

EIGENVALUES OF TRANSFER OPERATORS FOR DYNAMICAL SYSTEMS WITH HOLES

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ABSTRACT. For real-analytic expanding open dynamical systems in arbitrary finite dimension, we establish rigorous explicit bounds on the eigenvalues of the corresponding transfer operators acting on spaces of holomorphic functions. In dimension 1 the eigenvalue decay rate is exponentially fast, while in dimension d it is $O(\theta^{n^{1/d}})$ as $n \rightarrow \infty$ for some $0 < \theta < 1$.

1. INTRODUCTION

For an expanding map $T : X \rightarrow X$, the Perron-Frobenius operator \mathcal{P} defined by

$$\mathcal{P}f(x) = \sum_{Ty=x} \frac{f(y)}{|T'(y)|},$$

and more general transfer operators \mathcal{L} defined by

$$\mathcal{L}f(x) = \sum_{Ty=x} e^{\varphi(y)} f(y)$$

with potential function $\varphi : X \rightarrow \mathbb{R}$, are important objects in the thermodynamic formalism approach to ergodic theory.

Given a subset $H \subset X$, which we regard as a *hole* in X , it is natural to consider modified operators \mathcal{P}_H and \mathcal{L}_H , defined by $\mathcal{P}_H f = \mathcal{P}(f\chi_{X \setminus H})$ and $\mathcal{L}_H f = \mathcal{L}(f\chi_{X \setminus H})$, in view of their connections with escape rate (see, for example, [6, 9]) and various equilibrium measures supported by the survivor set $X_\infty = \bigcap_{n=0}^\infty T^{-n}(X \setminus H)$.

The purpose of this note is to describe, in the case where T is piecewise analytic, explicit estimates on the spectral asymptotics of \mathcal{P}_H and \mathcal{L}_H when acting on various Banach spaces of holomorphic functions.¹

Specifically, we take $X \subset \mathbb{R}^d$ to be compact and connected, and $\mathcal{X} = \{X_i\}_{i \in \mathcal{I}}$ a finite partition (consisting of non-empty pairwise disjoint

¹When acting on these spaces, \mathcal{P}_H has a strictly positive spectral radius δ , with $\delta > 0$ an eigenvalue such that $-\log \delta$ is the corresponding escape rate (see e.g. [14]); thus escape is at an exponential rate, rather than anything faster. Moreover, $\delta^{-n} \mathcal{P}_h^n 1 \rightarrow \varrho$, where ϱ is the density function for the Pianigiani-Yorke measure [15].

subsets of X , each one open in \mathbb{R}^d , whose union is dense in X). The map $T : X \rightarrow X$ is assumed Borel measurable, with $T(X_i)$ open in \mathbb{R}^d for each $i \in \mathcal{I}$, and $T|_{X_i} : X_i \rightarrow T(X_i)$ a C^1 diffeomorphism which can be extended to a C^1 map on $\overline{X_i}$. We assume that T is *full branch*, i.e. $\overline{T(X_i)} = X$ for all $i \in \mathcal{I}$, and *expanding*, i.e. there exists $\beta > 1$ such that if $x, y \in X_i$ for some $i \in \mathcal{I}$ then $\|T(x) - T(y)\| \geq \beta \|x - y\|$. Each $T|_{X_i}$ has an *inverse branch* T_i , defined so that $T \circ T_i$ is the identity on the interior of X , and $T_i \circ T$ the identity on X_i , and satisfying $\sup_{x \in \text{int}(X)} \|T_i'(x)\|_{L(\mathbb{R}^d)} \leq \beta^{-1}$ for all $i \in \mathcal{I}$, where $\|\cdot\|_{L(\mathbb{R}^d)}$ denotes the induced operator norm on $L(\mathbb{R}^d) = L((\mathbb{R}^d, \|\cdot\|))$. We assume that $T : X \rightarrow X$ is *real analytic*, i.e. there is a bounded connected open set $D \subset \mathbb{C}^d$, with $X \subset D$, such that each T_i has a holomorphic extension to D .

For simplicity we shall take the hole H to be some union of elements of \mathcal{X} . In fact with some extra effort, and more cumbersome notation, the techniques described here extend to the case where H is a union of members of some refinement $\bigvee_{i=0}^{n-1} T^{-i} \mathcal{X}$ (a so-called Markov hole). Let $\mathcal{J} \subset \mathcal{I}$ be such that $\bigcup_{i \in \mathcal{J}} X_i = X \setminus H$. Transfer operators \mathcal{L}_H for the open dynamical system $T|_{X \setminus H}$ then take the form

$$\mathcal{L}_H f = \sum_{i \in \mathcal{J}} w_i f \circ T_i, \quad (1)$$

where the *weight functions* w_i are related to the potential function φ by $w_i = \exp(\varphi \circ T_i)$ on X , and assumed to admit a holomorphic extension to D which in turn extends continuously to \overline{D} . In the particular case $\varphi = -\log |T'|$, when w_i are the holomorphic extensions to D of $|T_i'|$ on X , the corresponding transfer operator is precisely the modified Perron-Frobenius operator \mathcal{P}_H . We shall always assume that D has the property that the closure of $\bigcup_{i \in \mathcal{J}} T_i(D)$ lies inside D itself, referring to such domains D as being *admissible* for the map T ; this technical requirement, which we always assume without further comment, will ensure that \mathcal{L}_H preserves suitable Banach spaces of functions holomorphic on D .

The structure of the article is as follows. We begin in §2 by considering transfer operators \mathcal{L}_H acting on the Banach space² $U(D)$ of those holomorphic functions on D which extend continuously to \overline{D} equipped with the usual supremum norm $\|w\|_{U(D)} = \sup_{z \in D} |w(z)|$. We show (Theorems 1 and 2) that in dimension $d = 1$, the eigenvalues $\lambda_n(\mathcal{L}_H)$

²The study of transfer operators on this space $U(D)$ was inaugurated by Ruelle [18].

(arranged in order of decreasing modulus) converge to zero exponentially fast, deriving an explicit bound for $|\lambda_n(\mathcal{L}_H)|$. In higher dimensions $d \geq 2$ there are similar explicit bounds (see Theorem 3), though here the convergence to zero is³ $O(\theta^{n^{1/d}})$ as $n \rightarrow \infty$, for some $\theta \in (0, 1)$. In §3 we show that in fact the eigenvalues for $\mathcal{L}_H : U(D) \rightarrow U(D)$ are identical to those for \mathcal{L}_H acting on a variety of Banach spaces $A(D)$ of holomorphic functions. This suggests the possibility of improving the bounds of §2 by judicious choice of $A(D)$, a strategy we pursue in §4 where $A(D)$ is chosen to be Hilbert Hardy space $H^2(D)$, yielding Theorems 6 and 7.

2. EIGENVALUE ESTIMATES VIA WEYL'S INEQUALITY

We begin with an explicit estimate on the eigenvalues of the modified Perron-Frobenius operator in dimension $d = 1$:

Theorem 1. *For an expanding interval map, the eigenvalues of the modified Perron-Frobenius operator $\mathcal{P}_H : U(D) \rightarrow U(D)$ satisfy*

$$|\lambda_n(\mathcal{P}_H)| \leq \theta^{n-1} \sqrt{n} \sup_{z \in D} \sum_{i \in \mathcal{J}} |T'_i(z)| \quad \text{for all } n \geq 1, \quad (2)$$

provided each T'_i extends holomorphically to a disc $D \subset \mathbb{C}$, where $\theta < 1$ is such that $\cup_{i \in \mathcal{J}} T_i(D)$ is contained in the concentric disc whose radius is θ^2 times that of D .

The bounds in Theorem 1 are readily computed for specific maps T :

Example 1. As in [2], we consider the map

$$T(x) = \begin{cases} \frac{9x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{10} \\ 10x - i & \text{if } \frac{i}{10} < x \leq \frac{i+1}{10}, \text{ for } 1 \leq i \leq 9 \end{cases}$$

Note that the inverse branches $\{T_i\}_{0 \leq i \leq 9}$ are given by

$$T_0(x) = \frac{x}{9+x}$$

and

$$T_i(x) = (x+i)/10 \quad \text{for } 1 \leq i \leq 9.$$

Choosing Markov hole $H = [1/5, 1]$ corresponds to setting $\mathcal{J} = \{0, 1\}$.

³Ruelle [18], following Grothendieck [11], stated the asymptotics were $O(\theta^n)$ as $n \rightarrow \infty$, independent of the dimension d , though Fried [10] corrected this to $O(\theta^{n^{1/d}})$. One novelty of our results, relative to Fried and Ruelle, is that the constant θ , as well as the implicit constant in the big-O asymptotics, are rendered explicit.

We claim that the eigenvalues of the modified Perron-Frobenius operator $\mathcal{P}_H : U(D) \rightarrow U(D)$ are bounded by

$$|\lambda_n(\mathcal{P}_H)| \leq \frac{77}{320} \sqrt{5n} \left(\frac{1}{\sqrt{5}} \right)^n \quad \text{for all } n \geq 1. \quad (3)$$

In particular, note that the case $n = 1$ yields a bound on the escape rate γ (see, for example, [6]) for this open dynamical system, namely $\gamma \geq -\log 77/320$.

Let D be the disc of radius 1 centred at 0. Noting that $T_0(-1) = -1/8$, $T_1(-1) = 0$, $T_0(1) = 1/10$, and $T_1(1) = 1/5$, we see that $\cup_{i \in \mathcal{J}} T_i(D)$ is contained in the disc of radius $1/5$ centred at 0. This means we may set $\theta = 1/\sqrt{5}$ in Theorem 1. Note that $|T_0'(z)| + |T_1'(z)| = \frac{9}{|9+z|^2} + \frac{1}{10}$, and the supremum of this expression on D is the value $77/320$, attained (on the boundary of D) at $z = -1$. The bound (3) then follows from (2).

In fact Theorem 1 is a special case of the following one-dimensional result:

Theorem 2. *For an interval expanding map, the eigenvalues of the transfer operator $\mathcal{L}_H : U(D) \rightarrow U(D)$ satisfy:*

$$|\lambda_n(\mathcal{L}_H)| \leq \theta^{n-1} \sqrt{n} \sup_{z \in D} \sum_{i \in \mathcal{J}} |w_i(z)| \quad \text{for all } n \geq 1, \quad (4)$$

provided each w_i and T_i extend holomorphically to the disc $D \subset \mathbb{C}$, where $\theta < 1$ is such that $\cup_{i \in \mathcal{J}} T_i(D)$ is contained in the concentric disc whose radius is θ^2 times that of D .

Proof. Let D' denote the concentric disc whose radius is $r = \theta^2$ times that of D . First we observe that $\hat{\mathcal{L}}_H f := \sum_{i \in \mathcal{J}} w_i \cdot f \circ T_i$ defines a continuous operator $\hat{\mathcal{L}}_H : U(D') \rightarrow U(D)$. To see this, fix $f \in U(D')$ and note that $w_i \cdot f \circ T_i \in U(D)$ with $\|w_i \cdot f \circ T_i\|_{U(D)} \leq \|w_i\|_{U(D)} \|f\|_{U(D')}$ for every $i \in \mathcal{J}$. But $\|\hat{\mathcal{L}}_H f\|_{U(D)} \leq \sum_{i \in \mathcal{J}} \|w_i\|_{U(D)} \|f\|_{U(D')}$, so $\hat{\mathcal{L}}_H f \in U(D)$ and $\hat{\mathcal{L}}_H$ is continuous. Now $\|\hat{\mathcal{L}}_H\| \leq W =: \sup_{z \in D} \sum_{i \in \mathcal{J}} |w_i(z)|$, because for $f \in U(D')$ we have $|f(T_i(z))| \leq \|f\|_{U(D')}$ for every $z \in D$, $i \in \mathcal{J}$; thus by the maximum modulus principle $\|\hat{\mathcal{L}}_H f\|_{U(D)} = \sup_{z \in D} |(\hat{\mathcal{L}}_H f)(z)| \leq \sup_{z \in D} \sum_{i \in \mathcal{J}} |w_i(z)| |f(T_i(z))| \leq W \|f\|_{U(D')}$.

Recall that if $L : B_1 \rightarrow B_2$ is a continuous operator between Banach spaces then for $k \geq 1$, its k -th approximation number $a_k(L)$ is defined by $a_k(L) = \inf \{ \|L - K\| \mid K : B_1 \rightarrow B_2 \text{ linear with rank}(K) < k \}$, and in general $a_k(L_1 L_2) \leq \|L_1\| a_k(L_2)$ (see [16, 2.2]).

Now clearly $\mathcal{L}_H = \hat{\mathcal{L}}_H J$, where $J : U(D) \hookrightarrow U(D')$ denotes the canonical embedding, so

$$a_k(\mathcal{L}_H) \leq \|\hat{\mathcal{L}}_H\| a_k(J) \leq W a_k(J) \quad \text{for all } k \geq 1. \quad (5)$$

Moreover, it can be shown that \mathcal{L}_H is compact; in fact, it is of exponential class (see [3]), and in particular nuclear of any order.

Before proceeding recall that Weyl's inequality (see, for example, [12]) asserts that $\prod_{k=1}^n |\lambda_k(\mathcal{L}_H)| \leq n^{n/2} \prod_{k=1}^n a_k(\mathcal{L}_H)$ for every $n \in \mathbb{N}$.⁴

Together with (5) this yields the inequality

$$|\lambda_n(\mathcal{L}_H)| \leq W n^{1/2} \prod_{k=1}^n a_k(J)^{1/n} \quad \text{for all } n \geq 1, \quad (6)$$

because $|\lambda_n(\mathcal{L}_H)| \leq \prod_{k=1}^n |\lambda_k(\mathcal{L}_H)|^{1/n}$.

Using a result originally due to Babenko (see [1] or [17, Theorem VIII.2.1]) we see

$$a_l(J) \leq r^{l-1} \quad \text{for all } l \geq 1,$$

hence $\prod_{l=1}^n a_l(J)^{1/n} \leq r^{\frac{1}{n} \sum_{l=1}^n l-1} = r^{(n-1)/2}$, so (6) becomes

$$|\lambda_n(\mathcal{L}_H)| \leq W n^{1/2} r^{(n-1)/2},$$

which is the desired bound (4). \square

In higher dimension d the rate of eigenvalue decay is slower than exponential, and can be shown to be $O(\theta^{n^{1/d}})$ as $n \rightarrow \infty$, for some $\theta \in (0, 1)$. The main new ingredient in the following result, proved in [5], is an estimate due to Farkov [8] on the approximation numbers of the embedding operator J in higher dimensions, namely $a_l(J) \leq r^{t_l}$, where $t_l := k$ for $\binom{k-1+d}{d} < l \leq \binom{k+d}{d}$.

Theorem 3. *In dimension $d \geq 1$, suppose the Euclidean ball $D \subset \mathbb{C}^d$ is such that $\cup_{i \in \mathcal{J}} T_i(D)$ is contained in the concentric ball whose radius is $r < 1$ times that of D . Setting $W := \sup_{z \in D} \sum_{i \in \mathcal{J}} |w_i(z)|$, the eigenvalues of $\mathcal{L}_H : U(D) \rightarrow U(D)$ can be bounded by*

$$|\lambda_n(\mathcal{L}_H)| < \frac{W}{r^d} n^{1/2} r^{\frac{d}{d+1} (d!)^{1/d} n^{1/d}} \quad \text{for all } n \geq 1. \quad (7)$$

⁴This is a Banach space version of Weyl's original inequality [19] in Hilbert space; the constant $n^{n/2}$ is optimal (see [12]).

3. THE COMMON SPECTRUM

It turns out that a more oblique approach yields different, and sometimes better, bounds on the eigenvalues of $\mathcal{L}_H : U(D) \rightarrow U(D)$. This approach consists of varying the space upon which \mathcal{L}_H acts. Clearly, in general $U(D)$ is not the only function space preserved by a transfer operator \mathcal{L}_H , and we would expect the spectrum of \mathcal{L}_H to vary according to the space on which it acts. There is interest, however, in identifying a class of spaces $A(D)$ which are sufficiently closely related to $U(D)$ to ensure that the spectrum of \mathcal{L}_H on these spaces is precisely the same as that of $\mathcal{L}_H : U(D) \rightarrow U(D)$. This motivates the following definition:

Definition 1. For a non-empty open connected set $D \subset \mathbb{C}^d$, a Banach space $A(D)$ of holomorphic functions $f : D \rightarrow \mathbb{C}$ is called *favourable* if it contains $U(D)$, with the natural embedding $U(D) \hookrightarrow A(D)$ having norm 1, and if $f \mapsto f(z)$ is continuous on $A(D)$ for each $z \in D$.

Transfer operators \mathcal{L}_H can be shown (see [5]) to preserve all favourable spaces⁵ $A(D)$, with the eigenvalues of $\mathcal{L}_H : A(D) \rightarrow A(D)$ related to a certain entire function:

Theorem 4. *The transfer operator \mathcal{L}_H defined by (1) preserves every favourable space $A(D)$ of holomorphic functions on D . It has a well-defined spectral trace $\tau_{A(D)}(\mathcal{L}_H) = \sum_{n=1}^{\infty} \lambda_n(\mathcal{L}_H|_{A(D)})$ and spectral determinant $\det_{A(D)}$, related by*

$$\det_{A(D)}(I - z\mathcal{L}_H|_{A(D)}) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \tau_{A(D)}(\mathcal{L}_H^n)\right), \quad (8)$$

for all $z \in \mathbb{C}$ in a suitable neighbourhood of 0, and such that, counting multiplicities, the zeros of the entire function $z \mapsto \det(I - z\mathcal{L}_H|_{A(D)})$ are precisely the reciprocals of the eigenvalues of $\mathcal{L}_H : A(D) \rightarrow A(D)$.

Motivated by the possibility that the trace and determinant do not in fact vary with the choice of favourable space $A(D)$, we follow Ruelle [18] in considering the following function:

Definition 2. For given weight functions w_i , $i \in \mathcal{J}$, the associated *dynamical determinant* is the entire function $\Delta : \mathbb{C} \rightarrow \mathbb{C}$, defined for all z of sufficiently small modulus by

$$\Delta(z) = \exp\left(-\sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{i \in \mathcal{J}^n} \frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I - T'_{\underline{i}}(z_{\underline{i}}))}\right), \quad (9)$$

⁵As always, we are making the standing assumption that D is an admissible domain, i.e. that the closure of $\cup_{i \in \mathcal{J}} T_i(D)$ lies in D .

where for $\underline{i} = (i_1, \dots, i_n) \in \mathcal{J}^n$ we set $T_{\underline{i}} := T_{i_n} \circ \dots \circ T_{i_1}$, and $w_{\underline{i}} := \prod_{k=1}^n w_{i_k} \circ T_{P_{k-1}\underline{i}}$, where $P_k : \mathcal{J}^n \rightarrow \mathcal{J}^k$ denotes the projection $P_k \underline{i} = (i_1, \dots, i_k)$ with the convention that $T_{P_0 \underline{i}} = \text{id}$, and $z_{\underline{i}}$ denotes the (unique, by [7]) fixed-point of $T_{\underline{i}}$ in D .

Theorem 5. *For every favourable space $A(D)$, the determinant of the transfer operator $\mathcal{L}_H : A(D) \rightarrow A(D)$ defined by (1) is precisely the dynamical determinant Δ , and its eigenvalue sequence is precisely the same as for $\mathcal{L} : U(D) \rightarrow U(D)$.*

Proof. The common trace formula

$$\tau_{A(D)}(\mathcal{L}_H^n) = \sum_{\underline{i} \in \mathcal{I}^n} \frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I - T'_{\underline{i}}(z_{\underline{i}}))} \quad \text{for all } n \geq 1 \quad (10)$$

can be established (see [5]), valid for every favourable space $A(D)$ on which \mathcal{L}_H acts, so that equality of determinants follows from comparison of (8) and (9). The equality of the eigenvalue sequences follows from the fact that the determinants are spectral. \square

4. HILBERT HARDY SPACE

In view of Theorem 5 we are now at liberty to make particular choices of favourable spaces, in the hope of obtaining interesting new bounds on the eigenvalues of the transfer operator $\mathcal{L} : U(D) \rightarrow U(D)$.

For $p \in [1, \infty)$, the *Hardy space* $H^p(D)$ (see [13, Ch. 8.3]) is a favourable space, and we will be particularly interested in the Hilbert Hardy space $H^2(D)$.⁶ The following eigenvalue bounds, valid in dimension 1, are obtained by choosing favourable space $A(D) = H^2(D)$ for $D \subset \mathbb{C}$ a disc:

Theorem 6. *With the hypotheses and notation of Theorem 2,*

$$|\lambda_n(\mathcal{L}_H)| \leq \frac{W}{\sqrt{1 - \theta^4}} \theta^{n-1} \quad \text{for all } n \geq 1. \quad (11)$$

Proof. As in the proof of Theorem 3, let D' denote the concentric disc whose radius is $r = \theta^2$ times that of D , and let $J : H^2(D) \hookrightarrow H^\infty(D')$ denote canonical embedding. It can be shown that, for all $n \geq 1$,

$$|\lambda_n(\mathcal{L}_H)| \leq W \prod_{k=1}^n a_k(J)^{1/n}, \quad (12)$$

⁶If D has C^2 boundary then $H^2(D)$ can be identified with the $L^2(\partial D, \sigma)$ -closure of $U(D)$, where σ denotes $(2d - 1)$ -dimensional Lebesgue measure on the boundary ∂D , normalised so that $\sigma(\partial D) = 1$. The inner product in $H^2(D)$ is given (see [13, Ch. 1.5 and 8]) by $(f, g) = \int_{\partial D} f^* \bar{g}^* d\sigma$, where, for $h \in H^2(D)$, the symbol h^* denotes the corresponding nontangential limit function in $L^2(\partial D, \sigma)$.

an inequality which is superior to (6), by virtue of the original Hilbert space version of Weyl's inequality, namely $\prod_{k=1}^n |\lambda_k(L)| \leq \prod_{k=1}^n a_k(L)$ (see [16, 3.5.1], [19]). An argument (see [4]) exploiting the interplay between the reproducing kernel of $H^2(D)$, and an orthonormal basis for $H^2(D)$ then allows the estimate

$$a_n(J) \leq \frac{r^{n-1}}{\sqrt{1-r^2}}, \quad (13)$$

and substituting into (12) yields the result. \square

Example 2. Comparing (12) with (4), we see that Theorem 6 leads to improved eigenvalue bounds whenever $n > 1/(1-\theta^4)$. In Example 1 we can choose $\theta = 1/\sqrt{5}$, therefore for all $n \geq 2 > 25/24$, the estimate

$$|\lambda_n(\mathcal{P}_H)| \leq \frac{77}{320} \frac{5}{\sqrt{24}} \left(\frac{1}{\sqrt{5}} \right)^{n-1}$$

derived from (12) is sharper than the previous bound (3) on the eigenvalues of the modified Perron-Frobenius operator.

A more elaborate version of the proof of Theorem 6 (see [4] for details) gives the following higher dimensional analogue, which for sufficiently large values of n yields estimates which are superior to those of Theorem 3:

Theorem 7. *With the hypotheses and notation of Theorem 3,*

$$|\lambda_n(\mathcal{L}_H)| < \frac{W\sqrt{d}}{r^d(1-r^2)^{d/2}} n^{(d-1)/(2d)} r^{\frac{d}{d+1}(d!)^{1/d}n^{1/d}} \quad \text{for all } n \geq 1. \quad (14)$$

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