

Calibration to American Options: Numerical Investigation of the de–Americanization Method

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Abstract

American options are the reference instruments for the model calibration of a large and important class of single stocks. For this task, a fast and accurate pricing algorithm is indispensable. The literature mainly discusses pricing methods for American options that are based on Monte Carlo, tree and partial differential equation methods. We present an alternative approach that has become popular under the name *de–Americanization* in the financial industry. The method is easy to implement and enjoys fast run-times (compared to a direct calibration to American options). Since it is based on ad hoc simplifications, however, theoretical results guaranteeing reliability are not available. To quantify the resulting *methodological risk*, we empirically test the performance of the de–Americanization method for calibration. We classify the scenarios in which de–Americanization performs very well. However, we also identify the cases where de–Americanization oversimplifies and can result in large errors.

Keywords: American options, model calibration, binomial tree model, CEV model, Heston model, Lévy models, model reduction, variational inequalities

JEL classification: C51, C63

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1 Introduction

The most frequently traded single stock options are of American type. In general, there exist a variety of (semi-)closed pricing formulas for European options. However, for American options, there hardly exist any closed pricing formulas, and the pricing under advanced models relies on computationally expensive numerical techniques such as Monte Carlo simulation or partial (integro-) differential methods.

To tackle this core problem the financial industry has adopted the so-called de-Americanization approach which has become a market standard: American option prices are transferred into European prices before the calibration process itself is started. This is usually done by applying a relatively simple binomial tree. By replacing American options with European options, the complexity of the calibration problem is reduced and the computational costs are lowered significantly. The striking advantage of this procedure is that it enables to employ the advanced and standard tools for model calibration to European option data which are readily available and typically efficient. Figure 1 illustrates the scheme of the de-Americanization methodology.

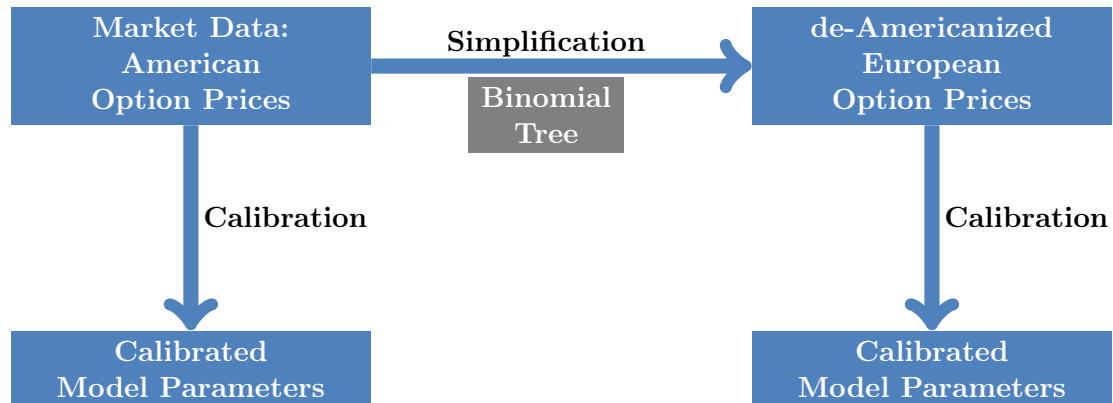


Figure 1 De-Americanization scheme: American option prices are transferred into European prices before the calibration process itself is started. We investigate the effects of de-Americanization by comparing the results to directly calibrating American options.

The de-Americanization methodology enjoys three attractive features,

- it delivers fast run-times compared to a direct calibration to American options,
- it is easy to implement,
- it can flexibly be integrated into the pricing and calibration toolbox at hand.

1 Introduction

One downside is that no theoretical error control is available. Therefore, it is important to empirically investigate the accuracy, the performance and the resulting methodological risk of the method.

The method is briefly mentioned by Carr and Wu (2010), who describe how their implied volatility data, stemming from the provider OptionMetrics, is obtained by applying exactly this de-Americanization scheme. To the best of our knowledge, the de-Americanization methodology has not been investigated deeply in the literature. We therefore devote the current paper to this task. In order to conduct a thorough investigation, we consider prominent models and identify relevant scenarios in which to perform extensive numerical tests. We focus on options on non-dividend-paying underlyings and explore the CEV model as an example of a local volatility model, the Heston model as a stochastic volatility model and the Merton model as a jump diffusion model. For all of these models, we implemented finite element solvers as benchmark method for pricing American options. The following questions serve as guidelines to specify decisive parameter settings within our studies.

1. Since American and European puts on non-dividend-paying underlyings coincide for zero interest rates, we analyze in particular the methodology for different interest rates.
2. Intuitively, with higher maturities, the early exercise feature of American options becomes more valuable and American and European option prices differ more significantly. Therefore, we investigate the following question: Does the accuracy of the de-Americanization methodology depend on the maturity and do de-Americanization errors increase with increasing maturities?
3. In-the-money and out-of-the-money options play different roles. First, out-of-the-money options are preferred by practitioners for calibration since they are more liquidly traded, see for instance Carr and Wu (2010). Second, in-the-money options are more likely to be exercised. How does the de-Americanization methodology perform for out-of-the-money options and for in-the-money options?
4. The difference between American and European options is model-dependent. Intuitively, (higher) jump intensities lead to higher values of early exercise features. How does the de-Americanization methodology perform for continuous models (CEV model and Heston model)? How does it perform for different jump intensities (Merton model)?

Our investigation is organized as follows. First, we introduce the de-Americanization methodology in Section 2. Then we briefly describe in Section 3 the models and the benchmark pricing methodology. Section 4 presents the numerical results: The accuracy of the calibration procedure obviously hinges on the accuracy of the underlying pricing routine. We therefore first specify the de-Americanization pricing routine and investigate its accuracy. Afterwards, we present the results of calibration to both synthetic data and market data. To conclude the numerical study, we present the effects of different

calibration results on the pricing of exotic options. We summarize our findings in Section 5.

1.1 Short literature overview on American options

For an overview of pricing American options, we refer to Barone-Adesi (2005). The problem of pricing an American put traces back to Samuelson (1965) and McKean (1965). Brennan and Schwartz (1977) were among the first to provide numerical solutions. Additionally, the binomial tree model of Cox et al. (1979) was developed to price American options. Broadie and Detemple (1996) described an approximation of the American put price by interpolating between an upper and lower bound. Longstaff and Schwartz (2001) combined American option pricing with Monte-Carlo techniques based on a polynomial interpolation of the continuation value. The problem of pricing American options can also be interpreted as a free boundary problem, see e.g. Kim (1990), or as an optimal stopping problem, see e.g. Peskir and Shiryaev (2006), and be formulated as a dynamic programming principle. Although Barone-Adesi (2005) concludes that the mainstream computational problems have been solved satisfactorily, by switching the focus on calibration, there are rather recent developments for calibrating American options. As examples, we state Haring and Hochreiter (2015), who apply a specific search algorithm in the calibration process, namely a Cuckoo search algorithm, and Ballestra and Cecere (2016). They provide a method to forecast the parameters of the constant elasticity of variance (CEV) model implied by American options in order to fit the model relatively quickly to market data. To summarize, calibrating American market data is a numerically challenging problem. The research in the literature puts the focus now on optimizing the calibration procedure to reduce the run-time. At its core, still path-dependent, rather complex, American options are priced.

2 De-Americanization methodology

In this section, we give a precise and detailed description of the methodology. The de-Americanization methodology is used to fit models to market data. The core idea of de-Americanization is to transfer the available American option data into pseudo-European option prices prior to calibration. This significantly reduces the computational time as well as the complexity of the required pricing technique. Basically, de-Americanization can be split into three parts. The first part consists in collecting the available market data. The currently observable price of the underlying S_0 , interest rate r and the available American option prices are collected. In the following, we will denote the American option price of the i -th observed option by V_A^i . We interpret the market data as the true option prices, thus we assume that the observed market prices V_A^i can be interpreted as $V_A^i = \sup_{t \in [0, T_i]} E[e^{-rt} \tilde{\mathcal{H}}_i(S_t) | \mathcal{F}_0]$, $i = 1, \dots, N$, where $\tilde{\mathcal{H}}_i$ is the i -th payoff function, T_i the maturity of the i -th option, and the expectations are taken under

a risk-neutral measure, \mathcal{F} is the natural filtration, and N denotes the total number of options. Up to this point, no approximation has been used.

The second step is the application of the binomial tree to create pseudo-European – so called de-Americanized – prices based on the observed American market data. In this step, we look at each American option individually and find the price of the corresponding European option with the same strike and maturity. This European option is found by fitting a binomial tree to the American option. The binomial tree was introduced by Cox et al. (1979) as follows. Starting at S_0 , at each time step and at each node, the underlying can either go up by a factor of u or down by a factor of $\frac{1}{u}$ and the risk-neutral probability of an upward movement is given by

$$p = \frac{e^{r\Delta t} - \frac{1}{u}}{u - \frac{1}{u}}. \quad (1)$$

Once the tree is set up, options can be valued by going backwards from each final node. Thus, path-dependent options can be evaluated easily. Since for each option i the American option price V_A^i is known, as well as S_0 and r , the only unknown parameter of the tree is the upward factor u . At this step, the upward factor u_i^* is determined such that the price of the American option in the binomial tree matches the observed market price. Thus, denoting $\{0 : \Delta t : T_i\} = \{0, \Delta t, 2\Delta t, \dots, T_i\}$, we have $\sup_{t \in \{0 : \Delta t : T_i\}} E[e^{-rt} \tilde{\mathcal{H}}_i(S_t^{u_i^*}) | \mathcal{F}_0] = V_A^i$, where $S_t^{u_i^*}$ denotes the underlying process described by a binomial tree with upward factor u_i^* . The early exercise feature of American options is reflected in the fact that the supremum is taken over all discrete time steps. A detailed description of pricing American options in a binomial tree model is given in Van der Hoek and Elliott (2006). Once $S_t^{u_i^*}$ is determined, the corresponding European option with the same strike and maturity as the American option is specified, $V_E^i = E[e^{-rT_i} \tilde{\mathcal{H}}_i(S_{T_i}^{u_i^*}) | \mathcal{F}_0]$. Note that fixing u_i^* also implicitly determines the implied volatility.

Then, for each American option V_A^i , a corresponding European option V_E^i has been found, and the actual model calibration can start. The goal is to fit a model M , depending on parameters $\mu \in \mathbb{R}^d$, where d denotes the number of parameters in the model, to the European option prices V_E^i , $i = 1, \dots, N$. Denote by $S_{T_i}^{M(\mu)}$ the underlying process in model M with parameters $\mu \in \mathbb{R}^d$. In the calibration, the parameter vector μ is determined by minimizing the objective function of the calibration. Algorithm 1 summarizes the de-Americanization methodology in detail.

Regarding the uniqueness of the factor u_i^* in the De-Americanization methodology described in Algorithm 1, we will first investigate the case of a European put option. Therefore, we interpret the risk-neutral probability in (1) as function of u , $p(u) = \frac{ue^{r\Delta t} - 1}{u^2 - 1}$. At each node in the binomial tree we have a two-point distribution, that we call Bernoulli distribution $X \sim QB(u)$, where the value u is taken with probability $p(u)$ and the value $\frac{1}{u}$ is taken with probability $(1 - p(u))$.

Algorithm 1 De-Americanization methodology

- 1: **procedure** COLLECTION OF OBSERVABLE DATA
 - 2: $S_0, r,$
 - 3: $V_A^i = \sup_{t \in [0, T_i]} E[e^{-rt} \tilde{\mathcal{H}}_i(S_t) | \mathcal{F}_0], \quad i = 1, \dots, N$
 - 4: **procedure** APPLICATION OF THE BINOMIAL TREE TO EACH OPTION INDIVIDUALLY
 - 5: **for** $i = 1 : N$
 - 6: Find u_i^* such that
 - 7: $\sup_{t \in \{0: \Delta t: T_i\}} E[e^{-rt} \tilde{\mathcal{H}}_i(S_t^{u_i^*}) | \mathcal{F}_0] = V_A^i$ where the supremum is taken over all stopping times t
 - 8: Derive the corresponding European option price with u_i^*
 - 9: $V_E^i = E[e^{-rT_i} \tilde{\mathcal{H}}_i(S_{T_i}^{u_i^*}) | \mathcal{F}_0]$
 - 10: **end**
 - 11: **procedure** CALIBRATION TO EUROPEAN OPTIONS
 - 12: Find μ such that the differences
 - 13: $E[e^{-rT_i} \tilde{\mathcal{H}}_i(S_{T_i}^{M(\mu)})] - V_E^i, \quad i = 1, \dots, N$
 - 14: are minimized according to the objective function
-

Proposition 2.1

For $i = 1, \dots, n$, let $X_i \sim QB(u)$ and $Y_i \sim QB(u')$. If $u \leq u'$, and $u, u' > e^{r\Delta t}$ then for any $K \in \mathbb{R}$

$$E \left[\left(K - \prod_{i=1}^n X_i \right)^+ \right] \leq E \left[\left(K - \prod_{i=1}^n Y_i \right)^+ \right].$$

Remark 2.2

In the implementation of the tree, we set the time step size $\Delta t \approx 0.0002$ and we use a simple bi-section approach as suggested by Van der Hoek and Elliott (2006) to find u^* . Thus, given a market price V_A , starting with an upper bound u_{ub} and a lower bound u_{lb} satisfying the conditions in Proposition 2.1 such that,

$$\begin{aligned} \sup_{t \in \{0: \Delta t: T_i\}} E[e^{-rt} \tilde{\mathcal{H}}_i(S_t^{u_{ub}}) | \mathcal{F}_0] &> V_A, \\ \sup_{t \in \{0: \Delta t: T_i\}} E[e^{-rt} \tilde{\mathcal{H}}_i(S_t^{u_{lb}}) | \mathcal{F}_0] &< V_A, \end{aligned}$$

the bi-section approach is started and the new candidate for u^* is $\hat{u} = \frac{u_{ub} + u_{lb}}{2}$. When $\sup_{t \in \{0: \Delta t: T_i\}} E[e^{-rt} \tilde{\mathcal{H}}_i(S_t^{\hat{u}}) | \mathcal{F}_0] > V_A$, we set $u_{ub} = \hat{u}$ for the next iteration, otherwise $u_{lb} = \hat{u}$. As stopping criterion, we choose

$$\left| \sup_{t \in \{0: \Delta t: T_i\}} E[e^{-rt} \tilde{\mathcal{H}}_i(S_t^{\hat{u}}) | \mathcal{F}_0] - V_A \right| \leq \varepsilon,$$

and set $\varepsilon = 10^{-5}$ in our implementation. In Proposition 2.1 we have investigated the European put case and can deduce from the convex ordering that the put prices are monotonically increasing in u . For a strict order, the u^* -value is thus uniquely determined.

3 Pricing Methodology

In our case, the u^ -value can be determined uniquely as minimum of all u values satisfying the stopping criterion. Moreover, this indicates that also the American put price in the binomial tree is increasing with increasing u . We validated this by numerical tests (not reported). This is in line with the recommendation in Van der Hoek and Elliott (2006). The only observed limitation is that the American put price can not given by an immediate exercise at the initial time. This is explained in detail in Remark 4.1.*

3 Pricing Methodology

In this section, we present a model formulation and numerical implementation of the three investigated models (CEV, Heston, Merton). To investigate the de-Americanization methodology, we need to price the American and European options. Our market data in the numerical study later on will be based on options on the Google stock (Ticker: GOOG). As Google does not pay dividends, we neglect dividend payments in our pricing methodology. Without dividend payments, for $r > 0$, it holds in general that American calls coincide with European calls and only American puts have to be treated differently. The opposite is true for $r < 0$, in which case American and European puts coincide and American and European calls have to be treated differently.

In general, for European options, there exists a variety of fast pricing methodologies such as Fast Fourier Transform (Carr and Madan (1999); Raible (2000)) or even closed-form solutions. The common approaches for pricing American options are P(I)DE methods using either the finite difference method (FDM) or a finite element method (FEM). We choose FEM since it is typically more flexible. For an introduction of FEM in finance, we refer to Achdou and Pironneau (2005), Achdou and Pironneau (2007) and Seydel (2012). To solve the resulting variational inequalities for American options, we use the Projected SOR Algorithm, Achdou and Pironneau (2005), Seydel (2012), for the CEV and Merton models, and the Primal Dual Active Set Strategy, Hintermüller et al. (2002), for the Heston model.

3.1 Option Pricing Models

We briefly present the models that we use for our study, namely the constant elasticity of variance model (CEV), the stochastic volatility Heston model, and the Merton model.

In all three of the models, the asset price dynamics S_τ are governed by a stochastic differential equation (SDE) of the form

$$dS_\tau = rS_\tau d\tau + \sigma(S, \tau)S_\tau dW_\tau + S_{\tau-} dJ_\tau, \quad S_0 = s \geq 0, \quad (2a)$$

$$J_\tau = \sum_{i=0}^{N_\tau} Y_i, \quad (2b)$$

3.2 Pricing P(I)DE

with W_τ a standard Wiener process, r the risk-free interest rate and a volatility function $\sigma(S, \tau)$. The jump part $(J_\tau)_{\tau \geq 0}$ is a compound Poisson process with intensity $\lambda \geq 0$ and independent identically distributed jumps Y_i , $i \in \mathbb{N}$, that are independent of the Poisson process $(N_\tau)_{\tau \geq 0}$. The Poisson process and the Wiener process are also independent.

As an example of a local volatility model, we begin by presenting the CEV model, which was introduced by Cox (1975). Here, the local volatility is assumed to be a deterministic function of the asset price for the process in (2), $\sigma(S, \tau) = \sigma S_\tau^\zeta$, $0 < \zeta < 1$, $\sigma > 0$ and $\lambda = 0$. In this paper, we use the original example of the CEV model by Cox (1975) in which the parameter constraint $0 < \zeta < 1$ is designed to capture the leverage effect. This effect describes the behaviour of an increasing volatility when the stock price decreases.

As an example of a stochastic volatility model, we use the model proposed by Heston (1993). In contrast to the CEV model, the stochastic volatility is driven by a second Brownian motion \widetilde{W}_τ whose correlation with W_τ is described by a correlation parameter $\rho \in [-1, 1]$, and the model is based on the dynamics of both the stock price (2), with jump intensity $\lambda = 0$, and the variance v_τ (3),

$$dv_\tau = \kappa(\gamma - v_\tau)dt + \xi\sqrt{v_\tau}d\widetilde{W}_\tau, \quad (3)$$

with $\sigma(S, \tau) = \sqrt{v_\tau}$, mean variance $\gamma > 0$, rate of mean reversion $\kappa > 0$ and volatility of volatility $\xi > 0$. Jumps are not included in either of the CEV or Heston models.

The Merton model includes jumps. The log-asset price process is not exclusively driven by a Brownian motion, but instead follows a jump-diffusion process. Thus, in the model of Merton (1976), the volatility of the asset process is still assumed to be constant, $\sigma(S, \tau) \equiv \sigma > 0$, $\forall S > 0, \forall \tau \geq 0$. But being a jump diffusion model, the jump intensity $\lambda > 0$ is positive and $N_t \sim \text{Poiss}(\lambda t)$. The jumps are taken to be independent normally distributed random variables, $Y_i \sim \mathcal{N}(\alpha, \beta^2)$ with expected jump size $\alpha \in \mathbb{R}$ and standard deviation $\beta > 0$.

3.2 Pricing P(I)DE

Denote by $t = T - \tau$ the time to maturity T , $T < \infty$ and by K the strike of an option. For the CEV model, we stay with the S variable, $S \in (0, \infty)$, for the Heston and Merton model we work with the log-transformed stock variable $x := \log\left(\frac{S}{K}\right)$, $x \in (-\infty, \infty)$. In the following, we will denote an American or European call or put price by $P_{call/put}^{Am/Eu}$. For the CEV model we have $P_{call/put}^{Am/Eu} : (0, T) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and for the Heston and Merton model we have $P_{call/put}^{Am/Eu} : (0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ ($n = 1$ (Merton), $n = 2$ (Heston)). The value of an option at $t = 0$ is given by the payoff function $\widetilde{\mathcal{H}}_{call/put}(\cdot)$, $P_{call/put}(0) = P_0 = \widetilde{\mathcal{H}}_{call/put}$ with $\widetilde{\mathcal{H}}_{call}(S) := (S - K)^+$ or $\widetilde{\mathcal{H}}_{put}(S) := (K - S)^+$ in the CEV model and $\widetilde{\mathcal{H}}_{call}(x) := (Ke^x - K)^+$, $(\widetilde{\mathcal{H}}_{put}(x) := (K - Ke^x)^+)$ in the Heston and Merton models.

3.2 Pricing $P(I)DE$

Then, to find the value of the European option $P_{call/put}^{Eu}$, paying $P_0^{call/put} = \tilde{\mathcal{H}}_{call/put}(S)$ ($P_0^{call/put} = \tilde{\mathcal{H}}_{call/put}(x)$) at $t = 0$ leads to solve the following initial boundary value problem

$$\frac{\partial P_{call/put}^{Eu}}{\partial t} - \mathcal{L}^s P_{call/put}^{Eu} = 0, \quad P_{call/put}^{Eu}(0) = P_0^{call/put}, \quad (4)$$

where the spatial partial (integro) differential operator \mathcal{L}^s , $s = \{\text{CEV}, \text{H}, \text{M}\}$ is determined by the model used to price the option. For the CEV, Heston and Merton, it is given by (5a), (5b) and (5c), respectively.

$$\mathcal{L}^{\text{CEV}} P_{call/put}^{Am/Eu} := \frac{\sigma S_t^{\zeta-1}}{2} S^2 \frac{\partial^2 P_{call/put}^{Am/Eu}}{\partial S^2} + r S \frac{\partial P_{call/put}^{Am/Eu}}{\partial S} - r P_{call/put}^{Am/Eu}, \quad (5a)$$

$$\begin{aligned} \mathcal{L}^{\text{H}} P_{call/put}^{Am/Eu} := & \frac{1}{2} v \frac{\partial^2 P_{call/put}^{Am/Eu}}{\partial x^2} + \xi v \rho \frac{\partial^2 P_{call/put}^{Am/Eu}}{\partial v \partial x} + \frac{1}{2} \xi^2 v \frac{\partial^2 P_{call/put}^{Am/Eu}}{\partial v^2} \\ & + \kappa(\gamma - v) \frac{\partial P_{call/put}^{Am/Eu}}{\partial v} + \left(r - \frac{1}{2}v\right) \frac{\partial P_{call/put}^{Am/Eu}}{\partial x} - r P_{call/put}^{Am/Eu}, \end{aligned} \quad (5b)$$

$$\begin{aligned} \mathcal{L}^{\text{M}} P_{call/put}^{Am/Eu} := & b \frac{P_{call/put}^{Am/Eu}}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 P_{call/put}^{Am/Eu}}{\partial x^2} P_{call/put}^{Am/Eu} \\ & + \int_{\mathbb{R}} (P_{call/put}^{Am/Eu}(x+z) - P_{call/put}^{Am/Eu}(x) - \frac{P_{call/put}^{Am/Eu}(x)}{\partial x} z) F(dz) - r P_{call/put}^{Am/Eu}, \end{aligned} \quad (5c)$$

where, for the Merton model, the jump measure F is given by

$$F(dz) = \frac{\lambda}{\sqrt{2\pi\beta^2}} \exp\left(-\frac{(z-\alpha)^2}{2\beta^2}\right) dz \quad (6)$$

and the drift $b \in \mathbb{R}$ is set to $b := r - \frac{1}{2}\sigma^2 - \lambda \left(e^{\alpha + \frac{\beta^2}{2}} - 1\right)$ due to the no-arbitrage condition.

Due to its early exercise possibility, pricing an American option (e.g., put) results in additional inequality constraints, and leads us to solve the following system of inequalities

$$\frac{\partial P_{call/put}^{Am}}{\partial t} - \mathcal{L}^s P_{call/put}^{Am} \geq 0, \quad P_{call/put}^{Am} - P_0^{call/put} \geq 0, \quad (7a)$$

$$\left(\frac{\partial P_{call/put}^{Am}}{\partial t} - \mathcal{L}^s P_{call/put}^{Am}\right) \cdot (P_{call/put}^{Am} - P_0^{call/put}) = 0. \quad (7b)$$

We denote the parameter vector by $\boldsymbol{\mu} := (\xi, \rho, \gamma, \kappa, r) \in \mathbb{R}^5$ for the Heston model, $\boldsymbol{\mu} := (\sigma, \zeta) \in \mathbb{R}^2$ for the CEV model and $\boldsymbol{\mu} := (\sigma, \alpha, \beta, \lambda) \in \mathbb{R}^4$ for the Merton model.

3.3 Variational Formulation

Then the problems (4), (7) are parametrized problems with $\boldsymbol{\mu} \in \mathcal{P}$, where $\mathcal{P} \subset \mathbb{R}^d$ is a parameter space. The solution can be written as $P = P(\boldsymbol{\mu})$. In some cases, for notational convenience, we will omit the parameter-dependence of P and related quantities.

3.3 Variational Formulation

3.3.1 Boundary Conditions

We tackle the non-homogeneous truncated Dirichlet boundary conditions by means of the lift function $u_L(t) = g(t)$ onto the domain. For all models, we consider only Dirichlet- or Neumann-type boundary conditions. For the European call in the Heston model, we specify them as follows according to Winkler et al. (2001),

$$\Gamma_1 : v = v_{\min} \quad P_{call}^{Eu}(t, v_{\min}, x) = Ke^x \Phi(d_1) - Ke^{-rt} \Phi(d_2), \quad (8a)$$

$$\Gamma_2 : v = v_{\max} \quad P_{call}^{Eu}(t, v_{\max}, x) = Ke^x, \quad (8b)$$

and we interpolate linearly on the boundaries $\Gamma_3 = \{x = x_{\min}\}$ and $\Gamma_4 = \{x = x_{\max}\}$. The cumulative distribution function $\Phi(\cdot)$ is defined in (10) and $d_{1,2} = \frac{x + (r \pm \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$ with $\sigma = \sqrt{v}$.

The boundary conditions for American put options in the Heston model are as follows, according to Clarke and Parrott (1999) and Düring and Fournié (2012),

$$\begin{aligned} P_{put}^{Am}(t, v, x) &= \tilde{\mathcal{H}}_{put}(x), \quad \text{on } \Gamma_3 \cup \Gamma_4, \\ \frac{\partial P_{put}^{Am}}{\partial v}(t, v_{\min}, x) &= 0, \quad \text{on } \Gamma_1, \quad \frac{\partial P_{put}^{Am}}{\partial v}(t, v_{\max}, x) = 0, \quad \text{on } \Gamma_2. \end{aligned}$$

For the CEV model, following Seydel (2012), we applied the boundary conditions

$$\begin{aligned} P_{call}^{Am/Eu}(t, S_{\min}) &= 0, \quad P_{call}^{Am/Eu}(t, S_{\max}) = S_{\max} - e^{-rt}K, & \text{for call options,} \\ P_{put}^{Eu}(t, S_{\min}) &= e^{-rt}K - S_{\min}, \quad P_{put}^{Eu}(t, S_{\max}) = 0, & \text{for European put options,} \\ P_{put}^{Am}(t, S_{\min}) &= K - S_{\min}, \quad P_{put}^{Am}(t, S_{\max}) = 0, & \text{for American put options.} \end{aligned}$$

In the Merton model, we subtract a function Ψ from the original pricing PIDE that approximately matches the behavior of P^{Merton} such that for all $t \in [0, T]$ we have $\tilde{P}^{\text{Merton}} = P^{\text{Merton}}(t, x) - \Psi(t, x) \rightarrow 0$ for $x \rightarrow \pm\infty$. We choose

$$\begin{aligned} \Psi^{\text{Am./Eu. call}}(t, x) &= (Ke^x - Ke^{-rt})\Phi(x), \\ \Psi^{\text{Am. put}}(t, x) &= (K - Ke^x)(1 - \Phi(x)), \end{aligned} \quad (9)$$

for European call and put options, respectively, where Φ is the cumulative distribution function of the normal distribution (10),

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}z^2} dz. \quad (10)$$

The transformation of the Merton model obtained by subtracting an appropriately chosen function Ψ as introduced in (9) results in zero boundary conditions in space, $u(t, x_{\min}) = u(t, x_{\max}) = 0$ for all $t \in [0, T]$.

4 Numerical Study of the effects of de-Americanization

Our main objective is to investigate the de-Americanization methodology with respect to the previously stated questions 1-4 on page 3. But before we look at these questions and the calibration results in detail, we describe the discretization of our FEM pricers followed by an investigation of the effects of de-Americanization on pricing. Then we switch to calibrating to synthetic data and, finally, to market data.

4.1 Discretization

We set up mesh sizes and time discretization in all three models such that the errors compared to benchmark solutions are roughly the same. In our test setting, we set $S_0 = 1$, $r = 0.07$, $T = \{0.5, 0.875, 1.25, 1.625, 2\}$ and K to 21 equally distributed values in $[0.5, 1.5]$. For the discretization, we choose $[S_{\min}, S_{\max}] = [0.01, 2]$ for the CEV model, $[v_{\min}, v_{\max}] = [10^{-5}, 3]$ and $[x_{\min}, x_{\max}] = [-5, 5]$ for the Heston model and for the Merton model we set $[x_{\min}, x_{\max}] = [-5, 5]$. Note that in order to avoid the degeneracy of the Heston PDE at $v = v_{\min}$, we set for the Heston model $v_{\min} = 10^{-5} > 0$. We set $\mathcal{N} = 1000$ for the CEV model, $\mathcal{N} = 49 \times 97 = 4753$ for the Heston model and $\mathcal{N} = 192$ for the Merton model, as well as $\Delta t = 0.008$ for all models. For the CEV model, we choose $\sigma = 0.15$ and $\zeta = 0.75$ and as benchmark solution we implement the semi-closed-form solution of the CEV model for European put and call prices as shown in Schroeder (1989). We use the semi-closed-form solution in Janek et al. (2011) for the Heston model as benchmark and as model parameters we use $\xi = 0.1$, $\rho = -0.5$, $\gamma = 0.05$, $\kappa = 1.2$ and $v_0 = 0.05$. In the Merton model, Fourier pricing is used as benchmark. The model is parametrized by setting $\sigma = 0.2$, $\alpha = -0.1$, $\beta = 0.1$ and $\lambda = 3$. Summarizing the results, we observe that for all models, with the introduced discretization, the absolute error between the benchmark and the FEM solution is in the region of 10^{-3} to 10^{-4} and, thus, the pricers for all three models have comparable accuracy.

4.2 Effects of de-Americanization on Pricing

First, we focus on pricing differences caused by de-Americanization. Therefore, we compare the de-Americanized American prices with the derived European option prices in the following way. Starting with a set of model parameters, we price the American and European options. Then, the binomial tree is applied to translate the American option prices into de-Americanized pseudo-European prices. Subsequently, we compare

4.2 Effects of de-Americanization on Pricing

the European and the pseudo-European *so called* de-Americanized prices to identify the effects of the de-Americanization methodology.

The advantage of this approach is that we can purely focus on de-Americanization, decoupled from calibration issues. In order to do so, we define the following test set for the range of investigated options. Here, we focus on put options due to the fact that American and European calls coincide for non-dividend-paying underlyings.

$$\begin{aligned}
 S_0 &= 1 \\
 K &= 0.80, 0.85, 0.90, 0.95, 1.00, 1.05, 1.10, 1.15, 1.20 \\
 T &= \frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{6}{12}, \frac{9}{12}, \frac{12}{12}, \frac{24}{12} \\
 r &= 0, 0.01, 0.02, 0.05, 0.07
 \end{aligned} \tag{11}$$

In each model, 5 parameter sets are investigated to cover the parameter range. These are summarized in Table 1.

Table 1 Overview of the parameter sets used for the CEV, Heston and Merton models

	CEV		Heston					Merton			
	σ	ζ	ξ	ρ	γ	κ	v_0	σ	α	β	λ
p_1	0.2	0.5	0.10	-0.20	0.07	0.1	0.07	0.20	-0.01	0.01	1
p_2	0.275	0.6	0.25	-0.50	0.10	0.4	0.10	0.15	-0.05	0.05	2
p_3	0.35	0.7	0.40	-0.50	0.15	0.6	0.15	0.20	-0.10	0.10	3
p_4	0.425	0.8	0.55	-0.45	0.20	1.2	0.20	0.10	-0.10	0.20	5
p_5	0.5	0.9	0.70	-0.80	0.30	1.4	0.30	0.10	-0.15	0.20	7

Motivation of the selected parameters for the CEV model The main feature of the CEV model is the elasticity of variance parameter ζ , which is combined with the level of the underlying to obtain a local volatility, namely $\sigma(S, t) = \sigma S^{\zeta-1}$, reflecting the leverage effect. In our example, we investigate American puts and the option-holder benefits from decreasing asset prices. In general, increasing the volatility leads to increasing option prices, but especially compared to the classical Black-Scholes model we are interested in the question of how strongly the incorporated leverage effect influences the put prices and whether the differences between American and European puts can be captured by the binomial tree. Thus, our selection for ζ in p_1 is 0.5, which strongly differs from the Black-Scholes model, and then ζ is further increased up to 0.9 within the scenarios. Additionally, we increase the values of σ .

Motivation of parameter selection for the Heston model Similar to the CEV model, the (American) put prices increase with increasing volatility. We try to cover this effect by increasing the volatility of the volatility parameters and the correlation between

4.2 Effects of de-Americanization on Pricing

the two stochastic processes. In general, for stocks, the correlation between the volatility and the underlying value is negative. Thus, in the de-Americanization study, we focus on negative correlation values ρ . Starting in p_1 with a relatively low volatility and a slightly negative correlation ρ , in p_2 to p_4 we increase the volatility of volatility parameter ξ , the mean reverting level γ and the mean reverting speed κ , and also investigate higher negative values for the correlation ρ . In all scenarios, the initial volatility v_0 is set to match the mean reverting level, i.e., $v_0 = \gamma$.

Motivation of parameter selection for the Merton model The Merton model is a jump diffusion model. Due to the early exercise feature of American options, the existence of jumps has a significant impact on American option prices. Consider for example an American put. Here, the option-holder benefits from decreasing asset prices. Consequently, when the possibility of negative jumps increases, the option price will increase as well. The jump intensity parameter λ therefore plays a decisive role in this de-Americanization study. The analogous reasoning holds for the expected jump size parameter α . In the upcoming numerical study, we try to incorporate these effects. The considered scenarios for the Merton model presented in Table 1 are chosen by this reasoning. Scenario p_1 describes a Black-Scholes-like market with a rather low presence of jump occurrences. In p_2 and p_3 , the jump feature appears more pronounced. Scenario p_4 and p_5 finally are encoded by rather jump-dominated parameter sets, which have an average number of 7 jumps per year with large expected negative jump sizes that appear highly volatile.

Remark 4.1

We price the put options in (11) for the parameter sets shown in Table 1. By including $K = 1$, at-the-money options are included in this analysis. For some parameters, especially for high interest rates combined with low volatility, it could occur that the price of an American put option equals exactly $K_i - S_0$, so that this American put option would be exercised immediately. In the following analysis, we excluded these cases because a unique European option price cannot be determined by applying the binomial tree. As illustrated in the following toy example in Figure 2, there are several possible values for u to replicate the American option price if the price of the American option is determined by immediately exercising it. In the example, a put option with strike $K = 120$ is priced. Here, $u = 1.04$ and $u = 1.11$ are possible solutions. To avoid this, we consequently only consider American put options in our analyses when $P_{put}^{Am} > (K - S_0)^+ \cdot (1 + \delta)$. Thus, the American put option price exceeds the immediate exercise price by a factor of δ . We set $\delta = 1\%$. In theory, we can not exclude that in all other cases the application of the binomial tree finds a unique u^ to replicate the American option price. However, in various empirical tests this has been the case and only when the price of an American put option equals exactly $K_i - S_0$, problems have been observed.*

In Tables 9 - 11 (CEV model), Tables 12 - 14 (Heston model) and Tables 19 - 21 (Merton model) in the appendix, we show in the appendix the pricing effects for the synthetic

4.2 Effects of de-Americanization on Pricing

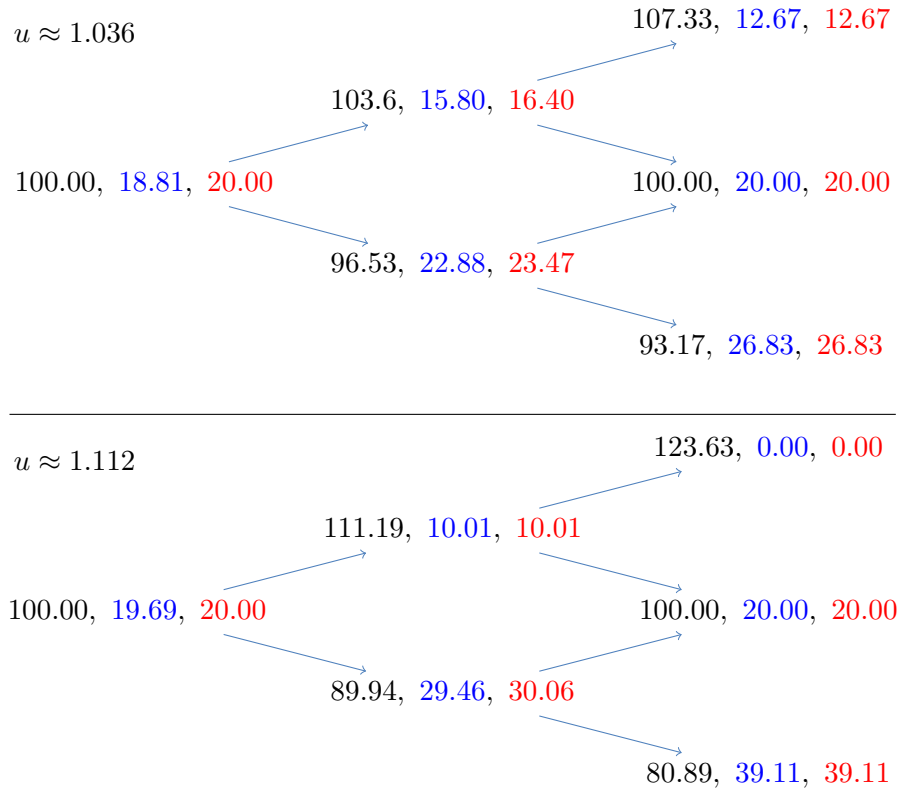


Figure 2 Given an American put option price of 20 with $S_0 = 100$, $K = 120$, $r = 0.01$, i.e., an American put option in the exercise region, a unique tree cannot be found to replicate this option. In this example, we show two binomial trees for $u \approx 1.036$ (top) as well as $u \approx 1.112$ (bottom). In each tree, we show the value of the underlying (black), the European put price (blue) and the American put price (red) at each node. Both trees replicate the American option price of 20.00 but result in different European put prices: 18.81 and 19.69.

4.2 Effects of de-Americanization on Pricing

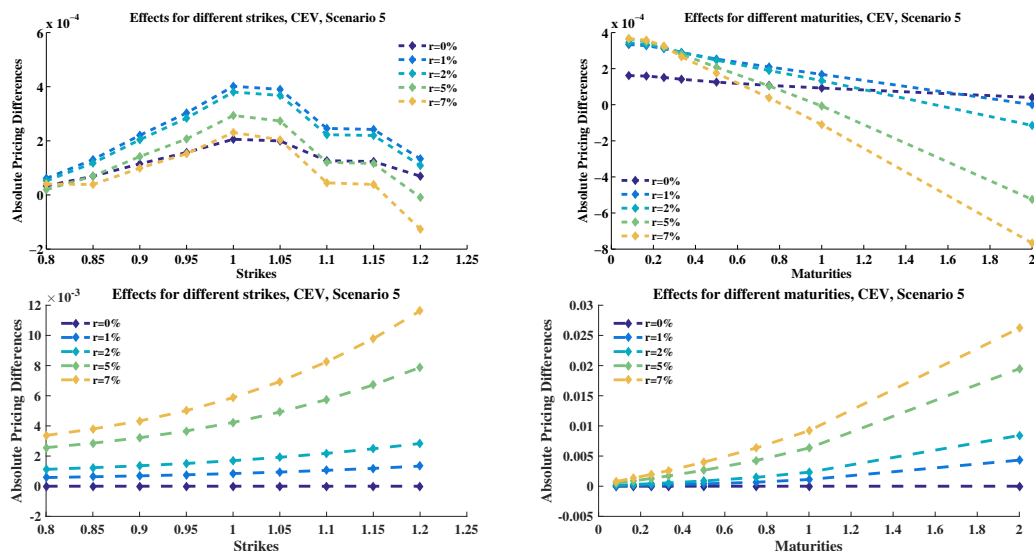


Figure 3 De-Americanization effects on pricing put options in the CEV model. As an example, the results are shown for p_5 for the average error between the de-Americanized and the European prices for each strike (top left) and each maturity (top right). The average differences of the corresponding American and European prices is shown for each strike (bottom left) and each maturity (bottom right).

prices in (11). For each scenario $p_i, i = 1, \dots, 5$, we present the average difference between the de-Americanized prices and the European prices for each maturity and each strike and accordingly show the maximal European price in this maturity to reflect the issue stated in Remark 4.1. Similar studies have been done for the maximal error at each strike and maturity and confirm the findings based on the average error presented in the following. In Figure 3, we highlight the results for scenario p_5 in the CEV model to illustrate the effects of de-Americanization in several interest rate environments for different maturities or different strikes. For p_5 in the Heston model and p_5 in the Merton model, the results are shown in Figure 4 and Figure 5, respectively. All of these figures clearly highlight the case $r = 0$ as having hardly any de-Americanization effects (Heston and Merton) or at least fewer such effects (CEV). For a better interpretability, in each of the figures the differences between the corresponding American and European option are shown, too.

In general, for the CEV model, we observe that for short maturities the de-Americanized prices seem to overprice the European prices, whereas for longer maturities they seem to underprice the European options. We see that with increasing σ and ζ parameters the maximal error increases and, overall, all parameter sets behave similarly. Focusing on the interest rate, we observe that for higher interest rates ($r = 5\%$ and $r = 7\%$) the average errors are higher or at least in a comparable region. Especially for higher interest rates, the maximal price has to be considered, because the higher the interest

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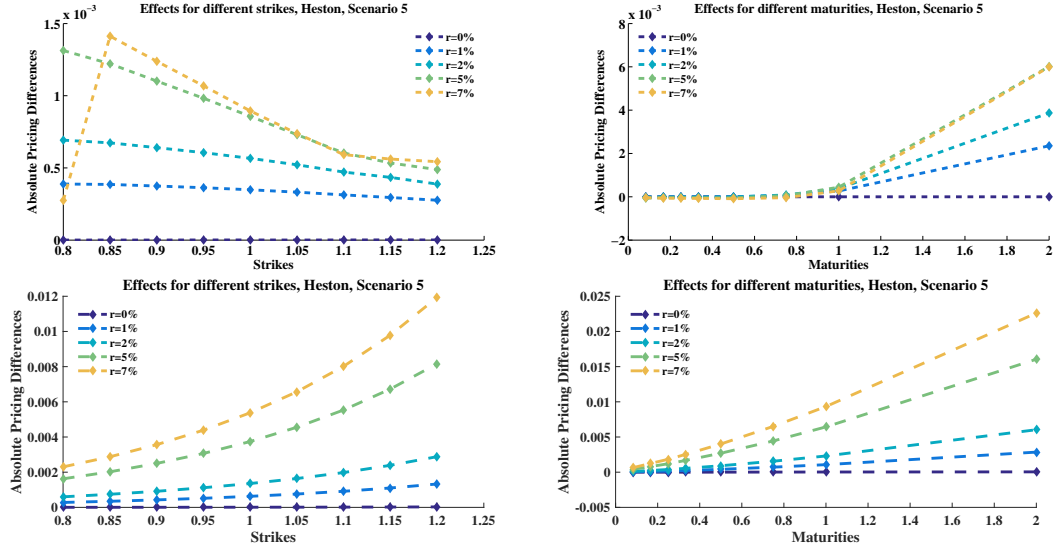


Figure 4 De-Americanization effects on pricing put options in the Heston model. As an example, the results are shown for p_5 for the average error between the de-Americanized and the European prices for each strike (top left) and each maturity (top right). The average differences of the corresponding American and European prices is shown for each strike (bottom left) and each maturity (bottom right).

rate, the higher the probability that we did not consider some in-the-money options due to Remark 4.1 and that the options with high prices are neglected in this setting. Thus we deduce that the error increases with increasing interest rates and that at high maturities the error increases for scenarios with higher volatility. For scenarios p_1 , p_2 and p_3 , we clearly observe that the effects of de-Americanization increase with increasing strikes. This means that for in-the-money options the de-Americanization effects tend to be stronger than for out-of-the-money options. This is consistent with the statements made by Carr and Wu (2010). However, for higher interest rates, the average error seems to decrease with increasing strikes. The test setting in (11) is defined for $S = 1$. In the CEV model, the volatility is scaled with $S^{\zeta-1}$. In additional test, we set in (11) $S = 100$ and scaled the strike values K with a factor of 100, too. Overall, we observe that the errors of the de-Americanization methodology give a similar picture to the results in Tables 9 - 11. Naturally, by investigating differences of prices, the absolute number of the error is higher by a factor of 10 to 100, which is tolerable when the underlying value is scaled with a factor of 100.

For the Heston model, we observe in general that the de-Americanization error increases with increasing interest rates within each parameter setting. Additionally we see that for $r = 0\%$ there is hardly any effect. By focusing on the scenarios with a higher volatility of volatility parameter (p_4 and p_5), we observe stronger de-Americanization effects at short and long maturities (T_1 and T_8). For short maturities, the de-Americanized price

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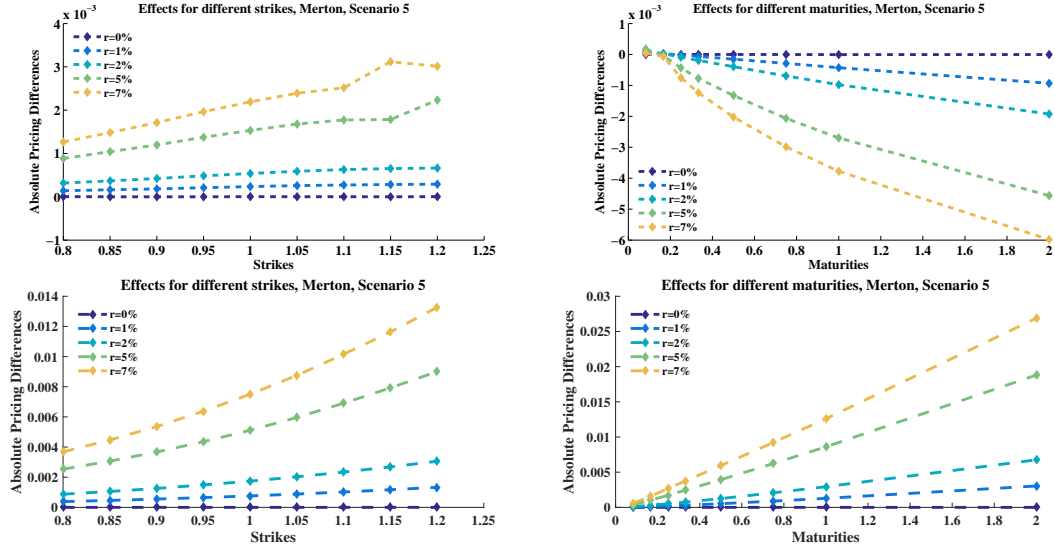


Figure 5 De-Americanization effects on pricing put options in the Merton model. As an example, the results are shown for p_5 for the average error between the de-Americanized and the European prices for each strike (top left) and each maturity (top right). The average differences of the corresponding American and European prices is shown for each strike (bottom left) and each maturity (bottom right).

is consistently lower than the corresponding European price throughout all scenarios, whereas for high maturities the de-Americanized price is higher than the corresponding European price. To highlight the in-the-money and out-of-the-money issue in Figure 4, note that in the Heston model the error is far smaller out-of-the-money than deep in-the-money. However, the highest errors tend to occur in the at-the-money and slightly in-the-money regions. For the Heston model, we did additional tests for parameter scenarios in which the initial value of the volatility v_0 does not equal the mean reverting level γ . We report these additional scenarios in Table 15. The results are reported in Tables 16 - 18. Overall, they are similar to the results presented in Tables 12 - 14. At this point, we want to highlight two observations. Comparing the results for p_2 to p_7 , for an increased initial volatility v_0 , the de-Americanization error is slightly higher. Additionally, by comparing p_6 with p_7 and p_8 with p_9 , the de-Americanization error is slightly higher in cases with an increased v_0 compared to cases with an increased γ .

For the Merton model, we observe similar, small effects for scenarios p_1 and p_2 , i.e., the scenarios with low jump intensity, whereas for the scenarios with increasing jump intensity (p_3 , p_4 and p_5) we observe stronger de-Americanization effects, especially for increasing maturities and interest rates. Figure 5 additionally shows that for increasing strikes the effect of de-Americanization increases slightly for lower interest rates and that for higher interest rates this error increases more strongly in the in-the-money region. Here, we focus on scenarios with $\alpha < 0$, i.e. the expected jump size is negative. In

4.3 Effects of de-Americanization on Calibration to Synthetic Data

additional tests, we investigated the five scenarios for the Merton model in Table 1 with the corresponding positive value of α . The results are similar to the results observed for negative values of α .

In addition to all of these de-Americanization effects in absolute terms, we checked the magnitude of the relative error for the 1-year at-the-money put option, i.e., the absolute difference between the European and the de-Americanized price divided by the European price. In the CEV model, the average relative error for this option in all scenarios and interest rate settings was 0.1% with a peak of 0.17% at scenario p_2 with $r = 1\%$. The average relative error for the Heston model was 0.17% with a peak of 0.83% in p_2 with $r = 7\%$. In the Merton model, the average relative error of the at-the-money put option with maturity of one year was 0.18% with a peak of 1.02% at p_5 and $r = 7\%$.

Summarizing the results,

- de-Americanization effects are sensitive to interest rate. The higher the interest rates, the higher the observable pricing differences,
- de-Americanization effects increase with increasing volatility and increasing maturities,
- de-Americanization effects tend to be stronger in-the-money,
- de-Americanization effects increase with higher jump intensities.

Overall, in the settings mentioned above, we observe a systematic effect caused by de-Americanization. In the next step, we are interested in finding out whether these effects are also reflected in the calibration results.

4.3 Effects of de-Americanization on Calibration to Synthetic Data

Here, we study the de-Americanization effect on synthetic American market data. To this effect, in a first step, we generate artificial market data using our FEM implementations of the three considered models. In a second step, we calibrate each model to the previously generated market data. This methodology allows us to disregard the noise affiliated with real market data is affiliated with and thus enables us to study the effect of de-Americanization exclusively.

4.3 Effects of de-Americanization on Calibration to Synthetic Data

Our artificial market data is specified as follows.

$$\begin{aligned}
 S_0 &= 1 \\
 r &= 7\% \\
 T_1 &= \frac{2}{12}, & K_1 &= \{0.95, 0.975, 1, 1.025, 1.05\}, \\
 T_2 &= \frac{6}{12}, & K_2 &= \{0.9, 0.925, K_1, 1.075, 1.1\}, \\
 T_3 &= \frac{9}{12}, & K_3 &= \{0.85, 0.875, K_2, 1.125, 1.15\}, \\
 T_4 &= 1, & K_4 &= \{0.8, 0.825, K_3, 1.175, 1.2\}, \\
 T_5 &= 2, & K_5 &= \{0.75, 0.775, K_4, 1.225, 1.25\}.
 \end{aligned} \tag{12}$$

While in the pricing analysis, we investigated only put options, we will now introduce two settings: one only involving put options, and one involving out-of-the money put and call options. Whereas by calibrating to put options only, in-the-money, at-the-money and out-of-the-money options are considered in the calibration process, by calibrating to out-of-the-money options only out-of-the-money options are considered. The motivation of focusing on out-of-the-money options is connected to observations of market practices. In the following section regarding the calibration to market data, we introduce the methodology of the creation of the VIX index as benchmark for handling available market data and for focussing on out-of-the-money options. As the data in (12) shows, we consider a high-interest market and a set of maturities ranging from rather short-term American options with 2 months maturity to long-term American products with 2 years maturity. Each maturity T_i is associated with a set of strikes K_i , $i \in \{1, \dots, 5\}$. To analyze the effects of de-Americanization on pricing, we price these options for the five parameter scenarios in Table 1. Regarding the calibration methodology, we have to make two choices. First, we have to decide which option types to include and, second, we need to determine the objective function.

Regarding the choice of options, we first consider only put options for the whole strike trajectory due to the fact that, in our setting of non-dividend paying underlyings, American and European calls coincide. Thus, we include in-the-money as well as out-of-the-money options. Second, motivated by the fact that the value of out-of-the-money options does not include any intrinsic value and is therefore supposed to better reflect the randomness of the market (as mentioned in Carr and Wu (2010)), we consider as a second approach that only includes out-of-the money puts and out-of-the money calls for the whole set of strikes and maturities. By calibrating later to market data, there was not given a strike exactly equal to S_0 , so we only have "almost" at-the-money options. Thus, calibrating to out-of-the-money options, we did not use the at-the-money option. Consequently, in this second study, for each $i \in \{1, \dots, 5\}$, we consider call option prices for maturities T_i and strikes $k \in K_i$ with $k > 1$ and put option prices for maturities T_i and strikes $k \in K_i$ with $k < 1$. At-the-money option data, i.e., options with strike $K = 1$, is neglected.

Once the synthetic American market data has been generated, we create associated

4.3 Effects of de-Americanization on Calibration to Synthetic Data

second synthetic market data by applying the de-Americanization routine using the binomial model.

Remark 4.1 and Figure 2 describe situations in which the de-Americanization routine yields non-unique results. In the calibration to de-Americanized prices, we exclude options that cannot be de-Americanized uniquely as explained by the following remark.

Remark 4.2 (Disregarding non-unique de-Americanized prices)

As outlined above, we artificially generate American market data for a calibration study on synthetic data. In a first step, we calibrate to the generated American prices directly. In a second step, we de-Americanize the option data and calibrate to the resulting quasi-European options. Here, we only consider option prices that admit a unique de-Americanized price. Consequently, all American put option prices that violate

$$P_{put}^{Am} > (K - S_0)^+ \cdot (1 + \delta), \quad \text{with } \delta = 1\%, \quad (13)$$

are not de-Americanized and thus are neglected in the second step.

The second crucial assumption is the objective function. A variety of objective functions are proposed in the literature, e.g., the root mean square error, the average absolute error as a percentage of the mean price, the average absolute error, the average relative percentage error, absolute price differences, relative price differences, absolute implied volatilities, relative implied volatilities (see for example Detlefsen and Haerdle (2006), Bauer (2012), Fengler (2005), Schoutens et al. (2004)).

We work directly with the observed prices and choose an objective function that considers prices, and due to the fact that the considered out-of-the-money option prices are rather small, we focus on absolute instead of relative differences. In the calibration, we take the absolute average squared error (aase) as the objective function and we minimize,

$$\text{aase} = \frac{1}{\#\text{options}} \sum_{\text{option}_k} |\text{Market price}_k - \text{Model price}_k|^2. \quad (14)$$

The results of the calibration to synthetic data are summarized in Table 2 for the CEV and Merton models and in Table 3 for the Heston model for calibrating to put options and calibrating to out-of-the-money options.

Overall, we see that for the CEV model the parameters match well when calibrating to American options. When calibrating to de-Americanized prices however, the volatility parameter σ is underestimated in most cases and this underestimation is counterbalanced by an overestimated ζ -value.

Focusing on the Heston model, we observe that in every calibration to American options the parameters are matched better than in the corresponding calibration to de-Americanized data. We clearly see that the three parameters γ , κ and v_0 are matched, but the remaining two parameters ξ and ρ show different results. When calibrating put

Table 2 Calibration results for calibrating to put options only and out-of-the-money options for the CEV model (left) and Merton model (right). Due to the effect of non-unique de-Americanization results, for the CEV model, some option prices have been neglected in the calibration to de-Americanized option data, as Remark 4.2 explains. In scenarios p_1 to p_5 , 5, 5, 10, 10 and 10 prices were excluded in the calibration to put options only. In scenarios p_1 and p_2 of the Merton model, 5 prices have been excluded in the calibration to put options only.

		CEV			Merton					
		σ	ζ	aase	σ	α	β	λ	aase	
p_1	true	0.2	0.5	—	0.20	-0.01	0.01	1	—	
	Put	Am	0.1977	0.4962	7.74e-6	0.20	0.01	0.05	0.29	1.07e-10
		DeAm	0.1894	0.4501	8.35e-5	0.20	-0.06	0.03	0.37	8.37e-8
	oom	Am	0.1997	0.4996	4.52e-6	0.20	0.00	0.05	0.32	1.35e-10
		DeAm	0.1793	0.9609	2.75e-4	0.20	-0.02	0.04	0.30	3.80e-9
	p_2	true	0.275	0.6	—	0.15	-0.05	0.05	2	—
Put		Am	0.2740	0.6004	4.98e-7	0.15	-0.06	0.05	1.55	2.69e-11
		DeAm	0.2607	0.7539	2.04e-6	0.14	-0.10	0.01	1.31	1.63e-7
oom		Am	0.2736	0.5978	1.91e-6	0.16	-0.10	0.04	0.66	1.91e-10
		DeAm	0.2484	0.5367	1.10e-5	0.15	-0.11	0.03	0.74	2.22e-8
p_3		true	0.35	0.7	—	0.20	-0.10	0.10	3	—
	Put	Am	0.3515	0.7576	4.37e-5	0.22	-0.19	0.07	1.31	8.83e-10
		DeAm	0.3272	0.8528	1.92e-4	0.16	-0.05	0.11	5.04	3.12e-8
	oom	Am	0.3476	0.6984	1.00e-4	0.22	-0.19	0.07	1.32	5.47e-10
		DeAm	0.3141	0.5527	5.99e-4	0.19	-0.17	0.09	1.88	3.33e-7
	p_4	true	0.425	0.8	—	0.10	-0.10	0.20	5	—
Put		Am	0.4258	0.7898	1.53e-6	0.10	-0.10	0.20	5.00	1.22e-13
		DeAm	0.3942	0.8755	7.30e-6	0.10	-0.10	0.20	5.00	6.29e-16
oom		Am	0.4262	0.7966	3.27e-6	0.09	-0.09	0.21	4.90	2.19e-7
		DeAm	0.3801	0.6009	1.96e-5	0.15	-0.14	0.22	3.63	6.53e-7
p_5		true	0.5	0.9	—	0.10	-0.15	0.20	7	—
	Put	Am	0.4982	0.9036	1.53e-6	0.10	-0.15	0.20	7.00	8.96e-13
		DeAm	0.4570	0.9192	1.02e-5	0.05	-0.11	0.21	7.95	1.73e-7
	oom	Am	0.4986	0.9036	4.02e-6	0.10	-0.15	0.20	7.00	5.00e-13
		DeAm	0.4430	0.6549	2.38e-5	0.05	-0.20	0.23	5.08	1.67e-6

options, as the volatility of volatility parameter ξ increases, the calibrated de-Americanized parameter overestimates the true parameter. When calibrating out-of-the-money options, we observe that ξ is underestimated for lower ξ values and overestimated for higher ξ values. Regarding ρ , as the volatility of volatility is increased, the de-Americanized parameter tends to underestimate the ρ value. Later, by focusing on pricing exotic options, we will see whether these two contrary effects cancel each other or lead

Table 3 Heston model: Calibration results for calibrating to put options only and out-of-the-money options.

			ξ	ρ	γ	κ	v_0	aase
p_1	Put	true	0.1	-0.2	0.07	0.1	0.07	—
		Am	0.1002	-0.1999	0.07	0.1026	0.07	1.43e-13
		DeAm	0.1	-0.4839	0.0651	0.5144	0.0695	1.50e-7
	oom	Am	0.1006	-0.1987	0.07	0.1049	0.07	4.73e-13
		DeAm	0.1	-0.1949	0.0665	0.2292	0.07	1.32e-8
p_2	Put	true	0.25	-0.5	0.1	0.4	0.1	—
		Am	0.25	-0.5	0.1	0.4	0.1	6.47e-23
		DeAm	0.2667	-0.5067	0.0978	0.4374	0.0992	3.28e-8
	oom	Am	0.25	-0.5	0.1	0.4	0.1	2.99e-17
		DeAm	0.2199	-0.5	0.0885	0.1618	0.1	5.76e-9
p_3	Put	true	0.4	-0.5	0.15	0.6	0.15	—
		Am	0.4	-0.5	0.15	0.6	0.15	7.15e-16
		DeAm	0.4684	-0.437	0.1544	0.6806	0.1495	6.04e-9
	oom	Am	0.4	-0.5	0.15	0.6	0.15	7.03e-18
		DeAm	0.3970	-0.5	0.1517	0.5413	0.1494	4.68e-9
p_4	Put	true	0.55	-0.45	0.2	1.2	0.2	—
		Am	0.55	-0.45	0.2	1.2	0.2	1.44e-17
		DeAm	0.5773	-0.4298	0.2046	1.1975	0.1986	2.86e-9
	oom	Am	0.55	-0.45	0.2	1.2	0.2	8.41e-22
		DeAm	0.5625	-0.4369	0.2035	1.2198	0.1988	4.50e-9
p_5	Put	true	0.7	-0.8	0.3	1.4	0.3	—
		Am	0.7	-0.8	0.3	1.4	0.3	1.68e-17
		DeAm	0.8504	-0.7057	0.3136	1.5832	0.2993	1.31e-8
	oom	Am	0.7	-0.8	0.3	1.4	0.3	8.58e-23
		DeAm	0.7763	-0.7602	0.3073	1.7021	0.2979	2.34e-8

to different exotic option prices.

For the Merton model, we observe that when calibrating American options σ is matched fairly accurately in most cases. For the other 3 parameters, we observe that whenever the jump intensity λ is underestimated, the corresponding mean α and standard deviation β of the jump are adjusted accordingly. Similar observations can be made for calibration to de-Americanized prices. Here, especially for calibrating out-of-the-money values in p_4 and p_5 , we observe that the σ value is also not matched.

Summarizing the results, we observe that when calibrating de-Americanized synthetic data in a high-interest-rate environment for the continuous CEV and Heston models, the main parameters driving the volatility of the underlying, ζ and σ (CEV) and ξ and ρ (Heston), are often not exactly matched. In these cases, the application of the binomial tree is not able to capture the volatility of the underlying exactly. For the jump model

4.4 Effects of de-Americanization on Calibration to Market Data

(Merton), we observe that due to the de-Americanization the jump intensity is (more strongly) mismatched than when directly calibrating to American options and in these cases the wrongly calibrated jump intensity parameter may be compensated by adjusting the other model parameters accordingly.

4.4 Effects of de-Americanization on Calibration to Market Data

In this section, we investigate the effects of de-Americanization by calibrating market data. The single stock of our choice is Google as an example of a non-dividend-paying stock. Table 4 gives an overview of the processed data for the calibration procedure. In total we obtained a data set containing 482 options, with slightly more puts than calls. The risk-free interest rate for maturities of 1 month, 3 months, 6 months, 1 year and 2 years are taken from the U.S. Department of the Treasury¹ and have been linearly interpolated whenever necessary.

Table 4 Processed Google option data for $t_0 = 02.02.2015$, $S_0 = 523.76$

	Maturity	T	# of options	r
T_1	27.02.2015	0.07	47	0.0001
T_2	20.03.2015	0.13	49	0.000129508
T_3	17.04.2015	0.20	52	0.00017541
T_4	19.06.2015	0.38	87	0.00046087
T_5	18.09.2015	0.62	98	0.000955435
T_6	15.01.2016	0.95	101	0.001602174
T_7	20.01.2017	1.97	48	0.004786339

In order to structure the available data, we follow the methodology applied for the volatility index (VIX) by the Chicago board of exchange (CBOE (2009)):

- Only out-of-the-money put and call options are used
- The midpoint of the bid-ask spread for each option with strike K_i is considered
- Only options with non-zero bid prices are considered
- Once two puts with consecutive strike prices are found to have zero bid prices, no puts with lower strikes are considered for inclusion (same for calls)

Basically, by this selection procedure, we only select out-of-the-money options that (due to non-zero bid prices) can be considered as liquid. In general, an option price consists of two components reflecting the time value and the intrinsic value of the option. By focusing on out-of-the-money options, the intrinsic value effects are mostly neglected and the highest option price will be at-the-money. Additionally, the highest market activity

¹<http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield>

Table 5 Calibration results for calibrating to out-of-the-money put and call options combined.

		CEV					
		σ	ζ			aase	
Google Data	Am	0.25	0.98			3	
	DeAm	0.25	0.97			3.32	
		Heston					
		ξ	ρ	γ	κ	v_0	aase
Google Data	Am	0.2290	-0.6854	0.0585	4.3186	0.0651	0.8464
	DeAm	0.2245	-0.6941	0.0586	4.1433	0.0647	0.8319
		Merton					
		σ	α	β	λ	aase	
Google Data	Am	0.1936	-0.2000	0.2194	0.2935	0.5813	
	DeAm	0.1935	-0.2000	0.2133	0.3014	0.6035	

is in the at-the-money and slightly out-of-the-money region. The calibration results are summarized in Table 5.

Here, we observe hardly any differences in the parameters. This is in line with our observations in Section 4.2 for low-interest-rate environments. In these settings, American and European puts almost coincide and, thus, there will hardly be any difference in the prices and it is only natural that we observe very similar calibration results. Interestingly, the *aase* value obtained by calibrating the Heston model is slightly lower when calibrating de-Americanized options than American options.

4.5 Effects of de-Americanization in pricing exotic options

Plain vanilla options are traded liquidly in the market and are used to calibrate models. Financial institutions use these calibrated models to price more exotic products such as barrier and lookback options. In this subsection, we analyze which influences different calibration results have on the accuracy of exotic option prices.

We analyze a down-and-out call option and a lookback option and hence translate differences in the calibrated model parameters into quantitative prices. The payoff $\tilde{\mathcal{H}}_{DOC}(S(T))$ of a down-and-out call option with barrier B is given by

$$\tilde{\mathcal{H}}_{DOC}(S(T)) = (S(T) - K)^+ \cdot \mathbb{1}_{\min_{t \leq T} S(t) \geq B}. \quad (15)$$

In our setting, we set $S_0 = 100$, the barrier B to 90% of the initial underlying value and the strike K to 105% of the underlying value. For the lookback option, we choose the same strike and the payoff $\tilde{\mathcal{H}}_{Lookback}(S(T))$ is

$$\tilde{\mathcal{H}}_{Lookback}(S(T)) = (\bar{S}(T) - K)^+, \quad \text{with } \bar{S}(T) = \max_{t \leq T} S(t). \quad (16)$$

4.5 Effects of de-Americanization in pricing exotic options

We price these two exotic options for the calibrated parameters in Tables 2, 3 and 5 via a standard Monte Carlo method with 10^6 sample paths, 400 time steps per year and antithetic variates as variance reduction technique. The results are shown in the following Table 6.

Table 6 Overview of prices for barrier and lookback options

		CEV		Heston		Merton		
		barrier	lookback	barrier	lookback	barrier	lookback	
p_1	Put	true	9.93	10.11	8.75	8.80	3.86	25.70
		Am	9.93	10.10	8.74	8.78	3.92	25.86
		DeAm	9.93	10.01	4.54	4.86	3.98	26.09
	oom	Am	9.93	10.11	8.73	8.78	3.93	25.86
		DeAm	10.13	11.13	8.27	8.37	3.88	25.77
p_2	Put	true	10.13	11.14	2.28	2.73	2.57	22.97
		Am	10.14	11.14	2.28	2.73	2.49	22.73
		DeAm	11.47	14.59	2.07	2.54	2.56	22.94
	oom	Am	10.12	11.10	2.27	2.73	2.64	23.24
		DeAm	9.95	10.40	4.32	4.57	2.43	22.75
p_3	Put	true	11.60	14.93	1.15	1.80	6.65	37.35
		Am	12.48	17.83	1.15	1.80	6.78	37.88
		DeAm	13.53	23.85	1.86	2.77	6.50	36.51
	oom	Am	11.56	14.81	1.14	1.80	6.76	37.85
		DeAm	10.07	10.91	1.40	2.08	6.17	37.63
p_4	Put	true	13.56	24.08	0.83	1.86	10.17	54.99
		Am	13.54	24.00	0.83	1.86	10.14	55.00
		DeAm	14.14	30.87	1.04	2.21	10.11	54.98
	oom	Am	13.51	23.81	0.83	1.86	10.18	54.99
		DeAm	10.60	12.37	0.94	2.05	9.42	55.54
p_5	Put	true	14.76	43.40	0.02	0.52	15.63	76.15
		Am	14.80	43.99	0.02	0.52	15.48	75.69
		DeAm	14.78	43.50	0.03	0.67	15.99	75.97
	oom	Am	14.74	42.81	0.02	0.52	15.58	76.06
		DeAm	11.68	15.13	0.02	0.56	13.98	75.01
Google data	Am	14.21	32.10	0.69	1.51	4.03	28.91	
	DeAm	14.12	30.70	0.67	1.47	4.03	28.93	

Overall, we observe a different picture in each of the three models. In p_1 and p_2 of the CEV model, the scenarios with relatively small volatility, we do not see any differences. Thus, in cases with small volatility and medium elasticity of variance ζ , de-Americanization seems to work. In the other scenarios, we observe that the calibration of de-Americanized prices leads to higher exotic option prices if we calibrate put options only and lower exotic option prices if we calibrate out-of-the-money options. Thus, the

4.6 Discussion of run-times

typically lower calibrated σ -value in combination with an increased ζ -value obtained by calibrating de-Americanized options has this effect on the pricing of exotic options.

For the Heston model, we see that in the cases where the calibration of de-Americanized data led to different ξ and ρ values there are differences in the exotic option prices. More precisely, in all these cases, the corresponding barrier and lookback prices are too high. This means that de-Americanization causes a systematic overpricing of exotic options.

For the Merton model, we see rather small differences for lookback options, but more interestingly, we observe differences for the down-and-out barrier option. This reflects the fact that the differently calibrated jump intensities and accordingly adjusted means and standard deviations of the jumps can buffer de-Americanization effects over paths where the option cannot vanish like in the barrier option case.

In high-interest-rate environments, the de-Americanization methodology leads to different exotic options prices in the CEV model when the volatility of the underlying is higher. When using only put options, the exotic option prices tend to be higher; when considering out-of-the-money options, the exotic option prices tend to be lower. In the Heston model, we observe a similar picture as in the CEV model, however here no general statement holds between higher and lower exotic option prices. Regarding the Merton model, the differences in the exotic option prices are more visible when considering the down-and-out barrier option.

4.6 Discussion of run-times

In the introduction, we described qualitatively why the de-Americanization methodology has a faster run-time and that run-time aspects are the driving factor behind this approach. Our investigations focus on the accuracy of the de-Americanization methodology. The numerical implementations are all made from scratch and not optimized with regards to run-time performance. With this in mind, we would like to present some run-time results for the Heston model to compare the run-times of calibrating directly to American options to applying the de-Americanization methodology. We compute the results on a PC with Intel(R) Xeon(R) CPU E31270 @ 3.40GHz (4 physical cores, 8 logical cores) and 16GB RAM. All codes are written in Matlab R2014a. In Table 7, the run-time results for calibrating the Heston model to the synthetic data are shown.

These results show the run-time advantages of calibrating to European options (using the FEM methodology) instead of calibrating to American options. Moreover, in the run-time comparison of both approaches, the preprocessing of the American option data by the application of the binomial tree has to be considered as well. Combining both, the run-times of the de-Americanization methodology are still advantageous compared to a direct calibration to American option data.

Table 7 Run-times for calibrating to synthetic market data in the Heston model. Here, we took the calibration to put options in scenario p_5 of the Heston model as an example.

	run-time
to American options (FEM)	8 hours
to European options (FEM)	9.96 min
preprocessing data (tree)	36.40 min

In Burkovska et al. (2016), we highlight another benefit of the de-Americanization methodology. By reducing the calibration problem to European options, the (semi-) closed-form solutions of the corresponding models, e.g. applying Fourier techniques for the Heston model, can be used in the calibration process. By calibrating to a similar set of Google market data, we observe the run-times as summarized in Table 8.

Table 8 Run-times for calibrating to a similar set of Google market data in the Heston model as shown in Burkovska et al. (2016). In this setting, the calibration has been applied to a set of 401 Google put option prices. The accuracy results are similar to the results reported in Table 5.

	run-time
to American options (FEM)	68.72 hours
to European options (closed-form)	4.96 min
preprocessing data (tree)	4.96 hours

Especially when using Fourier techniques in the calibration process for the European problem, the run-time of the de-Americanization method is mostly due to the processing of the American data. Although the numerical implementations have not been optimized with regards to run-time performances and the key focus in this paper is the accuracy, these observations highlight the attractive features of the de-Americanization methodology as stated in the introduction, namely

- it delivers fast run-times compared to a direct calibration to American options,
- it is easy to implement,
- it can flexibly be integrated into the pricing and calibration toolbox at hand.

5 Conclusion

In this paper, we investigate the de-Americanization methodology by performing accuracy studies to compare the empirical results of this approach to those obtained by

5 Conclusion

solving related variational inequalities for local volatility, stochastic volatility and jump diffusion models. On page 3, we pose key questions regarding the robustness of the de-Americanization methodology with regard to changes in the (i) interest rates, (ii) maturities, (iii) in-the-money and out-of-the-money options, and (iv) continuous and discontinuous models with increasing jump intensities.

First, focusing on pricing, we observe that de-Americanization causes larger errors (i) for higher interest rates, (ii) for higher maturities, (iii) in the in-the-money region and (iv) for continuous models in scenarios with higher volatility and/or correlation, as well as in jump models for higher jump intensities. Second, we investigate model calibration to synthetic data for a specified set of maturities and strikes in a high-interest-rate environment. Numerically, we observe noticeable differences in the calibration results of the de-Americanization methodology compared to the benchmark. For continuous models, the main difference lies in the resulting volatility parameters. For the jump model, the jump intensity is underestimated by the de-Americanization method, especially in settings with high jump intensities, whereas the mean and the standard deviation of the jumps are overestimated. When calibrating Google market data, hardly any differences occur, which can be explained by the very low-interest-rate environment. This is in line with the results for question (i). In the final step, we investigate the effects of de-Americanization in the model calibration on pricing exotic options. Here, exotic option prices play the role of a measure of the distance between differently calibrated model parameters. In most cases, we observe that exotic option prices are reasonably close to the benchmark prices. However, we observe severe outliers for all investigated models. We find scenarios in which the exotic option prices differ by roughly 50% in the CEV model (p_4) and the Heston model (p_1) and by roughly 10% in the Merton model (p_5); see Table 6. Whereas in the CEV model and the Merton model the differences tend to be higher when calibrating to out-of-the-money options instead of only to put options, in the Heston model we have a mixed picture for different scenarios.

First, the methodological risk of de-Americanization critically depends on the interest rate environment.

- For low-interest-environments, the errors caused by de-Americanization are negligibly small and the de-Americanization methodology can be employed when fast run-times are preferred.
- For higher-interest-rate environments, however, de-Americanization leads to uncontrollable outliers.

Intuitively, the higher the interest rate, the higher is the early exercise premium of an American put option and, hence, the higher are the differences between an American and a European put price. At its core, the de-Americanization methodology is replacing American options with European options. Thus, with higher differences between both of them, the potential of making an error increases. Second, we find that

- de-Americanization tends to lead to outliers in scenarios with a higher volatility and/or higher jump intensities.

5 Conclusion

In these scenarios, the model describing the evolution of the underlying differs strongly from the assumptions of the model, namely a constant volatility coefficient and the absence of jumps. The binomial tree is roughly a discrete version of the model and, therefore, the model independent approach of the binomial tree has its limitations the stronger the underlying process differs from the model. For this reason, and since the de-Americanization methodology does not provide an error control, we strongly recommend applying a pricing method in the calibration that is certified by error estimators.

However, also for scenarios in which a direct application of the de-Americanization methodology can lead to outliers, the method can be applied usefully:

- It can, for example, be used as first estimator for the model parameters and this estimate can then be used as initial guess in a calibration routine to reduce the run-time.

For further research, we see three main areas: an adjustment of the calibration routine to incorporate uncertainty in the data, an inclusion of dividend payments and an adjustment of the binomial tree as technique to translate American into European prices.

- Calibration: The effect of uncertainty in the data leads to an additional complexity of the calibration procedure. Calibration is not a deterministic inverse problem anymore, but can be formulated as a Bayesian inverse problem. For instance, Gupta and Reisinger (2014) present a robust calibration of financial models applying Bayesian estimators. Moreover, the impact of the de-Americanization step can be interpreted as distortion of the input data, which can also be considered from a Bayesian perspective.
- Dividend payments: The numerical results for the jump diffusion model and the sensitivity to interest rates indicate that discrete and continuous dividends may intensify the errors caused by the de-Americanization method.
- Binomial tree technique: The tree could be replaced by a more sophisticated, maybe not anymore model invariant, technique. Obviously, one has to keep in mind here that this leads to higher computational effort and some of the advantages of the de-Americanization methodology, such as faster run-times and model flexibility, could be reduced.

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6 Proof of Proposition 2.1

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6 Proof of Proposition 2.1

Proof

We can reduce this case to the one-dimensional case in the following way. Let $j \in \{1, \dots, n\}$ and denote by $P^j(\cdot|\cdot)$ the conditional probability given $\sigma(X_i, i \neq j)$. Then, by definition of the conditional probability

$$E \left[\left(K - \prod_{i=1}^n X_i \right)^+ \right] = \iint \left(K - X_j(\omega') \prod_{i \neq j} X_i(\omega) \right)^+ P^j(d\omega')(\omega) P(d\omega).$$

For all $j \in \{1, \dots, n\}$ and $\omega \in \Omega$ the function $x \mapsto (K - x \prod_{i \neq j} X_i(\omega))^+$ is convex. By the independence of X_j and $\{X_i, i \neq j\}$ a.s. $P^j(X_j \in \cdot) = P(X_j \in \cdot)$. This allows us to use the one-dimensional result from Lemma 6.1 and from $X_j \preceq_{cx} Y_j$ it follows

$$\int \left(K - X_j(\omega') \prod_{i \neq j} X_i \right)^+ P^j(d\omega') \leq \int \left(K - Y_j(\omega') \prod_{i \neq j} X_i \right)^+ P^j(d\omega') \text{ a.s.}$$

This yields

$$E \left[\left(K - \prod_{i=1}^n X_i \right)^+ \right] \leq E \left[\left(K - Y_j \prod_{i \neq j} X_i \right)^+ \right].$$

As a next step, conditioning on $(\sigma(Y_j, X_i, i \neq j, j_2))$ the same technique is applied to show

$$E \left[\left(K - X_{j_2} \prod_{i \neq j, j_2} X_i Y_j \right)^+ \right] \leq E \left[\left(K - Y_{j_2} \prod_{i \neq j, j_2} X_i Y_j \right)^+ \right]$$

and successively, the assertion of the proposition follows. \square

6 Proof of Proposition 2.1

Here, we present an additional lemma which will be used in the proof of Proposition 2.1.

Lemma 6.1

Focusing on one node in the binomial tree, let $X \sim QB(u)$ and $Y \sim QB(u')$ with $u' \geq u$. Let $u, u' > e^{r\Delta t}$ be satisfied. Then the random variable X is smaller than the random variable Y with respect to the convex order, i.e. $X \leq_{cx} Y$.

Proof

Following (Müller and Stoyan, 2002, Theorem 1.5.3 and Theorem 1.5.7) it suffices to show

1. $E[X] = E[Y]$
2. $E[(X - k)^+] \leq E[(Y - k)^+]$

Since p as in (1) is set up as risk-neutral probability, it holds for any u that $E[X] = e^{r\Delta t}$ and thus, the first condition is satisfied. Given a random variable X with a factor u and a random variable Y with factor $u' > u$, we distinguish regarding the second condition 5 cases.

Case 1: $\frac{1}{u'} \leq \frac{1}{u} \leq u \leq u' \leq k$

Obviously, in any case both options are out-of-the-money and $E[(X - k)^+] = 0 = E[(Y - k)^+]$.

Case 2: $\frac{1}{u'} \leq \frac{1}{u} \leq u \leq k \leq u'$.

Here, $E[(X - k)^+] = 0$ and hence, the second condition is satisfied.

Case 3: $\frac{1}{u'} \leq \frac{1}{u} \leq k \leq u \leq u'$.

In this case, we have

$$\begin{aligned} E[(Y - k)^+] - E[(X - k)^+] &= p(u')(u' - k) - p(u)(u - k) \\ &= (u')u' - p(u)u - k(p(u') - p(u)). \end{aligned}$$

The function $p(u) = \frac{ue^{r\Delta t} - 1}{u^2 - 1}$ is a monotonically decreasing function because the derivative $p'(u) = \frac{-e^{r\Delta t} - u^2 e^{r\Delta t} + 2u}{(u^2 - 1)^2}$ would have the roots $u_{a,b} = \frac{1 \pm \sqrt{1 - e^{2r\Delta t}}}{e^{r\Delta t}}$, but due to $e^{r\Delta t} \geq 1$ either the derivative has no roots or a root at $u = 1$ in the case $r = 0$. Thus, by assuming $u', u > e^{r\Delta t}$, $E[(Y - k)^+] - E[(X - k)^+]$ is monotone in k and in Case 4 for $k = \frac{1}{u}$ we show that $E[(Y - k)^+] - E[(X - k)^+] = 0$ holds.

Case 4: $\frac{1}{u'} \leq k < \frac{1}{u} \leq u \leq u'$.

Due to $k \leq \frac{1}{u}$ it follows $E[(X - k)^+] = E[X - k]$. Thus, $E[(Y - k)^+] - E[(X - k)^+] = E[(Y - k)^+] - E[X - k]$. Obviously, it holds

$$E[(Y - k)^+] \geq E[Y - k]$$

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and this leads to $E[(Y - k)^+] - E[X - k] \geq E[Y - k] - E[X - k] = 0$.

Case 5: $k \leq \frac{1}{u'} \leq \frac{1}{u} \leq u \leq u'$.

It holds $E[(Y - k)^+] = E[Y - k]$ and $E[(X - k)^+] = E[X - k]$. Thus, $E[(Y - k)^+] - E[(X - k)^+] = E[Y - k] - E[X - k] = 0$ follows. \square

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Table 9 De-Americanization effects on pricing put options in the CEV model - Average error between the de-Americanized and European prices for each maturity.

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
p_1	$r = 0\%$	1.E-4	1.E-4	1.E-4	1.E-4	1.E-4	1.E-4	9.E-5	6.E-5
	$r = 1\%$	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	2.E-4	1.E-4
	$r = 2\%$	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	2.E-4	8.E-5
	$r = 5\%$	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	1.E-4	-3.E-5
	$r = 7\%$	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	1.E-4	-9.E-5
p_2	$r = 0\%$	2.E-4	2.E-4	2.E-4	1.E-4	1.E-4	1.E-4	1.E-4	7.E-5
	$r = 1\%$	3.E-4	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	1.E-4
	$r = 2\%$	3.E-4	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	5.E-5
	$r = 5\%$	3.E-4	4.E-4	3.E-4	3.E-4	2.E-4	1.E-4	6.E-5	3.E-4
	$r = 7\%$	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	7.E-5	7.E-5	5.E-4
p_3	$r = 0\%$	2.E-4	2.E-4	2.E-4	1.E-4	1.E-4	1.E-4	1.E-4	6.E-5
	$r = 1\%$	3.E-4	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	1.E-4
	$r = 2\%$	3.E-4	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	6.E-5
	$r = 5\%$	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	1.E-4	8.E-5	-2.E-4
	$r = 7\%$	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	1.E-4	-1.E-7	-3.E-4
p_4	$r = 0\%$	2.E-4	1.E-4	1.E-4	1.E-4	1.E-4	1.E-4	8.E-5	8.E-5
	$r = 1\%$	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	2.E-4	2.E-4	3.E-4
	$r = 2\%$	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	1.E-4	1.E-4	3.E-4
	$r = 5\%$	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	1.E-4	8.E-5	1.E-4
	$r = 7\%$	4.E-4	3.E-4	3.E-4	2.E-4	2.E-4	1.E-4	7.E-5	2.E-4
p_5	$r = 0\%$	2.E-4	2.E-4	2.E-4	1.E-4	1.E-4	1.E-4	1.E-4	9.E-5
	$r = 1\%$	3.E-4	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	6.E-7
	$r = 2\%$	3.E-4	3.E-4	3.E-4	3.E-4	2.E-4	2.E-4	1.E-4	-1.E-4
	$r = 5\%$	4.E-4	3.E-4	3.E-4	3.E-4	2.E-4	1.E-4	-7.E-6	-5.E-4
	$r = 7\%$	4.E-4	4.E-4	3.E-4	3.E-4	2.E-4	4.E-5	-1.E-4	-8.E-4

Table 10 De-Americanization effects on pricing put options in the CEV model - Average error between the de-Americanized and European prices for each strike.

		0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20
p_1	$r = 0\%$	4.E-5	7.E-5	1.E-4	2.E-4	2.E-4	2.E-4	1.E-4	1.E-4	8.E-5
	$r = 1\%$	8.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	3.E-4	2.E-4	2.E-4
	$r = 2\%$	8.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	3.E-4	2.E-4	2.E-4
	$r = 5\%$	7.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	2.E-4	2.E-4	1.E-4
	$r = 7\%$	7.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	1.E-4	2.E-4	8.E-5
p_2	$r = 0\%$	5.E-5	9.E-5	1.E-4	2.E-4	2.E-4	2.E-4	1.E-4	1.E-4	8.E-5
	$r = 1\%$	9.E-5	2.E-4	3.E-4	3.E-4	4.E-4	4.E-4	3.E-4	3.E-4	2.E-4
	$r = 2\%$	8.E-5	2.E-4	2.E-4	3.E-4	4.E-4	4.E-4	3.E-4	2.E-4	2.E-4
	$r = 5\%$	6.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	3.E-4	3.E-4	2.E-4
	$r = 7\%$	6.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	3.E-4	3.E-4	2.E-4
p_3	$r = 0\%$	4.E-5	7.E-5	1.E-4	2.E-4	2.E-4	2.E-4	1.E-4	1.E-4	8.E-5
	$r = 1\%$	7.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	3.E-4	3.E-4	2.E-4
	$r = 2\%$	7.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	3.E-4	3.E-4	2.E-4
	$r = 5\%$	5.E-5	1.E-4	2.E-4	3.E-4	4.E-4	3.E-4	2.E-4	2.E-4	6.E-5
	$r = 7\%$	3.E-5	9.E-5	2.E-4	2.E-4	3.E-4	3.E-4	2.E-4	1.E-4	-2.E-5
p_4	$r = 0\%$	4.E-5	7.E-5	1.E-4	1.E-4	2.E-4	2.E-4	1.E-4	1.E-4	9.E-5
	$r = 1\%$	9.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	2.E-4	3.E-4	2.E-4
	$r = 2\%$	8.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	2.E-4	2.E-4	1.E-4
	$r = 5\%$	9.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	2.E-4	2.E-4	-3.E-5
	$r = 7\%$	9.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	2.E-4	2.E-4	-1.E-4
p_5	$r = 0\%$	3.E-5	7.E-5	1.E-4	2.E-4	2.E-4	2.E-4	1.E-4	1.E-4	7.E-5
	$r = 1\%$	6.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	2.E-4	2.E-4	1.E-4
	$r = 2\%$	5.E-5	1.E-4	2.E-4	3.E-4	4.E-4	4.E-4	2.E-4	2.E-4	1.E-4
	$r = 5\%$	2.E-5	7.E-5	1.E-4	2.E-4	3.E-4	3.E-4	1.E-4	1.E-4	-9.E-6
	$r = 7\%$	4.E-5	4.E-5	1.E-4	2.E-4	2.E-4	2.E-4	4.E-5	4.E-5	-1.E-4

Table 11 De-Americanization effects on pricing put options in the CEV model - Maximal European put prices.

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
p_1	$r = 0\%$	0.055	0.104	0.108	0.154	0.204	0.211	0.218	0.244
	$r = 1\%$	0.055	0.102	0.106	0.151	0.155	0.203	0.208	0.226
	$r = 2\%$	0.054	0.101	0.104	0.107	0.151	0.196	0.198	0.208
	$r = 5\%$	0.052	0.057	0.097	0.098	0.100	0.136	0.136	0.161
	$r = 7\%$	0.051	0.054	0.057	0.093	0.093	0.093	0.093	0.109
p_2	$r = 0\%$	0.103	0.154	0.204	0.208	0.217	0.230	0.242	0.283
	$r = 1\%$	0.103	0.152	0.201	0.204	0.212	0.223	0.233	0.265
	$r = 2\%$	0.102	0.109	0.154	0.201	0.207	0.216	0.224	0.248
	$r = 5\%$	0.099	0.104	0.147	0.150	0.192	0.195	0.198	0.203
	$r = 7\%$	0.058	0.101	0.105	0.144	0.146	0.182	0.181	0.176
p_3	$r = 0\%$	0.152	0.205	0.212	0.219	0.233	0.252	0.269	0.320
	$r = 1\%$	0.108	0.203	0.209	0.216	0.228	0.245	0.260	0.303
	$r = 2\%$	0.107	0.201	0.207	0.213	0.224	0.239	0.251	0.287
	$r = 5\%$	0.105	0.153	0.199	0.203	0.210	0.219	0.226	0.241
	$r = 7\%$	0.103	0.150	0.194	0.196	0.201	0.206	0.210	0.214
p_4	$r = 0\%$	0.156	0.212	0.223	0.233	0.252	0.276	0.297	0.352
	$r = 1\%$	0.155	0.210	0.220	0.230	0.247	0.270	0.288	0.336
	$r = 2\%$	0.154	0.208	0.218	0.227	0.243	0.263	0.279	0.319
	$r = 5\%$	0.152	0.203	0.210	0.217	0.229	0.244	0.255	0.275
	$r = 7\%$	0.150	0.200	0.206	0.211	0.221	0.232	0.239	0.247
p_5	$r = 0\%$	0.205	0.220	0.235	0.248	0.272	0.300	0.323	0.377
	$r = 1\%$	0.205	0.219	0.232	0.245	0.267	0.294	0.314	0.360
	$r = 2\%$	0.204	0.217	0.230	0.242	0.263	0.287	0.306	0.345
	$r = 5\%$	0.201	0.212	0.223	0.233	0.250	0.268	0.281	0.300
	$r = 7\%$	0.156	0.209	0.218	0.227	0.241	0.256	0.266	0.273

Table 12 De-Americanization effects on pricing put options in the Heston model - Average error between the de-Americanized and European prices for each maturity.

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
p_1	$r = 0\%$	1.E-8	2.E-7	2.E-7	1.E-7	2.E-7	1.E-7	3.E-7	-3.E-7
	$r = 1\%$	-7.E-5	-4.E-5	-5.E-5	-4.E-5	-4.E-5	-3.E-5	-3.E-5	6.E-6
	$r = 2\%$	-9.E-5	-7.E-5	-9.E-5	-6.E-5	-5.E-5	-5.E-5	-4.E-5	3.E-5
	$r = 5\%$	-3.E-4	-4.E-4	-1.E-4	-9.E-5	-9.E-5	-8.E-5	-7.E-5	3.E-5
	$r = 7\%$	-3.E-4	-3.E-4	-9.E-4	-1.E-4	-8.E-5	-1.E-4	-1.E-4	-4.E-5
p_2	$r = 0\%$	3.E-8	1.E-7	1.E-8	-2.E-7	3.E-7	4.E-7	-8.E-8	-3.E-8
	$r = 1\%$	-4.E-5	-3.E-5	-4.E-5	-3.E-5	-2.E-5	-1.E-5	-2.E-6	9.E-5
	$r = 2\%$	-1.E-4	-5.E-5	-6.E-5	-5.E-5	-4.E-5	-3.E-5	-8.E-6	1.E-4
	$r = 5\%$	-2.E-4	-7.E-5	-1.E-4	-9.E-5	-8.E-5	-9.E-5	-1.E-4	2.E-5
	$r = 7\%$	-2.E-4	-6.E-4	-1.E-4	-1.E-4	-1.E-4	-2.E-4	-2.E-4	-3.E-4
p_3	$r = 0\%$	3.E-9	4.E-8	-3.E-7	3.E-9	8.E-8	-2.E-7	5.E-7	1.E-7
	$r = 1\%$	-3.E-5	-2.E-5	-3.E-5	-2.E-5	-6.E-6	1.E-5	3.E-5	2.E-4
	$r = 2\%$	-6.E-5	-4.E-5	-4.E-5	-3.E-5	-7.E-6	2.E-5	6.E-5	3.E-4
	$r = 5\%$	-1.E-4	-7.E-5	-9.E-5	-5.E-5	-2.E-5	2.E-5	7.E-5	5.E-4
	$r = 7\%$	-4.E-4	-9.E-5	-1.E-4	-7.E-5	-4.E-5	-2.E-5	1.E-5	4.E-4
p_4	$r = 0\%$	2.E-8	-8.E-8	2.E-7	4.E-8	-5.E-7	-1.E-7	-4.E-7	2.E-7
	$r = 1\%$	-3.E-5	-2.E-5	-2.E-5	-1.E-5	2.E-6	2.E-5	5.E-5	2.E-4
	$r = 2\%$	-5.E-5	-3.E-5	-3.E-5	-1.E-5	1.E-5	5.E-5	9.E-5	4.E-4
	$r = 5\%$	-1.E-4	-5.E-5	-6.E-5	-1.E-5	4.E-5	1.E-4	2.E-4	8.E-4
	$r = 7\%$	-1.E-4	-6.E-5	-8.E-5	-1.E-5	4.E-5	1.E-4	2.E-4	9.E-4
p_5	$r = 0\%$	-1.E-7	-3.E-7	-3.E-7	2.E-7	5.E-8	8.E-7	1.E-6	-2.E-6
	$r = 1\%$	-2.E-5	-2.E-5	-2.E-5	-7.E-6	2.E-5	8.E-5	2.E-4	7.E-4
	$r = 2\%$	-4.E-5	-3.E-5	-4.E-5	-1.E-5	3.E-5	1.E-4	3.E-4	1.E-3
	$r = 5\%$	-1.E-4	-7.E-5	-1.E-4	-5.E-5	2.E-5	2.E-4	5.E-4	2.E-3
	$r = 7\%$	-1.E-4	-1.E-4	-1.E-4	-9.E-5	-1.E-5	2.E-4	5.E-4	3.E-3

Table 13 De-Americanization effects on pricing put options in the Heston model - Average error between the de-Americanized and European prices for each strike. The empty fields are due to Remark 4.1.

		0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20
p_1	$r = 0\%$	-4.E-9	5.E-8	1.E-8	-1.E-7	-6.E-8	-5.E-7	-3.E-8	-2.E-7	8.E-7
	$r = 1\%$	3.E-6	6.E-6	9.E-6	1.E-5	2.E-5	4.E-5	7.E-5	2.E-4	3.E-4
	$r = 2\%$	8.E-6	1.E-5	2.E-5	3.E-5	5.E-5	9.E-5	2.E-4	5.E-4	9.E-4
	$r = 5\%$	2.E-5	3.E-5	5.E-5	8.E-5	1.E-4	2.E-4	6.E-4	2.E-3	4.E-3
	$r = 7\%$	2.E-5	4.E-5	7.E-5	1.E-4	2.E-4	4.E-4	9.E-4	3.E-3	
p_2	$r = 0\%$	3.E-8	-6.E-8	8.E-9	1.E-7	-2.E-7	-4.E-7	-3.E-7	5.E-7	-6.E-7
	$r = 1\%$	-2.E-5	-1.E-5	-2.E-6	7.E-6	2.E-5	3.E-5	4.E-5	1.E-4	2.E-4
	$r = 2\%$	-2.E-5	-2.E-6	1.E-5	3.E-5	4.E-5	6.E-5	1.E-4	2.E-4	4.E-4
	$r = 5\%$	1.E-6	3.E-5	6.E-5	9.E-5	1.E-4	2.E-4	3.E-4	8.E-4	1.E-3
	$r = 7\%$	1.E-5	5.E-5	8.E-5	1.E-4	2.E-4	3.E-4	4.E-4	1.E-3	2.E-3
p_3	$r = 0\%$	7.E-8	8.E-7	7.E-7	8.E-7	7.E-7	2.E-6	7.E-7	3.E-6	2.E-6
	$r = 1\%$	2.E-6	4.E-6	9.E-6	1.E-5	2.E-5	3.E-5	4.E-5	6.E-5	1.E-4
	$r = 2\%$	5.E-6	1.E-5	2.E-5	4.E-5	6.E-5	8.E-5	9.E-5	2.E-4	3.E-4
	$r = 5\%$	2.E-5	5.E-5	9.E-5	1.E-4	2.E-4	2.E-4	4.E-4	8.E-4	1.E-3
	$r = 7\%$	4.E-5	9.E-5	1.E-4	2.E-4	3.E-4	3.E-4	7.E-4	1.E-3	2.E-3
p_4	$r = 0\%$	-2.E-8	-8.E-8	2.E-7	-1.E-8	1.E-7	-7.E-8	1.E-7	4.E-9	-2.E-7
	$r = 1\%$	-2.E-4	-2.E-4	-2.E-4	-2.E-4	-2.E-4	-2.E-4	-1.E-4	-9.E-5	-7.E-5
	$r = 2\%$	-4.E-4	-4.E-4	-3.E-4	-3.E-4	-2.E-4	-2.E-4	-1.E-4	-7.E-5	-9.E-6
	$r = 5\%$	-7.E-4	-6.E-4	-5.E-4	-4.E-4	-2.E-4	-1.E-4	-5.E-5	1.E-4	3.E-4
	$r = 7\%$	-7.E-4	-6.E-4	-5.E-4	-3.E-4	-2.E-4	-1.E-4	3.E-5	3.E-4	6.E-4
p_5	$r = 0\%$	2.E-7	2.E-7	4.E-7	7.E-7	7.E-7	4.E-7	8.E-7	1.E-6	9.E-7
	$r = 1\%$	4.E-4	4.E-4	4.E-4	4.E-4	3.E-4	3.E-4	3.E-4	3.E-4	3.E-4
	$r = 2\%$	7.E-4	7.E-4	6.E-4	6.E-4	6.E-4	5.E-4	5.E-4	4.E-4	4.E-4
	$r = 5\%$	1.E-3	1.E-3	1.E-3	1.E-3	9.E-4	7.E-4	6.E-4	5.E-4	5.E-4
	$r = 7\%$	3.E-4	1.E-3	1.E-3	1.E-3	9.E-4	7.E-4	6.E-4	6.E-4	5.E-4

Table 14 De-Americanization effects on pricing put options in the Heston model - Maximal European put prices.

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
p_1	$r = 0\%$	0.200	0.202	0.205	0.209	0.217	0.229	0.240	0.278
	$r = 1\%$	0.199	0.200	0.202	0.205	0.212	0.222	0.231	0.261
	$r = 2\%$	0.198	0.198	0.200	0.202	0.207	0.214	0.221	0.244
	$r = 5\%$	0.195	0.193	0.191	0.191	0.192	0.194	0.195	0.198
	$r = 7\%$	0.194	0.189	0.186	0.184	0.182	0.180	0.179	0.171
p_2	$r = 0\%$	0.201	0.204	0.208	0.214	0.224	0.238	0.251	0.293
	$r = 1\%$	0.200	0.202	0.206	0.210	0.219	0.231	0.242	0.277
	$r = 2\%$	0.199	0.200	0.203	0.207	0.214	0.224	0.233	0.261
	$r = 5\%$	0.196	0.195	0.195	0.197	0.200	0.204	0.208	0.217
	$r = 7\%$	0.194	0.191	0.190	0.190	0.191	0.192	0.193	0.192
p_3	$r = 0\%$	0.202	0.208	0.216	0.224	0.238	0.256	0.273	0.326
	$r = 1\%$	0.201	0.207	0.213	0.220	0.233	0.250	0.264	0.309
	$r = 2\%$	0.200	0.205	0.211	0.217	0.228	0.243	0.255	0.294
	$r = 5\%$	0.197	0.199	0.203	0.207	0.215	0.224	0.231	0.250
	$r = 7\%$	0.195	0.196	0.198	0.201	0.206	0.212	0.216	0.225
p_4	$r = 0\%$	0.204	0.214	0.224	0.234	0.252	0.275	0.295	0.362
	$r = 1\%$	0.203	0.212	0.221	0.231	0.247	0.268	0.287	0.345
	$r = 2\%$	0.202	0.210	0.219	0.227	0.243	0.262	0.278	0.329
	$r = 5\%$	0.199	0.205	0.211	0.218	0.229	0.243	0.255	0.286
	$r = 7\%$	0.197	0.202	0.207	0.212	0.221	0.231	0.240	0.259
p_5	$r = 0\%$	0.207	0.222	0.234	0.248	0.270	0.297	0.322	0.400
	$r = 1\%$	0.206	0.220	0.232	0.245	0.265	0.291	0.314	0.384
	$r = 2\%$	0.205	0.218	0.230	0.242	0.261	0.285	0.306	0.369
	$r = 5\%$	0.202	0.213	0.223	0.233	0.249	0.268	0.283	0.326
	$r = 7\%$	0.201	0.210	0.219	0.227	0.241	0.257	0.269	0.300

Table 15 Overview of additionally investigated parameter sets for the Heston model.

		Heston				
		ξ	ρ	γ	κ	v_0
p_6		0.25	-0.50	0.3	0.4	0.1
p_7	0.25	-0.50	0.1	0.4	0.3	
p_8		0.70	-0.80	0.3	1.4	0.1
p_9		0.70	-0.80	0.1	1.4	0.3

Table 16 De-Americanization effects on pricing put options in the Heston model - Average error between De-Americanized and European prices for each maturity in the additional scenarios.

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
p_6	$r = 0\%$	-5.E-9	-1.E-7	1.E-7	-1.E-7	1.E-7	4.E-7	3.E-7	-2.E-7
	$r = 1\%$	-8.E-5	-8.E-5	-1.E-4	-1.E-4	-1.E-4	-2.E-4	-2.E-4	-4.E-4
	$r = 2\%$	-2.E-4	-1.E-4	-2.E-4	-2.E-4	-3.E-4	-4.E-4	-5.E-4	-1.E-3
	$r = 5\%$	-4.E-4	-3.E-4	-4.E-4	-4.E-4	-7.E-4	-1.E-3	-1.E-3	-3.E-3
	$r = 7\%$	-5.E-4	-4.E-4	-5.E-4	-5.E-4	-8.E-4	-2.E-3	-2.E-3	-5.E-3
p_7	$r = 0\%$	-8.E-8	2.E-7	-2.E-8	-1.E-7	-4.E-7	-4.E-7	3.E-7	2.E-6
	$r = 1\%$	-4.E-5	-4.E-5	-4.E-5	-4.E-5	-3.E-5	8.E-6	6.E-5	5.E-4
	$r = 2\%$	-7.E-5	-6.E-5	-7.E-5	-5.E-5	-3.E-5	5.E-5	2.E-4	1.E-3
	$r = 5\%$	-2.E-4	-1.E-4	-1.E-4	-8.E-5	8.E-7	2.E-4	4.E-4	2.E-3
	$r = 7\%$	-2.E-4	-2.E-4	-2.E-4	-9.E-5	4.E-5	3.E-4	6.E-4	3.E-3
p_8	$r = 0\%$	2.E-5	3.E-5	3.E-5	3.E-5	3.E-5	3.E-5	3.E-5	5.E-5
	$r = 1\%$	-5.E-5	4.E-5	-1.E-5	-3.E-5	-5.E-5	-6.E-5	-4.E-5	3.E-4
	$r = 2\%$	-1.E-4	3.E-5	-6.E-5	-1.E-4	-2.E-4	-3.E-4	-2.E-4	2.E-4
	$r = 5\%$	-2.E-4	-3.E-5	-2.E-4	-4.E-4	-8.E-4	-1.E-3	-1.E-3	-9.E-4
	$r = 7\%$	-2.E-4	-1.E-4	-3.E-4	-5.E-4	-1.E-3	-2.E-3	-2.E-3	-2.E-3
p_9	$r = 0\%$	-4.E-7	1.E-6	7.E-6	2.E-5	6.E-5	2.E-5	3.E-5	6.E-5
	$r = 1\%$	-3.E-5	-9.E-6	2.E-5	7.E-5	2.E-4	5.E-4	8.E-4	1.E-3
	$r = 2\%$	-5.E-5	-2.E-5	2.E-5	1.E-4	3.E-4	7.E-4	1.E-3	2.E-3
	$r = 5\%$	-1.E-4	-3.E-5	4.E-5	2.E-4	6.E-4	9.E-4	1.E-3	4.E-3
	$r = 7\%$	-2.E-4	-3.E-5	3.E-5	3.E-4	7.E-4	1.E-3	2.E-3	4.E-3

Table 17 De-Americanization effects on pricing put options in the Heston model - Average error between De-Americanized and European prices for each strike in the additional scenarios.

		0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20
p_6	$r = 0\%$	-9.E-8	1.E-7	-3.E-8	-2.E-8	8.E-8	3.E-7	-4.E-7	3.E-7	4.E-7
	$r = 1\%$	-4.E-5	-6.E-5	-8.E-5	-1.E-4	-1.E-4	-2.E-4	-2.E-4	-3.E-4	-3.E-4
	$r = 2\%$	-9.E-5	-1.E-4	-2.E-4	-2.E-4	-3.E-4	-4.E-4	-5.E-4	-7.E-4	-8.E-4
	$r = 5\%$	-2.E-4	-3.E-4	-5.E-4	-6.E-4	-9.E-4	-1.E-3	-2.E-3	-2.E-3	-2.E-3
	$r = 7\%$	-3.E-4	-5.E-4	-7.E-4	-9.E-4	-1.E-3	-2.E-3	-2.E-3	-3.E-3	-2.E-3
p_7	$r = 0\%$	-5.E-8	5.E-7	3.E-7	-3.E-7	-8.E-8	3.E-7	7.E-7	7.E-7	-6.E-7
	$r = 1\%$	2.E-5	3.E-5	4.E-5	4.E-5	5.E-5	6.E-5	6.E-5	7.E-5	8.E-5
	$r = 2\%$	6.E-5	7.E-5	9.E-5	1.E-4	1.E-4	1.E-4	2.E-4	2.E-4	2.E-4
	$r = 5\%$	2.E-4	2.E-4	2.E-4	3.E-4	3.E-4	4.E-4	4.E-4	4.E-4	5.E-4
	$r = 7\%$	2.E-4	3.E-4	3.E-4	4.E-4	4.E-4	5.E-4	5.E-4	6.E-4	7.E-4
p_8	$r = 0\%$	3.E-6	4.E-6	5.E-6	8.E-6	1.E-5	2.E-5	4.E-5	7.E-5	1.E-4
	$r = 1\%$	1.E-5	8.E-6	4.E-6	-4.E-6	-2.E-5	-3.E-5	-1.E-5	-6.E-7	1.E-4
	$r = 2\%$	-2.E-6	-2.E-5	-3.E-5	-7.E-5	-1.E-4	-2.E-4	-2.E-4	-2.E-4	-5.E-5
	$r = 5\%$	-1.E-4	-2.E-4	-3.E-4	-4.E-4	-6.E-4	-8.E-4	-1.E-3	-1.E-3	-9.E-4
	$r = 7\%$	-2.E-4	-3.E-4	-5.E-4	-7.E-4	-1.E-3	-1.E-3	-2.E-3	-2.E-3	-2.E-3
p_9	$r = 0\%$	4.E-5	5.E-5	7.E-5	9.E-5	1.E-4	1.E-4	2.E-4	3.E-4	3.E-4
	$r = 1\%$	2.E-4	2.E-4	3.E-4	4.E-4	4.E-4	5.E-4	7.E-4	8.E-4	1.E-3
	$r = 2\%$	3.E-4	4.E-4	5.E-4	6.E-4	7.E-4	8.E-4	1.E-3	1.E-3	2.E-3
	$r = 5\%$	6.E-4	7.E-4	9.E-4	1.E-3	1.E-3	1.E-3	2.E-3	2.E-3	3.E-3
	$r = 7\%$	7.E-4	8.E-4	1.E-3	1.E-3	1.E-3	2.E-3	2.E-3	3.E-3	3.E-3

Table 18 De-Americanization effects on pricing put options in the Heston model - Maximal European put prices in the additional scenarios.

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
p_6	$r = 0\%$	0.201	0.204	0.209	0.216	0.229	0.249	0.269	0.343
	$r = 1\%$	0.200	0.202	0.207	0.212	0.224	0.242	0.260	0.327
	$r = 2\%$	0.199	0.201	0.204	0.209	0.219	0.236	0.252	0.311
	$r = 5\%$	0.196	0.195	0.196	0.199	0.205	0.216	0.227	0.266
	$r = 7\%$	0.194	0.192	0.191	0.192	0.196	0.204	0.212	0.240
p_7	$r = 0\%$	0.208	0.225	0.240	0.254	0.277	0.305	0.327	0.390
	$r = 1\%$	0.208	0.224	0.237	0.251	0.273	0.298	0.319	0.373
	$r = 2\%$	0.207	0.222	0.235	0.248	0.268	0.292	0.311	0.357
	$r = 5\%$	0.204	0.217	0.228	0.239	0.255	0.274	0.287	0.312
	$r = 7\%$	0.202	0.214	0.224	0.233	0.247	0.262	0.272	0.285
p_8	$r = 0\%$	0.200	0.203	0.208	0.215	0.230	0.255	0.280	0.366
	$r = 1\%$	0.199	0.201	0.205	0.211	0.226	0.249	0.272	0.350
	$r = 2\%$	0.198	0.199	0.203	0.208	0.221	0.243	0.264	0.335
	$r = 5\%$	0.195	0.194	0.195	0.198	0.208	0.224	0.241	0.293
	$r = 7\%$	0.193	0.190	0.190	0.192	0.199	0.213	0.226	0.268
p_9	$r = 0\%$	0.206	0.218	0.229	0.238	0.253	0.270	0.283	0.320
	$r = 1\%$	0.205	0.217	0.226	0.235	0.249	0.264	0.275	0.305
	$r = 2\%$	0.204	0.215	0.224	0.232	0.245	0.257	0.267	0.290
	$r = 5\%$	0.202	0.210	0.217	0.223	0.232	0.240	0.244	0.249
	$r = 7\%$	0.200	0.207	0.213	0.218	0.224	0.229	0.230	0.225

Table 19 De-Americanization effects on pricing put options in the Merton model - Average error between the de-Americanized and European prices for each maturity.

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
p_1	$r = 0\%$	-3.E-4	-3.E-4	-2.E-4	-2.E-4	-2.E-4	-2.E-4	-2.E-4	-1.E-4
	$r = 1\%$	-3.E-4	-3.E-4	-3.E-4	-3.E-4	-2.E-4	-2.E-4	-2.E-4	-2.E-4
	$r = 2\%$	-3.E-4	-3.E-4	-3.E-4	-3.E-4	-3.E-4	-2.E-4	-2.E-4	-2.E-4
	$r = 5\%$	-2.E-4	-2.E-4	-1.E-4	-2.E-4	-2.E-4	-3.E-5	-9.E-5	9.E-5
	$r = 7\%$	-2.E-4	-2.E-4	-4.E-5	-9.E-6	-5.E-5	-5.E-5	3.E-4	9.E-5
p_2	$r = 0\%$	-1.E-4	-1.E-4	-1.E-4	-1.E-4	-1.E-4	-1.E-4	-1.E-4	-8.E-5
	$r = 1\%$	-1.E-4	-2.E-4	-2.E-4	-2.E-4	-2.E-4	-2.E-4	-3.E-4	-3.E-4
	$r = 2\%$	-1.E-4	-1.E-4	-1.E-4	-2.E-4	-2.E-4	-3.E-4	-3.E-4	-5.E-4
	$r = 5\%$	-2.E-5	-3.E-5	-1.E-4	-3.E-5	-6.E-5	-3.E-4	-2.E-4	-5.E-4
	$r = 7\%$	-4.E-5	9.E-5	-5.E-6	-1.E-4	3.E-5	6.E-5	-2.E-4	-4.E-4
p_3	$r = 0\%$	-1.E-4	-9.E-5	-8.E-5	-7.E-5	-6.E-5	-6.E-5	-5.E-5	-4.E-5
	$r = 1\%$	-1.E-4	-1.E-4	-2.E-4	-2.E-4	-3.E-4	-3.E-4	-4.E-4	-6.E-4
	$r = 2\%$	-1.E-4	-2.E-4	-2.E-4	-3.E-4	-4.E-4	-6.E-4	-7.E-4	-1.E-3
	$r = 5\%$	-1.E-4	-2.E-4	-3.E-4	-4.E-4	-7.E-4	-1.E-3	-2.E-3	-2.E-3
	$r = 7\%$	-9.E-5	-2.E-4	-3.E-4	-5.E-4	-8.E-4	-1.E-3	-2.E-3	-3.E-3
p_4	$r = 0\%$	1.E-6	-2.E-6	-3.E-7	4.E-7	1.E-7	-3.E-7	3.E-6	-1.E-6
	$r = 1\%$	8.E-5	1.E-4	1.E-4	8.E-5	5.E-5	-8.E-6	-8.E-5	-4.E-4
	$r = 2\%$	2.E-4	2.E-4	2.E-4	1.E-4	2.E-5	-1.E-4	-3.E-4	-1.E-3
	$r = 5\%$	4.E-4	6.E-4	4.E-4	1.E-4	-3.E-4	-9.E-4	-1.E-3	-3.E-3
	$r = 7\%$	3.E-4	5.E-4	6.E-4	7.E-5	-6.E-4	-3.E-3	-2.E-3	-4.E-3
p_5	$r = 0\%$	-2.E-6	-8.E-7	-1.E-6	4.E-7	-2.E-7	-2.E-6	-4.E-6	-1.E-6
	$r = 1\%$	5.E-5	3.E-5	-2.E-5	-6.E-5	-2.E-4	-3.E-4	-4.E-4	-9.E-4
	$r = 2\%$	1.E-4	2.E-5	-9.E-5	-2.E-4	-4.E-4	-7.E-4	-1.E-3	-2.E-3
	$r = 5\%$	2.E-4	-4.E-5	-4.E-4	-8.E-4	-1.E-3	-2.E-3	-3.E-3	-5.E-3
	$r = 7\%$	4.E-5	-6.E-5	-8.E-4	-1.E-3	-2.E-3	-3.E-3	-4.E-3	-6.E-3

Table 20 De-Americanization effects on pricing put options in the Merton model - Average error between the de-Americanized and European prices for each strike. The empty fields are due to Remark 4.1.

		0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20
p_1	$r = 0\%$	8.E-05	1.E-04	2.E-04	3.E-04	3.E-04	3.E-04	2.E-04	2.E-04	3.E-04
	$r = 1\%$	8.E-05	1.E-04	2.E-04	3.E-04	3.E-04	3.E-04	3.E-04	3.E-04	2.E-04
	$r = 2\%$	7.E-05	1.E-04	2.E-04	3.E-04	3.E-04	3.E-04	3.E-04	2.E-04	2.E-04
	$r = 5\%$	5.E-05	9.E-05	2.E-04	2.E-04	2.E-04	2.E-04	3.E-05	-2.E-04	-6.E-04
	$r = 7\%$	3.E-05	7.E-05	1.E-04	2.E-04	2.E-04	8.E-05	-4.E-04	-8.E-04	0.E00
p_2	$r = 0\%$	2.E-05	4.E-05	7.E-05	1.E-04	2.E-04	2.E-04	2.E-04	2.E-04	1.E-04
	$r = 1\%$	3.E-05	6.E-05	1.E-04	2.E-04	3.E-04	4.E-04	4.E-04	4.E-04	4.E-04
	$r = 2\%$	4.E-05	7.E-05	1.E-04	2.E-04	3.E-04	4.E-04	4.E-04	4.E-04	3.E-04
	$r = 5\%$	5.E-05	1.E-04	2.E-04	3.E-04	3.E-04	4.E-04	3.E-04	-3.E-05	-4.E-04
	$r = 7\%$	6.E-05	1.E-04	2.E-04	3.E-04	4.E-04	8.E-05	-1.E-03	-1.E-03	0.E+00
p_3	$r = 0\%$	2.E-05	3.E-05	4.E-05	5.E-05	8.E-05	1.E-04	1.E-04	9.E-05	7.E-05
	$r = 1\%$	9.E-05	1.E-04	2.E-04	2.E-04	3.E-04	4.E-04	4.E-04	4.E-04	5.E-04
	$r = 2\%$	2.E-04	2.E-04	3.E-04	4.E-04	5.E-04	6.E-04	6.E-04	7.E-04	8.E-04
	$r = 5\%$	3.E-04	5.E-04	6.E-04	8.E-04	1.E-03	1.E-03	1.E-03	1.E-03	1.E-03
	$r = 7\%$	5.E-04	6.E-04	8.E-04	1.E-03	1.E-03	1.E-03	2.E-03	1.E-03	2.E-03
p_4	$r = 0\%$	-3.E-06	2.E-06	2.E-06	-3.E-06	3.E-06	-2.E-06	-3.E-07	-2.E-06	5.E-07
	$r = 1\%$	4.E-05	5.E-05	4.E-05	5.E-05	4.E-05	2.E-05	-9.E-06	-4.E-05	-9.E-05
	$r = 2\%$	1.E-04	1.E-04	2.E-04	2.E-04	2.E-04	1.E-04	8.E-05	5.E-06	-9.E-05
	$r = 5\%$	4.E-04	5.E-04	6.E-04	7.E-04	7.E-04	7.E-04	5.E-04	2.E-04	1.E-04
	$r = 7\%$	7.E-04	8.E-04	1.E-03	1.E-03	1.E-03	1.E-03	8.E-04	8.E-04	8.E-04
p_5	$r = 0\%$	2.E-06	-2.E-07	-6.E-09	-3.E-07	4.E-06	1.E-06	2.E-06	7.E-08	3.E-06
	$r = 1\%$	1.E-04	2.E-04	2.E-04	2.E-04	2.E-04	3.E-04	3.E-04	3.E-04	3.E-04
	$r = 2\%$	3.E-04	4.E-04	4.E-04	5.E-04	5.E-04	6.E-04	6.E-04	7.E-04	7.E-04
	$r = 5\%$	9.E-04	1.E-03	1.E-03	1.E-03	2.E-03	2.E-03	2.E-03	2.E-03	2.E-03
	$r = 7\%$	1.E-03	1.E-03	2.E-03	2.E-03	2.E-03	2.E-03	3.E-03	3.E-03	3.E-03

Table 21 De-Americanization effects on pricing put options in the Merton model - Maximal European put prices.

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
p_1	$r = 0\%$	0.200	0.200	0.201	0.203	0.207	0.214	0.221	0.248
	$r = 1\%$	0.199	0.198	0.198	0.199	0.201	0.206	0.211	0.230
	$r = 2\%$	0.198	0.196	0.195	0.195	0.196	0.199	0.202	0.212
	$r = 5\%$	0.195	0.190	0.187	0.184	0.180	0.176	0.174	0.165
	$r = 7\%$	0.193	0.186	0.181	0.177	0.170	0.163	0.157	0.138
p_2	$r = 0\%$	0.200	0.200	0.200	0.201	0.203	0.208	0.214	0.237
	$r = 1\%$	0.199	0.198	0.197	0.197	0.198	0.200	0.204	0.218
	$r = 2\%$	0.198	0.196	0.194	0.193	0.192	0.192	0.194	0.200
	$r = 5\%$	0.195	0.190	0.185	0.182	0.176	0.170	0.165	0.152
	$r = 7\%$	0.193	0.186	0.180	0.174	0.165	0.155	0.148	0.125
p_3	$r = 0\%$	0.200	0.201	0.205	0.210	0.221	0.237	0.252	0.302
	$r = 1\%$	0.199	0.199	0.202	0.206	0.216	0.230	0.243	0.284
	$r = 2\%$	0.198	0.197	0.200	0.203	0.211	0.223	0.234	0.268
	$r = 5\%$	0.195	0.192	0.191	0.193	0.197	0.203	0.209	0.223
	$r = 7\%$	0.193	0.188	0.186	0.186	0.188	0.191	0.194	0.197
p_4	$r = 0\%$	0.205	0.212	0.221	0.232	0.256	0.288	0.315	0.398
	$r = 1\%$	0.204	0.210	0.219	0.229	0.252	0.282	0.307	0.381
	$r = 2\%$	0.203	0.208	0.216	0.226	0.247	0.276	0.299	0.365
	$r = 5\%$	0.200	0.203	0.208	0.216	0.235	0.257	0.275	0.319
	$r = 7\%$	0.198	0.199	0.203	0.210	0.227	0.246	0.260	0.292
p_5	$r = 0\%$	0.205	0.217	0.238	0.260	0.295	0.336	0.370	0.474
	$r = 1\%$	0.204	0.215	0.236	0.258	0.291	0.330	0.362	0.457
	$r = 2\%$	0.203	0.213	0.234	0.255	0.286	0.324	0.354	0.441
	$r = 5\%$	0.200	0.208	0.228	0.247	0.274	0.306	0.330	0.394
	$r = 7\%$	0.198	0.205	0.224	0.241	0.267	0.294	0.315	0.366