Iterative Demodulation and Decoding Algorithm for 3GPP/LTE-A MIMO-OFDM Using Distribution Approximation

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Abstract—Soft iterative detection/decoding algorithms are fundamentally necessary for multiple-input multiple-output orthogonal frequency-division multiplexing (MIMO-OFDM) adopted in the 3rd Generation Long Term Evolution - Advanced (LTE-A) in order to increase the capacity and achieve high data rates. However, their high performance critically requires log likelihood ratio (LLR) computations with prohibitive complexity. This challenge will be addressed in this paper. We first use the assumption of Gaussian transmit symbols to show the equivalence among several existing algorithms. We next develop a non-Gaussian approximation for high-order constellations, which paves the way for interference cancellation based detectors. Based on both Gaussian and non-Gaussian approximations, we thus develop several capacity-achieving iterative MIMO-OFDM demodulation and decoding algorithms. To this end, we adopt K-best algorithms to take advantage of both types of approximations and the list decoder. Unlike existing algorithms, our proposed K-best algorithms make use of the a priori probabilities to generate the list. Simulations of standard-compliant LTE systems demonstrate that the proposed algorithms outperform existing ones.

Index Terms—Iterative decoding, MIMO-OFDM, K-best, Gaussian approximation, LTE-A, turbo principle, 3GPP.

I. INTRODUCTION

The 3rd Generation Partnership Project (3GPP) has standardized the Long Term Evolution - Advanced (LTE-A) in Release 10 [1] with ambitious data rates at more than 1 Gb/s in local areas and 100 Mb/s in wide areas. To meet these challenging requirements, multiple-input multiple-output (MIMO) techniques as well as high spectrum allocation of 100 MHz promise a linear increase of the wireless link capacity. To counteract the delay spread and to facilitate flexible resource allocation, in the downlink, orthogonal frequency-division multiplexing (OFDM) is combined with MIMO. However, to attain the capacity gains of MIMO-OFDM, one of the fundamental challenges is the complexity of the detection/decoding algorithms measured by decoding power consumption/silicon area [2]–[5].

Even for uncoded MIMO systems, the complexity of optimal maximum likelihood (ML) decoding is exponential in the number of antennas. Low complexity but sub-optimal receiver structures such as the vertical Bell Laboratories Layered Space-Time (V-BLAST) nulling/cancelling algorithm and many others are thus developed [6]–[10]. On the other hand, the complexity of ML decoding can be significantly reduced, especially in the high signal-to-noise ratio (SNR) region, by sphere decoding (SD) [11]. Nevertheless, if channel coding is added to MIMO, then the size of the search space is exponential in the product of the code blocklength and the number of antennas. The optimal joint detector/decoder is prohibitively complex notwithstanding the use of SD. For this reason, the main idea of near-capacity iterative detection/decoding method [12] is the iterative exchange between the MIMO detector and the channel decoder — the MIMO detector uses log likelihood ratio (LLR) information from the channel decoder, which then incorporates soft information from the MIMO detector. These iterations continue until desired performance is achieved. The decoder computing exact LLR is called maximum a posteriori (MAP) decoder. However, repeated LLR computations are essential.

The computation burden of LLR for each bit must be reduced. To achieve this goal, [12] modifies the standard SD to generate a list of candidates at the detector and only the contributions from the list are included in the LLR values needed for the iterative process. Moreover, the LLR computation is simplified via the max-log approximation. Although these two steps allow [12] to reach near-optimal performance with lower complexity, the complexity depending on SNR can nevertheless be high. Another idea is to avoid the rapid expansion of the search tree by retaining only a fixed number of nodes at each step. Following this idea, a K-best SD is proposed in [13] for both uncoded and coded MIMO systems, which has a constant complexity across the entire range of SNR. Another approach [14]–[17] to adopt soft information in iterative detection and decoding is to use nonlinear interference cancelation followed by a single-input single-output (SISO) iterative detector and decoder in [18], which is shown to have low complexity. Yet another idea is for the computation of the LLR of each bit, the other symbols are ranked according to

IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS, VOL. X, NO. X, XXX 201X

Manuscript received February 13, 2017; revised July 18, 2017, October 23, 2017; accepted November 16, 2017.
their contributions to the LLR of the said bit, and a subset of those symbols with less contributions are approximated as Gaussian variables [19].

In this paper, we consider efficient iterative detection/decoding for MIMO-OFDM based LTE-A systems. As in the aforementioned works, the burden of LLR computations is the critical factor. The method of LLR computation is what differentiates the various algorithms in [12], [14]–[17]. The algorithms in [14]–[16] are equivalent to using a Gaussian approximation. However, it is ineffective for high order modulations such as 64 quadrature amplitude modulation (QAM), which is needed in LTE-A. We thus propose non-Gaussian approximation by exploiting the probability mass function (PMF) of the constellation points into a unified form followed by relaxing the discrete variable to take continuous values. By integrating over the resultant non-Gaussian probability density function (PDF), we approximate the LLR value. Next, we propose combining the K-best algorithm and the Gaussian or non-Gaussian approximation to take advantage of both. In the proposed modified K-best algorithms, we generate a list of K lattice points. There are two major differences between our proposed algorithm and that in [12]. First, we use sum-log rather than max-log to compute the LLR value by summing over all lattice points in the list. Second, the list generation method is different. To generate the list, we adopt the a priori probability, which is approximated using Gaussian and non-Gaussian distributions. We also discuss several variations of the basic algorithms and efficient implementation in commercial MIMO-OFDM receivers. In addition to non-Gaussian approximation for high order constellations, we propose several list-decoding algorithms while all lattice points are used in [19] for the symbols not using Gaussian approximation. Different from [20] where a zero-forcing detector is used to approximate max-log LLR and Gaussian approximation is used to study the distribution of LLR, we use Gaussian and non-Gaussian distributions for sum-log listing decoding. Simulation results using an LTE simulator show that the proposed algorithms outperform the existing ones, especially for higher-order modulations.

The rest of the paper is organized as follows. In Section II, the MIMO-OFDM system model is presented. In Section III, the iterative receiver structure is outlined and several existing algorithms are reviewed. A non-Gaussian approximation based algorithm and several modified K-best algorithms are given in Section IV. Simulation results are provided in Section V and Section VI draws the conclusions.

II. SYSTEM MODEL

We consider a MIMO-OFDM system with M transmit and N receive antennas. The system has Ns subcarriers in an OFDM block. There are M data streams to be transmitted \(^1\). The constellation \(Q_m\) is applied on stream \(m\), where \(C_m\) is the number of bits per constellation symbol. The incoming bits of each stream \(m\) of length \(N_s C_m R_m, m=1,\ldots, M\), is encoded using a channel code (typically a convolutional or turbo code) of rate \(R_m\), resulting in a bit vector \(b_m\). The encoded bits are converted into symbols using a mapping function \(x_{i,m} = \mathcal{M}_m(b_m((i-1)C_m + 1; C_m))\) (e.g., Gray mapping and set partitioning mapping), \(i = 0, \ldots, N_s - 1\), where \(x_{i,m}\) is the symbol to be transmitted over subcarrier \(i\) and antenna \(m\). The inverse discrete Fourier transform (IDFT) of the data block \(x_{0,m}, \ldots, x_{N_s-1,m}\) yields the time domain sequence, i.e.,

\[ X_{j,m} = \frac{1}{\sqrt{N_s}} \sum_{i=0}^{N_s-1} x_{i,m} e^{j2\pi i/N_s}, j = 0, \ldots, N_s - 1. \]  

After adding cyclic prefix (CP), the received signal on carrier \(i\) after DFT can be written in vector form as

\[ y_i = H_i x_i + w_i, i = 0, \ldots, N_s - 1, \]  

where \(x_i = [x_{i,1}, \ldots, x_{i,M}]^T\) is the transmitted signal on \(M\) transmit antennas, \(y_i = [y_{i,1}, \ldots, y_{i,N}]^T\) and \(w_i = [w_{i,1}, \ldots, w_{i,N}]^T\) are the received signal and the additive noise on \(N\) receive antennas, respectively, and \(H_i\) denotes the \(N \times M\) channel matrix. Each entry in \(w_i\) has zero mean and variance \(\sigma^2\). The MIMO-OFDM system is depicted in Fig. 1. Each subcarrier (2) is a distinct MIMO system. In this paper except in Section IV-B, we will neglect the subscript \(i\) in (2) for brevity. We assume channel matrix \((H_i, i = 0, 1, \ldots, N_s - 1)\) is known at the receiver which can be estimated using algorithms in [21]–[26].

III. EXISTING ITERATIVE DETECTION AND DECODING ALGORITHMS

A. Basic Concepts

The optimal joint detector and decoder must compute the likelihood of each bit given the received signals \(y_0, \ldots, y_{N_s-1}\) on all subcarriers. However, this is computationally infeasible, the algorithms in [12], [14]–[16], [18] thus use the “turbo principle”, where information is exchanged between the detector and decoder iteratively. In this paper, we focus on how to generate extrinsic information at each subcarrier using the received signals on this subcarrier and the a priori information on each bit from the channel decoder. The generated extrinsic information on all subcarriers is then input to the soft-in-soft-out channel decoder for the next iteration of decoding and detection. We briefly review several such existing algorithms and unravel their relationships, which will motivate our algorithms in Section IV.
B. Iterative Detection and Decoding

The a priori probability (APP) is usually expressed as a LLR value ($L$-value). The sign of $L$-value indicates if a bit is zero or one. The magnitude of $L$-value indicates the reliability of a bit decision; e.g., magnitude close to zero shows an unreliable bit. We represent the logical zero for a bit by $b_i = -1$ and logical one by $b_i = +1$, respectively. Given the APP from the channel decoder, the a posteriori LLR value of the bit $b_i$ conditioned on the received vector $y$ is

$$L(b_i|y) = \log \frac{Pr(b_i = +1|y)}{Pr(b_i = -1|y)}.$$  \hspace{1cm} (3)

We can rewrite (3) using Bayes’ theorem as

$$L(b_i|y) = \log \left[ \frac{\sum_{j=0, j \neq i}^{M} \sum_{m=1}^{M} C_{m-1} \Pr(y|x) \prod_{j=0, j \neq i}^{M} \Pr(b_j = B_j(x))}{\sum_{j=0, j \neq i}^{M} \sum_{m=1}^{M} C_{m-1} \Pr(y|x) \prod_{j=0, j \neq i}^{M} \Pr(b_j = B_j(x))} \right]$$

$$+ \log \left[ \frac{\Pr(b_i = +1)}{\Pr(b_i = -1)} \right] = \log L_{E}(b_i|y)$$

where $X_{i,+1}$ and $X_{i,-1}$ are the set of $2 \times 2^{M-1}$ vectors such that the $i$-th bit is +1 or -1, respectively; Namely $X_{i,+1} = \{x|\mathcal{M}(b) = x, b_i = +1\}$, $b = \mathcal{B}(x)$ is the inverse mapping of $x = \mathcal{M}(b)$ and $B_j(x)$ is the $j$-th bit of $B(x)$.

In the case of Gaussian channel (2), we can further write $L(b_i|y)$ as

$$L(b_i|y) = \log \left[ \frac{\sum_{x \in X_{i,+1}} \exp \left( -\frac{||y - Hx||^2}{\sigma^2} + \sum_{j=0, j \neq i}^{M} \sum_{b_j = 1}^{M} L_{A}(b_j) \right) \right]$$

$$+ \log \left[ \frac{\Pr(b_i = +1)}{\Pr(b_i = -1)} \right] = \log L_{E}(b_i|y)$$

To find the $L$-value for each $b_i$ using (5), a search is needed over $2 \times 2^{M-1}$ terms, which is exponential in the total number of bits $\sum_{m=1}^{M} C_m$. Since this complexity is prohibitive, a list sphere decoder (LSD) is adopted in [12].

We consider a commercial iterative decoding and demodulation architecture for MIMO-OFDM in Fig. 2 adapted from [27]. $L_{E}(b_i|y)$ computation in (4) corresponds to the demodulator in Fig. 2. Besides iterative decoding in turbo decoder, there is an outerloop iterative procedure to exchange $L_{A}(b_i)$ between demodulator and turbo decoder. In LTE-A, rate matching is used to match the number of bits in transport block to the number of bits that can be transmitted in the given allocation [1]. Rate matching involves many things including sub-block interleaving, bit collection and pruning.

Derate matching is the reverse operation of rating matching. We can find the corresponding symbol $x$ for $b_i$ in (4) through rate matching.

C. Iterative Detection and Decoding with Gaussian Approximation

Here we review a Gaussian approximation for iterative detection and decoding and relate different existing algorithms, where we can gain insights to develop new algorithms. To this end, note that we can rewrite (4) as

$$L(b_i|y) = \log \left[ \frac{\sum_{x \in X_{i,+1}} \Pr(y|x_{m},x_{m}) \Pr(x_{m})}{\sum_{x \in X_{i,-1}} \Pr(y|x_{m},x_{m}) \Pr(x_{m})} \right]$$

where $x_m$ denotes the symbol that $b_i$ belongs to, $x_{-m}$ denotes the vector that contains all entries of $x$ except for the $m$-th entry, and $X_{i,+1} (X_{i,-1})$ is the set of $2^{M-1}$ vectors such that $b_i = +1$ (-1). For any given $x_m$, we need to compute $\sum_{x_{-m}} \Pr(y|x_{m},x_{m}) \Pr(x_{-m})$. A suboptimal approach is to replace the summation over $x_{-m}$ with an integration over a continuous distribution. One common choice is the Gaussian distribution. If we assume the entries of $x_{-m}$ are independent Gaussian random variables with mean and variance

$$\mu_{m} = E\{x_{m}\}, \nu_{m} = E\{|x_{m}|^2\} - E^2\{x_{m}\},$$

$m = 1, \ldots, M, m \neq m$ and Gaussian channel model (2) is used, then we have

$$\Pr(y|x_{m}) = \sum_{x_{m}} \Pr(y|x_{m},x_{m}) \Pr(x_{m})$$

$$\approx \int_{-\infty}^{+\infty} \Pr(y|x_{m},x_{m}) f(x_{m}) dx_{m}$$

$$\times \exp \left( - (y - H_{m} \mu_{m} - h_{m} x_{m})^T R_{m}^{-1} \left( y - H_{m} \mu_{m} - h_{m} x_{m} \right) \right),$$

where the integral is from $-\infty$ to $\infty$ in each dimension, $H_{m}$ contains the columns of $H$ except for the $m$-th column, $h_{m}$ is the $m$-th column of $H$, $\mu_{m} = [\mu_{1}, \ldots, \mu_{m-1}, \mu_{m+1}, \ldots, \mu_{M}]^T$, $R_{m} = H_{m} \text{diag}\{\nu_{1}, \ldots, \nu_{m-1}, \nu_{m+1}, \ldots, \nu_{M}\} H_{m}^H + \sigma^2 I_N$, and $I_N$ is an $N \times N$ identity matrix. To maximize $\Pr(y|x_{m})$ in (8) we only need to search through possible $x_{m}$s. The complexity of computing LLR reduces from $2^{M}$ possible constellation combinations to $2^{M}$. It can be shown that different algorithms [14]-[16] are equivalent to the Gaussian approximation (8).
IV. PROPOSED ITERATIVE DETECTION AND Decoding ALGORITHM

In this section, we assume all transmit symbols are from a square-QAM constellation, as in LTE-A [1], but our work can also readily extend to arbitrary constellations. To achieve a good performance-complexity tradeoff especially for high order constellations such as 64-QAM, we can try two different avenues. First, by following [12], we may compute L-values with the LSD. However, the number of summands in (4) is exponential. Since LSD depends on the maximally possible searching time, its detection throughput is variable, which would demand I/O buffers adding an extra overhead in a practical system [13]. Second, we may use the Gaussian approximation to get the closed-form (8) but its performance is not good for higher order modulations [16]. Due to these reasons, our goal is to combine LSD and continuous integral which may demand I/O buffers adding an extra overhead in a practical system [13]. However, our work would demand I/O buffers adding an extra overhead in a practical system [13]. Therefore, we approximate the PMF (12) with a second order polynomial in the exponent for any \( Q \), i.e.,

\[
\Pr(X = x) = \exp\left(-\left(c + 2rx + ax^2\right)\right), \tag{13}
\]

where \( c, r, a \) are constants. Note that the Gaussian distribution is a special case of (13), which contains a nonnegative \( a \). The coefficients \( a, r, c \) are found by solving

\[
\min_{a, r, c} \sum_i \omega_i \left(\exp\left(-\left(c + 2rx_i + ax_i^2\right)\right) - p_i\right)^2, \tag{14}
\]

where \( \omega_i \geq 0 \) is a weight for symbol \( x_i \). We may choose uniform weights or assign higher weights to the symbols with large probability. At each iteration of decoding, we can get \( p \) from \( L_A(b_i) \) in (4) and solve (14) using convex optimization.

The integration in (8) is from \(-\infty\) to \( +\infty\), which may distort the LLR value. Practical constellations typically contain finite alphabets, e.g., 2D-PAM is \([-2D+1, -2D+3, \ldots, 2D-3, 2D-1]\). We can integrate from \(-U\) to \( U\) instead. Some possible choices of \( U \) are \( 2D \) or \( 2D-1 + \sigma \). With (13) and \( U=2D \), we can write (8) as

\[
\Pr(y|x_m) \approx \left(a \sum_{x_m} \Pr(y|x_m, x_m) \prod_{k \neq m, k=1}^{2d_k-1} f(x_k)dx_k \right. \tag{a}
\]

\[
+ \int_U^{+U} \Pr(y|x_m, x_m)f(x_m)dx_m \tag{b}
\]

\[
\alpha \exp\left(-\frac{\|y - h_m x_m\|^2}{\sigma^2}\right) \times \int_U^{+U} \exp\left(-2 \left(\frac{r_{-m} - \frac{1}{\sigma^2} (y - h_m x_m)^T H_{-m}}{\sigma^2}\right)\right) x_m \tag{15}
\]

where \( r_{-m} = [r_1, \ldots, r_{m-1}, r_{m+1}, \ldots, r_M]^T \) and \( A_{-m} = \text{diag}\{a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_M\} \). \( r_m' \) and \( a_m' \) are obtained from (14). In (a), we have approximate the PMF using a finite integration over PDF as

\[
\Pr(x_k = 2dk - 1) \approx \int_{2dk-2}^{2dk-1} f(x_k)dx_k.
\]
In \( (b) \), we have applied the mean value theorem for definite integrals where
\[
\int_{2d_k-2}^{2d_k} \Pr(y|x_{-(m,k)},x_m,x_k) f(x_k) dx_k \\
= \Pr(y|x_{-(m,k)},x_m,\tilde{x}_k) \int_{2d_k-2}^{2d_k} f(x_k) dx_k \\
\approx \Pr(y|x_m,x_k) \int_{2d_k-2}^{2d_k} f(x_k) dx_k,
\]
where \( x_{-(m,k)} \) denotes the vector that contains all entries of \( x \) except for the \( m \)-th and \( k \)-th entries.

Comparing (15) with (8), we see that there are two main differences. First, \( r_{-m} \) and \( A_{-m} \) are not from the matched mean and variance but from matching the PMF directly. Second, the integral is from \( -U \) to \( U \).

Let the singular value decomposition of \( R_m \) be \( V^T \Lambda V \) and \( g(x_m) = Vb_{-m} \), where \( \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{M-1}\} \). We change variables by defining \( z = Vx_{-m} \). However, the integration region of \( z \) is an \( M-1 \) dimensional polytope, making the integral hard to compute. For simplicity, we neglect the integration region by setting a bound \( Z_i = U \sum_{j=1}^{M-1} |V_{i,j}| \) for dimension \( i \). We can then approximate (15) as
\[
\Pr(y|x_m) \propto \exp\left( -\frac{||y - h_m x_m||^2}{\sigma^2} \right) \\
\times \prod_{i=1}^{M-1} \int_{-Z_i}^{+Z_i} \exp\left( -2g_i(x_m)z_i - \lambda_i z_i^2 \right) dz_i,
\]

where \( g_i(x_m) \) is the \( i \)-th entry of \( g(x_m) = Vb_{-m} \). When \( \lambda_i > 0 \), we can rewrite the integral in (16) into a Q-function. When \( \lambda_i < 0 \), we can compute the integral using numerical methods.

To compare Gaussian and non-Gaussian approximations, we consider 4-PAM with \( Q = \{-3, -1, 1, 3\} \). Two bits \((b_1, b_2)\) are mapped to \( Q \) via gray mapping \((0, 1) \rightarrow -3, (0, 0) \rightarrow -1, (1, 0) \rightarrow 1, (1, 1) \rightarrow 3\). After several decoding iterations, we may have \( \Pr(b_1 = 1) = 0.6 \) and \( \Pr(b_2 = 1) = 0.8 \), which gives \( \Pr(X = -3) = 0.32 \), \( \Pr(X = -1) = 0.08 \), \( \Pr(X = 1) = 0.12 \), and \( \Pr(X = 3) = 0.48 \). We compare the PDF’s of non-Gaussian (14) and Gaussian approximations in Fig. 3. The areas between \( 2i-2 \) and \( 2i \) (as an approximation of the symbol probability \( P(X = 2i-1) \), \( i = -1, 0, 1, 2 \) are 0.31, 0.09, 0.10, 0.49 for the non-Gaussian approximation and 0.15, 0.29, 0.33, 0.23 for the Gaussian approximation. Clearly the Gaussian approximation does not match the discrete distribution when some bits are unreliable. This problem is a lot more severe when each symbol contains more than 2 bits, which may be the reason that the probabilistic data association (PDA) performs poorly for high order modulations as observed in [16].

For general bit mappings, due to the constraint on the second order polynomial, (13) may not fit the PMF for all symbols in large constellations. Moreover, it is observed in [18] that Gray mapping does not perform well in iterative joint detection and decoding. Other mappings such as set partitioning mapping may perform better. For set partitioning mapping, we can resolve the polynomial order constraint via constellation decomposition. Let \( b_i = +1 \) for logical one and \( b_i = -1 \) for logical zero. Let \( C \) be a positive integer. We can write the set partitioning mapping for \( 2^C \)-PAM \((Q = \{-2^C + 1, -2^C + 3, \ldots, 2^C - 3, 2^C - 1\}) \) as
\[
x = \sum_{i=0}^{C-1} 2^i b_i = d^T b,
\]

where \( d = [1, 2, \ldots, 2^C - 1]^T \) and \( b = [b_0, \ldots, b_{C-1}]^T \). Note that we have ignore the scaling factor in the modulation to keep unit average power. As each entry of \( b \) takes BPSK, the continuous approximation to the PMF is given in (11), where (13) is exact with
\[
a = c = -\frac{1}{4} \left( \log p + \log (1-p) \right), \quad b = -\frac{1}{4} \left( \log p - \log (1-p) \right).
\]

Define \( \tilde{H}_{-m} = H_{-m} \text{diag} \{d_1^T, \ldots, d_{M-1}^T\} \) and \( b_{-m} = [b_1^T, \ldots, b_{M-1}^T, b_{2M}^T, \ldots, b_{3M}^T]^T \). By replacing \( H_{-m} \) with \( \tilde{H}_{-m} \) and \( x_{-m} \) with \( b_{-m} \) in (15), we can obtain a similar form as (16). The only difference is that the new eigenvalue \( \tilde{\lambda}_i \) is nonnegative as \( a \) in (18) is nonnegative. We can thus rewrite (16) as
\[
\Pr(y|x_m) \propto \exp\left( -\frac{||y - h_m x_m||^2}{\sigma^2} \right) \\
\times \prod_{i=1}^{M-1} \left( Q\left( \sqrt{2\tilde{\lambda}_i} z_i + \frac{2}{\tilde{\lambda}_i} \tilde{g}_i(x_m) \right) \right) \\
- Q\left( -\sqrt{2\tilde{\lambda}_i} z_i + \frac{2}{\tilde{\lambda}_i} \tilde{g}_i(x_m) \right),
\]

where \( \tilde{\lambda}_i \) and \( \tilde{g}_i(x_m) \) are defined similarly as in (16) by replacing \( H_{-m} \) with \( \tilde{H}_{-m} \). This approach may be extended to other similar bit mappings resulting in constellation partitioning.
B. K-Best algorithm with Discrete Distribution Approximation

In [13], a K-best iterative MIMO receiver is proposed. The K-best algorithm offers the advantages of parallel implementation and fixed throughput regardless of the operating SNR. However, the original K-best [13] does not consider the impact of APP on the K best paths. Here, we consider a K-best algorithm which will generate different sets of K-best paths at each iteration by using a distributional approximation.

The LSD only considers the maximum term among all the $2^M$ terms in (4), and the list is generated by using $P_r(y|x_{1},...,x_{M})$ only without the \textit{a priori} information $P_r(x_m)$, $m'=1,...,M$. Moreover, when the LSD comes to the $i$-th data stream, it only checks the symbols satisfying

$$
(\bar{y}_i-R_{i,j,x_i}-\sum_{j=1+1}^{M} R_{i,j,x_j})^2 + \sum_{j=1+1}^{M} (\bar{y}_j-R_{i,x_j})^2 \leq \gamma^2,
$$

where $\bar{y}_i$ is the $i$-th entry of $Q^H y$, $R_{i,j}$ is the $(i,j)$-th entry of $R$ with QR decomposition $H=QR$, and $\bar{x}_j$ is the trial value of $x_j$. Using (20) is myopic and does not consider the effect of choosing $x_i$ on the data streams $1,...,i-1$. On the other hand, the Gaussian approximation algorithm (Section III-C) considers the summation in (4), but is not good for high-order constellations. We next develop an algorithm that unifies both approaches. The key idea is to use the Gaussian approximation or non-Gaussian approximation as a metric to guide the search, by taking into account the impact of stream $i$ on streams $1,...,i-1$.

As with LSD, we also want to find a list of $K$ lattice points. But there are three key differences from LSD. First, we try to find a list $L_{i,\pm 1}$ containing $K$ points for each $b_i=\pm 1$ rather than a common list for both $b_i$’s. The LLR value of $b_i$ in (4) is then approximated as

$$
L(b_i|x) \approx \log \frac{\sum_{x \in X_{i,\pm 1}} P_r(x|x)}{\sum_{x \in X_{i,-1}} P_r(x|x)},
$$

(21)

Second, we use sum-log rather than max-log used in LSD. The third difference lies in the way we generate the list. We want to find $K$ lattice points $x \in \mathcal{X}_{i,\pm 1}$ such that $P_r(x|x)$ is maximized rather than maximizing $P_r(x|x)$, where the \textit{a priori} information is exploited in the former case.

In the initial step of the proposed method, we assume that $b_i$ belongs to data stream $m$ and check each $\bar{x}_m \in \mathcal{X}_{i,\pm 1}$ to find the $K$ candidates such that $P_r(x_m|x)$ is maximized. Then we add $m$ into a set $\mathcal{V}$ containing the streams that have been checked. We can write $P_r(\bar{x}_m|x)$ as

$$
P_r(\bar{x}_m|x) \propto \sum_{x_m} P_r(y|x_m,\bar{x}_m) P_r(x_m).
$$

(22)

Direct computation of (22) requires $2^M$ summation, which may be computationally prohibitive. We replace the summation in (22) as an integral

$$
P_r(\bar{x}_m|x) \propto \int P_r(y|x_m,\bar{x}_m)f(x_m)dx_m,
$$

(23)

where $f(x_m)$ is the matched PDF of $x_m$, which could be either Gaussian or non-Gaussian. With Gaussian approximation (non-Gaussian approximation can be derived similarly), we have

$$
Pr(\bar{x}_m|x) \propto \exp \left(-\left(y-H_{m} \mu_m-h_m \bar{x}_m\right) \right)^H R_m^{-1} \left(y-H_{m} \mu_m-h_m \bar{x}_m\right),
$$

(24)

where $\mu_m$ and $R_m$ are defined in (8). The $K$ $\bar{x}_m$’s with the largest $Pr(\bar{x}_m|x)$ are added into a list $L$, which is initialized to be $\emptyset$.

The process then goes for $x_1,x_2,...,x_M$. Before it reaches $x_j$, $j \neq m$, we have $\mathcal{V} = \{m,1,...,j-1\}$ and the list $L$ contains $K$ candidates, each of which has the form $z_{\mathcal{V}} = [x_m,x_1,...,x_{j-1}]^T$. For each $z_{\mathcal{V}} \in L$ and $\bar{x}_j \in Q_j$, we compute $Pr(z_{\mathcal{V}},\bar{x}_j|x)$. Among the resultant $K|Q_j|$ candidates, we only choose $K$ of them such that $Pr(z_{\mathcal{V}},\bar{x}_j|x)$ is maximized, update the list $L$ with the $K$ chosen vectors, and add $j$ into $\mathcal{V}$. We can approximate $Pr(z_{\mathcal{V}},\bar{x}_j|x)$ in the same way as in (23). In case of Gaussian approximation, we have

$$
Pr(z_{\mathcal{V}},\bar{x}_j|x) \propto \exp \left(-\left(y-H_{\mathcal{V}} \mu_{\mathcal{V}}-h_{\mathcal{V}} \bar{x}_j\right) \right)^H R_{\mathcal{V}}^{-1} \left(y-H_{\mathcal{V}} \mu_{\mathcal{V}}-h_{\mathcal{V}} \bar{x}_j\right),
$$

(25)

where $\mu_{\mathcal{A}}$ constitutes the entries of $\mu$ that are not in $\mathcal{A}$, $H_{\mathcal{A}}$ consists of the columns of $H$ that are not in $\mathcal{A}$, and $R_{\mathcal{V}} = H_{\mathcal{V}} \mu_{\mathcal{V}} - h_{\mathcal{V}} \bar{x}_j$.

The process ends when $j = M$. After obtaining a list for $b_i = \pm 1$ respectively, we use (21) to compute the LLR value.

C. Extensions

The basic $K$-best algorithm (Section IV-B), named as sum-algorithm, can be extended in several ways.

\textbf{Max-Algorithm}: Different from the sum-algorithm where $Pr(x_{\mathcal{V}},\bar{x}_j|x)$ is maximized consecutively, in the max-algorithm, we maximize $Pr(x|x)$ directly. At the first step, for each $\bar{x}_m \in \mathcal{X}_{i,\pm 1}$, we find the corresponding $\bar{x}_m$ such that

$$
\bar{x}_m = \arg \max_{x_m} \Pr(\bar{x}_m, x_m | y)
$$

$$
= \arg \max_{x_m} \Pr(y | \bar{x}_m, x_m) \Pr(\bar{x}_m, x_m),
$$

(27)

where $\mathcal{X}^m$ includes all possible lattice points. We put $\bar{x}_m$ into the list $L$ such that $Pr(\bar{x}_m, \bar{x}_m | y)$ is the largest and add $m$ into a set $\mathcal{V}$. Similarly we replace $Pr(\bar{x}_m, x_m)$ with its continuous Gaussian or non-Gaussian approximations and relax the discrete set $\mathcal{X}^m$ into a continuous set $\mathcal{C}^m$.

When $\mathcal{C}^m$ is bounded, the boundary on $x_j$ is defined by the largest and the smallest elements in $Q_j$. For example, when $Q_j = \{-3,-1,3\}$, we choose $-3 \leq x_j \leq 3$. When the non-Gaussian approximation (13) is used, we need to solve

$$
\bar{x}_m = \arg \min_{x_m} \sum_{x_m} \| y - H_{m} x_m - h_{m} \bar{x}_m \|^2
$$

$$
+ 2\sigma^2 T_m x_m + 2\sigma^2 T_m A_m x_m.
$$

(28)
As (28) is quadratic in $x_m$, when the objective function of (28) is convex, $\hat{x}_m$ can be found using convex optimization tools. If not local minimum using numerical methods such as Newton’s method around

$$\arg\min_{x_m \in C^{-m}} \|y - H_m x_m - h_m \hat{x}_m\|^2.$$ 

We can set $\tilde{x}_m = \hat{x}_m$ or map $\hat{x}_m$ to the closest lattice point in $C^{-m}$. Comparing with (20), (28) uses the a priori information through $r_m$ and $A_{m}$, and it counts the impact of symbol $\tilde{x}_m$ on $Pr(\tilde{x}_m, \bar{x}_m | y)$

The process then goes to $x_1, x_2, \ldots, x_M$. Before it reaches $x_j$, $j \neq m$, $V = \{m, 1, \ldots, j - 1\}$, the list $L$ contains $K$ candidates, each of which has the form $z_V = [x_m, x_1, \ldots, x_{j-1}]^T$.

For each $z_V \in L$ and each $\bar{x}_j \in Q_j$, we find the corresponding $\tilde{x}_m$ such that

$$\tilde{x}_m = \arg\max_{x_m \in C^{-m}} Pr(z_V, \bar{x}_j, x_m \mid V) \cdot \tilde{x}_j.$$  

(29)

Among the resulting $K |Q_j| [\tilde{x}_j, \bar{x}_j]^T$, we only choose $K$ of them such that $Pr(z_V, \bar{x}_j, \tilde{x}_m \mid V)$ is maximized, update the list $L$ with the $K$ chosen vectors, and add $j$ into $V$. As in (28), we can approximate $\tilde{x}_m$ by solving

$$\tilde{x}_m = \arg\min_{x_m \in C^{-m}} \|y - H_m x_m - h_m \bar{x}_j\|^2$$

$$+ 2\sigma^2 r_{-m}^T x_m - \sigma^2 T_{-m} A_{-m} x_m - \bar{x}_j T,$$

(30)

where the notations are similar to those in (25) and (28).

Interestingly, when $C^{-m}$ is unbounded and the Gaussian approximation is used, we can show that the sum algorithm is equivalent to the max algorithm (see Appendix).

**Bit-wise Algorithm:** The sum-algorithm proceeds from symbol to symbol, which can also be applied on bits. For example, when set partitioning mapping is used, 2$^C$-PAM ($Q = \{-2^C + 1, -2^C + 3, \ldots, 2^C - 2^C + 1\}$) can be written as a weighted sum of bits using (17). Taking the sum-algorithm as an example, to compute $L(\bar{b}_i | y)$, we start with $\bar{b}_i$ and compute $Pr(b_i = \pm 1 | y) = \sum_{x_m \in \Lambda_{m, \pm 1}} Pr(x | y)$. In (23), we replace every $x_j$ except $x_m$ with a Gaussian or non-Gaussian continuous variable and $Pr(b_i = \pm 1 | y)$ is computed by summing over all possible $x_m$ in $\Lambda_{m, \pm 1}$. We can even approximate $x_m$ as a continuous variable. For example, when $x_m$ is assumed to be Gaussian, we can compute the matched mean and variance as

$$\mu_{m, i, \pm 1} = \sum_{x_m \in \Lambda_{m, \pm 1}} Pr(x_m) x_m$$

and

$$\nu_{m, i, \pm 1}^2 = \sum_{x_m \in \Lambda_{m, \pm 1}} Pr(x_m) |x_m| - |\mu_{m, i, \pm 1}|^2.$$  

(31)

(32)

When the non-Gaussian distribution is used, we can get the distribution by fitting the distribution over the symbols in $\Lambda_{m, \pm 1}$ only. We can obtain $Pr(b_i | y)$ as (24).

When the algorithm reaches bit $b_j$ and its corresponding symbol is $x_m$, let $b_j = [b_1, \ldots, b_j]^T$ where symbols $x_{m+1}, \ldots, x_{m-1}, x_{m+1}, \ldots, x_M$ have not been visited. For any $b_j$ from the list $L$, we can compute the matched mean and variance for $x_{m'}$ as

$$\mu_{m', b_j} = \sum_{x_{m'} \in \Lambda_{b_j, b_j}} Pr(x_{m'}) x_{m'}$$

and

$$\nu_{m', b_j}^2 = \sum_{x_{m'} \in \Lambda_{b_j, b_j}} Pr(x_{m'}) |x_{m'}|^2 - |\mu_{m', b_j}|^2.$$  

(33)

(34)

where $\Lambda_{b_j, b_j}$ is the set of constellation points for $x_{m'}$ such that the corresponding bits in $b_j$ is equal to $b_j$. The rest of the algorithm is identical to that of the symbol based algorithm.

The advantage of the bit-wise algorithm is that some symbols can be pruned early when the first few bits of the corresponding symbols are not chosen in the list with $K$ elements.

**Early Stopping and Varied $K$:** The sum-algorithm stops after reaching $x_M$. We can stop the algorithm at any $x_j$. In this case, we can compute the LLR value as (35).

We can then approximate

$$\sum_{x_j+1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_M} Pr(y | x)$$

using Gaussian or non-Gaussian approximation. When $j = m$, $K = |Q_m|$ and Gaussian approximation are used, the $K$-best algorithm with early stopping reduces to that in Section III-C or the PDA method in [16]. The stopping level gives a tradeoff between performance and complexity. This early stopping can also be used when some symbols are not reliable, e.g., every symbol in the constellation has roughly the same probability or LLR value is less than a threshold. In this case, different candidates may have roughly the same metric and choosing the best $K$ candidates may not be good. We can reorder the symbols such that the unreliable symbols correspond to the last few symbols and use early stopping when the algorithm reaches the unreliable symbols.

We can also vary the list size $K$ for different symbols. The list size $K_j$ can be chosen as $K_j$ after symbol $x_j$ is visited. For example, $K_j$ can be chosen to be a large value for the first few visited symbols as the choice of these symbols is important to the overall performance, and $K_j$ is chosen to be a small value when the algorithm is close to the end to save complexity.

**Remarks:**

- Note that the proposed algorithms are different from the $K$ best algorithm in [13] in the way how the $K$ best candidates are generated and updated and how the LLR value is computed. In [13], LLR value is computed using only the candidate from the list with the largest $Pr(x | y)$ while we consider all the candidates in the list.
- There are several interesting special cases of the proposed algorithms. When $K = 1$, the proposed algorithms become an improved soft version of V-BLAST algorithm. When $K = +\infty$, they reduce to the optimal MAP detection in Section III-B.
- The proposed $K$-best sum-algorithm can be considered as a combination of soft successive interference cancellation (SIC) and hard SIC, where a stream of data is canceled...
by hard decision on the bits of a symbol followed by re-
modulation of the hard decoded bits. The sum-algorithm uses soft SIC when choosing candidates from the list and uses hard SIC before it proceeds to the next symbol.

- Instead of using sum-log to compute the LLR value, we can also use max-log.
- In practical protocols there always exists some CRC check bits. When a particular data stream passes the CRC check, we do not need to include this data stream in the future iterative demodulation and decoding. We can cancel this data stream directly or hard SIC, which will reduce the complexity of the proposed K-best algorithms.

D. Complexity Reduction

Direct computation of (25) requires matrix inversion and matrix multiplication for every \( \hat{x}_j \in Q_j \). From the expression of \( R_{\{V,j\}} \) in (26) and the matrix inversion lemma [29], we have

\[
R_{\{V,j\}}^{-1} = (R_V + \nu_j^2 h_j h_j^H)^{-1} = R_V^{-1} - g_j (\nu_j^{-2} + h_j h_j^H) g_j^H
\]

(36)

where \( g_j = R_V^{-1} h_j \). Initially, we need to compute

\[
(H \text{diag}(\nu^2) H^H + \sigma^2 I_N)^{-1},
\]

which has a complexity \( O(N^2 370 + NM^2) \). Substituting (36) into (25), we obtain

\[
(y - H_\gamma \{V,j\} \mu_{\{V,j\}} - H_y x_V - h_j \hat{x}_j) H \cdot [V,j]^{-1}
\]

\[
= C - 2 \hat{x}_j A + \tilde{x}_j^2 A
\]

(37)

where

\[
A = h_j h_j^H g_j - h_j h_j^H (\nu_j^{-2} + h_j h_j^H) g_j^H h_j
\]

\[
B = \left(1 - h_j h_j^H (\nu_j^{-2} + h_j h_j^H) g_j^H\right) g_j^H
\]

\[
\times (y - H_\gamma \{V,j\} \mu_{\{V,j\}} - H_y x_V - h_j \hat{x}_j)
\]

\[
C = y^H R_V^{-1} y - y^H \left(g_j (\nu_j^{-2} + h_j h_j^H) g_j^H\right)^{-1} g_j^H y
\]

(38)

Computing \( h_j h_j^H \) and \( g_j^H y \) needs \( 2(N-1) \) additions and \( 2N \) multiplications. Moreover, \( y^H R_V^{-1} y \) and \( y - H_\gamma \{V,j\} \mu_{\{V,j\}} - H_y x_V \) are inherited from the previous step. \( H_y x_V \) is updated and stored in the list and the update needs \( KN \) multiplications and \( KN \) additions. Computing \( y - H_\gamma \mu_{\{V,j\}} - h_j \mu_j - H_y x_V \) needs \( N \) multiplications and \( 2N \) additions. The total number of additions to compute the coefficients \( A, B, C \) for all the elements in the list is \( 3(K+1)N + K - 2 \) and the total number of multiplications is \( (2K+3)N + 5 \). As (37) is a scalar function in \( \hat{x}_j \), we can search over \( Q_j \) for each \( x_V \) to find the \( K \) candidates with the maximum (25). This simple algorithm requires \( 2K|Q_j| \) multiplications and \( 2K|Q_j| \) additions.

Assuming exponential function can be computed by lookup table, Table I compares the complexity of the proposed K-Best algorithm, Gaussian approximation and solving (5) optimally denoted as MAP (maximum a posteriori) in a \( N \times M \) MIMO system with constellation size \( Q \), where we have assumed matrix inversion uses Gauss-Jordan elimination [29]. From the big-O notation, we can see that Gaussian approximation reduces the complexity from \( O(Q^3 NM) \) exponential in \( M \) to \( O(QNM^2) \). K-best algorithm can further reduce the complexity to \( O(KNM) \).

V. Simulation Results

In this section, we present simulation results to verify the effectiveness of the proposed algorithms. We consider a MIMO-OFDM system with 1024 subcarriers and 960 subcarriers are used for data transmission. Perfect knowledge of channel state information is assumed. Each transmit antenna is assigned power \( P \). The SNR is defined as \( P/N_0 \), where \( N_0 \) is the noise power. The simulations are conducted using an LTE simulator with the encoder, interleaver, rate matching procedure following LTE standard specified in [30]. We consider Extended Vehicular A model (EVA) [31] in this section with delay profile \([0 \ 30 \ 150 \ 310 \ 370 \ 710 \ 1090 \ 1730 \ 2510] \) ns and power profile \([0 \ -1.5 \ -1.4 \ -3.6 \ -0.6 \ -9.1 \ -7 \ -12 \ -16.9] \) dB. The channel power profile is normalized to unity. The scheme of turbo encoder is a Parallel Concatenated Convolutional Code (PCCC) with two 8-state constituent encoders and one turbo code internal interleaver. The coding rate of turbo encoder is 1/3. The transfer function of the PCCC is: \( G(D) = 1 + g_0(D) \) [30], where \( g_0(D) = 1 + D^2 + D^3 \). Eight iterations are performed within the turbo decoder. Finally, 64 QAM and Gray mapping are considered in this section.

The algorithm using (5) is denoted as MAP. LSD in [12] is denoted as LSD (List sphere decoder). The Gaussian approximation using (8) is denoted as Gaussian. Estimating \( \hat{x}_i \) from (2) using a minimum mean square error (MMSE) estimator followed by a turbo decoder without iterative demodulation and decoding is denoted as MMSE. In LSD, we choose the list size \( L = 512 \) to be consistent with [12]. The K-best algorithm using common list is denoted as K-best, while the K-Best algorithm using (25) for each bit is denoted as K-Best-Bit. The non-Gaussian approximation algorithm in Section IV-A is denoted as Non-Gaussian.

A. Bit Error Rate (BER) Comparison of Different Algorithms

We first consider fixed scheduling, where both data streams transmit using transport block size (TBS) 1916. The bit error rates of different algorithms after simulating 20000 subframes in a \( 2 \times 2 \) MIMO-OFDM system are shown in Fig. 4. The channel varies independently from subframe to subframe. All algorithms use 6 iterations. It is clear that all iterative algorithms benefit from the information exchange between the demapper and decoder as compared with MMSE. We can see
that Gaussian approximation incurs a 0.5 dB loss over MAP at BER = 10^{-3}. The Gaussian approximation only needs to sum over 64 terms while the MAP needs to compute 32 × 64 = 2048 terms in the numerator and denominator in (5), respectively. With the proposed K-best algorithms, K-Best, K = 64 has a 0.08 dB gain over Gaussian approximation, and K-Best-Bit, K = 4 has a 0.15 dB gain over Gaussian approximation at BER = 10^{-3}. K-Best-Bit, K = 4 only needs to sum over K = 4 terms in the LLR computation but with improved performance over Gaussian approximation. LSD with L = 512 incurs a 1 dB loss over MAP at BER = 10^{-3} but with a higher complexity than Gaussian approximation and the proposed K-best algorithms. The non-Gaussian approximation achieves a 0.3 dB gain over Gaussian approximation at BER = 10^{-3}. The proposed K-Best algorithms achieve good performance with reduced complexity.

B. Effects of the List Size K

In Fig. 5, the performance of the K-best algorithms are compared with different list size K in a 2 × 2 MIMO-OFDM system. We can see that for both K-Best and K-Best-Bit, doubling K gives a 0.5 dB gain at BER = 10^{-3}. In high SNR, the error floor is also reduced by increasing K. But doubling K also means that the complexity is roughly doubled. Moreover, more exponentials are needed to compute in the LLR computation, which is very expensive in hardware implementation.

C. Effects of the Number of Iterations

In practice, it is crucial that the decoding time satisfies the timeline constraint in LTE-A [1]. It is therefore important to understand the performance of the iterative decoders in terms of the number of iterations. Fig. 6 shows the BER performance of the proposed K-best algorithm with K = 64 and different number of iterations in a 2 × 2 MIMO-OFDM system (Fig. 6(a)) and in a 4 × 4 MIMO-OFDM system (Fig. 6(b)). We can see that marginal performance gain diminishes as the number of iterations increases. When comparing SNR = 8 dB in Fig. 6(a) with SNR = 6 dB in Fig. 6(b), we find that convergence speed of the proposed K-best algorithm decreases as the number of antennas increases. In general, three to four iterations is good enough in a 2 × 2 MIMO-OFDM system to achieve close to optimal performance using the proposed K-best algorithms, which means the proposed algorithms can be implemented with fixed complexity and fixed detection throughput.

D. Throughput Comparison of Different Algorithms

BER does not translate to throughput performance directly. A higher TBS may have higher BER even though data rate is higher. In Fig. 7, we compare the throughput gain of different algorithms over Gaussian approximation in a 2 × 2 MIMO-OFDM system. The throughput gain is obtained after averaging 50 channel realizations. For each channel realization, we find the maximum TBS pair on the two streams that can achieve average 10% block error rate (BLER) on both data

<table>
<thead>
<tr>
<th>MAP</th>
<th>Gaussian</th>
<th>K-Best</th>
<th>K-Best-Bit</th>
<th>LSD, L=512</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q^{ST}N(M+1) − 2</td>
<td>Q^{ST}(N(M+1)+Q−1)</td>
<td>M(3(K+1)N+2KQ+K−2) + 2KN(M+1) − 2</td>
<td>M((2K+3)N+2KQ+5) + 2K(N(M+1)+Q+1)</td>
<td>O(KNM)</td>
</tr>
</tbody>
</table>

**Table I** Complexity comparison of different algorithms to compute L(b_i|y).

![Fig. 4](image1.png) BER comparison of different algorithms in a MIMO-OFDM system with 2 transmit and 2 receive antennas over the EVA channel.

![Fig. 5](image2.png) K-best algorithm BER comparison with list size K in a MIMO-OFDM system with 2 transmit and 2 receive antennas over the EVA channel.
The number of iterations
BER

(a) A $2 \times 2$ MIMO-OFDM system

(b) A $4 \times 4$ MIMO-OFDM system

Fig. 6. BER comparison of $K$-best algorithm with $K=64$ and different number of iterations

Fig. 7. Throughput gain comparison of different algorithms over MMSE in a MIMO-OFDM system with 2 transmit and 2 receive antennas over the EV A channel.

VI. CONCLUSION

The design of low complexity MIMO-OFDM LTE-A receivers to meet the data rate requirement while achieving power consumption and silicon area savings is critically important. In this paper, we developed several such low-complexity iterative detection and decoding algorithms. Non-Gaussian approximation was proposed to enhance the performance of interference cancellation based detectors with large constellations. Several modified $K$-best algorithms were also developed to take advantages of both Gaussian or non-Gaussian approximation and the list decoder, which provide a flexible performance and complexity tradeoff. Simulation results demonstrated that the proposed low complexity algorithms can achieve a performance gain over existing ones for practical systems that use high-order constellations.

APPENDIX

THE EQUIVALENCE BETWEEN THE SUM AND MAX ALGORITHMS

In this appendix, we consider a general system

$$y = Hx + w,$$

where $x$ is of mean $\mu$ and covariance matrix $R_x$ and $w$ is of mean zero and covariance matrix $R_w$. We show that by using Gaussian approximation the sum algorithm (25) is equivalent to the max algorithm. To do so, we solve

$$\min_{\hat{x} \in C} \|y - H\hat{x}\|_2^2 + (\hat{x} - \mu)^H \Lambda (\hat{x} - \mu),$$

where $\Lambda = \text{diag} \{\nu_1^2, \nu_2^2, \ldots, \nu_M^2\}$. Define

$$f(y) = \exp \left( - (y - H\mu)^H (HR_xH^H + R_w)^{-1} (y - H\mu) \right)$$

(41)
which corresponds to (25). Define
\[
g(y) = \max_x \exp \left( -\|y - Hx\|^2 - (x - \mu)^H R_x^{-1} (x - \mu) \right)
\]
\[\propto \exp \left( -y^H R_w y + (H^H R_w^{-1} y + R_x^{-1} \mu)^H \right)
\times \left( R_w^{-1} + \mu R_x^{-1} \mu \right)^{-1} \left( H^H R_w^{-1} y + R_x^{-1} \mu \right),
\]
which corresponds to (40). From the definition of LLR value, we want to show that \( f(y) \) is proportional to \( g(y) \), i.e., \( f(y) \propto g(y) \).

By using the matrix inversion lemma [29], we have
\[
R_w - R_w^{-1} H (R_x^{-1} + H^H R_w^{-1} H)^{-1} H^H R_w^{-1} = (H R_x H^H + R_w)^{-1},
\]
which means that
\[
y^H R_w y - y^H R_w H (R_x^{-1} + H^H R_w^{-1} H)^{-1} H^H R_w^{-1} y
= y^H (H R_x H^H + R_w)^{-1} y.
\]
We also have
\[
\mu^H R_x^{-1} (R_x^{-1} + H^H R_w^{-1} H)^{-1} H^H R_w^{-1} y
= \mu^H R_x^{-1} \left( R_w - R_x H^H (H R_x H^H + R_w)^{-1} H R_x \right) H^H R_w^{-1} y
= \mu^H (I_N - H^H (H R_x H^H + R_w)^{-1} H R_x) H^H R_w^{-1} y
= \mu^H H (I_N - (H R_x H^H + R_w)^{-1} H R_x H^H) R_w^{-1} y
= \mu^H H (H R_x H^H + R_w)^{-1} y.
\]
Substituting (44) and (45) into (42) and comparing with (41), it can be readily seen that \( f(y) \propto g(y) \). Therefore, the sum algorithm (25) is equivalent to the max algorithm (40) under the Gaussian approximation.

REFERENCES
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