z}} \bar{\chi}^{i}_{(r)\,n} z^{-n-\frac{1}{2}}, \\ \tilde{\chi}^{i}_{(r)}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \tilde{\chi}^{i}_{(r)\,n} \bar{z}^{-n-\frac{1}{2}}, & \bar{\tilde{\chi}}^{i}_{(r)}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \bar{\tilde{\chi}}^{i}_{(r)\,n} \bar{z}^{-n-\frac{1}{2}}, \end{aligned} \tag{2.12}$$

whereas in the **NS sector** we have

$$\begin{aligned} \chi^{i}_{(r)}(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \chi^{i}_{(r)\,n} z^{-n-\frac{1}{2}}, & \bar{\chi}^{i}_{(r)}(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \bar{\chi}^{i}_{(r)\,n} z^{-n-\frac{1}{2}}, \\ \tilde{\chi}^{i}_{(r)}(\bar{z}) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{\chi}^{i}_{(r)\,n} \bar{z}^{-n-\frac{1}{2}}, & \bar{\tilde{\chi}}^{i}_{(r)}(\bar{z}) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \bar{\tilde{\chi}}^{i}_{(r)\,n} \bar{z}^{-n-\frac{1}{2}}. \end{aligned} \tag{2.13}$$

The OPEs (2.5) imply that the nonzero commutation relations of the bosonic modes are

$$\left[\alpha^{A\dot{A}}_{(r)\,n}, \alpha^{B\dot{B}}_{(s)\,m}\right] = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\, n\, \delta_{n+m,0}\, \delta_{rs}, \qquad \left[\tilde{\alpha}^{A\dot{A}}_{(r)\,n}, \tilde{\alpha}^{B\dot{B}}_{(s)\,m}\right] = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\, n\, \delta_{n+m,0}\, \delta_{rs}, \tag{2.14}$$

while while the nonzero anticommutation relations for the fermions (in both sectors) are

$$\left\{\psi^{1\dot{A}}_{(r)\,n}, \psi^{2\dot{B}}_{(s)\,m}\right\} = \epsilon^{\dot{A}\dot{B}}\, \delta_{n+m,0}\, \delta_{rs}, \qquad \left\{\tilde{\psi}^{\dot{1}\dot{A}}_{(r)\,n}, \tilde{\psi}^{\dot{2}\dot{B}}_{(s)\,m}\right\} = \epsilon^{\dot{A}\dot{B}}\, \delta_{n+m,0}\, \delta_{rs}. \tag{2.15}$$

The nonzero anticommutation relations for the modes of the χ fermions follow from those of the ψ fermions,

$$\left\{\chi^i_{(r)\,n}, \bar{\chi}^j_{(s)\,m}\right\} = \delta^{ij}\,\delta_{n+m,0}\,\delta_{rs}, \qquad \left\{\tilde{\chi}^i_{(r)\,n}, \bar{\tilde{\chi}}^j_{(s)\,m}\right\} = \delta^{ij}\,\delta_{n+m,0}\,\delta_{rs}. \tag{2.16}$$

From (2.14) we see that the rule for Hermitian conjugation of the modes of the bosons is

$$\left(\alpha^{A\dot{A}}_{(r)\,n}\right)^\dagger = \epsilon^{AB}\,\epsilon^{\dot{A}\dot{B}}\,\alpha^{B\dot{B}}_{(r)\,-n}, \qquad \left(\tilde{\alpha}^{A\dot{A}}_{(r)\,n}\right)^\dagger = \epsilon^{AB}\,\epsilon^{\dot{A}\dot{B}}\,\tilde{\alpha}^{B\dot{B}}_{(r)\,-n}, \tag{2.17}$$

while from (2.15) we see that for the fermions we have

$$\left(\psi^{\alpha\dot{A}}_{(r)\,n}\right)^\dagger = -\epsilon^{\alpha\beta}\,\epsilon^{\dot{A}\dot{B}}\,\psi^{\beta\dot{B}}_{(r)\,-n}, \qquad \left(\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)\,n}\right)^\dagger = -\epsilon^{\dot{\alpha}\dot{\beta}}\,\epsilon^{\dot{A}\dot{B}}\,\tilde{\psi}^{\dot{\beta}\dot{B}}_{(r)\,-n} \tag{2.18}$$

The rules for Hermitian conjugation of the fields are

$$\begin{aligned}
\left(\partial X^{A\dot{A}}_{(r)}(z)\right)^\dagger &= \epsilon^{AB}\,\epsilon^{\dot{A}\dot{B}}\,z^{-2}\partial X^{B\dot{B}}_{(r)}(1/z), \qquad &\left(\bar{\partial} X^{A\dot{A}}_{(r)}(\bar{z})\right)^\dagger &= \epsilon^{AB}\,\epsilon^{\dot{A}\dot{B}}\,\bar{z}^{-2}\partial X^{B\dot{B}}_{(r)}(1/\bar{z}),\\
\left(\psi^{\alpha\dot{A}}_{(r)}(z)\right)^\dagger &= -\epsilon_{\alpha\beta}\,\epsilon_{\dot{A}\dot{B}}\,z^{-1}\psi^{\beta\dot{B}}_{(r)}(1/z), \qquad &\left(\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z})\right)^\dagger &= -\epsilon_{\alpha\beta}\,\epsilon_{\dot{A}\dot{B}}\,\bar{z}^{-1}\tilde{\psi}^{\dot{\beta}\dot{B}}_{(r)}(1/\bar{z}),
\end{aligned} \tag{2.19}$$

The mode expansions for the fields on the cylinder won't be needed, but they can be obtained from the above knowing that the transformation from the complex plane to the cylinder is conformal and that the bosons and fermions are chiral primary operators, i.e. they transform under a conformal transformation $z \to w$ as

$$\mathcal{O}(w) = \left(\frac{\partial w}{\partial z}\right)^{-h} \mathcal{O}(z(w)), \qquad \mathcal{O}(\bar{w}) = \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \mathcal{O}(\bar{z}(\bar{w})), \tag{2.20}$$

where h is the left (holomorphic) conformal dimension of the field and $\bar{h}$ the right (antiholomorphic) one.

2.2.2 Vacuum states

The $D1D5$ CFT at the free orbifold point is a free theory, and this is reflected by the properties of the vacuum. In each CFT copy the vacuum state is the tensor product of a vacuum state for the bosons and one for the fermions, and each is in turn also a product of a vacuum state for the holomorphic sector and one for the antiholomorphic sector.

The bosonic vacuum state on the CFT copy (r), $|0\rangle_{(r)}$, is annihilated by all the positive modes of the bosons,

$$\alpha^{A\dot{A}}_{(r)\,n}|0\rangle_{(r)} = 0, \qquad \tilde{\alpha}^{A\dot{A}}_{(r)\,n}|0\rangle_{(r)} = 0, \quad \forall n \geq 0, \quad \forall A, \dot{A}. \tag{2.21}$$

It is not strictly necessary that the bosons' zero modes annihilate the vacuum, but if it wasn't so that would signal the presence of momentum along one of the directions of the T^4, which is a charge that we want to keep switched off.
As mentioned above, rigorously the two $|0\rangle$ states above are not the same state, as the one acted upon by the left-movers ($\alpha^{A\dot{A}}_{(r)\,n}$ modes) is the left bosonic vacuum while the one acted upon by the right-movers ($\tilde{\alpha}^{A\dot{A}}_{(r)\,n}$ modes) is the right bosonic vacuum. The left and right modes commute with each other and we will not distinguish between left and right vacuum, as it is implied the total vacuum is a product of the two. We assume the normalization

$$ {}_{(r)}\langle 0|0\rangle_{(s)} = \delta_{r,s}. \tag{2.22}$$

The properties of the vacuum state(s) for the fermions strongly differ between the R and NS sectors. In the **NS sector** the rule is analogous to what happens for bosons: the vacuum $|0\rangle_{(r),\mathrm{NS}}$ on the CFT copy (r) is annihilated by all the positive modes of the fermions

$$\psi^{\alpha\dot{A}}_{(r)\,n}|0\rangle_{(r),\mathrm{NS}} = 0, \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)\,n}|0\rangle_{(r),\mathrm{NS}} = 0, \quad \forall n>0, \quad \forall \alpha,\dot{\alpha},\dot{A}, \tag{2.23}$$

where again we don't make a distinction between left and right vacuum.
In the **R sector** things work differently, as the fermions do have zero modes in their expansions, and half of them annihilate the vacuum while the other half doesn't. As before, the vacuum $|0\rangle_{(r),\mathrm{R}}$ is annihilated by all the positive modes,

$$\psi^{\alpha\dot{A}}_{(r)\,n}|0\rangle_{(r),\mathrm{R}} = 0, \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)\,n}|0\rangle_{(r),\mathrm{R}} = 0, \quad \forall n>0, \quad \forall \alpha,\dot{\alpha},\dot{A}, \tag{2.24}$$

but this time we can define a Ramond vacuum state $|++\rangle_{(r)}$ such that

$$\psi^{1\dot{A}}_{(r)\,0}|++\rangle_{(r),\mathrm{R}} = 0, \qquad \tilde{\psi}^{\dot{1}\dot{A}}_{(r)\,0}|++\rangle_{(r),\mathrm{R}} = 0. \tag{2.25}$$

The fact that we can act with zero modes of the fermions $\psi^{2\dot{A}}$ and $\tilde{\psi}^{\dot{2}\dot{A}}$ on $|++\rangle_{(r)}$ without annihilating it means that we have a family of degenerate vacua (acting with negative modes on the other hand would raise the energy). The modes that create degenerate vacua are then

$$\psi^{2\dot{A}}_{(r)\,0}, \quad \tilde{\psi}^{\dot{2}\dot{A}}_{(r)\,0} \qquad \text{or} \qquad \bar{\chi}^{i}_{(r)\,0}, \quad \bar{\tilde{\chi}}^{i}_{(r)\,0}.$$

The total fermionic vacuum state for the CFT will have a structure like

$$\bigotimes_{r=1}^{N} |++\rangle_{(r)} \equiv |++\rangle^{N}. \tag{2.26}$$

We assume the vacuum states in the R and NS sectors are normalized,

$$
{}_{\mathrm{R},(r)}\langle + + | + + \rangle_{(s),\mathrm{R}} = \delta_{r,s}, \qquad {}_{\mathrm{NS},(r)}\langle 0|0\rangle_{(s),\mathrm{NS}} = \delta_{r,s}. \tag{2.27}
$$

2.2.3 Other operators

We want to focus on operators of small conformal dimension. Among the operators with total conformal dimension $h + \bar{h} = 1$ we have, on each strand of length 1, the current operators,

$$
\begin{aligned}
J^{\alpha\beta}_{(r)}(z) &= \frac{1}{2} : \psi^{\alpha\dot{A}}_{(r)}\, \epsilon_{\dot{A}\dot{B}}\, \psi^{\beta\dot{B}}_{(r)} : (z)\,, \\
\tilde{J}^{\dot{\alpha}\dot{\beta}}_{(r)}(\bar{z}) &= \frac{1}{2} : \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}\, \epsilon_{\dot{A}\dot{B}}\, \tilde{\psi}^{\dot{\beta}\dot{B}}_{(r)} : (\bar{z})\,,
\end{aligned} \tag{2.28}
$$

with dimensions $(h, \bar{h}) = (1, 0)$ and $(h, \bar{h}) = (0, 1)$ respectively. Here $: \mathcal{O}_1 \cdots \mathcal{O}_n :$ denotes the normal ordering of the product of n operators within respect to the $|{+}{+}\rangle_{(r)}$ vacuum. The current operators corresponding to the R-symmetry group $SU(2)_L \times SU(2)_R$ are just sums of the above ones on the N CFT copies[5],

$$
J^{\alpha\beta}(z) = \sum_{r=1}^{N} J^{\alpha\beta}_{(r)}(z), \qquad \tilde{J}^{\dot{\alpha}\dot{\beta}}(\bar{z}) = \sum_{r=1}^{N} J^{\dot{\alpha}\dot{\beta}}_{(r)}(\bar{z}). \tag{2.30}
$$

$J^{\alpha\beta}$ and $\tilde{J}^{\alpha\beta}$ have the same conformal dimensions as $J^{\alpha\beta}_{(r)}$ and $\tilde{J}^{\alpha\beta}_{(r)}$, respectively. The standard $SU(2)$ generators in the holomorphic part of the R sector are

$$
J^{+}_{(r)} = \frac{1}{2} : \psi^{1\dot{A}}_{(r)}\psi^{1\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}} := \mathrm{i} : \chi^{1}_{(r)}\chi^{2}_{(r)} :\equiv J^{1}_{(r)} + \mathrm{i}J^{2}_{(r)}, \tag{2.31a}
$$

$$
J^{-}_{(r)} = -\frac{1}{2} : \psi^{2\dot{A}}_{(r)}\psi^{2\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}} := \mathrm{i} : \bar{\chi}^{1}_{(r)}\bar{\chi}^{2}_{(r)} :\equiv J^{1}_{(r)} - \mathrm{i}J^{2}_{(r)}, \tag{2.31b}
$$

$$
J^{3}_{(r)} = -\frac{1}{2}\Big(: \psi^{1\dot{A}}_{(r)}\psi^{2\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}} : -1 \Big) = \frac{1}{2}\Big(: \chi^{1}_{(r)}\bar{\chi}^{1}_{(r)} : + : \chi^{2}_{(r)}\bar{\chi}^{2}_{(r)} : -1 \Big). \tag{2.31c}
$$

Analogous definitions hold for the antiholomorphic generators. The constant term in $J^3_{(r)}$ has been fixed in such a way that the $|{+}{+}\rangle_{(r)}$ state has quantum numbers $(1/2, 1/2)$ under (the zero modes of) $(J^3_{(r)}, \tilde{J}^3_{(r)})$. Denoting the current operators as $J^a_{(r)}(z), a = 1, 2, 3$, we have the OPE rule

$$
J^{a}_{(r)}(z)\, J^{b}_{(s)}(w) = \frac{\delta_{r,s}}{z - w}\, \mathrm{i}\, \epsilon^{abc}\, J^{c}_{(r)}(w) + [\text{reg.}], \tag{2.32}
$$

[5] In general, when an operator is written as a sum of operators acting on different copies, $\mathcal{O} = \sum_r \mathcal{O}_{(r)}$ we mean that $\mathcal{O}_r$ acts nontrivially on the r-th copy, and as the identity on all the other copies,

$$
\mathcal{O}_{(r)} \equiv \mathbb{1}_{(1)} \otimes \cdots \otimes \mathbb{1}_{(r-1)} \otimes \mathcal{O}_{(r)} \otimes \mathbb{1}_{(r+1)} \otimes \cdots \otimes \mathbb{1}_{(N)}. \tag{2.29}
$$

where ϵ^{abc} is totally antisymmetric and $\epsilon^{123}=1$. The mode expansions are

$$J^a_{(r)}(z)=\sum_{n\in\mathbb{Z}} J^a_{(r)\,n}\,z^{-n-1}, \tag{2.33}$$

and the OPE rule is reflected by the mode algebra

$$\left[J^a_{(r)\,n},J^b_{(s)\,m}\right]=\mathrm{i}\,\epsilon^{abc}\,J^c_{(r)\,n+m}\,\delta_{r,s}+\frac{c_{1\text{ copy}}}{12}\,n\,\delta^{ab}\,\delta_{r,s}\,\delta_{m+n,0}. \tag{2.34}$$

We use the zero modes of the currents to define other R vacua with different spin

$$|{-}{+}\rangle_{(r)}\equiv J^-_{(r)\,0}|{+}{+}\rangle_{(r)}\;,\quad |{+}{-}\rangle_{(r)}\equiv \tilde{J}^-_{(r)\,0}|{+}{+}\rangle_{(r)}\;,\quad |{-}{-}\rangle_{(r)}\equiv J^-_{(r)\,0}\tilde{J}^-_{(r)\,0}|{+}{+}\rangle_{(r)}\;. \tag{2.35}$$

Another family of operators of total conformal dimension 1 we are going to consider is given on a single strand by

$$O^{\alpha\dot{\alpha}}_{(r)}(z,\bar{z})\equiv\frac{-\mathrm{i}}{\sqrt{2}}:\psi^{\alpha\dot{A}}_{(r)}\,\epsilon_{\dot{A}\dot{B}}\,\tilde{\psi}^{\dot{\alpha}\dot{B}}_{(r)}:(z,\bar{z})=\sum_{n,m\in\mathbb{Z}} O^{\alpha\dot{\alpha}}_{(r)\,mn}\,z^{-n-\frac{1}{2}}\,\bar{z}^{-m-\frac{1}{2}}, \tag{2.36}$$

which correspond to the operators $O^{(1,1)}_{(1)1i}$ in the notation of [32] and have conformal dimension $(h,\bar{h})=(1/2,1/2)$. The action of the operator (2.36) on the $|{+}{+}\rangle$ state generates another R vacuum that plays an important role both in the examples discussed in [32] and in this work (see [1]),

$$|00\rangle_{(r)}\equiv\lim_{z\to 0}O^{2\dot{2}}_{(r)}(z,\bar{z})|{+}{+}\rangle_{(r)}=O^{2\dot{2}}_{(r)\,00}|{+}{+}\rangle_{(r)}=\frac{-\mathrm{i}}{\sqrt{2}}\,\psi^{2\dot{A}}_{(r)\,0}\,\epsilon_{\dot{A}\dot{B}}\,\tilde{\psi}^{\dot{2}\dot{B}}_{(r)\,0}|{+}{+}\rangle_{(r)}\;, \tag{2.37}$$

which has spin $(0,0)$ under $(J^3_{(r)},\tilde{J}^3_{(r)})$. As for the $SU(2)$ currents, the operator corresponding to (2.36) for the whole CFT (in the untwisted sector) is a sum over the copies,

$$O^{\alpha\dot{\alpha}}(z,\bar{z})=\sum_{r=1}^{N}O^{\alpha\dot{\alpha}}_{(r)}(z,\bar{z}). \tag{2.38}$$

The conjugation relations among the $O^{\alpha\dot{\alpha}}$ are

$$\left(O^{1\dot{1}}\right)^\dagger=O^{2\dot{2}},\qquad \left(O^{1\dot{2}}\right)^\dagger=-O^{2\dot{1}}. \tag{2.39}$$

The normalization of all the vacua obtained acting with operators on $|{+}{+}\rangle$ follows form that of $|{+}{+}\rangle$ itself and from the commutation relations of the operators,

$${}_{(r)}\langle S|S'\rangle_{(s)}=\delta_{r,s}\,\delta_{S,S'}, \tag{2.40}$$

where S,S' can take the values $\pm\pm$ or 00.

The stress-energy operator for the free orbifold theory is written in terms of bosons and the fermions,

$$T(z) = T_B(z) + T_F(z) = \sum_{r=0}^{N} T_{(r)}(z) = \sum_{n\in\mathbb{Z}} L_n \, z^{-n-2}, \tag{2.41}$$

with

$$T_{(r)}(z) = T^B_{(r)}(z) + T^F_{(r)}(z) = \sum_{n\in\mathbb{Z}} L_{(r)\,n} \, z^{-n-2}, \tag{2.42a}$$

$$T^B_{(r)}(z) = \frac{1}{2}\,\epsilon_{AB}\epsilon_{\dot{A}\dot{B}} : \partial X^{A\dot{A}}_{(r)}(z)\partial X^{B\dot{B}}_{(r)}(z) := \sum_{n\in\mathbb{Z}} L^B_{(r)\,n} \, z^{-n-2}, \tag{2.42b}$$

$$T^F_{(r)}(z) = \frac{1}{2}\,\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}} : \psi^{\alpha\dot{A}}_{(r)}(z)\partial\psi^{\beta\dot{B}}_{(r)}(z) := \sum_{n\in\mathbb{Z}} L^F_{(r)\,n} \, z^{-n-2}. \tag{2.42c}$$

The modes of $T_{(r)}(z)$ generate the Virasoro algebra on each CFT copy,

$$\left[L_{(r)\,n}, L_{(s)\,m}\right] = (n-m)\, L_{(r)\,n+m}\,\delta_{r,s} - \frac{c_{1\text{ copy}}}{12}\, n\left(n^2-1\right)\delta_{n+m,0}\,\delta_{r,s}. \tag{2.43}$$

Using the OPE rules (2.5) for the free bosons and the free fermions, $T(z)$ can be checked to have the usual OPE with itself,

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + [\text{reg.}]\,. \tag{2.44}$$

c is the central charge of the orbifold theory, and because $T(z)$ is the sum of operators $T_{(r)}(z)$ acting on one CFT copy each, it is given by the sum over the copies of the central charge of each copy,

$$c = [\#\text{ copies}] \times c_{1\text{ copy}} = 6N = 6\,n_1 n_5. \tag{2.45}$$

The explicit form of the modes is obtained using

$$L_{(r)\,n} = \oint_{z\sim 0} \frac{dz}{2\pi\mathrm{i}}\, z^{n+1}\, T_{(r)}(z), \tag{2.46}$$

where the contour runs around $z = 0$ counter-clockwise, and the result is

$$L_{(r)\,n} = L^B_{(r)\,n} + L^F_{(r)\,n}, \tag{2.47a}$$

$$L^B_{(r)\,n} = \oint_{z\sim 0} \frac{dz}{2\pi\mathrm{i}}\, z^{n+1}\, T^B_{(r)}(z) = \frac{1}{2}\,\epsilon_{AB}\epsilon_{\dot{A}\dot{B}} \sum_{m\in\mathbb{Z}} : \alpha^{A\dot{A}}_{(r)\,m}\alpha^{B\dot{B}}_{(r)\,n-m} :, \tag{2.47b}$$

$$L^F_{(r)\,n} = \oint_{z\sim 0} \frac{dz}{2\pi \mathrm{i}}\, z^{n+1}\, T^F_{(r)}(z) = \frac{1}{2}\,\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}} \sum_{m\in\mathbb{Z}} \left(-n+m-\frac{1}{2}\right) : \psi^{\alpha\dot{A}}_{(r)\,m}\psi^{\beta\dot{B}}_{(r)\,n-m} :, \tag{2.47c}$$

where we considered **R sector** fermions (for the NS sector ones we would just have a sum over $m \in \mathbb{Z} + 1/2$). The modes of the total stress energy operator (2.41) are obtained in terms of the above summing over the CFT copies,

$$L_n = \sum_{r=1}^{N} L_{(r)\,n} = \sum_{r=1}^{N} \left(L^B_{(r)\,n} + L^F_{(r)\,n}\right). \tag{2.48}$$

The currents and the stress-energy operator have a nontrivial OPE,

$$J^a(z)\, T(w) = \frac{J^a(w)}{z-w} + [\text{reg.}], \tag{2.49}$$

which generates nontrivial commutation relations among the modes,

$$\left[J^a_{(r)\,n}, L_{(s)\,m}\right] = n\, J^a_{n+m}\, \delta_{r,s}. \tag{2.50}$$

2.3 The twisted ($k > 1$) sector

2.3.1 Boundary conditions, mode expansions and mode algebras

So far we have looked at a collection of N strands of length $k = 1$, i.e. N independent CFT copies with fields' boundary conditions imposed as the cylinder coordinate undergoes the transformation $\sigma \to \sigma + 2\pi$ (or alternatively as the complex coordinate z is taken around the origin once). This is not the only possibility, as we can sew together k strands of length 1 in a single strand of length k, in which the fields' monodromies are nontrivial. In the most general case, we may have M strands with lengths $k_1, \ldots, k_M$ such that $\sum_{i=1}^{M} k_i = N$.

Let's consider a single strand of length k. Working on the complex plane, the rule is that the k CFT copies sewn together work as different Riemann sheets upon sending $z \to e^{2\pi\mathrm{i}}z$, so taking z around the origin once fields in one copy get mapped to fields in the adjacent copy among the k. For the bosons, this is

$$\partial X^{A\dot{A}}_{(r)}\left(e^{2\pi\mathrm{i}}z\right) = \partial X^{A\dot{A}}_{(r+1)}(z), \qquad \bar{\partial} X^{A\dot{A}}_{(r)}\left(e^{-2\pi\mathrm{i}}\bar{z}\right) = \bar{\partial} X^{A\dot{A}}_{(r+1)}(\bar{z}) \tag{2.51}$$

with the identification $\partial X^{A\dot{A}}_{(k+1)} \equiv \partial X^{A\dot{A}}_{(1)}$ and $\bar{\partial} X^{A\dot{A}}_{(k+1)} \equiv \bar{\partial} X^{A\dot{A}}_{(1)}$. The rule for antiholomorphic fields follows from the one for holomorphic fields knowing that in Euclidean signature $\bar{z}$ is actually the complex conjugate of z. Because of this, we impose that if for holomorphic fields sending $z \to e^{2\pi\mathrm{i}}z$ enforces the jump $(r) \to (r+1)$, than the same thing happens to antiholomorphic fields taking the "complex conjugate" of the same operation, i.e. sending $\bar{z} \to e^{-2\pi\mathrm{i}}\bar{z}$ enforces the copy jump $(r) \to (r+1)$.

We see that the boundary conditions are not diagonal, in the sense that upon taking z around the origin a field on the copy (r) is not mapped to something proportional to the field itself. Is it nevertheless possible to diagonalize the boundary conditions by taking linear combinations of the fields on different copies: it's essentially a change of basis from the basis labelled by $(r) = 1, \ldots, k$ to another basis labelled by another index $\rho = 0, \ldots, k-1$. For the bosons, this is

$$\partial X^{1\dot{1}}_{\rho}(z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\, \partial X^{1\dot{1}}_{(r)}(z), \quad \partial X^{2\dot{2}}_{\rho}(z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{r\rho}{k}}\, \partial X^{2\dot{2}}_{(r)}(z), \tag{2.52a}$$

$$\partial X^{1\dot{2}}_{\rho}(z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{r\rho}{k}}\, \partial X^{1\dot{2}}_{(r)}(z), \quad \partial X^{2\dot{1}}_{\rho}(z) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\, \partial X^{2\dot{1}}_{(r)}(z), \tag{2.52b}$$

$$\bar{\partial} X^{1\dot{1}}_{\rho}(\bar{z}) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{r\rho}{k}}\, \bar{\partial} X^{1\dot{1}}_{(r)}(\bar{z}), \quad \bar{\partial} X^{2\dot{2}}_{\rho}(\bar{z}) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\, \bar{\partial} X^{2\dot{2}}_{(r)}(\bar{z}), \tag{2.52c}$$

$$\bar{\partial} X^{1\dot{2}}_{\rho}(\bar{z}) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\, \bar{\partial} X^{1\dot{2}}_{(r)}(\bar{z}), \quad \bar{\partial} X^{2\dot{1}}_{\rho}(\bar{z}) = \frac{1}{\sqrt{k}} \sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{r\rho}{k}}\, \bar{\partial} X^{2\dot{1}}_{(r)}(\bar{z}), \tag{2.52d}$$

with the (diagonalized) monodromy conditions in the ρ basis now being

$$\partial X^{1\dot{1}}_{\rho}\left(e^{2\pi\mathrm{i}}z\right) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\, \partial X^{1\dot{1}}_{\rho}(z), \qquad \partial X^{2\dot{2}}_{\rho}\left(e^{2\pi\mathrm{i}}z\right) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\, \partial X^{2\dot{2}}_{\rho}(z), \tag{2.53a}$$

$$\partial X^{1\dot{2}}_{\rho}\left(e^{2\pi\mathrm{i}}z\right) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\, \partial X^{1\dot{2}}_{\rho}(z), \qquad \partial X^{2\dot{1}}_{\rho}\left(e^{2\pi\mathrm{i}}z\right) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\, \partial X^{2\dot{1}}_{\rho}(z), \tag{2.53b}$$

$$\bar{\partial} X^{1\dot{1}}_{\rho}(e^{-2\pi\mathrm{i}}\,\bar{z}) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\, \bar{\partial} X^{1\dot{1}}_{\rho}(\bar{z}), \qquad \bar{\partial} X^{2\dot{2}}_{\rho}(e^{-2\pi\mathrm{i}}\,\bar{z}) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\, \bar{\partial} X^{2\dot{2}}_{\rho}(\bar{z}), \tag{2.53c}$$

$$\bar{\partial} X^{1\dot{2}}_{\rho}(e^{-2\pi\mathrm{i}}\,\bar{z}) = e^{2\pi\mathrm{i}\frac{\rho}{k}}\, \bar{\partial} X^{1\dot{2}}_{\rho}(\bar{z}), \qquad \bar{\partial} X^{2\dot{1}}_{\rho}(e^{-2\pi\mathrm{i}}\,\bar{z}) = e^{-2\pi\mathrm{i}\frac{\rho}{k}}\, \bar{\partial} X^{2\dot{1}}_{\rho}(\bar{z}). \tag{2.53d}$$

The mode expansions for the bosons on a strand of length $k > 1$ are obtained from those in the untwisted sector by sending $n \to n \pm \rho/k$, with the sign depending on which field we are considering and following from (2.53),

$$\partial X^{1\dot{1}}_{\rho}(z) = \sum_{n\in\mathbf{Z}} \alpha^{1\dot{1}}_{\rho,n-\frac{\rho}{k}}\, z^{-n-1+\frac{\rho}{k}}, \qquad \partial X^{2\dot{2}}_{\rho}(z) = \sum_{n\in\mathbf{Z}} \alpha^{2\dot{2}}_{\rho,n+\frac{\rho}{k}}\, z^{-n-1-\frac{\rho}{k}}, \tag{2.54a}$$

$$\partial X^{1\dot{2}}_{\rho}(z) = \sum_{n\in\mathbf{Z}} \alpha^{1\dot{2}}_{\rho,n+\frac{\rho}{k}}\, z^{-n-1-\frac{\rho}{k}}, \qquad \partial X^{2\dot{1}}_{\rho}(z) = \sum_{n\in\mathbf{Z}} \alpha^{2\dot{1}}_{\rho,n-\frac{\rho}{k}}\, z^{-n-1+\frac{\rho}{k}}, \tag{2.54b}$$

$$\bar{\partial} X^{1\dot{1}}_{\rho}(\bar{z}) = \sum_{n\in\mathbf{Z}} \tilde{\alpha}^{1\dot{1}}_{\rho,n-\frac{\rho}{k}}\, \bar{z}^{-n-1+\frac{\rho}{k}}, \qquad \bar{\partial} X^{2\dot{2}}_{\rho}(\bar{z}) = \sum_{n\in\mathbf{Z}} \tilde{\alpha}^{2\dot{2}}_{\rho,n+\frac{\rho}{k}}\, \bar{z}^{-n-1-\frac{\rho}{k}}, \tag{2.54c}$$

$$\bar{\partial} X^{1\dot{2}}_{\rho}(\bar{z}) = \sum_{n\in\mathbf{Z}} \tilde{\alpha}^{1\dot{2}}_{\rho,n+\frac{\rho}{k}}\, \bar{z}^{-n-1-\frac{\rho}{k}}, \qquad \bar{\partial} X^{2\dot{1}}_{\rho}(\bar{z}) = \sum_{n\in\mathbf{Z}} \tilde{\alpha}^{2\dot{1}}_{\rho,n-\frac{\rho}{k}}\, \bar{z}^{-n-1+\frac{\rho}{k}} \tag{2.54d}$$

The nonzero commutation relations in the twisted sector are a generalization of the ones in the untwisted sector, (2.14) and (2.15), but they are naturally realized in the ρ basis. For the bosonic modes we have

$$\left[\alpha^{A\dot{A}}_{\rho_1,n},\alpha^{B\dot{B}}_{\rho_2,m}\right]=\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\,n\,\delta_{n+m,0}\,\delta_{\rho_1,\rho_2}\,,\qquad \left[\tilde{\alpha}^{A\dot{A}}_{\rho_1,n},\tilde{\alpha}^{B\dot{B}}_{\rho_2,m}\right]=\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\,n\,\delta_{n+m,0}\,\delta_{\rho_1,\rho_2}\,, \tag{2.55}$$

For the fermions we still have the distinction between the R and the NS sectors, but they work differently in comparison with the case of the untwisted CFT. In the **R sector** the monodromies of the fermions are

$$\psi^{\alpha\dot{A}}_{(r)}(e^{2\pi\mathrm{i}}z)=-\psi^{\alpha\dot{A}}_{(r+1)}(z),\qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}\left(e^{-2\pi\mathrm{i}}\bar{z}\right)=-\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r+1)}(\bar{z}), \tag{2.56}$$

again with the identifications $\psi^{\alpha\dot{A}}_{(k+1)}\equiv\psi^{\alpha\dot{A}}_{(1)}$ and $\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(k+1)}\equiv\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(1)}$, and can be diagonalized with a $(r)\to\rho$ change of basis, with $\rho=0,\dots,k-1$,

$$\psi^{1\dot{A}}_{\rho}(z)=\frac{1}{\sqrt{k}}\sum_{r=1}^{k}e^{2\pi\mathrm{i}\frac{r\rho}{k}}\;\psi^{1\dot{A}}_{(r)}(z),\qquad \psi^{2\dot{A}}_{\rho}(z)=\frac{1}{\sqrt{k}}\sum_{r=1}^{k}e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\;\psi^{2\dot{A}}_{(r)}(z), \tag{2.57a}$$

$$\tilde{\psi}^{\dot{1}\dot{A}}_{\rho}(\bar{z})=\frac{1}{\sqrt{k}}\sum_{r=1}^{k}e^{-2\pi\mathrm{i}\frac{r\rho}{k}}\;\tilde{\psi}^{\dot{1}\dot{A}}_{(r)}(\bar{z}),\qquad \tilde{\psi}^{\dot{2}\dot{A}}_{\rho}(\bar{z})=\frac{1}{\sqrt{k}}\sum_{r=1}^{k}e^{2\pi\mathrm{i}\frac{r\rho}{k}}\;\tilde{\psi}^{\dot{2}\dot{A}}_{(r)}(\bar{z}). \tag{2.57b}$$

In the ρ basis the (diagonalized) monodromy conditions are

$$\psi^{1\dot{A}}_{\rho}\left(e^{2\pi\mathrm{i}}\,z\right)=-e^{-2\pi\mathrm{i}\frac{\rho}{k}}\,\psi^{1\dot{A}}_{\rho}(z),\qquad \psi^{2\dot{A}}_{\rho}\left(e^{2\pi\mathrm{i}}\,z\right)=-e^{2\pi\mathrm{i}\frac{\rho}{k}}\,\psi^{2\dot{A}}_{\rho}(z), \tag{2.58a}$$

$$\tilde{\psi}^{\dot{1}\dot{A}}_{\rho}\left(e^{-2\pi\mathrm{i}}\bar{z}\right)=-e^{2\pi\mathrm{i}\frac{\rho}{k}}\tilde{\psi}^{\dot{1}\dot{A}}_{\rho}\left(\bar{z}\right),\qquad \tilde{\psi}^{\dot{2}\dot{A}}_{\rho}\left(e^{-2\pi\mathrm{i}}\bar{z}\right)=-e^{-2\pi\mathrm{i}\frac{\rho}{k}}\tilde{\psi}^{\dot{2}\dot{A}}_{\rho}\left(\bar{z}\right), \tag{2.58b}$$

and the mode expansions follow from (2.58),

$$\psi^{1\dot{A}}_{\rho}(z)=\sum_{n\in\mathbb{Z}}\psi^{1\dot{A}}_{\rho,n+\frac{\rho}{k}}\,z^{-n-\frac{1}{2}-\frac{\rho}{k}},\qquad \psi^{2\dot{A}}_{\rho}(z)=\sum_{n\in\mathbb{Z}}\psi^{2\dot{A}}_{\rho,n-\frac{\rho}{k}}\,z^{-n-\frac{1}{2}+\frac{\rho}{k}}, \tag{2.59a}$$

$$\tilde{\psi}^{\dot{1}\dot{A}}_{\rho}(\bar{z})=\sum_{n\in\mathbb{Z}}\tilde{\psi}^{\dot{1}\dot{A}}_{\rho,n+\frac{\rho}{k}}\,\bar{z}^{-n-\frac{1}{2}-\frac{\rho}{k}},\qquad \tilde{\psi}^{\dot{2}\dot{A}}_{\rho}(\bar{z})=\sum_{n\in\mathbb{Z}}\tilde{\psi}^{\dot{2}\dot{A}}_{\rho,n-\frac{\rho}{k}}\,\bar{z}^{-n-\frac{1}{2}+\frac{\rho}{k}}. \tag{2.59b}$$

Taking z around the origin k times we get

$$\psi^{\alpha\dot{A}}_{\rho}(e^{2\pi\mathrm{i}k}\,z)=(-1)^{k}\,\psi^{\alpha\dot{A}}_{\rho}(z),\qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{\rho}(e^{-2\pi\mathrm{i}k}\,\bar{z})=(-1)^{k}\,\tilde{\psi}^{\dot{\alpha}\dot{A}}_{\rho}(\bar{z}). \tag{2.60}$$

Fermions in the **NS sector** on a strand of length k have the following monodromy rules in the (r) basis,

$$\psi^{\alpha\dot{A}}_{(r)}(e^{2\pi\mathrm{i}}z) = \psi^{\alpha\dot{A}}_{(r+1)}(z), \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}\left(e^{-2\pi\mathrm{i}}\bar{z}\right) = \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r+1)}(\bar{z}), \tag{2.61}$$

this time with $\psi^{\alpha\dot{A}}_{(k+1)} \equiv (-1)^{k+1}\,\psi^{\alpha\dot{A}}_{(1)}$ and $\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(k+1)} \equiv (-1)^{k+1}\,\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(1)}$. The change of basis that diagonalizes the monodromy conditions is $(r) \to l$, with $l = -\frac{k-1}{2}, -\frac{k-1}{2}+1, \ldots, \frac{k-1}{2}$,

$$\psi^{1\dot{A}}_{l}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{rl}{k}}\;\psi^{1\dot{A}}_{(r)}(z), \qquad \psi^{2\dot{A}}_{l}(z) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{rl}{k}}\;\psi^{2\dot{A}}_{(r)}(z), \tag{2.62a}$$

$$\tilde{\psi}^{\dot{1}\dot{A}}_{l}(\bar{z}) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{-2\pi\mathrm{i}\frac{rl}{k}}\;\tilde{\psi}^{\dot{1}\dot{A}}_{(r)}(\bar{z}), \qquad \tilde{\psi}^{\dot{2}\dot{A}}_{l}(\bar{z}) = \frac{1}{\sqrt{k}}\sum_{r=1}^{k} e^{2\pi\mathrm{i}\frac{rl}{k}}\;\tilde{\psi}^{\dot{2}\dot{A}}_{(r)}(\bar{z}). \tag{2.62b}$$

In the l basis the (diagonalized) monodromy conditions are

$$\psi^{1\dot{A}}_{l}\left(e^{2\pi\mathrm{i}}\,z\right) = e^{-2\pi\mathrm{i}\frac{l}{k}}\,\psi^{1\dot{A}}_{l}(z), \qquad \psi^{2\dot{A}}_{l}\left(e^{2\pi\mathrm{i}}\,z\right) = e^{2\pi\mathrm{i}\frac{l}{k}}\,\psi^{2\dot{A}}_{l}(z), \tag{2.63a}$$

$$\tilde{\psi}^{\dot{1}\dot{A}}_{l}\left(e^{-2\pi\mathrm{i}}\,\bar{z}\right) = e^{2\pi\mathrm{i}\frac{l}{k}}\,\tilde{\psi}^{\dot{1}\dot{A}}_{l}\left(\bar{z}\right), \qquad \tilde{\psi}^{\dot{2}\dot{A}}_{l}\left(e^{-2\pi\mathrm{i}}\,\bar{z}\right) = e^{-2\pi\mathrm{i}\frac{l}{k}}\,\tilde{\psi}^{\dot{2}\dot{A}}_{l}\left(\bar{z}\right), \tag{2.63b}$$

and the mode expansions follow from (2.63),

$$\psi^{1\dot{A}}_{l}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{1\dot{A}}_{l,n+\frac{l}{k}}\;z^{-n-\frac{1}{2}-\frac{l}{k}}, \qquad \psi^{2\dot{A}}_{l}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{2\dot{A}}_{l,n-\frac{l}{k}}\;z^{-n-\frac{1}{2}+\frac{l}{k}}, \tag{2.64a}$$

$$\tilde{\psi}^{\dot{1}\dot{A}}_{l}(\bar{z}) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{\psi}^{\dot{1}\dot{A}}_{l,n+\frac{l}{k}}\;\bar{z}^{-n-\frac{1}{2}-\frac{l}{k}}, \qquad \tilde{\psi}^{\dot{2}\dot{A}}_{l}(\bar{z}) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{\psi}^{\dot{2}\dot{A}}_{l,n-\frac{l}{k}}\;\bar{z}^{-n-\frac{1}{2}+\frac{l}{k}}. \tag{2.64b}$$

Taking z around the origin k times we get

$$\psi^{\alpha\dot{A}}_{l}(e^{2\pi\mathrm{i}k}\,z) = (-1)^{k+1}\,\psi^{\alpha\dot{A}}_{l}(z), \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{l}(e^{-2\pi\mathrm{i}k}\,\bar{z}) = (-1)^{k+1}\,\tilde{\psi}^{\dot{\alpha}\dot{A}}_{l}(\bar{z}). \tag{2.65}$$

For the fermionic modes we have the nonzero anticommutation relations

$$\left\{\psi^{1\dot{A}}_{\rho_1,n}, \psi^{2\dot{B}}_{\rho_2,m}\right\} = \epsilon^{\dot{A}\dot{B}}\,\delta_{n+m,0}\,\delta_{\rho_1,\rho_2}, \qquad \left\{\tilde{\psi}^{\dot{1}\dot{A}}_{\rho_1,n}, \tilde{\psi}^{\dot{2}\dot{B}}_{\rho_2,m}\right\} = \epsilon^{\dot{A}\dot{B}}\,\delta_{n+m,0}\,\delta_{\rho_1,\rho_2}. \tag{2.66}$$

2.3.2 Vacuum states

Vacua in the twisted sector are analogous to the vacua in the untwisted one, apart from having the monodromy conditions discussed above for the fields. The key properties of the fields' modes are also analogous to the ones in the untwisted case, save the fact that they hold in the ρ basis, which is in a sense the most natural one due to the diagonal monodromy conditions. We denote the bosonic vacuum on a strand of length k as $|0\rangle_k$ and we impose that it is annihilated by all the nonnegative bosonic modes in the ρ

basis,

$$\alpha^{A\dot{A}}_{\rho,n}|0\rangle_k = 0, \qquad \tilde{\alpha}^{A\dot{A}}_{\rho,n}|0\rangle_k = 0, \qquad \forall n \geq 0, \qquad \forall A, \dot{A}. \tag{2.67}$$

The above generalization holds for the fermionic vacuum states as well. In the **R sector** we have the vacua

$$|\pm\pm\rangle_k, \quad |00\rangle_k, \tag{2.68}$$

with the spin highest weight state $|{+}{+}\rangle_k$ annihilated by all the fermions' positive modes in the ρ basis and by half of the zero modes,

$$\psi^{\alpha\dot{A}}_{\rho,n}|{+}{+}\rangle_k = 0, \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{\rho,n}|{+}{+}\rangle_k = 0, \quad \forall n > 0, \quad \forall \alpha, \dot{\alpha}, \dot{A}, \tag{2.69}$$

$$\psi^{1\dot{A}}_{\rho,0}|{+}{+}\rangle_k = 0, \qquad \tilde{\psi}^{\dot{1}\dot{A}}_{\rho,0}|{+}{+}\rangle_k = 0. \tag{2.70}$$

The other states are obtained from $|{+}{+}\rangle_k$ acting with the $J^-, \tilde{J}^-$ and O^{--} operators defined on a length-k strand (see next section) and the left and right spins of all the states (2.68) are the eigenvalues of the $J^3, \tilde{J}^3$ operators, again defined on a strand of length k.

For fermions in the **NS sector** we only have one vacuum state

$$|0\rangle_k, \tag{2.71}$$

which is annihilated by all the fermionic positive modes in the ρ basis,

$$\psi^{\alpha\dot{A}}_{\rho,n}|0\rangle_k = 0, \qquad \tilde{\psi}^{\dot{\alpha}\dot{A}}_{\rho,n}|0\rangle_k = 0, \qquad \forall n > 0, \quad \forall \alpha, \dot{\alpha}, \dot{A}. \tag{2.72}$$

The NS vacuum with twist k is a scalar of the $SU(2)_L \times SU(2)_R$ R-symmetry group, as it was in the untwisted case.

2.3.3 Other operators in the twisted sector

Let's look at the spin operators and at $O^{\alpha\dot{\alpha}}$, this time defined on a strand of length k. In the cases of operators that are written as sums over different copies, they keep their form, but this time the sum goes up to k instead of N. Moreover, if the operators can be written in terms of elementary fields (the bosons or the fermions), a change of basis

$(r) \to \rho$ can be performed. For the spin operators we have

$$J^{+} = \sum_{r=1}^{k} J^{+}_{(r)} = \frac{1}{2}\sum_{r=1}^{k} :\psi^{1\dot{A}}_{(r)}\psi^{1\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{1}{2}\left(:\psi^{1\dot{A}}_{\rho=0}\psi^{1\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{1\dot{A}}_{\rho}\psi^{1\dot{B}}_{k-\rho}\epsilon_{\dot{A}\dot{B}}: \right), \tag{2.73a}$$

$$J^{-} = \sum_{r=1}^{k} J^{-}_{(r)} = -\frac{1}{2}\sum_{r=1}^{k} :\psi^{2\dot{A}}_{(r)}\psi^{2\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = -\frac{1}{2}\left(:\psi^{2\dot{A}}_{\rho=0}\psi^{2\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{2\dot{A}}_{\rho}\psi^{2\dot{B}}_{k-\rho}\epsilon_{\dot{A}\dot{B}}: \right), \tag{2.73b}$$

$$J^{3} = \sum_{r=1}^{k} J^{3}_{(r)} = -\frac{1}{2}\sum_{r=1}^{k} \Big(:\psi^{1\dot{A}}_{(r)}\psi^{2\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: -1 \Big) = -\frac{1}{2}\sum_{\rho=1}^{k-1} \Big(:\psi^{1\dot{A}}_{\rho}\psi^{2\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}: -1 \Big), \tag{2.73c}$$

where we used the inverse of the transformation (2.57) (which are obtained switching the sign of the phases) and the orthonormality condition

$$\sum_{r=1}^{k} e^{2\pi \mathrm{i}\frac{r}{k}(\rho_1+\rho_2)} = k\,\delta_{\rho_1+\rho_2,0}. \tag{2.74}$$

The right spin operators are totally analogous.

For the $O^{\alpha\dot{\alpha}}$ operators we have

$$O^{1\dot{1}} = \sum_{r=1}^{k} O^{1\dot{1}}_{(r)} = \frac{-\mathrm{i}}{\sqrt{2}}\sum_{r=1}^{k} :\psi^{1\dot{A}}_{(r)}\tilde{\psi}^{\dot{1}\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{-\mathrm{i}}{\sqrt{2}}\sum_{\rho=0}^{k-1} :\psi^{1\dot{A}}_{\rho}\tilde{\psi}^{\dot{1}\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}:, \tag{2.75a}$$

$$O^{2\dot{2}} = \sum_{r=1}^{k} O^{2\dot{2}}_{(r)} = \frac{-\mathrm{i}}{\sqrt{2}}\sum_{r=1}^{k} :\psi^{2\dot{A}}_{(r)}\tilde{\psi}^{\dot{2}\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{-\mathrm{i}}{\sqrt{2}}\sum_{\rho=0}^{k-1} :\psi^{2\dot{A}}_{\rho}\tilde{\psi}^{\dot{2}\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}:, \tag{2.75b}$$

$$O^{1\dot{2}} = \sum_{r=1}^{k} O^{1\dot{2}}_{(r)} = \frac{-\mathrm{i}}{\sqrt{2}}\sum_{r=1}^{k} :\psi^{1\dot{A}}_{(r)}\tilde{\psi}^{\dot{2}\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{-\mathrm{i}}{\sqrt{2}}\left(:\psi^{1\dot{A}}_{\rho=0}\tilde{\psi}^{\dot{2}\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{1\dot{A}}_{\rho}\tilde{\psi}^{\dot{2}\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}: \right), \tag{2.75c}$$

$$O^{2\dot{1}} = \sum_{r=1}^{k} O^{2\dot{1}}_{(r)} = \frac{-\mathrm{i}}{\sqrt{2}} :\psi^{2\dot{A}}_{(r)}\tilde{\psi}^{\dot{2}\dot{B}}_{(r)}\epsilon_{\dot{A}\dot{B}}: = \frac{-\mathrm{i}}{\sqrt{2}}\left(:\psi^{2\dot{A}}_{\rho=0}\tilde{\psi}^{\dot{1}\dot{B}}_{\rho=0}\epsilon_{\dot{A}\dot{B}}: + \sum_{\rho=1}^{k-1} :\psi^{2\dot{A}}_{\rho}\tilde{\psi}^{\dot{1}\dot{B}}_{\rho}\epsilon_{\dot{A}\dot{B}}: \right). \tag{2.75d}$$

The vacuum states in the twisted sector (2.68) can be obtained starting from $|{+}{+}\rangle_k$ and acting with the appropriate operators as in (2.35) and (2.37), the only difference being that the operators are now defined on a strand of length k.

The stress-energy operator on a strand of length k is

$$T(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2} = T_B(z) + T_F(z) = \sum_{r=1}^{k} \left(T^B_{(r)}(z) + T^F_{(r)}(z) \right), \tag{2.76}$$

where $T^B_{(r)}$ and $T^F_{(r)}$ have the forms in (2.42b) and (2.42c) respectively. Using (2.52) and (2.57) (e.g. for **R sector** fermions) we can change from the (r) to the ρ basis obtaining

$$T_B(z) = \sum_{n\in\mathbb{Z}} L^B_n\, z^{-n-2} = \sum_{r=1}^{k} T^B_{(r)}(z) = \sum_{\rho=0}^{k-1} T^B_\rho(z), \tag{2.77a}$$

$$T_F(z) = \sum_{n\in\mathbb{Z}} L^F_n\, z^{-n-2} = \sum_{r=1}^{k} T^F_{(r)}(z) = \sum_{\rho=0}^{k-1} T^F_\rho(z), \tag{2.77b}$$

with

$$T^B_\rho(z) = \frac{1}{2}\,\epsilon_{AB}\epsilon_{\dot A\dot B} : \partial X^{A\dot A}_\rho\, \partial X^{B\dot B}_\rho : (z), \tag{2.78a}$$

$$T^F_\rho(z) = \frac{1}{2}\,\epsilon_{\alpha\beta}\epsilon_{\dot A\dot B} : \psi^{\alpha\dot B}_\rho\, \partial\psi^{\beta\dot A}_\rho : (z). \tag{2.78b}$$

Switching to the ρ basis is essential to be able to use the mode expansions (2.54) and (2.59), exploiting which we obtain

$$L^B_n = \sum_{\rho=0}^{k-1} \frac{1}{2}\,\epsilon_{AB}\epsilon_{\dot A\dot B} \sum_{m\in\mathbb{Z}} : \alpha^{A\dot A}_{\rho,m\pm\frac{\rho}{k}}\,\alpha^{B\dot B}_{\rho,n-m\mp\frac{\rho}{k}} :, \tag{2.79a}$$

$$L^B_n = \sum_{\rho=0}^{k-1} \frac{1}{2}\,\epsilon_{\alpha\beta}\epsilon_{\dot A\dot B} \sum_{m\in\mathbb{Z}} \left(-n + m - \frac{1}{2} \pm \frac{\rho}{k} \right) : \psi^{\alpha\dot A}_{\rho,m\pm\frac{\rho}{k}}\, \psi^{\beta\dot B}_{\rho,n-m\mp\frac{\rho}{k}}, \tag{2.79b}$$

where the $\pm$ signs depend on the specific indices $A, B, \dot A, \dot B, \alpha, \beta$ of the fields (but the relative signs are those shown). The modes L_n realise the Virasoro algebra (2.43) on the length-k strand, and the OPE of $T(z)$ with itself is again (2.44), with the central charge given by $k \times c_{1\ \mathrm{copy}}$.

Let's now introduce another family of operators that play an important role in this work. Strands of length k were defined as CFT copies obtained by sewing together k untwisted copies and on which fields acquire the monodromy conditions studied in section (2.3.1). The sewing operation is carried out by operators that act on a tensor product of k untwisted vacua to give a single twisted one: these are the **twist operators**. We have twist operators σ^X_k and $\tilde\sigma^X_k$ for the left and the right bosonic vacua, twist operators Σ_k for the fermionic vacua in the NS sector and twist operators $\Sigma^{s\dot s}_k$ for the fermionic vacua in the R sector.

The σ_k^X and $\tilde{\sigma}_k^X$ operators create bosonic ground states of length k as

$$\lim_{z\to 0} \sigma_k^X(z)\,\tilde{\sigma}_k^X(\bar{z})\left[\otimes_{r=1}^{k}|0\rangle_{(r)}\right] = |0\rangle_k, \tag{2.80}$$

and have conformal dimension

$$(h,\bar{h}) = \left(\frac{k^2-1}{6k}, \frac{k^2-1}{6k}\right). \tag{2.81}$$

σ_k^X and its antiholomorphic counterpart can be written as a product of operators acting on each of the original k CFT copies in the ρ basis,

$$\sigma_k^X = \otimes_{\rho=0}^{k-1}\sigma_\rho^X, \qquad \tilde{\sigma}_k^X = \otimes_{\rho=0}^{k-1}\tilde{\sigma}_\rho^X, \tag{2.82}$$

where σ_ρ^X and $\tilde{\sigma}_\rho^X$ act nontrivially on the copy ρ and as the identity on all the other copies. Since for a product of k vacua of length 1 the $(r)\to\rho$ change of basis is trivial,

$$\otimes_{r=1}^{k}|0\rangle_{(r)} = \otimes_{\rho=0}^{k-1}|0\rangle_\rho, \tag{2.83}$$

(2.80) can also be written as

$$|0\rangle_k = \lim_{z\to 0}\otimes_{\rho=0}^{k-1}\sigma_\rho^X(z)\,\tilde{\sigma}_\rho^X(\bar{z})|0\rangle_\rho. \tag{2.84}$$

The bosonic twist fields are scalars under all the internal symmetries. For each value of ρ, the conformal dimension of $\sigma_\rho^X(x)$ is

$$h_{\sigma_\rho^X} = \frac{\rho}{k}\left(1-\frac{\rho}{k}\right), \tag{2.85}$$

and the total conformal dimension (2.81) is obtained as a sum over $\rho = 0,\ldots,k-1$. The same value corresponds to the antiholomorphic conformal dimension of $\tilde{\sigma}_\rho^X(\bar{x})$.

Let's now consider the twist field $\Sigma_k(z,\bar{z})$ for the fermions in the **NS** sector. They are scalars under $SU(2)_L\times SU(2)_R$ and their conformal dimension is

$$(h,\bar{h}) = \left(\frac{1}{12}\left(k-\frac{1}{k}\right), \frac{1}{12}\left(k-\frac{1}{k}\right)\right). \tag{2.86}$$

The action of Σ_k is analogous to the one of the bosonic twist fields, and again they are written as a product of operators acting on a single CFT copy in the ρ basis,

$$\Sigma_k(z,\bar{z}) = \otimes_{\rho=0}^{k-1}\Sigma_\rho(z,\bar{z}). \tag{2.87}$$

As in the bosonic case, for a product of k vacua of length 1 the $(r)\to\rho$ change of basis is trivial,

$$\otimes_{r=1}^{k} \; |0\rangle_{(r),\mathrm{NS}} = \otimes_{\rho=0}^{k-1} |0\rangle_{\rho,\mathrm{NS}}, \tag{2.88}$$

so the fermionic vacuum of length k in the NS sector is obtained as

$$\lim_{z\to 0} \Sigma_k(z,\bar z) \otimes_{r=1}^{k} |0\rangle_{(r),\mathrm{NS}} = \lim_{z\to 0} \otimes_{\rho=0}^{k-1} \Sigma_\rho(z,\bar z)|0\rangle_{\rho,\mathrm{NS}} = |0\rangle_k. \tag{2.89}$$

Let's now consider the twist field $\Sigma_k^{s_1\dot s_2}(z,\bar z)$ for the fermions in the **R sector**. The indices s_1 and $\dot s_2$ transform under a representation of spin $((k-1)/2,(k-1)/2)$ under $SU(2)_L\times SU(2)_R$, corresponding respectively to the left spin operators (2.73) and their right counterparts. The conformal dimension of $\Sigma_k^{s_1\dot s_2}$ is

$$(h,\bar h) = \left(\frac{(k-1)(2k-1)}{6k}, \frac{(k-1)(2k-1)}{6k}\right). \tag{2.90}$$

If $k=2$ the spin indices are in the fundamental representation of the $SU(2)_L\times SU(2)_R$ R-symmetry group, so we have the usual $\alpha,\dot\alpha$ and the operators are $\Sigma_2^{\alpha\dot\alpha}$. The conjugation relations among these four operators are

$$\left(\Sigma_2^{1\dot 1}\right)^\dagger = \Sigma_2^{2\dot 2}, \qquad \left(\Sigma_2^{1\dot 2}\right)^\dagger = -\Sigma_2^{2\dot 1}. \tag{2.91}$$

As in the other cases, $\Sigma_k^{s_1\dot s_2}$ is written as the product

$$\Sigma_k^{s_1\dot s_2}(z,\bar z) = \otimes_{\rho=0}^{k-1} \Sigma_\rho^{s_1\dot s_2}(z,\bar z). \tag{2.92}$$

In the R sector we have to be more careful because there is more than one twisted vacuum, and whether or not these can be generated acting with $\Sigma_k^{s_1\dot s_2}$ on products of length-1 vacua also depends on the spin (i.e. the choice for s_1 and $\dot s_2$). The next section contains examples that will hopefully help clarify this point. For each value of $\rho = 0,\dots,k-1$ the conformal dimension of $\Sigma_\rho^{s_1\dot s_2}$ is

$$h = \bar h = \frac{\rho^2}{k^2}, \tag{2.93}$$

and the total conformal dimension (2.90) is obtained summing over ρ.

In general, a twist operator can be seen as introducing a branch cut from its insertion point z to infinity: the effect of taking one of the elementary fields around z is that of crossing the branch cut and landing on the next (counter-clockwise) or previous (clockwise) Riemann sheet, i.e. on the next or previous copy of the CFT among the k glued together. We'll always consider twist operators that are products of a bosonic and a fermionic part so they introduce the same twist in the bosonic and in the fermionic sector of the theory: if we picture the fields as living on strands with various possible lengths, it wouldn't make sense to have a different length for the bosons and for the fermions. In this view, it's useful to write the operator that carries out the twist for the bosons and the fermions in the R sector at the same time,

$$\sigma_k^X\,\tilde{\sigma}_k^X\,\Sigma_k^{s_1\dot{s}_2}=\bigotimes_{\rho=0}^{k-1}\sigma_\rho^X\,\tilde{\sigma}_\rho^X\,\Sigma_\rho^{s_1\dot{s}_2}. \tag{2.94}$$

The total conformal dimension of the operator above is

$$h=\bar{h}=\sum_{\rho=0}^{k-1}\left[\frac{\rho}{k}\left(1-\frac{\rho}{k}\right)+\frac{\rho^2}{k^2}\right]=\frac{k-1}{2}, \tag{2.95}$$

which is of course just the sum of (2.81) and (2.90), but written in a way that highlights the contribution of each value of ρ.

2.3.4 Example of a strand of length $k = 2$

As an example, let's consider the $N = 2$ case and the gluing of two vacuum states in the R sector of the fermions (in the following we will ignore the action of the twist field on the bosonic part of the theory, but in general it's still there),

$$|{+}{+}\rangle_{(r=1)}\otimes|{+}{+}\rangle_{(r=2)}.$$

If $N = 2$ the $J^3, \tilde{J}^3$ spin operators read

$$J^3=J^3_{(1)}\otimes\mathbb{1}_{(2)}+\mathbb{1}_{(1)}\otimes J^3_{(2)},\qquad \tilde{J}^3=\tilde{J}^3_{(1)}\otimes\mathbb{1}_{(2)}+\mathbb{1}_{(1)}\otimes\tilde{J}^3_{(2)},$$

and we have

$$\begin{aligned}
J^3\left(|{+}{+}\rangle_{(1)}\otimes|{+}{+}\rangle_{(2)}\right)&=\left(J^3_{(1)}|{+}{+}\rangle_1\right)\otimes|{+}{+}\rangle_2+|{+}{+}\rangle_1\otimes\left(J^3_{(2)}|{+}{+}\rangle_2\right)\\
&=\frac{1}{2}\Big(|{+}{+}\rangle_{(1)}\otimes|{+}{+}\rangle_{(2)}\Big),\\
\tilde{J}^3\left(|{+}{+}\rangle_{(1)}\otimes|{+}{+}\rangle_{(2)}\right)&=\left(\tilde{J}^3_{(1)}|{+}{+}\rangle_1\right)\otimes|{+}{+}\rangle_2+|{+}{+}\rangle_1\otimes\left(\tilde{J}^3_{(2)}|{+}{+}\rangle_2\right)\\
&=\frac{1}{2}\Big(|{+}{+}\rangle_{(1)}\otimes|{+}{+}\rangle_{(2)}\Big).
\end{aligned}$$

Let's now act on the state with $\Sigma_2^{\alpha\dot{\alpha}}$ in order to create a single strand of length 2. The length-1 vacuum states $|{+}{+}\rangle_{(r)}$ are spin highest weight state with respect to both J^3 and $\tilde{J}^3$. The length-2 vacuum state we are going to generate has two key properties:

- Its total left (J^3) spin is the sum of the left spin of the two length-1 vacuum states and of the twist field, and the same holds for its right ($\tilde{J}^3$) spin.

- It's still in the fundamental representation of the left and right spin, so its eigenvalues under J^3 and $\tilde{J}^3$ cannot be greater than 1/2.

The only possibility in order for

$$\Sigma_2^{\pm\pm}\Big(|{+}{+}\rangle_{(1)} \otimes |{+}{+}\rangle_{(2)}\Big)$$

not to be zero is therefore that of acting with Σ_2^{--}, which gives a state with spin $(2 \times 1/2 - 1/2, 2 \times 1/2 - 1/2) = (1/2, 1/2)$, still a highest weight state, denoted as

$$\lim_{z\to 0} \Sigma_2^{--}(z,\bar{z})\Big(|{+}{+}\rangle_{(1)} \otimes |{+}{+}\rangle_{(2)}\Big) \equiv |{+}{+}\rangle_{k=2}.$$

Acting with any other of the $\Sigma_2^{\pm\pm}$ would annihilate the state.

2.3.5 Twist fields acting on N strands

In the generic case, we will start from N strands of length 1 and act with a twist field $\Sigma_2^{s_1\dot{s}_2}$. As an example, let's consider the tensor product of N vacua in the fermionic R sector,

$$\bigotimes_{r=1}^{N} |{+}{+}\rangle_{(r)} = |{+}{+}\rangle_{(1)} \otimes |{+}{+}\rangle_{(2)} \otimes \cdots \otimes |{+}{+}\rangle_{(N)} \tag{2.96}$$

and the twist field Σ_2^{--}. Again, the action of any other of the $\Sigma_2^{\pm\pm}$ would annihilate the state.

Acting with Σ_2^{--} on (2.96) generates the sum of all possible states in which two of the length-1 states have been glued together to give a length-2 state. Formally, we could write the twist field as

$$\Sigma_2^{--} = \sum_{r=1}^{N-1} \sum_{s>r} \sigma_{(rs)}^{--}, \tag{2.97}$$

where $\sigma_{(rs)}^{--}$ is an operator that takes the length-1 states in the product (2.96) and maps them into a $|{+}{+}\rangle_2$ state. The sums in (2.97) give a total of of $N(N-1)/2$ terms, so

$$\Sigma_2^{--} \bigotimes_{r=1}^{N} |{+}{+}\rangle_{(r)}$$

is a sum of $N(N-1)/2$ states, each with one strand of length 2 and $N-2$ strands of length 1. Due to the fact that every possible pair of length-1 strands have been glued, the state generated still satisfies the requirement of being symmetric under exchange of any two states in the tensor product.

In the language of permutations, $\sigma_{(rs)}^{\pm\pm}$ can be seen as creating a cycle from the objects in positions r and s in the sequence

$$(1)(2)\cdots(r)\cdots(s)\cdots(N),$$

i.e. (assuming the spin properties do not annihilate the state) we have

$$\sigma^{\pm\pm}_{(rs)}\left[(1)(2)\cdots(r)\cdots(s)\cdots(N)\right]=(1)(2)\cdots(rs)\cdots(N), \tag{2.98}$$

with cycles of length k corresponding to strands of length k.
$\sigma^{\pm\pm}_{(rs)}$ can also break a cycle (rs) of length 2 into its constituent cycles of length 1, giving $(r)(s)$. This is best understood looking at specific examples, e.g.

$$\sigma_{(rs)}\left[(12\cdots)\cdots(r)(s)\cdots(N)\right]=(12\cdots)\cdots(rs)\cdots N.$$

If we act p times (having $p<N/2$) with Σ_2^{--} on (2.96) we will get a complicated sum of states, not all with the maximal number p of strands of length 2, which is due to the fact that once a strand of winding 2 has been created, it can be destroyed acting again with the twist field. If we denote as $|2^p\rangle$ the sum of all possible products of p strands $|{+}{+}\rangle_{k=2}$ and $(N-2p)$ strands $|{+}{+}\rangle_{k=1}$, we define

$$\left(\Sigma_2^{--}\right)^p\left[\bigotimes_{r=1}^{N}|{+}{+}\rangle_{(r)}\right]=p!\,|2^p\rangle+\cdots, \tag{2.99}$$

where $p!$ was added for convenience and the dots contain sum of states with less than p strands of length 2.
It is useful to compute how many states are contained in $|2^p\rangle$. Acting once with Σ_2^{--} on (2.96) we can form one strand of length 2 starting from N strands of length one, which can be done in $\binom{N}{2}$ inequivalent ways. Acting with Σ_2^{--} a second time we can form another strand of length 2, this time starting from $N-2$ strands of length one (and disregarding the possibility of disrupting the cycle formed with the first action, which would give states that are not in $|2^p\rangle$), which can be done in $\binom{N-2}{2}$ inequivalent ways, and so on. In the end the number of states is the product of the inequivalent ways in which each Σ_2^{--} can create a new strand of length 2, which is

$$\binom{N}{2}\binom{N-2}{2}\cdots\binom{N-2(p-1)}{2}\frac{1}{p!}=\frac{N!}{(N-2p)!p!2^p}, \tag{2.100}$$

where the factor $1/p!$ comes from dividing by the permutations of the p operators. From this, we can fix the normalization of the state: the inner product of $|2^p\rangle$ and $|2^q\rangle$ will be proportional to $\delta_{p,q}$, with the proportionality constant given by the number of states in $|2^p\rangle$ (because in the product each state in the ket will give zero unless it encounters its homologous state in the bra), so

$$\langle 2^q | 2^p \rangle = \frac{N!}{(N-2p)!p!2^p}\,\delta_{k,p}. \tag{2.101}$$

Relation analogous to the ones appearing in this section can be written for the action of a generic twist field $\Sigma_k^{s_1\dot{s}_2}$.

2.4 Bosonization rules

2.4.1 Bosonization of the fermions

The properties of the fermions can be realized writing them in terms of a set of bosons with a technique called **bosonization** which will prove to be very useful for the computations of CFT correlators. On strands of length 1 we introduce bosons $H_{(r)}(z), K_{(r)}(z)$ in the holomorphic sector and $\tilde{H}_{(r)}(\bar{z}), \tilde{K}_{(r)}(\bar{z})$ in the antiholomorphic one with the OPE rules

$$H_{(r)}(z)\,H_{(s)}(w) = -\delta_{r,s}\,\log(z-w) + [\text{reg.}], \tag{2.102a}$$
$$K_{(r)}(z)\,K_{(s)}(w) = -\delta_{r,s}\,\log(z-w) + [\text{reg.}], \tag{2.102b}$$
$$\tilde{H}_{(r)}(\bar{z})\,\tilde{H}_{(s)}(\bar{w}) = -\delta_{r,s}\,\log(\bar{z}-\bar{w}) + [\text{reg.}], \tag{2.102c}$$
$$\tilde{K}_{(r)}(\bar{z})\,\tilde{K}_{(s)}(\bar{w}) = -\delta_{r,s}\,\log(\bar{z}-\bar{w}) + [\text{reg.}]. \tag{2.102d}$$

The fermions can be written in terms of the bosons[6] as

$$\psi^{1\dot{1}}_{(r)} = \mathrm{i} : e^{\mathrm{i}H_{(r)}} :, \qquad \psi^{2\dot{2}}_{(r)} = \mathrm{i} : e^{-\mathrm{i}H_{(r)}} :, \tag{2.103a}$$
$$\psi^{1\dot{2}}_{(r)} =: e^{\mathrm{i}K_{(r)}} :, \qquad \psi^{2\dot{1}}_{(r)} =: e^{-\mathrm{i}K_{(r)}} :, \tag{2.103b}$$
$$\tilde{\psi}^{\dot{1}\dot{1}}_{(r)} = \mathrm{i} : e^{\mathrm{i}\tilde{H}_{(r)}} :, \qquad \tilde{\psi}^{\dot{2}\dot{2}}_{(r)} = \mathrm{i} : e^{-\mathrm{i}\tilde{H}_{(r)}} :, \tag{2.103c}$$
$$\tilde{\psi}^{\dot{1}\dot{2}}_{(r)} =: e^{\mathrm{i}\tilde{K}_{(r)}} :, \qquad \tilde{\psi}^{\dot{2}\dot{1}}_{(r)} =: e^{-\mathrm{i}\tilde{K}_{(r)}} :, \tag{2.103d}$$

which is more simply expressed using the χ fermions as

$$\chi^1_{(r)} =: e^{\mathrm{i}H_{(r)}} :, \qquad \bar{\chi}^1_{(r)} =: e^{-\mathrm{i}H_{(r)}} :, \tag{2.104a}$$
$$\chi^2_{(r)} =: e^{\mathrm{i}K_{(r)}} :, \qquad \bar{\chi}^2_{(r)} =: e^{-\mathrm{i}K_{(r)}} :, \tag{2.104b}$$
$$\tilde{\chi}^1_{(r)} =: e^{\mathrm{i}\tilde{H}_{(r)}} :, \qquad \bar{\tilde{\chi}}^1_{(r)} =: e^{-\mathrm{i}\tilde{H}_{(r)}} :, \tag{2.104c}$$

[6] Notice that this way of writing the fermions doesn't naturally implement the fact that some of them anticommute, e.g.

$$\left\{\psi^{1\dot{1}}_{(r)\,n}, \psi^{1\dot{2}}_{(r)\,m}\right\} = 0, \qquad \forall n, m \in \mathbb{Z}.$$

This has been taken into account in all the computations performing the anticommutations among operators before switching to the bosonized language. It is possible to define bosonized fermions with the correct anticommutation relations more rigorously through the use of cocycles [73].

$$\tilde{\chi}^2_{(r)} =: e^{\mathrm{i}\tilde{K}_{(r)}} :, \qquad\qquad \bar{\tilde{\chi}}^2_{(r)} =: e^{-\mathrm{i}\tilde{K}_{(r)}} : . \tag{2.104d}$$

In general, if $X(z)$ is an operator with the same OPE rules as H or K, the operator

$$: e^{\mathrm{i}\alpha X(z)} :$$

has conformal dimension $(h, \bar{h}) = (\alpha^2/2, 0)$ and spin $(J^3, \tilde{J}^3) = (\alpha/2, 0)$ (had X been an antiholomorphic boson, the roles of the left and right conformal dimensions and spins would have been exchanged). The OPE of two operators of the above form is

$$\begin{aligned} : e^{\mathrm{i}\alpha X(z)} :: e^{\mathrm{i}\beta X(w)} : &=: \exp\left((\mathrm{i}\alpha)(\mathrm{i}\beta) \overline{X(z) X}(w) + \mathrm{i}\alpha X(z) + \mathrm{i}\beta X(w) \right) : \\ &= (z-w)^{\alpha\beta} \; : \exp\left(\mathrm{i}(\alpha+\beta) X(w) + \sum_{n=1}^{+\infty} \frac{(z-w)^n}{n!} \, \partial^n X(w) \right) :, \end{aligned} \tag{2.105}$$

where in the last step we used the contraction rule for X with itself (which gives the divergent part of the $X(z)X(w)$ OPE) and expanded $X(z)$ around w. Knowing this it's easy to see that the contraction rules (2.102) and the definitions (2.103) imply the OPE rules (2.5). As an example, let's consider $\psi^{1\dot{1}}_{(r)}(z)\, \psi^{2\dot{2}}_{(r)}(w)$:

$$\begin{aligned} \psi^{1\dot{1}}_{(r)}(z) \psi^{2\dot{2}}_{(r)}(w) &= - : e^{\mathrm{i}H_{(r)}(z)} :: e^{-\mathrm{i}H_{(r)}(w)} : \\ &= - : \exp\left(\overline{H(z) H}(w) + \sum_{n=1}^{+\infty} \partial^n \frac{(z-w)^n}{n!} H(w) \right) : \\ &= -\frac{1}{z-w} : \exp\left((z-w)\, \partial H(w) + \frac{1}{2} (z-w)^2\, \partial^2 H(w) + O\left((z-w)^3 \right) \right) : \\ &= -\frac{1}{z-w} - \partial H(w) - \frac{1}{2}(z-w) \left(\partial^2 H(w) + : (\partial H(w))^2 \right) : + O\left((z-w)^2 \right), \end{aligned}$$

which agrees with (2.5).

2.4.2 Spectral flow

Another interesting reason to define the H and K bosons and their antiholomorphic counterparts is that they can be used to define an operator that maps the fermions' NS vacuum to the R vacuum $|{+}{+}\rangle$, an operation called **spectral flow**. On a length-1 strand we have

$$|{+}{+}\rangle_{(r)} = \lim_{z\to 0} e^{\frac{\mathrm{i}}{2}\left(H_{(r)}(z) + K_{(r)}(z) + \tilde{H}_{(r)}(\bar{z}) + \tilde{K}_{(r)}(\bar{z}) \right)} |0\rangle_{(r),\mathrm{NS}}. \tag{2.106}$$

The generalization to N length-1 strands is straightforward: the operator is just a tensor product over the copies, and of course the state is the tensor product of N states

$|0\rangle_{(r),\mathrm{NS}}$, so we have

$$\bigotimes_{r=1}^{N} |{+}{+}\rangle_{(r)} = \bigotimes_{r=1}^{N} \left[\lim_{z\to 0} e^{\frac{\mathrm{i}}{2}\left(H_{(r)}(z)+K_{(r)}(z)+\tilde{H}_{(r)}(\bar{z})+\tilde{K}_{(r)}(\bar{z})\right)}|0\rangle_{(r),\mathrm{NS}}\right]. \tag{2.107}$$

The left and right conformal dimensions of the spectral flow operator in (2.107) are given by the sums of left and right conformal dimensions of each term in the tensor product, so we have

$$h = \bar{h} = \sum_{r=1}^{N} \frac{1}{2}\left(\frac{1}{2}\right)^2 2 = \frac{N}{4} = \frac{c}{24}, \tag{2.108}$$

where we used that

$$c = 6n_1 n_5 = 6N. \tag{2.109}$$

Since the NS vacuum has zero conformal dimension, we argue that the state (2.107) has conformal dimension (2.108). It's possible to check that the insertion of the spectral flow operator on a length-1 strand in the NS vacuum, (2.106), indeed changes the boundary conditions of the fermions living on that strand from NS to R.

2.4.3 Twist operators

The H and K bosons can also be used to define the fermionic part of the twist operators. Among the operators in the $\Sigma_k^{s_1\dot{s}_2}$ multiplet, we will only need the lowest weight state, which we will denote as $\Sigma_k^{-\frac{k-1}{2},-\frac{k-1}{2}}(z,\bar{z})$, the analogous for generic k of Σ_2^{--}. Before writing down the operator, though, a few remarks are in order.

In the twisted sector, on a strand of length k, the bosonization of the fermions is naturally written in the basis in which the monodromy conditions are diagonal, the ρ basis. Therefore we have

$$\psi_\rho^{1\dot{1}} = \mathrm{i} :e^{\mathrm{i}H_\rho}:, \qquad \psi_\rho^{2\dot{2}} = \mathrm{i} :e^{-\mathrm{i}H_\rho}:, \tag{2.110a}$$
$$\psi_\rho^{1\dot{2}} =: e^{\mathrm{i}K_\rho}:, \qquad \psi_\rho^{2\dot{1}} =: e^{-\mathrm{i}K_\rho}:, \tag{2.110b}$$
$$\tilde{\psi}_\rho^{\dot{1}\dot{1}} = \mathrm{i} :e^{\mathrm{i}\tilde{H}_\rho}:, \qquad \tilde{\psi}_\rho^{\dot{2}\dot{2}} = \mathrm{i} :e^{-\mathrm{i}\tilde{H}_\rho}:, \tag{2.110c}$$
$$\tilde{\psi}_\rho^{\dot{1}\dot{2}} =: e^{\mathrm{i}\tilde{K}_\rho}:, \qquad \tilde{\psi}_\rho^{\dot{2}\dot{1}} =: e^{-\mathrm{i}\tilde{K}_\rho}:, \tag{2.110d}$$

with the OPE rules for the bosons

$$H_{\rho_1}(z)\, H_{\rho_2}(w) = -\delta_{\rho_1,\rho_2}\, \log(z-w) + [\text{reg.}], \tag{2.111a}$$
$$K_{\rho_1}(z)\, K_{\rho_2}(w) = -\delta_{\rho_1,\rho_2}\, \log(z-w) + [\text{reg.}], \tag{2.111b}$$
$$\tilde{H}_{\rho_1}(\bar{z})\, \tilde{H}_{\rho_2}(\bar{w}) = -\delta_{\rho_1,\rho_2}\, \log(\bar{z}-\bar{w}) + [\text{reg.}], \tag{2.111c}$$
$$\tilde{K}_{\rho_1}(\bar{z})\, \tilde{K}_{\rho_2}(\bar{w}) = -\delta_{\rho_1,\rho_2}\, \log(\bar{z}-\bar{w}) + [\text{reg.}]. \tag{2.111d}$$

The bosonization technique can therefore be used to compute correlators more easily also in the twisted sector, provided we write everything, operators and states, in the ρ basis.
The second piece of information we need, in analogy with (2.83) and (2.88), is the fact that for the tensor product of k length-1 vacua the $(r) \to \rho$ change of basis is trivial,

$$\bigotimes_{r=1}^{k} |{+}{+}\rangle_{(r)} = \bigotimes_{\rho=0}^{k-1} |{+}{+}\rangle_{\rho}. \tag{2.112}$$

This writing is useful to understand the action of the twist fields, as they are naturally written in the ρ basis. It also means that

$$\begin{aligned}\bigotimes_{r=1}^{k} |{+}{+}\rangle_{(r)} &= \bigotimes_{r=1}^{k} \left[\lim_{z\to 0} e^{\frac{\mathrm{i}}{2}\left(H_{(r)}(z)+K_{(r)}(z)+\tilde{H}_{(r)}(\bar{z})+\tilde{K}_{(r)}(\bar{z})\right)}|0\rangle_{(r),\mathrm{NS}}\right]\\ &= \bigotimes_{\rho=0}^{k-1} \left[\lim_{z\to 0} e^{\frac{\mathrm{i}}{2}\left(H_{\rho}(z)+K_{\rho}(z)+\tilde{H}_{\rho}(\bar{z})+\tilde{K}_{\rho}(\bar{z})\right)}|0\rangle_{\rho,\mathrm{NS}}\right].\end{aligned} \tag{2.113}$$

We are now ready to write the fermionic part of the lowest-weight state in the $\Sigma_k^{s_1\dot{s}_2}$ multiplet in terms of the $H, K, \tilde{H}, \tilde{K}$ bosons. It has the form

$$\begin{aligned}\Sigma_k^{-\frac{k-1}{2},-\frac{k-1}{2}} &= \bigotimes_{\rho=0}^{k-1} e^{-\mathrm{i}\frac{\rho}{k}H_\rho}\, e^{-\mathrm{i}\frac{\rho}{k}K_\rho}\, e^{-\mathrm{i}\frac{\rho}{k}\tilde{H}_\rho}\, e^{-\mathrm{i}\frac{\rho}{k}\tilde{K}_\rho}\ ,\\ &= \bigotimes_{\rho=0}^{k-1} e^{-\mathrm{i}\frac{\rho}{k}\left(H_\rho+K_\rho+\tilde{H}_\rho+\tilde{K}_\rho\right)}\ ,\end{aligned} \tag{2.114}$$

which in the notation of (2.92) means that

$$\Sigma_\rho^{-\frac{k-1}{2},-\frac{k-1}{2}} = e^{-\mathrm{i}\frac{\rho}{k}\left(H_\rho+K_\rho+\tilde{H}_\rho+\tilde{K}_\rho\right)}. \tag{2.115}$$

All this allows us to write explicitly the action of the twist fields: the length-k state $|{+}{+}\rangle_k$ is produced acting with (2.114) on (2.113),

$$\begin{aligned}|{+}{+}\rangle_k &= \lim_{z\to 0} \Sigma^{-\frac{k-1}{1},-\frac{k-1}{2}}(z,\bar{z}) \bigotimes_{\rho=0}^{k-1} |{+}{+}\rangle_\rho\\ &= \lim_{\substack{z\to 0\\ w\to 0}} \bigotimes_{\rho=0}^{k-1} \left[e^{-\mathrm{i}\frac{\rho}{k}\left(H_\rho(z)+K_\rho(z)+\tilde{H}_\rho(\bar{z})+\tilde{K}_\rho(\bar{z})\right)}\, e^{\frac{\mathrm{i}}{2}\left(H_\rho(w)+K_\rho(w)+\tilde{H}_\rho(\bar{w})+\tilde{K}_\rho(\bar{w})\right)}|0\rangle_{\rho,\mathrm{NS}}\right]\\ &= \lim_{\substack{z\to 0\\ w\to 0}} |z-w|^{-(k-1)} \bigotimes_{\rho=0}^{k-1} \left[e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)\left(H_\rho(w)+K_\rho(w)+\tilde{H}_\rho(\bar{w})+\tilde{K}_\rho(\bar{w})\right)}|0\rangle_{\rho,\mathrm{NS}}\right],\end{aligned} \tag{2.116}$$

where $\forall\rho$ we computed the OPE between the the ρ part of the twist operator and the ρ part of the spectral flow operator. For each term involving an holomorphic boson we have a structure like

$$e^{-\mathrm{i}\frac{\rho}{k}H_\rho(z)}e^{\frac{\mathrm{i}}{2}H_\rho(w)} = (z-w)^{-\frac{\rho}{2k}} \exp\left[\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)H_\rho(w) + O(z-w)\right]. \qquad (2.117)$$

Chapter 3

The bulk gravitational theory

The bulk dual interpretation of the CFT described in the previous chapter was already pointed out in [24] and has been the starting point for the research on AdS_3 holography. This was later applied to black hole physics, especially in the context of the fuzzball proposal [14, 60] in which classes of black hole microstate geometries were described in terms of their boundary CFT dual states [1, 32, 33, 37, 38, 48, 58, 62, 63]. The purpose of this chapter is that of describing the geometric setting of the bulk gravitational theory, so that the statement of the various dualities between geometries and states will be stated as clearly as possible.

3.1 The 3-charge geometry

Let's start from type IIB supergravity on a spacetime with topology $\mathcal{M}_5 \times S^1 \times T^4$. Assuming invariance under translation in the T^4 directions, the resulting solution can be written [74] in the string frame as

$$\begin{aligned}
ds^2_{(10)} &= -\frac{2\alpha}{\sqrt{Z_1 Z_2}}\,(dv+\beta)\left[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\right]+\sqrt{Z_1 Z_2}\,ds^2_4+\sqrt{\frac{Z_1}{Z_2}}\,d\hat{s}^2_4\,, && \text{(3.1a)}\\
e^{2\phi} &= \alpha\,\frac{Z_1}{Z_2}\,, && \text{(3.1b)}\\
B &= -\frac{\alpha\,Z_4}{Z_1 Z_2}\,(du+\omega)\wedge(dv+\beta)+a_4\wedge(dv+\beta)+\delta_2\,, && \text{(3.1c)}\\
C_0 &= \frac{Z_4}{Z_1}\,, && \text{(3.1d)}\\
C_2 &= -\frac{\alpha}{Z_1}\,(du+\omega)\wedge(dv+\beta)+a_1\wedge(dv+\beta)+\gamma_2\,, && \text{(3.1e)}\\
C_4 &= \frac{Z_4}{Z_2}\,\hat{\mathrm{vol}}_4-\frac{\alpha\,Z_4}{Z_1 Z_2}\,\gamma_2\wedge(du+\omega)\wedge(dv+\beta)+x_3\wedge(dv+\beta)\,, && \text{(3.1f)}
\end{aligned}$$

where

$$\alpha=\frac{Z_1 Z_2}{\mathcal{P}}\,,\qquad \mathcal{P}=Z_1 Z_2-Z_4^2\,. \tag{3.2}$$

The flat metric of T^4 has been denoted as $d\hat{s}_4^2$, while ds_4^2 corresponds to a 4-dimensional Euclidean line element that reduces asymptotically to flat $\mathbb{R}^4$. t and y are respectively the time coordinate and the coordinate along the S^1, with radius R_y, and we also defined the light-cone coordinates

$$u = \frac{t-y}{\sqrt{2}}\,, \quad v = \frac{t+y}{\sqrt{2}}\,. \tag{3.3}$$

The solution depends on the following objects: four scalar functions, $Z_1, Z_2, Z_4, \mathcal{F}$; four 1-forms on $\mathbb{R}^4$, β, ω, a_1, a_4; two 2-forms on $\mathbb{R}^4$, γ_2, δ_2; one 3-form on $\mathbb{R}^4$, x_3. All these objects depend in general on v and on the $\mathbb{R}^4$ coordinates x_i. Supersymmetry and the equations of motion are satisfied if the conditions in [74] are satisfied. We will call (3.1) the **3-charge solution**. Sometimes it will be convenient do define the 1-forms

$$A \equiv -\frac{\beta+\omega}{\sqrt{2}}, \qquad B \equiv -\frac{\beta-\omega}{\sqrt{2}}. \tag{3.4}$$

3.1.1 Reduction to $5d$ Strominger-Vafa black hole

The 3-charge solution is a generalization of the supergravity solution obtained by wrapping a number n_1 of $D1$-branes around the S^1, a number n_5 of $D5$-branes along bith the S^1 and the T^4 and allowing a number n_p of momentum modes to propagate along the $D1$'s (see section 5.1 of [60] and references therein, or [75]), which corresponds to a 3-charge (Strominger-Vafa) black hole in 5 dimensions, upon reduction over $S^1 \times T^4$. The corresponding metric will be called 3-charge *naive geometry* and has the form (3.1a) with

$$\begin{aligned} Z_1 &= 1 + \frac{Q_1}{r^2}, & Z_2 &= 1 + \frac{Q_5}{r^2}, & \mathcal{F} &= -\frac{2Q_p}{r^2}, \\ Z_4 &= 0, & \beta &= \omega = 0, & ds_4^2 &= dx_i\, dx_j\, \delta^{ij}. \end{aligned} \tag{3.5}$$

The charges can be expressed in terms of the D-branes data n_1, n_5, n_p as

$$Q_1 = \frac{n_1 g_s (\alpha')^3}{V}\ , \quad Q_5 = g_s n_5 \alpha'\ , \quad Q_p = \frac{n_p g_s^2 (\alpha')^4}{V R_y^2}\ , \tag{3.6}$$

where g_s is the string coupling constant, α' is the Regge slope and V parametrizes the volume of the T^4 as below (3.19).

Let's consider the general Ansatz (3.1a) and evaluate it on the solution (3.5). The first step is the dimensional reduction of the metric on the T^4. The metric is globally a product and no off-diagonal blocks are present, so the $6d$ metric in the string frame is just the $6d$ block of the $10d$ one: let's call it G_{MN}, with line element

$$\begin{aligned} ds^2 &= G_{MN}\, dx^M dx^N \\ &= -\frac{1}{\sqrt{Z_1 Z_2}}\left(1+\frac{\mathcal{F}}{2}\right)dt^2 + \frac{1}{\sqrt{Z_1 Z_2}}\left(1-\frac{\mathcal{F}}{2}\right)dy^2 - \frac{\mathcal{F}}{\sqrt{Z_1 Z_2}}\,dt dy + \sqrt{Z_1 Z_2}\left(dr^2 + r^2 d\Omega_3^2\right), \end{aligned} \tag{3.7}$$

where we wrote the flat $\mathbb{R}^4$ part in spherical coordinate, $d\Omega_3^2$ being the metric of S^3 with unit radius,

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2. \tag{3.8}$$

Let's now rename as $\mu, \nu, \ldots$ the indices along the direction transverse to the S^1 and as α the indices along it (so we have $\alpha \equiv y$): this way the 6-dimensional indices are split into $5+1$ as $\{M\}_{M=t,r,y,\theta,\phi,\psi} = \{\mu, \alpha\}_{\mu=t,r,\theta,\phi,\psi;\, \alpha=y} = \{\mu, y\}_{\mu=t,r,\theta,\phi,\psi}$. We can now perform Kaluza-Klein dimensional reduction on the S^1. Following the recipe in [76] we define a 6-dimensional Vielbein $\hat{e}^{\hat{A}}_M$, where $\hat{A}, \hat{B}, \ldots = 0, \ldots, 5$ are six flat indices with Lorentz signature raised and lowered using the 6-dimensional Minkowski metric $\eta_{\hat{A}\hat{B}} = \mathrm{diag}(-1, 1, \ldots, 1)$, in terms of which the 6-dimensional metric is obtained as

$$G_{MN} = \hat{e}^{\hat{A}}_M \hat{e}^{\hat{B}}_N\, \eta_{\hat{A}\hat{B}}. \tag{3.9}$$

Since the ty block of the metric is the only one with off-diagonal components, we can restrict the analysis to that 2×2 block,

$$[G_{MN}]_{\text{ty block}} = \begin{bmatrix} G_{tt} & G_{ty} \\ G_{yt} & G_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{Z_1 Z_2}}\left(1+\frac{\mathcal{F}}{2}\right) & -\frac{\mathcal{F}}{2\sqrt{Z_1 Z_2}} \\ -\frac{\mathcal{F}}{2\sqrt{Z_1 Z_2}} & \frac{1}{\sqrt{Z_1 Z_2}}\left(1-\frac{\mathcal{F}}{2}\right) \end{bmatrix}. \tag{3.10}$$

Using local Lorentz invariance, again restricting ourselves to the ty block, we can put the 6-dimensional Vielbein in triangular form,

$$\left[\hat{e}^{\hat{A}}_M\right] = \begin{bmatrix} \hat{e}^0_t & \hat{e}^1_t \\ 0 & \hat{e}^1_y \end{bmatrix}, \tag{3.11}$$

and using (3.9) we get

$$\begin{aligned} &\hat{e}^0_t = (Z_1 Z_2)^{-1/4}\left(1-\frac{\mathcal{F}}{2}\right)^{-1/2}, \quad \hat{e}^1_t = -\frac{\mathcal{F}}{2}\,(Z_1 Z_2)^{-1/4}\left(1-\frac{\mathcal{F}}{2}\right)^{-1/2}, \\ &\hat{e}^1_y = (Z_1 Z_2)^{-1/4}\left(1-\frac{\mathcal{F}}{2}\right)^{1/2}. \end{aligned} \tag{3.12}$$

We can now define a 5-dimensional reduced metric $g_{\mu\nu}$ with Vielbein e^A_μ ($A, B, \ldots = 0, 2, 3, 4, 5$) and a metric $g_{\alpha\beta}$ on the compact space with Vielbein E^a_α ($a = 1$), with indices $A, B, \ldots$ raised and lowered with the 5-dimensional Minkowski metric $\eta_{AB} =$

diag$(-1,1,\ldots,1)$ and indices $a,b,\ldots$ (in this case just one) raised and lowered with the Euclidean metric δ_{ab},

$$g_{\mu\nu} = e^A_\mu e^B_\nu \, \eta_{AB}, \qquad g_{yy} = E^a_y E^b_y \, \delta_{ab} = \left(E^1_y\right)^2 . \tag{3.13}$$

We can write the 6-dimensional Vielbein in terms of the above ones in the ty block as

$$\left[\hat{e}^{\hat{A}}_M\right] = \left[\begin{array}{c|c} e^A_\mu & A^\alpha_\mu E^a_\alpha \\ \hline 0 & E^a_\alpha \end{array}\right] = \begin{bmatrix} e^0_t & A^y_t E^1_y \\ 0 & E^1_y \end{bmatrix}, \tag{3.14}$$

and comparing with (3.11) and (3.12) we get

$$\begin{aligned} E^1_y &= \hat{e}^1_y = (Z_1 Z_2)^{-1/4} \left(1 - \frac{\mathcal{F}}{2}\right)^{1/2}, \qquad e^0_t = \hat{e}^0_t = (Z_1 Z_2)^{-1/4} \left(1 - \frac{\mathcal{F}}{2}\right)^{-1/2}, \\ A^y_t &= \frac{\hat{e}^1_t}{E^1_y} = -\frac{\mathcal{F}/2}{\left(1 - \frac{\mathcal{F}}{2}\right)}. \end{aligned} \tag{3.15}$$

The metric in 5 dimensions is obtained from the Vielbein e^A_μ, but with a subtlety: if we worked at the level of the action, we would have seen that the action in 5 dimensions has the canonical form only if we rescale the metric by a factor

$$\left|\det\left(g_{yy}\right)\right|^{\frac{1}{d-2}}\Big|_{d=5} = g_{yy}^{1/3} = \left(E^1_y\right)^{2/3} = (Z_1 Z_2)^{1/6} \left(1 - \frac{\mathcal{F}}{2}\right)^{1/3} . \tag{3.16}$$

This rescaling affects all the components of the 5-dimensional metric. The result is

$$\begin{aligned} g_{tt} &= e^A_t e^B_t \, \eta_{AB} \;\left|\det\left(g_{yy}\right)\right|^{\frac{1}{d-2}}\Big|_{d=5} = -\left(Z_1 Z_2\right)^{-2/3} \left(1 - \frac{\mathcal{F}}{2}\right)^{-2/3}, \\ g_{rr} &= G_{rr} \left|\det\left(g_{yy}\right)\right|^{\frac{1}{d-2}}\Big|_{d=5} = \left(Z_1 Z_2\right)^{1/3} \left(1 - \frac{\mathcal{F}}{2}\right)^{1/3}, \\ g_{\theta\theta} &= G_{\theta\theta} \left|\det\left(g_{yy}\right)\right|^{\frac{1}{d-2}}\Big|_{d=5} = r^2 \left(Z_1 Z_2\right)^{1/3} \left(1 - \frac{\mathcal{F}}{2}\right)^{1/3}, \\ g_{\phi\phi} &= G_{\phi\phi} \left|\det\left(g_{yy}\right)\right|^{\frac{1}{d-2}}\Big|_{d=5} = r^2 \sin^2\theta \, \left(Z_1 Z_2\right)^{1/3} \left(1 - \frac{\mathcal{F}}{2}\right)^{1/3}, \\ g_{\psi\psi} &= G_{\psi\psi} \left|\det\left(g_{yy}\right)\right|^{\frac{1}{d-2}}\Big|_{d=5} = r^2 \cos^2\theta \, \left(Z_1 Z_2\right)^{1/3} \left(1 - \frac{\mathcal{F}}{2}\right)^{1/3} . \end{aligned} \tag{3.17}$$

Substituting the forms (3.5) for $Z_1, Z_2, \mathcal{F}$, the corresponding line element is

$$\begin{aligned} ds^2_{5d} = &-\frac{dt^2}{\left(1+\frac{Q_1}{r^2}\right)^{2/3}\left(1+\frac{Q_5}{r^2}\right)^{2/3}\left(1+\frac{Q_p}{r^2}\right)^{2/3}}+ \\ &+\left(1+\frac{Q_1}{r^2}\right)^{1/3}\left(1+\frac{Q_5}{r^2}\right)^{1/3}\left(1+\frac{Q_p}{r^2}\right)^{1/3}\left[dr^2+r^2 d\Omega_3^2\right], \end{aligned} \tag{3.18}$$

which is the metric of for the $5d$ 3-charge black hole in a set of coordinates where the horizon is at $r=0$. In this case the Einstein and the string frame metrics are related just by a rescaling by a constant: if we look at (3.1) and call $d\hat{s}_4^2 = \hat{g}_{ab}\, dx^a dx^b$ the line element of the T^4, we have that after dimensional reduction to $6d$ the dilaton becomes

$$e^{-2\phi_6} = e^{-2\phi_{10}}\,\mathrm{Vol}\left(T^4\right) = \frac{Z_2}{\alpha Z_1}\left(\det\left(\sqrt{\frac{Z_1}{Z_2}}\,\hat{g}_{ab}\right)\right)^{1/2} = \frac{(2\pi)^4\, V}{\alpha}, \tag{3.19}$$

where we defined V through $(\det(\hat{g}_{ab}))^{1/2} \equiv (2\pi)^4\, V$. Further reducing on S^1 we get that the five-dimensional dilaton is

$$e^{-2\phi_5} = e^{-2\phi_6}\,\mathrm{Vol}\left(S^1\right) = \frac{(2\pi)^4\, V}{\alpha}\, 2\pi R_y. \tag{3.20}$$

This holds in general for the Ansatz (3.1): specifying to the case of the naive metric (3.5) we have that since $Z_4 = 0$ then $\alpha = 1$, so

$$e^{-2\phi_5}|_{\text{naive}} = (2\pi)^5\, V R_y, \tag{3.21}$$

which is a constant. As the string frame ($g^S_{\mu\nu}$) and Einstein frame metrics ($g^E_{\mu\nu}$) are related by

$$g^E_{\mu\nu} = e^{-\frac{4}{d-2}\,\phi_d}\, g^S_{\mu\nu}, \tag{3.22}$$

(3.21) allows us to check that the $5d$ result (3.18) agrees up to a constant factor with the one given in chapter 11 of [77], which is in the Einstein frame. As a last remark, if we take the asymptotic ($r \to \infty$) limit we immediately see that the geometry reduces to 5-dimensional Minkowski spacetime: the naive 3-charge geometry is asymptotically flat. This will hold also for the 3-charge microstate geometries.

The above results give us one of the main motivations behind this work, because they show that on the bulk side of the AdS/CFT correspondence we indeed have a $5d$ black hole. In the following we will consider generalizations of the naive geometry (3.7), i.e. more complicated choices of the functions the Ansatz (3.1) depends upon, and in the spirit of the fuzzball proposal we will interpret them as microstates of the $5d$ black hole. Furthermore, black hole microstates will be put in correspondence with CFT states.

The Strominger-Vafa black hole is a BPS supergravity solution preserving 1/8 of the supersymmetries, with finite horizon area and therefore nonzero Bekenstein-Hawking entropy. Being a BPS solution, it corresponds to an extremal black hole, so the mass can be expressed as a function of the other charges as [77]

$$M = M_1 + M_2 + M_3, \qquad M_i = \frac{\pi Q_i}{4G_5}, \quad i = 1, 5, p, \tag{3.23}$$

where G_5 is Newton's constant reduced to 5 dimensions. As usual for extremal black holes, the surface gravity is zero and therefore the Hawking temperature is also zero.

The horizon area of the Strominger-Vafa black hole can be computed in $10d$ as in [14], and in the Einstein frame we get

$$A_H = \left(2\pi^2\right)(2\pi R_y)\left((2\pi)^4 V\right)(Q_1 Q_5 Q_p)^{1/2}. \tag{3.24}$$

Newton's constant in $5d$ is given by

$$G_5 = \frac{G_{10}}{(2\pi R_y)\left((2\pi)^4 V\right)}, \tag{3.25}$$

so the Bekenstein-Hawking entropy is

$$S_{\text{B-H}} = \frac{A_H}{4G_{10}} = \frac{2\pi^2 (Q_1 Q_5 Q_p)^{1/2}}{4G_5}. \tag{3.26}$$

G_{10} can be expressed in terms of the string coupling constant g_s and of the Regge slope α' as

$$G_{10} = 8\pi^6 g_s^2 (\alpha')^4, \tag{3.27}$$

so using (3.6) we get

$$S_{\text{B-H}} = 2\pi (n_1 n_5 n_p)^{1/2}. \tag{3.28}$$

The fact that all the moduli of the theory (the volumes of the compact spaces, the string coupling constant and the Regge slope) cancel out to give a result depending only on three integer numbers was one of the ideas from which the whole black hole microstate study originated. The microstate counting can also be performed microscopically: this is performed in [14] in a different duality frame, and the result matches exactly (3.28).

The naive 3-charge metric and the general 3-charge geometries are all 1/8-BPS solutions, provided momentum modes along the $D1$-branes (which give the charge Q_p) are either only left-moving or right-moving: modes moving in opposite directions break orthogonal sectors of the supersymmetry and having both would break supersymmetry completely.

3.1.2 Dimensional reduction on T^4 and asymptotic form of the metric

As was mentioned above, reducing (3.1a) on T^4 we get the $6d$ string frame metric

$$ds^2_{S,(6)} = -\frac{2\alpha}{\sqrt{Z_1 Z_2}}\,(dv+\beta)\left[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\right]+\sqrt{Z_1 Z_2}\,ds^2_4\,. \tag{3.29}$$

In order to get the Einstein frame metric we use (3.22) with $d=6$ and ϕ_6 given by (3.19). Setting $V=1$ we get

$$ds^2_{E,(6)} = -\frac{2}{\sqrt{\mathcal{P}}}\,(dv+\beta)\left[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\right]+\sqrt{\mathcal{P}}\,ds^2_4\,. \tag{3.30}$$

The first operation we have to perform on this geometry is the **decoupling limit** (or near-horizon limit),

$$Q_p \ll r^2 \ll Q_1, Q_5. \tag{3.31}$$

In this case the functions Z_1, Z_2 in the naive 3-charge geometries are approximated by

$$Z_1 \simeq \frac{Q_1}{r^2}, \qquad Z_2 \simeq \frac{Q_5}{r^2}. \tag{3.32}$$

In the more general cases of 3-charge microstates, Z_1 and Z_2 will get other contributions in the form of terms with higher and higher negative powers of r. The important fact contained in (3.32) is that the constant terms can be neglected: they cause the geometry to be asymptotically flat, so in the decoupling limit asymptotic flatness is lost. If we then take the $r\to\infty$ limit again, starting from (3.30), writing

$$ds^2_4 = dr^2 + r^2\,\mathrm{d}\Omega^2_3, \tag{3.33}$$

we get

$$ds^2_{E,(6)} \simeq \sqrt{Q_1 Q_5}\left[\frac{dr^2}{r^2}+\frac{r^2}{Q_1 Q_5}\left(-dt^2+dy^2\right)\right]+\sqrt{Q_1 Q_5}\,d\Omega^2_3. \tag{3.34}$$

Defining

$$u \equiv \frac{r}{\sqrt{Q_1 Q_5}} \tag{3.35}$$

the metric becomes

$$ds^2_{E,(6)} \simeq \sqrt{Q_1 Q_5}\left[\frac{du^2}{u^2}+u^2\left(-dt^2+dy^2\right)\right]+\sqrt{Q_1 Q_5}\,d\Omega^2_3, \tag{3.36}$$

which is the metric for $AdS_3\times S^3$ with the AdS factor written in Poincaré coordinates and with radii

$$R_{AdS_3} = \sqrt{Q_1 Q_5} = R_{S^3}. \tag{3.37}$$

In the case of the microstate geometries this asymptotic limit is the same, as terms with higher negative powers of r in Z_1 and Z_2 will be subleading as $r \to \infty$. The decoupling limit and the asymptotic geometry we obtained are crucial in order for the holographic construction to hold: the AdS_3 factor is what allows us to exploit AdS_3 holography and characterize the microstate geometries in terms of CFT states.

3.2 The 2-charge geometry

If we start from (3.1) and we set $\mathcal{F} = 0$ we get a solution called **2-charge geometry**, which in the string frame is

$$ds^2_{(10)} = -\frac{2\alpha}{\sqrt{Z_1 Z_2}} \left(dv + \beta\right) \left(du + \omega\right) + \sqrt{Z_1 Z_2}\, ds^2_4 + \sqrt{\frac{Z_1}{Z_2}}\, d\hat{s}^2_4 \,. \tag{3.38}$$

This solution is again parametrized by the same objects as in (3.1), apart from $\mathcal{F}$, and has a "naive counterpart" obtained starting from the 3-charge one (3.5) and setting $Q_p = 0$, which corresponds to having the same $D1$ and $D5$-branes as in the 3-charge case, but with no momentum modes along the $D1$-branes: because of this, less supersymmetry is broken, and indeed 1/4 of the supercharges are preserved. Also, in 2-charge geometries the functions Z_1, Z_2 and Z_4 do not depend on the coordinate v. As we see from (3.24), the horizon area is zero: the solution is a degenerate black hole for which no Bekenstein-Hawking entropy is present[7]. Surprisingly, this doesn't correspond to zero microstate entropy, as counted using the D-branes setup: the microscopic computation [14] gives a nonzero result (in agreement with [78]). The 2-charge solution also has a description in terms of (1/4-BPS) microstates, dual to CFT states in which the charge corresponding to the momentum modes along the $D1$-brane is turned off. As in the 3-charge case, the 2-charge geometry is asymptotically flat.

In the case of 2-charge geometries the decoupling limit still exists, and we set $Q_p = 0$. Taking $r^2 \gg Q_p$ in the 3-charge case allowed us to neglect all the term containing Q_p in the geometry: with 2-charges they are simply not there to begin with, and the result is the same. AdS_3 holography is indeed applied both to 1/8- and to 1/4-BPS microstates, the only difference being in what CFT dual states we have.

3.3 General structure of the geometries

In the previous sections we saw the construction and the characteristics of families of solutions, giving some motivations about why we are considering them. Having exposed

[7] One should be careful about this kind of statement, as this analysis relies on an extrapolation to strongly coupled string theory. What is known, though, is that if higher derivative corrections to the Einstein-Hilbert action are considered, then the 2-charge black hole has nonzero entropy as well, if the internal space is $K3$ [78].

all the ingredients, we can now look at the structure of the geometries in a broader perspective, clarifying what we really mean by microstate geometries. Consider the 5-dimensional Strominger-Vafa (i.e. 3-charge) black hole: according to the Bekenstein-Hawking formula it has an entropy proportional to the area of its horizon. Moreover, away from extremality it also has a temperature set by the values of the mass and charges, the Hawking temperature given by the surface gravity. We may therefore try to describe the black hole as a statistical mix of microstates in a canonical ensemble at a certain temperature. The fuzzball proposal is indeed an attempt in this direction, and it postulates that at least some of the entropy is given by microstates which are smooth, horizonless geometries with the same charges as the black hole and which asymptotically become indistinguishable from the black hole geometry itself. Looking at the previous section, we identify the black hole as the naive 3-charge geometry and the microstates as the Ansatz (3.1), with some suitable choice of the functions $Z_1, Z_2, Z_4, \ldots$ it is parametrized by. The same things happens in the case with 2 charges, the only difference being that we describe the microstates of a somewhat degenerate black hole solution.

The geometry can be seen schematically as follows:

- For $r \gg \sqrt{Q_{1,5}}$ we are in the **asymptotically flat regime**: if we either consider the black hole or a microstate, at infinity we still get Minkowski spacetime in $5d$.
- As r decreases, we encounter a region called **neck** in which the functions Z_1, Z_2 etc. do not differ between the naive and the microstate case: here a microstate geometry and a black hole are still indistinguishable.
- For $|g| \ll r \ll \sqrt{Q_{1,5}}$, where $|g|$ is an estimate of the order of some function(s) g that gives the terms of order $r^{-3}, r^{-4}, \ldots$ in Z_1, Z_2, Z_4 (i.e. the deviation from the naive geometry), we are in a region called **throat** and we can forget the 1's in Z_1, Z_2, Z_4, obtaining a geometry that reduces asymptotically (i.e. if we then take the $r \to \infty$ limit again) to $AdS_3 \times S^3 \times T^4$. In the case of 2-charge geometries, the functions g are called **shape functions** and exactly parametrize the geometry, while in the 3-charge case the the parametrization is not as easy. The shape functions give the deviation of a microstate geometry from that of a black hole.
- For $r \sim |g|$ the microstate geometry deviates from that of a black hole and depends strongly on the shape functions. Since microstate geometries are smooth and horizonless, as r decreases we don't encounter any coordinate or curvature singularity: as $r \to 0$ the geometry ends in a **cap** whose shape is given by the shape functions. The shape of the cap is what distinguishes one microstates from another. It is possible to give an estimate of the typical scale at which the microstate and the naive geometry start to differ [14]: this happens at the

horizon scale. Indeed the fuzzball proposal is among the class of theories for which nontrivial effects take place already at the horizon scale, where for big enough black holes general relativity would suggest nothing happens: fuzzballs are in fact additional structure at the horizon.

For 2-charge geometries, there is an algorithmic procedure to associate shape functions g and CFT states, at least if they have a certain structure. In the 3-charge case this is not possible and the geometry-state matching must be done case by case. Finally, notice that in taking the decoupling limit we lose information about the asymptotically flat behaviour of the geometry: if we start already in the decoupling limit (e.g. we work out the gravity-CFT matching starting from a CFT state), it may not be trivial to extend the microstate geometry we obtain to the asymptotically flat regime [59]. Finally, we want to stress that the horizon scale is only a typical scale for stringy effects to kick in: indeed we will consider nontypical geometries which, in the decoupling limit, differ from the asymptotic $AdS_3 \times S^3 \times T^4$ geometry already near the AdS_3 boundary.

Chapter 4

$1/4$ and $1/8$-BPS precision holography

This chapter relies on the results of [1] and is dedicated to the identification of the precision dictionary between geometries (in both the 1/4 and 1/8-BPS cases) and CFT states. The states will be superpositions of vacua in the Ramond sectors of the CFT copies, or of excitations thereof, and they will be matched to specific forms for the functions and 1-forms the microstate geometries are parametrized by, see (3.1).
In practice, this amounts to matching the first nontrivial terms inside $Z_1, Z_2, Z_4, \ldots$ to the VEVs of CFT operators of small dimension on the state dual to the geometry considered. In the 1/4-BPS case, for the classes of geometries and states considered here there will be an algorithmic way to perform the matching. Precision holography is also tested using entanglement entropy as an observable: here we have both a CFT prescription [70] and an holographic (bulk) one [71] to perform the calculation, so once we have a guess for a geometry-state matching we can compute the entanglement entropy on both sides and check they agree.

4.1 Generalities about the AdS/CFT correspondence

The statement of the AdS/CFT correspondence [24–26] is that a dictionary exists between string theory in asymptotically AdS space and a gauge theory in flat space living on its boundary, and quantities can be computed equivalently on either side. Regimes amenable to direct computations are classical (super)gravity on the bulk side and the perturbative (weak coupling) regime on the boundary side. The regimes of validity of these approximations, in the case of AdS_5/CFT_4 holography, are detailed in [79]. The string scale is set by the Regge slope α' as

$$\alpha' = l_s^2, \tag{4.1}$$

where l_s is the string length. Stringy effects kick in when the string scale is comparable to the curvature scale R of spacetime, $\alpha'/R^2 \sim 1$, so strings can be ignored if α'/R^2 is small. On the bulk side we have another parameter governing the magnitude of quantum effects, the string coupling g_s. α' and g_s are independent: if they are both big we have to consider quantum strings (string scattering, etc.), but it could also be the case that only one is big, in which case we can have classical strings or quantum point particles. The most tractable case on the bulk side definitely corresponds to having both parameters small, in fact this means we have classical fields, and the full string theory reduces to classical supergravity. On the boundary CFT side we also have two parameters, the Yang-Mills coupling constant g_{YM} and the number of colors N of the $SU(N)$ gauge group[8]. We can trade g_{YM} for the 't Hooft coupling

$$\lambda \equiv g_{YM}^2 N. \tag{4.2}$$

The large N expansion corresponds to an expansion in the topology of the scattering diagrams, with the leading $N \to \infty$ term corresponding to planar diagrams. Fixing a topology, we can then perform an expansion in λ. The perturbative regime on the CFT side corresponds to having λ small. The key relations that AdS/CFT provides are

$$g_s = g_{YM}^2 = \frac{\lambda}{N}, \qquad \alpha' = \frac{R^2}{\sqrt{\lambda}}. \tag{4.3}$$

In order to get classical supergravity on the bulk side we can take the following limits on the parameters: first we keep λ fixed and we send $N \to \infty$. This way we have that $g_s \to 0$, and thus no quantum effects are expected to appear. In this regime α' can still have any value, so in general we'll have a theory of classical strings. Then we can also send $\lambda \to \infty$, which corresponds to $\alpha' \to 0$, and we get a bulk theory of classical fields, supergravity. This is the bulk regime we want, but we can achieve it only at the price of having a strong coupling for the boundary CFT. Also, by the above reasoning we see that in the large N limit, corrections in $1/N$ represent quantum corrections in the bulk theory.

In AdS_3 holography the dual field theory description is a 2-dimensional conformal field theory, and the role of the parameter N is played by the central charge c (actually $N^2 - 1 \sim N^2 \leftrightarrow c$). In the case of the D1D5 CFT described in Chapter 2 the total central charge is[9]

$$c = \tilde{c} = 6\, n_1 n_5. \tag{4.4}$$

The theory described in Chapter 2 is a free CFT, but we can imagine to deform it

[8]Here we have in mind the $AdS_5 \times S^5$ case; the AdS_3 case will be detailed below.

[9]Unfortunately we use the symbol N to refer both to the gauge group of $SU(N)$ gauge theories and to the number $n_1 n_5$ of CFT copies. We refer to $SU(N)$ only in this section, while in all other parts of this work we only mean $N = n_1 n_5$.

turning on some coupling: the classical gravity regime would then correspond to having a strongly coupled theory and to sending the number of CFT copies $N \to \infty$. This is confirmed by the link between Newton's constant in $3d$ and the central charge [79],

$$c = \frac{3R}{2\,G_N^{(3)}}, \tag{4.5}$$

where in our case the curvature radius is the AdS_3 (and S^3) radius, $R = R_{AdS_3} = R_{S^3}$: if we want $G_N^{(3)}$ to be small we need N to be large. While working at large N, we'll still work at the **orbifold point** of the CFT where the couplings are all set to zero and all we are left with is a free theory: this can be done as long as we look at **protected quantities**, i.e. observables that do not depend on the point in moduli space (values of the couplings) we consider. Protected quantities can therefore be computed in the free theory and then extrapolated to strong coupling. In particular, a certain class of 3-point functions[10] are known to be protected by non-renormalization theorems [27]: these are precisely the objects we will be looking at.

One of the aims of our analysis is to show that it is possible to use the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality to study the microstates of the Strominger-Vafa black hole, which carry D1, D5 and momentum charges. On the CFT side, the microstates that can have a dual geometric description in classical supergravity are the BPS semiclassical states with the charges of the black hole. The expectation values of the BPS operators in a semiclassical state $|s_i\rangle$ of this type give direct information on the structure of the bulk solution corresponding to $|s_i\rangle$: by using the standard AdS/CFT dictionary, each BPS operator[11] corresponds to a supergravity mode and so, roughly speaking, its expectation value determines a particular deviation of the microstate solution from $\mathrm{AdS}_3 \times S^3$.

This approach was pioneered for the D1D5 CFT in [32, 63, 80] where it was applied to 1/4-BPS configurations, which correspond on the bulk side to the microstates of a black hole of vanishing horizon area in the supergravity limit. As we will discuss in detail, much of the technology developed in those works can be directly used also in the 1/8-BPS case. In order to illustrate the method we will focus on the expectation values of the simplest class of BPS operators, i.e. those of (total) dimension one. The main stumbling block preventing the generalization of [32, 63, 80] to the 1/8-BPS case has been the absence of a rich enough class of geometries with a known CFT dual. The geometries obtained by spectral flow in [37, 38, 41] have a too simple structure to highlight the general pattern, while we do not know an explicit CFT dual for the general multicentre solutions [40, 55–57]. However, recently a new class of 1/8-BPS

[10] We will often call VEVs our 3-point functions, as the two operators corresponding to the bulk geometry are the same and are taken to be asymptotic (in and out) states.

[11] As explained below, for the time being we focus on operators of low dimension, even though it would be very interesting to extend the analysis further.

solutions was derived in [58] with an explicit proposal for the dual semiclassical CFT states. We will focus on "atypical" states in this class which differ from $\mathrm{AdS}_3 \times S^3$ already very close to the AdS boundary. Then, we have non-trivial expectation values already for BPS operators of dimension one and we can show that they match in a non-trivial way the supergravity results. This provides a strong check for the proposed dictionary between 3-charge geometries and semiclassical states.

Another interesting way to reconstruct the spacetime structure from CFT data is to study the Entanglement Entropy (EE) both on the CFT side [70] and by using the holographic prescription of [71,72]. In particular the EE of a space interval in the CFT_2 probes the metric of the dual space-time deeper in the holographic direction as the interval becomes bigger. The application of this approach to the 1/4-BPS geometries that are (small) black hole microstates was first discussed in [81] focusing on the first terms in the limit of small l/R_y (l is the size of the EE interval and R_y is the radius of the space direction of the CFT). Again, in order to have a non-trivial match between bulk and CFT results at this order one needs to focus on "atypical" geometries, however this is sufficient to highlight the general issues that need to be understood in order to use the EE as a tool to characterise the microstate geometries. In general these geometries are not a metric product of (deformed) AdS_3 and a compact 3d space, so one needs to reformulate the extremization problem of [71,72] in terms of a codimension 2 submanifold of a 6D geometry that is asymptotically $\mathrm{AdS}_3 \times S^3$. A proposal on how to do this in a computationally efficient way is discussed in [81]. Here we show that this proposal is equivalent to the general covariant prescription of [82] and, as an explicit application to 1/8-BPS configurations, we test this holographic prescription for the EE in the case of the superstrata geometries derived in [58].

4.2 Gravity-CFT map for D1-D5 states

The aim of this section is to precisely characterize the semiclassical states that are dual to the class of superstrata constructed in [58]. We first review the CFT/geometry dictionary in the 1/4-BPS sector by summarising the results of [32, 63] in the language of orbifold CFT. Then we turn our attention to the 1/8-BPS sector relevant for the superstrata.

4.2.1 Gravity-CFT map in a 1/4-BPS sector

In Chapter 2 we introduced the concept of strands which can be used to define the states in the D1D5 CFT at the orbifold point. The RR ground state of each strand is denoted by $|s\rangle_k$, where $s = (0,0), (\pm,\pm)$ runs over one of the five[12] possible spin states

[12] We restrict here to bosonic states which are invariant under rotations of the internal space T^4. Hence our results trivially extend to the D1-D5 system compactified on K_3. If one included all the

and k is the length, or winding number, of the strand. A ground state of the D1D5 orbifold theory is obtained by taking the tensor product of $N_k^{(s)}$ copies of the strand $|s\rangle_k$, with the constraint that the total winding number be $N = n_1 n_5$. Thus a ground state is specified by a partition $\{N_k^{(s)}\}$ of N:

$$\psi_{\{N_k^{(s)}\}} \equiv \prod_{k,s} (|s\rangle_k)^{N_k^{(s)}}\,, \qquad \sum_{s,k} k\, N_k^{(s)} = N\,. \tag{4.6}$$

By convention we relate the norm of these states to the number of ways, $\mathcal{N}(\{N_k^{(s)}\})$, the strand configuration determined by the partition $\{N_k^{(s)}\}$ can be obtained starting from the state $\prod_{r=1}^{N} |{+}{+}\rangle_{(r)} \equiv |{+}{+}\rangle^N$:

$$(\psi_{\{N_k^{(s)}\}}, \psi_{\{N_k'^{(s)}\}}) = \delta_{\{N_k^{(s)}\},\{N_k'^{(s)}\}}\, \mathcal{N}(\{N_k^{(s)}\})\,. \tag{4.7}$$

To compute the combinatoric factor $\mathcal{N}(\{N_k^{(s)}\})$, consider the action of the twist field $\Sigma_k^{\pm\pm}$ on N copies of the CFT, to produce a strand of length k: there are $\frac{N!}{(N-k)!\,k}$ ways in which the twist field can act, corresponding to the possible choices of k among N copies, up to cyclic permutations [83]. The full state $\psi_{\{N_k^{(s)}\}}$ is obtained by acting repeatedly with twist fields, so that the total number of terms produced is

$$\frac{N!}{(N-k_1)!\,k_1}\frac{(N-k_1)!}{(N-k_1-k_2)!\,k_2}\cdots = \frac{N!}{\prod_{k,s} k^{N_k^{(s)}}}\,. \tag{4.8}$$

For strands with multiplicity $N_k^{(s)} > 1$, the order by which the $N_k^{(s)}$ twist operators act is immaterial, and one should hence divide by $N_k^{(s)}!$. Since each term produced by the action of twist operators has unit norm, one finds

$$\mathcal{N}(\{N_k^{(s)}\}) = \frac{N!}{\prod_{k,s} N_k^{(s)}!\, k^{N_k^{(s)}}}\,. \tag{4.9}$$

At the orbifold point, also the action of the operators on the CFT states contains a combinatoric part. Again this can be described in terms of permutations. The untwisted operators correspond to the identity permutation and act equally on each copy of the CFT. For instance, the total angular momenta are

$$J^3 = \sum_{r=1}^{N} J^3_{(r)}\,, \quad \tilde{J}^3 = \sum_{r=1}^{N} \tilde{J}^3_{(r)} \tag{4.10}$$

and, by construction, the states $\psi_{\{N_k^{(s)}\}}$ are eigenstates of the zero-modes of J^3 and $\tilde{J}^3$

bosonic states, one would have 3 extra states for the theory on T^4 and 19 extra states for K_3. On T^4 there are also 8 fermionic states, while there are no fermionic states for K_3.

with eigenvalues $\sum_{k,s} s\, N_k^{(s)}$. In general the action of an operator on a D1-D5 state involves the composition of the permutation defining the operator and the permutation defining the state. Twisted operators correspond to permutations containing cycles of length $k > 1$. For instance, in Section 4.3 we will consider the chiral primary operators with a cycle of length 2 and all others of length 1. We will still indicate them with the same symbol used in Chapter 2, $\Sigma_2^{\pm\pm}$, understanding that one has to sum over the contributions coming from any pair of the N CFT copies since the full operator contains a sum over all permutations with a single length 2 cycle.

The geometries dual to coherent superpositions of RR ground states have been constructed in [32–35]: as anticipated they are completely specified in terms of a closed curve in $\mathbb{R}^5$, $g_A(v')$ $(A = 1, \dots, 5)$. The parameter along the curve, v', has periodicity $L = 2\pi \frac{Q_5}{R_y}$, where Q_5 is the D5 charge and R_y is the radius of the S^1 on which the branes are wrapped. The equations that allow to construct the geometry given the profile $g_A(v')$, are listed in Appendix A. The map between geometries and states can however be expressed solely in terms of the profile: the general idea is that the 5 spin states s are related to the 5 components of $g_A(v')$, the length of each strand is related to the harmonic number in the Fourier expansion of $g_A(v')$, and the magnitude of each harmonic mode specifies the number of strands of each type. More precisely, define the Fourier expansions

$$\begin{aligned} g_1(v') + \mathrm{i}\, g_2(v') &= \sum_{n\neq 0} \frac{a_n^{(1)}}{n}\, \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,n}{L} v'} \,, \quad g_3(v') + \mathrm{i}\, g_3(v') \;= \sum_{n\neq 0} \frac{a_n^{(2)}}{n}\, \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,n}{L} v'} \\ g_5(v') &= -\mathrm{Im}\Big[\sum_{k=1}^{\infty} \frac{a_k^{(00)}}{k}\, \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,k}{L} v'}\Big] \,, \end{aligned} \tag{4.11}$$

where, for later convenience, we rename

$$a_{k>0}^{(1)} = a_k^{(++)} \,, \quad a_{k<0}^{(1)} = -a_{|k|}^{(--)} \,, \quad a_{k>0}^{(2)} = a_k^{(+-)} \,, \quad a_{k<0}^{(2)} = -a_{|k|}^{(-+)} \,, \tag{4.12}$$

and where we highlight the contribution to the $(J^3, \tilde{J}^3)$ quantum numbers of each excitation. The Fourier coefficients $a_k^{(s)}$ are in general complex and satisfy a constraint

$$\sum_k \Big[|a_k^{(++)}|^2 + |a_k^{(--)}|^2 + |a_k^{(+-)}|^2 + |a_k^{(-+)}|^2 + \frac{1}{2}|a_k^{(00)}|^2\Big] = \frac{Q_1 Q_5}{R_y^2} \,. \tag{4.13}$$

The dual CFT state is more naturally expressed in terms of dimensionless coefficients $A_k^{(s)}$:

$$A_k^{(\pm\pm)} \equiv R_y \sqrt{\frac{N}{Q_1 Q_5}}\, a_k^{(\pm\pm)} \,, \quad A_k^{(00)} \equiv R_y \sqrt{\frac{N}{2\, Q_1 Q_5}}\, a_k^{(00)} \,, \tag{4.14}$$

which satisfy

$$\sum_{k,s} |A_k^{(s)}|^2 = N\,. \tag{4.15}$$

A given set of Fourier coefficients $\{A_k^{(s)}\}$ specifies a profile $g_A(v')$ and hence a geometry; the CFT state dual to this geometry is [32, 62, 63]

$$\psi(\{A_k^{(s)}\}) = {\sum_{\{N_k^{(s)}\}}}' (\prod_{k,s} A_k^{(s)})^{N_k^{(s)}} \psi_{\{N_k^{(s)}\}} = {\sum_{\{N_k^{(s)}\}}}' \prod_{k,s} (A_k^{(s)}\, |s\rangle_k)^{N_k^{(s)}}\,, \tag{4.16}$$

where again the sum ${\sum}'_{\{N_k^{(s)}\}}$ is restricted to

$$\sum_{s,k} k\, N_k^{(s)} = N\,. \tag{4.17}$$

Eq. (4.16) gives the explicit map between gravity and CFT for states with D1, D5 charges. Notice that the states dual to geometries, $\psi(\{A_k^{(s)}\})$, are generically superpositions of angular momentum eigenstates $\psi_{\{N_k^{(s)}\}}$. The only exception is when a single Fourier coefficient $A_k^{(s)}$ is different from zero, and hence the CFT state is composed of N/k equal strands. The states whose dual geometries are well described in the classical supergravity limit are the ones in which the average numbers of strands of each type $(\overline{N}_k^{(s)})$ is very large: $\overline{N}_k^{(s)} \gg 1$. In this limit the sum over $\{N_k^{(s)}\}$ which appears in the definition of the state $\psi(\{A_k^{(s)}\})$ is peaked over the average numbers $\overline{N}_k^{(s)}$, which are determined by the magnitudes of the Fourier coefficients $A_k^{(s)}$. To see this, consider the norm of the state $\psi(\{A_k^{(s)}\})$:

$$|\psi(\{A_k^{(s)}\})|^2 = {\sum_{\{N_k^{(s)}\}}}' \mathcal{N}(\{N_k^{(s)}\}) \prod_{k,s} |A_k^{(s)}|^{2N_k^{(s)}}\,, \tag{4.18}$$

where we have used the orthogonality of the states $\psi_{\{N_k^{(s)}\}}$ (4.7). One can now study where the sum over $\{N_k^{(s)}\}$ in (4.18) is peaked in the large $N_k^{(s)}$ limit. Using the leading Stirling approximation for factorials, $\log N_k^{(s)}! \approx (N_k^{(s)} + 1/2) \log N_k^{(s)} - N_k^{(s)}$, the saddle point values $\overline{N}_k^{(s)}$ are the stationary points of the function

$$S(\{N_k^{(s)}\}) = \sum_{k,s} N_k^{(s)} \log |A_k^{(s)}|^2 - N_k^{(s)} \log N_k^{(s)} + N_k^{(s)} - N_k^{(s)} \log k\,, \tag{4.19}$$

with the constraint $\sum_{s,k} k\, N_k^{(s)} = N$. One finds

$$k\, \overline{N}_k^{(s)} = |A_k^{(s)}|^2\,, \tag{4.20}$$

which is consistent with (4.15).

In conclusion, in the state dual to the geometry specified by the Fourier coefficients $\{A_k^{(s)}\}$, the average number of strands of type $|s\rangle_k$ is $|A_k^{(s)}|^2/k$. We will see that some properties of the geometry are sensitive not only to the average numbers $\overline{N}_k^{(s)}$, but also to the form of the state in (4.16): in particular, the fact that the state $\psi(\{A_k^{(s)}\})$ is a superposition of angular momentum eigenstates $\psi_{\{N_k^{(s)}\}}$ will be crucial in the following.

4.2.2 Gravity-CFT map in a 1/8-BPS sector

We saw that the profile $g_A(v')$ provides a direct link between the 1/4-BPS geometries and the corresponding semiclassical states in the CFT. In the 1/8-BPS sector, we do not have a complete classification of the gravitational solutions dual to states and so it is not possible to construct an exhaustive dictionary. Here we focus on the class of 1/8-BPS geometries recently derived in [58] by exploiting the linear structure of the supersymmetry equations [54].

It is possible to construct a gravity-CFT map in this sector by relating each term in the scalar function Z_4 that appears in the general 1/8-BPS Ansatz (see equation (3.1)) to the type of strands defining the dual state. From this point of view then Z_4 plays the same role as the profile (4.11) for the 1/4-BPS case. We refer to Eq. (3.20) of [58] for the explicit expression of Z_4 in this class of solution, while here it is sufficient to say that each term in Z_4 is labeled by a pair (k, m_k) of integer numbers satisfying $k > 1$ and $0 \leq m_k \leq k$ and is completely determined by a positive number b_{k,m_k} and a phase η_{k,m_k}. The combination $b_{k,m_k} \mathrm{e}^{\mathrm{i}\eta_{k,m_k}}$ plays the same role as $a_k^{(00)}$ in (4.11).

In analogy with the discussion of the 1/4-BPS case, we define the following eigenstates of total angular momenta (4.10)

$$\psi_{\{N_{k,m_k}^{(s)}\}} \equiv \prod_{s=1}^{4} \prod_{k} (|s\rangle_k)^{N_k^{(s)}} \prod_{k,m_k} \left(\frac{(J_{-1}^{+})_k^{m_k}}{m_k!} |00\rangle_k \right)^{N_{k,m_k}^{(00)}} \tag{4.21}$$

where $s = 1, \ldots, 4$ corresponds to the strands $|\pm\pm\rangle_k$, $(J_n^+)_k$ is n-th mode of the $SU(2)_L$ current acting on a strand of length k and, as before, the sum is constrained by (4.17). The states represent a generalization of the 1/4-BPS building block in (4.6) because we now allow for the presence of RR ground states $|00\rangle_k$ excited with $m_k \leq k$ insertions of $(J_{-1}^+)_k$ (it can be checked by using the free field representation of the operators that m_k cannot be greater than k otherwise the state vanishes). Then the $(0, 0)$ strands in (4.21) have eigenvalue m_k for both $(L_0)_k$ and $(J_0^3)_k$. The normalization $\mathcal{N}(\{N_{k,m_k}^{(s)}\})$ of these states is related to the combinatoric properties of the permutation $\{N_{k,m_k}^{(s)}\}$

but contains also an extra factor derived from the contractions of the $(J^+_{-1})_k$ insertions

$$\mathcal{N}(\{N^{(s)}_{k,m_k}\}) = \left(\frac{N!}{\prod_{s=1}^4 \prod_k N^{(s)}_k!\, k^{N^{(s)}_k}}\right)\left(\frac{1}{\prod_{k,m_k} N^{(00)}_{k,m_k}!\, k^{N^{(00)}_{k,m_k}}}\right)\prod_{k.m_k}\binom{k}{m_k}^{N^{(00)}_{k,m_k}} . \tag{4.22}$$

Then we can define the states $\psi(\{A^{(s)}_k, B_{k,m_k}\})$ as follows

$$\psi(\{A^{(s)}_k, B_{k,m_k}\}) = \sum_{\{N^{(s)}_{k,m_k}\}}{}' \left[\prod_{s=1}^4 \prod_k (A^{(s)}_k |s\rangle_k)^{N^{(s)}_k} \prod_{k,m_k}\left(B_{k,m_k}\frac{(J^+_{-1})^{m_k}_k}{m_k!}|00\rangle_k\right)^{N^{(00)}_{k,m_k}}\right] , \tag{4.23}$$

with norm

$$|\psi(\{A^{(s)}_k, B_{k,m_k}\})|^2 = \sum_{\{N^{(s)}_{k,m_k}\}}{}' \mathcal{N}(\{N^{(s)}_{k,m_k}\})\left(\prod_{s=1}^4 \prod_k |A^{(s)}_k|^{2N^{(s)}_k}\right)\left(\prod_{k,m_k} |B_{k,m_k}|^{2N^{(00)}_{k,m_k}}\right) . \tag{4.24}$$

The numbers of strands $\overline{N}^{(s)}_{k,m_k}$ on which the sum in (4.24) is peaked are the stationary points of the function

$$\begin{aligned} S(\{N^{(s)}_{k,m_k}\}) =& \sum_{s=1}^4 \sum_k \Big[N^{(s)}_k \log|A^{(s)}_k|^2 - N^{(s)}_k \log N^{(s)}_k + N^{(s)}_k - N^{(s)}_k \log k\Big] + \\ &+ \sum_{k,m_k}\Big[N^{(00)}_{k,m_k}\log|B_{k,m_k}|^2 - N^{(00)}_{k,m_k}\log N^{(00)}_{k,m_k} + N^{(00)}_{k,m_k} - N^{(00)}_{k,m_k}\log k + \\ &+ N^{(00)}_{k,m_k}\log\binom{k}{m_k}\Big] , \end{aligned} \tag{4.25}$$

again with the constraint $\sum_{s,k} k\, N^{(s)}_k + \sum_{k,m_k} k N^{(00)}_{k,m_k} = N$. One finds

$$k\,\overline{N}^{(s)}_k = |A^{(s)}_k|^2, \qquad k\,\overline{N}^{(00)}_{k,m_k} = \binom{k}{m_k}|B_{k,m_k}|^2 . \tag{4.26}$$

We can relate the coefficients $A^{(s)}_k$ with $s = (\pm,\pm)$ to the supergravity parameters $a^{(s)}_k$ by using (4.14), while for $s = (00)$ we have

$$B_{k,m_k} \equiv R_y \sqrt{\frac{N}{2\,Q_1 Q_5}}\binom{k}{m_k}^{-1} b_{k,m_k} e^{i\eta_{k,m_k}} . \tag{4.27}$$

Note that the gravity parameters $a \equiv a^{(++)}_1$ and b_{k,m_k} satisfy the constraints (6.10) in [58], which generalizes the constraint (4.13) valid for two-charge geometries. When translated in terms of the CFT parameters $A^{(s)}_k$ and B_{k,m_k}, using the above dictionary, the constraint becomes

$$\sum_{s=1}^{4}\sum_{k}|A_k^{(s)}|^2+\sum_k\binom{k}{m_k}|B_{k,m_k}|^2=N\,, \tag{4.28}$$

which generalizes (4.15).

4.3 CFT 1-point functions and holography

Holography allows to extract the 1-point functions of chiral primary operators in 1/4 and 1/8 BPS states from the asymptotic expansion of the dual geometries. As these 1-point functions are protected, they should match the VEVs computed at the free orbifold point of the CFT. We concentrate in this section on chiral primaries of dimension 1 and work out a series of examples that confirm the gravity-CFT map defined in the previous section.

We start by recalling the connection between the geometry and the VEVs of CFT operators for a general D1-D5-P microstate [32, 63]. The 6D Einstein frame metric for such a microstate can be written [74] in the form (3.30), where all the objects are defined in (3.1).

As we saw in Section (3.1.2), at leading order in the large distance expansion the metric (3.30) reduces to $\mathrm{AdS}_3\times S^3$. To extract the VEVs of operators of dimension 1, it is enough to keep the first non-trivial corrections around $\mathrm{AdS}_3\times S^3$, which have the form

$$Z_1=\frac{Q_1}{r^2}\left(1+\frac{f^1_{1i}}{r}Y^i_1+O(r^{-2})\right)\,,\quad Z_2=\frac{Q_5}{r^2}\left(1+\frac{f^5_{1i}}{r}Y^i_1+O(r^{-2})\right)\,, \tag{4.29a}$$

$$Z_4=\frac{\sqrt{Q_1Q_5}}{r^3}\mathcal{A}_{1i}Y^i_1+O(r^{-4})\,,\quad \mathcal{F}=-\frac{2\,Q_p}{r^2}+O(r^{-3})\,,\quad ds_4^2=dx^idx^i+O(r^{-4})\,, \tag{4.29b}$$

$$\beta=-\frac{\sqrt{2\,Q_1Q_5}}{r^2}a_{\alpha-}Y_1^{\alpha-}+O(r^{-3})\,,\quad \omega=-\frac{\sqrt{2\,Q_1Q_5}}{r^2}a_{\alpha+}Y_1^{\alpha+}+O(r^{-3})\,. \tag{4.29c}$$

It is always possible to pick coordinates in such a way that

$$f^1_{1i}+f^5_{1i}=0\,, \tag{4.30}$$

and we will always assume this gauge choice in the following. We have denoted by Y^i_1 the $l=1$ scalar spherical harmonics on $\mathbb{R}^4$, and by $Y_1^{\alpha\pm}$ the $l=1$ vector spherical harmonics; their expressions are

$$Y^i_1=2\frac{x^i}{r}\,,\quad Y_1^{\alpha+}=\frac{\eta^\alpha_{ij}dx^ix^j}{r^2}\,,\quad Y_1^{\alpha-}=\frac{\bar\eta^\alpha_{ij}dx^ix^j}{r^2}\,, \tag{4.31}$$

where $\eta^\alpha_{ij} = \delta_{\alpha i}\delta_{4j} - \delta_{\alpha j}\delta_{4i} + \epsilon_{\alpha ij4}$ and $\bar\eta^\alpha_{ij} = \delta_{\alpha i}\delta_{4j} - \delta_{\alpha j}\delta_{4i} - \epsilon_{\alpha ij4}$ (with $\alpha = 1,2,3$) are the 't Hooft symbols. The D1, D5 and P charges Q_1, Q_5 and Q_p are quantized in terms of the D-branes data n_1, n_5, n_p as in (3.6). The coefficients f^1_{1i}, $\mathcal{A}_{1i}$, $a_{\alpha\pm}$ are constants for 2-charge geometries but might depend on the light-cone coordinate v for 3-charge states. They capture the VEVs of the chiral primaries of conformal dimension 1.

These chiral primaries comprise the $SU(2)_L \times SU(2)_R$ currents J^3 and $\tilde J^3$ (which have dimensions $(1,0)$ and $(0,1)$), and the operators of dimension $(1/2,1/2)$, $\Sigma_2^{\alpha\dot\alpha}$ and $O^{\alpha\dot\alpha}$, introduced in Sections 2.2.3 and 2.3.3; it is understood that these operators contain a sum over all copies of the CFT, as in (4.10). The same operators where introduced in [32], where they were denoted by $O^{(0,0)}_{(2)i}$ and $O^{(1,1)}_{(1)1i}$; (2.36), (2.75) and (2.114) give an explicit representation of the operators at the free orbifold point of the CFT. The precise relation between our operators and the operators of [32] is[13]

$$\Sigma_2^{++} = O^{(0,0)}_{(2)1} + \mathrm{i}O^{(0,0)}_{(2)2}\,, \quad \Sigma_2^{--} = (\Sigma_2^{++})^\dagger = O^{(0,0)}_{(2)1} - \mathrm{i}O^{(0,0)}_{(2)2}\,, \tag{4.32a}$$

$$\Sigma_2^{+-} = O^{(0,0)}_{(2)3} + \mathrm{i}O^{(0,0)}_{(2)4}\,, \quad \Sigma_2^{-+} = -(\Sigma_2^{+-})^\dagger = -\left(O^{(0,0)}_{(2)3} - \mathrm{i}O^{(0,0)}_{(2)4}\right)\,, \tag{4.32b}$$

and similarly

$$O^{++} = O^{(1,1)}_{(1)11} + \mathrm{i}O^{(1,1)}_{(1)12}\,, \quad O^{--} = (O^{++})^\dagger = O^{(1,1)}_{(1)11} - \mathrm{i}O^{(1,1)}_{(1)12}\,, \tag{4.33a}$$

$$O^{+-} = O^{(1,1)}_{(1)13} + \mathrm{i}O^{(1,1)}_{(1)14}\,, \quad O^{-+} = -(O^{+-})^\dagger = -\left(O^{(1,1)}_{(1)13} - \mathrm{i}O^{(1,1)}_{(1)14}\right)\,. \tag{4.33b}$$

The relation between the 1-point functions of these operators in a state $|s\rangle$ and the dual geometry was worked out in [32, 63], and it is given by

$$\langle s|J^\alpha|s\rangle = c_J\, a_{\alpha+}, \qquad \langle s|\tilde J^\alpha|s\rangle = c_{\tilde J}\, a_{\alpha-}\,, \tag{4.34a}$$

$$\langle s|O^{(0,0)}_{(2)i}|s\rangle = c_{O^{(0,0)}}\, f^1_{1i}, \qquad \langle s|O^{(1,1)}_{(1)1i}|s\rangle = c_{O^{(1,1)}}\, \mathcal{A}_{1i}\,. \tag{4.34b}$$

The coefficients c_j, $c_{\tilde J}$, $c_{O^{(0,0)}}$, $c_{O^{(1,1)}}$ are constants *independent of the state*; their value is difficult to determine a priori, and hence we will fix them by comparison with some particular simple state. We will see that consistency between the CFT and the holographic computations of the entanglement entropy in the D1-D5 microstates provides a non-trivial check on the values of these coefficients. In [81] this consistency relation was used to fix some of these coefficients:

$$c_J = -c_{\tilde J} = \frac{N R_y}{\sqrt{Q_1 Q_5}}, \qquad c_{O^{(1,1)}} = \frac{\sqrt{2}\, N R_y}{\sqrt{Q_1 Q_5}}\,; \tag{4.35}$$

as expected, they only depend on the asymptotic moduli.

[13] The minus sign in the second equations in (4.32b) and (4.33b) is imposed by consistency with the $SU(2)$ algebra.

All microstates considered in previous works had vanishing VEVs of the twist operators $\Sigma_2^{\alpha\dot{\alpha}}$, and hence the coefficient $c_{O^{(0,0)}}$ was left undetermined. One of the purposes of the next section is to fill this gap, by considering a microstate where the VEV of $\Sigma_2^{\alpha\dot{\alpha}}$ is non-trivial.

4.3.1 Switching on the twist fields' VEVs

In this section we analyze the simplest D1-D5 microstate in which the VEV of the twist field $\Sigma_2^{\alpha\dot{\alpha}}$ is non-vanishing. Since the twist field can join two strands of winding one into a strand of winding two (or split a doubly wound strand into two singly wound strands), see Section 2.3.5, a state which contains both strands of winding one and two has a non-trivial $\Sigma_2^{\alpha\dot{\alpha}}$ VEV. A more general situation, in which the twist field joins strands of winding k_1 and k_2 into a strand of winding $k_1 + k_2$ will be considered in Appendix B.

The building blocks of the state we consider here are the strands $|{+}{+}\rangle_{k=1}$ and $|{+}{+}\rangle_{k=2}$, where $|{+}{+}\rangle_k$ is defined explicitly in (2.116). As we explained in Section 4.2, to have a state which is well described by a classical geometry one needs to take a linear superposition of states of the form (4.16), where now only the coefficients $A_1^{(++)}$ and $A_2^{(++)}$ are non-vanishing; $N_1^{(++)}$ and $N_2^{(++)}$ denote the numbers of strands of type $|{+}{+}\rangle_{k=1}$ and $|{+}{+}\rangle_{k=2}$. To lighten the notation, in this section we rename $A_1^{(++)} \equiv A_1$, $A_2^{(++)} \equiv A_2$ and $N_2^{(++)} \equiv p$. Then the constraint (4.17) implies $N_1^{(++)} = N - 2p$. The state we consider is then

$$\psi(A_1, A_2) = \sum_{p=1}^{N/2} \big(A_1|{+}{+}\rangle_1\big)^{N-2p} \big(A_2|{+}{+}\rangle_2\big)^p \,, \tag{4.36}$$

where

$$|A_1|^2 + |A_2|^2 = N \,, \tag{4.37}$$

as a consequence of (4.15). We know from (4.20) that the sum in (4.36) is peaked over

$$\bar{p} \equiv \overline{N}_2^{(++)} = \frac{|A_2|^2}{2} \quad \Rightarrow \quad \overline{N}_1^{(++)} = N - 2\bar{p} = |A_1|^2 \,. \tag{4.38}$$

Note that the state $\psi(A_1, A_2)$ is not normalized, but its norm is

$$|\psi(A_1, A_2)|^2 = \sum_{p=1}^{N/2} \mathcal{N}(p)\, |A_1|^{2(N-2p)} |A_2|^{2p} \quad \text{with} \quad \mathcal{N}(p) = \frac{N!}{(N-2p)!\, p!\, 2^p} \,, \tag{4.39}$$

where we have used (4.18).

By conservation of the angular momenta J^3 and $\tilde{J}^3$ it is easy to determine which of the operators $\Sigma_2^{\alpha\dot{\alpha}}$ acquire a VEV in the above state. When $\Sigma_2^{\alpha\dot{\alpha}}$ acts on two strands of type $|{+}{+}\rangle_{k=1}$, it produces a state with winding two and angular momenta $(1+\alpha/2, 1+\dot\alpha/2)$, with $\alpha, \dot\alpha = \pm 1$; for the VEV of the twist field to be non-zero, this latter state has to overlap with the state $|{+}{+}\rangle_{k=2}$, whose spin is $(1/2, 1/2)$. One thus needs $\alpha = \dot\alpha = -1$, which means that Σ_2^{--} acquires VEV in the state (4.36). Since $\Sigma_2^{++} = (\Sigma_2^{--})^\dagger$, the VEV of Σ_2^{++} must also be non-zero: this VEV originates from the process in which Σ_2^{++} acts on the doubly wound strand $|{+}{+}\rangle_{k=2}$ to produce two copies of the singly wound strand, $(|{+}{+}\rangle_{k=1})^2$.

Consider first the VEV of Σ_2^{--}: the relevant contribution comes from the process in which the twist field lowers by two the number of length one strands and increases by one the number of length two strands, which is represented by

$$\Sigma_2^{--}\left[\big(|{+}{+}\rangle_1\big)^{N-2p}\big(|{+}{+}\rangle_2\big)^p\right] = (p+1)\left[\big(|{+}{+}\rangle_1\big)^{N-2(p+1)}\big(|{+}{+}\rangle_2\big)^{p+1}\right]. \tag{4.40}$$

The combinatorial factor $p+1$ can be understood as follows. The twist field Σ_2^{--} can act on any one of the $\binom{N-2p}{2}$ copies of length one strands in the state $\left[\big(|{+}{+}\rangle_1\big)^{N-2p}\big(|{+}{+}\rangle_2\big)^p\right]$, which is made of $\mathcal{N}(p)$ terms; the total number of terms on the l.h.s. and the r.h.s. of (4.40) matches if

$$\binom{N-2p}{2}\mathcal{N}(p) = (p+1)\,\mathcal{N}(p+1)\,, \tag{4.41}$$

which is verified using the expression for $\mathcal{N}(p)$ in (4.39).

From the basic action (4.40), one therefore has

$$\Sigma_2^{--}\psi(A_1,A_2) = \sum_{p=1}^{N/2} A_1^{N-2p} A_2^p\,(p+1)\big(|{+}{+}\rangle_1\big)^{N-2(p+1)}\big(|{+}{+}\rangle_2\big)^{p+1} \tag{4.42}$$

The VEV of Σ_2^{--} over $\psi(A_1,A_2)$ is then computed as

$$\begin{aligned}\langle\Sigma_2^{--}\rangle &\equiv |\psi(A_1,A_2)|^{-2}\,\langle\psi(A_1,A_2)|\Sigma_2^{--}|\psi(A_1,A_2)\rangle\\ &= \frac{A_1^2}{A_2}\,|\psi(A_1,A_2)|^{-2}\sum_{p=1}^{N/2}\big(|A_1|^2\big)^{N-2p}\big(|A_2|^2\big)^p\,p\,\mathcal{N}(p) = \frac{A_1^2}{A_2}\,\bar p = \frac{A_1^2\bar A_2}{2}\,,\end{aligned} \tag{4.43}$$

where, in the last step, we have used (4.38).

For consistency, we should also verify that the VEV of Σ_2^{++} is the complex conjugate of the VEV in (4.43). The relevant action of Σ_2^{++} is given by

$$\Sigma_2^{++}\left[\big(|{+}{+}\rangle_1\big)^{N-2p}\big(|{+}{+}\rangle_2\big)^p\right] = \frac{(N-2p+1)(N-2p+2)}{2}\left[\big(|{+}{+}\rangle_1\big)^{N-2p+2}\big(|{+}{+}\rangle_2\big)^{p-1}\right], \tag{4.44}$$

where the combinatorial factor follows from the identity

$$p\,\mathcal{N}(p) = \frac{(N-2p+1)(N-2p+2)}{2}\,\mathcal{N}(p-1)\,, \tag{4.45}$$

which can be derived by following steps similar to those explained after (4.40); note that the factor p on the l.h.s. of the above equation comes from the p possible ways in which Σ_2^{++} can act on the p strands of type $|{+}{+}\rangle_2$. It follows by comparison of (4.40) and (4.44), and by the identity (4.45), that

$$(\psi(p+1), \Sigma_2^{--}\,\psi(p)) = (\Sigma_2^{++}\,\psi(p+1), \psi(p))\,, \tag{4.46}$$

where for brevity we have denoted

$$\psi(p) \equiv \Big[\big(|{+}{+}\rangle_1\big)^{N-2p}\big(|{+}{+}\rangle_2\big)^p\Big]\,. \tag{4.47}$$

This proves that indeed $\Sigma_2^{++} = (\Sigma_2^{--})^\dagger$ and it implies that

$$\langle\Sigma_2^{++}\rangle = \langle\Sigma_2^{--}\rangle^* = \frac{\bar{A_1}^2 A_2}{2}. \tag{4.48}$$

The only other operators of dimension one that have a non-vanishing VEV in the state $\psi(A_1, A_2)$ are the currents J^3, $\tilde{J}^3$. These VEVs can be straightforwardly computed, as they are only sensitive to the average numbers of strands of length one and two, which both carry spin $(1/2, 1/2)$. Using (4.38) one then finds

$$\langle J^3\rangle = \langle\tilde{J}^3\rangle = \frac{1}{2}\left(\overline{N}_1^{(++)} + \overline{N}_2^{(++)}\right) = \frac{1}{2}\left(|A_1|^2 + \frac{|A_2|^2}{2}\right)\,. \tag{4.49}$$

We now compare the 1-point functions computed in the CFT with the ones extracted from the dual geometry. This is the geometry associated with a profile whose only two excited modes are $a_1^{(++)}$ and $a_2^{(++)}$, in the notation of (4.11). For notational simplicity we abbreviate $a_1^{(++)} \equiv a_1$ and $a_2^{(++)} \equiv a_2$. The relation between a_1, a_2 and A_1, A_2 is given in (4.14):

$$a_i = \frac{A_i}{R_y}\sqrt{\frac{Q_1 Q_5}{N}} \quad (i = 1, 2)\,. \tag{4.50}$$

The parameters which encode the asymptotic behavior of the geometry, defined in general in (4.29), take the following values for our microstate (see Appendix A):

$$f^1_{11} - \mathrm{i} f^1_{12} = \frac{R_y^2}{2\,Q_1 Q_5} a_1^2 \bar{a}_2\,, \quad \mathcal{A}_{1i} = 0\,, \quad a_{3+} = -a_{3-} = \frac{R_y}{2\sqrt{Q_1 Q_5}}\left(|a_1|^2 + \frac{|a_2|^2}{2}\right)\,. \tag{4.51}$$

Using the dictionary in (4.34a), with the c_J and $c_{\tilde{J}}$ of (4.35), one readily verifies that the VEVs of J_3 and $\tilde{J}_3$ computed in (4.49) agree with their holographically derived values. For the VEV of Σ_2^{--}, the first of (4.34b), together with (4.32), gives

$$\langle \Sigma_2^{--} \rangle = c_{O^{(0,0)}} \, (f^1_{11} - \mathrm{i} f^1_{12}) \,. \tag{4.52}$$

Comparison of the CFT (4.43) and gravity (4.51) results fixes the value of the unknown coefficient $c_{O^{(0,0)}}$:

$$c_{O^{(0,0)}} = \frac{N^{3/2} R_y}{\sqrt{Q_1 Q_5}}. \tag{4.53}$$

The fact that $c_{O^{(0,0)}}$ is independent of a_1, a_2 represents already a non-trivial check; we will see that the precise numerical value of $c_{O^{(0,0)}}$ is checked also by the computation of the entanglement entropy in the state $\psi(A_1, A_2)$.

4.3.2 3 charges and two kinds of strands

We now extend the holographic computation of 1-point functions of dimension 1 chiral primaries to the class of three-charge microstates introduced in Section 4.2.2. Consider first a simple D1-D5-P state containing only two types of strands: strands of type $|{+}{+}\rangle_{k=1}$ in their ground state and strands of type $|00\rangle_{k=1}$ acted upon by the current J^+_{-1}, which carries momentum. The geometry dual to this state was first constructed in [48, 74]. The CFT state has the form (4.21) where the non-vanishing coefficients are $A_1^{(++)}$ and $B_{1,1}$; renaming $A_1^{(++)} \equiv A$, $B_{1,1} \equiv B$ and $N_1^{(++)} \equiv p$, and using the constraint (4.6), we get

$$\psi(A, B) \equiv \sum_{p=0}^{N} (A|{+}{+}\rangle_{k=1})^p \left(B \ J^+_{-1}|00\rangle_{k=1}\right)^{N-p} . \tag{4.54}$$

The constraint (4.28) now reads

$$|A|^2 + |B|^2 = N. \tag{4.55}$$

and from (4.26) we have

$$\overline{N}_1^{(++)} = \bar{p} = |A|^2, \qquad \overline{N}_1^{(00)} = N - \bar{p} = |B|^2. \tag{4.56}$$

These relations immediately give the VEVs of the angular momentum operators:

$$\langle J^3 \rangle = \frac{\overline{N}_1^{(++)}}{2} + \overline{N}_1^{(00)} = \frac{|A|^2}{2} + |B|^2, \qquad \langle \tilde{J}^3 \rangle = \frac{\overline{N}_1^{(++)}}{2} = \frac{|A|^2}{2} \,, \tag{4.57}$$

since the strands $|{+}{+}\rangle_{k=1}$ and $J^+_{-1}|00\rangle_{k=1}$ carry angular momenta $(1/2,1/2)$ and $(1,0)$ respectively. We can also read off the average value of momentum:

$$\langle\tilde{T}\rangle = 0\,,\quad \langle T\rangle = (N-\bar{p}) = |B|^2 \quad\Rightarrow\quad n_p \equiv \langle L_0 - \tilde{L}_0\rangle = |B|^2\,, \tag{4.58}$$

since every strand $J^+_{-1}|00\rangle_{k=1}$ carries 1 unit of momentum.

Consider now the operator $O^{\alpha\dot{\alpha}}$. As one sees from (2.37) the operator $O^{2\dot{2}}$ transform the strand $|{+}{+}\rangle_{k=1}$ into $|00\rangle_{k=1}$; in our state, the $|00\rangle_{k=1}$ strand is acted upon by J^+_{-1}, and thus, to determine the action of $O^{\alpha\dot{\alpha}}$ on the state $\psi(A,B)$ we need to know the commutation properties of $O^{\alpha\dot{\alpha}}$ with the $SU(2)$ current algebra. As the index α transforms in the fundamental representation of $SU(2)$ (which we represent by the matrices $\tau^i = \sigma^i/2$), one has the following nontrivial commutator[14]

$$\left[\left(J^i_n\right)^{\alpha\beta}, O^{\beta\dot{\alpha}}(v,u)\right] = \frac{1}{2}\,\mathrm{e}^{\mathrm{i}n\frac{\sqrt{2}v}{R}}\left(\sigma^i\right)^{\alpha\beta} O^{\beta\dot{\alpha}}(v,u), \tag{4.59}$$

where the v-dependent factor comes from the fact that we are considering the n-th mode of current $J^i(v,u)$. Hence if we use (2.37), the commutator (4.59) and the fact that positive modes of the currents annihilate the vacuum strands ($(J^i_n)_k|s\rangle_k = 0$ for $n>0$), we obtain the following VEVs for individual strands

$${}_{k=1}\langle 00|J^-_{+1}\,O^{1\dot{2}}(v,u)|{+}{+}\rangle_{k=1} = \mathrm{e}^{\mathrm{i}\frac{\sqrt{2}v}{R}}\,,\qquad {}_{k=1}\langle{+}{+}|O^{2\dot{1}}(v,u)\,J^+_{-1}|00\rangle_{k=1} = -\mathrm{e}^{-\mathrm{i}\frac{\sqrt{2}v}{R}}\,, \tag{4.60}$$

which are consistent with the hermiticity property $O^{2\dot{1}} = -\left(O^{1\dot{2}}\right)^\dagger$. Note that it is important that the operators $O^{1\dot{2}}$ and $O^{2\dot{1}}$ are inserted at a generic worldsheet point (v,u) and that, due to the presence of the current J^-_{+1}, J^+_{-1}, the non-zero-mode part of the operators contributes to the correlator; if only zero-modes had contributed, $O^{2\dot{1}}$ would have annihilated the state $|{+}{+}\rangle_1$ because of (2.25).

The action of $O^{2\dot{1}}$ on angular momentum eigenstates is obtained by combining the above result with the appropriate combinatorial factor[15]

$$O^{2\dot{1}}(v,u)\left[(|{+}{+}\rangle)^p_{k=1}\left(J^+_{-1}|00\rangle_{k=1}\right)^{N-p}\right] = -\mathrm{e}^{-\mathrm{i}\frac{\sqrt{2}v}{R}}\,(p{+}1)\left[(|{+}{+}\rangle_{k=1})^{p+1}\left(J^+_{-1}|00\rangle_{k=1}\right)^{N-p-1}\right]. \tag{4.61}$$

The VEV of $O^{2\dot{1}}$ on the state $\psi(A,B)$ is then

$$\langle O^{2\dot{1}}(v,u)\rangle = -\mathrm{e}^{-\mathrm{i}\frac{\sqrt{2}v}{R}}\,\frac{B}{A}\,\bar{p} = -\mathrm{e}^{-\mathrm{i}\frac{\sqrt{2}v}{R}}\,\bar{A}B. \tag{4.62}$$

[14] A similar relation holds for the modes of $\tilde{J}^3$, with the difference that the dotted index gets rotated.

[15] $O^{2\dot{1}}$ can act on the $N-p$ strands of type $J^+_{-1}|00\rangle_1$, producing $(N-p)\binom{N}{p} = (p+1)\binom{N}{p+1}$ terms, which matches the number of terms on the r.h.s. of (4.61).

Because $O^{2\dot{1}} = -\left(O^{1\dot{2}}\right)^{\dagger}$, the VEV of $O^{1\dot{2}}$ is

$$\langle O^{1\dot{2}}(v,u)\rangle = \mathrm{e}^{\mathrm{i}\frac{\sqrt{2}v}{R}} A\bar{B}. \tag{4.63}$$

This example highlights a new feature of three-charge microstates: the VEVs of some operators, like $O^{2\dot{1}}$ and $O^{1\dot{2}}$ in our example, are v-dependent. This v-dependence originates from the presence of momentum charge (carried in our case by the current J^{+}_{-1}) and from the fact that states dual to geometries are not eigenstates of the momentum operator. Since holography relates the VEVs of operators with the coefficients of the metric expanded around the AdS_3 boundary, this implies that three-charge microstate geometries are generically v-dependent.

The geometry dual to $\psi(A,B)$ is given in eqs. (5.2)-(5.3) of [48]. At the first non-trivial order in the asymptotic expansion around the AdS boundary, this three-charge solution admits an expansion of the form (4.29), where the only non-trivial metric functions are

$$Z_4 \approx R\,a\,b\,\frac{\cos\theta}{r^3}\,\cos\left(\frac{\sqrt{2}v}{R}-\psi\right)\,,\quad \mathcal{F}\approx -\frac{b^2}{r^2}\,, \tag{4.64a}$$

$$\beta \approx \frac{R\,a^2}{\sqrt{2}}\,\frac{\sin^2\theta\,d\phi-\cos^2\theta\,d\psi}{r^2}\,,\quad \omega\approx\frac{R\,(a^2+b^2)}{\sqrt{2}}\,\frac{\sin^2\theta\,d\phi+\cos^2\theta\,d\psi}{r^2}\,; \tag{4.64b}$$

the coefficients a and b are taken to be real. The gravity coefficients extracted from this geometry are then

$$\mathcal{A}_{13}+\mathrm{i}\mathcal{A}_{14} = \frac{R\,a\,b}{2\sqrt{Q_1Q_5}}\,\mathrm{e}^{\mathrm{i}\frac{\sqrt{2}v}{R}}\,,\quad Q_p=\frac{b^2}{2}\,,\quad a_{3+}=\frac{R\,(a^2+b^2)}{2\sqrt{Q_1Q_5}}\,,\quad a_{3-}=-\frac{R\,a^2}{2\sqrt{Q_1Q_5}}\,. \tag{4.65}$$

Using the dictionary (4.14), (4.27), along with (3.6), (4.34), and (4.35), we find agreement with the CFT results for the 1-point functions $\langle O^{2\dot{1}}\rangle$, $\langle O^{1\dot{2}}\rangle$, $\langle J^3\rangle$, $\langle \tilde{J}^3\rangle$, $\langle \tilde{L}_0 - L_0\rangle$.

4.3.3 3 charges and three kinds of strands

The state analyzed in the previous section is a very particular three-charge state: as explained in [48], that state can be generated by acting on the two-charge state with strands $|{+}{+}\rangle_{k=1}$ and $|00\rangle_{k=1}$ with the symmetry operator $\mathrm{e}^{\frac{\pi}{2}(J^{+}_{-1}-J^{-}_{+1})}$. We call such states descendants. We consider in this section a simple state which is *not* a descendant. This state has also the property that the VEVs of all the dimension one operators are non-trivial and it will allow us to provide a CFT derivation of a numerical coefficient which was fixed in [58] by a non-trivial regularity requirement.

The state we consider has the form (4.21) with three type of strands: $|{+}{+}\rangle_{k=1}$,

$J^+_{-1}|00\rangle_{k=2}$, $|00\rangle_{k=1}$. We rename the associated coefficients as $A^{(++)}_1 \equiv A, B_{2,1} \equiv B_1, B_{1,0} \equiv B_2$ and the respective numbers of strands as $N^{(++)}_1 \equiv N - 2p_1 - p_2$, $N^{(00)}_{2,1} \equiv p_1, N^{(00)}_{1,0} \equiv p_2$, so that the state can be written as

$$\psi(A, B_1, B_2) = \sum_{p_1=0}^{N/2} \sum_{p_2=0}^{N-2p_1} (A|{+}{+}\rangle_{k=1})^{N-2p_1-p_2} \left(B_1 J^+_{-1}|00\rangle_{k=2}\right)^{p_1} (B_2|00\rangle_{k=1})^{p_2} \, . \tag{4.66}$$

It is important to keep in mind that the state $J^+_{-1}|00\rangle_{k=2}$ has norm 2

$$_{k=2}\langle 00|J^-_{+1}J^+_{-1}|00\rangle_{k=2} = 2 \,, \tag{4.67}$$

as a consequence of the fractional mode contributions which appear when J^+_{-1} acts on a strand of length 2: $(J^+_{-1})_2 = \psi^{1\dot{1}}_{-1}\psi^{1\dot{2}}_0 + \psi^{1\dot{1}}_0\psi^{1\dot{2}}_{-1} + \psi^{1\dot{1}}_{-1/2}\psi^{1\dot{2}}_{-1/2}$. The same mechanism gives rise, for generic k and m_k, to the factor $\binom{k}{m_k}$ in (4.22). We can then borrow the general result (4.26) to obtain the average numbers of strands in our state:

$$\bar{p}_1 = |B_1|^2, \qquad \bar{p}_2 = |B_2|^2, \qquad N - 2\bar{p}_1 - \bar{p}_2 = |A|^2, \tag{4.68}$$

where the constraint among the coefficients is now

$$|A|^2 + 2|B_1|^2 + |B_2|^2 = N. \tag{4.69}$$

Since the strands $|{+}{+}\rangle_{k=1}$, $J^+_{-1}|00\rangle_{k=2}$ and $|00\rangle_{k=1}$ carry spin $(1/2, 1/2)$, $(1, 0)$ and $(0, 0)$, the VEVs of the angular momentum operators are

$$\langle J^3 \rangle = \frac{\overline{N}^{(++)}_1}{2} + \overline{N}^{(00)}_{2,1} = \frac{|A|^2}{2} + |B_1|^2, \qquad \langle \tilde{J}^3 \rangle = \frac{\overline{N}^{(++)}_1}{2} = \frac{|A|^2}{2}. \tag{4.70}$$

Momentum is carried only by the $J^+_{-1}|00\rangle_{k=2}$ strands, and thus

$$\langle \tilde{T} \rangle = 0 \;, \quad \langle T \rangle = \overline{N}^{(00)}_{2,1} = |B_1|^2 \quad \Rightarrow \quad n_p = \langle L_0 - \tilde{L}_0 \rangle = |B_1|^2 \,. \tag{4.71}$$

The presence of $|00\rangle_{k=1}$ and $|{+}{+}\rangle_{k=1}$ strands signals that some of the $O^{\alpha\dot{\alpha}}$ operators can have a nonzero VEV, while the presence of strands of length 2 with non-zero modes of the current acting on them implies a nonzero v-dependent VEV for the twist operators $\Sigma^{\alpha\dot{\alpha}}_2$. On the gravity side, the VEV of $\Sigma^{\alpha\dot{\alpha}}_2$ corresponds to a term of order r^{-3} in the metric function Z_1 (see Eq. (4.29a)); it was shown in [58] that such a term is needed to ensure regularity of the metric. We will verify that the precise numerical coefficient derived in [58] matches the CFT prediction.

Consider first the VEV of $O^{\alpha\dot{\alpha}}$. By angular momentum conservation only $O^{2\dot{2}}$ and $O^{1\dot{1}}$ can acquire a VEV; in particular one can consider the process in which $O^{2\dot{2}}$ converts a $|{+}{+}\rangle_{k=1}$ into a $|00\rangle_{k=1}$ strand:

$$O^{2\dot{2}}\Big[(|{+}{+}\rangle_{k=1})^{N-2p_1-p_2}\left(J^+_{-1}|00\rangle_{k=2}\right)^{p_1}(|00\rangle_{k=1})^{p_2}\Big] =$$
$$= (p_2+1)\Big[(|{+}{+}\rangle_{k=1})^{N-2p_1-p_2-1}\left(J^+_{-1}|00\rangle_{k=2}\right)^{p_1}(|00\rangle_{k=1})^{p_2+1}\Big]. \qquad (4.72)$$

This gives rise to the VEV

$$\langle O^{2\dot{2}}\rangle = \frac{A}{B_2}\,\bar{p}_2 = A\bar{B}_2\,. \qquad (4.73)$$

By hermiticity, $O^{1\dot{1}} = \left(O^{2\dot{2}}\right)^\dagger$, one also obtains the VEV

$$\langle O^{1\dot{1}}\rangle = \langle O^{2\dot{2}}\rangle^* = \bar{A}B_2. \qquad (4.74)$$

Consider now $\Sigma_2^{\alpha\dot{\alpha}}$. The twist operator can join two strands of length one into a length two strand of type $J^+_{-1}|00\rangle_{k=2}$; by angular momentum conservation, the two starting strands have to be $|{+}{+}\rangle_{k=1}$ and $|00\rangle_{k=1}$ and the operator acting on them Σ_2^{+-}. We thus expect the basic correlator

$$_{k=2}\langle 00|\,J^-_{+1}\,\Sigma_2^{+-}\Big(|++\rangle_{k=1}\otimes|00\rangle_{k=1}\Big)_{\text{Symm.}} \qquad (4.75)$$

to be non-vanishing, where we have denoted by

$$\Big(|++\rangle_{k=1}\otimes|00\rangle_{k=1}\Big)_{\text{Symm.}} \equiv |++\rangle_{(r=1)}\otimes|00\rangle_{(r=2)} + |00\rangle_{(r=1)}\otimes|++\rangle_{(r=2)} \qquad (4.76)$$

the product of the two states $|{+}{+}\rangle_{k=1}$ and $|00\rangle_{k=1}$ symmetrized over two copies ($r = 1,2$) of the CFT[16]. Note that

$$\left\|\Big(|++\rangle_{k=1}\otimes|00\rangle_{k=1}\Big)_{\text{Symm.}}\right\|^2 = 2. \qquad (4.77)$$

To compute (4.75) we need to know the commutator of the doublet of operators $\Sigma_2^{\alpha\dot{\alpha}}$ with the currents J^i_n; this has a form analogous to (4.59):

$$\Big[\left(J^i_n\right)^{\alpha\beta}, \Sigma_2^{\beta\dot{\alpha}}(v,u)\Big] = \frac{1}{2}\mathrm{e}^{\mathrm{i}n\frac{\sqrt{2}v}{R}}\left(\sigma^i\right)^{\alpha\beta}\Sigma_2^{\beta\dot{\alpha}}(v,u)\,. \qquad (4.78)$$

[16] This a basic requirement of the dual CFT: being an orbifold theory it requires invariance over permutations of the copies, which translates into the fact that all the states must be symmetric upon exchanging any pair of copies.

Thus we find

$$
\begin{aligned}
{}_{k=2}\langle 00|\, J^{-}_{+1}\Sigma_2^{+-}\Big(|++\rangle_{k=1}\otimes|00\rangle_{k=1}\Big)_{\text{Symm.}} &= \mathrm{e}^{\mathrm{i}\frac{\sqrt{2}v}{R}}\ {}_{k=2}\langle 00|\,\Sigma_2^{--}\Big(|++\rangle_{k=1}\otimes|00\rangle_{k=1}\Big)_{\text{Symm.}} \\
&= \mathrm{e}^{\mathrm{i}\frac{\sqrt{2}v}{R}}\,, \qquad (4.79)
\end{aligned}
$$

where we have used

$$
\Sigma_2^{--}\Big(|++\rangle_{k=1}\otimes|00\rangle_{k=1}\Big)_{\text{Symm.}} = |00\rangle_{k=2}\,. \qquad (4.80)
$$

We can now include the combinatorial factors[17] and obtain the action of Σ_2^{+-} on the full ensemble of strands:

$$
\begin{aligned}
&\Sigma_2^{+-}\Big[(|{+}{+}\rangle_{k=1})^{N-2p_1-p_2}\,\big(J^{+}_{-1}|00\rangle_{k=2}\big)^{p_1}\,(|00\rangle_{k=1})^{p_2}\Big] = \\
&= \frac{\mathrm{e}^{\mathrm{i}\frac{\sqrt{2}v}{R}}}{2}(p_1+1)\Big[(|{+}{+}\rangle_{k=1})^{N-2p_1-p_2-1}\,\big(J^{+}_{-1}|00\rangle_{k=2}\big)^{p_1+1}\,(|00\rangle_{k=1})^{p_2-1}\Big]. \qquad (4.81)
\end{aligned}
$$

Hence we obtain the VEV

$$
\langle\Sigma_2^{+-}(v,u)\rangle = \frac{\mathrm{e}^{\mathrm{i}\frac{\sqrt{2}v}{R}}}{2}\,\frac{AB_2}{B_1}\,\bar{p}_1 = \frac{\mathrm{e}^{\mathrm{i}\frac{\sqrt{2}v}{R}}}{2}A\,\bar{B}_1B_2. \qquad (4.82)
$$

Of course one can also consider the opposite process in which Σ_2^{-+} acts on $J^{+}_{-1}|00\rangle_{k=2}$ to produce singly wound strands $|{+}{+}\rangle_{k=1}\otimes|00\rangle_{k=1}$. This is captured by

$$
\Sigma_2^{-+}J^{+}_{-1}|00\rangle_{k=2} = -\mathrm{e}^{-\mathrm{i}\frac{\sqrt{2}v}{R}}\,\Sigma_2^{++}|00\rangle_{k=2} = -\frac{\mathrm{e}^{-\mathrm{i}\frac{\sqrt{2}v}{R}}}{2}\Big(|{+}{+}\rangle_{k=1}\otimes|00\rangle_{k=1}\Big)_{\text{Symm.}}\,, \qquad (4.83)
$$

where we have used

$$
\Sigma_2^{++}|00\rangle_{k=2} = \frac{1}{2}\Big(|++\rangle_{k=1}\otimes|00\rangle_{k=1}\Big)_{\text{Symm.}}\,. \qquad (4.84)
$$

Together with (4.77) this implies

$$
\Big({}_{k=1}\langle ++|\otimes\ {}_{k=1}\langle 00|\Big)_{\text{Symm.}}\,\Sigma_2^{-+}J^{+}_{-1}|00\rangle_{k=2} = -\mathrm{e}^{-\mathrm{i}\frac{\sqrt{2}v}{R}}\,, \qquad (4.85)
$$

[17]The number of ways Σ_2^{+-} can act on the strands $(|{+}{+}\rangle_1)^{N-2p_1-p_2}\,(|00\rangle_1)^{p_2}$ is $\frac{(N-2p_1-p_2)\,p_2}{2}$; we divide by 2 because we have already taken into account the exchange of a $|{+}{+}\rangle_1$ and a $|00\rangle_1$ strand in the symmetrized combination (4.76).

which is consistent with (4.79) and the property $\Sigma_2^{-+} = -\left(\Sigma_2^{+-}\right)^\dagger$. Thus

$$\langle\Sigma_2^{-+}(v,u)\rangle = -\langle\Sigma_2^{+-}(v,u)\rangle^* = -\frac{\mathrm{e}^{-\mathrm{i}\frac{\sqrt{2}v}{R}}}{2}\bar{A}\,B_1\bar{B}_2. \tag{4.86}$$

Let us now compare with the dual geometry: it can be read off from Section 5.2 of [58] taking $k_1 = 2$, $m_1 = 1$. We focus in particular on the metric functions Z_1, Z_2 and Z_4, which determine the VEVs of $O^{\alpha\dot{\alpha}}$ and $\Sigma_2^{\alpha\dot{\alpha}}$ (the gravity values of the momentum and of the angular momenta are given in (6.11) and (6.15) of [58] and are easily seen to agree with the CFT values computed above). At the relevant order in the $1/r$ expansion, the gravity solution is characterized by

$$Z_1 = \frac{Q_1}{r^2} + \frac{R_y^2}{2\,Q_5}\,a\,b_1b_2\,\cos\left(\frac{\sqrt{2}v}{R_y} - \psi\right)\frac{\cos\theta}{r^3} + O(r^{-4})\,, \quad Z_2 = \frac{Q_5}{r^2} + O(r^{-4})\,, \tag{4.87a}$$

$$Z_4 = R_y\,a\,b_2\,\frac{\sin\theta\,\cos\phi}{r^3} + O(r^{-4})\,, \qquad \mathcal{F} = -\frac{b_1^2}{4r^2}\,, \tag{4.87b}$$

$$\beta = \frac{R_y a^2}{\sqrt{2}r^2}\left(\sin^2\theta\,d\phi - \cos^2\theta\,d\psi\right), \quad \omega = \frac{R_y\left(a^2 + \frac{b_1^2}{4}\right)}{\sqrt{2}r^2}\left(\sin^2\theta\,d\phi + \cos^2\theta\,d\psi\right) \tag{4.87c}$$

where the parameters a, b_1, b_2 are real. The order r^{-3} term of Z_1 is necessary for having a regular metric: its numerical value is determined by the constant c given in Eq. (5.15) of [58], which in our case reads $c = 1/2$. Transforming into a coordinate system where (4.30) is satisfied, one finds

$$f_{13}^1 + \mathrm{i} f_{14}^1 = \frac{R_y^2}{Q_1Q_5}\frac{a\,b_1b_2}{8}\,\mathrm{e}^{\mathrm{i}\frac{\sqrt{2}v}{R}}\,, \qquad \mathcal{A}_{11} = \frac{R_y\,ab_2}{2\sqrt{Q_1Q_5}}\,, \tag{4.88a}$$

$$a_{3+} = \frac{R_y}{\sqrt{Q_1Q_5}}\frac{1}{2}\left(a^2 + \frac{b_1^2}{4}\right), \qquad a_{3-} = -\frac{R_y}{\sqrt{Q_1Q_5}}\frac{a^2}{2}\,, \quad Q_p = \frac{b_1^2}{8}. \tag{4.88b}$$

Using the holographic dictionary (4.34), the values of the constant $c_{O^{(1,1)}}$ and $c_{O^{(0,0)}}$ in (4.35) and (4.53), and the relations (4.14) and (4.27), which give

$$A = a\frac{R_y\sqrt{N}}{\sqrt{Q_1Q_5}}, \qquad B_1 = \frac{b_1}{2\sqrt{2}}\frac{R_y\sqrt{N}}{\sqrt{Q_1Q_5}}, \qquad B_2 = \frac{b_2}{\sqrt{2}}\frac{R_y\sqrt{N}}{\sqrt{Q_1Q_5}}, \tag{4.89}$$

one verifies that the gravity result (4.88) matches, including all numerical factors, with the CFT VEVs (4.70), (4.71), (4.73), (4.74), (4.82) and (4.86).

4.3.4 3 charges and two kinds of strands with different modes

Let's consider another three-charge state, this time not of the class (4.21) and given by superpositions of strands of length 1, $|{+}{+}\rangle_{k=1}$ and $\left(L_{-1} - J_{-1}^3\right)|00\rangle_{k=1}$, again with coefficients called A and B,

$$\chi(A,B) \equiv \sum_{p=0}^{N} \left(A|{+}{+}\rangle_{k=1}\right)^p \left(B\left(L_{-1}-J^3_{-1}\right)|00\rangle_{k=1}\right)^{N-p} \equiv \sum_{p=0}^{N} A^p B^{N-p}|p\rangle, \quad (4.90)$$

where $|p\rangle$ is the state given by the tensor product of p states $|{+}{+}\rangle_{k=1}$ and $(N-p)$ states $\left(L_{-1}-J^3_{-1}\right)|00\rangle_{k=1}$, symmetrized under the exchanged of any two CFT copies. All the strands have length one, and in this section we omit the twist subscript, so that $|{+}{+}\rangle \equiv |{+}{+}\rangle_{k=1}$ and $|00\rangle \equiv |00\rangle_{k=1}$.

As a first step, let's compute the norms of the building blocks $\chi(A,B)$ is given by. We have

$$\||{+}{+}\rangle\|^2 = \langle + + | + +\rangle = 1, \quad (4.91)$$

by normalization of the Ramond vacua in the untwisted sector. Moreover,

$$\begin{aligned} \|\left(L_{-1}-J^3_{-1}\right)|00\rangle\|^2 &= \langle 00|\left[L_1,L_{-1}\right]|00\rangle + \langle 00|\left[J^3_1,J^3_{-1}\right]|00\rangle + \\ &\quad - \langle 00|\left[L_1,J^3_{-1}\right]|00\rangle - \langle 00|\left[J^3_1,L_{-1}\right]|00\rangle \\ &= \langle 00|\left[L_1,L_{-1}\right]|00\rangle + \langle 00|\left[J^3_1,J^3_{-1}\right]|00\rangle \\ &= \frac{1}{2}+\frac{1}{2} \\ &= 1, \end{aligned} \quad (4.92)$$

where we used the mode algebra (2.34), (2.43) and (2.50), the fact that $|00\rangle$ is normalized, the fact that $J^3_0|00\rangle = 0$, i.e. that $|00\rangle$ has spin $(J^3,\tilde{J}^3) = (0,0)$, and finally the fact that positive modes of T and J^a annihilate the Ramond vacua.

We can now compute the norm of $|p\rangle$. This state is explicitly written as

$$|p\rangle = \underbrace{|{+}{+}\rangle\cdots|{+}{+}\rangle}_{p \text{ times}} \underbrace{\left(L_{-1}-J^3_{-1}\right)|00\rangle\cdots\left(L_{-1}-J^3_{-1}\right)|00\rangle}_{(N-p)\text{ times}} + [\text{perm.}], \quad (4.93)$$

and is a sum of $\binom{N}{p}$ terms. When we compute the norm of $|p\rangle$, we get nonzero contributions to the inner products coming only from the states in which we have the same object in all the CFT copies: this reduces two sums to just one, and the result is a product of the norms of the building block states (always 1 in our case), taken the appropriate number of times, multiplied by the number of states in $|p\rangle$,

$$\langle p|p\rangle = \binom{N}{p}. \quad (4.94)$$

Moreover states with different p are orthogonal, so

$$\langle p_1|p_2\rangle = \binom{N}{p_1}\,\delta_{p_1,p_2}. \tag{4.95}$$

We are finally ready compute the norm of the full state $\chi(A,B)$,

$$\begin{aligned}\|\chi(A,B)\|^2 &= \sum_{p,q=0}^{N} \bar{A}^q \bar{B}^{N-q} A^p B^{N-p} \langle q|p\rangle \\ &= \sum_{p=0}^{N} \left(|A|^2\right)^p \left(|B|^2\right)^{N-p} \binom{N}{p} \\ &= \left(|A|^2+|B|^2\right)^N .\end{aligned} \tag{4.96}$$

We impose

$$|A|^2+|B|^2 = N, \tag{4.97}$$

so that the normalization condition becomes

$$\|\chi(A,B)\|^2 = N^N. \tag{4.98}$$

We can now compute the VEV of O^{++} starting from the building blocks. This operator connects states with spin $J^3=0$ and $J^3=1/2$, so let's work on a single CFT copy, suppressing copy indices, and let's consider

$$\langle + + |O^{++}(v,u)\left(L_{-1}-J^3_{-1}\right)|00\rangle, \tag{4.99}$$

where we chose to write everything in coordinates on the cylinder so the result can be compared directly with the gravity computation. Using (4.59) we have

$$\begin{aligned}\langle + + |O^{++}(v,u)\,J^3_{-1}|00\rangle &= -\langle + + |\left[J^3_{-1},O^{++}(v,u)\right]|00\rangle \\ &= -\frac{e^{-\mathrm{i}\frac{\sqrt{2}}{R_y}v}}{2}\,\langle + + |O^{++}(v,u)|00\rangle \\ &= -\frac{e^{-\mathrm{i}\frac{\sqrt{2}}{R_y}v}}{2}.\end{aligned} \tag{4.100}$$

The second ingredient we need is the commutator of a primary operator with the modes L_n. The stress-energy operator

$$T(z) = \sum_{n\in\mathbb{Z}} L_n\,z^{-n-2} \tag{4.101}$$

generates infinitesimal transformations

$$z \to w(z) = z + \epsilon(z), \tag{4.102}$$

and in particular its n-th mode L_n generates the transformation with $\epsilon(z) = z^{n+1}$. In general for a holomorphic operator $\mathcal{O}(z)$ of conformal dimension h, under a conformal transformation $z \to w(z)$ we have

$$\mathcal{O}(z) \to \mathcal{O}'(w) = \left(\frac{\partial w}{\partial z}\right)^{-h} \mathcal{O}(z)|_{z=z(w)}. \tag{4.103}$$

For the infinitesimal transformation (4.102) this gives

$$\delta\mathcal{O}(z) \equiv \mathcal{O}'(z) - \mathcal{O}(z) = -h\,\partial_z\epsilon(z)\,\mathcal{O}(z) - \epsilon(z)\,\partial_z\mathcal{O}(z), \tag{4.104}$$

and specifying to $\epsilon(z) = z^{n+1}$ we get

$$\delta\mathcal{O}(z) = \big[\mathcal{O}(z), L_n\big] = -h\,(n+1)\,z^n\,\mathcal{O}(z) - z^{n+1}\,\partial_z\,\mathcal{O}(z). \tag{4.105}$$

Transforming both the LHS and the RHS to the (v,u) coordinates (in Minkowskian signature) using

$$z = e^{\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}, \qquad \bar{z} = e^{\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,u}, \tag{4.106}$$

we get

$$[\mathcal{O}(v), L_n] = \delta\mathcal{O}(v) = -n\,h\,e^{\mathrm{i}n\,\frac{\sqrt{2}}{R_y}\,v}\,\mathcal{O}(v) + \mathrm{i}\,\frac{R_y}{\sqrt{2}}\,e^{\mathrm{i}n\,\frac{\sqrt{2}}{R_y}\,v}\,\partial_v\mathcal{O}(v). \tag{4.107}$$

For the term with L_{-1} therefore we have

$$\begin{aligned}\langle + + |O^{++}(v,u)\,L_{-1}|00\rangle &= \langle + + |\,\big[O^{++}(v,u), L_{-1}\big]\,|00\rangle \\ &= \frac{e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}}{2}\,\langle + + |O^{++}(v,u)|00\rangle + \\ &\quad + \mathrm{i}\,\frac{R_y}{\sqrt{2}}\,e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}\,\langle + + |\partial_v O^{++}(v,u)|00\rangle,\end{aligned} \tag{4.108}$$

where we used the fact that negative modes of $T(z)$ annihilate R vacua from the right. The mode expansion of an operator in coordinates on the cylinder is the "natural" one,

$$\mathcal{O}(v,u) = \sum_{m,n\in\mathbb{Z}} \mathcal{O}_{nm}\,e^{-\mathrm{i}nv-\mathrm{i}mu}, \tag{4.109}$$

and because positive modes of O^{++} annihilate R vacua from the left and negative modes annihilate R vacua from the right we have

$$\begin{aligned}\langle + + | O^{++}(v,u) | 00\rangle &= \langle + + | O^{++}_{00} | 00\rangle \\ &= \langle + + | ++\rangle \\ &= 1,\end{aligned} \tag{4.110}$$

where we recognized the definition in of $|++\rangle$ in terms of $|00\rangle$. In the mode expansion in (v,u) coordinates, the zero mode is constant, so when we act with a derivative as in $\partial_v O^{++}(v,u)$ the zero mode doesn't appear in the expansion: having only nonzero modes, the VEV of $\partial_v O^{++}(v,u)$ on R vacua is zero,

$$\langle + + | \partial_v O^{++}(v,u) | 00\rangle = 0. \tag{4.111}$$

The term with L_{-1} is therefore

$$\langle + + | O^{++}(v,u)\, L_{-1} | 00\rangle = \frac{e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}}{2}. \tag{4.112}$$

Putting these result together, on a single CFT copy and reinstating the copy indices we have

$${}_{(r)}\langle + + | O^{++}_{(r)}(v,u) \left(L_{(r)\,-1} - J^3_{(r)\,-1}\right) | 00\rangle_{(r)} = e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}. \tag{4.113}$$

Let's now consider all the N CFT copies. As usual the total operator O^{++} is defined as

$$O^{++} = \sum_{r=1}^{N} O^{++}_{(r)}. \tag{4.114}$$

From (4.113) we can write the action of $O^{++}_{(r)}$ on $\left(L_{-1} - J^3_{-1}\right)|00\rangle_{(r)}$,

$$O^{++}_{(r)}(v,u)\left(L_{-1} - J^3_{-1}\right)|00\rangle_{(r)} = e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}\,|++\rangle_{(r)} + [\cdots], \tag{4.115}$$

where $[\cdots]$ contain states with zero overlap with ${}_{(r)}\langle + + |$ and that therefore give no contribution to the VEV. Let's now look at the action of the symmetrized operator O^{++} on the states $|p\rangle$. $|p\rangle$ is the sum of $\binom{N}{p}$ states, each of which is a tensor product with $(N-p)$ states $\left(L_{-1} - J^3_{-1}\right)|00\rangle_{(r)}$: acting on each of these tensor product states, O^{++} generates states that give a contribution to the VEV only when it acts on these $(N-p)$ factors (and this is true for every state in the sum $|p\rangle$ is given by). Moreover, each time $O^{++}_{(r)}$ acts on a factor $\left(L_{-1} - J^3_{-1}\right)|00\rangle_{(r)}$, it gives a tensor product state with $(p+1)$ factors $|++\rangle$ (plus other states with no overlap with the bras). Therefore the number of states $|{+}{+}\rangle + [\cdots]$ that O^{++} generates is

$$
\begin{aligned}
&[\#\text{ states in } |p\rangle] \times \big[\#\text{ of copies on which } O^{++} \text{ acts nontrivially in the tensor product states}\big] = \\
&\qquad = \binom{N}{p}(N-p) \\
&\qquad = [\#\text{ states in } |p+1\rangle] \times (p+1),
\end{aligned}
\tag{4.116}
$$

which suggests that the action is

$$
O^{++}(v,u)\,|p\rangle = e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}\,(p+1)\,|p+1\rangle + [\cdots],
\tag{4.117}
$$

where again $[\cdots]$ contains states that have no overlap with any bra state $\langle q|$.

We can now compute the VEV of O^{++} on the state $\chi(A,B)$,

$$
\begin{aligned}
\langle O^{++}(v,u)\rangle_\chi &= \sum_{p,q=0}^{N} \bar{A}^q \bar{B}^{N-q} A^p B^{N-p} \langle q|O^{++}(v,u)|p\rangle \\
&= \sum_{p,q=0}^{N} \bar{A}^q \bar{B}^{N-q} A^p B^{N-p}\, e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}\,(p+1)\langle q|p+1\rangle \\
&= \sum_{p,q=0}^{N} \bar{A}^q \bar{B}^{N-q} A^p B^{N-p}\, e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}\,(p+1)\binom{N}{q}\,\delta_{q,p+1} \\
&= \frac{B}{A}\, e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v} \sum_{q=0}^{N} \left(|A|^2\right)^q \left(|B|^2\right)^{(N-q)}\, q\binom{N}{q}.
\end{aligned}
\tag{4.118}
$$

Calling $x \equiv |A|^2$ we have

$$
\begin{aligned}
\langle O^{++}(v,u)\rangle_\chi &= \frac{B}{A}\, e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}\, x\,\partial_x \sum_{q=0}^{N} x^q \left(|B|^2\right)^{N-q}\binom{N}{q} \\
&= \frac{B}{A}\, e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}\, x\,N\left(x+|B|^2\right)^{N-1} \\
&= \frac{B}{A}\, e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}\, |A|^2\,N\left(|A|^2+|B|^2\right)^{N-1} \\
&= \bar{A}BN^N\, e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v},
\end{aligned}
\tag{4.119}
$$

where we used the normalization condition (4.97). Dividing by the norm $\|\chi(A,B)\|^2$ we get

$$
\frac{\langle O^{++}(v_E,u_E)\rangle_\chi}{\|\chi(A,B)\|^2} = \bar{A}B\, e^{-\mathrm{i}\,\frac{\sqrt{2}}{R_y}\,v}.
\tag{4.120}
$$

The operator $O^{--} = (O^{++})^\dagger$ also acquires a VEV on $\chi(A,B)$, given by the complex conjugate of the above.

Let's now consider the J^3 and $\tilde{J}^3$ operators. In order to compute their VEVs on $\chi(A,B)$ we need to remember that the positive modes of these operators annihilate R vacua from the left, while the negative modes annihilate R vacua from the right. Moreover, R vacua are eigenstates of the zero modes on one CFT copy $J^3_{(r)\,0}$ and $\tilde{J}^3_{(r)\,0}$, with the eigenvalues corresponding respectively to their left and right spin. Using this and using the mode algebras (2.34) and (2.50) we see that on a single CFT copy the only nonzero terms when considering $\langle q|J^3|p\rangle$ or $\langle q|\tilde{J}^3|p\rangle$ are

$$ {}_{(r)}\langle ++|J^3_{(r)}(v)|++\rangle_{(r)} = \frac{1}{2} = {}_{(r)}\langle ++|\tilde{J}^3_{(r)}(v)|++\rangle_{(r)}. \tag{4.121} $$

We can then write

$$ J^3(v)|p\rangle = \frac{1}{2}\,|p\rangle, \tag{4.122a} $$

$$ \tilde{J}^3(v)|p\rangle = \frac{1}{2}\,|p\rangle. \tag{4.122b} $$

Because of this, the VEVs of J^3 and $\tilde{J}^3$ on $\chi(A,B)$ are the same, and, e.g. for J^3, we have

$$ \begin{aligned} \langle J^3(v)\rangle_\chi &= \sum_{p,q=0}^{N} \bar{A}^q\bar{B}^{N-q}A^pB^{N-p}\langle q|J^3(v_E)|p\rangle \\ &= \sum_{p,q=0}^{N} \bar{A}^q\bar{B}^{N-q}A^pB^{N-p}\left(\frac{p}{2}\right)\langle q|p\rangle \\ &= \frac{1}{2}\sum_{p=0}^{N}\left(|A|^2\right)^p\left(|B|^2\right)^{N-p}\,p\binom{N}{p} \\ &= \frac{1}{2}\,x\,\partial_x\sum_{p=0}^{N}(x)^p\left(|B|^2\right)^{N-p}\binom{N}{p}\Bigg|_{x=|A|^2} \\ &= \frac{1}{2}\,x\,\partial_x\left(x+|B|^2\right)^N\Big|_{x=|A|^2} \\ &= \frac{|A|^2}{2}\,N\left(|A|^2+|B|^2\right)^{N-1} \\ &= \frac{|A|^2}{2}\,N^N, \end{aligned} \tag{4.123} $$

where we used the inner product (4.95) and the normalization condition (4.97). The result for $\tilde{J}^3$ is the same, and dividing by the norm (4.98) we get

$$ \frac{\langle J^3(v)\rangle_\chi}{\|\chi(A,B)\|^2} = \frac{|A|^2}{2}, \tag{4.124a} $$

$$ \frac{\langle \tilde{J}^3(v)\rangle_\chi}{\|\chi(A,B)\|^2} = \frac{|A|^2}{2}. \tag{4.124b} $$

Finally, we can consider the momentum modes. The units of momentum are counted by

$$n_p = \langle L_0 - \tilde{L}_0 \rangle, \tag{4.125}$$

and correspond to the momentum charge Q_p through (3.6), which gives

$$Q_p = \frac{Q_1 Q_5}{N R_y^2}\, n_p. \tag{4.126}$$

To compute n_p, let's start working on a single strand and act on the states with L_0. We have

$$L_{(r)\,0} |{+}{+}\rangle_{(r)} = 0, \tag{4.127}$$

and

$$\begin{aligned} L_{(r)\,0}\left(L_{(r)\,-1} - J^3_{(r)\,-1}\right)|00\rangle_{(r)} &= \left(\left[L_{(r)\,0}, L_{(r)\,-1}\right] - \left[L_{(r)\,0}, J^3_{(r)\,-1}\right]\right)|00\rangle_{(r)} \\ &= \left(L_{(r)\,-1} - J^3_{(r)\,-1}\right)|00\rangle_{(r)}, \end{aligned} \tag{4.128}$$

where we used the fact that $|{+}{+}\rangle$ and $|00\rangle$ are ground states in the Ramond sector and we exploited the mode algebras (2.43) and (2.50). When we consider N strands, with L_n given by the sum over copies (2.48), acting on states $|p\rangle$, L_0 just reads the number of states $\left(L_{-1} - J^3_{-1}\right)|00\rangle$ in the tensor product,

$$L_0|p\rangle = (N-p)|p\rangle, \tag{4.129}$$

which intuitively corresponds to counting the units of momentum in the state. The computation is analogous to that for the VEV of J^3 and gives

$$\begin{aligned} \langle L_0 \rangle_\chi &= \sum_{p,q=0}^{N} \bar{A}^q \bar{B}^{N-q} A^p B^{N-p} \langle q|L_0|p\rangle \\ &= \sum_{p=0}^{N} \left(|A|^2\right)^p \left(|B|^2\right)^{N-p} (N-p)\binom{N}{p} \\ &= \left(N - |A|^2\right) N^2 \\ &= |B|^2 N^N, \end{aligned} \tag{4.130}$$

so

$$\frac{\langle L_0 \rangle_\chi}{\|\chi(A,B)\|^2} = N - |A|^2 = |B|^2 = n_p, \tag{4.131}$$

which corresponds to the average number of $\left(L_{-1} - J^3_{-1}\right)|00\rangle$. There are no excitations for the right momentum modes, $\langle \tilde{L}_0 \rangle_\chi = 0$.

The geometry dual to the state $\chi(A,B)$ was found in [59] and corresponds to selecting the values for the k, n, m parameters

$$k = 1, \qquad m = 0, \qquad n = 1. \tag{4.132}$$

The full solution is complicated, but we can restrict to just the objects we need. We have

$$Z_4 = b_{k,m,n}\, R_y\, \frac{\Delta_{k,m,n}}{\Sigma}\, \cos \hat{v}_{k,m,n}, \tag{4.133}$$

where

$$\Delta_{k,m,n} = a^k r^n \left(r^2 + a^2\right)^{-\frac{k+n}{2}} \cos^m \theta\, \sin^{k-m} \theta, \tag{4.134}$$

$$\Sigma = r^2 + a^2\, cos^2\theta, \tag{4.135}$$

$$\hat{v}_{k,m,n} = \frac{\sqrt{2}}{R_y}\, (m+n)\, v + (k-m)\, \phi - m\, \phi. \tag{4.136}$$

The parameter a will be related to the parameter A in the CFT state, while $b_{k,m,n}$ will be related to the parameter B. Specifying to the $k = 1, m = 0, n = 1$ case we get

$$\Delta_{1,0,1} = a\, \frac{r}{r^2 + a^2}\, \sin \theta, \tag{4.137}$$

$$\hat{v}_{1,0,1} = \frac{\sqrt{2}}{R_y}\, v + \phi. \tag{4.138}$$

The object $b_{k,m,n}$ has inside Z_4 has a complicated expression, but it our case it can be simply treated as another parameter, and we put

$$b_{1,0,1} \equiv b. \tag{4.139}$$

Given the above definitions we get

$$Z_4 = \frac{a\, b\, r\, R_y}{(r^2 + a^2)\, (r^2 + a^2 \cos^2 \theta)}\, \cos\left(\frac{\sqrt{2}\, v}{R_y} + \phi\right), \tag{4.140}$$

which has the large-r expansion

$$Z_4 = \frac{1}{r^3}\left\{\sqrt{2}\, a\, R_y \sqrt{\frac{Q_1 Q_5}{R_y} - a^2}\, \cos\left(\frac{\sqrt{2}\, v}{R_y} + \phi\right) \sin \theta\right\} + O(r^{-4}). \tag{4.141}$$

From the supergravity solution we also have the regularity condition

$$a^2 + \frac{x_{k,m,n}\, b_{k,m,n}}{2} = \frac{Q_1 Q_5}{R_y^2}, \tag{4.142}$$

where the only information we need is that

$$x_{1,0,1} = 1, \tag{4.143}$$

(see [59] for the general definition). In the $k = 1, m = 0, n = 1$ case this reduces to

$$a^2 + \frac{b^2}{2} = \frac{Q_1 Q_5}{R_y^2}. \tag{4.144}$$

The angular dependence of the leading term in (4.141) can be rewritten as

$$\begin{aligned} \cos\left(\frac{\sqrt{2}\,v}{R_y} + \phi\right) \sin\theta &= \cos\left(\frac{\sqrt{2}\,v}{R_y}\right) \cos\phi \,\sin\theta - \sin\left(\frac{\sqrt{2}\,v}{R_y}\right) \sin\phi\,\sin\theta \\ &= \frac{1}{2}\cos\left(\frac{\sqrt{2}\,v}{R_y}\right) Y_1^1 - \frac{1}{2}\sin\left(\frac{\sqrt{2}\,v}{R_y}\right) Y_1^2, \end{aligned} \tag{4.145}$$

so using (4.29) we can read off

$$\mathcal{A}_{11} = \frac{a\,b\,R_y}{2\sqrt{Q_1 Q_5}} \cos\left(\frac{\sqrt{2}\,v}{R_y}\right), \qquad \mathcal{A}_{12} = -\frac{a\,b\,R_y}{2\sqrt{Q_1 Q_5}} \sin\left(\frac{\sqrt{2}\,v}{R_y}\right), \tag{4.146}$$

where we used the constraint (4.144). Using (4.33), (4.34b) and (4.35) the gravity prediction for the VEV of O^{++} is

$$\begin{aligned} \Big(\langle O^{++}(v,u)\rangle_\chi\Big)_{\text{Gravity}} &= c_{O(1,1)}\Big(\mathcal{A}_{11} + \mathrm{i}\,\mathcal{A}_{12}\Big) \\ &= \frac{N\,a\,b\,R_y^2}{\sqrt{2}\,Q_1 Q_5}\, e^{-\mathrm{i}\,\frac{\sqrt{2}\,v}{R_y}} \\ &= \left(a\,\frac{R_y\sqrt{N}}{\sqrt{Q_1 Q_5}}\right)\left(b\,\frac{R_y\sqrt{N}}{\sqrt{2Q_1 Q_5}}\right) e^{-\mathrm{i}\,\frac{\sqrt{2}\,v}{R_y}}. \end{aligned} \tag{4.147}$$

The result above agrees with the CFT computation (4.120) upon setting

$$A = a\,\frac{R_y\sqrt{N}}{\sqrt{Q_1 Q_5}}, \qquad B = b\,\frac{R_y\sqrt{N}}{\sqrt{2Q_1 Q_5}}, \tag{4.148}$$

assuming that $A, B \in \mathbb{R}$.

We can now compute the VEVs of J^3 and $\tilde{J}^3$ on the gravity side and compare them with the CFT results (4.124). According to the holographic prescription (4.34a) and (4.29c), the VEVs of J^a and $\tilde{J}^a$ are encoded in the leading term ($\sim r^{-2}$) of the asymptotic expansion of the 1-forms β and ω, which are part of the supergravity solution. Taking the solution found in [59] with $k = 1$, $m = 0$, $n = 1$ and expanding for large r we get

$$\begin{aligned}\beta &= \frac{R_y\, a^2}{\sqrt{2}\, r^2}\left(\sin^2\theta\, d\phi - \cos^2\theta\, d\psi\right) + O(r^{-3}),\\ &= \frac{R_y\, a^2}{\sqrt{2}\, r^2}\, Y_1^{3-}(\theta,\phi) + O(r^{-3}), \qquad (4.149a)\\ \omega &= \frac{R_y\, a^2}{\sqrt{2}\, r^2}\left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right) + O(r^{-3}),\\ &= \frac{R_y\, a^2}{\sqrt{2}\, r^2}\, Y_1^{3+}(\theta,\phi) + O(r^{-3}), \qquad (4.149b)\end{aligned}$$

from which, using (4.29c) we read off

$$a_{3+} = \frac{R_y\, a^2}{2\sqrt{Q_1 Q_5}}, \qquad a_{3-} = -\frac{R_y\, a^2}{2\sqrt{Q_1 Q_5}}\,. \qquad (4.150)$$

Using (4.34a) and (4.35) we finally get

$$\begin{aligned}\Big(\langle s|J^3(v_E)|s\rangle\Big)_{\text{Gravity}} &= \frac{N\, R_y^2\, a^2}{2\, Q_1 Q_5} = \frac{A^2}{2}, \qquad (4.151a)\\ \Big(\langle s|\tilde{J}^3(v_E)|s\rangle\Big)_{\text{Gravity}} &= \frac{N\, R_y^2\, a^2}{2\, Q_1 Q_5} = \frac{A^2}{2}, \qquad (4.151b)\end{aligned}$$

where we used (4.148). This agrees with the CFT results (4.124) upon taking A and B to be real and is an independent check that the relation (4.148) among the parameters appearing in the CFT state and in the supergravity solution is correct.

Let's finally compute Q_p. From [59] we have

$$\mathcal{F} = b_{k,m,n}^2\, \mathcal{F}_{k,m,n}, \qquad \mathcal{F}_{1,0,n} = -\frac{1}{a^2}\left(1 - \frac{r^{2n}}{(r^2+a^2)^n}\right), \qquad (4.152)$$

where for $k=1, m=0, n=1$ we have the identification (4.139) and

$$\mathcal{F}_{1,0,1} = -\frac{1}{a^2}\left(1 - \frac{r^2}{r^2+a^2}\right), \qquad (4.153)$$

which give the asymptotic expansion

$$\mathcal{F} = -\frac{b^2}{r^2} + O\left(r^{-4}\right). \qquad (4.154)$$

Using (4.29) we read off the momentum charge

$$Q_p = \frac{b^2}{2}. \qquad (4.155)$$

The gravity prediction for n_p is thus

$$(n_p)_{\text{gravity}} = \frac{N R_y^2}{Q_1 Q_5} Q_p = B^2, \tag{4.156}$$

where we used (4.148), finding agreement with the CFT result (4.131) upon choosing $B \in \mathbb{R}$.

4.4 Entanglement entropy in D1-D5 microstates

The entanglement entropy of a space domain A in a given microstate represents a useful observable to characterize the microstate itself. The investigation of this observable, for a domain A composed of a single interval of length l in a two-charge microstate, was initiated in [81], where by following [84, 85] it was shown that the EE admits an expansion for small l in terms of the VEVs of operators whose dimensions increase with the order of the l-expansion. If only chiral primary operators (and their descendants) are kept in this expansion, the resulting EE coincides with the one evaluated at the gravity point in the CFT moduli space, which, on the other hand, can be holographically computed via the Ryu-Takayanagi formula [71] (and its generalizations [82]). Hence the EE provides an alternative handle to compare the VEVs of chiral operators in D1-D5 microstates in the CFT and the gravity pictures. We extend here the results of [81] by considering more general two-charge microstates, with non-vanishing VEVs for the twist operators, and also a class of three-charge microstates.

Before analyzing particular examples, we describe a general approach for the holographic and the CFT derivations of the EE in microstate geometries.

4.4.1 Holographic computation at the first non-trivial order

The original formalism of Ryu-Takayanagi applies to static asymptotically AdS geometries; as microstate geometries are not static, the appropriate formulation is the covariant one developed in [82]: the EE is given by the area of the co-dimension two surface that extremizes the area functional and is homotopic to the entangling domain A, seen as a submanifold of the AdS boundary. Our situation has a further complication, in that microstate geometries are asymptotically $\text{AdS}_3 \times S^3$ (as the compact space trivially decouples for our class of microstates, we directly work in the 6D Einstein geometry (3.30) obtained by compactification on T^4); moreover generic microstates have a product structure only at the boundary, and there is no invariant way to decouple the AdS_3 and the S^3 part in the spacetime interior. In [81] a recipe was given to write the 6D space as an S^3 part fibered over an asymptotically AdS_3 space (mathematically, to define an almost product structure); the recipe was based on the introduction of a special set of coordinates, defined, at the first non-trivial order in the expansion around the AdS boundary, by a de Donder gauge condition. This recipe allows to define a 3D asymptotically AdS metric, to which the formulas of [71, 82] can be directly applied; moreover, reducing the problem from 6D to 3D drastically simplifies the computation of the EE.

Though the recipe used in [81] correctly reproduces the CFT result at the first non-trivial order in the small l expansion, it would be desirable to have an a priori justification for the gauge choice defining the $\mathrm{AdS}_3 \times S^3$ split. An alternative, geometrically natural, procedure[18] to holographically compute the EE in spaces that are asymptotically $\mathrm{AdS}_3 \times S^3$, is to consider, as suggested by [82], an extramal co-dimension two surface in the full 6D space that reduces at the boundary to $\partial A \times S^3$. We will show here the equivalence between the invariant 6D and the gauge-fixed 3D recipes, at the first non-trivial order in the expansion around the AdS boundary (which coincides with the small l expansion). The extension to higher orders remains an interesting open problem.

The 6D[19] Einstein metric can, in full generality, be written in the form

$$ds_6^2 \equiv G_{MN}\, dx^M dx^N = g_{\mu\nu}\, dx^\mu dx^\nu + G_{\alpha\beta}(dx^\alpha + A^\alpha_\mu\, dx^\mu)(dx^\beta + A^\beta_\nu\, dx^\nu)\,, \qquad (4.157)$$

so that one has

$$A^\alpha_\mu = G^{\alpha\beta} G_{\mu\beta}\ , \quad g_{\mu\nu} = G_{\mu\nu} - G^{\alpha\beta} G_{\mu\alpha} G_{\nu\beta}\ . \qquad (4.158)$$

The coordinates are chosen in such a way that x^μ are AdS_3 coordinates at the boundary and x^α are S^3 coordinates at the boundary; the continuation of these coordinates to the interior of the space is, a priori, arbitrary. In [81] this arbitrariness was (partly) fixed by requiring that the gauge fields A^α_μ satisfy the gauge condition

$$\nabla^0_\alpha A^\alpha_\mu = 0\,, \qquad (4.159)$$

with ∇^0_α the covariant derivative with respect to round metric of S^3. We will see that this gauge choice simplifies the covariant EE computation and reduces the problem to the 3D one solved in [81].

In this 6D geometry, consider a co-dimension two submanifold which at the boundary reduces to S^3 times a co-dimension two submanifold in AdS_3 given by the boundary of the entangling domain $A = [0, l]$. We can parametrize its worldvolume by x^α plus a parameter λ, so that the parametric representation of the submanifold is $(x^\mu(\lambda, x^\alpha), x^\alpha)$.

[18]We thank R. Emparan, V. Hubeny and J. Simon for drawing our attention to this point.

[19]Though we specify to geometries that are asymptotically $\mathrm{AdS}_3 \times S^3$, as they are the ones relevant for the D1-D5 system, the argument can be readily extended to $\mathrm{AdS}\times S$-type of spaces in arbitrary dimension.

The metric induced on the submanifold is

$$ds_*^2 = g_{\mu\nu}dx_*^\mu dx_*^\nu + G_{\alpha\beta}(dx^\alpha + A_\mu^\alpha dx_*^\mu)(dx^\beta + A_\nu^\beta dx_*^\nu) \equiv g_{IJ}^* d\lambda^I d\lambda^J \,, \tag{4.160}$$

with

$$dx_*^\mu = \dot{x}^\mu d\lambda + \partial_\alpha x^\mu dx^\alpha \,, \tag{4.161}$$

and $\lambda^I \equiv (\lambda, x^\alpha)$. According to the recipe of [82], this submanifold should extremize the area functional:

$$\frac{\partial}{\partial x^\mu}\sqrt{\det g^*} - \frac{\partial}{\partial\lambda}\frac{\partial}{\partial\dot{x}^\mu}\sqrt{\det g^*} - \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial\partial_\alpha x^\mu}\sqrt{\det g^*} = 0\,, \tag{4.162}$$

where we abbreviate $\dot{x}^\mu \equiv \partial_\lambda x^\mu$. These are complicated partial differential equations for the unknowns $x^\mu(\lambda, x^\alpha)$. However, in the limit of small l, the extremal surface probes only a region of spacetime very near the AdS boundary, and, at least at leading order in this asymptotic expansion, the extremality equations can be reduced to simpler ordinary differential equations for the functions $X^\mu(\lambda) \equiv \int dx^\alpha \sqrt{\det G^0}\, x^\mu(\lambda, x^\alpha)$. To perform this perturbative analysis, we introduce a parameter ϵ that controls the expansion away from the AdS boundary; the first non-trivial corrections to the metric have the form

$$g_{\mu\nu} \equiv g_{\mu\nu}^0 + \epsilon\, \delta g_{\mu\nu}^1 + \epsilon^2\, \delta g_{\mu\nu}^2 \,, \quad G_{\alpha\beta} \equiv G_{\alpha\beta}^0 + \epsilon\, \delta G_{\alpha\beta}^1 + \epsilon^2\, \delta G_{\alpha\beta}^2 \,, \quad A_\mu^\alpha \equiv \epsilon\, \delta A_\mu^\alpha \,, \tag{4.163}$$

where $g_{\mu\nu}^0$ is the AdS$_3$ metric, which only depends on x^μ, and $G_{\alpha\beta}^0$ is the S^3 metric, which only depends on x^α; the correction terms, $\delta g_{\mu\nu}^i$, $\delta G_{\alpha\beta}^i$, δA_μ^α, depend both on x^μ and x^α. Correspondingly the functions describing the submanifold can be expanded as

$$x^\mu(\lambda, x^\alpha) = x_0^\mu(\lambda) + \epsilon\, x_1^\mu(\lambda, x^\alpha) + \epsilon^2\, x_2^\mu(\lambda, x^\alpha) + O(\epsilon^3)\,, \tag{4.164}$$

where $x_0^\mu(\lambda)$ is an extremal surface in AdS$_3$. The expansion (4.163) descends from the asymptotic expansion (4.29), where one should think of f_{1i}^I, $\mathcal{A}_{1i}$, $a_{\alpha\pm}$ as being proportional to ϵ, while Q_p is proportional to ϵ^2. One can then verify that, for our geometries, the first order corrections to the AdS$_3$ and the S^3 metrics vanish: $\delta g_{\mu\nu}^1 = \delta G_{\alpha\beta}^1 = 0$. Since, as we will see, the gauge fields A_μ^α only contribute quadratically, this implies that the first non-trivial corrections to the extremal surface $x^\mu(\lambda, x^\alpha)$ and to the EE appear at order ϵ^2. Here we will limit our analysis to these first non-trivial corrections..

In Appendix C we provide the proof of the following facts:

(i) in the gauge (4.159), the first order corrections to the extremal surface vanish: $x_1^\mu(\lambda, x^\alpha) = 0$;

(ii) at order ϵ^2 the area of the extremal surface, and hence the EE, only depends on the S^3 integral of the extremal surface: $X^\mu(\lambda) \equiv \int dx^\alpha \sqrt{\det G^0}\, x^\mu(\lambda, x^\alpha)$;

(iii) the extremality equations for $X^\mu(\lambda)$ are the geodesic equations for a curve in a reduced 3D metric

$$\tilde{g}_{\mu\nu} \equiv g^0_{\mu\nu} + \epsilon^2 \int dx^\alpha \sqrt{\det G^0} \left(\delta g^2_{\mu\nu} + \frac{1}{3} g^0_{\mu\nu}\, G_0^{\alpha\beta} \delta G^2_{\alpha\beta}\right). \tag{4.165}$$

These are precisely the equations considered in [81].

4.4.2 CFT computation at the first non-trivial order

The CFT result for the EE for a single interval A of length l at order $\sim l^2$ is

$$S_A = \Bigg[2N \log\left(\frac{l}{R_y}\right) - \frac{l^2}{12R_y^2}\Big(-2\langle T\rangle + \mathcal{N}_J^{-1}\langle J^\alpha\rangle^2 + \mathcal{N}_{\tilde{J}}^{-1}\langle \tilde{J}^\alpha\rangle^2 + \\ + \mathcal{N}^{-1}_{O^{(1,1)}}\langle O^{(1,1)}_{(1)1i}\rangle^2 + \mathcal{N}^{-1}_{O^{(0,0)}}\langle O^{(0,0)}_{(2)i}\rangle^2\Big) + O\left((l/R_y)^3\right)\Bigg], \tag{4.166}$$

where the $\mathcal{N}$ coefficients are the normalizations of the two-point functions of the operators

$$\langle 0|J^\alpha(1)J^\beta(0)|0\rangle = \mathcal{N}_J\,\delta^{\alpha\beta}, \qquad \langle 0|\tilde{J}^\alpha(1)\tilde{J}^\beta(0)|0\rangle = \mathcal{N}_{\tilde{J}}\,\delta^{\alpha\beta}, \\ \langle 0|O^{(1,1)}_{(1)1i}O^{(1,1)}_{(1)1j}|0\rangle = \mathcal{N}_{O^{(1,1)}}\,\delta_{ij}, \qquad \langle 0|O^{(0,0)}_{(2)i}O^{(0,0)}_{(2)j}|0\rangle = \mathcal{N}_{O^{(0,0)}}\,\delta_{ij}\,, \tag{4.167}$$

with values

$$\mathcal{N}_J = \mathcal{N}_{\tilde{J}} = \mathcal{N}_{O^{(1,1)}} = \frac{n_1 n_5}{2}. \tag{4.168}$$

Part of this result was found in [81], the only difference being that here we need to compute the explicit value of $\mathcal{N}_{O^{(0,0)}}$ and we have an extra term coming from the VEV of the stress-energy operator.

The computation of $\mathcal{N}_{O^{(0,0)}}$ is straightforward: it is sufficient to consider a state $(|{+}{+}\rangle_{k=1})^N$ and compute the VEV of $\Sigma_2^{++}\Sigma_2^{--}$ on it, which by (4.40) and (4.44) yields

$$\left({}_{k=1}\langle + + |\right)^N \Sigma_2^{++}\Sigma_2^{--}\left(|{+}{+}\rangle_{k=1}\right)^N = \frac{N(N-1)}{2} \simeq \frac{N^2}{2}. \tag{4.169}$$

Writing the operators $\Sigma_2^{\pm\pm}$ in terms of $O^{(0,0)}_{(2)i}$ as in (4.32) we get an extra factor 1/2 ($\mathcal{N}_{O^{(0,0)}}$ is defined starting from the real operators) which gives

$$\mathcal{N}_{O^{(0,0)}} \simeq \frac{N^2}{4}. \tag{4.170}$$

As explained in [81], all the terms but the one related to T come from contributions of 2-point functions of the CFT primaries: the contributions of the 1-point functions of primaries give zero, and in the case analyzed there no descendants had a nonzero VEV. In the present case, though, T is a descendant of the identity operator, has a nonzero VEV and because of its conformal dimension it gives a contribution of the same order in l/R_y as the 2-point functions. This new contribution can be computed exploiting the procedure followed in [84, 85]. The EE for a single interval A in the dual CFT can be written as

$$S_A = -\frac{\partial}{\partial n} S_n|_{n=1}, \qquad S_n = \langle s|\mathcal{T}_n(z,\bar{z})\mathcal{T}_{-n}(w,\bar{w})|s\rangle, \tag{4.171}$$

with

$$\mathcal{T}_n(z,\bar{z})\mathcal{T}_{-n}(w,\bar{w}) = |z-w|^{-4\Delta_n}\left(1+\sum_K\sum_{j=1}^{n}(z-w)^{\Delta_K}(\bar{z}-\bar{w})^{\bar{\Delta}_K} d_K^{(j)} O_K^{(j)} + \cdots\right), \tag{4.172}$$

where we have written only the contribution of single CFT operators acting nontrivially on one copy of the CFT (not tensor products or two or more of them) and $\Delta_n = \bar{\Delta}_n = \frac{c}{24}(n-\frac{1}{n})$ is the conformal dimension of the twist fields $\mathcal{T}_{\pm n}$. We can isolate the contribution given by T multiplying both sides by $T(u)$, taking the VEV on the vacuum[20] $|0\rangle$ and comparing the terms in $\sim (u-w)^{-4}$ as $z \to w$. From the OPE

$$T(z)\,T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{4.173}$$

we have that the relevant part of the RHS is

$$|z-w|^{-4\Delta_n}(z-w)^2 d_T \frac{c/2}{(u-w)^4}, \tag{4.174}$$

where we have used the fact that the 2-point function of T with itself brings a $\delta^{j,j'}$ and, since the constant doesn't actually depend on j we set $d_K^{(j)} \equiv d_T$. For the LHS we have

$$\langle 0|\mathcal{T}_n(z,\bar{z})\mathcal{T}_{-n}(w,\bar{w})T(u)|0\rangle = \frac{\langle 0|\mathcal{T}_{-n}(\infty)T(1)\mathcal{T}_n(0)|0\rangle}{(z-w)^{2\Delta_n-2}(z-u)^2(w-u)^2(\bar{z}-\bar{w})^{2\Delta_n}}, \tag{4.175}$$

and we immediately see that as $z \to w$ this is exactly of the same order as (4.174). Therefore we have

$$d_T = \frac{2}{c}\,\langle 0|\mathcal{T}_{-n}(\infty)T(1)\mathcal{T}_n(0)|0\rangle. \tag{4.176}$$

The twist fields $\mathcal{T}_{\pm n}$ introduce a branch cut from 0 to ∞, with the effect that taking an operator around the origin makes it jump from one Riemann sheet to the another

[20] Here by $|0\rangle$ we mean the vacuum given by the tensor product of the bosonic vacuum and of the NS vacuum both in the left and right fermionic sectors. The crucial property we want to exploit is that the VEV on this state of any primary operator is zero.

among the n. We can get rid of the twist fields altogether by performing the conformal transformation

$$z \to \tilde{z} = z^{1/n}, \tag{4.177}$$

which maps the n Riemann sheets to a single copy of the complex plane. In doing so the $\mathcal{T}_{\pm n}$ disappear and we only have to worry about the transformation of T. It's now clear why we are considering the contribution to the EE of the 1-point function of the stress-energy operator and not of primary operators of small conformal dimension. Had we had a primary instead of T, the conformal transformation we are using to get rid of the twist fields would have given an object proportional to the 1-point function of the primary itself in the vacuum $|0\rangle$, which is zero. We can only hope to obtain a nonzero contribution from 1-point functions of descendants, and among these the only one contributing at the order in l/R_y we are interested in is exactly T. The reason why the contribution of T is nonzero comes from the fact that it transforms as

$$T(z) \to T(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \left[T(z) - \frac{c}{12}\, S(\tilde{z}, z)\right], \tag{4.178}$$

where

$$S(\tilde{z}, z) = \left(\frac{\partial^3 \tilde{z}}{\partial z^3}\right)\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left[\left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1}\right]^2 \tag{4.179}$$

is the *Schwarzian derivative* of the transformation. Performing the conformal transformation $u \to \tilde{u} = u^{1/n}$ we get a term proportional to $\langle 0|T(\tilde{u})|0\rangle$, which is zero because T has no VEV on $|0\rangle$, plus another term that gives

$$d_T = \lim_{u\to 1} \frac{2}{c}\,\frac{c}{12}\, S(\tilde{u}, u) = \frac{1}{6}\lim_{u\to 1} S(\tilde{u}, u). \tag{4.180}$$

The Scwarzian derivative reads

$$S(\tilde{u}, u) = \frac{1}{2}\, u^{-2}\, \frac{1}{n}\left(n - \frac{1}{n}\right) = \frac{12}{c}\, u^{-2}\left(\frac{\Delta_n}{n}\right), \tag{4.181}$$

which gives

$$d_T = \left(\frac{2}{c}\right)\frac{\Delta_n}{n} = \frac{1}{12}\left(1 - \frac{1}{n^2}\right). \tag{4.182}$$

Notice that within respect to [70], our Δ_n is defined after summing over j, which gives an extra factor n.

The result for the contribution to S_n (where again the sum over j brings just a factor n) is

$$S_{n,T} = |z - w|^{-4\Delta_n}\,(z - w)^2\,\frac{1}{12}\left(n - \frac{1}{n}\right)\langle s|T(w)|s\rangle, \tag{4.183}$$

which gives a contribution to the EE

$$S_{A,T} = -\frac{1}{6}(z-w)^2 \langle s|T(w)|s\rangle. \tag{4.184}$$

Notice that up to this point the coordinates $(z, \bar{z})$, $(w, \bar{w})$ are generic; now we have to specify a choice of coordinates in order to set a fixed value for the time and an interval in the spatial circle for the entangling domain A. In doing this we have to be careful, as the fact that the stress-energy operator is not a primary causes its VEV to depend on the choice of coordinates, i.e. under a conformal transformation $z \to \tilde{z}$, due to (4.178), T and its antiholomorphic counterpart $\tilde{T}$ can acquire a VEV even if it was zero in the original coordinates,

$$\langle T(z)\rangle \to \langle T(\tilde{z})\rangle = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \left[\langle T(z)\rangle - \frac{c}{12}\, S(\tilde{z}, z)\right]. \tag{4.185}$$

In the preceding section all the VEVs have been computed in coordinates on the complex plane (proportional to $\sim t \pm \mathrm{i}y$), and in particular it is with this choice that $\langle T\rangle = 0$ on 2-charge states, i.e. that L_0 reads zero conformal dimension for ground states in the Ramond sector. Consistency with this requires that the coordinates in (4.184) be identified with

$$z \equiv \frac{t+\mathrm{i}y}{R_y}, \qquad \bar{z} \equiv \frac{t-\mathrm{i}y}{R_y}. \tag{4.186}$$

Notice that the coordinates in (4.184) are generic and must not be confused with the coordinates on the cylinder with the same name as in (4.106): the one above is an identification, not a coordinate transformation. This is also consistent with Eq. (4.3) of [81]. We can further identify z with the choice $(t, y) = (t, l)$ and w with $(t, y) = (t, 0)$ where 0 and l correspond to the boundary of the entangling domain $A = [0, l]$ and where the value of t is arbitrary, getting

$$z = \frac{t+\mathrm{i}l}{R_y}, \qquad w = \frac{t}{R_y}. \tag{4.187}$$

The contribution of T to the EE at order $\sim l^2/R_y^2$ then becomes

$$S_{A,T} = \frac{l^2}{6R_y^2}\, \langle s|T(t/R_y)|s\rangle. \tag{4.188}$$

As a final remark, we'd like to stress the fact that the EE at a given order in the small l/R_y expansion is independent of the choice of coordinates, even though $\langle T\rangle$ is not. This comes from the fact that when changing coordinates, additional terms come both from the transformation of $\langle T\rangle$ (and $\langle\tilde{T}\rangle$) and from the universal logarithmic leading term. It can be seen, e.g. at order $\sim l^2/R_y^2$, that the additional terms in fact cancel out.

4.4.3 Entanglement Entropy of three-charge states

We now want to compare the CFT prediction for the single interval EE derived in Section 4.4.2, with the holographic computation outlined in Section 4.4.1. For generic D1-D5-P states, we immediately face the difficulty that we do not know the general expression of the dual geometry. We have however verified, through the examples of Sections 4.3.2 and 4.3.3, that the 3-charge solutions found in [58] have an asymptotic expansion of the form (4.29). We conjecture that this is true for all three-charge states. The knowledge of the expansion (4.29) is enough to compute the EE up to order $\sim (l/R_y)^2$, and hence compare with the CFT result (4.166).

Starting with the 6D metric given in (3.30) with the metric coefficients expanded as in (4.29), one derives the reduced 3D metric defined in (4.165):

$$\tilde{g}_{tt} = -\frac{r^2}{\sqrt{Q_1Q_5}}\left[1 + 2\delta\mathcal{P} + \frac{1}{r^2}\left((a_+)^2 + (a_-)^2 - Q_p\right)\right] + O(r^{-2})\,, \tag{4.189}$$

$$\tilde{g}_{yy} = \frac{r^2}{\sqrt{Q_1Q_5}}\left[1 + 2\delta\mathcal{P} - \frac{1}{r^2}\left((a_+)^2 + (a_-)^2 - Q_p\right)\right] + O(r^{-2})\,, \tag{4.190}$$

$$\tilde{g}_{rr} = \frac{r^2}{\sqrt{Q_1Q_5}}\left[1 + 4\delta\mathcal{P}\right] + O(r^{-2})\,, \tag{4.191}$$

$$\tilde{g}_{ty} = \frac{r^2}{\sqrt{Q_1Q_5}}\left[-\frac{1}{r^2}\left((a_+)^2 - (a_-)^2 - Q_p\right)\right] + O(r^{-2})\,, \tag{4.192}$$

with

$$\delta\mathcal{P} = -\frac{1}{2}\frac{(f_1^1)^2}{r^2} - \frac{1}{2}\frac{(\mathcal{A}_1^1)^2}{r^2}\,, \tag{4.193}$$

where

$$(a_\pm)^2 \equiv \sum_{\alpha=1}^{3}(a_{\alpha\pm})^2\,, \quad (f_1^1)^2 \equiv \sum_{i=1}^{4}(f_{1i}^1)^2\,, \quad (\mathcal{A}_1)^2 \equiv \sum_{i=1}^{4}(\mathcal{A}_{1i})^2\,. \tag{4.194}$$

The gauge fields coming from the reduction on S^3 (4.157) are

$$A_v^\alpha = \sqrt{2}\,G_0^{\alpha\beta}a_{\gamma+}(Y_1^{\gamma+})_\beta + O(r^{-2})\,, \quad A_u^\alpha = \sqrt{2}\,G_0^{\alpha\beta}a_{\gamma-}(Y_1^{\gamma-})_\beta + O(r^{-2})\,, \tag{4.195}$$

with $G_0^{\alpha\beta}$ the inverse of the round S^3 metric. They satisfy the gauge condition (4.159) because the vector spherical harmonics are divergence-less:

$$\nabla^\alpha(Y_1^{\gamma\pm})_\alpha = 0\,. \tag{4.196}$$

As explained in Section 4.4.1, we can thus apply the Ryu-Takayanagi procedure to the reduced 3D metric $\tilde{g}_{\mu\nu}$ and we obtain the result:

$$S_A = 2n_1n_5\left[\log\left(\frac{r_0 l}{\sqrt{Q_1Q_5}}\right) - \frac{l^2}{12Q_1Q_5}\left(-Q_p + (a_+)^2 + (a_-)^2 + 2(\mathcal{A}_1)^2 + 2(f_1^1)^2\right) + O(l^3)\right], \tag{4.197}$$

where r_0 is an IR cutoff corresponding to the AdS_3 boundary (see [81]). One immediately recognizes a structure similar to (4.166): the term $(f_1^1)^2$ corresponds to the contribution given by $O^{(0,0)}_{(2)i} \equiv \Sigma_2^{\alpha\dot{\alpha}}$, the term $(\mathcal{A}_1)^2$ to $O^{(1,1)}_{(1)1i} \equiv O^{\alpha\dot{\alpha}}$, the terms $(a_\pm)^2$ to J^α and $\tilde{J}^\alpha$ and the term Q_p to $\langle L_0 - \tilde{L}_0\rangle$. To verify that also the numerical coefficients match, one uses the relations between the gravity parameters f^1_{1i}, $\mathcal{A}_{1i}$, $a_{\alpha\pm}$, Q_p and the CFT VEVs given in (4.34) with the coefficients $c_{O^{(0,0)}}$, $c_{O^{(1,1)}}$, c_J, $c_{\tilde{J}}$ specified in (4.53) and (4.35), and the values of the normalization constants $\mathcal{N}$ in (4.168) and (4.170). One can check that these substitutions map precisely the gravity result (4.197) into the CFT formula (4.166). Part of this match was already performed in [81]; what is new here is the momentum contribution proportional to $Q_p \sim \langle L_0 - \tilde{L}_0\rangle$ and the verification of the numerical factor in front of the twist field term proportional to $(f_1^1)^2 \sim \langle\Sigma_2\rangle^2$. Note that this provides an independent non-trivial check of the coefficient $c_{O^{(0,0)}}$, which was fixed in Section 4.3.1 by requiring the CFT-gravity consistency for one particular microstate.

The contribution of T also agrees with the expansion for small L of equation (3.11) of [86] with $r_0^2 = Q_p$.

4.5 1-point functions and Entaglement Entropy: discussion and outlook

The 1-point functions of BPS operators and the single interval EE are useful observables to establish a link between microstates and the dual geometries, and to enlighten the emergence of the spacetime from the CFT. Even if the computations of this chapter were limited to chiral primaries of dimension 1 and to the first non-trivial corrections to the EE in the small interval limit, the detailed match between gravity and CFT results provides a quite impressive verification of the map between 1/4-BPS states and two-charge geometries proposed in [32,63], and of its extension to the 1/8-BPS states of [58]. In examples like the one worked out in Appendix B, a relatively simple gravity result is matched against a very non-trivial CFT computation, which uses the correlators of twist operators[21] derived in [88]. In other examples, like the one of Section 4.3.3, the presence of a particular term in the geometry follows, in the gravity picture, from a quite involved regularity analysis [58], while it is implied quite straightforwardly by the non-vanishing of a twist operator VEV, in the CFT picture. This last phenomenon is

[21]The techniques for handling twist operator insertions in orbifold CFTs have been developed in a long series of papers [87, 88]; the effects of these insertions on the EE have been investigated in [89].

surprising, because the analysis of regularity requires the knowledge of the geometry in the interior of the spacetime, while the CFT picture only involves operators of small dimension (one, in our case), which are associated with deformations of the geometry close to the AdS boundary. This example highlights the power of the CFT in predicting non-trivial features of the dual spacetime.

Hence, a natural extension of our work consists in extracting from the CFT analysis the necessary information to construct the geometries dual to a larger and more generic family of three-charge states than the one known at present, possibly capturing a finite fraction of the D1-D5-P entropy. In the three-charge microstates of [58], the momentum is carried by the current J^+_{-1} acting on strands with spin $(0, 0)$; when spectrally flowed to the NS sector, J^+_{-1} becomes J^+_0 [44]. Together with L_0, $L_{\pm1}$,[22] the modes J^α_0 form the rigid subsector of the CFT chiral algebra, and states where momentum is carried by these rigid generators constitute the so-called "graviton gas" contribution to the D1-D5-P elliptic genus [90, 91]. The full elliptic genus includes states where momentum is carried by fractional-moded currents acting on strands of winding greater than one: indeed these states dominate the entropy in the limit of large charges. Constructing the geometries dual to such states[23] is crucial for the advancement of the fuzzball program [7, 60, 93], which aims at providing a geometric description of black hole microstates in terms supergravity (or more generally string theory) configurations without horizons. For the purpose of this construction, the information provided by the VEVs of BPS operators of dimension larger than one, which determine the higher orders in the asymptotic expansion (4.29), could be essential. Extending the holographic analysis to higher dimension operators could pose technical hurdles (like the operator mixing phenomenon discussed in [80]), but the general methods developed in [94–96] should allow progress in this direction.

Having higher dimension operators under control would also be necessary for understanding how a thermal behavior emerges from typical black hole microstates and to quantify the deviations between typical pure states and statistical ensembles [97–99]. The states we consider in this chapter are not generic representatives of the ensemble giving rise to the black hole entropy, and indeed the VEVs of simple, low dimension operators, which are non-vanishing in our states, are expected to be suppressed in the large charge limit for typical microstates. But more complex, higher dimension operators can have non-trivial VEVs also in typical states. At least for BPS operators, the free orbifold CFT picture described in Chapter 2 offers a precise tool to characterize and estimate the correlators which can distinguish generic states among themselves and from the maximally mixed state. The holographic dictionary will then allow to

[22] Geometries dual to states where momentum is carried by L_{-1} were constructed at linear level in [61] and can be extended to nonlinear level using methods similar to [58].

[23] Particular states in this class have already been constructed in [92].

determine if and how these differences manifest themselves in the classical geometry.

Similar questions could be addressed by using the single interval EE as a probe of the microstate geometry. As we have seen, when the length of the interval is small, the EE only probes the region of spacetime close to the boundary, and is only sensitive to operators of small dimension. But as the length increases, the entangling curve reaches deeper in the bulk, possibly exploring the whole spacetime[24]. It has been argued [65, 101, 102] that in the limit of large central charge, the EE in a typical pure state is dominated by the conformal block of the identity, and hence it reproduces the thermal answer associated with the BTZ black hole [86]. On the other hand we have seen that in our atypical states, the EE receives contributions also from the conformal blocks of non-trivial chiral primaries. It would be interesting to quantify the contribution of non-trivial primaries to the EE in typical states, and evaluate the induced deviations from the thermal behaviour. Chapter 5 moves in this direction giving a conformal and affine block analysis of 2-point functions of operators of small conformal dimension in CFT states corresponding to (simple) 2- and 3-charge microstates.

[24]The regions of the geometry that are not swept by the entangling curve are called entanglement shadows. The existence of shadows in geometries containing conical defects was pointed out in [100]; in the D1D5 CFT, these geometries are dual to pure states containing multiply wound strands with spin $(\pm 1/2, \pm 1/2)$.

Chapter 5

HH-LL 4-point functions and holography

As we have seen in Chapter 4, on the CFT side the black hole microstates correspond to 1/8-BPS "heavy" states in the Ramond-Ramond sector which have conformal dimension of order c. If we think for example of the simple $(|{+}{+}\rangle_{k=1})^N$ state, we see that it is the product of N Ramond vacua obtained after spectral flow from N Neveu-Schwarz ones as in (2.107): its dimension is equal to the dimension (2.108) of the spectral flow operator itself and it is therefore of order $\sim c$. All of the R vacua are obtained starting from the NS vacuum by spectral flow (and then acting with some other operator, depending on the specific state considered), so all of them have dimensions of order $\sim c$.

In this chapter we extend the study of the 1/4 and 1/8-BPS states in the $(4,4)$ CFT and their dual asymptotically $\mathrm{AdS}_3 \times S^3 \times \mathcal{M}$ geometries by studying the correlators of (two) light operators in a heavy state[25]. In the OPE limit in which the light operators are close, the correlator effectively resums an infinite series of vev's, and hence it represents an observable that can probe the bulk of the space-time. Our approach is based on very standard techniques: on the CFT side we need to calculate a 4-point function with two heavy and two light operators, while on the bulk side we study the wave equation of a light field in the dual non-trivial geometry. The main goal is to understand in some detail how the large c limit of the CFT correlator reproduces the result obtained in the gravitational description. This heavy-light, large c limit has been analyzed in several papers: an explicit expression for the Virasoro blocks in this limit was derived in [64, 65] and a dual interpretation of this result in pure AdS_3 gravity was discussed in [105–109]. Here we will apply the same approach to the simplest possible

[25] Correlators involving two light twist operators that induce a transition between two different heavy states have been computed in [103], with the purpose of studying absorption and emission of quanta from a D1D5 bound state. A computation of two-point functions of primary operators in heavy excited states at large c has also been recently performed in [104].

heavy operators in the $(4,4)$ CFT that have a dual geometric description in type IIB supergravity.

One of the main features of our analysis is that the full higher dimensional geometry is important in the bulk calculation, which is reflected on the CFT side by the contribution of operators that are not Virasoro descendants of the identity. This is a pattern that already emerged in Chapter 4 in the study of the 1-point functions and the entanglement entropy [1, 81] and, of course, it is particularly evident in our calculations because we chose very peculiar and simple heavy operators (i.e. very atypical states in the black hole ensemble). However, these examples show that pure heavy states are not directly described by the 3D geometry of the BTZ solution and that, on the CFT side, Virasoro primaries different from the identity can play an important role also in the large c limit. In particular, in the correlators we consider, the singularities due to the large c Virasoro block of the identity are resolved by the contributions of new primaries that are non-trivial already at the leading order in the limit $c \gg 1$. So in this case the pattern is different from the one discussed in [68][26], where it is argued that $1/c$ corrections are crucial to restore unitarity. In the simple cases we investigate, this mechanism is visible already at the supergravity level as the relevant new Virasoro primaries are actually affine descendants of the identity. For more general correlators the contribution of primary operators that are not captured in the supergravity approximation will most likely be crucial to avoid the appearance of spurious singularities when $c \to \infty$. We also present an argument based on crossing symmetry supporting the idea that the heavy-light correlators have in general a regular large c limit if the contribution of all primaries is considered. Thus, even if the results for the correlators we studied cannot be directly extrapolated to typical black hole microstates, we suggest that the absence of large c spurious singularities in the heavy-light correlators is generic and that it might be seen as a CFT feature supporting the fuzzball proposal [14, 30]. The results in this chapter were found in [2].

5.1 The CFT picture

In this section we discuss some simple examples of four-point correlators in the D1D5 CFT. In particular we are interested in correlators with two heavy (O_H) operators, which have conformal dimension of order c, and two light (O_L) operators, which have conformal dimension of order one. Thus the structure of the correlators we consider is

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \frac{1}{z_{12}^{2h_H}\, z_{34}^{2h_L}}\frac{1}{\bar{z}_{12}^{2\bar{h}_H}\, \bar{z}_{34}^{2\bar{h}_L}}\mathcal{G}(z,\bar{z}) \; , \tag{5.1}$$

[26] See [66, 102, 110] for a detailed discussion of the Virasoro blocks beyond the leading term in the $c \to \infty$ expansion and [111] for the possible relevance of $1/c$ corrections in black hole collapse.

where, as usual, $z_{ij} = z_i - z_j$ and

$$z = \frac{z_{14}z_{23}}{z_{13}z_{24}}\,, \tag{5.2}$$

while $(h_H, \bar{h}_H)$ and $(h_L, \bar{h}_L)$ are the holomorphic/antiholomorphic conformal dimensions of the heavy and light operators respectively.

As in the previous chapter, we take two main simplifying assumptions. First we focus on highly supersymmetric operators. The light operators we use are chiral primaries both in the left and in the right sector of the CFT. Instead the heavy operators are in the Ramond-Ramond sector of the CFT, but are related to chiral primaries by a chiral algebra transformation that acts only on the left sector (hence they generically preserve half of the CFT supercharges). Second, we work again at the free orbifold point of the CFT moduli space, so all the technology of Chapter 2 applies.

5.1.1 Simple correlators in the untwisted sector

We first focus on operators in the untwisted sector of the symmetric orbifold, which means that they are written as combinations of operators acting on each copy. The symmetry under permutations among the copies is realised differently in the light and the heavy operators: the light operators act trivially on all the strands but one[27] (see (2.29)), while the heavy ones are constructed by multiplying N copies of the same operator, each copy acting on a different strand:

$$O_L = \frac{1}{\sqrt{N}}\sum_{r=1}^{N} O^L_{(r)}\,, \qquad\qquad O_H = \otimes_{r=1}^{N} O^H_{(r)}\,. \tag{5.3}$$

In the language of the previous chapters, the structures (5.3) are the key distinguishing feature between what we generically called "CFT operators" and what we called "vacuum states". In this chapter we concentrate on light operators of dimension $h_L = \bar{h}_L = 1/2$ constructed with the fermions; in concrete we take

$$O^L_{(r)} = -\frac{\mathrm{i}}{\sqrt{2}}\,\psi^{1\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{\dot{1}\dot{B}}_{(r)} \equiv O^{++}_{(r)}\,, \qquad \bar{O}^L_{(r)} = -\frac{\mathrm{i}}{\sqrt{2}}\,\psi^{2\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{\dot{2}\dot{B}}_{(r)} \equiv O^{--}_{(r)}\,. \tag{5.4}$$

All the operators $O^H_{(r)}$ we are going to consider in the untwisted sector have right conformal dimension $\bar{h}_{(r)} = 1/4$ and right spin $\tilde{J}^3_{(r)} = 1/2$, which gives a total right conformal dimension for the heavy operators $\bar{h}_H = N/4$, so we can distinguish the heavy operators by their left conformal dimension and left spin. The heavy operators we choose in the untwisted sector are characterised by an integer s determining the

[27] In an unfortunate mismatch of conventions, in this chapter we found more useful to add a $1/\sqrt{N}$ normalization to the definition of the light operators, see (2.38) for a comparison. This only changes a numerical factor in the correlators.

number of J^+ excitations acting on a ground state in each copy; their explicit expression is more easily written in the bosonized language of Section 2.4 (see also below), and their left conformal dimension and spin are given by

$$h_H = N\left(s+\frac{1}{2}\right)^2, \qquad J^3_H = N\left(s+\frac{1}{2}\right). \tag{5.5}$$

We therefore denote the single copy operators making up the heavy states as $O^H_{(r)}(s)$ and the same notation will be adopted for the correlators, which are denoted as $\mathcal{G}\left(s;z,\bar{z}\right)$.

As a first concrete example we consider the heavy operator corresponding to $s=0$; it is written in terms of the spin fields $S^{\dot{A}}_{(r)}$ twisting the elementary fermions $\psi^{\alpha\dot{A}}_{(r)}$ (and $\tilde{S}^{\dot{A}}_{(r)}$ twisting $\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}$)

$$O^H_{(r)}(s=0) = S^{\dot{1}}_{s=0,(r)} S^{\dot{2}}_{s=0,(r)} \tilde{S}^{\dot{1}}_{s=0,(r)} \tilde{S}^{\dot{2}}_{s=0,(r)}\ , \tag{5.6}$$

where the definition for generic s is

$$S^{\dot{1}}_{s,(r)} \equiv e^{\mathrm{i}\left(s+\frac{1}{2}\right)H_{(r)}}, \quad S^{\dot{2}}_{s,(r)} \equiv e^{\mathrm{i}\left(s+\frac{1}{2}\right)K_{(r)}}, \quad \tilde{S}^{\dot{1}}_{s,(r)} \equiv e^{\mathrm{i}\left(s+\frac{1}{2}\right)\tilde{H}_{(r)}}, \quad \tilde{S}^{\dot{2}}_{s,(r)} \equiv e^{\mathrm{i}\left(s+\frac{1}{2}\right)\tilde{K}_{(r)}}. \tag{5.7}$$

The $s=0$ case corresponds to the operators generating the states (2.106). Let us comment on the AdS-dual interpretation of the operators entering this correlator. For $s=0$ the heavy state is the Ramond-Ramond ground state with the highest value for the left and right spins, $(|{+}{+}\rangle_{k=1})^N$. This state can be obtained by starting from the $SL(2,\mathbb{C})$ invariant vacuum and performing a spectral flow to the Ramond-Ramond sector as in (2.106), which means that the dual supergravity solution[28] is locally isometric to $\mathrm{AdS}_3\times S^3$. The light operator (5.4) is a supersymmetric fluctuation of the B-field and the axion [32] around the geometry dual to O_H. We can calculate the correlator at the orbifold point of the CFT moduli space by using the standard bosonization approach and the free field contractions in the bosonic language. We collect in Appendix D a brief derivation of the result:

$$\mathcal{G}\big(s=0;z,\bar{z}\big) = \frac{1}{|z|}\ . \tag{5.8}$$

A simple generalization of (5.8) is to consider the correlator with the same light states, but heavy states corresponding to generic s, which contain excited spin fields in the holomorphic sector

$$O^H_{(r)}\left(s;z,\bar{z}\right) = S^{\dot{1}}_{s,(r)} S^{\dot{2}}_{s,(r)} \tilde{S}^{\dot{1}}_{s=0,(r)} \tilde{S}^{\dot{2}}_{s=0,(r)}\ , \tag{5.9}$$

[28]It is possible to extend this solution to an asymptotically flat type IIB supergravity background, which then represents a (very special) microstate for the Strominger-Vafa black hole [112, 113].

where $S^{\dot{A}}_{s,(r)}$ has conformal weight $(s+1/2)^2/2$. Again by using the bosonized language it is straightforward to calculate the correlator (see Appendix D for some detail)

$$\mathcal{G}\left(s; z, \bar{z}\right) = \frac{1}{z^{s+\frac{1}{2}}\,\bar{z}^{\frac{1}{2}}} \tag{5.10}$$

Note that the new heavy state is an affine descendant of the Ramond-Ramond ground state (5.6) and so the dual description can be locally mapped to $\mathrm{AdS}_3 \times S^3$ with a change of coordinates that encode (at the boundary) the action of the superalgebra on (5.6). Thus, as discussed later in Section 5.1.2, this new correlator inherits several properties from the previous example in (5.8).

5.1.2 Simple correlators in the twisted sector

We now consider correlators in the twisted sector of the CFT. In analogy to what we did in the previous section, the heavy operators are constructed by taking N/k identical strands of length k. The antiholomorphic conformal dimension of our heavy operators on each strand is always $\bar{h}_{H,1\text{ strand}} = k/4$ and their right spin is $\tilde{J}^3_{H,1\text{ strand}} = 1/2$. As before, we consider s momentum-carrying excitations in the holomorphic sector, so we characterize the heavy operators by two integers s and k, and their left conformal dimension and spin read

$$h_{H,1\text{ strand}} = \frac{N}{k}\left(\frac{k}{4} + \frac{s(s+1)}{k}\right), \qquad J^3_{1\text{ strand}} = \frac{N}{k}\left(s + \frac{1}{2}\right). \tag{5.11}$$

The operators are denoted as $O_H(s,k)$ and the correlators as $\mathcal{G}(s,k;z,\bar{z})$.

The first kind of heavy operators we consider corresponds to $s = 0$ and generic k and is a generalization to strands of length k of (5.6): on each strand we have k operators $S^{\dot{A}}_{k,\rho}$ and k operators $\tilde{S}^{\dot{A}}_{k,\rho}$ and the total heavy operator is

$$O_H(s=0,k) = \left[\otimes_{\rho=0}^{k-1}\, \sigma^X_\rho\, \tilde{\sigma}^X_\rho\, S^{\dot{1}}_{k,s=0,\rho} S^{\dot{2}}_{k,s=0,\rho} \tilde{S}^{\dot{1}}_{k,s=0,\rho} \tilde{S}^{\dot{2}}_{k,s=0,\rho}\right]^{N/k}, \tag{5.12}$$

where σ^X_ρ and $\tilde{\sigma}^X_\rho$ are the twist fields acting on the bosonic sector of the CFT (see (2.82)) while the $S^{\dot{\alpha}}_{k,s=0,\rho}$ are the operators in (2.116), i.e.

$$\begin{aligned} S^{\dot{1}}_{k,s=0,\rho} &\equiv e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)H_\rho}, \quad S^{\dot{2}}_{k,s=0,\rho} \equiv e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)K_\rho}, \\ \tilde{S}^{\dot{1}}_{k,s=0,\rho} &\equiv e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)\tilde{H}_\rho}, \quad \tilde{S}^{\dot{2}}_{k,s=0,\rho} \equiv e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)\tilde{K}_\rho}. \end{aligned} \tag{5.13}$$

The correlator is obtained again through bosonization in the twisted sector (the derivation is sketched in appendix D) and reads

$$\mathcal{G}\,(s=0,k;z,\bar{z}) = \frac{1/k}{|z|}\,\frac{1-|z|^2}{1-|z|^{2/k}}, \tag{5.14}$$

where the $1/k$ factor comes from having the same contribution from each of the N/k strands and from the normalization chosen for the light operators in (5.3).

The second kind of heavy operator we consider corresponds to nonzero s and k and is a generalization to strands of length k of (5.9). These states have $s(s+1)/k$ units of momentum on each strand, and since the number of momentum units must be integer, assuming k is a prime number for simplicity, we have that either $s = pk$ or $s = pk-1$, with $p \in \mathbb{N}$. In the $s = pk$ case, in the left sector of each strand we have k operators $S^{\dot{A}}_{k,s,\rho}$, and another k operators $\tilde{S}^{\dot{A}}_{k,\rho}$ live in the right sector. The total heavy operator is

$$O_H(s=pk,k) = \left[\otimes_{\rho=0}^{k-1}\,\sigma^X_\rho\,\tilde{\sigma}^X_\rho\,S^{\dot{1}}_{k,s=pk,\rho}S^{\dot{2}}_{k,s=pk,\rho}\tilde{S}^{\dot{1}}_{k,s=0,\rho}\tilde{S}^{\dot{2}}_{k,s=0,\rho}\right]^{N/k}. \tag{5.15}$$

Notice that since $h_{H,1\text{ strand}}$ depends on s, for $s>0$ we have $h_{H,1\text{ strand}} \neq \bar{h}_{H,1\text{ strand}}$ and so heavy states carry non-vanishing momentum; the explicit definition of the heavy operators for $s = pk$ is, in the bosonized language,

$$\begin{aligned} S^{\dot{1}}_{k,s=pk,\rho} &\equiv e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+\frac{s}{k}\right)H_\rho}, \quad S^{\dot{2}}_{k,s=pk,\rho} \equiv e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+\frac{s}{k}\right)K_\rho},\\ \tilde{S}^{\dot{1}}_{k,s=pk,\rho} &\equiv e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+\frac{s}{k}\right)\tilde{H}_\rho}, \quad \tilde{S}^{\dot{2}}_{k,s=pk,\rho} \equiv e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+\frac{s}{k}\right)\tilde{K}_\rho}, \end{aligned} \tag{5.16}$$

and the correlator reads

$$\mathcal{G}\,(s=pk,k;z,\bar{z}) = \frac{1/k}{|z|}\,\frac{1-|z|^2}{1-|z|^{2/k}}\,z^{-p}. \tag{5.17}$$

When $s = pk-1$ the heavy operator differs from the previous case only in the $\rho = 0$ sector, and has the form

$$O_H(s=pk-1,k) = \left[S^{\dot{1}}_{k,s=pk-1,\rho=0}S^{\dot{2}}_{k,s=pk-1,\rho=0}\otimes_{\rho=1}^{k-1}\,\sigma^X_\rho\,\tilde{\sigma}^X_\rho\,S^{\dot{1}}_{k,s=pk,\rho}S^{\dot{2}}_{k,s=pk,\rho}\tilde{S}^{\dot{1}}_{k,s=0\rho}\tilde{S}^{\dot{2}}_{k,s=0,\rho}\right]^{N/k}, \tag{5.18}$$

where $S^{\dot{\alpha}}_{k,s=pk-1,\rho\neq 0}$ and $\tilde{S}^{\dot{\alpha}}_{k,s=pk-1,\rho\neq 0}$ have the same forms as in the $s = pk$ case, while for $\rho = 0$ they read

$$\begin{aligned} S^{\dot{1}}_{k,s=pk-1,\rho=0} &\equiv e^{\mathrm{i}\left(-\frac{1}{2}+p\right)H_{\rho=0}}, \quad S^{\dot{2}}_{k,s=pk-1,\rho=0} \equiv e^{\mathrm{i}\left(-\frac{1}{2}+p\right)K_{\rho=0}},\\ \tilde{S}^{\dot{1}}_{k,s=pk-1,\rho=0} &\equiv e^{\mathrm{i}\left(-\frac{1}{2}+p\right)\tilde{H}_{\rho=0}}, \quad \tilde{S}^{\dot{2}}_{k,s=pk-1,\rho=0} \equiv e^{\mathrm{i}\left(-\frac{1}{2}+p\right)\tilde{K}_{\rho=0}}. \end{aligned} \tag{5.19}$$

The difference between the $s = pk$ and the $s = pk-1$ cases is that the heavy operators in the first are obtained from the one in the latter by acting with the mode $2p$ of the

J^- current acting on the length-k strand, J^-_{2p}; this action only changes the operators in the $\rho = 0$ sector. The correlator reads

$$\mathcal{G}(s = kp - 1, k; z, \bar{z}) = \frac{1/k}{|z|}\, z^{-p} \left(z + \frac{|z|^{2/k} - |z|^2}{1 - |z|^{2/k}} \right) . \tag{5.20}$$

5.2 Conformal blocks decomposition

In this section we analyze the correlators obtained above in terms of Virasoro and affine conformal blocks, exploiting the underlying $SU(2)$ R-symmetry. In the channel where the two light operators approach each other ($z_3 \to z_4$), the cross-ratio z tends to 1 and we can expand the function $\mathcal{G}$ in (5.1) to extract the Virasoro or affine primary operators entering in the decomposition:

$$\mathcal{G} = (1 - z)^{2h_L}(1 - \bar{z})^{2\bar{h}_L} \sum_{O_p} C_{HHO_p} C_{LLO_p} \mathcal{V}_{V,A}(h_p, h_H, h_L, z) \bar{\mathcal{V}}_{V,A}(\bar{h}_p, \bar{h}_H, \bar{h}_L, \bar{z}) \, , \tag{5.21}$$

where the sum is over all Virasoro or affine primaries O_p, $\mathcal{V}_V$ and $\mathcal{V}_A$ are the Virasoro or affine blocks, C_{HHO_p} are the structure constants between O_p and the heavy operators and C_{LLO_p} are the structure constants between O_p and the light operators.

5.2.1 Virasoro blocks decomposition

For the description in terms of the Virasoro blocks we focus on the large c limit where it is possible to use the results of [64, 65]. In this limit the contribution of the Virasoro descendants of a primary of weight h_p is captured by the block whose holomorphic part is[29]

$$\mathcal{V}_V(h_p, h_H, h_L, z) = z^{h_L(\alpha - 1)} \left(\frac{1 - z^\alpha}{\alpha} \right)^{h_p - 2h_L} {}_2F_1\left(h_p, h_p; 2h_p; 1 - z^\alpha\right) \, , \tag{5.22}$$

where $\alpha = \sqrt{1 - \frac{24 h_H}{c}}$. Some of the heavy states we consider have conformal dimension $h_H = c/24$ (they are the ones corresponding to tensor products of ground states, without any excitation); in this case the large c limit of the Virasoro block is captured by the $\alpha \to 0$ limit[30] of (5.22)

$$\mathcal{V}_V(h_p, h_H \to c/24, h_L, z) = z^{-h_L} \left(-\ln z\right)^{h_p - 2h_L} \, . \tag{5.23}$$

In all amplitudes analyzed in the previous section, the first primary entering the

[29] We normalize the conformal block so that the first term of the $z \to 1$ expansion is $(1 - z)^{h_p - 2h_L}$.

[30] It is also possible to follow a similar derivation as in [65] with $h_H = c/24$ and show that the result agrees with the $\alpha \to 0$ limit of the formula above.

$z \to 1$ decomposition is the identity. If we consider only the contribution of its Virasoro block, for instance in the simplest case (5.8), we have

$$\mathcal{G}\big(s=0; z, \bar{z}\big) = \frac{1}{|z|}\frac{|1-z|^2}{|\ln z|^2} + \dots \ , \tag{5.24}$$

where we used (5.21) and (5.23) with $h_p = 0$, $h_L = 1/2$, and the analogous expression for the antiholomorphic sector with $\bar{h}_p = 0$, $\bar{h}_L = 1/2$. Focusing on the holomorphic dependence, there is a mismatch between (5.24) and (5.8) already at the order $(1-z)$, which signals that primaries of conformal dimension $(h_p, \bar{h}_p) = (1, 0)$ must contribute to the correlator (5.8). It is straightforward to see that in the OPE of the two light operators O_L, $\bar{O}_L$ the first (normalized) Virasoro primaries are

$$\begin{aligned} O_{(1,0)} &= \sqrt{\frac{2}{N}}\sum_{r=1}^{N} J^3_{(r)} \ , \\ O_{(2,0)} &= \frac{1}{\sqrt{6N}}\sum_{r=1}^{N}\left(-\partial\psi^{\alpha\dot{A}}_{(r)}\psi^{\beta\dot{B}}_{(r)}\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}} + \frac{1}{2}\partial X^{A\dot{A}}_{(r)}\partial X^{B\dot{B}}_{(r)}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\right) \ . \end{aligned} \tag{5.25}$$

We can straightforwardly compute the three-point correlators between these primaries and the heavy or the light operators so to extract the structure constants entering in the decomposition (5.22). For later convenience, we summarize the results involving the light and the heavy operators in (5.9) for generic s:

$$\begin{aligned} C_{LLO_{(1,0)}} &= \frac{1}{\sqrt{2}} \ , \quad C_{HHO_{(1,0)}} = \sqrt{2}\left(s+\frac{1}{2}\right) \ , \\ C_{LLO_{(2,0)}} &= \frac{1}{\sqrt{6}} \ , \quad C_{HHO_{(2,0)}} = \frac{(1+2s)^2}{2\sqrt{6}} \ . \end{aligned} \tag{5.26}$$

Thus one can improve on the decomposition (5.24) by adding the Virasoro blocks for the operators in (5.25)

$$\mathcal{G}\big(s=0; z, \bar{z}\big) = \frac{1}{|z|}\frac{|1-z|^2}{|\ln z|^2}\left(1 - \frac{1}{2}\ln z + \frac{1}{12}(\ln z)^2 + \dots\right) \ , \tag{5.27}$$

which reproduces (5.8) to the leading order in the $\bar{z} \to 1$ and to second order in $z \to 1$ limits.

We can proceed with the same analysis for the remaining correlator (5.10) in the untwisted sector. One now has $h_H = \frac{c}{6}\left(s+\frac{1}{2}\right)^2, \bar{h}_H = \frac{c}{24}$, and we have to use the large c Virasoro blocks (5.22) for the holomorphic part and (5.23) for the antiholmorphic one. The contribution of the identity gives

$$\mathcal{G}\big(s; z, \bar{z}\big) = -\frac{|1-z|^2}{\sqrt{\bar{z}}\log(\bar{z})}\ \frac{\alpha\ z^{\frac{\alpha-1}{2}}}{1-z^{\alpha}} + \cdots \ , \tag{5.28}$$

where $\alpha = \sqrt{1-4\left(s+\frac{1}{2}\right)^2}$. Again, the expansion of the expression above for $z \to 1$ already disagrees with the exact result (5.10) at order $(1-z)$ and, as before, we need to add the Virasoro blocks of other primaries. By using the s-dependent structure constants in (5.26), we have

$$\mathcal{G}\big(s;z,\bar{z}\big) = \frac{|1-z|^2}{\sqrt{\bar{z}}\log(\bar{z})}\,\frac{\alpha\, z^{\frac{\alpha-1}{2}}}{z^\alpha-1}\left[1-\frac{1+2s}{2}\log z-\frac{(1+2s)^2}{2\alpha^2}\left(2+\frac{1+z^\alpha}{1-z^\alpha}\log z^\alpha\right)+\dots\right]. \tag{5.29}$$

As in the $s=0$ case, the expression above agrees with the exact result (5.10) up to order $(1-z)^2(1-\bar{z})^0$ in the $z\to 1$ expansion.

5.2.2 Affine blocks decomposition

In all of our examples the light operator (5.4) used to probe the heavy states is written just in terms of the elementary fermions of the orbifold CFT. This suggests that it is convenient to study the decomposition of this type of correlators in terms of affine blocks related to the $SU(2)_L$ current algebra generated by (2.31). As this symmetry is part of the chiral superalgebra we can use this analysis to argue that the correlators considered in the previous section are protected by supersymmetry, and then, in the next section, to match the free CFT result with supergravity calculations. Also, in contrast to the pure Virasoro case, the results for the affine blocks are exact in c and so we can use them to understand the effect of resumming the large c limit of the blocks of all Virasoro primaries: we will see that the singularities due to each Virasoro block [68] disappear even at large c. This is reminiscent of what happens in some out-of-time-ordered correlators in $SU(N)_k$ WZW models [114].

We start from the simplest example discussed in (5.8) and analyze it in two slightly different ways. First we observe that the correlator is purely fermionic and that it is given by a sum over the N strands of correlators that involve non-trivially only the fields on one strand at a time. We can then effectively restrict to two free complex fermions on a length one strand, which realize a $SU(2)_{k=1}\times U(1)$ WZW model[31] (see for instance [115]). Note that the $SU(2)_{k=1}$ factor is identified with the R-symmetry $SU(2)_L$, and is thus a symmetry of the CFT at a generic point in the moduli space; the $U(1)$ symmetry, instead, disappears away from the free orbifold point. The non-trivial 4-point function to compute is the one appearing in the first line of (D.6) for $s=0$; with respect to the $SU(2)_{k=1}$ subsector of the WZW model, all the four operators involved are $SU(2)_{k=1}$ primaries of spin 1/2. Though the light operators also carry a $U(1)$

[31] This approach is similar to one adopted in [114] in the study of quantum chaos in rational CFT. Notice however that in that analysis the large central charge limit is obtained by studying the WZW model $SU(N)_k$ in the limit $N,k\to\infty$ with N/k fixed, instead of using the symmetric orbifold of many copies of $SU(2)_{k=1}$, as relevant for our case.

charge, the heavy states are scalars under this $U(1)$, and thus the correlator reduces to a trivial 2-point function in the $U(1)$ sector. This means that it should be possible to write the amplitude (5.8) by using the classic result of [116] for the affine blocks of $SU(N)_k$ WZW models in the special case where $N = 2$ and $k = 1$. This model has only two primaries (the identity and the spin 1/2 primary) and so the only $SU(2)_{k=1}$ primary appearing in the OPE of two spin 1/2 operators has to be the identity. So in this case the affine decomposition (5.21) contains just one term, given by the $SU(2)_{k=1}$ block of the identity: since $SU(2)_{k=1}$ is part of the superconformal algebra, this shows that the amplitude (5.8) can be written in terms of protected quantities.

It is straightforward to check that the hypergeometric describing the $SU(N)_k$ blocks reduce to elementary functions for the identity block with $N = 2$ and $k = 1$; by adapting the results summarized in [115] to our notations we have[32]

$$\mathcal{V}_{SU(2)_1} = (1-z)^{-2h_L} \begin{pmatrix} F_1^- \\ F_2^- \end{pmatrix} = (1-z)^{-2h_L} \begin{pmatrix} z^{-\frac{1}{2}} \\ z^{\frac{1}{2}} \end{pmatrix} , \qquad (5.30)$$

where the component F_1^- (F_2^-) contributes if the operators in z_1 and z_4 (z_2 and z_4) have opposite spin. In our case (D.6) F_1^- enters in the decomposition of (5.8) and reproduces directly the whole amplitude.

The simple result in (5.30) suggests that only a subsector of the full $SU(2)_{k=1}$ affine blocks contributes to our correlator. This is indeed the case and the amplitude is saturated just considering the affine descendants obtained by acting with the modes of the currents J^3 (and $\tilde{J}^3$) on the identity. Focusing on this $U(1)_L$ subgroup, the affine block of the identity reads[33]

$$\mathcal{V}_{U(1)}(q_H, q_L, z) = (1-z)^{-2h_L} z^{2q_H q_L} , \qquad (5.31)$$

where the q_H and q_L are identified with the J^3 quantum numbers of the operators $\bar{O}^H_{(r)}(z_2)$ and $O^L_{(r)}(z_3)$ (note that, with this identification, the level of the $U(1)_L$ current algebra is $k = 1/2$, in the conventions of [65]). Then, by using $q_H = -1/2 - s$ and $q_L = 1/2$, we immediately reproduce not just (5.8) but also (5.10).

The correlators involving states in the twisted sector can also be described in terms of $U(1)_L$ affine blocks. From (D.10a) the generator J^3 on a strand of length k splits into the sum of k $U(1)$'s labelled by $\rho = 0, \ldots, k-1$. While the charge of the light operator is still $q_L = 1/2$ for any ρ, the charge of the heavy operators is ρ-dependent, as can be seen from (5.16) and (5.19). So the contribution to the block decomposition of each ρ-sector is given by (5.31) with the values for the q's that can be read off from

[32]In order to translate the choice of the z_i^D's of [115] into ours it is sufficient to take $z^D_{i=1,3} = z_{i=1,3}$, $z_2^D = z_4$, and $z_4^D = z_2$; notice also that the blocks in [115] have a different normalization and that the hypergeometric appearing in Eq.(15.170) of [115] should read ${}_2F_1\left(\frac{\kappa+1}{\kappa}, \frac{\kappa-1}{\kappa}, \frac{2\kappa-N}{\kappa}, x\right)$.

[33]See [65] for a recent discussion of the $U(1)$ blocks in the context of the heavy-light large c limit.

(5.16) and (5.19); after performing the sum over ρ, one can check that the correlators (5.17) and (5.20) are reproduced by (5.21) with only the inclusion of the $U(1)_L$ affine block of the identity.

5.3 The gravity picture

Let $|s,k\rangle$ denote the pure states generated by the action of the heavy operators on the conformal invariant vacuum:

$$|s,k\rangle \equiv \lim_{z,\bar{z}\to 0} O_H(s,k;z,\bar{z})|0\rangle\,. \tag{5.32}$$

Since operators of conformal dimension of order c backreact strongly on the geometry and generate a non-trivial gravity background, these states admit a dual gravity description. The four-point correlators computed in the previous section can thus be thought as two-point functions of light correlators in a non-trivial geometry:

$$\langle s,k|O_L(1)\bar{O}_L(z)|s,k\rangle = \frac{1}{|1-z|^{4h_L}}\,\mathcal{G}(z,\bar{z})\,. \tag{5.33}$$

In the limit of large central charge this geometry is well approximated by a solution in supergravity[34]. In this section we will compute this two-point function at the point in the CFT moduli space where supergravity is weakly coupled, i.e. higher curvature corrections are negligible.

This point in moduli space differs from the free orbifold point, where the CFT correlators have been computed. While the light operators we consider are chiral primaries both in the left and right sector and the heavy operators are chiral at least in the right sector, their four-point correlators are generically expected to receive corrections when one deforms the free orbifold theory towards the point in moduli space corresponding to weakly coupled supergravity. This is made evident by the decomposition (5.21), which generically contains also non-chiral primaries (and their descendants). For the particular correlators we consider in this work, we have however shown in Section 5.2.2 that the expansion (5.21) only contains the identity operator and its super-descendants with respect to a $U(1)$ subgroup of the superconformal algbera. This implies that CFT and gravity results must agree. In this section we verify this expectation.

[34]The requirement of large central charge limit is however not sufficient to guarantee the dual bulk state is described by a supergravity solution, as it's possible to construct CFT states with no dual classical geometry even in the $c\to\infty$ limit.

5.3.1 The 6D geometries

As we saw, the D1D5 CFT is dual to a gravity theory on spaces that are asymptotically[35] $\mathrm{AdS}_3 \times S^3$. We stress that the S^3 factor is necessary to geometrically implement the $SU(2)_L \times SU(2)_R$ R-symmetry of the CFT. The geometries dual to generic heavy operators are complicated 6D spaces, which only asymptotically factorize into the product of AdS_3 and S^3. All these geometries are known when the heavy operators are chiral primaries both on the left and the right sector [32,34,35]; a subset of geometries is known for heavy operators that are chiral only on the right sector [37,41,50,55,56,58,92,117], or are not chiral on either sector [118,119].

In this chapter we concentrate on a particularly simple set of BPS states, whose dual geometries are *locally* isometric to $\mathrm{AdS}_3 \times S^3$ via a diffeomorphism that does not vanish at the boundary (which suggests that some transformation also takes places in the boundary CFT). The 6D Einstein metric for these states can be written in the form

$$ds^2 = \sqrt{Q_1 Q_5}\,(ds^2_{AdS_3} + ds^2_{S^3})\,, \tag{5.34a}$$

$$ds^2_{AdS_3} = \frac{dr^2}{a^2 k^{-2} + r^2} - \frac{a^2 k^{-2} + r^2}{Q_1 Q_5} dt^2 + \frac{r^2}{Q_1 Q_5} dy^2\,, \tag{5.34b}$$

$$ds^2_{S^3} = d\theta^2 + \sin^2\theta\, d\hat{\phi}^2 + \cos^2\theta\, d\hat{\psi}^2\,. \tag{5.34c}$$

As usual, the coordinates t, y are identified with the time and space coordinates of the CFT, and we take y to parametrize an S^1 of radius R_y; $\hat{\phi}$ and $\hat{\psi}$ are some linear combinations of the S^3 Cartan's angles ϕ, ψ and the CFT coordinates t, y; the particular linear combination depends on the state and will be given below. This is just a particular case of the general six-dimensional Einstein metric (3.30). The parameter a is linked to the D-brane charges and the S^1 radius by

$$a = \frac{\sqrt{Q_1 Q_5}}{R_y}\,. \tag{5.35}$$

Finally k is a positive integer which introduces a conical defect in the geometry $ds^2_{AdS_3}$: indeed this space represents a $\mathbb{Z}_k$ orbifold of AdS_3.

The gravity solution also includes a RR 2-form, whose field strength is

$$F_3 = 2\, Q_5\, (-\mathrm{vol}_{AdS_3} + \mathrm{vol}_{S^3})\,, \tag{5.36a}$$

$$\mathrm{vol}_{AdS_3} = \frac{r}{Q_1 Q_5}\, dr \wedge dt \wedge dy\,, \quad \mathrm{vol}_{S^3} = \sin\theta\cos\theta\, d\theta \wedge d\hat{\phi} \wedge d\hat{\psi}\,. \tag{5.36b}$$

[35] To describe generic states one should consider the full 10D geometry, which asymptotes $\mathrm{AdS}_3 \times S^3 \times M$, with M either T^4 or $K3$. For the class of states we consider, the M factor is irrelevant and we restrict to the 6D part of the geometry.

The 3-form field strength is anti-self-dual in the 6D Einstein metric

$$*_6\, F_3 = -F_3\,, \tag{5.37}$$

where $*_6$ is the Hodge star with respect to ds^2 and we choose the orientation $\epsilon_{rty\theta\hat{\phi}\hat{\psi}} = +1$.

The two-charge states

The states $|s=0,k\rangle$ have $h_H = \bar{h}_H = \frac{c}{24} = \frac{N}{4}$ and thus carry D1 and D5 charges but no momentum charge. The geometries dual to these states were found in [34] and can be written in the form (5.34) with

$$\hat{\phi} = \phi - \frac{t}{R_y\, k}\,, \quad \hat{\psi} = \psi - \frac{y}{R_y\, k}\,. \tag{5.38}$$

Note that the original set of coordinates (t, y, ϕ, ψ) is subject to the identifications

$$(t, y, \phi, \psi) \sim (t,\, y + 2\pi\, l\, R_y,\, \phi + 2\pi\, m,\, \psi + 2\pi\, n)\,, \tag{5.39}$$

with $l, m, n \in \mathbb{Z}$. Only when $k = 1$ eq. (5.38) defines a new set of coordinates $(t, y, \hat{\phi}, \hat{\psi})$ which satisfy analogous identifications

$$(t, y, \hat{\phi}, \hat{\psi}) \sim (t,\, y + 2\pi\, l\, R_y,\, \hat{\phi} + 2\pi\, m,\, \hat{\psi} + 2\pi\, n)\,, \quad (k = 1)\,. \tag{5.40}$$

In this case the coordinate transformation $(t, y, \phi, \psi) \to (t, y, \hat{\phi}, \hat{\psi})$ realizes the spectral flow from the state $|s = 0, k = 1\rangle$ to the SL(2,$\mathbb{C}$)-invariant vacuum, whose dual geometry is (5.34) with the identifications (5.40), i.e. global AdS$_3 \times S^3$. For $k > 1$ the identifications induced on the $(t, y, \hat{\phi}, \hat{\psi})$ coordinates are more complicated:

$$(t, y, \hat{\phi}, \hat{\psi}) \sim \left(t,\, y + 2\pi\, l\, R_y,\, \hat{\phi} + 2\pi\, m,\, \hat{\psi} - 2\pi\, \frac{l}{k} + 2\pi\, n\right)\,. \tag{5.41}$$

The geometry dual to the state $|s = 0, k\rangle$ is given by (5.34) expressed in the (t, y, ϕ, ψ) coordinate system via (5.38): geometrically it represents a $\mathbb{Z}_k$ orbifold of AdS$_3 \times S^3$. For $k > 1$ there is no state in the D1D5 CFT dual to the geometry (5.34) with the identifications (5.40).

The three-charge states

The states $|s, k\rangle$ have $h_H = \frac{N}{4} + \frac{N\, s(s+1)}{k^2}$, $\bar{h}_H = \frac{N}{4}$ and thus carry momentum $n_p = h - \bar{h} = \frac{N\, s(s+1)}{k^2}$. The dual geometries have been found in [92] and are of the form

(5.34) with

$$\hat{\phi} = \phi - \frac{t}{R_y\, k} - s\,\frac{t+y}{R_y\, k}\,, \quad \hat{\psi} = \psi - \frac{y}{R_y\, k} - s\,\frac{t+y}{R_y\, k} \quad (s \in \mathbb{Z})\,. \tag{5.42}$$

As in the previous example, this coordinate redefinition preserves the simple periodic identifications only for $k = 1$. For $k > 1$ the geometry is again a $\mathbb{Z}_k$ orbifold of $\mathrm{AdS}_3 \times S^3$, though the orbifold group, determined by the coordinate redefinition (5.42), acts differently than in the previous example. It is important to keep in mind that the integers s and k must be such that the momentum on each strand $s(s+1)/k$ be integer[36]. This allows for non-integer s/k; states with s/k integer are particularly simple, as they are obtained from the 2-charge states with $s = 0$ by a global chiral algebra transformation.

We note that setting $s = 0$ the D1-D5-P states specified by eq. (5.42) reduce to the D1D5 states corresponding to (5.38). In the following we will thus work with the more general class of states described by (5.42).

5.3.2 The holographic two-point function

We want to compute the correlator of the light operators $O_L \equiv O^{++}$ and $\bar{O}_L \equiv O^{--}$ in the states $|s,k\rangle$, whose dual geometries are specified by (5.34),(5.36) and (5.42). We will do this by computing the vev of the operator $\bar{O}_L$ in the presence of a source for the operator O_L, and then differentiating the vev with respect to the source to obtain the two-point correlator:

$$\langle s,k|O_L(0,0)\bar{O}_L(t,y)|s,k\rangle = \mathrm{i}\,\frac{\delta\langle\bar{O}_L(t,y)\rangle_J}{\delta\bar{J}_L(0,0)}\Big|_{J=0}\,, \tag{5.43}$$

where $\bar{J}_L$ is the source coupling to O_L and the correlator is computed on the cylinder parametrized by t and y. The vev $\langle\bar{O}_L(t,y)\rangle_J$ is extracted from the supergravity field dual to $\bar{O}_L$. The above correlator is in Lorentzian signature and is time-ordered, which allows Wick rotation to Euclidean signature and comparison with the CFT results.

In 6D[37] the fields dual to the chiral primary operators $O^{\pm\pm}$ are a scalar w and a 2-form B_2, which satisfy a coupled system of differential equations. The linearization of these equations around the background given by (5.34) and (5.36) gives [44, 120]

$$dB_2 - *_6 dB_2 = 2\,w\,F_3\,, \quad d *_6 dw = \frac{Q_1}{Q_5}\,dB_2 \wedge F_3\,. \tag{5.44}$$

[36] This condition only holds when n_1 and n_5 are coprime and a more general condition applies if n_1 and n_5 share a common divisor [92]; for simplicity we will assume n_1 and n_5 coprime in this work.

[37] When lifted to the 10D IIB duality frame, B_2 is the NSNS 2-form and w is the component of the RR 4-form along the compact space M.

The factorised form of the background (when expressed in $\hat{\phi}$, $\hat{\psi}$ coordinates) allows to reduce the 6D equations (5.44) to two sets of decoupled equations on AdS$_3$ and S^3. To this purpose one can make the Ansatz [53]

$$w = Y\,B\,, \qquad B_2 = \gamma\,(Y\;*_{AdS_3} dB - B\;*_{S^3} dY)\,, \tag{5.45}$$

where Y is a function of $\theta, \hat{\phi}, \hat{\psi}$, B is a function of r, t, y, $*_{AdS_3}$ and $*_{S^3}$ are the Hodge duals with respect to $ds^2_{AdS_3}$ and $ds^2_{S^3}$ and γ is a constant that will be determined shortly. It is straightforward to verify that this Ansatz satisfies (5.44) if Y and B are eigenfunctions of the respective Laplacians:

$$\Box_{AdS_3} B = \ell(\ell-2)\,B\,, \quad \Box_{S^3} Y = -\ell(\ell+2)\,Y\,, \tag{5.46}$$

and if $\gamma = \frac{Q_5}{\ell}$. Then Y is a scalar harmonic on S^3 of order ℓ, with ℓ a positive integer; B is a minimally coupled scalar in AdS$_3$ with mass $m^2 = \ell(\ell-2)$.

As the CPO's $O^{\pm\pm}$ form a multiplet with $SU(2)_L \times SU(2)_R$ charges $j = \bar{j} = 1/2$, the gravity dual field must have spin 1, and hence we should look for solutions for B and Y with $\ell = 1$. This follows from the fact that the above multiplet contains four elements and forms a vector representation of $SO(4) = SU(2) \times SU(2)$. The vev of O^{--} is encoded in the component of the field w proportional to the spherical harmonic $Y_1^{++} = \sin\theta\, e^{\mathrm{i}\phi}$ (see (4.33) and (4.34)). Thus we look for a solution of the form

$$w = B(t,y,r)\,\sin\theta\, e^{\mathrm{i}\hat{\phi}} = B(t,y,r)\, e^{-\mathrm{i}\,\frac{t}{R_y k} - \mathrm{i}\,s\,\frac{t+y}{R_y k}}\,\sin\theta\, e^{\mathrm{i}\phi}\,, \tag{5.47}$$

where $B(t,y,r)$ solves the AdS$_3$ Laplace equation (5.46) with $\ell = 1$. Note that the phase $e^{-\mathrm{i}\,s\,\frac{y}{R_y k}}$ is not globally well-defined on the circle $y \sim y + 2\pi\,R_y$ when s/k is fractional. Thus, for w to be a globally defined field, we need to require that the function $B(t,y,r)$ has an appropriate monodromy when going around the S^1 to cancel that of the phase:

$$B(y, y+2\pi R_y, r) = B(t,y,r)\, e^{\mathrm{i}\,\frac{\hat{s}}{k}\,2\pi}\,, \tag{5.48}$$

where $\hat{s} = s \bmod k$ and we choose $0 \le \hat{s} < k$.

Since the non-normalizable and normalizable solutions of the AdS$_3$ wave equation go like $r^{-1}\log r$ and r^{-1}, the usual AdS/CFT prescription implies that the asymptotic behaviour of the field w has the form

$$w \approx \frac{\bar{J}_L(t,y)\,\log r + \langle \bar{O}_L(t,y)\rangle_J}{r}\,\sin\theta\, e^{\mathrm{i}\phi}\,. \tag{5.49}$$

Requiring that w is finite in the interior of space links the normalizable and non-normalizable terms of the solution. In accordance with (5.43), the two point function of $O_L(0,0)$ and $\bar{O}_L(t,y)$ is given by the vev $\langle \bar{O}_L(t,y)\rangle_J$ when the source for O_L is a delta-function: $\bar{J}_L(t,y) = \delta(t,y)$.

In summary, one looks for a solution of the equation (5.46) for B with $\ell = 1$ which is *regular* in the bulk, has the monodromy (5.48), and with the leading behavior at large r

$$B(t,y,r) \approx \delta(t,y)\,\frac{\log r}{r} + b_1(t,y)\,\frac{1}{r}\,. \tag{5.50}$$

AdS solutions with monodromies like in (5.48) are not usually considered in the literature. In Appendix E we will derive the solution of the wave equation in $\mathrm{AdS}_3/\mathbb{Z}_k$ with the boundary conditions prescribed above by generalising the computations in [121,122]. One finds

$$b_1(t,y) = -\mathrm{i}\frac{e^{\mathrm{i}\hat{s}\frac{y}{R_y\,k}}}{e^{\mathrm{i}\frac{t}{R_y\,k}} - e^{-\mathrm{i}\frac{t}{R_y\,k}}}\left[\frac{e^{\mathrm{i}\frac{t-y}{R_y}}}{e^{\mathrm{i}\frac{t-y}{R_y}}-1}e^{-\mathrm{i}\hat{s}\frac{t}{R_y\,k}} + \frac{1}{e^{\mathrm{i}\frac{t+y}{R_y}}-1}e^{\mathrm{i}\hat{s}\frac{t}{R_y\,k}}\right]\,. \tag{5.51}$$

The two-point correlator of the light operators in the state $|s,k\rangle$ is given by

$$\langle s,k|O_L(0,0)\,\bar{O}_L(t,y)|s,k\rangle = \mathrm{i}\,b_1(t,y)\,e^{-\mathrm{i}\,\frac{t}{R_y k}-\mathrm{i}\,s\,\frac{t+y}{R_y k}}\,. \tag{5.52}$$

To compare the bulk result (5.52) with the CFT, one should transform from the Minkowskian cylinder coordinates t and y to the Euclidean plane coordinates[38] $z, \bar{z}$:

$$z = e^{\mathrm{i}\,\frac{t+y}{R_y}}\,, \quad \bar{z} = e^{\mathrm{i}\,\frac{t-y}{R_y}}\,, \tag{5.53}$$

and remember that

$$O_L(z,\bar{z}) = (z\bar{z})^{-1/2}\,O_L(t,y)\,, \tag{5.54}$$

(and the same for $\bar{O}_L$) since $O_L(z,\bar{z})$ is a primary of dimension $h_L = \bar{h}_L = 1/2$. The coordinate transformation above is just (2.3) in Lorentzian signature with the identifications[39] $\tau \equiv t/R_y$ and $\sigma \equiv y/R_y$. The gravity result for the correlator on the plane is then

$$\langle s,k|O_L(1)\,\bar{O}_L(z,\bar{z})|s,k\rangle = \frac{z^{\frac{\hat{s}-s}{k}}}{|z|\,|1-z|^2}\,\frac{1-|z|^{2(1-\frac{\hat{s}}{k})} + \bar{z}\,(|z|^{-2\frac{\hat{s}}{k}}-1)}{1-|z|^{\frac{2}{k}}}\,. \tag{5.55}$$

One can check that when $s = kp$ (and thus $\hat{s} = 0$) the previous result reduces to the CFT expression (5.17), and when $s = kp - 1$ (and thus $\hat{s} = k-1$) one recovers (5.20), up to overall numerical coefficients that have not been kept in the gravity derivation.

[38] This is different from what is done when the thermal results are extracted from the Euclidean correlators. Of course in the thermal case, one needs to perform the Wick rotation so as to identify the compact coordinate with time and, on the bulk side, the four point correlators are compared with the wave equation on a BTZ black hole.

[39] Correctly, the periodicity of the CFT σ coordinate, $\sigma \sim \sigma + 2\pi$, corresponds to the periodicity $y \sim y + 2\pi R_y$ of the spacetime coordinate y along the S^1.

5.4 HH-LL 4-point functions: discussion and outlook

It is well known that symmetric orbifolds provide a prototypical example of CFTs that have a sparse spectrum, which is a necessary condition to have a dual gravitational description in terms of a string or supergravity theory [123]. We focused on the best known example of such orbifold theories, the D1D5 CFT at the free point. In Section 5.1 we calculated on the CFT side a very special class of 4-point correlators among BPS operators, where two states are heavy (i.e. have conformal dimension of order c), while the other two are light (i.e. their conformal dimension is of order 1). These correlators are essentially combination of the free-fermion result and, in the $(O_H O_H)(O_L O_L)$ OPE, are completely saturated by the affine identity block of a $U(1)$ subgroup of the $SU(2)$ symmetry of the theory. This suggests that they are protected by supersymmetry and motivates the supergravity analysis of Section 5.3. Again thanks to the simplicity of our external states, also the gravity calculation is easy and, in this case, the basic ingredient is obtained by studying the scalar wave equation in $\mathrm{AdS}_3/\mathbb{Z}_k$. Then in order to obtain the full correlator it is important to know how the 3D result is uplifted to the full 10D geometry. In all examples under analysis, we find agreement with the free CFT result, even if this description is valid in a different point of the moduli space, thus confirming the expectations based on supersymmetry as mentioned above.

Of course, in the Euclidean case, the correlators we studied are singular only in the OPE limits. One of the main features of our result is that, for the whole correlator, this holds even at the leading order in the large c limit, while, in the same limit, the contribution of the Virasoro identity block in the $(O_H O_H)(O_L O_L)$ OPE develops spurious singularities [66, 68]. In other words, the $c \to \infty$ limit of the correlators studied here is not captured by the contribution of the identity Virasoro block in the heavy-light channel. This is reflected by the gravity calculations: the 2-point functions of the light operators in the near-horizon limit of the Strominger-Vafa black hole (which is the extremal BTZ) captures just the identity Virasoro block, while the same calculation in the microstate geometry dual to the heavy state reproduces the whole 4-point correlators, including the contributions of the higher order Virasoro primaries. This supports the intuition that the black hole geometry describes the correlators in a statistical ensemble, while each individual microstate yields correlators that deviate from the statistical answer before one reaches singularities that are usually related to the presence of a horizon.

In our case, due to the simple form of the heavy states, these deviations are present even at distances larger than the Schwarzschild radius, and in particular near the AdS boundary. On the CFT side, this means that, in the $(HH)(LL)$ OPE, there are contributions of non-trivial Virasoro primaries with small conformal dimension (order 1). The pattern discussed above is different from the one advocated in [68], where it is suggested that quantum (i.e. $1/c$ corrections) are needed to resolve the spurious singularities of the statistical/black hole result. Thus it is natural to ask whether the regularity of our Euclidean correlators in the large c regime is due to some peculiar feature of the D1D5 CFT under analysis and/or is a consequence of the very special operators considered. We believe that this is actually a general property as argued below.

The absence of spurious singularities at finite values of the central charge c is a direct consequence of the convergence of the OPE expansion in unitary CFT and of the basic properties of the Hilbert space structure of the spectrum [124]. In a nutshell, in the radial quantization, one can separate the four operators in the correlator by a sphere of radius r, with $|z_4| < |z_3| < r < |z_2| < |z_1|$. Then the convergence of the OPE ensures that the operators O_1 and O_2 in the external region produce a new state $|\phi_e\rangle$ on the sphere and the same happens, in the internal region, for the operators O_3 and O_4 that produce $|\phi_i\rangle$ (of course if $z_1 \to \infty$, $z_2 = 1 > z_3 > z_4 = 0$, $|\phi_i\rangle$ depends on $z = 1 - z_3$). So the 4-point correlator reduces to the scalar product $\langle\phi_e|\phi_i(z)\rangle$ which is finite for any value of z in the interval $0 < |z| < 1$. In [68] it was noted that it is not straightforward to take the $c \to \infty$ limit in this argument if one identifies O_1, O_2 with the heavy operators and O_3, O_4 with the light ones. We can see this directly in the simplest one of our examples, i.e. the correlator with the operators (5.4) and (5.6). The OPE between the light operators reads

$$O_L(w)\bar{O}_L(0) = \frac{1}{|w|^2} + \frac{1}{N}\sum_r \left(\frac{J^3_{(r)}}{\bar{w}} + \frac{\tilde{J}^3_{(r)}}{w}\right) + \frac{1}{N}\sum_{r\neq s} O^L_{(r)} O^L_{(s)} + \dots \tag{5.56}$$

In the large c limit, normally one would discard the contribution of the terms with the currents, as their norm is of order $1/N$. However the OPE between the heavy operators produces terms, again proportional to the currents, that are divergent in the $N \to \infty$ limit

$$O_H(w)\bar{O}_H(0) = \frac{1}{|w|^{2h_H}}\left(1 + w\sum_r J^3_{(r)} + \bar{w}\sum_r \tilde{J}^3_{(r)} + \dots\right). \tag{5.57}$$

Such non-normalizable terms can combine with the currents that appear in (5.56) to give non-negligible contributions to the block decomposition of the correlator; moreover their presence invalidates the regularity argument based on the existence of a well-defined scalar product, and is probably responsible for the singular behaviour of the heavy-light Virasoro blocks.

At the level of the correlators one can repeat the same derivation focusing on the OPE channel where the light operators are close to the heavy ones. In this case the intermediate states are normalizable even in the $c \to \infty$ limit and so the argument discussed above shows that the large c Euclidean correlators should not have spurious singularities. Of course this does not provide any information on the identity Virasoro block nor other $(HH)(LL)$ blocks because they do not appear in the $(HL)(HL)$ decomposition. However once the regularity of the large c limit of the correlators is established, we know that there is an infinite number of Virasoro primaries contributing to the $(HH)(LL)$ OPE. In the simple cases considered in this work, it turns out that these primaries are protected, as they are affine descendants of the identity operator. Thus the correlator we compute at the CFT orbifold point reproduces the one extracted from the dual geometry: in these instances then correlators are regular already at the level of supergravity. In general the OPE argument in the $(HL)(HL)$ channel predicts that correlators be regular in the large c limit *at a generic point* in the CFT moduli space. We do not expect, however, that all the operators ensuring the absence of spurious singularities at large c will be captured in the supergravity approximation. It would be an important progress to identify explicitly the CFT operators that are relevant to the $(HH)(LL)$ decomposition of a more general correlator. This could help to understand from a CFT perspective what contributions survive in the large c limit beside those that reproduce the thermal behaviour.

It is of course very interesting to elucidate the meaning of this pattern on the dual gravity side, where the main question is whether there are effects that modify the standard general relativity picture at the scales of Schwarzschild radius R_s in the limit where R_s is large in Planck units. Scenarios that fall in this class are the fuzzball [14,30] and the firewall [15] proposals. In situations that can be studied within the AdS/CFT duality, one could rephrase these ideas by saying that the heavy-light correlators, in a pure heavy state should differ from the ones calculated in a statistical ensemble even in the $c \to \infty$ limit. This is exactly the behaviour we observe in the simple correlators analyzed in this work. Of course, even if this is a general pattern as suggested above, there are several points that need to be understood in order to have a complete picture on the gravitational side. These include the following questions: what are the non-trivial operators that generically appear in the $(HH)(LL)$ decomposition of a *typical* heavy states? Is it possible to associate a scale in the radial direction to these contributions and show it is of the same order of R_s? For which correlators are the contributions from non-trivial conformal blocks negligible and in which cases is the result well approximated by the thermal correlator? Posing such questions in this framework might help to clarify some aspects of the "fuzzball complementarity" conjecture [12,125].

We conclude by discussing some less speculative and more concrete possible developments. Of course it would be interesting to consider 4-point correlators that are not related by a change of coordinates to 2−point functions in $\mathrm{AdS}_3/\mathbb{Z}_k$. In the same spirit, also changing the form of the light operators could provide new information on how different objects probe the heavy backgroud. Both these generalizations would allow to compare the bulk and the CFT results in examples with a richer structure. Finally it would be interesting to analyze heavy-light 4-point correlators in other CFTs that have a holographic interpretation at large c, so as to check or disprove the generality of the pattern suggested by the analysis for the D1D5 CFT.

Chapter 6

More correlators in the twisted sector and supersymmetric Ward identities

The aim of this chapter is that of providing a check of the CFT techniques developed up to this point giving alternative derivations of results known in the literature. In particular we want to compute 4-point functions involving bosonic or fermionic[40] light operators and in which the heavy states are given by twist fields. To this aim we will make use of both the bosonization technique presented in section (2.4) and the mode expansions and mode algebra in the twisted sector presented in section (2.3.1). The correlators we want to compute are the ones contained in [126], which are also used as building blocks for the analysis performed in [127].

In addition to the aforementioned points, we also write the supersymmetry transformations between the bosons and fermions of the D1D5 CFT at the free orbifold point, writing also the supersymmetric Ward identities that connect correlators with light operators given in terms of bosons to correlators with operators given in terms of fermions. Always working at the free orbifold point, we are able to explicitly compute the correlators connected by the Ward identities and to check that these are indeed verified.

6.1 Fermionic light operators

In the first correlator we consider the light operators are two of the CFT fermions, $\psi^{1\dot{1}}$ and $\psi^{2\dot{2}}$, and we compute their 4-point function with two heavy operators as their

[40]In a little abuse of language we call "bosonic" and "fermionic" the operators written in terms of free bosons or free fermions respectively, even though e.g. composite operators given by two fermions are indeed bosons.

2-point function between asymptotic states $|{++}\rangle_k$, generated starting from the NS vacuum $|0\rangle_{\rm NS}$ by acting with the operator in (2.116), which we will call

$$\mathcal{O}_k^{++}(z,\bar z) \equiv \prod_{\rho=0}^{k-1} e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)\left(H_\rho(z)+K_\rho(z)+\tilde H_\rho(\bar z)+\tilde K_\rho(\bar z)\right)}. \tag{6.1}$$

To be precise, the one above is the composition of the twist operator $\Sigma^{-\frac{k-1}{2},-\frac{k-1}{2}}$ with (tensor product of k times) the operator that implements the spectral flow $|0\rangle_{\rm NS} \to |{++}\rangle_{k=1}$. We consider the light operators with specific copy indices in the (r) basis so that we have light operators $\psi^{1\dot 1}_{(r)}$ and $\psi^{2\dot 2}_{(s)}$ acting nontrivially only on one of the CFT copies glued together into the length-k strand by the twist field[41]. The correlator explicitly reads

$$\begin{aligned}\mathcal{G}^F(z_1,z_2) &= {}_k\langle{+}{+}|\psi^{1\dot 1}_{(r)}(z_1)\,\psi^{2\dot 2}_{(s)}(z_2)|{++}\rangle_k\\ &= \lim_{\substack{u_1\to\infty\\ u_2\to 0}} u_1^{2h_H}\,\bar u_1^{2\bar h_H}\langle\left(\mathcal{O}_k^{++}\right)^\dagger(u_1,\bar u_1)\,\psi^{1\dot 1}_{(r)}(z_1)\,\psi^{2\dot 2}_{(s)}(z_2)\,\mathcal{O}_k^{++}(u_2,\bar u_2)\rangle,\end{aligned} \tag{6.2}$$

where in the last line the 4-point function is computed in the tensor product of k copies of the NS vacuum $|0\rangle_{\rm NS}$. h_H and $\bar h_H$ are the left and right conformal dimensions of $\mathcal{O}_k^{++}$, which can be computed by looking at (6.1) as

$$h_H = \bar h_H = 2\sum_{\rho=0}^{k-1}\frac{1}{2}\left(-\frac{\rho}{k}+\frac{1}{2}\right)^2 = \frac{k^2+2}{12k}\,. \tag{6.3}$$

The first thing we do is performing the $(r)\to\rho$ change of basis inverting the relations in (2.57), which gives

$$\mathcal{G}^F(z_1,z_2) = \frac{1}{k}\sum_{\rho_1,\rho_2=0}^{k-1} {}_k\langle{+}{+}|\psi^{+\dot 1}_{\rho_1}(z_1)\,e^{-2\pi\mathrm{i}\frac{r\rho_1}{k}}\,\psi^{-\dot 2}_{\rho_2}(z_2)\,e^{2\pi\mathrm{i}\frac{s\rho_2}{k}}|{+}{+}\rangle_k. \tag{6.4}$$

Looking at the correlator above, we see that (by angular momentum conservation) we have a nonzero result only if the fermions can have nontrivial contractions among themselves, i.e. if $\rho_1=\rho_2$, so

$$\mathcal{G}^F(z_1,z_2) = \frac{1}{k}\sum_{\rho=0}^{k-1} e^{2\pi\mathrm{i}(s-r)\frac{\rho}{k}}\,{}_k\langle{+}{+}|\psi^{+\dot 1}_{\rho}(z_1)\,\psi^{-\dot 2}_{\rho}(z_2)|{+}{+}\rangle_k. \tag{6.5}$$

[41]There is another abuse of language here: light operators were previously defined as *sums* over operators acting nontrivially on a different CFT copy in order to fulfil the orbifold requirement of symmetry under the exchange of any two copies. In this section instead we consider operators acting only on *one specific copy*: rigorously this would be forbidden by the orbifold CFT constraint, but we will proceed anyway, keeping in mind that at this stage this is only, possibly, just a building block for a more meaningful quantity we may want to compute. The operators may be called light anyway, as their conformal dimension is of order $\sim N^0$

We are ready to compute the 4-point function within the sum. The heavy operators have both a holomorphic and a antiholomorphic part: the correlator will be a product of a holomorphic and an antiholomorphic term, with the latter being given just by the contraction of the heavy operators between themselves (the dependence on u_1, u_2 and their complex conjugates must of course disappear in the $u_1 \to \infty$ and $u_2 \to 0$ limit). Using the bosonized form (2.110) of the fermions, the complete expression of the correlator in the sum is

$$
\begin{aligned}
&{}_k\langle + + | \psi_\rho^{+\dot{1}}(z_1)\, \psi_\rho^{-\dot{2}}(z_2) | + + \rangle_k = \\
&\quad = - \lim_{\substack{u_1 \to \infty \\ u_2 \to 0}} u_1^{2h_H}\, \bar{u}_1^{2\bar{h}_H} \langle e^{-\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)(H_\rho(u_1)+K_\rho(u_1))} e^{\mathrm{i}H_\rho(z_1)} e^{-\mathrm{i}H_\rho(z_2)} e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)(H_\rho(u_2)+K_\rho(u_2))} \rangle \times \\
&\qquad \times \langle e^{-\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)\left(\tilde{H}_\rho(\bar{u}_1)+\tilde{K}_\rho(\bar{u}_1)\right)} e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}\right)\left(\tilde{H}_\rho(\bar{u}_2)+\tilde{K}_\rho(\bar{u}_2)\right)} \rangle \times \\
&\qquad \times \prod_{\substack{\rho'=0 \\ \rho'\neq\rho}}^{k-1} \Big[\langle e^{-\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}\right)\left(H_{\rho'}(u_1)+K_{\rho'}(u_1)\right)} e^{\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}\right)\left(H_{\rho'}(u_2)+K_{\rho'}(u_2)\right)} \rangle \times \\
&\qquad \times \langle e^{-\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}\right)\left(\tilde{H}_{\rho'}(\bar{u}_1)+\tilde{K}_{\rho'}(\bar{u}_1)\right)} e^{\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}\right)\left(\tilde{H}_{\rho'}(\bar{u}_2)+\tilde{K}_{\rho'}(\bar{u}_2)\right)} \rangle \Big].
\end{aligned}
\tag{6.6}
$$

The above object may look complicated, but in fact is just the product of simple building blocks: due to the nature of the heavy operator, the full object is a product over $\rho' = 0, \dots, k-1$ and has been split into the $\rho' = \rho$ term times all the other ones (which are all the same, because the fermions appear only for $\rho' = \rho$). Everything was then also split between an holomorphic and an antiholomorphic part, with the fermions appearing only in the first, being holomorphic (the antiholomorphic part of the $\rho' = \rho$ term actually reconstructs a full product over all values of ρ'). Performing all the possible contractions using the rules (2.105) for the bosonized fields we get

$$
\begin{aligned}
&{}_k\langle + + | \psi_\rho^{+\dot{1}}(z_1)\, \psi_\rho^{-\dot{2}}(z_2) | + + \rangle_k = \\
&\quad = -z_{12}^{-1} \lim_{\substack{u_1 \to \infty \\ u_2 \to 0}} u_1^{2h_H}\, \bar{u}_1^{2\bar{h}_H} \left(\frac{z_1 - u_2}{z_2 - u_2}\right)^{\left(-\frac{\rho}{k}+\frac{1}{2}\right)} \left(\frac{u_1 - z_2}{u_1 - z_1}\right)^{\left(-\frac{\rho}{k}+\frac{1}{2}\right)} \times \\
&\qquad \times u_{12}^{-2\left(-\frac{\rho}{k}+\frac{1}{2}\right)^2} \bar{u}_{12}^{-2\left(-\frac{\rho}{k}+\frac{1}{2}\right)^2} \prod_{\substack{\rho'=0 \\ \rho'\neq\rho}}^{k-1} u_{12}^{-2\left(-\frac{\rho'}{k}+\frac{1}{2}\right)^2} \bar{u}_{12}^{-2\left(-\frac{\rho'}{k}+\frac{1}{2}\right)^2} \qquad (6.7) \\
&\quad = -z_{12}^{-1} \left(\frac{z_1}{z_2}\right)^{\left(-\frac{\rho}{k}+\frac{1}{2}\right)},
\end{aligned}
$$

where we defined $u_{12} \equiv u_1 - u_2$ and used the fact that

$$\begin{aligned} u_{12}^{-2\left(-\frac{\rho}{k}+\frac{1}{2}\right)^2} \bar{u}_{12}^{-2\left(-\frac{\rho}{k}+\frac{1}{2}\right)^2} \prod_{\substack{\rho'=0 \\ \rho'\neq\rho}}^{k-1} u_{12}^{-2\left(-\frac{\rho'}{k}+\frac{1}{2}\right)^2} \bar{u}_{12}^{-2\left(-\frac{\rho'}{k}+\frac{1}{2}\right)^2} &= \\ &= u_{12}^{-2\sum_{\rho'=0}^{k-1}\left(-\frac{\rho'}{k}+\frac{1}{2}\right)^2} \bar{u}_{12}^{-2\sum_{\rho'=0}^{k-1}\left(-\frac{\rho'}{k}+\frac{1}{2}\right)^2} \\ &= u_{12}^{-2h_H} \bar{u}_{12}^{-2\bar{h}_H}. \end{aligned} \tag{6.8}$$

The full correlator is then computed performing the sum over ρ,

$$\begin{aligned} \mathcal{G}^F(z_1, z_2) &= -\frac{1}{k}\frac{1}{z_{12}}\left(\frac{z_1}{z_2}\right)^{1/2} \sum_{\rho=0}^{k-1}\left[\left(\frac{z_2}{z_1}\right)^{1/k} e^{2\pi \mathrm{i}\frac{s-r}{k}}\right]^{\rho} \\ &= -\frac{1}{k}\frac{1}{z_{12}}\left(\frac{z_1}{z_2}\right)^{1/2} \frac{1-\left(\frac{z_2}{z_1}\right) e^{2\pi \mathrm{i}(s-r)}}{1-\left(\frac{z_2}{z_1}\right)^{1/k} e^{2\pi \mathrm{i}\,\frac{s-r}{k}}}. \end{aligned} \tag{6.9}$$

Let's now change coordinates as

$$z_i = e^{\mathrm{i} w_i}. \tag{6.10}$$

We have

$$\begin{aligned} \mathcal{G}^F(w_1, w_2) &= \left(\frac{dz_1}{dw_1}\right)^{1/2}\left(\frac{dz_2}{dw_2}\right)^{1/2} \mathcal{G}(z_1(w_1), z_2(w_2)) \\ &= -\frac{e^{\mathrm{i}\left(\frac{w_1-w_2}{2k}+\frac{\pi}{k}(r-s)\right)}}{2k\,\sin\left(\frac{w_1-w_2}{2k}+\frac{\pi}{k}(r-s)\right)}, \end{aligned} \tag{6.11}$$

which agrees with equation (A.40) of [126], upon defining

$$w \equiv w_1 - w_2 + 2\pi(r-s), \qquad \bar{w} \equiv \bar{w}_1 - \bar{w}_2 + 2\pi(r-s). \tag{6.12}$$

The extra sign we get is accounted for by the fact that in (A.40) of [126] the correlator considered is defined with the light operators in the opposite order.

6.2 Bosonic light operators

In the previous chapters we studied correlators (3- and 4-point functions) involving light operators written in terms of the free fermions of the theory at the orbifold point, but nothing prevents us from considering light operators composed of the bosons. Although this may be less relevant for the direct study of microstate geometries (in which the deviations from $AdS_3 \times S^3$ are captured by the VEVs of operators written in terms of the fermions), the 4-point functions have their own relevance. This also allows us to check the technology developed in Chapter 2 for the bosons in the twisted sector. We consider again a strand of length k with the k length-1 bosonic vacua glued together by the twist operators $\sigma_k^X(z)$ and $\tilde{\sigma}_k^X(\bar{z})$ (these are heavy operators because they are written as products over copies, as in (2.82)). The light bosonic operators are

$$\left(\partial X^{1\dot{1}}_{(r)}\bar{\partial}X^{1\dot{1}}_{(r)}\right)(z,\bar{z}), \qquad \left(\partial X^{2\dot{2}}_{(r)}\bar{\partial}X^{2\dot{2}}_{(r)}\right)(z,\bar{z}), \tag{6.13}$$

where as a first step in analogy with the previous section we consider operators acting on one single CFT copy (in the (r) basis, with $r = 1,\dots,k$). The correlator is

$$\mathcal{G}^B_{r,s}(z_i,\bar{z}_i) = {}_k\langle 0|\left(\partial X^{1\dot{1}}_{(r)}\bar{\partial}X^{1\dot{1}}_{(r)}\right)(z_1,\bar{z}_1)\left(\partial X^{2\dot{2}}_{(s)}\bar{\partial}X^{2\dot{2}}_{(s)}\right)(z_2,\bar{z}_2)|0\rangle_k. \tag{6.14}$$

In order to compute $\mathcal{G}^B$ we change from the (r) to the ρ basis inverting (2.52), we use the the mode expansions (2.54) and then we perform the sums over ρ. The full correlator is the product of a holomorphic and a antiholomorphic part, and we will focus only on the holomorphic one, the other being obtained from it just by sending $z_i \to \bar{z}_i$,

$$\mathcal{G}^B_{r,s}(z_i,\bar{z}_i) = \mathcal{G}^B_{r,s}(z_i)\,\bar{\mathcal{G}}^B_{r,s}(\bar{z}_i). \tag{6.15}$$

The key point to compute the correlator is the action of the modes of the operators on the twisted vacuum states: as we see from (2.67) apart from the shift in the modes, this is analogous to what happens in the untwisted sector, with positive modes annihilating the vacuum states. Changing from the (r) to the ρ basis the left part of the correlator reads

$$\begin{aligned}\mathcal{G}^B_{r,s}(z_i) &= {}_k\langle 0|\partial X^{1\dot{1}}_{(r)}(z_1)X^{2\dot{2}}_{(s)}(z_2)|0\rangle_k \\ &= \frac{1}{k}\sum_{\rho_1,\rho_2=0}^{k-1} e^{2\pi\mathrm{i}\,\frac{r\rho_1}{k}}e^{-2\pi\mathrm{i}\,\frac{s\rho_2}{k}}\,{}_k\langle 0|\partial X^{1\dot{1}}_{\rho_1}(z_1)\partial X^{2\dot{2}}_{\rho_2}(z_2)|0\rangle_k \\ &= \frac{1}{k}\sum_{\rho=0}^{k-1} e^{2\pi\mathrm{i}\,\frac{r-s}{k}\,\rho}\,{}_k\langle 0|\partial X^{1\dot{1}}_{\rho}(z_1)\partial X^{2\dot{2}}_{\rho}(z_2)|0\rangle_k,\end{aligned} \tag{6.16}$$

where in the last step we used the fact that if the bosons act on different CFT copies (either in the (r) or in the ρ basis) the correlator is zero. Expanding the bosons in modes we get that the correlator inside the sum reads

$$\begin{aligned}
&{}_k\langle 0|\partial X^{1\dot{1}}_\rho(z_1)\partial X^{2\dot{2}}_\rho(z_2)|0\rangle_k = \\
&\quad = \sum_{\substack{n_1\in\mathbf{Z}\\ n_1-\frac{\rho}{k}>0}} \sum_{\substack{n_2\in\mathbf{Z}\\ n_2+\frac{\rho}{k}<0}} z_1^{-n_1-1+\frac{\rho}{k}} z_2^{-n_2-1-\frac{\rho}{k}}\, {}_k\langle 0|\alpha^{1\dot{1}}_{\rho,n_1-\frac{\rho}{k}}, \alpha^{2\dot{2}}_{\rho,n_2+\frac{\rho}{k}}|0\rangle_k \\
&\quad = \sum_{\substack{n_1\in\mathbf{Z}\\ n_1-\frac{\rho}{k}>0}} \sum_{\substack{n_2\in\mathbf{Z}\\ n_2+\frac{\rho}{k}<0}} z_1^{-n_1-1+\frac{\rho}{k}} z_2^{-n_2-1-\frac{\rho}{k}}\, {}_k\langle 0|\left[\alpha^{1\dot{1}}_{\rho,n_1-\frac{\rho}{k}}, \alpha^{2\dot{2}}_{\rho,n_2+\frac{\rho}{k}}\right]|0\rangle_k \\
&\quad = (z_1z_2)^{-1}\left(\frac{z_1}{z_2}\right)^{\rho/k}\sum_{n=1}^{+\infty}\left(\frac{z_2}{z_1}\right)^n\left(n-\frac{\rho}{k}\right) \\
&\quad = (z_1z_2)^{-1}\left(\frac{z_1}{z_2}\right)^{\rho/k}\left\{\frac{\left(\frac{z_2}{z_1}\right)}{\left[1-\left(\frac{z_2}{z_1}\right)\right]^2} - \frac{\rho}{k}\frac{\left(\frac{z_2}{z_1}\right)}{1-\left(\frac{z_2}{z_1}\right)}\right\},
\end{aligned} \tag{6.17}$$

where we used that

$$\sum_{n=1}^{+\infty} A^n = \frac{A}{1-A}, \tag{6.18a}$$

$$\sum_{n=1}^{+\infty} n\, A^n = A\,\partial_A\left[\sum_{n=1}^{+\infty} A^n\right] = \frac{A}{(1-A)^2}. \tag{6.18b}$$

The total correlator reads

$$\begin{aligned}
\mathcal{G}^B_{r,s}(z_i) = \frac{1}{k}\,(z_1z_2)^{-1}&\left\{\frac{\left(\frac{z_2}{z_1}\right)}{\left[1-\left(\frac{z_2}{z_1}\right)\right]^2}\sum_{\rho=0}^{k-1}\left[e^{2\pi\mathrm{i}\,\frac{r-s}{k}}\left(\frac{z_1}{z_2}\right)^{1/k}\right]^\rho + \right. \\
&\left. -\frac{1}{k}\frac{\left(\frac{z_2}{z_1}\right)}{1-\left(\frac{z_2}{z_1}\right)}\sum_{\rho=0}^{k-1}\rho\left[e^{2\pi\mathrm{i}\,\frac{r-s}{k}}\left(\frac{z_1}{z_2}\right)^{1/k}\right]^\rho\right\}.
\end{aligned} \tag{6.19}$$

We can split $\mathcal{G}^B_{r,s}(z_i)$ into the sum of two terms corresponding to the terms in the curly brackets,

$$\mathcal{G}^B_{r,s}(z_i) = \mathcal{G}^B_{r,s,1}(z_i) + \mathcal{G}^B_{r,s,2}(z_i), \tag{6.20}$$

and using that

$$\sum_{\rho=0}^{k-1} A^\rho = \frac{1-A^k}{1-A}, \tag{6.21a}$$

$$\sum_{\rho=0}^{k-1}\rho\, A^\rho = A\,\partial_A\left[\sum_{\rho=0}^{k-1} A^\rho\right] = \frac{1}{(1-A)^2}\Big(A - kB^k + (k-1)B^{k+1}\Big), \tag{6.21b}$$

we get that

$$\begin{aligned}
\mathcal{G}^B_{r,s,1}(z_i) &= \frac{1}{k}\,(z_1z_2)^{-1}\frac{\left(\frac{z_2}{z_1}\right)}{\left[1-\left(\frac{z_2}{z_1}\right)\right]^2}\,\frac{1-e^{2\pi\mathrm{i}(r-s)}\left(\frac{z_1}{z_2}\right)}{1-e^{2\pi\mathrm{i}\,\frac{r-s}{k}}\left(\frac{z_1}{z_2}\right)^{1/k}}\\
&= \frac{1}{k}\,(z_1z_2)^{-1}\frac{x^{-1}}{(1-x^{-1})^2}\,\frac{1-x}{1-x^{1/k}}, \qquad (6.22a)\\
\mathcal{G}^B_{r,s,2}(z_i) &= -\frac{1}{k^2}\,(z_1z_2)^{-1}\frac{\left(\frac{z_2}{z_1}\right)}{1-\left(\frac{z_2}{z_1}\right)}\,\frac{1}{\left(1-e^{2\pi\mathrm{i}\,\frac{r-s}{k}}\left(\frac{z_1}{z_2}\right)^{1/k}\right)^2}\times\\
&\quad\times\left\{e^{2\pi\mathrm{i}\,\frac{r-s}{k}}\left(\frac{z_1}{z_2}\right)^{1/k}-k\,e^{2\pi\mathrm{i}(r-s)}\frac{z_1}{z_2}+(k-1)\,e^{2\pi\mathrm{i}(r-s)\frac{k+1}{k}}\left(\frac{z_1}{z_2}\right)^{\frac{k+1}{k}}\right\}\\
&= -\frac{1}{k^2}\,(z_1z_2)^{-1}\frac{x^{-1}}{1-x^{-1}}\,\frac{1}{\left(1-x^{1/k}\right)^2}\left\{x^{1/k}-kx+(k-1)x^{1+\frac{1}{k}}\right\} \qquad (6.22b)
\end{aligned}$$

where $x \equiv e^{2\pi\mathrm{i}(r-s)}z_1/z_2$. The total result simplifies into

$$\begin{aligned}
\mathcal{G}^B_{r,s}(z_i) &= \frac{1}{k^2}\,(z_1z_2)^{-1}\frac{1}{\left(x^{1/2k}-x^{-1/2k}\right)^2}\\
&= \frac{1}{k^2}\,(z_1z_2)^{-1}\frac{1}{\left(e^{2\pi\mathrm{i}\,\frac{r-s}{2k}}e^{\mathrm{i}\,\frac{w_1-w_2}{2k}}-e^{-2\pi\mathrm{i}\,\frac{r-s}{2k}}e^{-\mathrm{i}\,\frac{w_1-w_2}{2k}}\right)^2},
\end{aligned} \qquad (6.23)$$

and changing coordinates as $z_i \to w_i$ using (6.10) we get

$$\begin{aligned}
\mathcal{G}^B_{r,s}(w_i) &= \left(\frac{dz_1}{dw_1}\right)\left(\frac{dz_2}{dw_2}\right)\mathcal{G}^B_{r,s}(z_i(w_i)) =\\
&= \frac{1}{4k^2}\frac{1}{\sin^2\left(\frac{w_1-w_2}{2k}+\frac{2\pi}{2k}(r-s)\right)}.
\end{aligned} \qquad (6.24)$$

The antiholomorphic part is just the complex conjugate of this, so the complete correlator in w_i coordinates is

$$\mathcal{G}^B_{r,s}(w_i,\bar{w}_i) = \frac{1}{4k^2\sin^2\left(\frac{w_1-w_2}{2k}+\frac{2\pi}{2k}(r-s)\right)}\,\frac{1}{4k^2\sin^2\left(\frac{\bar{w}_1-\bar{w}_2}{2k}+\frac{2\pi}{2k}(r-s)\right)}, \qquad (6.25)$$

which agrees[42] with (A.33) of [126].

In order to obtain something that doesn't contain free copy indices, we have to sum over all the copies, which boils down to a sum over the difference $r-s=0,\dots,k-1$: this operation is performed between (4.9) and (4.11) of [126] and it's rather difficult. In the following, we will take another path, passing to the ρ basis and doing the sum over

[42]Notation: we label the length of a strand as k while in [126] it's labelled as n. What is called k in [126] is the difference $r-s$.

r as our first step. Summing over copies also satisfies the orbifold CFT requirement of symmetry among all the copies, so the object we compute is actually a sensible quantity to consider. Let's consider the complete correlator

$$\mathcal{G}^B(z_i,\bar{z}_i) = {}_k\langle 0| \left(\partial X^{1\dot{1}}\bar{\partial}X^{1\dot{1}}\right)(z_1,\bar{z}_1)\left(\partial X^{2\dot{2}}\bar{\partial}X^{2\dot{2}}\right)(z_2,\bar{z}_2)|0\rangle_k, \tag{6.26}$$

with implicit sums over the k copies,

$$\left(\partial X^{A\dot{A}}\bar{\partial}X^{A\dot{A}}\right)(z,\bar{z}) = \sum_{r=1}^{k} \partial X^{A\dot{A}}_{(r)}(z)\bar{\partial}X^{A\dot{A}}_{(r)}(\bar{z}). \tag{6.27}$$

Again we can perform the $(r) \to \rho$ change of basis, but this time the sum over (r) makes the phases disappear,

$$\begin{aligned}
\left(\partial X^{1\dot{1}}\bar{\partial}X^{1\dot{1}}\right)(z,\bar{z}) &= \frac{1}{k}\sum_{r=1}^{k}\sum_{\rho_1,\rho_2=0}^{k-1} e^{2\pi \mathrm{i}\,\frac{\rho_1-\rho_2}{k}r}\partial X^{1\dot{1}}_{\rho_1}(z)\bar{\partial}X^{1\dot{1}}_{\rho_2}(\bar{z})\\
&= \frac{1}{k}\sum_{\rho_1,\rho_2=0}^{k-1}\partial X^{A\dot{A}}_{\rho_1}(z)\bar{\partial}X^{1\dot{1}}_{\rho_2}(\bar{z})\, k\,\delta_{\rho_1,\rho_2}\\
&= \sum_{\rho=0}^{k-1}\partial X^{1\dot{1}}_{\rho}(z)\bar{\partial}X^{1\dot{1}}_{\rho}(\bar{z}),
\end{aligned} \tag{6.28}$$

$$\left(\partial X^{2\dot{2}}\bar{\partial}X^{2\dot{2}}\right)(z,\bar{z}) = \sum_{\rho=0}^{k-1}\partial X^{2\dot{2}}_{\rho}(z)\bar{\partial}X^{2\dot{2}}_{\rho}(\bar{z}). \tag{6.29}$$

The correlator becomes

$$\begin{aligned}
\mathcal{G}^B(z_i,\bar{z}_i) &= \sum_{\rho_1,\rho_2=0}^{k-1} {}_k\langle 0|\partial X^{1\dot{1}}_{\rho_1}(z_1)\partial X^{2\dot{2}}_{\rho_2}(z_2)|0\rangle_k\, {}_k\langle 0|\bar{\partial}X^{1\dot{1}}_{\rho_1}(\bar{z}_1)\bar{\partial}X^{2\dot{2}}_{\rho_2}(\bar{z}_2)|0\rangle_k\\
&= \sum_{\rho=0}^{k-1} {}_k\langle 0|\partial X^{1\dot{1}}_{\rho}(z_1)\partial X^{2\dot{2}}_{\rho}(z_2)|0\rangle_k\, {}_k\langle 0|\bar{\partial}X^{1\dot{1}}_{\rho}(\bar{z}_1)\bar{\partial}X^{2\dot{2}}_{\rho}(\bar{z}_2)|0\rangle_k\\
&\equiv \sum_{\rho=0}^{k-1}\mathcal{G}^B_\rho(z_i)\,\mathcal{G}^B_\rho(\bar{z}_i),
\end{aligned} \tag{6.30}$$

where we called $\mathcal{G}^B_\rho(z_i)$ and $\mathcal{G}^B_\rho(\bar{z}_i)$ respectively the holomorphic and the antiholomorphic parts of the object within the sum. Following the same steps used to compute the first bosonic correlator we get

$$\begin{aligned}\mathcal{G}^B(z_i) &\equiv {}_k\langle 0|\partial X^{1\dot{1}}_\rho(z_1)\partial X^{2\dot{2}}_\rho(z_2)|0\rangle_k \\ &= (z_1 z_2)^{-1}\left(\frac{z_1}{z_2}\right)^{\rho/k}\left(-\frac{\rho}{k}\frac{\left(\frac{z_2}{z_1}\right)}{1-\left(\frac{z_2}{z_1}\right)}+\frac{\left(\frac{z_2}{z_1}\right)}{\left[1\left(\frac{z_2}{z_1}\right)\right]^2}\right),\end{aligned} \tag{6.31}$$

while the antiholomorphic part is obtained taking the above and replacing $z_i \to \bar{z}_i$. In total we get

$$\mathcal{G}^B(z_i,\bar{z}_i) = \frac{1}{|z_1 z_2|^2}\sum_{\rho=0}^{k-1}\left|\frac{z_1}{z_2}\right|^{\frac{2\rho}{k}}\left(-\frac{\rho}{k}\frac{\left(\frac{z_2}{z_1}\right)}{1-\left(\frac{z_2}{z_1}\right)}+\frac{\left(\frac{z_2}{z_1}\right)}{\left[1-\left(\frac{z_2}{z_1}\right)\right]^2}\right)\left(-\frac{\rho}{k}\frac{\left(\frac{\bar{z}_2}{\bar{z}_1}\right)}{1-\left(\frac{\bar{z}_2}{\bar{z}_1}\right)}+\frac{\left(\frac{\bar{z}_2}{\bar{z}_1}\right)}{\left[1-\left(\frac{\bar{z}_2}{\bar{z}_1}\right)\right]^2}\right). \tag{6.32}$$

If we define

$$z \equiv \frac{z_2}{z_1}, \qquad \bar{z} \equiv \frac{\bar{z}_2}{\bar{z}_1}, \tag{6.33}$$

the result above is more usefully rewritten as

$$\begin{aligned}\mathcal{G}^B(z,\bar{z}) = \frac{1}{|z_1 z_2|^2}\frac{|z|^2}{|1-z|^2}\Bigg\{&\frac{1}{|1-z|^2}\sum_{\rho=0}^{k-1}|z|^{-\frac{2\rho}{k}} - \frac{1}{k}\left(\frac{1}{1-z}+\frac{1}{1-\bar{z}}\right)\sum_{\rho=0}^{k-1}\rho|z|^{-\frac{2\rho}{k}} + \\ &+\frac{1}{k^2}\sum_{\rho=0}^{k-1}\rho^2|z|^{-\frac{2\rho}{k}}\Bigg\}.\end{aligned} \tag{6.34}$$

The prefactor $1/|z_1 z_2|^2$ is compensated by the Jacobians when passing to the w_i coordinates. Looking at (4.9) and (4.11) of [126] we see that the heavy state there is more complicated: what we are computing corresponds to the generic term of their sum, and to our aims it's just sufficient to ignore the sum over n (more precisely, we will ignore what in their notation is $1/N\sum_{n=1}^{+\infty} nN_n \cdots$), so we get that the analogous of $\mathcal{G}^B(w_i)$ is

$$\mathcal{G}^B_{\rm BKS}(w,\bar{w}) = -\frac{C}{16k^2\sin^2\left(\frac{w-\bar{w}}{2}\right)}\left[\frac{1}{\sin^2\left(\frac{w}{2}\right)}+\frac{1}{\sin^2\left(\frac{\bar{w}}{2}\right)}-\frac{2\sin\left(\frac{w-\bar{w}}{2}\right)}{k\tan\left(\frac{w-\bar{w}}{2k}\right)\sin\left(\frac{w}{2}\right)\sin\left(\frac{\bar{w}}{2}\right)}\right]. \tag{6.35}$$

The original definition of w and $\bar{w}$ is $w \equiv w_1 - w_2 + 2\pi(r-s)$ and $\bar{w} \equiv \bar{w}_1 - \bar{w}_2 + 2\pi(r-s)$ but since the sums over the copies have already been done nothing in the above expression should depend on r and s. This is indeed the case: the dependence disappears from the combination $w-\bar{w}$ as we have that

$$\begin{aligned}\sin^2\left(\frac{w}{2}\right) &= \sin^2\left(\frac{w_1-w_2}{2}+\pi(r-s)\right)=\sin^2\left(\frac{w_1-w_2}{2}\right),\\ \sin\left(\frac{w}{2}\right)\sin\left(\frac{\bar{w}}{2}\right) &= \sin\left(\frac{w_1-w_2}{2}\right)\sin\left(\frac{\bar{w}_1-\bar{w}_2}{2}\right).\end{aligned} \tag{6.36}$$

Using (6.10) and (6.33) we have

$$z=e^{-\mathrm{i}(w_1-w_2)}=e^{-\mathrm{i}w}, \qquad \bar{z}=e^{\mathrm{i}(\bar{w}_1-\bar{w}_2)}=e^{\mathrm{i}\bar{w}}, \tag{6.37}$$

and the result is rewritten as

$$\begin{aligned}\mathcal{G}^B(w,\bar{w}) = \frac{|z|^2}{|1-z|^2}\Bigg\{&\frac{1}{|1-z|^2}\sum_{\rho=0}^{k-1}|z|^{-\frac{2\rho}{k}}-\frac{1}{k}\left(\frac{1}{1-z}+\frac{1}{1-\bar{z}}\right)\sum_{\rho=0}^{k-1}\rho|z|^{-\frac{2\rho}{k}}+\\ &+\frac{1}{k^2}\sum_{\rho=0}^{k-1}\rho^2|z|^{-\frac{2\rho}{k}}\Bigg\}_{\substack{z=e^{-\mathrm{i}w}\\ \bar{z}=e^{\mathrm{i}\bar{w}}}}\end{aligned} \tag{6.38}$$

Performing the sums is not difficult, but the result is not illuminating: we can make *Mathematica* do it, write the results in the w coordinates and compare the expressions, the final answer being that $\mathcal{G}^B(w,\bar{w})$ and $\mathcal{G}^B_{\mathrm{BKS}}(w,\bar{w})$ are in agreement.

6.3 Connecting fermionic and bosonic operators: Ward identities

In addition to conformal symmetry, the D1D5 CFT also enjoys supersymmetry, with transformations connecting the bosons and the fermions of the theory. This fact can be used to derive Ward identities relating 4-point functions with fermionic light operators to 4-point functions with bosonic light operators. To start, let's consider a single strand of length 1 (we will omit the copy label in the following) and let's define the left and right supercurrents,

$$G^{\alpha}_{A}(z)\equiv\left(\partial X_{A\dot{A}}\psi^{\alpha\dot{A}}\right)(z), \qquad \tilde{G}^{\dot{\alpha}}_{A}(z)\equiv\left(\bar{\partial}X_{A\dot{A}}\tilde{\psi}^{\dot{\alpha}\dot{A}}\right)(\bar{z}), \tag{6.39}$$

where the indices of the bosons have been lowered[43] using ϵ_{AB} and $\epsilon_{\dot{A}\dot{B}}$,

$$\partial X_{A\dot{A}}=\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\,\partial X^{B\dot{B}}, \qquad \bar{\partial}X_{A\dot{A}}=\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\,\bar{\partial}X^{B\dot{B}}. \tag{6.40}$$

[43]For convenience, notice that the operation of raising the indices works the same way,

$$\partial X^{A\dot{A}}=\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\,\partial X_{B\dot{B}}, \qquad \bar{\partial}X^{A\dot{A}}=\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\,\bar{\partial}X_{B\dot{B}}.$$

The action of the supercurrent $G^2_A(w)$ on a fermion $\psi^{1\dot{A}}(z)$ is a contour integral in which w is taken on a counter-clockwise path around z,

$$\begin{aligned}\oint_{w\sim z}\frac{dw}{2\pi\mathrm{i}}\,G^2_A(w)\psi^{1\dot{A}}(z) &= \oint_{w\sim z}\frac{dw}{2\pi\mathrm{i}}\,\partial X_{A\dot{B}}(w)\psi^{2\dot{B}}(w)\psi^{1\dot{A}}(z)\\ &= \oint_{w\sim z}\frac{dw}{2\pi\mathrm{i}}\,\partial X_{A\dot{B}}(w)\left(-\frac{\epsilon^{\dot{B}\dot{A}}}{w-z}+[\text{reg.}]\right)\\ &= -\partial X_{A\dot{B}}(z)\,\epsilon^{\dot{B}\dot{A}},\end{aligned}\tag{6.41}$$

where the $SU(2)_L$ indices for the supercurrent and for the fermion were chosen so that they have a nontrivial OPE (we used (2.5a)) and the result follows from the fact that $\partial X_{A\dot{A}}(w)$ has no singularities at $w=z_1$, so the only singular term is the one brought by the OPE of the fermions. An analogous fact holds for the antiholomorphic supercurrent and fermions,

$$\oint_{\bar{w}\sim\bar{z}}\frac{d\bar{w}}{2\pi\mathrm{i}}\,\tilde{G}^{\dot{2}}_A(\bar{w})\tilde{\psi}^{\dot{1}\dot{A}}(\bar{z}) = -\bar{\partial}X_{A\dot{B}}(\bar{z})\,\epsilon^{\dot{B}\dot{A}}.\tag{6.42}$$

These are the supersymmetry transformations of the fields. Looking at the previous sections, we see that we can apply this to transform the light bosonic operators into the light fermionic ones, as

$$\begin{aligned}&\oint_{w\sim z}\frac{dw}{2\pi\mathrm{i}}\oint_{\bar{w}\sim\bar{z}}\frac{d\bar{w}}{2\pi\mathrm{i}}\,G^2_A(w)\tilde{G}^{\dot{2}}_B(\bar{w})\psi^{1\dot{C}}(z)\tilde{\psi}^{\dot{1}\dot{D}}(\bar{z}) =\\ &\qquad = -\oint_{w\sim z}\frac{dw}{2\pi\mathrm{i}}\,G^2_A(w)\psi^{1\dot{C}}(z)\oint_{\bar{w}\sim\bar{z}}\frac{d\bar{w}}{2\pi\mathrm{i}}\,\tilde{G}^{\dot{2}}_B(\bar{w})\tilde{\psi}^{\dot{1}\dot{D}}(\bar{z})\\ &\qquad = -\partial X_{A\dot{A}}(z)\,\epsilon^{\dot{A}\dot{C}}\,\bar{\partial}X_{B\dot{B}}(\bar{z})\,\epsilon^{\dot{B}\dot{D}},\end{aligned}\tag{6.43}$$

which can be recast into

$$\partial X_{A\dot{A}}(z)\,\bar{\partial}X_{B\dot{B}}(\bar{z}) = -\epsilon_{\dot{A}\dot{C}}\epsilon_{\dot{B}\dot{D}}\oint_{w\sim z}\frac{dw}{2\pi\mathrm{i}}\oint_{\bar{w}\sim\bar{z}}\frac{d\bar{w}}{2\pi\mathrm{i}}\,G^2_A(w)\tilde{G}^{\dot{2}}_B(\bar{w})\psi^{1\dot{C}}(z)\tilde{\psi}^{\dot{1}\dot{D}}(\bar{z}).\tag{6.44}$$

The light operators considered in the previous sections can readily be obtained by choosing the appropriate values for the indices of the bosons,

$$\begin{aligned}\partial X^{1\dot{1}}(z)\bar{\partial}X^{1\dot{1}}(\bar{z}) &= \partial X_{2\dot{2}}(z)\bar{\partial}X_{2\dot{2}}(\bar{z})\\ &= -\oint_{w\sim z}\frac{dw}{2\pi\mathrm{i}}\oint_{\bar{w}\sim\bar{z}}\frac{d\bar{w}}{2\pi\mathrm{i}}\,G^2_2(w)\tilde{G}^{\dot{2}}_2(\bar{w})\psi^{1\dot{1}}(z)\tilde{\psi}^{\dot{1}\dot{1}}(\bar{z}).\end{aligned}\tag{6.45}$$

Always working on a strand of length 1 and suppressing the copy indices for simplicity, we have that the analogous to the correlator with bosonic light operators (6.26) is

$$
\begin{aligned}
&\langle H| \left(\partial X^{1\dot{1}}\bar{\partial} X^{1\dot{1}}\right)(z_1,\bar{z}_1)\left(\partial X^{2\dot{2}}\bar{\partial} X^{2\dot{2}}\right)(z_2,\bar{z}_2)|H\rangle = \\
&\quad = \langle H| \left(\partial X_{2\dot{2}}\bar{\partial} X_{2\dot{2}}\right)(z_1,\bar{z}_1)\left(\partial X_{1\dot{1}}\bar{\partial} X_{1\dot{1}}\right)(z_2,\bar{z}_2)|H\rangle = \\
&\quad = -\oint_{w\sim z_1}\frac{dw}{2\pi \mathrm{i}}\oint_{\bar{w}\sim\bar{z}_1}\frac{d\bar{w}}{2\pi\mathrm{i}}\,\langle H|G^2_2(w)\tilde{G}^{\dot{2}}_2(\bar{w})\psi^{1\dot{1}}(z_1)\tilde{\psi}^{\dot{1}1}(\bar{z}_1)\left(\partial X_{1\dot{1}}\bar{\partial} X_{1\dot{1}}\right)(z_2,\bar{z}_2)|H\rangle,
\end{aligned}
\tag{6.46}
$$

where we considered generic asymptotic heavy states $|H\rangle$. In order to compute the integrals above we could deform the contour of the dw integral, picking contributions for all the points where we can have poles of order 1, except for z_1 (we also do the same for the $d\bar{w}$ integral). Naively, these points will be where the supercurrent $G^2_2(w)$ can have nontrivial contractions with other operators, i.e. at $w = 0, \infty, z_2$, while for the antiholomorphic integral we have to look at $\tilde{G}^{\dot{2}}_2(\bar{w})$ and the points will be $\bar{w} = 0, \infty, \bar{z}_2$. If the heavy states in the fermionic sector are R vacua, though, we cannot push the contours as easily, as they introduce a branch cut corresponding to the antiperiodic boundary conditions of the fermions. The branch cut has the nature of a square root, as going around the origin once the fermions in the R sector get a minus sign (this can be seen from the mode expansions (2.10)). We can nevertheless introduce a factor in the integrand that cancels the branch cut without altering the value of the integral itself, this factor being $\sqrt{w}$ (and $\sqrt{\bar{w}}$ for the antiholomorphic integral). Indeed we have

$$
\begin{aligned}
\oint_{w\sim z_1}\frac{dw}{2\pi\mathrm{i}}\,\sqrt{w}\,G^2_2(w)\psi^{1\dot{A}}(z_1) &= \oint_{w\sim z_1}\frac{dw}{2\pi\mathrm{i}}\,\sqrt{w}\,\partial X_{2\dot{B}}(w)\psi^{2\dot{B}}(w)\psi^{1\dot{A}}(z_1) \\
&= -\sqrt{z_1}\,\partial X_{2\dot{B}}(z_1)\,\epsilon^{\dot{B}\dot{A}},
\end{aligned}
\tag{6.47}
$$

where we just had to evaluate $\sqrt{w}\,\partial X_{2\dot{B}}(w)$ at $w = z_1$ since it contains no singularities at that point. For the antiholomorphic integral we have the analogous relation

$$
\oint_{\bar{w}\sim\bar{z}_1}\frac{d\bar{w}}{2\pi\mathrm{i}}\,\sqrt{\bar{w}}\,\tilde{G}^{\dot{2}}_2(\bar{w})\tilde{\psi}^{\dot{1}\dot{A}}(\bar{z}_1) = -\sqrt{\bar{z}_1}\,\bar{\partial} X_{2\dot{B}}(\bar{z}_1)\,\epsilon^{\dot{B}\dot{A}}. \tag{6.48}
$$

Noticing that introducing a factor $\sqrt{w\,\bar{w}}$ brings an extra factor $\sqrt{z_1\bar{z}_1} = |z_1|$ we have to divide by, we then have

$$
\begin{aligned}
&\langle H| \left(\partial X^{1\dot{1}}\bar{\partial} X^{1\dot{1}}\right)(z_1,\bar{z}_1)\left(\partial X^{2\dot{2}}\bar{\partial} X^{2\dot{2}}\right)(z_2,\bar{z}_2)|H\rangle = \\
&\quad = -\frac{1}{|z_1|}\oint_{w\sim z_1}\frac{dw}{2\pi\mathrm{i}}\oint_{\bar{w}\sim\bar{z}_1}\frac{d\bar{w}}{2\pi\mathrm{i}}\,\sqrt{w\,\bar{w}}\,\langle H|G^2_2(w)\tilde{G}^{\dot{2}}_2(\bar{w})\psi^{1\dot{1}}(z_1)\tilde{\psi}^{\dot{1}1}(\bar{z}_1)\left(\partial X_{1\dot{1}}\bar{\partial} X_{1\dot{1}}\right)(z_2,\bar{z}_2)|H\rangle.
\end{aligned}
\tag{6.49}
$$

We can obtain a Ward identity relating the correlators with bosonic and fermionic light operators by computing the integrals, which can now be done pushing the contour. The full correlator is again the product of an holomorphic and and antiholomorphic 4-point function, so we can write the last line of (6.49) as

$$\frac{1}{|z_1|}\oint_{w\sim z_1}\frac{dw}{2\pi\mathrm{i}}\sqrt{w}\,\langle H|G^2_2(w)\psi^{1\dot{1}}(z_1)\partial X_{1\dot{1}}(z_2)|H\rangle\oint_{\bar{w}\sim\bar{z}_1}\frac{d\bar{w}}{2\pi\mathrm{i}}\sqrt{\bar{w}}\,\langle\tilde{H}|\tilde{G}^{\dot{2}}_2(\bar{w})\tilde{\psi}^{1\dot{1}}(\bar{z}_1)\bar{\partial}X_{1\dot{1}}(\bar{z}_1)|\tilde{H}\rangle, \tag{6.50}$$

where again with an abuse of notation we denoted $|H\rangle$ as the holomorphic part of the asymptotic state and $|\tilde{H}\rangle$ as its antiholomorphic part. Let's consider the holomorphic term. As we said before, pushing the contour we get a contour integral with a path going around the only possible points where we could have singularities coming from the contraction of $G^2_2(w)$ with other operators: in total the integral becomes a sum of three integrals, with w going around $0,\infty$ and z_2. The paths go around these points clockwise, so we get an extra minus sign to bring them back into counter-clockwise orientation. If w goes around 0, singularities can arise from the contraction of G^2_2 with $|H\rangle$ and we have

$$\begin{aligned}-\oint_{w\sim 0}\frac{dw}{2\pi\mathrm{i}}\sqrt{w}\,G^2_2(w)|H\rangle &= -\sum_{n\in\mathbb{Z}}\oint_{w\sim 0}\frac{dw}{2\pi\mathrm{i}}\sqrt{w}\,G^2_{2,n}\,z^{-n-3/2}|H\rangle\\ &= -G^2_{2,0}|H\rangle\\ &= 0,\end{aligned} \tag{6.51}$$

where we expanded the supercurrent in modes $G^2_{2,n}$ and assumed the fact that the heavy state is invariant under supersymmetry (which is true if we choose it to be e.g. a Ramond vacuum)[44]. Analogously, from the term in which w goes around ∞ we get

$$\begin{aligned}-\oint_{w\sim\infty}\frac{dw}{2\pi\mathrm{i}}\sqrt{w}\,\langle H|G^2_2(w) &= -\sum_{n\in\mathbb{Z}}\oint_{w\sim\infty}\frac{dw}{2\pi\mathrm{i}}\sqrt{w}\,\langle H|G^2_{2,n}\,z^{-n-3/2}\\ &= \oint_{u\sim 0}\frac{du}{2\pi\mathrm{i}}\,u^{-2}\langle H|G^2_{2,n}\,u^{n+1}\\ &= \langle H|G^2_{2,0}\\ &= 0,\end{aligned} \tag{6.52}$$

where we changed variables in the integral as $w\to u=1/2$ and again we assumed the heavy state is invariant under supersymmetry. The only point that can give contribu-

[44]The modes $G^\alpha_{A,n}$ of the supercurrents on a strand of length 1 are written in terms of the modes of the bosons and fermions as

$$G^\alpha_{A,n}=\sum_{m\in\mathbb{Z}}\alpha_{A\dot{A},m}\,\psi^{\alpha\dot{A}}_{n-m}\,,$$

so we immediately see that if $|H\rangle$ is a Ramond vacuum all the nonzero modes of G^2_2 certainly annihilate it.

tions is then $w = z_2$. In this case we have

$$\begin{aligned}
-\oint_{w\sim z_2}\frac{dw}{2\pi\mathrm{i}}\,\sqrt{w}\,\langle H|G^2_2(w)\psi^{1\dot{1}}(z_1)\partial X_{1\dot{1}}(z_2)|H\rangle = \\
&= \oint_{w\sim z_2}\frac{dw}{2\pi\mathrm{i}}\,\sqrt{w}\,\langle H|\psi^{1\dot{1}}(z_1)G^2_2(w)\partial X_{1\dot{1}}(z_2)|H\rangle \\
&= \oint_{w\sim z_2}\frac{dw}{2\pi\mathrm{i}}\,\sqrt{w}\,\langle H|\psi^{1\dot{1}}(z_1)\psi^{2\dot{A}}(w)\partial_{2\dot{A}}(w)\partial X_{1\dot{1}}(z_2)|H\rangle \\
&= \oint_{w\sim z_2}\frac{dw}{2\pi\mathrm{i}}\,\langle H|\left(\sqrt{z_2}+\frac{1}{2\sqrt{z_2}}(w-z_2)+O((w-z_2)^2)\right)\times \\
&\times\left(\psi^{1\dot{1}}(z_2)+(w-z_2)\partial\psi^{1\dot{1}}(z_2)+O((w-z_2)^2)\right)\times \\
&\times\left(\frac{\epsilon_{21}\epsilon_{\dot{A}2}}{(w-z_2)^2}+[\mathrm{reg.}]\right)|H\rangle \\
&= \langle H|\psi^{1\dot{1}}(z_1)\left(\frac{1}{2\sqrt{z_2}}\,\psi^{2\dot{2}}+\sqrt{z_2}\,\partial\psi^{2\dot{2}}(z_2)\right)|H\rangle \\
&= \partial_{z_2}\left\{\sqrt{z_2}\langle H|\psi^{1\dot{1}}(z_1)\psi^{2\dot{2}}(z_2)|H\rangle\right\}.
\end{aligned} \tag{6.53}$$

With an analogous computations we get that the antiholomorphic part of the correlator is

$$-\oint_{\bar{w}\sim\bar{z}_2}\frac{dw}{2\pi\mathrm{i}}\,\sqrt{\bar{w}}\,\langle\tilde{H}|\tilde{G}^{\dot{2}}_2(\bar{w})\tilde{\psi}^{1\dot{1}}(\bar{z}_1)\bar{\partial}X_{1\dot{1}}(\bar{z}_2)|\tilde{H}\rangle = \partial_{\bar{z}_2}\left\{\sqrt{\bar{z}_2}\langle\tilde{H}|\tilde{\psi}^{1\dot{1}}(\bar{z}_1)\tilde{\psi}^{\dot{2}\dot{2}}(\bar{z}_2)|\tilde{H}\rangle\right\}. \tag{6.54}$$

Substituting this into (6.49) we get the full Ward identity. We will write it reinstating the copy indices for the fields: working on N strands of length 1, we get the Ward identity

$$\begin{aligned}
&\langle H|\left(\partial X^{1\dot{1}}\bar{\partial}X^{1\dot{1}}\right)_{(r)}(z_1,\bar{z}_1)\left(\partial X^{2\dot{2}}\bar{\partial}X^{2\dot{2}}\right)_{(s)}(z_2,\bar{z}_2)|H\rangle = \\
&= \frac{\delta_{r,s}}{|z_1|}\,\partial_{z_2}\partial_{\bar{z}_2}\left\{|z_2|\langle H|\psi^{1\dot{1}}_{(r)}(z_1)\psi^{2\dot{2}}_{(s)}(z_2)|H\rangle\langle\tilde{H}|\tilde{\psi}^{1\dot{1}}_{(r)}(\bar{z}_1)\tilde{\psi}^{\dot{2}\dot{2}}_{(s)}(\bar{z}_2)|\tilde{H}\rangle\right\},
\end{aligned} \tag{6.55}$$

where we assumed the fact that if the two operators act on different CFT copies we get zero (this happens e.g. if the heavy states are R vacuum states, as in Chapter 5, as opposed to superpositions, as in Chapter 4).

Let's consider the heavy states $|H\rangle = |{+}{+}\rangle_{k=1}$ and see if the Ward identity is satisfied. The state $|{+}{+}\rangle_{k=1}$ only refers to the fermionic sector: when computing the bosonic correlator it has to be read as

$$|{+}{+}\rangle_{k=1}\otimes|0\rangle_{k=1}, \tag{6.56}$$

so the 4-point function reads

$$\begin{aligned} &{}_{k=1}\langle 0| \left(\partial X^{1\dot{1}}_{(r)}\bar{\partial} X^{1\dot{1}}_{(r)}\right)(z_1,\bar{z}_1)\left(\partial X^{2\dot{2}}_{(s)}\bar{\partial} X^{2\dot{2}}_{(s)}\right)(z_2,\bar{z}_2)|0\rangle_{k=1} \times {}_{k=1}\langle ++|++\rangle_{k=1} = \\ &\quad = \frac{\delta_{r,s}}{|z_1-z_2|^4}, \end{aligned} \tag{6.57}$$

where the result follows from (6.19) setting $k=1$ (or easily by expanding the operators in modes and using the mode algebra) and we used the fact that the fermionic ground states are normalized,

$$ {}_{k=1}\langle ++|++\rangle_{k=1} = 1. \tag{6.58}$$

The result for the fermionic correlator can be read from (6.9), again setting $k=1$,

$$ {}_{k=1}\langle ++|\psi^{1\dot{1}}_{(r)}(z_1)\psi^{2\dot{2}}_{(s)}(z_2)|++\rangle_{k=1} = -\frac{\delta_{r,s}}{z_1-z_2}\left(\frac{z_1}{z_2}\right)^{1/2}, \tag{6.59}$$

$$ {}_{k=1}\langle ++|\tilde{\psi}^{\dot{1}\dot{1}}_{(r)}(\bar{z}_1)\tilde{\psi}^{\dot{2}\dot{2}}_{(s)}(\bar{z}_2)|++\rangle_{k=1} = -\frac{\delta_{r,s}}{\bar{z}_1-\bar{z}_2}\left(\frac{\bar{z}_1}{\bar{z}_2}\right)^{1/2}, \tag{6.60}$$

where the correlator for antiholomorphic fields is obtained from the first one by replacing $z_i \to \bar{z}_i$. Let's evaluate the RHS of the Ward identity (6.55),

$$\begin{aligned} &\frac{\delta_{r,s}}{|z_1|}\,\partial_{z_2}\partial_{\bar{z}_2}\left\{|z_2|\,{}_{k=1}\langle ++|\psi^{1\dot{1}}_{(r)}(z_1)\psi^{2\dot{2}}_{(s)}(z_2)|++\rangle_{k=1}\,{}_{k=1}\langle ++|\tilde{\psi}^{\dot{1}\dot{1}}_{(r)}(\bar{z}_1)\tilde{\psi}^{\dot{2}\dot{2}}_{(s)}(\bar{z}_2)|++\rangle_{k=1}\right\} = \\ &\quad = \delta_{r,s}\,\partial_{z_2}\left(\frac{1}{z_1-z_2}\right)\partial_{\bar{z}_2}\left(\frac{1}{\bar{z}_1-\bar{z}_2}\right) \\ &\quad = \frac{\delta_{r,s}}{|z_1-z_2|^4}, \end{aligned} \tag{6.61}$$

which matches the bosonic correlator (6.57), satisfying (6.55) and giving a specific example of how the Ward identity is satisfied.

Let's now consider the twisted sector. The definition of the supercurrents on a strand of length k is

$$ G^{\alpha}_{A}(z) \equiv \sum_{r=1}^{k}\psi^{\alpha\dot{A}}_{(r)}(z)\,\partial X^{(r)}_{A\dot{A}}(z), \qquad \tilde{G}^{\dot{\alpha}}_{A}(\bar{z}) \equiv \sum_{r=1}^{k}\tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z})\,\bar{\partial} X^{(r)}_{A\dot{A}}(\bar{z}). \tag{6.62}$$

Following the same steps as in the untwisted sector we get that the Ward identity is

$$\begin{aligned} &\langle H|\left(\partial X^{1\dot{1}}\bar{\partial}X^{1\dot{1}}\right)(z_1,\bar{z}_1)\left(\partial X^{2\dot{2}}\bar{\partial}X^{2\dot{2}}\right)(z_2,\bar{z}_2)|H\rangle = \\ &\quad = \frac{1}{|z_1|}\,\partial_{z_2}\partial_{\bar{z}_2}\left\{|z_2|\langle H|\psi^{1\dot{1}}(z_1)\psi^{2\dot{2}}(z_2)|H\rangle\langle\tilde{H}|\tilde{\psi}^{\dot{1}\dot{1}}(\bar{z}_1)\tilde{\psi}^{\dot{2}\dot{2}}(\bar{z}_2)|\tilde{H}\rangle\right\}, \end{aligned} \tag{6.63}$$

where now the operators are sums over the k CFT copies, as in (6.27) for the bosons, while for the fermions this reads

$$\left(\psi^{\alpha\dot{A}}\tilde{\psi}^{\dot{\alpha}\dot{B}}\right)(z,\bar{z}) = \sum_{r=1}^{k} \psi^{\alpha\dot{A}}_{(r)}(z)\,\tilde{\psi}^{\dot{\alpha}\dot{B}}_{(r)}(\bar{z}). \tag{6.64}$$

Let's put $|H\rangle = |{+}{+}\rangle_k$ for generic k and check that the Ward identity is satisfied. For the bosonic correlator on the LHS the result is $\mathcal{G}^B(z_i,\bar{z}_i)$ as in (6.34), with z and $\bar{z}$ written in terms of $z_1, z_2, \bar{z}_1$ and $\bar{z}_2$ using the definitions (6.33). For the RHS of (6.63) we have that the fermionic correlators read

$$\begin{aligned}
&{}_k\langle + + |\psi^{1\dot{1}}(z_1)\psi^{2\dot{2}}(z_2)|{+}{+}\rangle_k\,{}_k\langle + + |\tilde{\psi}^{\dot{1}\dot{1}}(\bar{z}_1)\tilde{\psi}^{\dot{2}\dot{2}}(\bar{z}_2)|{+}{+}\rangle_k = \\
&\quad = \sum_{\rho_1,\rho_2=0}^{k-1} {}_k\langle + + |\psi^{1\dot{1}}_{\rho_1}(z_1)\psi^{2\dot{2}}_{\rho_2}(z_2)|{+}{+}\rangle_k\,{}_k\langle + + |\tilde{\psi}^{\dot{1}\dot{1}}_{\rho_1}(\bar{z}_1)\tilde{\psi}^{\dot{2}\dot{2}}_{\rho_2}(\bar{z}_2)|{+}{+}\rangle_k \\
&\quad = \sum_{\rho=0}^{k-1} {}_k\langle + + |\psi^{1\dot{1}}_{\rho}(z_1)\psi^{2\dot{2}}_{\rho}(z_2)|{+}{+}\rangle_k\,{}_k\langle + + |\tilde{\psi}^{\dot{1}\dot{1}}_{\rho}(\bar{z}_1)\tilde{\psi}^{\dot{2}\dot{2}}_{\rho}(\bar{z}_2)|{+}{+}\rangle_k,
\end{aligned} \tag{6.65}$$

where we used the fact that nonzero contributions only come from the cases in which the fermions can have nontrivial contractions among themselves, which happens only if $\rho_1 = \rho_2$. The holomorphic factor inside the sum can be read off from (6.7) while the antiholomoprhic one is obtained by sending $z_i \to \bar{z}_i$. In total we have

$$\begin{aligned}
&{}_k\langle + + |\psi^{1\dot{1}}(z_1)\psi^{2\dot{2}}(z_2)|{+}{+}\rangle_k\,{}_k\langle + + |\tilde{\psi}^{\dot{1}\dot{1}}(\bar{z}_1)\tilde{\psi}^{\dot{2}\dot{2}}(\bar{z}_2)|{+}{+}\rangle_k = \\
&\qquad = \frac{1}{|z_1-z_2|^2}\left|\frac{z_1}{z_2}\right|\sum_{\rho=0}^{k-1}\left|\frac{z_2}{z_1}\right|^{2\rho/k} \\
&\qquad = \frac{\left|\frac{z_1}{z_2}\right|}{|z_1-z_2|^2}\,\frac{1-\left|\frac{z_2}{z_1}\right|^2}{1-\left|\frac{z_2}{z_1}\right|^{2/k}}.
\end{aligned} \tag{6.66}$$

In order to check the Ward identity, the result above must be inserted into (6.63). As for the computation of the bosonic correlator, the result is not illuminating by itself. Nevertheless it's easy to check with *Mathematica* that (6.63) is indeed satisfied.

6.4 Correlators involving more complicated operators

In this section we consider a 4-point function on one strand of length 1 (untwisted sector), but with more complicated operators. Inspired by the analysis carried out in Chapter 5, we cosider $O^{\alpha\dot{\alpha}}$ as the light operators, while the role of the heavy states is now played by $|00\rangle_{k=1}$ (which for simplicity we will just denote as $|00\rangle$ here). A we are working on a single strand in the untwisted sector, the heavy operators are not "really heavy” in the sense that we would need to actually consider $N = c/6$ singly-wound strands to have $h \sim c$, but we keep the name by analogy nevertheless. Also, the indices over the CFT copies won't be needed in this section. The correlator is

$$\mathcal{G}_{00}(z_i, \bar{z}_i) \equiv \langle 00|O^{++}(z_1, \bar{z}_1)\, O^{--}(z_2, \bar{z}_2)|00\rangle. \tag{6.67}$$

In order to use the bosonization technique, we first need to write the heavy state in the bosonized language, which we haven't done yet. By the definition (2.37) we have

$$\begin{aligned}
|00\rangle &= \lim_{u_1\to 0} u_1^h \bar{u}_1^{\bar{h}}\, O^{--}(u_1, \bar{u}_1)\, |{+}{+}\rangle_{k=1} \\
&= -\frac{\mathrm{i}}{\sqrt{2}} \lim_{\substack{u_1\to u_2\\ u_2\to 0}} (u_1 - u_2)^{1/2} (\bar{u}_1 - \bar{u}_2)^{1/2}\, \epsilon_{\dot{A}\dot{B}}\, \psi^{2\dot{A}}(u_1) \tilde{\psi}^{\dot{2}\dot{B}}(\bar{u}_1) \times \\
&\quad \times e^{\frac{\mathrm{i}}{2}\left(H(u_2)+K(u_2)+\tilde{H}(\bar{u}_2)+\tilde{K}(\bar{u}_2)\right)} |0\rangle_{\mathrm{NS}} \\
&= \frac{1}{\sqrt{2}} \lim_{\substack{u_1\to u_2\\ u_2\to 0}} |u_1| \left[e^{-\mathrm{i}\left(K(u_1)+\tilde{H}(\bar{u}_1)\right)} - e^{-\mathrm{i}\left(H(u_1)+\tilde{K}(\bar{u}_1)\right)} \right] e^{\frac{\mathrm{i}}{2}\left(H(u_2)+K(u_2)+\tilde{H}(\bar{u}_2)+\tilde{K}(\bar{u}_2)\right)} |0\rangle_{\mathrm{NS}} \\
&= \frac{1}{\sqrt{2}} \lim_{u\to 0} \left[e^{\frac{\mathrm{i}}{2}\left(H(u)-K(u)-\tilde{H}(\bar{u})+\tilde{K}(\bar{u})\right)} - e^{\frac{\mathrm{i}}{2}\left(-H(u)+K(u)+\tilde{H}(\bar{u})-\tilde{K}(\bar{u})\right)} \right] |0\rangle_{\mathrm{NS}},
\end{aligned} \tag{6.68}$$

where $h = \bar{h} = 1/2$ are the left and right conformal dimensions of $O^{\alpha\dot{\alpha}}$ and we used the rules in (2.105). $\mathcal{G}_{00}$ splits into the sum of four terms,

$$\mathcal{G}_{00}(z_i, \bar{z}_i) = \sum_{n=1}^{4} \mathcal{G}_{00,n}(z_i, \bar{z}_i), \tag{6.69}$$

where

$$\mathcal{G}_{00,1}(z_i, \bar{z}_i) \equiv \langle 00|\psi^{1\dot{1}}(z_1) \tilde{\psi}^{\dot{1}\dot{2}}(\bar{z}_1) \psi^{2\dot{1}}(z_2) \tilde{\psi}^{\dot{2}\dot{2}}(\bar{z}_2)|00\rangle, \tag{6.70a}$$
$$\mathcal{G}_{00,2}(z_i, \bar{z}_i) \equiv \langle 00|\psi^{1\dot{2}}(z_1) \tilde{\psi}^{\dot{1}\dot{1}}(\bar{z}_1) \psi^{2\dot{2}}(z_2) \tilde{\psi}^{\dot{2}\dot{1}}(\bar{z}_2)|00\rangle, \tag{6.70b}$$
$$\mathcal{G}_{00,3}(z_i, \bar{z}_i) \equiv -\langle 00|\psi^{1\dot{2}}(z_1) \tilde{\psi}^{\dot{1}\dot{1}}(\bar{z}_1) \psi^{2\dot{1}}(z_2) \tilde{\psi}^{\dot{2}\dot{2}}(\bar{z}_2)|00\rangle, \tag{6.70c}$$
$$\mathcal{G}_{00,4}(z_i, \bar{z}_i) \equiv -\langle 00|\psi^{1\dot{1}}(z_1) \tilde{\psi}^{\dot{1}\dot{2}}(\bar{z}_1) \psi^{2\dot{2}}(z_2) \tilde{\psi}^{\dot{2}\dot{1}}(\bar{z}_2)|00\rangle. \tag{6.70d}$$

Because $|00\rangle$ is the sum of two terms, each of the $\mathcal{G}_{00,n}$ is in turn the sum of four terms. Not all of them contribute, though: in the bosonized language, in order to have a nonzero correlator the coefficient of the bosons in the exponential must add up to zero. For $\mathcal{G}_{00,1}$ and $\mathcal{G}_{00,2}$ we have

$$\begin{aligned}
\mathcal{G}_{00,1}(z_i,\bar{z}_i) &= -\frac{1}{4}\lim_{\substack{u_1\to\infty\\u_2\to 0}}|u_1|\,{}_{\mathrm{NS}}\langle 0|\left[e^{-\frac{\mathrm{i}}{2}\left(H-K-\tilde{H}+\tilde{K}\right)}-e^{-\frac{\mathrm{i}}{2}\left(-H+K+\tilde{H}-\tilde{K}\right)}\right](u_1,\bar{u}_1)\times\\
&\times e^{\mathrm{i}H(z_1)}e^{-\mathrm{i}K(z_2)}e^{\mathrm{i}\tilde{K}(\bar{z}_1)}e^{-\mathrm{i}\tilde{H}(\bar{z}_2)}\left[e^{\frac{\mathrm{i}}{2}\left(H-K-\tilde{H}+\tilde{K}\right)}-e^{\frac{\mathrm{i}}{2}\left(-H+K+\tilde{H}-\tilde{K}\right)}\right](u_2,\bar{u}_2)|00\rangle_{\mathrm{NS}}\\
&= \frac{1}{4}\lim_{\substack{u_1\to\infty\\u_2\to 0}}|u_1|\,{}_{\mathrm{NS}}\langle 0|e^{-\frac{\mathrm{i}}{2}\left(H(u_1)-K(u_1)-\tilde{H}(\bar{u}_1)+\tilde{K}(\bar{u}_1)\right)}e^{\mathrm{i}H(z_1)}e^{-\mathrm{i}K(z_2)}\times\\
&\times e^{\mathrm{i}\tilde{K}(\bar{z}_1)}e^{-\mathrm{i}\tilde{H}(\bar{z}_2)}e^{\frac{\mathrm{i}}{2}\left(H(u_2)-K(u_2)-\tilde{H}(\bar{u}_2)+\tilde{K}(\bar{u}_2)\right)}|0\rangle_{\mathrm{NS}}\\
&= \frac{1}{4}\frac{1}{|z_1||z_2|}=\mathcal{G}_{00,2}(z_i,\bar{z}_i),
\end{aligned}\tag{6.71}$$

while for $\mathcal{G}_{00,3}$ and $\mathcal{G}_{00,4}$ we have

$$\begin{aligned}
\mathcal{G}_{00,3}(z_i,\bar{z}_i) &= \frac{1}{4}\lim_{\substack{u_1\to\infty\\u_2\to 0}}|u_1|\,{}_{\mathrm{NS}}\langle 0|\left[e^{-\frac{\mathrm{i}}{2}\left(H-K-\tilde{H}+\tilde{K}\right)}-e^{-\frac{\mathrm{i}}{2}\left(-H+K+\tilde{H}-\tilde{K}\right)}\right](u_1,\bar{u}_1)\times\\
&\times e^{\mathrm{i}K(z_1)}e^{-\mathrm{i}K(z_2)}e^{\mathrm{i}\tilde{H}(\bar{z}_1)}e^{-\mathrm{i}\tilde{H}(\bar{z}_2)}\left[e^{\frac{\mathrm{i}}{2}\left(H-K-\tilde{H}+\tilde{K}\right)}-e^{\frac{\mathrm{i}}{2}\left(-H+K+\tilde{H}-\tilde{K}\right)}\right](u_2,\bar{u}_2)|00\rangle_{\mathrm{NS}}\\
&= \frac{1}{4}\lim_{\substack{u_1\to\infty\\u_2\to 0}}|u_1|\Big\{{}_{\mathrm{NS}}\langle 0|e^{-\frac{\mathrm{i}}{2}\left(H(u_1)-K(u_1)-\tilde{H}(\bar{u}_1)+\tilde{K}(\bar{u}_1)\right)}e^{\mathrm{i}K(z_1)}e^{-\mathrm{i}K(z_2)}e^{\mathrm{i}\tilde{H}(\bar{z}_1)}e^{-\mathrm{i}\tilde{H}(\bar{z}_2)}\times\\
&\times e^{\frac{\mathrm{i}}{2}\left(H(u_2)-K(u_2)-\tilde{H}(\bar{u}_2)+\tilde{K}(\bar{u}_2)\right)}|0\rangle_{\mathrm{NS}}+\\
&+{}_{\mathrm{NS}}\langle 0|e^{-\frac{\mathrm{i}}{2}\left(-H(u_1)+K(u_1)+\tilde{H}(\bar{u}_1)-\tilde{K}(\bar{u}_1)\right)}e^{\mathrm{i}K(z_1)}e^{-\mathrm{i}K(z_2)}e^{\mathrm{i}\tilde{H}(\bar{z}_1)}e^{-\mathrm{i}\tilde{H}(\bar{z}_2)}\times\\
&\times e^{\frac{\mathrm{i}}{2}\left(-H(u_2)+K(u_2)+\tilde{H}(\bar{u}_2)-\tilde{K}(\bar{u}_2)\right)}|0\rangle_{\mathrm{NS}}\Big\}\\
&= \frac{1}{4}\frac{1}{|z_1-z_2|^2}\left\{\left|\frac{z_2}{z_1}\right|+\left|\frac{z_1}{z_2}\right|\right\}=\mathcal{G}_{00,4}(z_i,\bar{z}_i).
\end{aligned}\tag{6.72}$$

The total result is

$$\mathcal{G}_{00}(z_i,\bar{z}_i)=\frac{1}{2}\frac{1}{|z_1z_2|}+\frac{1}{2}\frac{1}{|z_1-z_2|^2}\left\{\left|\frac{z_2}{z_1}\right|+\left|\frac{z_1}{z_2}\right|\right\}.\tag{6.73}$$

The next step in our analysis will be that of writing a Ward identity between $\mathcal{G}_{00}$, which is a correlator containing light operators written in terms of the free fermions, and another correlator with light operators written in terms of the free bosons, much in the same spirit as in the above sections. With passages analogous to the ones followed previously it's easy to get

$$\oint_{w\sim z}\frac{dw}{2\pi\mathrm{i}}\sqrt{w}\,G^{\alpha}_{A}(w)\partial X^{B\dot{B}}(z)=\delta^B_A\partial_z\left(\sqrt{z}\,\psi^{\alpha\dot{B}}(z)\right),\tag{6.74a}$$

$$\oint_{\bar{w}\sim\bar{z}}\frac{d\bar{w}}{2\pi\mathrm{i}}\sqrt{\bar{w}}\,\tilde{G}^{\dot{\alpha}}_{A}(\bar{w})\bar{\partial}X^{B\dot{B}}(\bar{z})=\delta^B_A\partial_{\bar{z}}\left(\sqrt{\bar{z}}\,\tilde{\psi}^{\dot{\alpha}\dot{B}}(\bar{z})\right),\tag{6.74b}$$

where as usual the strategy is to take the OPE between the supercurrent and the bosonic field and use the residue theorem. Knowing this we can write

$$
\begin{aligned}
&\delta^B_A\delta^D_C\,\partial_z\partial_{\bar z}\left\{\left(-\frac{\mathrm{i}}{\sqrt{2}}\right)|z|\,\psi^{1\dot A}(z)\tilde\psi^{\dot 1\dot B}(\bar z)\epsilon_{\dot A\dot B}\right\}=\\
&\qquad=\delta^B_A\delta^D_C\,\partial_z\partial_{\bar z}\Big(|z|\,O^{++}(z,\bar z)\Big)\\
&\qquad=\left(-\frac{\mathrm{i}}{\sqrt{2}}\right)\oint_{w\sim z}\frac{dw}{2\pi\mathrm{i}}\sqrt{w}\,G^1_A(w)\partial X^{B\dot A}(z)\oint_{\bar w\sim\bar z}\frac{d\bar w}{2\pi\mathrm{i}}\sqrt{\bar w}\,\tilde G^{\dot 1}_C(\bar w)\bar\partial X^{D\dot B}(\bar z)\epsilon_{\dot A\dot B}.
\end{aligned}
\tag{6.75}
$$

We then choose $A=B=C=D=1$, multiply by O^{--} and consider the 4-point function with heavy states $|H\rangle$, getting

$$
\begin{aligned}
&\partial_{z_1}\partial_{\bar z_1}\Big\{|z_1|\langle H|O^{++}(z_1,\bar z_1)O^{--}(z_2,\bar z_2)|H\rangle\Big\}=\\
&\quad=\left(-\frac{\mathrm{i}}{\sqrt{2}}\right)^2\oint_{w\sim z_1}\frac{dw}{2\pi\mathrm{i}}\oint_{\bar w\sim\bar z_1}\frac{d\bar w}{2\pi\mathrm{i}}\,|w|\,G^1_1(w)\tilde G^{\dot 1}_1(\bar w)\langle H|\partial X^{1\dot A}(z_1)\bar\partial X^{1\dot B}(\bar z_1)\epsilon_{\dot A\dot B}\times\\
&\quad\times\psi^{2\dot C}(z_2)\tilde\psi^{\dot 2\dot D}(\bar z_2)\epsilon_{\dot C\dot D}|H\rangle.
\end{aligned}
\tag{6.76}
$$

The $\sqrt{w}$ and $\sqrt{\bar w}$ factors were considered from the beginning so that now we can freely deform the contour as in (6.53) (again assuming the heavy state is such that no contributions arise from $w=0$ and $w=\infty$, as in (6.51) and (6.52)). The result is the Ward identity for correlators involving O^{++} and O^{--},

$$
\begin{aligned}
&\partial_{z_1}\partial_{\bar z_1}\Big\{|z_1|\langle H|O^{++}(z_1,\bar z_1)O^{--}(z_2,\bar z_2)|H\rangle\Big\}=\\
&\quad=\frac{|z_2|}{2}\langle H|\left(\partial X^{1\dot A}\bar\partial X^{1\dot B}\epsilon_{\dot A\dot B}\right)(z_1,\bar z_1)\left(\partial X^{2\dot C}\bar\partial X^{2\dot D}\epsilon_{\dot C\dot D}\right)(z_2,\bar z_2)|H\rangle.
\end{aligned}
\tag{6.77}
$$

We can verify the above identity is satisfied for the choice $|H\rangle=|00\rangle$. Denoting

$$
\mathcal{G}^F_{00}(z_i,\bar z_i)\equiv\mathcal{G}_{00}(z_i,\bar z_i),\tag{6.78}
$$

$$
\mathcal{G}^B_{00}(z_i,\bar z_i)\equiv\langle 00|\partial\left(X^{1\dot A}\bar\partial X^{1\dot B}\epsilon_{\dot A\dot B}\right)(z_1,\bar z_1)\left(\partial X^{2\dot C}\bar\partial X^{2\dot D}\epsilon_{\dot C\dot D}\right)(z_2,\bar z_2)|00\rangle,\tag{6.79}
$$

(6.77) reads

$$
\partial_{z_1}\partial_{\bar z_1}\Big(|z_1|\,\mathcal{G}^F_{00}(z_i,\bar z_i)\Big)=\frac{|z_2|}{2}\,\mathcal{G}^B_{00}(z_i,\bar z_i).\tag{6.80}
$$

We already have the result for $\mathcal{G}^F_{00}(z_i,\bar z_i)$, while for $\mathcal{G}^B_{00}(z_i,\bar z_i)$ we observe as before that the different choices of $|H\rangle$ only involve the fermionic sector, while the asymptotic states in the bosonic one are just the vacuum, giving

$$
\mathcal{G}^B_{00}(z_i,\bar z_i)=\frac{2}{|z_1-z_2|^4},\tag{6.81}
$$

the factor 2 coming from the fact that $\mathcal{G}^B_{00}(z_i,\bar{z}_i)$ is the sum of four terms, two of which are zero (indices must be such that all the bosons can have nontrivial contractions). Inserting (6.73) and (6.81) it's easy to verify that (6.80) is indeed satisfied.

Chapter 7

Conclusion

First developed as a tentative description of hadronic resonances, string theory became one of the most promising candidates as a theory of quantum gravity upon the realization, in the mid '70s, that the spectrum of the closed bosonic string contains an excitation that has all the characteristics required to be the graviton, the quantum of the gravitational interaction. Since then, much progress has been made, from superstrings to M-theory, D-branes and the AdS/CFT conjecture.

It is both natural and tempting to try to put together all the pieces of the puzzle and try to solve the long-standing questions that the problem of quantum gravity brings up. In particular, the description of black holes, the understanding of the origin of their thermodynamic properties and the resolution of the information paradox are perhaps the most important and ambitious topics on which a theory of quantum gravity must be tested. This becomes even more pressing if we think that these problems lead to theoretical inconsistencies between two theories, GR and quantum field theory, that are well established and incredibly successful experimentally, in their domains of validity.

Mathur's fuzzball proposal [14] is a genuine description of black holes, and it's well motivated from fundamental principles in string theory. It incorporates successfully the most important tools at our disposal: the supergravity black hole solutions are understood in terms of D-brane configurations and AdS/CFT is naturally implemented in the near-horizon limit, with a CFT description given by the D1D5 CFT. This last point allows to shed a light on the thermodynamics of black holes, as statistical field theory is rigorously established, as opposed to the thermodynamic description emerging on the gravity side. We finally seem to have an explicit form for classes of black hole microstates, both as CFT states and as bulk geometries: questions like distinguishability of microstates and typicality can be assessed more precisely.

One of the topics of this thesis is that of establishing precisely the holographic duality between CFT states and their dual bulk geometries. In doing so, it's important to choose relevant observables that are able to capture the differences between microstates. In particular, we focused on the VEVs of (light) CFT operators of small conformal dimensions taken on the (heavy) states dual to the bulk geometries: these correspond holographically [32, 62, 63] to deviations from $AdS_3 \times S^3$ geometry already near the AdS_3 boundary. The strategy was that of computing the 1-point functions on the CFT side (exploiting non-renormallization theorems [27] to work with the D1D5 CFT at the free orbifold point) and the deviation from $AdS_3 \times S^3$ geometry of the corresponding microstate metric, establishing the match.

Another observable we considered is Entanglement Entropy: on the one hand this can be computed on the CFT side using the replica trick [70], while on the other hand it corresponds to the area of a co-dimension 2 extremal surface that reduces to the boundary of the entangling domain in the CFT spatial circle (and wraps the whole S^3) asymptotically [71, 72, 82]. For a sufficiently small entangling domain, the extremal surface doesn't extend much away from the boundary and only the leading order deviations from $AdS_3 \times S^3$ geometry are relevant, which again correspond to 1-point function of CFT operators of small conformal dimension. The bulk-boundary correspondence was tested between classes of 2- and 3-charge microstates.

Another topic that was investigated is that of HHLL CFT 4-point functions, where the heavy states are dual to a particular class of microstate geometries and the light operators correspond to supergravity fields probing the bulk geometry. Using O^{++} as the light operator and 2- and 3-charge states as the heavy states, we computed the 4-point functions in the D1D5 CFT at the free orbifold point and, following the holographic prescription, in the bulk, solving the wave equation for the appropriate supergravity field probing the background geometry dual to the heavy states. The agreement between the CFT and bulk result came somewhat as a surprise, but was justified upon considering the affine block decomposition of the HHLL correlator: the affine $SU(2)_L \times SU(2)_R$ symmetry is indeed present at any point of the moduli spaces, and correlators entirely saturated by affine blocks are therefore protected.

A final direction of work involved CFT computations at the free orbifold point prominently. In particular, it was important to verify that the techniques used in the previously mentioned parts of the work were actually able to reproduce results from the literature, and in particular we focused on the correlator contained in [126]. Moreover, exploiting the fact that the D1D5 CFT enjoys supersymmetry, we wrote the supersymmetric Ward identities connecting correlators with light operators given in terms of bosons at the free orbifold point and correlators where those operators are given in terms of fermions. The validity of the Ward identities was verified explicitly, as the correlators can be computed explicitly at the free orbifold point.

HHLL have recently been connected to the black hole information loss problem, seen from a dual CFT point of view [64–69]. In particular, a connection was made between "spurious singularities" in Euclidean time appearing in the Virasoro block of the identity in the $c \to \infty$ limit and how one expects the correlator to behave [68]. HHLL 4-point functions can be seen as 2-point functions computed in the background given by the heavy operators, and there must be a qualitative difference between a thermal 2-point function, corresponding holographically to probes moving in a black hole background, and a 2-point function computed in a pure state.

In a thermal 2-point function (thermal) Euclidean time is periodic, and in the CFT computation this creates infinitely many images of the OPE singularities. This is exactly what one gets from the Virasoro block of the identity as $c \to \infty$. The puzzle appears when we consider the same 2-point function in a pure state: with pure states time is not periodic anymore and a CFT correlator in Euclidean signature should have only singularities corresponding to the different OPE limits that can be taken. If the correlator is saturated by the Virasoro block of the identity, then it develops infinitely many singularities in the $c \to \infty$ limit, as in the thermal case, which is not acceptable. Mechanisms to avoid this considering each Virasoro block separately have been suggested, referring to sub-leading corrections in the $1/c$ expansion and to "non-perturbative" features [65, 66, 68, 128]. Our result, on the contrary, moves in the other direction: having a specific CFT and dual supergravity theory to work with, we know exactly which Virasoro blocks appear in the correlator, and, with our choice of light and heavy operators, we observe that spurious singularities cancel out among the different blocks even in the $c \to \infty$ limit.

The appearance of spurious singularities in HHLL correlators brings up questions that still have to be settled. In particular, in [128] it is argued that in the cases considered here spurious singularities cancel out between different Virasoro blocks at $c \to \infty$ because of the presence of affine symmetry, i.e. because we are actually focusing on an integrable sector of the theory. While it's true that we are focusing on a particularly simple sector, the argument given in Section 5.4 should be valid irrespective of the presence of an affine symmetry, which ends up being just the particular mechanism that enforces the more general feature of singularity cancellation between different Virasoro blocks as $c \to \infty$ in the specific cases considered. Whether and how cancellations happen in correlators that are not protected by affine symmetry is one of the main motivations for studying the more complicated 4-point functions considered in [129].

Another future line of work to address the above problem avoiding the complications faced in [129] could be that of exploiting the Ward identities presented in Chapter 6: it should be possible to start from non-protected 4-point functions involving light operators written in terms of fermions and map them to correlators with light operators written in terms of bosons, with the advantage that the latter class of operators, of the form $\sim \partial X^{A\dot{A}} \bar{\partial} X^{B\dot{B}}$ (like the ones considered in [126]), have indices only along the internal T^4 directions, thus giving a simpler wave equations to solve when computing the 4-point function holographically.

Even though fifteen years of research have brought much progress towards the explanation of black hole physics through the fuzzball proposal, a lot remains to be done. The known classes of microstates do not account for the totality of the entropy of (3-charge) black holes, and, much in the spirit of [58], an active branch of research consists in finding more bulk microstate geometries and their dual CFT states. Moreover, research is being carried out about 4-point functions in which the CFT result at the free orbifold point and in the bulk differ [129]. Another interesting direction of work involves considering deformation of the D1D5 CFT away from the free orbifold point that still preserve supersymmetry, and generalizing the construction to non-supersymmetric fuzzballs.

Finally, we would like to mention that the fuzzball proposal gives an explicit and rigorous realization of holography: a lot of effort in the literature is being devoted to connecting black holes to topics like quantum chaos, the SYK model and Random Matrix theory, see e.g. [130], and one should expect that all the features pointed out using those approaches are actually reproduced using the fuzzball description and the D1D5 CFT in particular, as was checked in [127].

More than forty years have passed since Hawking's computation of black hole radiation and since theoretical physicists realized the best theories they had to describe the gravitational interaction on one side and all other particle interactions on the other side don't fit well in a unified picture. In this context, black holes have represented the best theoretical laboratory in which to test ideas about quantum gravity. String theory in particular has performed well in this direction, with an unparalleled richness of concepts and techniques that revealed to be useful to tackle one of the most difficult problems theoretical physics has ever had to face. On the other hand, black hole physics has been a constant theme within string theory for the last twenty years and has guided string theorists towards new discoveries (such as holography), in an enriching feedback loop of ideas.

Appendix A

General gravity results for D1-D5 geometries

We will now give some general results for the objects Z_1, Z_2 and Z_4 for 2-charge geometries up to order $\sim 1/r^3$. First we define

$$h_1(v') \equiv g_1(v') + \mathrm{i} g_2(v'), \qquad h_2(v') \equiv g_3(v') + \mathrm{i} g_4(v'), \tag{A.1}$$

where $g_1, \ldots, g_5$ have the expansions (4.11).

We have

$$Z_1 \equiv 1 + \frac{Q_5}{L}\int_0^L dv \frac{|\dot{h}_1|^2 + |\dot{h}_2|^2 + |\dot{g}_5|^2}{|x_i - g_i|^2}, \quad Z_2 \equiv 1 + \frac{Q_5}{L}\int_0^L dv \frac{1}{|x_i - g_i|^2}, \tag{A.2}$$

$$\mathcal{A} \equiv -\frac{Q_5}{L}\int_0^L dv \frac{\dot{g}_5}{|x_i - g_i|^2}, \qquad A \equiv -\frac{Q_5}{L}\int_0^L dv \frac{\dot{g}_j dx^j}{|x_i - g_i|^2}, \tag{A.3}$$

where the denominator can also be rewritten as

$$|x_i - g_i|^2 \equiv \sum_{i=1}^{4} (x_i - g_i)^2 = |(x_1 + \mathrm{i} x_2) - h_1|^2 + |(x_3 + \mathrm{i} x_4) - h_2|^2, \tag{A.4}$$

and where A is the 1-form defined in (3.4).

The result for Z_1 at order $\sim 1/r^3$ is

$$\begin{aligned} Z_1 \simeq\; & \frac{4\pi^2 Q_5}{L^2} \sum_{k\neq 0} \left\{ |a_k^{(1)}|^2 + |a_k^{(2)}|^2 + \frac{1}{4}|a_{|k|}^{(00)}|^2 \right\} + \\ & + \frac{4\pi^2 Q_5}{L^2}\frac{1}{r^3} \sum_{k,l,n\neq 0} \frac{1}{n} \left\{ \left(\frac{x_1 - \mathrm{i}x_2}{r}\right) \left(a_k^{(1)} \bar{a}_l^{(1)} a_n^{(1)} \delta_{k-l+n} + a_k^{(2)} \bar{a}_l^{(2)} a_n^{(1)} \delta_{k-l+n} \right) + \right. \\ & \left. + \left(\frac{x_3 - \mathrm{i}x_4}{r}\right) \left(a_k^{(1)} \bar{a}_l^{(1)} a_n^{(2)} \delta_{k-l+n} + a_k^{(2)} \bar{a}_l^{(2)} a_n^{(2)} \delta_{k-l+n} \right) + [\text{c.c.}] \right\} + \end{aligned}$$

$$+\frac{\pi^2 Q_5}{L^2}\frac{1}{r^3}\sum_{k,l=0}^{+\infty}\sum_{n\neq 0}\frac{1}{n}\left\{\left(\frac{x_1-\mathrm{i}x_2}{r}\right)\left(a_k^{(00)}a_l^{(00)}a_n^{(1)}\delta_{k+l+n}+2a_k^{(00)}\bar{a}_l^{(00)}a_n^{(1)}\delta_{k-l+n}+\right.\right.$$
$$\left.+\bar{a}_k^{(00)}\bar{a}_l^{(00)}a_n^{(1)}\delta_{-k-l+n}\right)+$$
$$+\left(\frac{x_3-\mathrm{i}x_4}{r}\right)\left(a_k^{(00)}a_l^{(00)}a_n^{(2)}\delta_{k+l+n}+2a_k^{(00)}\bar{a}_l^{(00)}a_n^{(2)}\delta_{k-l+n}+\right.$$
$$\left.\left.+\bar{a}_k^{(00)}\bar{a}_l^{(00)}a_n^{(2)}\delta_{-k-l+n}\right)+[\text{c.c.}]\right\}, \tag{A.5}$$

where $\delta_m \equiv \delta_{m,0}$ and where we put $a_{k<0}^{(00)}=0$.

The result for Z_2 does not contain terms of order $\sim 1/r^3$, thus

$$Z_2 = 1+\frac{Q_5}{r^2}+O\left(\frac{1}{r^4}\right), \tag{A.6}$$

while for Z_4 we have

$$Z_4 = \frac{\pi Q_5}{L}\frac{1}{r^3}\sum_{k=1}^{+\infty}\frac{1}{k}\left\{\left(\frac{x_1-\mathrm{i}x_2}{r}\right)\left(a_k^{(00)}a_{-k}^{(1)}+\bar{a}_k^{(00)}a_k^{(1)}\right)+\right.$$
$$\left.+\left(\frac{x_3-\mathrm{i}x_4}{r}\right)\left(a_k^{(00)}a_{-k}^{(2)}+\bar{a}_k^{(00)}a_k^{(2)}\right)+[\text{c.c.}]\right\}. \tag{A.7}$$

The 1-form $A = A_i dx^i$ can be written at order $\sim 1/r^3$ as

$$A_i = -2Q_5 f_{ij}\frac{x_j}{r^4}, \qquad f_{ij}\equiv \frac{1}{L}\int_0^L dv\ \dot{g}_i g_j = -f_{ji}. \tag{A.8}$$

We can switch to complex coordinates

$$z_1 \equiv x_1+\mathrm{i}x_2, \qquad \bar{z}_1 \equiv x_1-\mathrm{i}x_2, \tag{A.9}$$
$$z_2 \equiv x_3+\mathrm{i}x_4, \qquad \bar{z}_2 \equiv x_3-\mathrm{i}x_4, \tag{A.10}$$

and define indices $z^a, z^b, \ldots$ such that $z^a = (z_1,\bar{z}_1,z_2,\bar{z}_2)$ and so on to get

$$A_{z^a} = -2Q_5 f_{z^a z^b}\frac{dz^b}{r^4}. \tag{A.11}$$

We have

$$f_{z_1\bar{z}_1} = \frac{2\pi i}{L}\sum_{n\neq 0}a_n^{(1)}\frac{\bar{a}_n^{(1)}}{n}, \qquad f_{z_1 z_2} = -\left(f_{\bar{z}_1\bar{z}_2}\right)^* = -\frac{2\pi i}{L}\sum_{n\neq 0}a_n^{(1)}\frac{a_{-n}^{(2)}}{n}, \tag{A.12}$$

$$f_{z_2\bar{z}_2} = \frac{2\pi i}{L}\sum_{n\neq 0}a_n^{(2)}\frac{\bar{a}_n^{(2)}}{n}, \qquad f_{z_1\bar{z}_2} = -\left(f_{\bar{z}_1,z_2}\right)^* = \frac{2\pi i}{L}\sum_{n\neq 0}a_n^{(1)}\frac{\bar{a}_n^{(2)}}{n}. \tag{A.13}$$

The components of the 1-form B at order $\sim 1/r^3$ are obtained in the coordinates x_i as

$$B_i = -Q_5 \; \epsilon_{ijkl} f_{kl} \frac{x_j}{r^4}. \tag{A.14}$$

Appendix B

General D1-D5 state with twist field VEV

In general the twist field $\Sigma^{s_1\dot{s}_2}_{k_1+k_2}$ can join two strands of length k_1 and k_2 into a strand or length k_1+k_2 (or vice versa split the (k_1+k_2)-long strand into k_1 and k_2 pieces). Thus a state with three different strands of lengths k_1, k_2 and $k_3 = k_1 + k_2$ will have a non-vanishing VEV for $\Sigma^{s_1\dot{s}_2}_{k_1+k_2}$. For simplicity we take the spin state of all the strands to be $(++)$, so our building blocks are $|{++}\rangle_{k_i}$, with $i = 1, 2, 3$. In Section 4.3.1 we have considered the particular case with $k_1 = k_2 = 1$, $k_3 = 2$. The interest of the more general case relies on the fact that the action of the twist field on strands of length greater than one is quite subtle, and it produces a non-trivial numerical factor which was computed by CFT methods in [88] (see Eq. (5.25) there). We will show that holography provides a non-trivial check for this coefficient.

The state we consider has the form (4.16) where the only non-trivial coefficients are $A^{(++)}_{k_1} \equiv A_1$, $A^{(++)}_{k_2} \equiv A_2$, $A^{(++)}_{k_3} \equiv A_3$; for brevity, we also rename $N^{(++)}_{k_1} \equiv p_1$, $N^{(++)}_{k_2} \equiv p_2$, $N^{(++)}_{k_3} \equiv p_3$; these numbers are subject to the constraint $k_1p_1 + k_2p_2 + k_3p_3 = N$. The state is then

$$\psi(A_1, A_2, A_3) \equiv \sum_{p_1=0}^{N/k_1} \sum_{p_2=0}^{\frac{N-k_1p_1}{k_2}} (A_1|{++}\rangle_{k_1})^{p_1} (A_2|{++}\rangle_{k_2})^{p_2} (A_3|{++}\rangle_{k_3})^{\frac{N-k_1p_1-k_2p_2}{k_3}} . \tag{B.1}$$

Its norm is

$$|\psi(A_1, A_2, A_3)|^2 = \sum_{p_1=0}^{N/k_1} \sum_{p_2=0}^{\frac{N-k_1p_1}{k_2}} A_1^{p_1} A_2^{p_2} A_3^{\frac{N-k_1p_1-k_2p_2}{k_3}} \mathcal{N}(p_1, p_2) , \tag{B.2}$$

with

$$\mathcal{N}(p_1,p_2) = \frac{N!}{p_1!\,p_2!\,(\frac{N-k_1p_1-k_2p_2}{k_3})!\,k_1^{p_1}\,k_2^{p_2}\,k_3^{\frac{N-k_1p_1-k_2p_2}{k_3}}}\,. \tag{B.3}$$

According to the general result (4.20) the sum in (B.1) is peaked around the average values

$$\bar{p}_i = \frac{|A_i|^2}{k_i} \qquad (i=1,2,3)\,. \tag{B.4}$$

We can now consider the action of the twist field on the state $\psi(A_1,A_2,A_3)$. By angular momentum conservation, only the operator $\Sigma_{k_1+k_2}^{-\frac{k_1+k_2-1}{2},-\frac{k_1+k_2-1}{2}}$ can glue two strands and only $\Sigma_{k_1+k_2}^{\frac{k_1+k_2-1}{2},\frac{k_1+k_2-1}{2}}$ can split one strand[45]. The novelty with respect to the state with $k_1=k_2=1$ is that when $\Sigma_{k_1+k_2}^{-\frac{k_1+k_2-1}{2},-\frac{k_1+k_2-1}{2}}$ glues two strands of windings $k_1,k_2>1$, the final state is multiplied by the factor

$$c_{k_1k_2} = \frac{k_1+k_2}{2k_1k_2}\,. \tag{B.5}$$

Note that $c_{1,1}=1$, and thus this effect was invisible in the computation of Section 4.3.1. This factor was derived via a non-trivial CFT computation in [88]; we will import their result here, and show that it is necessary for consistency with the holographic computation of the VEV. One has moreover to include the usual combinatorial factors which arise when one has multiple strands of the same type, so the total action of the twist field is

$$\begin{aligned}
&\Sigma_{k_1+k_2}^{-\frac{k_1+k_2-1}{2},-\frac{k_1+k_2-1}{2}}\left[(|{+}{+}\rangle_{k_1})^{p_1}\,(|{+}{+}\rangle_{k_3})^{p_2}\,(|{+}{+}\rangle_{k_3})^{p_3}\right] = \\
&\quad = c_{k_1k_2}\,(p_3+1)\,k_3\left[(|{+}{+}\rangle_{k_1})^{p_1-1}\,(|{+}{+}\rangle_{k_2})^{p_2-1}\,(|{+}{+}\rangle_{k_3})^{p_3+1}\right].
\end{aligned} \tag{B.6}$$

The combinatorics is explained as follows: there are p_1 (p_2) ways to pick one strand of length k_1 (k_2); moreover on a strand of length k_1 (k_2), the gluing action of $\Sigma_{k_1+k_2}^{-\frac{k_1+k_2-1}{2},-\frac{k_1+k_2-1}{2}}$ can be applied at k_1 (k_2) positions within the strand. Thus the number of terms appearing on the l.h.s. of (B.6) is

$$p_1\,p_2\,k_1\,k_2\,\mathcal{N}(p_1,p_2) = (p_3+1)\,k_3\,\mathcal{N}(p_1-1,p_2-1)\,, \tag{B.7}$$

where we have used (B.3). Since this equals the number of terms present on the r.h.s. of (B.6) (up to the factor c_{k_1,k_2}), this justifies the combinatorial factors in that equation.

The calculation for the VEV of $\Sigma_{k_1+k_2}^{-\frac{k_1+k_2-1}{2},-\frac{k_1+k_2-1}{2}}$ on $\psi(A_1,A_2,A_3)$ now proceeds

[45] Although the notation is not incredibly clear, these twist operators are simply, respectively, the ones with minimal and maximal (left and right) spins, given that they create/split strands of length k_1+k_2.

along similar lines as in Eq. (4.43), and one obtains

$$\langle\Sigma_{k_1+k_2}^{-\frac{k_1+k_2-1}{2},-\frac{k_1+k_2-1}{2}}\rangle \equiv |\psi(A_1,A_2,A_3)|^{-2}\,\langle\psi(A_1,A_2,A_3)|\Sigma_2^{--}|\psi(A_1,A_2,A_3)\rangle = c_{k_1k_2}\frac{A_1A_2}{A_3}\bar{p}_3 = \frac{k_1+k_2}{2\,k_1k_2}A_1A_2\bar{A}_3\,. \tag{B.8}$$

Analogous arguments determine the action of $\Sigma_{k_1+k_2}^{\frac{k_1+k_2-1}{2},\frac{k_1+k_2-1}{2}}$, when it splits a strand of winding k_1+k_2 into pieces of length k_1 and k_2:

$$\Sigma_{k_1+k_2}^{\frac{k_1+k_2-1}{2},\frac{k_1+k_2-1}{2}}\Big[(|{+}{+}\rangle_{k_1})^{p_1}\,(|{+}{+}\rangle_{k_3})^{p_2}\,(|{+}{+}\rangle_{k_3})^{p_3}\Big] = = c_{k_1k_2}\,(p_1+1)\,k_1\,(p_2+1)\,k_2\Big[(|{+}{+}\rangle_{k_1})^{p_1+1}\,(|{+}{+}\rangle_{k_2})^{p_2+1}\,(|{+}{+}\rangle_{k_3})^{p_3-1}\Big]. \tag{B.9}$$

One can again check that, thanks to the identity (B.7), the action of $\Sigma_{k_1+k_2}^{\frac{k_1+k_2-1}{2},\frac{k_1+k_2-1}{2}}$ is consistent with hermitian conjugation and thus

$$\langle\Sigma_{k_1+k_2}^{\frac{k_1+k_2-1}{2},\frac{k_1+k_2-1}{2}}\rangle = \langle\Sigma_{k_1+k_2}^{-\frac{k_1+k_2-1}{2},-\frac{k_1+k_2-1}{2}}\rangle^* = \frac{k_1+k_2}{2\,k_1k_2}\bar{A}_1\bar{A}_2A_3\,. \tag{B.10}$$

The VEVs of the angular momentum operators are determined by the average numbers of strands, and are given by

$$\langle J^3\rangle = \langle\tilde{J}^3\rangle = \frac{1}{2}\,(\bar{p}_1+\bar{p}_2+\bar{p}_3) = \frac{1}{2}\left(\frac{|A_1|^2}{k_1}+\frac{|A_2|^2}{k_2}+\frac{|A_3|^2}{k_1+k_2}\right). \tag{B.11}$$

On the gravity side, the dual geometry is associated with the profile with modes $a_{k_1}^{(++)}\equiv a_1$, $a_{k_2}^{(++)}\equiv a_3$ and $a_{k_3}^{(++)}\equiv a_3$, related with the CFT parameters as

$$a_i = \frac{A_i}{R}\sqrt{\frac{Q_1Q_5}{N}}\qquad (i=1,2,3)\,. \tag{B.12}$$

The gravity coefficients determining the VEVs are

$$f^1_{11}-\mathrm{i}f^1_{12} = \frac{R^2}{Q_1Q_5}\,\frac{k_1+k_2}{2\,k_1k_2}\,a_1\,a_2\,\bar{a}_3\,,\quad \mathcal{A}_{1i}=0\,, \tag{B.13}$$

$$a_{3+} = -a_{3-} = \frac{R}{2\sqrt{Q_1Q_5}}\left(\frac{|a_1|^2}{k_1}+\frac{|a_2|^2}{k_2}+\frac{|a_3|^2}{k_3}\right). \tag{B.14}$$

The angular momenta derived from a_{3+}, a_{3-} are easily seen to match with the CFT values (B.11). Using the coefficient $c_{O^{(00)}}$ given in (4.53), the gravity prediction for the VEV of Σ_2^{--} is

$$\langle\Sigma_{k_1+k_2}^{-\frac{k_1+k_2-1}{2},-\frac{k_1+k_2-1}{2}}\rangle_{\text{Grav.}} = c_{O^{(0,0)}}\,(f^1_{11}-\mathrm{i}f^1_{12}) = \frac{N^{3/2}\,R^3}{(Q_1Q_5)^{3/2}}\,\frac{k_1+k_2}{2\,k_1k_2}a_1\,a_2\,\bar{a}_3\,, \tag{B.15}$$

which matches with the CFT prediction (B.8) in view of (B.12).

Appendix C

Entanglement Entropy and reduced metric

Below we sketch the proof for the statements at the end of Section 4.4.1.

(i) Consider the extremality equation (4.162) at first order in ϵ. Since $\partial_\alpha x^\mu$ starts at order ϵ, and in (4.162) there appears the first derivative of $\sqrt{\det g^*}$ with respect to $\partial_\alpha x^\mu$, it is enough to compute $\sqrt{\det g^*}$ at second order in $\partial_\alpha x^\mu$. This can be done by doing an expansion around $\partial_\alpha x^\mu = 0$, where the induced metric g^* greatly simplifies. Indeed when $\partial_\alpha x^\mu = 0$ one has

$$g^*_{\lambda\lambda} = \hat{g}_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu \,, \quad g^*_{\lambda\alpha} = G_{\alpha\beta}\, A^\beta_\mu\, \dot{x}^\mu \,, \quad g^*_{\alpha\beta} = G_{\alpha\beta} \,, \quad (\partial_\alpha x^\mu = 0) \tag{C.1}$$

with

$$\hat{g}_{\mu\nu} \equiv g_{\mu\nu} + G_{\alpha\beta} A^\alpha_\mu A^\beta_\nu \,. \tag{C.2}$$

Then the inverse of the induced metric is

$$g_*^{\lambda\lambda} = g^{\lambda\lambda} \,, \quad g_*^{\lambda\alpha} = -g^{\lambda\lambda}\, A^\alpha_\mu\, \dot{x}^\mu \,, \quad g_*^{\alpha\beta} = G^{\alpha\beta} + g^{\lambda\lambda}\, A^\alpha_\mu A^\beta_\nu\, \dot{x}^\mu \dot{x}^\nu \,, \quad (\partial_\alpha x^\mu = 0) \tag{C.3}$$

where $g^{\lambda\lambda}$ is the inverse of

$$g_{\lambda\lambda} \equiv g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu \,. \tag{C.4}$$

Using this observation, one can compute the expansion of $\sqrt{\det g^*}$ up to the first order in $\partial_\alpha x^\mu$:

$$\sqrt{\det g^*}\Big|_{\partial_\alpha x^\mu = 0} = \sqrt{g_{\lambda\lambda}\, \det G} \,, \tag{C.5}$$

$$\frac{\partial \sqrt{\det g^*}}{\partial \partial_\alpha x^\mu}\Big|_{\partial_\alpha x^\mu = 0} = \sqrt{g_{\lambda\lambda}\, \det G}\, (A^\alpha_\mu - g^{\lambda\lambda} A^\alpha_\sigma g_{\mu\rho} \dot{x}^\rho \dot{x}^\sigma) \,. \tag{C.6}$$

When evaluating the first two terms in the extremality equation (4.162) at first order in ϵ, one only needs (C.5); moreover, due to the absence of first order corrections to $g_{\mu\nu}$ and $G_{\alpha\beta}$, one can approximate

$$\sqrt{\det g^*} = \sqrt{g^0_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,\sqrt{\det G^0} + O(\epsilon^2)\,. \tag{C.7}$$

Substituting the expansion (4.164) for $x^\mu(\lambda, x^\alpha)$ in the above equation, one immediately concludes that, at first order in ϵ, the first two terms in (4.162) give a linear and homogeneous equation for x_1^μ. Consider now the last term in (4.162): the only contribution that is not homogeneous in x_1^μ comes from (C.6). At our order of approximation such a term is

$$-\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial\partial_\alpha x^\mu}\sqrt{\det g^*} = -\epsilon\sqrt{g^0_{\lambda\lambda}\det G^0}\, g_0^{\lambda\lambda}(\nabla^0_\alpha\delta A^\alpha_\mu\, g^0_{\rho\sigma} - \nabla^0_\alpha\delta A^\alpha_\sigma\, g^0_{\mu\rho})\,\dot{x}^\rho_0\dot{x}^\sigma_0 + O(\epsilon^2)\,, \tag{C.8}$$

where

$$g^0_{\lambda\lambda} \equiv g^0_{\mu\nu}\,\dot{x}^\mu_0\dot{x}^\nu_0 \tag{C.9}$$

does not depend on x^α. This term vanishes thanks to the de Donder gauge condition (4.159). We thus conclude that the equation for x_1^μ is linear and homogeneous and hence it admits the solution $x_1^\mu = 0$.

(ii) Consider now the contributions of order ϵ^2 to the area of the extremal surface

$$A = \int d\lambda dx^\alpha\sqrt{\det g^*}\,, \tag{C.10}$$

which gives the EE. We notice that to compute $\sqrt{\det g^*}$ up to order ϵ^2 one can set $\partial_\alpha x^\mu = 0$: indeed, having shown that $x_1^\mu = 0$, we know that $\partial_\alpha x^\mu$ starts at order ϵ^2; moreover (C.6) implies that the first derivative of $\sqrt{\det g^*}$ with respect to $\partial_\alpha x^\mu$ is at least of order ϵ; thus the contributions from $\partial_\alpha x^\mu$ to $\sqrt{\det g^*}$ are at least of order ϵ^3. For the computation of A we can then use the simplified expression (C.5), and obtain

$$A = \int d\lambda dx^\alpha\sqrt{g_{\lambda\lambda}\det G} + O(\epsilon^3) = A_0 + \epsilon^2\int d\lambda\sqrt{g^0_{\lambda\lambda}}\, g_0^{\lambda\lambda}\, g^0_{\mu\nu}\,\dot{x}^\mu_0\,\dot{X}^\nu_2 + \ldots + O(\epsilon^3)\,, \tag{C.11}$$

where A_0 is the order zero term, X_2^μ is the S^3 integral of x_2^μ

$$X_2^\mu \equiv \int dx^\alpha\sqrt{\det G^0}\, x_2^\mu\,, \tag{C.12}$$

and the dots in (C.11) are terms of order ϵ^2 that do not depend on x_2^μ (but are proportional to $\delta g^2_{\mu\nu}$ and $\delta G^2_{\alpha\beta}$). We conclude that to compute A at second order we do not need to know $x_2^\mu(\lambda, x^\alpha)$ but only its integral $X_2^\mu(\lambda)$.

(iii) We now want to derive a differential equation for $X_2^\mu(\lambda)$, or equivalently for $X^\mu(\lambda)$. Since the extremality equation (4.162) at order ϵ^2 is of course linear in x_2^μ, we can derive an equation for its S^3-integral by integrating (4.162) on S^3; the last term in

(4.162), being a total derivative with respect to x^α, drops out of the integral; so we get the equation

$$\int dx^\alpha \Big[\frac{\partial}{\partial x^\mu}\sqrt{\det g^*} - \frac{\partial}{\partial\lambda}\frac{\partial}{\partial\dot{x}^\mu}\sqrt{\det g^*}\Big] = 0\,, \tag{C.13}$$

where we can use the approximation (C.5) for $\sqrt{\det g^*}$.

We thus see that the problem reduces to that of finding an extremal surface in the "reduced 3D" metric $g^E_{\mu\nu} \equiv g_{\mu\nu}\,(\det G)$. Note that $g^E_{\mu\nu}$ would be the Einstein metric in 3D if it were independent of x^α. In this extremality problem the variables x^α appear as external parameters, i.e. the equation does not contain derivatives with respect to x^α. At the end of the computation one should integrate over x^α. Alternatively one can perform the integral over x^α before solving the equations and define an x^α-independent 3D metric

$$\tilde{g}_{\mu\nu} \equiv g^0_{\mu\nu} + \epsilon^2 \int dx^\alpha \sqrt{\det G^0}\left(\delta g^2_{\mu\nu} + \frac{1}{3} g^0_{\mu\nu}\, G_0^{\alpha\beta}\delta G^2_{\alpha\beta}\right). \tag{C.14}$$

(Note: we are assuming the normalization $\int dx^\alpha \sqrt{\det G^0} = 1$). The equations that determine $X^\mu(\lambda) \equiv \int dx^\alpha \sqrt{\det G^0}\, x^\mu(\lambda, x^\alpha)$ are the geodesic equations for a curve in the metric $\tilde{g}_{\mu\nu}$.

Appendix D

CFT computations of the HH-LL 4-point functions

In this appendix we will study in more detail the CFT computations that lead to the results for the HH-LL correlators in section (5.1). Since the theory enjoys an $SU(2)_1 \times U(1)$ affine symmetry on each strand, in addition to the $SU(2)_L$ generators we also have a $U(1)$ generator J^0 defined as

$$J^0(z) = \sum_{r=1}^{N} J^0_{(r)}(z)\,, \qquad J^0_{(r)} = -\frac{C}{2}\psi^{\alpha\dot{1}}_{(r)}\psi^{\beta\dot{2}}_{(r)}\epsilon_{\alpha\beta}\,, \tag{D.1}$$

where C is a constant that is not fixed by the algebra (corresponding to the fact that the level of the $U(1)$ factor inside a $SU(2)_k \times U(1)$ affine algebra is undetermined). The $J^a_{(r)}$ generators of $SU(2)_L$ can also be written in terms of the bosons H and K, noticing that

$$\psi^{+\dot{1}}_{(r)}\psi^{-\dot{2}}_{(r)} = -\mathrm{i}\partial H_{(r)}, \qquad \psi^{+\dot{2}}_{(r)}\psi^{-\dot{1}}_{(r)} = \mathrm{i}\partial K_{(r)}, \tag{D.2}$$

as

$$J^3_{(r)} = \frac{\mathrm{i}}{2}\big(\partial H_{(r)} + \partial K_{(r)}\big), \qquad J^+_{(r)} = \mathrm{i}e^{\mathrm{i}\left(H_{(r)}+K_{(r)}\right)}, \qquad J^-_{(r)} = \mathrm{i}e^{-\mathrm{i}\left(H_{(r)}+K_{(r)}\right)}. \tag{D.3}$$

The light operators we consider are (on a strand)

$$O^L_{(r)} = -\frac{\mathrm{i}}{\sqrt{2}}\psi^{+\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{+\dot{B}}_{(r)} \equiv O^{++}_{(r)}\,, \qquad \bar{O}^L_{(r)} = -\frac{\mathrm{i}}{\sqrt{2}}\psi^{-\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{-\dot{B}}_{(r)}, \tag{D.4}$$

while the ones acting on the product theory are given by the sum over copies in (5.3). In all the cases considered we will get the same result for each strand, so we can just work on a copy and (in the untwisted sector) multiply by N. The states corresponding to the heavy operators (5.9) we consider in the untwisted sector are

$$\begin{aligned} |s,k=1\rangle &\equiv \lim_{z,\bar z\to 0} O^H(s,k=1;z,\bar z)|0\rangle_{\rm NS} \\ &= \otimes_{r=1}^N \left[(J^+_{-2s})_{(r)}\cdots(J^+_{-2})_{(r)} \lim_{z,\bar z\to 0} O^H_{(r)}(s=0,k=1;z,\bar z)\right]|0\rangle_{\rm NS}\,. \end{aligned} \tag{D.5}$$

The left and right parts of the four-point function (5.3) factorize and we need to evaluate correlators of the form

$$\begin{aligned} F^{\dot A\dot C}_{s,(r)}(z_i) \equiv \langle e^{\mathrm{i}\left(s+\frac12\right)\left(H_{(r)}(z_1)+K_{(r)}(z_1)\right)} e^{-\mathrm{i}\left(s+\frac12\right)\left(H_{(r)}(z_2)+K(z_2)_{(r)}\right)}\psi^{+\dot A}_{(r)}(z_3)\psi^{-\dot C}_{(r)}(z_4)\rangle\times \\ \times\prod_{r'\neq r}\langle e^{-\mathrm{i}\left(s+\frac12\right)\left(H_{(r)}(z_1)+K_{(r)}(z_1)\right)} e^{\mathrm{i}\left(s+\frac12\right)\left(H_{(r)}(z_2)+K_{(r)}(z_2)\right)}\rangle. \end{aligned} \tag{D.6}$$

The right part is completely analogous, with the exception that in the right sector we always have $s=0$. Notice that in principle the light operators acting on the product theory bring two sums over strands. Despite this, by spin conservation, the only nonzero contributions come from the cases in which both light operators act on the same strand, which reduces the full correlator to just one sum over copies. Moreover, since the heavy operators are product over copies, the term relative to the r-th copy is multiplied by the two-point functions of the heavy operators on all the copies $r'\neq r$. The full correlation function reads

$$\langle O_H(z_1)\bar O_H(z_2)O_L(z_3)\bar O_L(z_4)\rangle = \sum_{r=1}^N \frac12\, F^{\dot A\dot C}_{s,(r)}(z_i)F^{\dot B\dot D}_{0,(r)}(\bar z_i)\,\epsilon_{\dot A\dot B}\epsilon_{\dot C\dot D}. \tag{D.7}$$

$F^{\dot A\dot C}_{s,(r)}(z_i)$ is nonzero only if the two fermions can have a nontrivial contraction, which selects the cases $(\dot A,\dot B)=(\dot 1,\dot 2)$ and $(\dot A,\dot B)=(\dot 2,\dot 1)$. In the first case, using (2.105) to contract each possible pair of fields, we get

$$F^{\dot 1\dot 2}_{s,(r)}(z_i) = -\frac{z_{13}^{s+\frac12}z_{24}^{s+\frac12}}{z_{12}^{2h}z_{14}^{s+\frac12}z_{23}^{s+\frac12}z_{34}} = -\frac{1}{z_{12}^{2h}z_{34}}\,z^{-s-\frac12}\,, \tag{D.8}$$

where $h=\left(s+\frac12\right)^2$. The second case is analogous, giving $F^{\dot 2\dot 1}_{s,(r)}(z_i)=-F^{\dot 1\dot 2}_{s,(r)}(z_i)$. The antiholomorphic parts are obtained from these setting $s=0$ and replacing $z_i\to\bar z_i$ and $h\to\bar h=1/4$. Putting everything together we get

$$\langle O_H(z_1)\bar O_H(z_2)O_L(z_3)\bar O_L(z_4)\rangle = \frac{1}{z_{12}^{2h_H}\bar z_{12}^{2\bar h_H}|z_{34}|^2}\,|z|^{-1}\,z^{-s}\,; \tag{D.9}$$

a factor N would come from the fact that each term of the sum over r gives N times the same contribution, but this is cancelled by the normalization (5.3) of O_L. The first correlator we compute in the untwisted sector corresponds to $s = 0$, while the second to generic s.

Let's consider the twisted sector. In this case we have N/k strands of length k and, working on one strand, the current operators become

$$J^3 = -\frac{1}{2}\sum_{\rho=0}^{k-1} \psi_\rho^{+\dot{A}} \psi_\rho^{-\dot{B}} \epsilon_{\dot{A}\dot{B}}, \tag{D.10a}$$

$$J^+ = \frac{1}{2}\left(\psi_{\rho=0}^{+\dot{A}}\psi_{\rho=0}^{+\dot{B}}\epsilon_{\dot{A}\dot{B}} + \sum_{\rho=1}^{k-1}\psi_\rho^{+\dot{A}}\psi_{k-\rho}^{+\dot{B}}\epsilon_{\dot{A}\dot{B}}\right), \tag{D.10b}$$

$$J^- = \frac{1}{2}\left(\psi_{\rho=0}^{-\dot{A}}\psi_{\rho=0}^{-\dot{B}}\epsilon_{\dot{A}\dot{B}} + \sum_{\rho=1}^{k-1}\psi_\rho^{-\dot{A}}\psi_{k-\rho}^{-\dot{B}}\epsilon_{\dot{A}\dot{B}}\right). \tag{D.10c}$$

Switching from the (r) to the ρ basis using (2.57) the light operators are rewritten as

$$\sum_{r=1}^{k} O_{(r)}^{++} = \sum_{\rho=0}^{k-1} O_\rho^{++}, \qquad O_\rho^{++} \equiv -\frac{\mathrm{i}}{\sqrt{2}}\,\psi_\rho^{+\dot{A}}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}_\rho^{+\dot{B}}, \tag{D.11}$$

where $\bar{O}_L$ is the complex conjugate of this. In the $s = pk$ case the states generated by the heavy operators (5.15) can be expressed in terms of the operators (5.13) corresponding the $s = 0$ case plus the action of modes of J^+,

$$|s,k\rangle \equiv \left[\left(J^+_{-2s/k}\dots J^+_{-2/k}\right)\lim_{z,\bar{z}\to 0}\otimes_{\rho=0}^{k-1}\,\sigma_\rho^X\tilde{\sigma}_\rho^X S^{\dot{1}}_{k,s=0,\rho}S^{\dot{2}}_{k,s=0,\rho}\tilde{S}^{\dot{1}}_{k,s=0,\rho}\tilde{S}^{\dot{2}}_{k,s=0,\rho}\right]^{N/k}|0\rangle_{\mathrm{NS}}\,. \tag{D.12}$$

Following the same logic as in the untwisted sector, the correlator is given in terms of functions

$$\begin{aligned}F^{\dot{A}\dot{C}}_{k,s=pk,\rho}(z_i) \equiv{}& \langle e^{\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+p\right)(H_\rho(z_1)+K_\rho(z_1))}e^{-\mathrm{i}\left(-\frac{\rho}{k}+\frac{1}{2}+p\right)(H_\rho(z_2)+K(z_2)_\rho)}\psi_\rho^{+\dot{A}}(z_3)\psi_\rho^{-\dot{C}}(z_4)\rangle\times\\ &\times\prod_{\rho'\neq\rho}\langle e^{\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_1)+K_{\rho'}(z_1)\right)}e^{-\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_2)+K_{\rho'}(z_2)\right)}\rangle\times\\ &\times\prod_{\rho''=0}^{k-1}\left[\langle\sigma^X_{\rho''}(z_1)\sigma^X_{\rho''}(z_2)\rangle\langle\tilde{\sigma}^X_{\rho''}(\bar{z}_1)\tilde{\sigma}^X_{\rho''}(\bar{z}_2)\rangle\right]\end{aligned} \tag{D.13}$$

as

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \frac{1}{k}\sum_{\rho=0}^{k-1}\frac{1}{2}\,F^{\dot{A}\dot{C}}_{k,s=pk,\rho}(z_i)F^{\dot{B}\dot{D}}_{k,s=0,\rho}(\bar{z}_i)\;\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}, \tag{D.14}$$

where the $1/k$ factor takes care of the fact that we have the same contribution for each length-k strand (it would be N/k, but the N factor in the numerator cancels out because of the normalization of the light operators). As in the untwisted sector, $F^{\dot{A}\dot{C}}_{k,s=pk,\rho}(z_i)$ is nonzero only if $(\dot{A},\dot{C})$ take values $(\dot{1},\dot{2})$ or $(\dot{2},\dot{1})$, and we have

$$F^{\dot{1}\dot{2}}_{k,s=pk,\rho}(z_i) = -\frac{z_{13}^{-\frac{\rho}{k}+\frac{1}{2}+p}\, z_{24}^{-\frac{\rho}{k}+\frac{1}{2}+p}}{z_{12}^{2h}\, z_{14}^{-\frac{\rho}{k}+\frac{1}{2}+p}\, z_{23}^{-\frac{\rho}{k}+\frac{1}{2}+p}\, z_{34}} = -\frac{1}{z_{12}^{2h}\, z_{34}}\, z^{\frac{\rho}{k}-\frac{1}{2}-p}\,, \tag{D.15}$$

where $h = \frac{k}{4} + \frac{s(s+1)}{k}$, $F^{\dot{2}\dot{1}}_{k,s,\rho}(z_i) = -F^{\dot{1}\dot{2}}_{k,s,\rho}(z_i)$ and z is defined in (5.2). The antiholomorphic part is again obtained taking the holomorphic one, setting $s = 0$ (i.e. $p = 0$) and replacing $z_i \to \bar{z}_i$ and $h \to \bar{h} = k/4$. Putting everything together we get

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \frac{1/k}{z_{12}^{2h_H}\,\bar{z}_{12}^{2\bar{h}_H}\,|z_{34}|^2}\,\frac{z^{-p}}{|z|}\,\frac{1-|z|^2}{1-|z|^{\frac{2}{k}}}. \tag{D.16}$$

In the $s = pk-1$ case, with the same procedure as before we have

$$\begin{aligned} F^{\dot{A}\dot{C}}_{k,s=pk-1,\rho=0}(z_i) = {} & \langle e^{\mathrm{i}\left(-\frac{1}{2}+p\right)(H_0(z_1)+K_0(z_1))} e^{-\mathrm{i}\left(-\frac{1}{2}+p\right)(H_0(z_2)+K_0(z_2))} \psi_0^{+\dot{A}}(z_3)\psi_0^{-\dot{C}}(z_4)\rangle \times \\ & \times \prod_{\rho'=1}^{k-1} \langle e^{\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_1)+K_{\rho'}(z_1)\right)} e^{-\mathrm{i}\left(-\frac{\rho'}{k}+\frac{1}{2}+p\right)\left(H_{\rho'}(z_2)+K_{\rho'}(z_2)\right)} \\ & \times \Big[\langle \sigma^X_{\rho=0}(z_1)\sigma^X_{\rho=0}(z_2)\rangle\langle\tilde{\sigma}^X_{\rho=0}(\bar{z}_1)\tilde{\sigma}^X_{\rho=0}(\bar{z}_2)\rangle\Big], \end{aligned} \tag{D.17}$$

while for $\rho \neq 0$ (and in the whole right sector) we have the same functions as in (D.13), i.e. $F^{\dot{A}\dot{C}}_{k,s=pk-1,\rho\neq 0} = F^{\dot{A}\dot{C}}_{k,s=pk,\rho\neq 0}$. The correlator takes again the form (D.14) and the only new object to compute is

$$F^{\dot{1}\dot{2}}_{k,s=pk-1,\rho=0}(z_i) = -\frac{z_{13}^{-\frac{1}{2}+p}\, z_{24}^{-\frac{1}{2}+p}}{z_{12}^{2h}\, z_{14}^{-\frac{1}{2}+p}\, z_{23}^{-\frac{1}{2}+p}\, z_{34}} = -\frac{1}{z_{12}^{2h}\, z_{34}}\, z^{\frac{1}{2}-p}, \tag{D.18}$$

where again $h = \frac{k}{4} + \frac{s(s+1)}{k}$ and $F^{\dot{2}\dot{1}}_{k,s=pk-1,\rho=0}(z_i) = -F^{\dot{1}\dot{2}}_{k,s=pk-1,\rho=0}(z_i)$. The full correlator in the $s = pk-1$ case reads

$$\langle O_H(z_1)\bar{O}_H(z_2)O_L(z_3)\bar{O}_L(z_4)\rangle = \frac{1/k}{z_{12}^{2h_H}\,\bar{z}_{12}^{2\bar{h}_H}\,|z_{34}|^2}\, z^{-p} \left(\left(\frac{z}{\bar{z}}\right)^{\frac{1}{2}} + \frac{1}{|z|}\,\frac{|z|^{\frac{2}{k}}-|z|^2}{1-|z|^{\frac{2}{k}}} \right). \tag{D.19}$$

The first correlator considered in the twisted sector corresponds to choosing $s = 0$, while the second and the third correspond respectively to the $s = pk$ and the $s = pk-1$ case.

Appendix E

Wave equation in $\mathbf{AdS}_3/\mathbb{Z}_k$

In this section we solve the wave equation (5.46) for a scalar field of dimension 1, in the geometry written in (5.34), with the monodromy (5.48) and the boundary condition (5.50). We will follow a route similar to the one employed in [121, 122], and our result generalises the one obtained in the previous works to the case with non-trivial monodromy ($\hat{s} \neq 0$). The boundary CFT lives on the cylinder and to induce the appropriate geometry on the boundary we will work in global AdS coordinates; we will keep careful track of the periodicity of the spatial circle, which is crucial to distinguish geometries with different values of the conical defect and to properly implement the monodromy condition. More general discussions about the dynamics of a scalar field in Lorentzian AdS of general conformal dimension, the interpretation of the normalizable modes solution, and the difference between different choice of patch can be found in [131].

The AdS part of the geometry in (5.34b) can be simplified by the redefinitions:

$$t = k\,\frac{\sqrt{Q_1 Q_5}}{a}\,\tau \qquad y = k\,\frac{\sqrt{Q_1 Q_5}}{a}\,\sigma\,, \quad r = \frac{a}{k}\tan\rho\,, \tag{E.1}$$

where the new coordinates τ, σ, ρ have the following domains[46]

$$\rho \in \left[0, \frac{\pi}{2}\right]\,, \quad \sigma \in \left[0, \frac{2\pi}{k}\right]\,, \quad \tau \in [0, +\infty)\,. \tag{E.2}$$

After this change the metric takes the form

$$ds^2_{AdS_3} = \frac{1}{\cos^2\rho}\left(-d\tau^2 + d\rho^2 + \sin^2\!\rho\, d\sigma^2\right) \tag{E.3}$$

with the boundary located at $\rho = \frac{\pi}{2}$.

The most general solution with the prescribed monodromies involves an arbitrary sum

[46] The σ coordinate defined here should not be confused with the CFT spatial coordinate with the same name.

over Fourier modes:

$$B(\tau,\sigma,\rho) = \frac{1}{(2\pi)^2}\, e^{\mathrm{i}\hat{s}\sigma} \sum_{l\in\mathbb{Z}} \int d\omega\, e^{\mathrm{i}\omega\tau} e^{\mathrm{i}lk\sigma} g(l,\omega)\chi_{l,\omega}(\rho)\,, \tag{E.4}$$

where the choice of the function $g(l,\omega)$ encodes a particular boundary data and we assume $0 \le \hat{s} < k$. Substituting into the wave equation we obtain a differential equation for $\chi_{l,\omega}(\rho)$ that reads

$$\chi''_{l,\omega}(\rho) + \csc\rho \sec\rho \chi'_{l,\omega}(\rho) + \left(\omega^2 - (lk+\hat{s})^2 \csc^2\rho + \ell(\ell-2)\right)\chi_{l,\omega}(\rho) = 0\,. \tag{E.5}$$

This is an hypergeometric equation, as it is made evident by the change $x = \sin^2\rho$:

$$\chi''_{l,\omega}(x) + \frac{1}{x}\chi'_{l,\omega}(x) + \frac{1}{4}\left(\frac{\omega^2}{x(1-x)} - \frac{(lk+\hat{s})^2}{x^2(1-x)} + \frac{1}{x(1-x)^2}\right)\chi_{l,\omega}(x) = 0\,. \tag{E.6}$$

The solution that is finite everywhere in the bulk[47] is

$$\chi_{l,\omega}(x) = x^{\frac{|lk+\hat{s}|}{2}}(1-x)^{\frac{1}{2}}\, {}_2F_1\left(\frac{1}{2}(1+|lk+\hat{s}|-\omega), \frac{1}{2}(1+|lk+\hat{s}|+\omega), 1+|lk+\hat{s}|, x\right)\,. \tag{E.7}$$

From the expansion of this solution near the boundary ($x = 1$) one can extract the non-normalizable and the normalizable modes

$$\begin{aligned} \chi_{l,\omega}(x) \approx\ & \frac{\Gamma(1+|lk+\hat{s}|)}{\Gamma(\frac{1}{2}(1+|lk+\hat{s}|-\omega))\Gamma(\frac{1}{2}(1+|lk+\hat{s}|+\omega))}\times \\ & \left\{\left[2\gamma_E + \psi(\frac{1}{2}(1+|lk+\hat{s}|-\omega)) + \psi(\frac{1}{2}(1+|lk+\hat{s}|+\omega))\right](1-x)^{\frac{1}{2}}\right. \\ & \left. + \left[\log(1-x)\right](1-x)^{\frac{1}{2}}\right\}, \end{aligned} \tag{E.8}$$

with the digamma function defined as $\psi(z) \equiv \frac{d}{dz}\log(\Gamma(z))$, and γ_E the Euler constant. The non-normalizable mode (the source) is the coefficient of the $[\log(1-x)](1-x)^{\frac{1}{2}}$ term and the normalizable mode (the VEV) is the term proportional to $(1-x)^{\frac{1}{2}}$. Reverting to the original coordinates, these two terms correspond to the ones shown in (5.50). A delta function source at the boundary is obtained by tuning the function $g(l,\omega)$ in (E.4) in such a way that the non-normalizable term has constant Fourier transform; this is achieved setting

$$g(l,\omega) = \frac{\Gamma(\frac{1}{2}(1+|lk+\hat{s}|-\omega))\,\Gamma(\frac{1}{2}(1+|lk+\hat{s}|+\omega))}{\Gamma(1+|lk+\hat{s}|)}\,. \tag{E.9}$$

The coefficient of the normalizable term, denoted as $b_1(\tau,\sigma)$ in (5.50), is then found

[47] The form of the other independent solution can be found, for example, in [131]. It can be shown to contain divergences for $x \to 0$ (i.e. $r \to 0$).

from (E.8) to be

$$b_1(\tau,\sigma) = \sum_{l\in\mathbb{Z}} \int \frac{d\omega}{(2\pi)^2}\, e^{\mathrm{i}\omega\tau+\mathrm{i}(lk+\hat{s})\sigma}\Big[\psi(\frac{1}{2}(1+|lk+\hat{s}|-\omega))+\psi(\frac{1}{2}(1+|lk+\hat{s}|+\omega))+2\gamma_E\Big]\,. \tag{E.10}$$

In order to perform the sum we use the series representation of the digamma function

$$\psi(z) = -\gamma_E + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right)\,. \tag{E.11}$$

Separating the term with $l = 0$ in the sum, and forgetting contact terms coming from summation over constants Fourier modes we have

$$\begin{aligned} b_1(\tau,\sigma) =& \sum_{n=0}^{\infty}\Bigg[\sum_{l=0}^{\infty}\int \frac{d\omega}{(2\pi)^2}\, e^{\mathrm{i}\omega\tau+\mathrm{i}(lk+\hat{s})\sigma}\left(\frac{2}{\omega-(lk+\hat{s})-1-2n} - \frac{2}{\omega+(lk+\hat{s})+1+2n}\right) \\ &+ \sum_{l=1}^{\infty}\int \frac{d\omega}{(2\pi)^2}\, e^{\mathrm{i}\omega\tau-\mathrm{i}(lk-\hat{s})\sigma}\left(\frac{2}{\omega-(lk-\hat{s})-1-2n} - \frac{2}{\omega+(lk-\hat{s})+1+2n}\right)\Bigg]\,. \end{aligned} \tag{E.12}$$

As usual, to define the ω-integral one has to pick the integration contour: we choose the Feynman prescription, which allows the Wick rotation to Euclidean space and hence comparison with the CFT correlator, which is evaluated on the Euclidean plane. The integral is thus readily computed and yields

$$\begin{aligned} b_1(\tau,\sigma) &= -\frac{\mathrm{i}}{2\pi}\sum_{n=0}^{\infty}\left[\sum_{l=0}^{\infty} e^{\mathrm{i}(lk+\hat{s})\sigma} e^{-\mathrm{i}(lk+\hat{s}+1+2n)\tau} + \sum_{l=1}^{\infty} e^{-\mathrm{i}(lk-\hat{s})\sigma} e^{-\mathrm{i}(lk-\hat{s}+1+2n)\tau}\right] \\ &= -\frac{\mathrm{i}}{2\pi}\frac{e^{\mathrm{i}\hat{s}\sigma}}{e^{\mathrm{i}\tau}-e^{-\mathrm{i}\tau}}\left[\frac{e^{-\mathrm{i}\hat{s}\tau}}{1-e^{\mathrm{i}k(\sigma-\tau)}} + \frac{e^{\mathrm{i}\hat{s}\tau}}{e^{\mathrm{i}k(\sigma+\tau)}-1}\right]\,. \end{aligned} \tag{E.13}$$

Re-expressing the result in the original physical coordinates defined in (E.1), and suppressing the overall numerical coefficient (which is not meaningful as we did not keep track of the normalization of the operators), we finally obtain

$$\begin{aligned} b_1(t,y) &= -\mathrm{i}\frac{e^{\mathrm{i}\hat{s}\frac{y}{R_y\,k}}}{e^{\mathrm{i}\frac{t}{R_y\,k}} - e^{-\mathrm{i}\frac{t}{R_y\,k}}}\left[\frac{e^{\mathrm{i}\frac{t-y}{R_y}}}{e^{\mathrm{i}\frac{t-y}{R_y}}-1}e^{-\mathrm{i}\hat{s}\frac{t}{R_y\,k}} + \frac{1}{e^{\mathrm{i}\frac{t+y}{R_y}}-1}e^{\mathrm{i}\hat{s}\frac{t}{R_y\,k}}\right] \\ &= -\mathrm{i}\left(\frac{z}{\bar{z}}\right)^{\frac{\hat{s}}{2k}}\frac{1}{|z|^{\frac{1}{k}}-|z|^{-\frac{1}{k}}}\left[\frac{\bar{z}}{\bar{z}-1}|z|^{-\frac{\hat{s}}{k}} + \frac{1}{z-1}|z|^{\frac{\hat{s}}{k}}\right]\,. \end{aligned} \tag{E.14}$$

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