

Periodic points, escape rates and escape measures

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ABSTRACT. For piecewise real analytic expanding Markov maps, with Markov hole, it is shown that the escape rate and corresponding escape measure can be rapidly approximated using periodic points.

1. Introduction

For a dynamical system $T : X \rightarrow X$, a non-empty subset $H \subset X$ induces an *escape time function*

$$e(x) = e_H(x) = \min \{n \geq 0 : T^n(x) \in H\} ,$$

the nomenclature motivated by interpreting H as a *hole* in phase space X , through which points may escape under iteration. The sequence of super-level sets $E_n = \{x \in X : e(x) > n\}$ decreases with n , and for a probability measure m on X it is often the case that $m(E_n) \rightarrow 0$ as $n \rightarrow \infty$.

If T is a suitable hyperbolic map and m is for example Lebesgue measure, then the $m(E_n)$ approach zero at an exponential rate. In this case the exponential decay rate

$$\delta = \delta(T, H, m) = \lim_{n \rightarrow \infty} m(E_n)^{1/n}$$

is a quantity of interest; indeed

$$\varepsilon = \varepsilon(T, H, m) = -\log \delta(T, H, m)$$

is commonly referred to as the *escape rate*, and has been widely studied (see e.g. [1, 2, 7, 8, 9, 10, 16]). In certain special cases $\delta(T, H, m)$ can be found exactly¹, though in general this is not feasible, so there is interest in developing methods for its efficient approximation.

The purpose of this note is to describe, in the context of analytic expanding maps T , a method for rapidly approximating $\delta = \delta(T, H, m)$. It relies on locating all periodic

¹e.g. if $T(x) = 3x \pmod{1}$ on the interval $X = [0, 1]$, and H is the ‘middle third’ $(1/3, 2/3)$, then $m(E_n) = (2/3)^n$ for each n , so that $\delta(T, H, m) = 2/3$.

points of T , up to a certain period N , say. This yields (see §3 for further details) an approximation $\delta_N \approx \delta$, where the error satisfies

$$|\delta - \delta_N| \leq C\theta^{N^2}$$

for some $\theta \in (0, 1)$, $C \geq 0$; in particular, the δ_N approximate δ *super-exponentially* fast.

For example if the map $T : [0, 1] \rightarrow [0, 1]$ is defined, as in [2], by

$$T(x) = \begin{cases} \frac{9x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{10} \\ 10x - i & \text{if } \frac{i}{10} < x \leq \frac{i+1}{10} \text{ for } 1 \leq i \leq 9, \end{cases}$$

and $H = [\frac{9}{10}, 1]$, we derive (see §5 for further details) the successive approximations

$$\begin{aligned} \delta_2 &= 0.899376191482276109518851011534 \\ \delta_3 &= 0.901142928953763644891210358737 \\ \delta_4 &= 0.901139819292137417448614669069 \\ \delta_5 &= 0.901139820047631592907392158902 \\ \delta_6 &= 0.901139820047605710579196990120 \\ \delta_7 &= 0.901139820047605710706369756237 \end{aligned}$$

In fact our techniques also yield a means of rapidly approximating the corresponding *escape measure* μ , the T -invariant measure supported on the *survivor set* $E_\infty = e_H^{-1}(\infty)$ and maximizing the quantity $h(m) - \int_{E_\infty} \log |T'| dm$ over all T -invariant probability measures m , where $h(\cdot)$ denotes metric entropy (see e.g. [8, 10, 15]). For example μ is completely determined by its sequence of n -th moments $\mu(n) = \int x^n d\mu(x)$, which in general are not known exactly, but the periodic points of T can again be used (see §4 for the method, and §5 for an example) to derive a sequence $\mu_N(n)$, where $|\mu(n) - \mu_N(n)| = O(\theta^{N^2})$ as $N \rightarrow \infty$.

This article is organised as follows. After some preliminaries on transfer operators and their determinants in §2, the method for approximating the escape rate is described in §3, and for the escape measure in §4. In the final §5, the speed of convergence of these methods is illustrated using the map T and hole H defined above.

2. Transfer operators and determinants

Suppose the unit interval² $I = [0, 1]$ is partitioned as $I = I_1 \cup \dots \cup I_d$, $d \geq 2$, where the I_i are closed intervals with pairwise disjoint interiors. We shall assume that $T : I \rightarrow I$ is such that $T|_{I_i}$ is real analytic, for each i , and *expanding* in the sense that $\min \{|T'(x)| : x \in I_i, 1 \leq i \leq d\} > 1$. We say that T is *Markov* if for each $1 \leq i \leq d$

²For simplicity of exposition we restrict attention to one-dimensional dynamical systems, though in fact similar results apply to real analytic expanding Markov maps in higher dimensions. In dimension D the rate of convergence (of δ_N to δ , and of $\mu_N(n)$ to $\mu(n)$) can be shown to be $O(\theta^{N^{1+D^{-1}}})$ as $N \rightarrow \infty$, for some $0 < \theta < 1$; in particular it is super-exponential.

the closure of $T(I_i)$ is a union of elements of the partition $\alpha = \{I_1, \dots, I_d\}$, in which case α is referred to as the *Markov partition*. For each $n \geq 1$, define the usual refined partition $\alpha^{(n)} = \bigvee_{i=0}^{n-1} T^{-i}\alpha$. By a *Markov hole* we mean a union of members of $\alpha^{(n)}$, for some $n \geq 1$. The fact that T is expanding ensures that any sub-interval $H \subset I$ can be approximated arbitrarily well by a Markov hole³.

Although the techniques described below apply, with slight modification, to general Markov holes H for Markov maps T , for simplicity of exposition we shall henceforth assume that for each $1 \leq i \leq d$ the closure of $T(I_i)$ equals I (the so-called *Bernoulli* case), and that the hole $H \subset I$ is a member of α .

We denote by $T_i : I \rightarrow I_i$ ($1 \leq i \leq d$) the contractions which are inverse branches to T . By the implicit function theorem the maps T_i are real analytic, since each $T|_{I_i}$ is real analytic. In particular, we can choose a bounded open neighbourhood $U \subset \mathbb{C}$ containing I such that

$$\overline{\bigcup_{i=1}^d T_i U} \subset U, \quad (2.1)$$

where here T_i denotes the relevant holomorphic extension to U .

Let $A^2(U)$ denote the Hilbert space of analytic functions $f : U \rightarrow \mathbb{C}$ which are square-integrable with respect to 2-dimensional Lebesgue measure on U equipped with the usual inner product.

We may now define a *transfer operator* \mathcal{L} acting on $A^2(U)$ by

$$\mathcal{L}f(z) = \sum_{i=1}^d \epsilon_i T_i'(z) f(T_i z) \text{ where } f \in A^2(U). \quad (2.2)$$

Here $\epsilon_i \in \{-1, 1\}$ denotes the sign of the derivative of T_i on I .

Using (2.1) it is not difficult to see that \mathcal{L} maps $A^2(U)$ continuously into itself. In fact, on this space the transfer operator has strong spectral properties, which will be crucial for the results to follow. The spectral properties are conveniently described in terms of the theory of exponential classes developed in [3], which we briefly recall. Given positive real numbers a and γ , a bounded operator L on a Hilbert space is said to belong to the exponential class $E(a, \gamma)$ if

$$\sup_{n \in \mathbb{N}} s_n(L) \exp(an^\gamma) < \infty,$$

where $s_n(L) = \inf \{ \|L - K\| : \text{rank}(K) < n \}$ denotes the n -th *approximation number* of L . We now have the following result.

PROPOSITION 2.1. *The transfer operator $\mathcal{L} : A^2(U) \rightarrow A^2(U)$ given in (2.2) belongs to the exponential class $E(a, 1)$ for some $a > 0$. In particular, \mathcal{L} is trace class. Moreover, its eigenvalues decay at an exponential rate.*

³This suggests the possibility of efficiently approximating the escape rate for non-Markov holes H , by using the methods of this paper for a sequence of Markov holes approximating H .

PROOF. The first assertion follows from [5, Theorem 5.9]. The second now follows since the approximation numbers of \mathcal{L} are summable. The statement about the eigenvalue decay follows from [5, Lemma 5.11]. \square

Given a hole $H \in \alpha$, without loss of generality assume that $H = I_d$. In order to analyse the corresponding escape rate we consider the following modified operator:

DEFINITION 2.2. Define \mathcal{L}_H by

$$\mathcal{L}_H f(z) = \sum_{i=1}^{d-1} \epsilon_i T'_i(z) f(T_i z) \text{ where } f \in A^2(U) \text{ and } z \in U. \quad (2.3)$$

Equivalently, we can think of \mathcal{L}_H as the original transfer operator \mathcal{L} with the term corresponding to H removed. As a result, the modified transfer operator enjoys the same strong spectral properties as the original transfer operator.

PROPOSITION 2.3. *The modified transfer operator $\mathcal{L}_H : A^2(U) \rightarrow A^2(U)$ given in (2.3) belongs to the exponential class $E(a, 1)$ for some $a > 0$. In particular, \mathcal{L}_H is trace class. Moreover, its eigenvalues decay at an exponential rate.*

PROOF. See the proof of Proposition 2.1 \square

Since \mathcal{L}_H is trace class, it has a well-defined trace. Moreover, there is an explicit expression for the trace of any power of \mathcal{L}_H in terms of fixed points of the iterates of the map:

PROPOSITION 2.4. *For any $n \in \mathbb{N}$ we have*

$$\mathrm{tr}(\mathcal{L}_H^n) = \sum_{x \in \mathrm{Fix}_H(T^n)} \frac{\mathrm{sgn}((T^n)'(x))}{(T^n)'(x) - 1},$$

where $\mathrm{Fix}_H(T^n) = \{x \in [0, 1] : T^n x = x, T^k x \notin H \text{ for } 0 \leq k < n\}$ and $\mathrm{sgn}(\xi) \in \{-1, 1\}$ denotes the sign of $\xi \in \mathbb{R}$.

PROOF. This follows from [6, Theorem 4.2]. \square

The traces can now be used to calculate the determinant of the operator \mathcal{L}_H .

PROPOSITION 2.5. *The function $z \mapsto \det(1 - z\mathcal{L}_H)$ given for z of sufficiently small modulus by*

$$\det(I - z\mathcal{L}_H) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(\mathcal{L}_H^n)\right) \quad (2.4)$$

extends to an entire function, the zeros of which are exactly the reciprocals of the eigenvalues of \mathcal{L}_H (counting algebraic multiplicities).

The Taylor coefficients c_n of

$$\det(I - z\mathcal{L}_H) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (2.5)$$

satisfy both the recurrence relation

$$c_n = -\frac{1}{n} \sum_{l=0}^{n-1} c_l \operatorname{tr}(\mathcal{L}_H^{n-l}) \text{ for } n \geq 1 \quad (2.6)$$

with $c_0 = 1$, and Plemelj's formula

$$c_n = \frac{(-1)^n}{n!} \det \begin{pmatrix} \operatorname{tr}(\mathcal{L}_H) & 1 & & & 0 \\ \operatorname{tr}(\mathcal{L}_H^2) & \operatorname{tr}(\mathcal{L}_H) & 2 & & \\ \vdots & \vdots & & \ddots & \\ \operatorname{tr}(\mathcal{L}_H^{n-1}) & \operatorname{tr}(\mathcal{L}_H^{n-2}) & \cdots & \operatorname{tr}(\mathcal{L}_H) & n-1 \\ \operatorname{tr}(\mathcal{L}_H^n) & \operatorname{tr}(\mathcal{L}_H^{n-1}) & \cdots & \operatorname{tr}(\mathcal{L}_H^2) & \operatorname{tr}(\mathcal{L}_H) \end{pmatrix}. \quad (2.7)$$

Moreover, we have

$$|c_n| = O(\theta^{n^2}) \text{ as } n \rightarrow \infty, \quad (2.8)$$

for some $0 < \theta < 1$.

PROOF. For the recurrence formula and Plemelj's formula see [17, Theorem 4.4.10]. The decay estimate for the Taylor coefficients is proved in [5, Theorem 6.1]. The remaining assertions follow from Lidskii's Trace Theorem (see, for example, [11, Theorem 8.4, Chapter III]). \square

REMARK 2.6. Explicit estimates for θ , in terms of geometric properties of $T_i(U)$, can be found in [5, Theorem 6.1].

PROPOSITION 2.7. *The following hold:*

- (a) *The operator \mathcal{L}_H has a simple eigenvalue $\delta \in (0, 1]$, strictly larger in modulus than all other eigenvalues, with corresponding eigenfunction $\varrho \in A^2(U)$, which is positive on I .*
- (b) *There exists a probability measure ν supported on the survivor set E_∞ satisfying*

$$\int_{E_\infty} \mathcal{L}_H f \, d\nu = \delta \int_{E_\infty} f \, d\nu \text{ for all } f \in A^2(U).$$

- (c) *The probability measure $\mu = \varrho\nu$ supported on the survivor set E_∞ is T -invariant and coincides with the escape measure.*
- (d) *The escape rate with respect to Lebesgue measure m satisfies*

$$\varepsilon(T, H, m) = -\log \delta.$$

PROOF. The assertions in (a), (b) and (c) follow from results in [15]. To be precise, the existence of the eigenmeasure ν in (b) follows immediately from Theorem A in [15]. For (a) observe that (b) together with the compactness of \mathcal{L}_H imply the existence of an eigenvector $\varrho \in A^2(U)$ corresponding to δ , which, by the positivity arguments used for

the proof of Theorem A in [15], must have the stated properties. The same theorem also yields (c). Finally, (d) follows from the fact that

$$m(E_n) = \int_{I \setminus H} \mathcal{L}_H^n 1 \, dm$$

together with the spectral properties of \mathcal{L}_H given in (a). \square

3. Determining the escape rate

The results of §2 mean we can find the value $0 < \delta(T, H, m) \leq 1$ by considering the determinant:

PROPOSITION 3.1. *The smallest zero (in modulus) of $z \mapsto \det(I - z\mathcal{L}_H)$ is simple, real, and equal to $\delta(T, H, m)^{-1}$.*

PROOF. By Proposition 2.7 the value $\delta(T, H, m)$ is a simple eigenvalue of the transfer operator \mathcal{L}_H and also the largest in modulus. Combining this with Proposition 2.5 the assertions follow. \square

Setting $\delta = \delta(T, H, m)$, the expansion (2.5) now gives

$$0 = 1 + \sum_{n=1}^{\infty} c_n \delta^{-n} = 1 + \sum_{n=1}^N c_n \delta^{-n} + O(\theta^{N^2}).$$

leading naturally to the following definition:

DEFINITION 3.2. For each $N \geq 1$ define δ_N to be the largest value (in modulus) such that

$$0 = 1 + \sum_{n=1}^N c_n \delta_N^{-n}.$$

This brings us to the first main result:

THEOREM 3.3. *The values δ_N converge to δ at a super-exponential rate; more precisely,*

$$\delta_N = \delta + O(\theta^{N^2}) \quad \text{as } N \rightarrow \infty.$$

PROOF. By construction the sequence δ_N converges to δ . In order to estimate the speed of convergence, fix $N \geq 1$ and write

$$\Delta(z) = \det(1 - z\mathcal{L}_H) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

$$\Delta_N(z) = 1 + \sum_{n=1}^N c_n z^n.$$

By the mean value theorem, there is t_N on the line segment joining δ^{-1} and δ_N^{-1} such that

$$(\delta^{-1} - \delta_N^{-1}) \Delta'_N(t_N) = \Delta_N(\delta^{-1}) - \Delta_N(\delta_N^{-1}) = \Delta_N(\delta^{-1}) = \Delta_N(\delta^{-1}) - \Delta(\delta^{-1}).$$

But since $\Delta'_N(t_N) \rightarrow \Delta'(\delta^{-1}) \neq 0$ by Proposition 3.1 it follows that $|\Delta'_N(t_N)|$ is bounded away from zero. Thus

$$|\delta^{-1} - \delta_N^{-1}| \leq \frac{1}{|\Delta'_N(t_N)|} \sum_{n=N+1}^{\infty} |c_n| \delta^{-n} = O(\theta^{N^2}) \quad \text{as } N \rightarrow \infty$$

for some $0 < \theta < 1$. \square

REMARK 3.4. The implied constant in Theorem 3.3 can if necessary be explicitly estimated, using bounds on the Taylor coefficients c_n .

4. Determining the escape measure

In order to approximate the escape measure, we first introduce the following weighted transfer operator.

DEFINITION 4.1. Let ϕ be a bounded holomorphic function on U . For t in a bounded neighbourhood V of $0 \in \mathbb{C}$, define the *weighted transfer operator* $\mathcal{L}_{H,t}$ by

$$\mathcal{L}_{H,t}f(z) = \sum_{i=1}^{d-1} \epsilon_i T'_i(z) e^{t\phi(z)} f(T_i z) \quad \text{where } f \in A^2(U) \text{ and } z \in U.$$

We now have analogues of the results from §2:

PROPOSITION 4.2. *The operators $\mathcal{L}_{H,t} : A^2(U) \rightarrow A^2(U)$ satisfy:*

- (a) *For each $t \in V$ the operator $\mathcal{L}_{H,t}$ belongs to the exponential class $E(a, 1)$ for some $a > 0$.*
- (b) *The mapping $t \mapsto \mathcal{L}_{H,t}$ is holomorphic in the trace-class operator topology; in particular, the function $(z, t) \mapsto \det(1 - z\mathcal{L}_{H,t})$ is holomorphic on $\mathbb{C} \times V$.*
- (c) *For any $t \in V$ and any $n \in \mathbb{N}$ we have*

$$\text{tr}(\mathcal{L}_{H,t}^n) = \sum_{x \in \text{Fix}_H(T^n)} \frac{\text{sgn}((T^n)'(x)) e^{t\phi^{(n)}(x)}}{(T^n)'(x) - 1}.$$

where $\text{Fix}_H(T^n) = \{x \in [0, 1] : T^n x = x, T^k x \notin H \text{ for } 0 \leq k < n\}$ and $\phi^{(n)} = \sum_{k=0}^{n-1} \phi \circ T^k$.

- (d) *The Taylor coefficients $c_{n,\phi}(t)$ of the determinant*

$$\det(I - z\mathcal{L}_{H,t}) = 1 + \sum_{n=1}^{\infty} c_{n,\phi}(t) z^n$$

satisfy $\sup_{t \in V} |c_{n,\phi}(t)| = O(\theta^{n^2})$ as $n \rightarrow \infty$ for some $0 < \theta < 1$.

PROOF. Assertion (a) follows from [5, Theorem 5.9], while assertion (b) follows from (a) and [11, Section 1.9, Chapter IV]. The formula for the traces in (c) is a consequence of [6, Theorem 4.2], and (d) follows from [5, Theorem 6.1]. \square

REMARK 4.3. Setting $t = 0$ we see that $\mathcal{L}_{H,0} = \mathcal{L}$, hence $c_{n,\phi}(0) = c_n$ for all $n \geq 1$.

It turns out that the escape measure can be expressed as a quotient of the partial derivatives of $(z, t) \mapsto \det(I - z\mathcal{L}_{H,t})$. The proof of this relies on a formula for the derivative of a determinant which we briefly recall. Let $D \subset \mathbb{C}$ be an open neighbourhood of 0 and suppose that $D \ni s \mapsto L(s)$ is an operator-valued function which is holomorphic in the trace-class topology. If $\det(I - L(0)) \neq 0$, then

$$\frac{d}{ds} \det(I - L(s))|_{s=0} = -\det(I - L(0)) \operatorname{tr}(\dot{L}(0)(I - L(0))^{-1}), \quad (4.1)$$

where $\dot{L}(0) = \frac{d}{ds} L(s)|_{s=0}$. For a proof see [17, 4.3.1.9 Proposition] or [11, Section 1.9, Chapter IV].

The calculation of the escape measure relies on the following result.

PROPOSITION 4.4. *We have*

$$\int_{E_\infty} \phi d\mu = \delta \frac{\frac{\partial}{\partial t} \det(I - z\mathcal{L}_{H,t})|_{t=0, z=1/\delta}}{\frac{\partial}{\partial z} \det(I - z\mathcal{L}_{H,t})|_{t=0, z=1/\delta}}.$$

PROOF. Let D be a small punctured neighbourhood of $1/\delta$ such that $\det(1 - \zeta\mathcal{L}_{H,0}) \neq 0$ for $\zeta \in D$. Using (4.1) twice we obtain for any $\zeta \in D$

$$\frac{\partial}{\partial t} \det(I - \zeta\mathcal{L}_{H,t})|_{t=0} = -\det(I - \zeta\mathcal{L}_{H,0}) \operatorname{tr}(\zeta \dot{\mathcal{L}}_{H,0} (I - \zeta\mathcal{L}_{H,0})^{-1})$$

where $\dot{\mathcal{L}}_{H,0} = \frac{d}{dt} \mathcal{L}_{H,t}|_{t=0}$, and

$$\frac{\partial}{\partial z} \det(I - z\mathcal{L}_{H,0})|_{z=\zeta} = -\det(I - \zeta\mathcal{L}_{H,0}) \operatorname{tr}(\mathcal{L}_{H,0} (I - \zeta\mathcal{L}_{H,0})^{-1}).$$

We now observe that $\frac{d}{dt} \mathcal{L}_{H,t}|_{t=0} = M_\phi \mathcal{L}_{H,0}$ where $M_\phi : A^2(U) \rightarrow A^2(U)$ is the operator of multiplication by ϕ , that is, $M_\phi f = \phi f$ for $f \in A^2(U)$. Moreover, for $\zeta \in D$ we can write

$$\mathcal{L}_{H,0} (1 - \zeta\mathcal{L}_{H,0})^{-1} = \frac{\delta}{1 - \zeta\delta} \Pi + Q(\zeta),$$

where $\Pi f = \int_{E_\infty} f d\nu \cdot \varrho$ denotes the spectral projection associated to the eigenvalue δ and Q is a trace-class operator valued holomorphic function on D . Thus,

$$\begin{aligned} \frac{\frac{\partial}{\partial t} \det(I - z\mathcal{L}_{H,t})|_{t=0, z=1/\delta}}{\frac{\partial}{\partial z} \det(I - z\mathcal{L}_{H,t})|_{t=0, z=1/\delta}} &= \lim_{\zeta \rightarrow 1/\delta} \frac{\frac{\partial}{\partial t} \det(I - \zeta\mathcal{L}_{H,0})|_{t=0}}{\frac{\partial}{\partial z} \det(I - z\mathcal{L}_{H,0})|_{z=\zeta}} \\ &= \lim_{\zeta \rightarrow 1/\delta} \zeta \frac{\delta \operatorname{tr}(M_\phi \Pi) + (1 - \zeta\delta) \operatorname{tr}(M_\phi Q(\zeta))}{\delta \operatorname{tr}(\Pi) + (1 - \zeta\delta) \operatorname{tr}(Q(\zeta))} = \frac{1}{\delta} \frac{\operatorname{tr}(M_\phi \Pi)}{\operatorname{tr}(\Pi)} \end{aligned}$$

and the result follows by noting that $\text{tr}(\Pi) = \int_{E_\infty} \varrho \, d\nu = 1$ and

$$\text{tr}(M_\phi \Pi) = \int_{E_\infty} \phi \varrho \, d\nu = \int_{E_\infty} \phi \, d\mu.$$

□

Using Proposition 4.4 we can write

$$\int_{E_\infty} \phi \, d\mu = \delta \frac{\sum_{n=0}^{\infty} c'_{n,\phi}(0) \delta^{-n}}{\sum_{n=0}^{\infty} n c_{n,\phi}(0) \delta^{-(n-1)}} = \frac{\sum_{n=0}^N c'_{n,\phi}(0) \delta^{1-n}}{\sum_{n=0}^N n c_{n,\phi}(0) \delta^{1-n}} + O(\theta^{N^2}), \quad (4.2)$$

for some $0 < \theta < 1$. Here we have used the fact that $c'_{n,\phi}(0) = O(\theta^{n^2})$ as $n \rightarrow \infty$ for some $0 < \theta < 1$, which follows from Proposition 4.2 (d) and Cauchy's formula.

This leads naturally to the following definition:

DEFINITION 4.5. For each $N \geq 1$, define $I_N(\phi)$ by

$$I_N(\phi) = \frac{\sum_{n=0}^N c'_{n,\phi}(0) \delta_N^{1-n}}{\sum_{n=0}^N n c_{n,\phi}(0) \delta_N^{1-n}} = \frac{\sum_{n=1}^N c'_{n,\phi}(0) \delta_N^{1-n}}{\sum_{n=1}^N n c_n \delta_N^{1-n}}.$$

This brings us to the second main result:

THEOREM 4.6. *The values $I_N(\phi)$ converge to $\int_{E_\infty} \phi \, d\mu$ at a super-exponential rate; more precisely,*

$$I_N(\phi) = \int_{E_\infty} \phi \, d\mu + O(\theta^{N^2}) \quad \text{as } N \rightarrow \infty$$

for some $0 < \theta < 1$.

PROOF. This follows from (4.2) and Theorem 3.3. □

REMARK 4.7. Similar approximating formulae, in the context of invariant measures equivalent to Lebesgue measure, have been derived in [12, 13, 14] using a slightly different approach.

Importantly, it is possible to efficiently calculate each $c'_{n,\phi}(0)$ using periodic points:

PROPOSITION 4.8. *Setting*

$$b_{\phi,n} = \frac{1}{n} \sum_{x \in \text{Fix}_H(T^n)} \frac{\text{sgn}((T^n)'(x)) \phi^{(n)}(x)}{(T^n)'(x) - 1}, \quad (4.3)$$

we have

$$c'_{\phi,n}(0) = - \sum_{i=1}^n b_{\phi,i} c_{n-i} \quad \text{for all } n \geq 1. \quad (4.4)$$

PROOF. Let z belong to a sufficiently small disc centred at the origin. Then we have

$$\begin{aligned} \frac{\partial}{\partial t} \det(1 - z\mathcal{L}_{H,t})|_{t=0} &= -\det(1 - z\mathcal{L}_{H,0}) \sum_{m=1}^{\infty} \frac{z^m}{m} \frac{\partial}{\partial t} \operatorname{tr}(\mathcal{L}_{H,t}^m)|_{t=0} \\ &= -(1 + \sum_{n=1}^{\infty} c_n z^n) \sum_{m=1}^{\infty} b_{\phi,m} z^m. \end{aligned} \quad (4.5)$$

On the other hand,

$$\frac{\partial}{\partial t} \det(1 - z\mathcal{L}_{H,t})|_{t=0} = 1 + \sum_{n=1}^{\infty} c'_n(0) z^n, \quad (4.6)$$

and the result now follows by comparing coefficients in (4.5) and (4.6). \square

5. An example

As in [2], we consider the map

$$T(x) = \begin{cases} \frac{9x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{10} \\ 10x - i & \text{if } \frac{i}{10} < x \leq \frac{i+1}{10} \text{ for } 1 \leq i \leq 9 \end{cases}$$

and $H = [\frac{9}{10}, 1]$.

Note that the inverse branches $\{T_i\}_{0 \leq i \leq 9}$ are given by

$$T_0(x) = \frac{x}{9+x}$$

and

$$T_i(x) = (x+i)/10 \quad \text{for } 1 \leq i \leq 9.$$

Writing

$$a_n = \frac{1}{n} \operatorname{tr}(\mathcal{L}_H^n) = \frac{1}{n} \sum_{x \in \operatorname{Fix}_H(T^n)} \frac{1}{(T^n)'(x) - 1},$$

these a_n can be computed by locating the members of $\operatorname{Fix}_H(T^n)$, all of which are quadratic numbers.

For example there are 9 members of $\operatorname{Fix}_H(T)$, denoted x_0, x_1, \dots, x_8 , say. For each $1 \leq i \leq 8$ we see that

$$\frac{1}{T'(x_i) - 1} = \frac{1}{10 - 1} = \frac{1}{9},$$

whereas

$$\frac{1}{T'(x_0) - 1} = \frac{1}{9 - 1} = \frac{1}{8}.$$

Therefore

$$a_1 = \sum_{i=0}^8 \frac{1}{T'(x_i) - 1} = \frac{1}{8} + \frac{8}{9} = \frac{73}{72}.$$

The computation of a_2 is only slightly more involved. For the fixed point 0 we have

$$\frac{1}{(T^2)'(0) - 1} = \frac{1}{81 - 1} = \frac{1}{80},$$

whereas for those 64 period-2 points $x_{ij} = (T_i \circ T_j)(x_{ij})$ with $1 \leq i, j \leq 8$, we have

$$\frac{1}{(T^2)'(x_{ij}) - 1} = \frac{1}{100 - 1} = \frac{1}{99}.$$

It remains to consider the 8 period-2 points of the form $x_{0i} = (T_0 \circ T_i)(x_{0i})$, and the 8 period-2 points of the form $x_{i0} = (T_i \circ T_0)(x_{i0})$, for $1 \leq i \leq 8$. In fact since $(T^2)'(x_{0i}) = (T^2)'(x_{i0})$, it suffices to consider the points x_{0i} , and a calculation gives

$$(T_0 \circ T_i)(x) = \frac{x + i}{x + 90 + i}, \quad (T_0 \circ T_i)'(x) = \frac{90}{(x + 90 + i)^2},$$

$$x_{0i} = 5 \left(\sqrt{\left(9 + \frac{i-1}{10}\right)^2 + \frac{i}{25}} - 9 - \frac{i-1}{10} \right),$$

from which we compute

$$a_2 = \frac{1}{2} \left(\frac{1}{80} + \frac{64}{99} + 2 \sum_{i=1}^8 \frac{1}{(T^2)'(x_{0i}) - 1} \right) = 0.410995345836251121588654162858 \dots$$

Subsequent values a_n can be computed similarly, for example:

$$\begin{aligned} a_3 &= 0.244247986872392594300895837121 \dots \\ a_4 &= 0.164881484924536515073990416986 \dots \\ a_5 &= 0.118849630250109944686793773181 \dots \\ a_6 &= 0.089248843422890449580723889612 \dots \\ a_7 &= 0.068936195289851448498303594869 \dots \end{aligned}$$

5.1. The escape rate. We are now in a position to compute the power series coefficients c_i of the determinant $\det(I - z\mathcal{L}_H) = 1 + \sum_{i=1}^{\infty} c_i z^i$. Specifically, the formulae of Proposition 2.5 give

$$\begin{aligned} c_1 &= -a_1 \\ c_2 &= -a_2 + \frac{a_1^2}{2} \\ c_3 &= -a_3 + a_1 a_2 - \frac{a_1^3}{6} \\ c_4 &= -a_4 + \frac{a_2^2}{2} + a_1 a_3 - \frac{a_1^2 a_2}{2} + \frac{a_1^4}{24} \\ c_5 &= -a_5 + a_1 a_4 + a_2 a_3 - \frac{a_1^2 a_3}{2} - \frac{a_1 a_2^2}{2} + \frac{a_1^3 a_2}{6} - \frac{a_1^5}{120} \end{aligned}$$

$$\begin{aligned}
c_6 &= -a_6 + \frac{a_3^2}{2} + a_1a_5 + a_2a_4 - \frac{a_1^2a_4}{2} - a_1a_2a_3 - \frac{a_2^3}{6} + \frac{a_1^3a_3}{6} + \frac{a_1^2a_2^2}{4} - \frac{a_1^4a_2}{24} + \frac{a_1^6}{720} \\
c_7 &= -a_7 + a_1a_6 + a_2a_5 + a_3a_4 - \frac{a_1^2a_5}{2} - a_1a_2a_4 - \frac{a_1a_3^2}{2} - \frac{a_2^2a_3}{2} \\
&\quad + \frac{a_1^3a_4}{6} + \frac{a_1^2a_2a_3}{2} + \frac{a_1a_2^3}{6} - \frac{a_1^4a_3}{24} - \frac{a_1^3a_2^2}{12} + \frac{a_1^5a_2}{120} - \frac{a_1^7}{5040}.
\end{aligned}$$

Substituting the above numerical values⁴ of a_n into the formulae for the c_i then gives

$$\begin{aligned}
c_1 &= -\frac{73}{72} \\
c_2 &= 0.102989993669921717917518676648\dots \\
c_3 &= -0.001252380603001953819578039057\dots \\
c_4 &= 1.994754501536932614209760476393\dots \times 10^{-6} \\
c_5 &= -4.367117910658311343671035602900\dots \times 10^{-10} \\
c_6 &= 1.348215512356863399693187985465\dots \times 10^{-14} \\
c_7 &= -5.969559406869561159884947613741\dots \times 10^{-20}.
\end{aligned}$$

These values of c_i allow us to form, for $1 \leq N \leq 7$, the degree- N polynomial approximation

$$\Delta_N(z) = 1 + \sum_{i=1}^N c_i z^i$$

to the determinant. The smallest root z_N of Δ_N can then be computed as follows:

$$\begin{aligned}
z_1 &= 72/73 = 0.986301369863013698630136\dots \\
z_2 &= 1.111881779249553184201012015076\dots \\
z_3 &= 1.109701877327063363180409111227\dots \\
z_4 &= 1.109705706696569182143392132129\dots \\
z_5 &= 1.109705705766218331774455583303\dots \\
z_6 &= 1.109705705766250204483482219528\dots \\
z_7 &= 1.109705705766250204326875729570\dots
\end{aligned}$$

and inverting these gives the same sequence of approximations $\delta_N = z_N^{-1}$ to $\delta(T, H, m)$ as listed in §1.

⁴Of course we use higher precision for the a_n , ensuring that the values c_i are correct to the precision given.

5.2. The escape measure. The escape measure μ is completely determined by its set of *moments* $\int_{E_\infty} x^n d\mu(x)$, $n \geq 0$. Each n -th moment can be rapidly approximated by setting $\phi(x) = x^n$, then using the approach described in §4. Here we shall illustrate this in the case $n = 1$: the first moment $\mu(1) = \int_{E_\infty} x d\mu(x)$ is often called the *barycentre*, or *resultant*, of the measure μ .

Since $\phi(x) = x$ is fixed, we write $b_n = b_{\phi,n}$ (see (4.3)), so that

$$b_n = \frac{1}{n} \sum_{x \in \text{Fix}_H(T^n)} \frac{\sum_{i=0}^{n-1} T^i x}{(T^n)'(x) - 1}.$$

We find that

$$\begin{aligned} b_1 &= 4/9 = 0.4444444444444444444444444444 \dots \\ b_2 &= 0.363146979940866817710676390686 \dots \\ b_3 &= 0.323945697078082902031586942946 \dots \\ b_4 &= 0.291597918113354097600085433302 \dots \\ b_5 &= 0.262738636423342281952526356399 \dots \\ b_6 &= 0.236761095523224368789249278048 \dots \\ b_7 &= 0.213354539113042148099894783840 \dots \end{aligned}$$

Recall that the coefficients $d_i = c'_{\phi,i}(0)$ (where $\phi(x) = x$) are given by formula (4.4). It follows that, for example, the first four⁵ d_i are given by:

$$\begin{aligned} d_1 &= -b_1 \\ d_2 &= a_1 b_1 - b_2 \\ d_3 &= -b_3 + b_1 a_2 + a_1 b_2 - \frac{a_1^2 b_1}{2} \\ d_4 &= a_2 b_2 + b_1 a_3 + a_1 b_3 - b_4 - a_1 b_1 a_2 - \frac{a_1^2 b_2}{2} + \frac{a_1^3 b_1}{6} \end{aligned}$$

Substituting the numerical values of a_n, b_n into the formulae for the d_i gives us:

$$\begin{aligned} d_1 &= -4/9 = -0.4444444444444444444444444444 \dots \\ d_2 &= 0.087470304009750466239940893264 \dots \\ d_3 &= -0.00152833960244703092715945867 \dots \\ d_4 &= 3.133193453094917698092477916170 \dots \times 10^{-6} \\ d_5 &= -8.40390182408161094002529348420 \dots \times 10^{-10} \\ d_6 &= 3.090985019372664486353921814698 \dots \times 10^{-14} \\ d_7 &= -1.60253894897971331452691425140 \dots \times 10^{-19} \end{aligned}$$

⁵In the calculation that follows we use d_i for $1 \leq i \leq 7$, though the algebraic formulae for d_i in terms of a_n, b_n are a little long to conveniently give here (e.g. the analogous expression for d_7 consists of a sum of 30 terms).

The approximations

$$\mu_N(1) = \frac{\sum_{n=1}^N d_n z_N^{n-1}}{\sum_{n=1}^N n c_n z_N^{n-1}}$$

to the integral $\mu(1) = \int_{E_\infty} x d\mu(x)$ are then:

$$\mu_2(1) = 0.442354383674664532214929145156 \dots$$

$$\mu_3(1) = 0.442135977598196893113667748055 \dots$$

$$\mu_4(1) = 0.442136676297808722065125231922 \dots$$

$$\mu_5(1) = 0.442136676053865369048181249845 \dots$$

$$\mu_6(1) = 0.442136676053875847256104872452 \dots$$

$$\mu_7(1) = 0.442136676053875847197526214497 \dots$$

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