Collective action in networks: communication, cooperation and redistribution

A thesis submitted in partial fulfilment of the requirements of the degree of Doctor of Philosophy (Ph.D.) in Economics

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Abstract

A person's friends, neighbours and other social relationships can have a large impact on their economic outcomes. We examine three important ways that networks can affect people's lives: when networks describe who they communicate with, who they can trust, and who benefits from their public good provision. We analyse information transmission in networks in a new, intuitive way which removes the problematic redundancy of double counting the signals that travel through more than one walk between nodes. Two-connectedness and cycles of length four play an important role in whether players are 'visible', which means that other players can communicate about them.

Next, using this approach to network communication, we investigate cooperation and punishment in a society where information flows about cheating are determined by an arbitrary fixed network. We identify which players can trust and cooperate with each other in a repeated game where members of a community are randomly matched in pairs. Our model shows how two aspects of trust depend on players' network position: they are 'trusting' if they are more likely to receive information about other players' types; and they are 'trusted' if others

can communicate about them, giving them strong incentives not to deviate.

Lastly, in networks with private provision of public goods, we show that a 'neutral' policy corresponds to a switch in the direction of the impact of income redistribution. Where redistribution is non-neutral, we can identify the welfare effects of transfers, including whether or not Pareto-improving transfers are possible. If not, we find the implicit welfare weights of the original equilibrium. In this setting, we also identify a transfer paradox, where, counter-intuitively, a transfer of wealth between economic agents can result in the giver being better off at the new Nash equilibrium, while the recipient is worse off.

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This thesis is dedicated to my parents and to Andy and Annie, who make everything possible.

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Introduction

In our first chapter, we identify a new approach to finding the probabilities of information transmission between nodes in a fixed network, which removes the problem of double counting of signals. Our approach, using De Morgan's laws, is a closed-form function that is simple to calculate and can use any value for the primitive probability that a signal passes between neighbouring nodes in the network. We introduce the concept of 'obstruction' by allowing nodes to decline to pass on the signals that may travel through the network links. If a node cannot prevent any other nodes from communicating by obstructing signals, we call that node 'visible'. A sufficient condition for a node to be visible is that it is a member of a cycle of length four. We also find new centrality measures that depend on the word-of-mouth probabilities and obstruction.

In our second chapter, we investigate cooperation in networks. Community enforcement is an important device for sustaining efficiency in some repeated games of cooperation. We investigate cooperation when information about players' reputations spreads to their future partners through links in the social network. We find that information supports cooperation by increasing trust between players, and obtain the 'radius of trust': an endogenous network listing the potentially cooperative relationships between pairs of players in a community. We identify two aspects of trust. Players are trusted if others can communicate about them, which we link to 2-connectedness of the network and the length of cycles within it. Players are trusting if they are more likely to receive information from others through their network connections; this is linked to word-of-mouth centrality.

In our final chapter, we move from a decentralised model to a centralised one, investigating the welfare implications of income redistribution on an economy with privately provided public goods in networks. First, we provide a new perspective on the neutrality result by showing that it corresponds to a change of policy direction. Next, we characterise the effects of income redistribution on social welfare, identifying conditions for Pareto-improving transfers. If these are not available, we find the implicit welfare weights of the initial equilibrium. We illustrate our results using some example core-periphery networks.

Chapter 1

Word-of-mouth communication in networks

1.1 Introduction

There are many different ways that information can flow in networks (Borgatti, 2005), depending on the nature of the information and the updating approach used by individuals. While Bayesian updating is the standard approach in complete networks, for arbitrary networks the inferences become rather complex, and behavioural approaches are often used.[1] We tackle this complexity by focusing on the transmission of signals, rather than beliefs.[2] This approach allows us to find the probabilities of information transmission between the nodes of any network, allowing Bayesian updating even in complex networks.

This chapter develops a new closed-form expression for the probabilities of node-to-node information transmission by *diffusion*, where neighbours may or may not pass signals to each other along walks of limited length (Banerjee, Chandrasekhar, Duflo and Jackson, 2013). We use the term *word-of-mouth* for our probabilities of information transmission between nodes, capturing the intuitive concept whereby information travels within a community via conversations between players and their connections (Ahn and Suominen, 2001; Lippert and Spagnolo, 2011).

We also show how the word-of-mouth probabilities are affected if players choose not to pass on messages, an action which we call obstruction. To do so we assume that information is 'hard' or 'evidentiary' (Nava, 2016; Wolitzky, 2014), so that nodes can choose whether or not to pass on messages. We find

[1] Degroot (1974); Golub and Jackson (2012); Mueller-Frank and Neri (2015); Goyal (2016); Levy and Razin (2014).

[2] In other settings, Hagenbach and Koessler (2010), Galeotti et al. (2013) and Acemoglu et al. (2014) also focus on transmission of signals in networks, using the cheap talk framework of Crawford and Sobel (1982).

that a node who is part of a cycle of length four is 'visible' in the sense that he cannot prevent any two other nodes in the network from communicating, if he chooses to obstruct messages.

A closely related measure to word-of-mouth centrality is *diffusion centrality*, developed by Banerjee et al. (2013) as an approximation of their epidemiological simulation measure, communication centrality. As a simple approximation, diffusion centrality implicitly assumes that the probabilities that a signal travels along each walk in the network are independent. With diffusion, a player may receive the same signal more than once along different walks in the network, and diffusion centrality measures the *expected total number of times* information is transmitted between nodes. As an aggregate measure, diffusion centrality suffers from the problem of double counting: the same signal is counted again when it is transmitted along different walks.[3] While maintaining the independence assumption, our word-of-mouth approach presents an improvement on diffusion centrality, using De Morgan's laws to remove the problem of double counting. This means that we can describe *whether or not* a signal is received along these different walks, allowing for Bayesian updating in networks.

Diffusion centrality has been found to have empirical relevance (Breza and Chandrasekhar, 2015; Fafchamps and Labonne, 2016), and our word-of-mouth probabilities may be easier to work with in an empirical context since they lie between zero and one (so no transformation is required) and they can be calculated for any value of the probability that two neighbours talk. For diffusion centrality, this value is given by the inverse of the largest eigenvalue of the

[3]This problem is related to but distinct from correlation neglect, where players observe the same signal more than once through different walks in the network and erroneously treat each report as a distinct signal.

network's adjacency matrix.

We find that removing double counting reduces the inequality of centrality between nodes in the network. Central nodes have high diffusion centrality because they receive so many signals. But word-of-mouth centrality takes account of the fact many of these signals may be redundant, because the information could already have been received by another route. By reducing the relative centrality of the most central nodes in a network, the word-of-mouth approach reduces the overall inequality in centrality. Arguably, counting the total number of times a signal is transmitted, may be more appropriate in investigating questions of influence, rather than information.

The chapter is structured as follows. Section 1.2 defines the probabilities of information transmission in networks, and Section 1.3 illustrates the approach on networks with homophily and segregation. Section 1.4 introduces the concept of obstruction and its consequences, and Section 1.5 provides some new centrality measures based on the word-of-mouth probabilities. Section 1.6 describes the literature on information transmission in networks and possible applications.

1.2 Word-of-mouth probabilities

In this section, we show how the fixed information network that connects the players can provide us with the node-to-node probabilities of information transmission.

1.2.1 The information network

The n players in $N = \{1, ..., n\}$ occupy the nodes of a fixed undirected unweighted information network $\mathbf{g}$ such that $\{i, j\} \in \mathbf{g}$ if i and j are neighbours. A walk of length a between two nodes i and j in network $\mathbf{g}$ is a sequence of nodes $(i = x_0, x_1, ..., x_{a-1}, x_a = j)$ such that for every $r \in \{1, 2, ..., a\}$, we have that $\{x_{r-1}, x_r\} \in \mathbf{g}$. If the nodes are distinct, the sequence is a path, and if in addition $i = j$, it is a cycle. Let $\mathbf{G} = [g_{ij}]$ be the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} = 1$ indicates that players i and j are neighbours so $\{i, j\} \in \mathbf{g}$, and $g_{ij} = 0$ otherwise (and $g_{ii} = 0 \;\; \forall i \in N$ by convention). The network $\mathbf{G}$ is common knowledge; all players know each other's network positions. Let $\mathcal{N}_i = \{j : g_{ij} = 1\}$ be the set of player i's neighbours and $|\mathcal{N}_i|$ be i's degree.

Let $d_{ij}(\mathbf{G})$ be the *social distance* — the length of the shortest path — between two players i and j in the network $\mathbf{G}$. Let $D_{\mathbf{G}} = \max\{d_{ij}(\mathbf{G})\}$ be the diameter of the network $\mathbf{G}$: the length of the longest shortest path between any two players. Two players are *connected* if there exists a path of finite length between them, and a network is connected if all players are connected to each other. If $\mathbf{G}$ is not connected, its diameter is infinite. Let $\mathbf{G}_{-k}$ be the adjacency matrix of the network with player k removed — that is, the $n \times n$ adjacency matrix created when all the entries in the kth row and column of $\mathbf{G}$ are set to zero. If $\mathbf{G}_{-k}$ is connected then the network is *2-connected with respect to k*; the network is *2-connected* if it is 2-connected with respect to all players.

1.2.2 Information transmission by diffusion

Diffusion is a structure of information transmission defined by Banerjee et al. (2013), where a signal flows through each link in the information network with a fixed probability, up to a maximum number of links.

Definition 1. Diffusion (Banerjee et al., 2013) is a process whereby information flows through the network with probability $p \in (0, 1]$ along each link, up to a maximum T links. The probability of information flowing along each walk in the network is independent.[4]

The parameter p denotes how likely players are to meet and/or exchange information with their neighbours. For example, if $p = 1$ and $T = 1$, information is passed with certainty only to a player's direct neighbours. Let $\mathbf{\Omega}$ denote the information structure of the network such that $\mathbf{\Omega} = \{p, T, \mathbf{G}\}$.

1.2.3 Word-of-mouth probabilities

We focus on the simple case of just one binary signal, which is either transmitted or not — and either received or not. In this section we do not allow nodes to conceal signals by not passing them on to their neighbours; this is considered in the next section.

Let $s_i \in \{\{1\}, \emptyset\}$ be the signal emitted by node i, and $\rho_j \in \{\{1\}, \emptyset\}$ be the signal received by node j. We define the *word-of-mouth probability* $w_{ij}(\mathbf{\Omega})$ as the

[4]This independence is implicitly assumed in Banerjee et al. (2013) and is achieved with the following assumptions: a signal is emitted by the source in each round of information transmission; players pass on each signal they receive independently of whether they receive any other signals; and players do not store information after passing it on to their neighbours, so that the only information players recall after the information transmission process has ended, is that which arrived in the final round of information transmission.

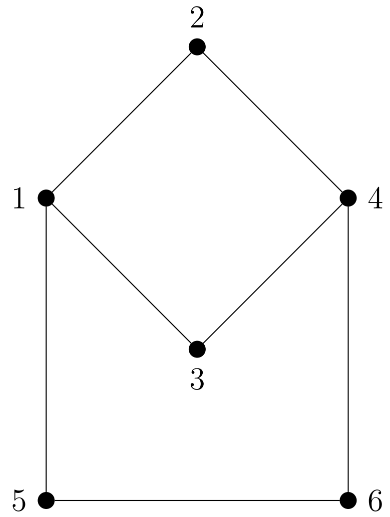


Figure 1.1: Example of diffusion

probability that a signal emitted by i will reach j.

$$\Pr\left[\rho_j = 1 \mid s_i\right] = \begin{cases} w_{ij}(\mathbf{\Omega}) & \text{if } s_i = 1 \\ 0 & \text{if } s_i = \emptyset \end{cases} \quad \forall i, j \in N \tag{1.2.1}$$

To calculate these probabilities, we need a way to account for the fact that a pair of players may be connected to each other by several walks in the network and as such, may transmit a signal via more than one of these walks. To deal with this issue, we use De Morgan's laws of duality (Fuente, 2000). One of these laws states that for a family of sets $\mathbb{A} = \{A_i; i \in I\}$ in the universal set X, where I is some index set, we have that $\sim(\cup_{i\in I} A_i) = \cap_{i\in I}(\sim A_i)$.[5] In other words, the complement of $w_{ij}(\mathbf{\Omega})$ is given by the probability that j does *not* hear a signal from i along *any* of the walks that connect i and j in $\mathbf{G}$.

Take for example the network in Figure 1.1, where we would like to find w_{14},

[5] With two events and using logic notation, this law can be written as $\neg(A \cup B) = \neg A \cap \neg B$.

the probability that a signal emitted by node 1 will be received by node 4. Let us set $T = 2$. There are two walks of length $\leq T$ that a signal could pass from 1 to 4: those via 2 and 3. Each are of length 2, so a signal can pass along each of them with probability p^2. Adding these two probabilities together would give the bilateral[6] entry of diffusion centrality: $d_{14} = p^2 + p^2$. But this is not a probability of node-to-node information transmission, because we need to take account of the fact that the signal could travel along both walks. This happens with probability p^4 because the probability of information flowing along each of these walks is independent. So we find that $w_{14} = p^2 + p^2 - p^4$. We can also get this result using De Morgan's laws where $1 - w_{14} = (1 - p^2)(1 - p^2)$.

For a more general formula, we know that for each walk of length τ, the probability that the signal does not travel along all the links in that walk by diffusion is $1 - p^\tau$. For a signal *not* to travel from i to j, we need a signal *not* to travel along every possible walk that connects i to j in $\mathbf{G}$, of length $\leq T$.[7] Let $l_{ij}(\tau, \mathbf{G})$ be the number of walks between i and j of length τ in the network $\mathbf{G}$.

Definition 2. ***Word-of-mouth probability*** given by $w_{ij}(\mathbf{\Omega})$ is the probability that a signal passes from i to j by diffusion, given in Definition 1. For any $\mathbf{\Omega}$, that is, for any $p \in (0, 1]$, any T and any $\mathbf{G}$, we have that

$$\begin{aligned} w_{ij}(\mathbf{\Omega}) &= \Pr[\rho_j = 1 \mid s_i = 1] \\ &= 1 - \prod_{\tau=1}^{T} [1 - p^\tau]^{l_{ij}(\tau, \mathbf{G})} \end{aligned}$$

[6]Diffusion centrality Banerjee et al. (2013) is given by $d_i = \left[\sum_{\tau=1}^{T} (p\mathbf{G})^\tau \mathbf{1}\right]_i$.

[7]In parallel work, Ambrus, Chandrasekhar and Elliott (2014) use the inclusion-exclusion principle to tackle a similar problem in a different context.

Where $l_{ij}(\tau, \mathbf{G}) = [\mathbf{G}^\tau]_{ij}$. This applies $\forall i \neq j \in N$, while $w_{ii}(\mathbf{\Omega}) = 1$.

Note that we have assumed that $\mathbf{G}$ is symmetric, which means that ingoing and outgoing probabilities of word-of-mouth communication are identical. This could be easily modified for a directed network. Let us say that players can *communicate* if there is a positive probability that a signal sent by one of them will be received by the other. Let us also say that a network is *informative* if every pair of players in the network can communicate.

1.2.3.1 Independence

Note that Definition 2 only holds if information flows as set out in Definition 1 apply; in particular, if the assumption that the probability of a signal travelling along each walk in the network is independent holds. Let us take this opportunity to reflect on the independence assumption. Consider the network in Figure 1.1, with node 6 removed, and w_{45}, the probability that a signal emitted by node 4 is received by node 5. There are two walks of length 3 between them: $\{4, 2, 1, 5\}$ and $\{4, 3, 1, 5\}$. With the independence assumption, the two walks can be treated independently, so we have that $w_{15} = 1 - (1 - p^3)(1 - p^3) = 2p^3 - p^6$.

We can see that node 1 could receive the signal from 4 through two walks, either via nodes 2 or 3. With independence, he would treat these two signals separately. But a more accurate specification might be that there would only be one opportunity for 1 to pass this signal to 5 or not. In this case, $w_{45} = pw_{41} = p(1-(1-p^2)(1-p^2)) = 2p^3 - p^5$. This result can also be found as the probability that the 3 links in each walk are activated separately, minus the probability that all 5 links in both walks are activated. This would be the kind of result generated by Banerjee et al. (2013)'s algorithm of communication centrality. This example shows that there is a small loss in precision due to the independence assumption. Arguably, this is offset by a much easier and quicker computation of the word-of-mouth probabilities as an approximation of this communication mechanism.

1.2.4 The information structure

Next we examine how the three aspects of the information structure $\boldsymbol{\Omega}$ affect the probabilities of information transmission.

Proposition 1. *There are complementarities between the three aspects of the information structure $\boldsymbol{\Omega}$ for the word-of-mouth probabilities given in Definition 2. In particular, it holds that*

1. $w_{ij}(\boldsymbol{\Omega})$ *is increasing in* p *if and only if* $\exists\ \tau \leq T$ *such that* $l_{ij}(\tau, \boldsymbol{G}) \geq 1$
2. $w_{ij}(\boldsymbol{\Omega})$ *increases as* T *increases to* $T+1$ *if and only if* $l_{ij}(T+1, \boldsymbol{G}) \geq 1$
3. $w_{ij}(\boldsymbol{\Omega})$ *increases as a link is added to* $\boldsymbol{G}$ *if and only if the new link leads to an increase in* $l_{ij}(\tau, \boldsymbol{G})$ *for any* $\tau \leq T$

An increase in p *or* T *or an additional link in* $\boldsymbol{G}$ *cannot lead to a decrease in any obstructed word-of-mouth probabilities.*

Proof. See Appendix. ■

Proposition 1 shows that, as would be expected, a network with more links, greater probability of players transmitting messages to their neighbours, and more rounds for information to travel, could have higher probabilities of information transmission. The following Remark shows how the players' communication depends on social distance, the diameter of the network and the furthest distance that information can travel.

Remark 1.2.1. *It holds that:*

- *for i, j such that $d_{ij}(\boldsymbol{G}) \leq T$, we have $w_{ij}(\boldsymbol{\Omega}) \geq p^{d_{ij}(\boldsymbol{G})} > 0$; and*

- *For i, j such that $d_{ij}(\boldsymbol{G}) > T$, we have $w_{ij}(\boldsymbol{\Omega}) = 0$.*

This implies that:

- *If $\boldsymbol{G}$ is connected and $D_{\boldsymbol{G}} \leq T$, then $w_{ij}(\boldsymbol{\Omega}) \geq p^{D_{\boldsymbol{G}}} > 0 \quad \forall i, j$ and the network is informative;*

- *If, in addition $p = 1$, then $w_{ij}(\boldsymbol{\Omega}) = 1 \ \forall \ i, j$, and there is perfect information; and*

- *If $\boldsymbol{G}$ is not connected, then $\forall i \ \exists \ j$ such that $w_{ij}(\boldsymbol{\Omega}) = 0$.*

- *If $d_{ij} > T \ \forall \ i, j$ then $w_{ij}(\boldsymbol{\Omega}) = 0 \quad \forall i, j$*

Corollary 1. *For any pair of nodes $\{i, j\}$ in any network, if $p < 1$, there exists an upper bound $P*_{ij} < 1$ on their word-of-mouth probability, with respect to T.*

Proof. See Appendix. ∎

The intuition behind this result is that as T increases, it becomes possible for a signal to travel between two nodes using longer and longer walks. But the probability of information travelling through all the links in a long walk is low, because it depends on p^T. In fact, the incremental effect of increasing T on the word-of-mouth probabilities converges to zero at high T. So the most important factor affecting the magnitude of the word-of-mouth probabilities is the number of short walks between players.

Following on from this, the existence of a lower bound $p^{d_{ij}(\mathbf{G})}$ begs the question as to what might lead to a divergence between this lower bound and $w_{ij}(\mathbf{\Omega})$. If $d_{ij}(\mathbf{G}) = T$, then $w_{ij}(\mathbf{\Omega}) = p^{d_{ij}(\mathbf{G})}$ if there is only one walk of length T between i and j, and increases above the lower bound as the number of walks of length T between i and j increases. Next, if the social distance $p^{d_{ij}(\mathbf{G})} < T$, there will probably be other walks between i and j of length less than T. More walks of length $\leq T$ will increase the divergence between $w_{ij}(\mathbf{\Omega})$ and $p^{d_{ij}(\mathbf{G})}$. The greater the difference between T and the social distance between i and j, the more likely there will be more shorter walks between them. In all cases, the more walks there are of lengths less than T between nodes, the more ways there are for those nodes to transmit information between them, and the greater divergence expected between the lower bound $p^{d_{ij}(\mathbf{G})}$ and $w_{ij}(\mathbf{\Omega})$. As described above, shorter walks have higher probability of communication and so have a bigger impact on this divergence.

1.2.5 Example networks: line and star

Next, let us show the of word-of-mouth probabilities in two example networks, the line and the star, shown in Figure 1.2. With parameters of $p = 0.15$ and $T = 6$,

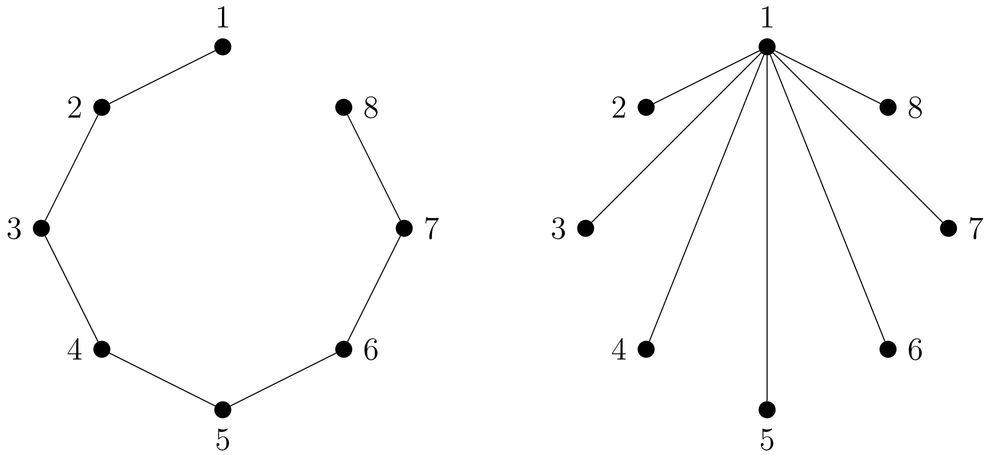


Figure 1.2: Line and star networks of eight nodes

the word-of-mouth probabilities between the eight nodes in the line network are:

$$w_{ij}(\mathbf{\Omega}) = \begin{bmatrix} 1 & 0.156 & 0.024 & 0.004 & 0.001 & 0^+ & 0^+ & 0 \\ 0.156 & 1 & 0.159 & 0.025 & 0.004 & 0.0010 & 0^+ & 0^+ \\ 0.024 & 0.159 & 1 & 0.159 & 0.025 & 0.004 & 0.001 & 0^+ \\ 0.004 & 0.025 & 0.159 & 1 & 0.159 & 0.025 & 0.004 & 0.001 \\ 0.001 & 0.004 & 0.025 & 0.159 & 1 & 0.159 & 0.025 & 0.004 \\ 0^+ & 0.001 & 0.004 & 0.025 & 0.159 & 1 & 0.159 & 0.024 \\ 0^+ & 0^+ & 0.001 & 0.004 & 0.025 & 0.159 & 1 & 0.156 \\ 0 & 0^+ & 0^+ & 0.001 & 0.004 & 0.024 & 0.156 & 1 \end{bmatrix} \tag{1.2.2}$$

As would be expected, these decrease for further-away nodes, becoming negligible at a social distance of five links. Meanwhile the probabilities are higher than the lower bound determined by the social distance between nodes, e.g. the probability of communication between nodes 2 and 3 is 0.159, higher than p, which is 0.15. The reverberation of the message in the network increases the probability that it will eventually arrive.

Due to symmetry, there are only two word-of-mouth probabilities in the star. The probability that a signal emitted by the centre of the star will be received by a node on the periphery is 0.173. The probability that a signal emitted by one periphery node will be received by the other is 0.027. Note that the highest probability that a signal will travel a walk of length 1 is 0.159 in the line and 0.173 in the star; and for a walk of length 2 it is 0.025 and 0.027 respectively. With $p = 0.15$ and so $p^2 = 0.0225$, we can see that there is greater divergence between $w_{ij}(\mathbf{\Omega})$ and $p^{d_{ij}(\mathbf{G})}$ in the star network than the line. This is because, as discussed above, the network structure of the star allows for a greater number of short walks between nodes.

1.3 Word-of-mouth probabilities in networks with homophily

Having found the word-of-mouth probabilities in the general case, next we can illustrate the case when networks exhibit homophily (Currarini et al., 2009; Golub and Jackson, 2012). Homophily, which is closely related to segregation, is when nodes are more likely to be connected to others who are similar to them — members of their own group — rather than to members of other groups. To examine homophily in our case, we use the approach of equitable partition and the quotient graph.

1.3.1 Equitable partition and quotient graph

Definition 3. Equitable partition (Powers and Sulaiman, 1982). Each node in one group must have the same number of links to nodes in the other group. Recall that $\mathcal{N}_i = \{j : g_{ij} = 1\}$ be the set of player i's neighbours and $|\mathcal{N}_i|$ be i's degree. A partition $\pi = \{C_1, ..., C_H\}$ of agents in a network is an equitable partition if, when agents i and j belong to the same type h,

$$| \mathcal{N}_i \cap C_h | = | \mathcal{N}_j \cap C_h | \quad \forall \;\; h = 1, ..., H$$

Definition 4. Quotient graph (Powers and Sulaiman, 1982). An equitable partition can be represented by a quotient graph $\mathbf{g}_\pi$ and its $H \times H$ adjacency matrix $\mathbf{G}_\pi$, whose ij th entry is $| \mathcal{N}_i \cap C_j |$. The *indicator matrix* $\mathbf{C}$ is an $n \times H$ matrix, which denotes the membership of each of the n agents in each of the H groups with a 1 if they are a member and a zero if not. $\mathbf{G}$ and $\mathbf{G}_\pi$ are related by $\mathbf{GC} = \mathbf{CG}_\pi$.

The quotient graph plays an important role in the study of the main part of the spectrum $\mathcal{M}$ since it holds that

$$\mathcal{M} \subset \text{spec}(\mathbf{G}_\pi) \subset \text{spec}(\mathbf{G}) \tag{1.3.1}$$

Godsil (1993) (Lemma 2.2, Chapter 5) shows that if Y is an equitable partition of Z, then if $(\lambda, \mathbf{x})$ is an eigenpair of Y, then $(\lambda, \mathbf{Cx})$ is an eigenpair of Z. This is known as 'lifting' the eigenvectors. We note that this approach could be used to transform any $n \times 1$ vector of characteristics of each agent to a $H \times 1$ vector of characteristics of each partitioned group.

If there are just two groups labelled A and B, of sizes a and b respectively, then the adjacency matrix $\mathbf{G}$ can be partitioned as follows, where $\mathbf{G}_{AA}$ is the submatrix reflecting only the links between nodes in group A etc.

$$\mathbf{G} = \left[\begin{array}{c|c} \mathbf{G}_{AA} & \mathbf{G}_{AB} \\ \hline \mathbf{G}_{BA} & \mathbf{G}_{BB} \end{array}\right] \tag{1.3.2}$$

Meanwhile the entries in the quotient graph $\mathbf{G}_\pi$ denote how many links their are between nodes in the two different groups. That is,

$$\mathbf{G}_\pi = \begin{bmatrix} \alpha & \gamma \\ \delta & \beta \end{bmatrix} \tag{1.3.3}$$

Where α and β are the number of links within each group A and B respectively, γ is the number of links that any node in A has to nodes in group B, and δ is the number of links that any node in B has to nodes in group A (γ and δ are not necessarily the same, even in an undirected matrix, as long as a and b are different).

From Definition 1.3.3 we can also observe that the exponent of the adjacency matrix, which we have used to identify the number of walks of different lengths, can also be found from the exponent of quotient graph and the indicator matrix. That is, from $\mathbf{GC} = \mathbf{CG}_\pi$ we have that $\mathbf{G}^\tau\mathbf{C} = \mathbf{CG}_\pi^\tau$. Hence, we can see that the entries in $\mathbf{G}_{\pi}^{\tau}$ are

$$\mathbf{G}_{\pi}^{\tau} = \left[\begin{array}{c|c} \sum_{j \in A} l_{Aj}(\tau, \mathbf{G}) & \sum_{j \in B} l_{Aj}(\tau, \mathbf{G}) \\ \hline \sum_{j \in A} l_{Bj}(\tau, \mathbf{G}) & \sum_{j \in B} l_{Bj}(\tau, \mathbf{G}) \end{array} \right] \tag{1.3.4}$$

where $\sum_{j \in A} l_{Aj}(\tau, \mathbf{G}) = \sum_{j \in A} l_{ij}(\tau, \mathbf{G})$ for each $i \in A$, that is, the total number of walks of length τ from any node in A to all other nodes in A. Due to equitable partition this total is the same for every member of A. Note this is a total number of walks: when comparing different pairs of nodes in the same or different groups, they may have more or fewer walks between them, but the total number of walks of each length that a node has to all nodes in a particular group is the same, when aggregating over all the nodes in the relevant group. This is due to the assumption of equitable partition: different nodes may or may not be connected to each other in $\mathbf{G}$ but the total number of connections that a node has to a particular group is the same.

1.3.2 Homophily

Now we can use the quotient graph approach to examine the word-of-mouth probabilities in an example case of homophily with two groups. Let us consider a network with symmetry in relation to the number of links between and within groups. That is, from (1.3.3) we have that $\alpha = \beta$ is the number of links within groups and $\gamma = \delta$ is the number of links between groups. If there is homophily and segregation, nodes are more likely to have links with their own group than with outsiders, and we have that $\alpha > \gamma$. We can now observe that the exponent of the quotient graph can be written as

$$\mathbf{G}_\pi^\tau = \begin{bmatrix} \alpha & \gamma \\ \gamma & \alpha \end{bmatrix}^\tau = \frac{1}{2}\left((\alpha+\gamma)^\tau \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (\alpha-\gamma)^\tau \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \tag{1.3.5}$$

Hence in this symmetric case, the total number of walks within and between groups are given respectively as

$$\sum_{j\in A} l_{Aj}(\tau,\mathbf{G}) \;\; = \textstyle\sum_{j\in B} l_{Bj}(\tau,\mathbf{G}) \;\; = \frac{1}{2}\left[(\alpha+\gamma)^\tau + (\alpha-\gamma)^\tau\right] \tag{1.3.6}$$

$$\sum_{j\in B} l_{Aj}(\tau,\mathbf{G}) \;\; = \textstyle\sum_{j\in A} l_{Bj}(\tau,\mathbf{G}) \;\; = \frac{1}{2}\left[(\alpha+\gamma)^\tau - (\alpha-\gamma)^\tau\right] \tag{1.3.7}$$

According to the equations above, the number of walks between each pair of nodes depends on both α and γ, both within and between groups. We noted above that homophily implies that $\alpha > \gamma$. As the difference between α and γ increases, we find that the number of links between groups becomes less important in determining the number of walks in the network. In particular, let $\gamma = x\alpha$, with $x < 1$. In this case we can rewrite the above equations, and observe that as x gets very small and the network becomes more segregated, we can approximate the number of walks of length τ as follows.

$$\frac{1}{2}\left[\alpha^\tau(1+x)^\tau + \alpha^\tau(1-x)^\tau\right] \;\approx\; \alpha^\tau \tag{1.3.8}$$

$$\frac{1}{2}\left[\alpha^\tau(1+x)^\tau - \alpha^\tau(1-x)^\tau\right] \;\approx\; 0 \tag{1.3.9}$$

So for a segregated society, the number of walks of each length, and hence the word-of-mouth probabilities, can be approximated by looking only at the number of walks within each group. This recalls a result by Allouch (2017) that nodes' Bonacich centrality (Bonacich, 1987) in segregated networks can be approximated by their Bonacich centrality within their own group, because adding an extra step to a walk inside a group will most likely reach a member of the same group.

1.3.3 Complete bipartite networks

In the complete bipartite network, the two groups, A and B, are not connected within the group and are completely connected to all nodes in the other group. For example, the star network results when one of the groups contains only one node. As described in the previous section, the divergence between the word-of-mouth probabilities and their lower bound is due to the number of short walks. For complete bipartite networks, the number of walks of different lengths depends on the group sizes.

As above the two groups have sizes a and b respectively, so that $n = a + b$. Now in (1.3.3) we have that $\alpha = \beta = 0$, $\gamma = b$ and $\delta = a$. As before, we can find the total number of walks between and within groups as follows. As may be expected, there is a repeating pattern for walks of even and odd lengths. Let

$R = T/2$ if T is even, and $R = (T-1)/2$ if T is odd. We now have that

$$\mathbf{G}_\pi^\tau = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}^\tau = \begin{cases} \mathbf{G}_\pi^{2r} & = \begin{bmatrix} (ab)^r & 0 \\ 0 & (ab)^r \end{bmatrix} \text{ if } \tau \text{ is even} \\ \mathbf{G}_\pi^{2r+1} & = \begin{bmatrix} 0 & a^r b^{r+1} \\ a^{r+1} b^r & 0 \end{bmatrix} \text{ if } \tau \text{ is odd} \end{cases}, r \in [0, R] \tag{1.3.10}$$

These are the total number of walks that a node in one group has with all nodes in a particular group, as shown in (1.3.4). Our formula for the word-of-mouth probabilities requires the number of walks of length τ between any two nodes i and j. To find this from the matrices using the quotient graph in (1.3.10) in a complete bipartite network, we need to divide by a if j is in group A (the first column) and divide by b if j is in group B (the second column). From (1.3.10) we can obtain the following results for walks of odd length, where $\tau = 2r+1$.

- If i, j are in the same group, $l_{ij}(\tau, \mathbf{G}) = 0$
- If i, j are in different groups, $l_{ij}(\tau, \mathbf{G}) = (ab)^r = (ab)^{\frac{\tau-1}{2}}$

Whereas for walks of even length where $\tau = 2r$, (1.3.10) shows that

- If i, j are in different groups, $l_{ij}(\tau, \mathbf{G}) = 0$
- If i, j are in group A, $l_{ij}(\tau, \mathbf{G}) = a^{r-1} b^r = a^{\frac{\tau-2}{2}} b^{\frac{\tau}{2}}$
- If i, j are in group B, $l_{ij}(\tau, \mathbf{G}) = a^r b^{r-1} = a^{\frac{\tau}{2}} b^{\frac{\tau-2}{2}}$

We can now calculate the word-of-mouth probabilities. Let $w_{ab}(\mathbf{\Omega})$ be the probability of information transmission by word-of-mouth between distinct nodes in groups A and B, and $w_{aa}(\mathbf{\Omega})$ and $w_{bb}(\mathbf{\Omega})$ be the probabilities of information transmission between distinct nodes within groups A and B respectively. The word-of-mouth probabilities are:

$$w_{ab}(\mathbf{\Omega}) = 1 - [1-p]\prod_{r=1}^{R}\left[1-p^{2r+1}\right]^{(ab)^r} \tag{1.3.11}$$

$$w_{aa}(\mathbf{\Omega}) = 1 - \prod_{r=1}^{R}\left[1-p^{2r}\right]^{a^{r-1}b^r} \tag{1.3.12}$$

$$w_{bb}(\mathbf{\Omega}) = 1 - \prod_{r=1}^{R}\left[1-p^{2r}\right]^{a^r b^{r-1}} \tag{1.3.13}$$

1.3.3.1 Comparative statics

The multiplicative formulation for the number of walks suggests that for a given overall population, there are more walks of each length if group sizes are more similar to each other, than if the group sizes are very unequal, e.g. in the star network. We can check this using the above formulation. This gives us the following result.

Remark 1.3.1. *The total number of walks in a complete bipartite network of two equal-sized groups is strictly greater than that in a star network, for any walk length and $n > 2$.*

Proof. See Appendix. ■

Given these results and Remark 1.3.1, we would expect that the probabilities of information transmission by word-of-mouth would be higher in the case of two equal-sized groups than in the star. We can easily find the expressions for these probabilities of word-of-mouth information transmission in these two example networks as follows. For two equal groups of size $\frac{n}{2}$, we have two probabilities of information transmission: between distinct nodes in the same group, $w_s(\mathbf{\Omega})$ and between nodes in different groups $w_d(\mathbf{\Omega})$. For the star, the two probabilities of information transmission are that between the core and periphery nodes $w_{cp}(\mathbf{\Omega})$, and that between distinct periphery nodes, $w_{pp}(\mathbf{\Omega})$.

$$w_s(\mathbf{\Omega}) = 1 - \prod_{r=1}^{R} \left[1 - p^{2r}\right]^{\left(\frac{n}{2}\right)^{(2r-1)}} \quad (1.3.14)$$

$$w_d(\mathbf{\Omega}) = 1 - [1 - p] \prod_{r=1}^{R} \left[1 - p^{2r+1}\right]^{\left(\frac{n}{2}\right)^{2r}} \quad (1.3.15)$$

$$w_{pp}(\mathbf{\Omega}) = 1 - \prod_{r=1}^{R} \left[1 - p^{2r}\right]^{(n-1)^{(r-1)}} \quad (1.3.16)$$

$$w_{cp}(\mathbf{\Omega}) = 1 - [1 - p] \prod_{r=1}^{R} \left[1 - p^{2r+1}\right]^{(n-1)^{r}} \quad (1.3.17)$$

We can now illustrate the effects on these probabilities as T or n increases. Figure 1.3 shows these probabilities in the two networks, with the chart on the left showing the word-of-mouth probabilities of information transmission between nodes with social distance one: that is, between the two groups, or between the core and periphery of the star, and the lower bound of p. The chart on the right shows the probabilities between nodes with social distance two: that is, within the same group and between periphery nodes in the star, as well as the lower bound of p^2. It is clear that word-of-mouth probabilities in the complete bipartite network are indeed higher for the two equal groups than for the star.

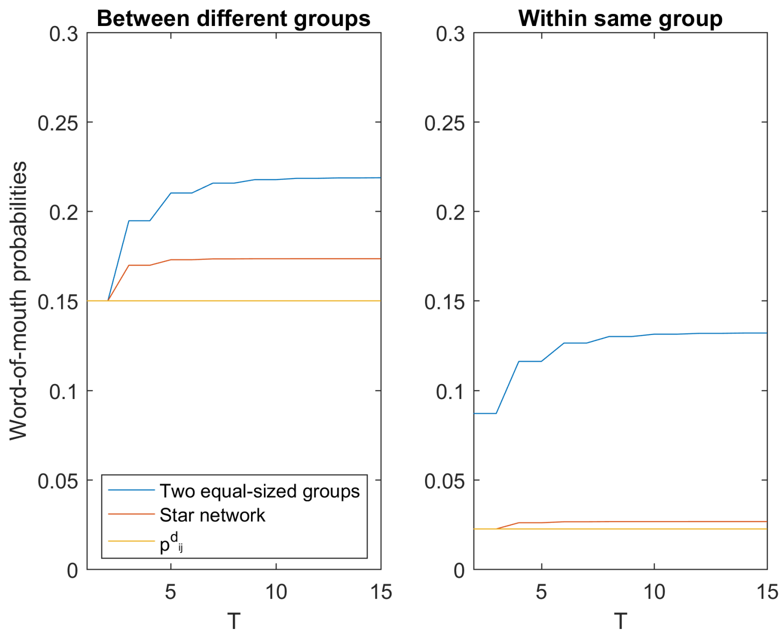


Figure 1.3: Word-of-mouth probabilities for two types of complete bipartite network with $n = 8$, as total number of rounds of information transmission (T) increases.

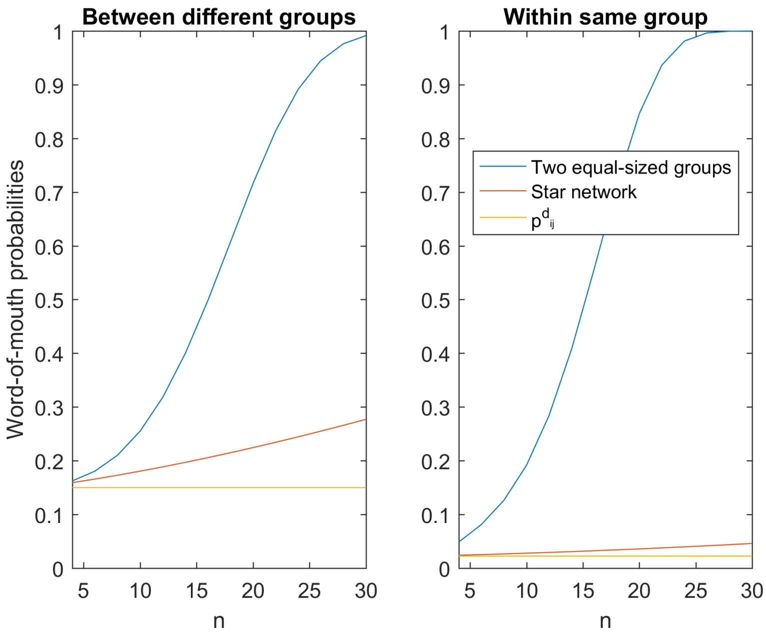


Figure 1.4: Word-of-mouth probabilities for two types of complete bipartite network with $T = 6$, as total number of nodes n increases.

The divergence between $w_{ij}(\mathbf{\Omega})$ and $p^{d_{ij}(\mathbf{G})}$ is lower in the star. Figure 1.3 also shows the probabilities converge to some upper bound for large T, as described in Corollary 1.

The proof of Remark 1.3.1 shows that the divergence in the number of walks between the star and the network of two equal groups follows a quadratic function in n, the total number of nodes. Figure 1.4 shows the divergence between the word-of-mouth probabilities in the two difference types of network increasing with a quadratic shape as n increases. As would be expected, the probabilities converge to 1 as n increases: the limits of the probabilities as $n \to \infty$ are 1. This does not imply that larger group sizes are better for word-of-mouth information transmission though: we are adding many more links than nodes as n increases due to the complete bipartite structure. In particular, if two new nodes are added, this requires n new links in the network. If the numbers of links change at a very different rate to the size of the population, arguably it is not possible to draw a comparison between group sizes.

1.4 Obstruction and visibility

In this section we build on the word-of-mouth probabilities by allowing nodes to choose whether to pass on information or not. To model this we assume that information is 'hard' or 'evidentiary' (Nava, 2016; Wolitzky, 2014), in the sense that nodes can choose to conceal information, but not modify it. If they do not pass it on, we call this *obstruction*.

Definition 5. Obstructed diffusion is a process whereby information flows through the network with adjacency matrix $\mathbf{G}$ with probability $p \in (0, 1]$ along each link, up to a maximum T links. Information is evidentiary and the probability of information flowing along each walk in the network is independent. The network links, through which a signal can flow by obstructed diffusion, include only those nodes who choose not to obstruct it in that round of information transmission.

The wider structure of a network game, where nodes can either obstruct information or not, would provide the incentives for choosing obstruction. Nodes may have incentives to obstruct signals in different rounds of the information transmission process – so a different subset of nodes could choose to obstruct certain signals in each of the T rounds. We show here the simplest example, of obstruction by a single player in all rounds of the game — but the same approach could be used for more complex obstruction patterns (shown in the Appendix). In this chapter we do not focus on the incentives in a game that would lead a player to obstruct the signal — simply on the probabilities that would result if he did so. Chapter 2 provides an interesting application of such a case.

1.4.1 Obstructed word-of-mouth probabilities

Let $p_{ij}(k, \mathbf{\Omega})$ be the *obstructed word-of-mouth probability* that a signal emitted by player i will be received by player j, when it is obstructed by player k. We calculate these probabilities in the same way as the word-of-mouth probabilities, except we use the network $\mathbf{G}_{-k}$ to calculate the number of walks, since player k will not pass on the signal. Let $l_{ij}(\tau, \mathbf{G}_{-k})$ be the number of walks between i and j of length τ in the network $\mathbf{G}_{-k}$.

Let $s_i(k) \in \{\{1\}, \emptyset\}$ be the signal that player i sends that is obstructed by player k, and let $\rho_j(k) \in \{\{1\}, \emptyset\}$ be the signal that j receives that is obstructed by k. We define the *probability of information transmission* $p_{ij}(k)$ as the probability that a signal emitted by i obstructed by k will reach j.

$$\Pr\left[\rho_j(k) = 1 \mid s_i(k)\right] = \begin{cases} p_{ij}(k) & \text{if } s_i(k) = 1 \\ 0 & \text{if } s_i(k) = \emptyset \end{cases} \quad \forall i, j, k \in N \tag{1.4.1}$$

Definition 6. Obstructed word-of-mouth probability given by $p_{ij}(k, \mathbf{\Omega})$ is the probability that a signal passes from i to j by obstructed diffusion given in Definition 5, when the signal is obstructed by k. For any $\mathbf{\Omega}$, that is, for any

$p \in [0,1]$, any T and any $\mathbf{G}$, we have that

$$\begin{aligned} p_{ij}(k,\mathbf{\Omega}) &= \Pr[\rho_j(k) = 1 \mid s_i(k) = 1] \\ &= 1 - \prod_{\tau=1}^{T} [1 - p^\tau]^{l_{ij}(\tau,\mathbf{G}_{-k})} \end{aligned}$$

Where $l_{ij}(\tau, \mathbf{G}_{-k}) = [\mathbf{G}_{-k}^\tau]_{ij}$. This applies $\forall i \neq j, k \in N$, while $p_{ii}(k,\mathbf{\Omega}) = 1$.

Note that $p_{ij}(k,\mathbf{\Omega}) = 0$ if $i = k$ or $j = k$. Proposition 1 and Remark 1.2.1, which describe the effects of the information structure $\mathbf{\Omega}$ and social distance on information transmission probabilities, both apply to the case with obstruction, if we replace $\mathbf{G}$ with $\mathbf{G}_{-k}$. Since $\mathbf{G}$ has weakly more links than $\mathbf{G}_{-k}$, from Proposition 1, we have that $w_{ij}(\mathbf{\Omega}) \geq p_{ij}(k,\mathbf{\Omega})$ $\forall i, j, k$.

1.4.2 Visibility

We have shown that if player i emits a signal that is obstructed by player k, the probabilities that the signal is received by other nodes are determined by player i's position in $\mathbf{G}_{-k}$, the network omitting k. Recall that nodes can *communicate* if there is a positive probability that a signal sent by one of them will be received by the other. We now define the concept of visibility in networks.

Definition 7. A node is **visible** if and only if everyone can still communicate, even when he is obstructing the signal. That is, player k is visible iff $p_{ij}(k,\mathbf{\Omega}) > 0 \;\; \forall \; i, j \in N \setminus k$.

We can rewrite Remark 1.2.1 for the case when player k obstructs the signal.

Remark 1.4.1. *It holds that:*

- *For i, j such that $d_{ij}(\boldsymbol{G}_{-k}) \leq T$, we have that $p_{ij}(k, \boldsymbol{\Omega}) \geq p^{d_{ij}(\boldsymbol{G}_{-k})} > 0$*
- *For i, j such that $d_{ij}(\boldsymbol{G}_{-k}) > T$, we have that $p_{ij}(k, \boldsymbol{\Omega}) = 0$*

This implies that:

- *If $\boldsymbol{G}_{-k}$ is connected and $D_{\boldsymbol{G}_{-k}} \leq T$, then $p_{ij}(k, \boldsymbol{\Omega}) \geq p^{D_{\boldsymbol{G}_{-k}}} > 0 \;\; \forall i, j$ and player k is visible.*
- *If the network is 2-connected and $\max_{k \in N}\{D_{\boldsymbol{G}_{-k}}\} \leq T$, all nodes are visible.*
- *If, in addition, $p = 1$, then $p_{ij}(k, \boldsymbol{\Omega}) = 1 \; \forall \; i, j, k$, and there is perfect information.*
- *If $\boldsymbol{G}_{-k}$ is not connected, then $\forall i \; \exists \; j$ such that $p_{ij}(k, \boldsymbol{\Omega}) = 0$*

Visibility depends on what the network looks like when a node is absent i.e. the structure of $\mathbf{G}_{-k}$. It requires two things for k to be visible. First, $\mathbf{G}_{-k}$ must be connected, because a visible player does not disconnect the network by his absence, i.e. if $\mathbf{G}_{-k}$ is connected then $\mathbf{G}$ is 2-connected with respect to k. Secondly, the diameter of $\mathbf{G}_{-k}$ must be not be greater than T, the maximum distance a signal can travel.[8]

1.4.3 Obstructiveness

A comparison between word-of-mouth probabilities and obstructed word-of-mouth probabilities gives us a measure of the effect of each player's obstruction.

[8]This implies 2-connectedness because a network with a finite diameter is connected.

Definition 8. A node is **obstructive** if and only if his obstruction means that two or more nodes who could previously communicate no longer can. That is, player k is obstructive iff $\exists\, i, j \in N$ such that $w_{ij}(\mathbf{\Omega}) > 0$ and $p_{ij}(k, \mathbf{\Omega}) = 0$.

Obstruction is linked to social distance, because player k is not obstructive if for all $\{i, j\}$ with $d_{ij}(\mathbf{G}) \leq T$ we also have that $d_{ij}(\mathbf{G}_{-k}) \leq T$. That is to say, player k is not obstructive if he does not increase the social distances too much by his absence from the information network. We collect the conditions for obstructiveness in the following Remark, highlighting the link between obstructiveness and the length of the *cycles* that include a player.

Remark 1.4.2. *If k has only one neighbour, k is not obstructive. If k has more than one neighbour, then k is not obstructive if and only if the following. For each pair of nodes l, m with $d_{lm}(\boldsymbol{G}) \leq T$ and for whom the sequence (i, k, j) is part of the shortest path(s) between them (implying that $i, j \in \mathcal{N}_k$), we require that $d_{lm}(\boldsymbol{G}) - 2 + d_{ij}(\boldsymbol{G}_{-k}) \leq T$, or equivalently that there exists a cycle in $\boldsymbol{G}$ including the sequence (i, k, j) with length $\leq T + 4 - d_{lm}(\boldsymbol{G})$.*

More generally, if a player k has more than one neighbour, a sufficient condition for him not to be obstructive is if, for each pair of k's neighbours $i, j \in \mathcal{N}_k$, there is a cycle of length ≤ 4 including the sequence (i, k, j). This implies that for all $l, m \in N$ who have the sequence (i, k, j) as part of the shortest path(s) between them, we have that $d_{lm}(\boldsymbol{G}_{-k}) = d_{lm}(\boldsymbol{G})$.

Figure 1.5 shows an example of this result. In both networks in the Figure, $\mathbf{G}$ is made up of all the solid and dashed links, while $\mathbf{G}_{-k}$ includes only the solid links. In the network on the left, k is in a cycle of four in network $\mathbf{G}$, and $d_{lm}(\mathbf{G}) = 4$. If node k is removed, and we examine $\mathbf{G}_{-k}$, then the social

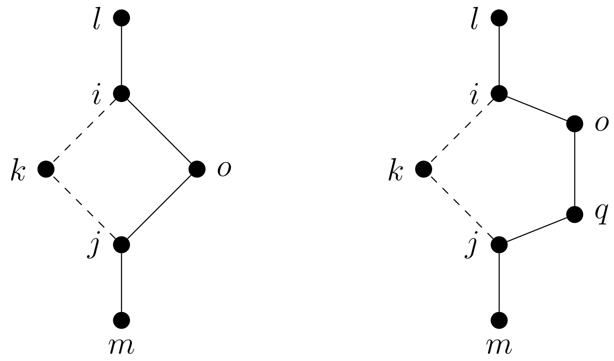


Figure 1.5: Examples of obstructiveness

distance between l and m is unchanged — it remains 4. This is because there is an alternative route between l and m via node o. Node o is connected to k's neighbours i and j, so any other walks in a wider network which include the sequence $\{i, k, j\}$ could instead include the sequence $\{i, o, j\}$, which is the same length. So if k is in a cycle of length 4, his removal from the network does not increase the social distances of any other pairs of nodes. So k cannot be obstructive, no matter what the value of T is. On the other hand, the network on the right shows a case where k could be obstructive, depending on the value of T. Now in $\mathbf{G}_{-k}$, the social distance between l and m has increased to 5 links, whereas before when k was present it was only 4. So if $T = 4$, with k passing on signals in the network it would be possible for l and m to communicate. Without him, they cannot: k is obstructive.

The length of the cycle determines whether or not k's neighbours can still communicate, even when k is obstructing those signals. The importance of cycles of length four, our sufficient condition for a player not to be obstructive, recalls well-known results on the importance of network cycles of length three: Coleman's (1988) *closure*; and Jackson et al.'s (2012) *support*.

In Remark 1.2.1 we said that a network is *informative* if everyone can communicate without obstruction. Now we can make a link between obstructiveness and visibility.

Proposition 2. *Player k is visible if and only if the network is informative and player k is not obstructive.*

Proof. See Appendix. ■

Proposition 2 shows that there are two possible reasons why a player may not be visible: firstly, if the network is not informative, so that even without obstruction some nodes cannot communicate; and secondly, if the network is informative but a player is obstructive, in that he can prevent some nodes from communicating if he does not pass signals. Obstructiveness sheds further light on the star network.

Remark 1.4.3. *For a given number of nodes in a connected information network that is informative and a tree, the star network has the most visible nodes of any tree configuration.*

Proof. See Appendix. ■

1.5 Word-of-mouth centrality

It is useful to rank nodes by their capacity to send or receive signals in the network. We do so by constructing a centrality measure from the probabilities of information transmission, which we call *word-of-mouth centrality*.

Definition 9. Word-of-mouth centrality is the average probability of information transmission by diffusion for each player in a network and is given by

$$w_i(\mathbf{\Omega}) \quad = \quad \frac{1}{n-1}\sum_{j\neq i} w_{ij}(\mathbf{\Omega})$$

We have assumed an unweighted, undirected network so that ingoing and outgoing measures are symmetric, but alternatives are easily computed. Let $W(\mathbf{\Omega}) = \frac{1}{n}\sum_{i\in N} w_i(\mathbf{\Omega})$ be the average word-of-mouth centrality in a network.

There are several related measures of centrality, in particular diffusion centrality and communication centrality (Banerjee et al., 2013, 2014), Bonacich centrality (Bonacich, 1987), information centrality (Stephenson and Zelen, 1989), random walk closeness centrality (Noh and Rieger, 2004), cascade centrality (Teytelboym et al., 2015), and percolation centrality (Moore and Newman, 2000; Piraveenan et al., 2013) in the epidemiological literature. As far as we are aware, no measure uses probabilities of information travelling by diffusion between two nodes.

1.5.1 Comparison with diffusion centrality

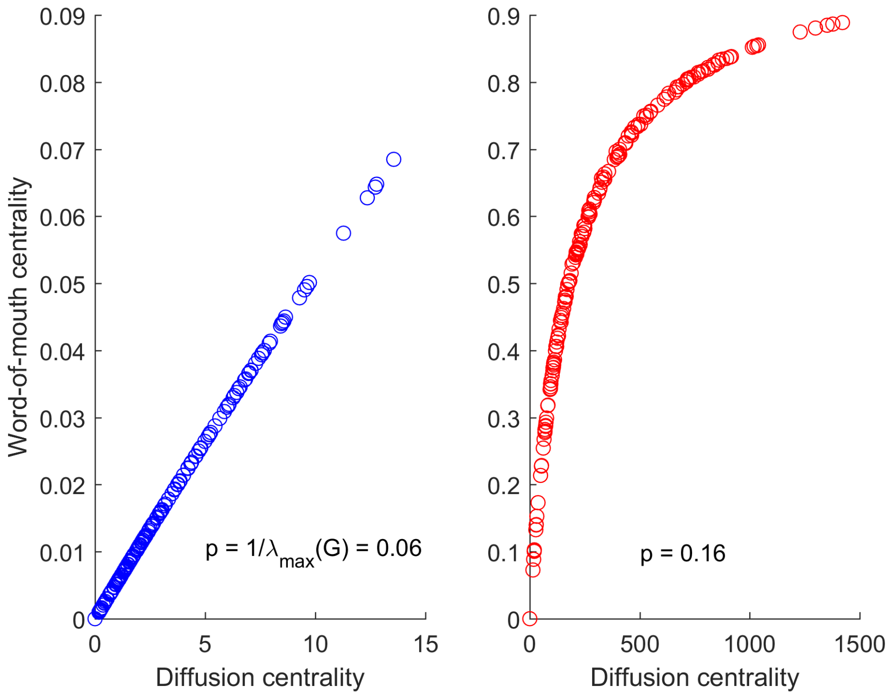
Of particular interest is the relationship between our measure and diffusion centrality. Banerjee et al. (2013) empirically investigate the effects of information in social networks on the decisions of individuals to take up a microfinance opportunity in villages in India. They develop diffusion centrality as an approximation of *communication centrality*, a simulated measure linked to the Susceptible, Infected, Recovered model (Kermack and McKendrick, 1927; Bailey, 1957), which runs for a finite period of time and allows for non-participants to pass on the message. Diffusion centrality has a 0.86 correlation with communication centrality, and is given by $d_i = \left[\sum_{\tau=1}^{T}(p\mathbf{G})^{\tau}\mathbf{1}\right]_i$, and p is equal to the inverse of the largest eigenvalue of the adjacency matrix, $\lambda_{max}(\mathbf{G})$. This is because as T tends to infinity, diffusion centrality becomes proportional to either Bonacich centrality or eigenvector centrality, depending on whether p is smaller or larger than $1/\lambda_{max}(\mathbf{G})$, respectively. Thus they choose this critical value of $1/\lambda_{max}(\mathbf{G})$ for their p, since *"the entries of $p\boldsymbol{G}^T$ tend to 0 as T grows if $p < 1/\lambda_{max}(\boldsymbol{G})$, and some entries diverge if $p > 1/\lambda_{max}(\boldsymbol{G})$"*. But diffusion centrality does not take account of double counting, measuring instead the total amount of information travelling between nodes in a network — rather than the probability that information flows. This means that it overemphasises the benefit of hearing a lot of information, because at some point extra information is redundant if these signals are likely to have already been received via other walks in the network.

Figure 1.6 compares diffusion centrality and word-of-mouth centrality, using data from one of the Indian villages studied by Banerjee et al. (2013). Centralities are calculated at the household level.[9] The value of $p = 1/\lambda_{max}(\mathbf{G})$ is used to compare the two centralities in the left chart, and there is clearly a strong relationship between the two measures. In fact, at this value of p, a linear transformation of diffusion centrality would be a good approximation of the node-to-node probabilities, and this approach is used by Breza and Chandrasekhar (2015). The chart suggests a transformation factor of 0.06.

The chart on the right shows the comparison between the two measures when

[9]Village number 1; $n = 182$. For this comparison we use $w_i = \frac{1}{n}\sum_{j \in N} w_{ij}$ (i.e. not excluding w_{ii} compared to the formula in Definition 9). This makes it more comparable with diffusion centrality, which includes the diagonal entries of the matrix in its sum.

Figure 1.6: Comparing centrality measures in a village



we use a larger p, and while ranking seems to be generally preserved, there is some divergence in the relative magnitude of the measures, as the word-of-mouth centralities converge towards 1. So if there is any reason to suspect that p differs from $1/\lambda_{max}(\mathbf{G})$, word-of-mouth centrality may be useful for calculating probabilities of information transmission by diffusion in a network. In particular, we can observe that the level of inequality in diffusion centrality between the nodes in a network is higher than when word-of-mouth centrality is used. This is because central nodes who receive a lot of information have extremely high values of diffusion centrality. But with word-of-mouth centrality, the effect of this extra information is discounted due to the fact that it is probably redundant. Central nodes have most likely received the signal via other walks already. This analysis suggests that diffusion centrality overemphasises the benefit of a central network position in relation to information transmission.

1.5.2 Obstruction and centrality

We can also calculate centrality measures that take account of obstruction.

Definition 10. Obstructed centrality $P_i(\mathbf{\Omega})$ is the average probability a signal emitted by a player will be received by others, averaged over the possible obstruction of that signal by any of the other nodes in the network.

$$P_i(\mathbf{\Omega}) \quad = \quad \frac{1}{(n-1)^2} \sum_{k \neq i} \sum_{j \neq i} p_{ij}(k, \mathbf{\Omega})$$

A player's obstructed centrality could be significantly lower than their word-of-mouth centrality if the network architecture means it is easy for other nodes to obstruct their signals.[10] Next we examine the effect that a node's obstruction can have on the communication of other nodes, in a similar vein to betweenness centrality (Freeman, 1977).

Definition 11. A node's **obstructing centrality** $O_k(\mathbf{\Omega})$ is the average probability other players can communicate if that node obstructs the signals.

[10]Note that, like diffusion and word-of-mouth centrality, this measure varies with the primitives of information transmission p and T, because these parameters weight the importance of walks of different lengths for information transmission in the network. To abstract from this, we could examine a measure which focuses only on the structure of the network links in $\mathbf{G}$ by using $p = 1$ and $T = D_{\mathbf{G}}$, the diameter of the network. But this would be less interesting because it would not weigh the paths by their length nor whether there is double counting of paths — it would just be a measure of the size of the connected components in the network when each node is removed.

Node	B_i	d_i	w_i	P_i	O_k	b_i
1,8	1.6663	0.6485	0.0735	0.0573	0.1217	0
2,7	2.2211	1.1871	0.1281	0.1024	0.0991	12
3,6	2.4040	1.3589	0.1486	0.1168	0.0896	20
4,5	2.4588	1.4073	0.1550	0.1206	0.0866	24
Average	2.1876	1.1505	0.1443	0.0993	0.0993	14.0

(a) Line network

Node	B_i	d_i	w_i	P_i	O_k	b_i
1	8.3784	5.5334	0.4870	0.3907	0	42
2-8	3.5135	1.8850	0.2157	0.1512	0.2070	0
Average	4.1216	2.3411	0.2853	0.1811	0.1811	5.25

(b) Star network

Table 1.1: Comparison of centrality measures for two networks of eight nodes ($p = 0.3$, $T = 6$)

$$O_k(\mathbf{\Omega}) = \frac{1}{(n-1)^2} \sum_{i \neq k} \sum_{j \neq i} p_{ij}(k, \mathbf{\Omega})$$

Table 1.1 shows the different centrality measures for the line and the star example networks in Figure 1.2. For each node we compare Bonacich centrality B_i, diffusion centrality d_i, betweenness centrality b_i, word-of-mouth centrality w_i, obstructed centrality P_i and obstructing centrality O_k. On almost all measures, centrality increases towards the centre of the line, and the periphery nodes in the star are always less central than the centre. Meanwhile obstructing centrality is higher at the end of the line and the periphery of the star, showing that if those players obstruct signals, the probability of information transmission between the other players is closer to w_i. Obstructing centrality for the centre of the star is zero, because if he obstructs signals, no-one else in the network can communicate.

1.6 Applications

These probabilities of information transmission could be used in many types of games on networks. They are particularly relevant for games of imperfect private monitoring, where nodes do not necessarily receive the same signals about past play (Kandori, 2002; Sekiguchi, 1997; Bhaskar and Obara, 2002; Chen, 2010). In Chapter 2 we study the case of cooperation in networks where the information on past transgressions flows through the network by word-of-mouth. We find two different aspects of trust that each relate to obstructed and obstructing centrality measures.

As described above, diffusion centrality was developed by Banerjee et al. (2013) as an approximation of communication centrality, which shows how overall participation in a microfinance scheme depends on the centrality of the injection points of information about it. As we have shown, the problem of double counting with diffusion centrality is exacerbated if the parameter p varies significantly from the inverse of the largest eigenvalue of the adjacency matrix. It would be interesting to see how well word-of-mouth centrality, which takes account of double counting, would fare as an approximation of communication centrality.

As well as Banerjee et al. (2013), two other recent empirical papers have used diffusion centrality as an approximation of the probability of information transmission. Breza and Chandrasekhar (2015) use it to investigate how monitoring by different members of a community in villages in India incentivises individuals to save for the future. They find that a savings monitor with higher diffusion centrality in the village network significantly increases savings. Fafchamps and Labonne (2016) use family network data from the Philippines to test whether households' centrality affects whether they receive services from the municipal government. They find that betweenness centrality is more important than their measure of information diffusion, highlighting the importance of coalition-building in politics. In both these cases, double counting the transmission of signals may be an issue, and so it would be interesting to compare these results to those using the word-of-mouth probabilities. In addition, the role of obstruction could be important. For example, obstructing centrality might be relevant for the analysis in Fafchamps and Labonne (2016), since it depends on the number of walks between other nodes that a node occupies (similar to betweenness centrality), but also measures the importance of those walks for communication between other nodes. Meanwhile in Breza and Chandrasekhar (2015)'s context, obstruction might be important because people who had not fulfilled their commitments to increased saving may not wish to pass on information about this failure.

1.7 Conclusion

We have shown a new way to calculate the probabilities of node-to-node information transmission in networks. This simple measure can be used to allow Bayesian updating in network games where signals, rather than posterior beliefs or actions, are transmitted through the network. We also allowed for the case where nodes may conceal information travelling through the network. This approach gives us several new centrality measures, which tell us the average probabilities of information flowing through the network in different cases and may be useful in applications.

Appendix

1.A Proofs

1.A.1 Proof of Proposition 1

Obstructed word-of-mouth probabilities and the information structure $\mathbf{\Omega}$.

1. **Increasing** p: We have that $\frac{\partial w_{ij}(p,T,\mathbf{G})}{\partial p} \geq 0$ with a strict inequality if and only if $l_{ij}(\tau, \mathbf{G}) > 0$ for any $\tau \leq T$, that is, if and only if i and j are connected by one or more walks of length $\leq T$ in the network $\mathbf{G}$.

2. **Increasing** T **to** $T+1$: Let

$$F_{ij}(p, T, \mathbf{G}) = 1 - w_{ij}(\mathbf{\Omega}) = \prod_{\tau=1}^{T} [1 - p^{\tau}]^{l_{ij}(\tau, \mathbf{G})} \tag{1.A.1}$$

If T increases to $T+1$, we have that

$$F_{ij}(p, T+1, \mathbf{G}) = F_{ij}(p, T, \mathbf{G})(1 - p^{(T+1)})^{l_{ij}(T+1, \mathbf{G})} \tag{1.A.2}$$

So $F_{ij}(p, T+1, \mathbf{G}) < F_{ij}(p, T, \mathbf{G})$ if and only if $l_{ij}(T+1, \mathbf{G}) \geq 1$; otherwise they are equal. The same argument holds for any further increases in T.

Hence $F_{ij}(\mathbf{\Omega})$ is weakly monotonically decreasing in T and so $w_{ij}(\mathbf{\Omega})$ is weakly monotonically increasing in T.

3. **Adding a link to the information network**: Suppose that we add a link to the network $\mathbf{G}$, creating the network $\mathbf{G}'$, which has an additional walk of length τ between i and j so that $l_{ij}(\tau, \mathbf{G}') = l_{ij}(\tau, \mathbf{G}) + 1$. Now we want to compare $F_{ij}(p, T, \mathbf{G}')$ and $F_{ij}(p, T, \mathbf{G})$. Since

$$F_{ij}(p, T, \mathbf{G}') = F_{ij}(p, T, \mathbf{G})(1 - p^{\tau}) \tag{1.A.3}$$

We have that $F_{ij}(p, T, \mathbf{G}') < F_{ij}(p, T, \mathbf{G})$ as required. Meanwhile any change to the network that leaves $l_{ij}(\tau)$ unchanged $\forall \tau \leq T$ leaves the probabilities unchanged.

1.A.2 Proof of Corollary 1

From (1.A.2) we have that $F_{ij}(\mathbf{\Omega})$ is decreasing in T since $(1-p^{(T+1)})^{l_{ij}(T+1,\mathbf{G}_{-k})} < 1$. Observe also that $F_{ij}(\mathbf{\Omega})$ is bounded from below by 0. This means that $F_{ij}(\mathbf{\Omega})$ converges to a lower bound. This lower bound is strictly positive, $\lim_{T\to\infty} F_{ij}(\mathbf{\Omega}) = F*_{ij}(\mathbf{\Omega}) > 0$, because $p > 0$ and at the limit, the factor by which $F_{ij}(\mathbf{\Omega})$ decreases is $\lim_{T\to\infty}(1 - p^{(T+1)})^{l_{ij}(T+1,\mathbf{G}_{-k})} = 1$. As $w_{ij}(\mathbf{\Omega})$ is the complement of $F_{ij}(\mathbf{\Omega})$, this implies that $w_{ij}(\mathbf{\Omega})$ converges to an upper bound $P*_{ij} < 1$.

1.A.3 Proof of Remark 1.3.1

For any r, the total number of walks of even length is $(ab)^r$ (since each walk is counted twice — incoming and outgoing — in (1.3.10)). The total number of walks of even length is $\frac{1}{2}(ab)^r(a+b)$, and so the total number of walks of any length for each r is $\frac{1}{2}(ab)^r(2+a+b)$. If group sizes are equal then $a = b = \frac{n}{2}$ (assuming n is even), while with the star we have that $a = 1$ and $b = n - 1$. Comparing the number of walks in these two cases, we find that the number of walks is strictly higher with two equal groups if $n^2 - 4n + 4 > 0$, which is the case for $n > 2$.

1.A.4 Proof of Proposition 2

By Remark 1.4.1, the network is informative if and only if $\mathbf{G}$ is connected and $D_{\mathbf{G}} \leq T$, and player k is visible if and only if $\mathbf{G}_{-k}$ is connected and $D_{\mathbf{G}_{-k}} \leq T$. Since removing a node from the network cannot connect a disconnected network, and can only increase social distances so that $d_{ij}(\mathbf{G}_{-k}) \geq d_{ij}(\mathbf{G})\ \ \forall\, i, j, k \in N$, if $\mathbf{G}_{-k}$ is connected and $D_{\mathbf{G}_{-k}} \leq T$, this implies that $\mathbf{G}$ is connected and $D_{\mathbf{G}} \leq T$: i.e. a player's visibility implies the original network $\mathbf{G}$ is informative. When the network is informative, $w_{ij}(\mathbf{G}) > 0\ \forall\, i, j \in N$, and so k is visible if and only if he is not obstructive.

1.A.5 Proof of Remark 1.4.3

There are $\frac{n!}{2(n-2)!}$ pairs in a network of n nodes, and if L is the number of visible nodes, there are $\frac{L!}{2(L-2)!}$ pairs where both partners are visible. In an informative tree network, all nodes except the leaf nodes are obstructive, because leaf nodes are the only ones who would not disconnect the network by their absence. Nodes are defined as visible if they are not obstructive in an informative network, so only the leaf nodes in a tree network are visible. All nodes except the centre of the star are leaf nodes, and so the star has the maximum number of visible nodes for any tree: $L = n - 1$.

1.B Obstruction by subsets of nodes in different rounds of information transmission

Let X_τ be the subset of nodes who obstruct a signal in round τ of information transmission. Let $\mathbf{X} = \{X_\tau \subset N,\ 1 \leq \tau \leq T\}$ be the set of those subsets. Let $l_{ij}(\tau, \mathbf{G}, \mathbf{X})$ be the number of walks between i and j of length τ when the set of obstructing nodes is $\mathbf{X}$.

To calculate this, recall that nodes only remember the information they receive in the last round. So longer walks will not connect to other nodes, if the links it would traverse are those which connect a node who is obstructing in the relevant round of information transmission. So we have that, for example, $l_{ij}(1, \mathbf{G}, \mathbf{X}) = [\mathbf{G}_{-X_1}]_{ij}$ where $\mathbf{G}_{-X_1}$ is the network $\mathbf{G}$ with those nodes in X_1 removed. Then we have that $l_{ij}(2, \mathbf{G}, \mathbf{X}) = [\mathbf{G}_{-X_2}\mathbf{G}_{-X_1}]_{ij}$ and $l_{ij}(3, \mathbf{G}, \mathbf{X}) = [\mathbf{G}_{-X_3}\mathbf{G}_{-X_2}\mathbf{G}_{-X_1}]_{ij}$, and so on. In general, $l_{ij}(\tau, \mathbf{G}, \mathbf{X}) = [\prod_{\tau=1}^{\tau} \mathbf{G}_{-X_\tau}]_{ij}$, ensuring that the ordering of matrices $\mathbf{G}$ is preserved.

Chapter 2

Cooperation in networks

2.1 Introduction

The extent to which individuals can trust and cooperate with each other in the absence of formal enforcement has important effects on economic outcomes and has long been a topic of scholarly interest.[11] Trust consists of "placing valued outcomes at risk of others' malfeasance"(Tilly, 2004) and empirical research on the topic often begins with the survey question: "generally speaking, would you say that most people can be trusted, or that you can't be too careful in dealing with people?" But where does this trust come from? In economic models, there are two main reasons why one person might trust another, even in the face of temptation. Firstly, because their partner could face *punishment* for behaving badly, shifting their *incentives* away from shirking. In this case, if a player knows the expected punishment facing their partner, they can decide whether to trust them on not depending on whether they think the punishment is strong enough to incentivise good behaviour. Secondly, their partner's type could determine their action: they may be a *good type* who is immune to temptation — or a *bad type* who will always cheat no matter what. In this case, if a player knows his partner's type, then he knows whether to trust him or not.

In this chapter we build a model which examines both drivers of trust - incentives and types - and show how each aspect depends on the structure of a communication network which connects players. These links could depend on many factors: family and kin relationships; friendships or trading relationships; or proximity given by physical geography such as roads, rivers or the streets of

[11] Coleman (1988); Ostrom (1990); Fukuyama (1996); Putnam et al. (1994); Knack and Keefer (1997); Leeson (2005).

a town. We use a two-sided trust model that allows players to cooperate in a prisoner's dilemma game if and only if they both trust each other. Cooperation is supported by community enforcement: the threat of punishment by other members of the group for any deviation that is detected.[12]

This gives us our main result: a *'cooperation network'* showing who within a community can cooperate with whom, which is endogenous to the communication network and the other parameters of the model. This network of cooperation may be quite different from the original communication network, and shows how certain network structures can be more or less supportive of overall levels of cooperation, and hence lead to higher (or lower) payoffs.

We find that a pair of players can cooperate if and only if they can both trust each other not to deviate. In particular, a player is *trusted* to cooperate if his position in the network means that other players are able to communicate about him. When trust depends on incentives, what matters is players' expectations of the likelihood of detection and hence punishment. This in turn is linked to *obstructing centrality* given in Chapter 1: the average probability that other players can communicate about someone, if he tries to obstruct the message.

We also identify a second aspect of trust: a player is *trusting* if his network position means he is likely to detect deviations by others. When trust depends on knowledge of players' types, payoffs are linked to *obstructed centrality*, which gives the average probability that a player will receive messages that have been obstructed by others. Players who have better knowledge of past play because of

[12]Community enforcement can support cooperation in many settings (Greif, 1993), and its reliance on information transmission between players has been highlighted by Kandori (1992):

> "*In small communities where members can observe each other's behaviour [...] the crux of the matter is information transmission among the community members.*"

their central network position can be more trusting.

We find that both aspects of trust increase with greater probabilities of information transmission, expanding the number of players who can cooperate, and leading to (weakly) higher welfare as information flow increases. These two centrality measures, and hence the two aspects of trust, do not necessarily move together, leading to some surprising results in certain networks. For example, we might expect the centre of a star to be very trusted, because everyone can observe him. Not so. In fact we find that for most parameters in a star network, players on the periphery can cooperate with each other, but the centre is excluded from cooperation. This is because if the centre deviates, the periphery players cannot inform each other, because they are completely dependent upon him for their communication (his obstructing centrality is zero, as shown in Chapter 1). Hence they cannot trust him not to deviate. We also find cases where the two aspects of trust are diametrically opposed to each other. In a line network, players in the centre of the line cannot be trusted because those at the two ends of the line cannot communicate with each other about the centre's bad behaviour. On the other hand, these central players are very trusting because their network position means they are highly likely to receive signals about others' deviations. In fact, in the line network, those players who are neither in the centre nor the end have the highest capacity to cooperate, echoing the concept of *middle-status conformity* (Phillips and Zuckerman, 2001).[13]

[13] I am grateful to Birger Wernerfelt for providing this reference.

2.1.1 Related literature

The important effect of networks in shaping and influencing economic outcomes has long been emphasised, most recently by Jackson (2008) and Goyal (2007). In particular, there is now a growing literature examining repeated games where interactions and/or monitoring are influenced by a network connecting players, surveyed comprehensively by Nava (2016).[14]

Dixit (2003b) looks at community enforcement when community members are distant from each other. He allows distance to affect three things: the probability players meet, the payoffs if they do, and also the probability they can exchange information. In his model, a continuum of players are arranged around a circle, and cooperation between pairs of players, who are matched in the first period of a two-period game, can be supported by punishment in the final period. A player's incentive to deviate depends on the probability that a true signal emitted by the victim of his deviation will be received by other players: his potential future partners. Dixit finds the 'size of the trading world', an arc of his circle which shows the greatest distance possible between cooperating players, and beyond which they shirk – a similar concept to Fukuyama's (2001) 'radius of trust'. Dixit finds that honesty prevails in a small enough world, and self-enforcing honesty decreases as size increases. He also compares community enforcement to global enforcement with different-sized worlds.

In this chapter, we modify Dixit's continuous model of community

[14]Nava and Piccione (2014) examine the case of local public goods, where a player takes the same action with respect to each of his neighbours, while Wolitzky (2013) finds a new centrality measure that can influence a player's robust maximum contribution to global public goods. Karlan et al. (2009), Breza and Chandrasekhar (2015) and Annen (2003) investigate the role of network links in supporting commitment in different settings.

enforcement to apply in a network setting with discrete players. To do so to specify the process by which information is transmitted within the network, using the probabilities of node-to-node information transmission we developed in Chapter 1. These *word-of-mouth probabilities* are based on *diffusion* (Banerjee et al., 2013, 2014), where information flows through a limited number of links, each link with a decay factor that represents the probability that two neighbouring players in the information network pass a signal between them. The signal that flows through our network is a player's 'bad reputation', which Kandori (1992) shows is adequate to ensure that a deviator will be punished by other members of the community. To make our diffusion model more tractable, and following Dixit (2003b), we make arguably rather strong assumptions about truth-telling, abstracting from interesting issues around the possibilities of cheap talk and fabricating rumours, which are studied in detail elsewhere (Ahn and Suominen, 2001; Bloch et al., 2014; Annen, 2011).

There are many possible ways information can flow in networks (Borgatti, 2005) and many sophisticated information structures have been proposed in this field, and we believe ours is the first model to use probabilistic information flow. Balmaceda and Escobar (2013) and Raub and Weesie (1990) model information as flowing along one link in the network. Renault and Tomala (1998) and Wolitzky (2014) let information flow through all links in a connected component, finding that the potential for cooperation depends on whether the network is 2-connected. Lippert and Spagnolo (2011) allow information to travel through network links with a delay and highlight the importance of *gatekeeping* for cooperation, while in an alternative model with delay Kinateder (2008) finds the *diameter* of the network plays an important role. Bloch, Genicot and Ray (2008), Laclau (2014) and Larson (2014, 2017) allow messages to be passed to a subset of players, while in a different setting, Gallo (2014) models information flow in a network as a random walk process.

The communication network in our model means that different players may have different beliefs about past play, depending on the signals they receive from each other through the network, and so our repeated game falls within the class of games of imperfect private monitoring (Kandori, 2002; Sekiguchi, 1997; Bhaskar and Obara, 2002; Chen, 2010). Like Dixit (2003b), our solution concept is perfect Bayesian equilibrium because our players use Bayesian updating when they receive signals through the network. As described in Chapter 1, our probabilistic information flow of signals (not beliefs) allows for Bayesian updating in networks, in contrast to behavioural approaches often used in networks (Degroot, 1974; Golub and Jackson, 2012).

In common with much of the literature on cooperation, we model pairwise interactions between players where the stage game is the prisoners' dilemma, as do Lippert and Spagnolo (2011), Ali and Miller (2013), Bloch, Genicot and Ray (2008) and Laclau (2012). In those papers, a network of relationships determines *both* the interaction possibilities and the information flows between players. In contrast, we allow interactions and monitoring relationships to be unrelated to each other - players can play the stage game with partners with whom they do not exchange information and vice versa, as is the case for Fainmesser (2012) and Fainmesser and Goldberg (2012), although, different to them, our networks are common knowledge. This allows us to highlight the importance of two distinct aspects of enforcement: *monitoring* via the information network; and *sanctioning* via the matching probabilities (Ostrom, 1990; Sobel, 2002). Like Kandori (1992) and Ellison (1994), each pair of players in our model has a given probability of matching, and this is independent across periods.

There are two key cooperation-supporting punishment strategies seen most frequently in the literature: contagion, used by Kandori (1992), Ali and Miller (2013) and Jackson, Rodriguez-Barraquer and Tan (2012); and grim trigger or ostracism (sometimes with forgiveness) used by Ahn and Suominen (2001), Raub and Weesie (1990) and Ali and Miller (2016). In common with Dixit (2003b), we apply a different approach: an incomplete information game where players behave cooperatively in order to avoid being mistaken for a *bad type* whom future partners would ostracise. This means that the punishment is renegotiation-proof (Farrell and Maskin, 1989; Benoit and Krishna, 1993; Jackson et al., 2012), and in our setting entails a finitely repeated game (Benoit and Krishna, 1985), in contrast to much of the literature where infinite repetition is used. The bad type also means that this is a game of reputation (Samuelson and Mailath, 2006) and also allows us to pin down expectations off the equilibrium path.

The outline of the chapter is as follows. Section 2.2 describes the network connecting players and the repeated game, and Section 2.3 describes the equilibrium of interest. Sections 2.4 and 2.5 show how levels of trust, cooperation and payoffs depend on the structure of the network. Section 2.6 concludes the chapter.

2.2 The network and the cooperation game

In this section, we begin the outline of our model by showing how information flows in the network, and how matching probabilities and payoffs can also depend on the network. The N players occupy the nodes of a fixed undirected unweighted network $\mathbf{g}$ such that $\{i, j\} \in \mathbf{g}$ if i and j are neighbours. A walk of length a between two nodes i and j in network $\mathbf{g}$ is a sequence of nodes $(i = x_0, x_1, ..., x_{a-1}, x_a = j)$ such that for every $r \in \{1, 2, ..., a\}$, we have that $\{x_{r-1}, x_r\} \in \mathbf{g}$. If the nodes are distinct, the sequence is a path, and if in addition $i = j$, it is a cycle. Let $\mathbf{G} = [g_{ij}]$ be the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} = 1$ indicates that players i and j are neighbours so $\{i, j\} \in \mathbf{g}$, and $g_{ij} = 0$ otherwise (and $g_{ii} = 0 \;\; \forall i \in N$ by convention). The network $\mathbf{G}$ is common knowledge; all players know each other's network positions. Let $\mathcal{N}_i = \{j : g_{ij} = 1\}$ be the set of player i's neighbours and $|\mathcal{N}_i|$ be i's degree.

As usual, $d_{ij}(\mathbf{G})$ is the length of the shortest path between two players i and j in the network $\mathbf{G}$, which is known as the *social distance*. Let $D_{\mathbf{G}} = \max\{d_{ij}(\mathbf{G})\}$ be the diameter of the network $\mathbf{G}$: the length of the longest shortest path. Two players are *connected* if there exists a path of finite length between them, and a network is connected if all players are connected to each other. If $\mathbf{G}$ is not connected, its diameter is infinite. Let $\mathbf{G}_{-k}$ be the adjacency matrix of the network with player k removed - that is, the $n \times n$ adjacency matrix created when all the entries in the kth row and column of $\mathbf{G}$ are set to zero. If $\mathbf{G}_{-k}$ is connected then the network is *2-connected with respect to* k; the network is *2-connected* if it is 2-connected with respect to all players.

2.2.1 Information transmission

In this chapter we make use of the obstructed probabilities of information diffusion that were developed in Chapter 1, based on the process of obstructed diffusion given in Definition 5. With obstructed diffusion, a signal flows through each link in the network with a fixed probability p, up to a maximum number of links T. The parameter p denotes how likely players are to meet and/or exchange information with their neighbours. For example, if $p = 1$ and $T = 1$, information is passed with certainty only to a player's direct neighbours. When information is transmitted by obstructed diffusion, players can choose whether or not to pass on signals they receive, but cannot fabricate them. Passing information on or concealing it are both costless.

2.2.2 Matching probabilities

The repeated game of cooperation has three periods: $t \in \{1, 2, 3\}$. In periods 1 and 3, players are matched in pairs and the stage game is played. In period 2, information is transmitted in the network. There are n players in $N = \{1, ..., n\}$ where $n > 2$ and even. Let μ_i^t denote player i's partner in periods $t \in \{1, 3\}$, and μ^t list the partnerships in each period, with $\mu = (\mu^1, \mu^3)$. *Matching probabilities*, m_{ij} $\forall i, j \in N$, determine the likelihood that any two players are matched as partners, and these are independent across periods: [15]

$$\Pr\{\mu_i^t = j\} = m_{ij} = m_{ji} = \Pr\{\mu_j^t = i\} \quad \forall t \in \{1, 3\} \tag{2.2.1}$$

[15] This is for simplicity; matching probabilities do not need to be constant across both periods for our results to go through.

Let $m_{ij} > 0 \ \forall\ j \neq i$, so that there is a positive possibility of meeting any player. Also let $m_{ii} \geq 0$, which, if positive, signifies that a player is 'sitting out' of the stage game without a partner; that is, we do not require *perfect matching* where everyone has a partner. Players know the identity of their own partner, but not who anyone else has matched with, as in Kandori (1992). The matching probabilities are collected in the symmetric doubly stochastic matrix $\mathbf{M} = [m_{ij}]$.

Our baseline model allows for any matching probabilities, but it may often be the case that *local matching* would apply, where matching probabilities depend on the network. In particular, local matching could mean that probabilities decrease with social distance, and that there is a parameter, T_m, which gives the maximum social distance over which players have a non-negligible probability of matching. We consider this case in an example network in Section 2.4 and Appendix 2.D.

2.2.3 The stage game

The stage game is the prisoners' dilemma with exit (Benoit and Krishna, 1985), augmented by an additional 'dangerous' action B_i, which is very damaging for a player's partner: e.g. 'steal everything'. The action space for each player $i \in N$ is $A_i = \{C_i, D_i, O_i, B_i\}$ where C_i is cooperation and D_i is defection, O_i is exit and B_i is the dangerous action. Table 2.1 shows how payoffs depend on actions.

2.2.3.1 Player types

There are two types of player in the set $\Xi = \{\text{S-type, B-type}\}$. Most of the players are strategic or 'S-type'; but there are a few bad apples — 'B-types' — who lurk in the population. This is a game of incomplete information as

	$C_{\mu_i^t}$	$D_{\mu_i^t}$	$O_{\mu_i^t}$	$B_{\mu_i^t}$
C_i	1	$-\beta$	0	$-\beta$
D_i	α	σ	0	$-\beta$
O_i	0	0	0	0
B_i	x_i	x_i	$x_i/2$	x_i

Table 2.1: The stage game between player i and his partner μ_i^t. Payoffs are symmetric, and the entries denote the payoffs of the row player.

players do not observe each other's types — as well as imperfect monitoring as described earlier. The bad type is included in the game in order to make punishment renegotiation-proof (Samuelson and Mailath, 2006), and to pin down expectations off the equilibrium path. These bad or 'inept' types are sometimes called *commitment types* because they are committed to a certain action. In our case, we use a simple specification for the payoffs of the bad type.

We assume that $x_i = -x$ for S-types and $x_i = x$ for B-types, and that $x > \beta > \alpha > 1 > \sigma > 0$ and $2 > \alpha - \beta$. This implies that the dangerous action B_i is strictly dominant for a B-type and strictly dominated for the S-type. In turn, these assumptions mean that the stage game between two S-types is the usual prisoners' dilemma with exit, which has two Nash equilibria in pure strategies: mutual defection $(D_i, D_{\mu_i^t})$ or mutual exit $(O_i, O_{\mu_i^t})$. For a game between an S-type player i and a B-type, the only Nash equilibrium is $(O_i, B_{\mu_i^t})$.

We make the following assumption about ϕ, an S-type player's prior belief that another player is a B-type.

Assumption 1 $\sigma(1-\phi) - \beta\phi > 0$

This assumption ensures that an S-type player will not exit against an unknown player he expects to defect, in either period when the stage game is played.

2.2.3.2 Payoffs and the network

We also assume that the payoffs in Table 2.1 are multiplied by the factor $\gamma_{i\mu_i^t} > 0$ to give the overall payoffs for player i facing $\mu_i^t \neq i$, his match in period t.[16] These factors signify the possibility of higher or lower payoffs when facing different partners, and are collected in the matrix $\mathbf{\Gamma} = [\gamma_{ij}]$. As Dixit (2003b) describes, different levels of payoffs with different partners could reflect, for example, complementarities in production with players who have different skills or resources. These complementarities could be greater with players who are at greater social distance, and we examine this case in Section 2.4.

A player's payoff in each period depends only on his own action, that of his partner, and on the identity of his match. Let $a_i^t \in A_i$ and $U_i(a_i^t, a_{\mu_i^t}^t, \gamma_{i\mu_i^t})$ respectively be i's action and payoff in the stage game in period $t \in \{1, 3\}$, and let a^t list the actions in period t.

2.2.4 Reputation and community enforcement

A player k gets a bad reputation (Kandori, 1992), $r^k = 1$, if and only if their partner in period 1 received a negative payoff.

$$r^k = 1 \iff U^1_{\mu_k^1} < 0 \tag{2.2.2}$$

Otherwise, they have $r^k = 0$. So a B-type player will always get a bad reputation (as long as he did not 'sit out' with no partner), as will an S-type player who

[16] If $\mu_i^t = i$, a player 'sits out' of the stage game with no partner, payoffs are zero and $\gamma_{ii} = 0$.

defected against a cooperating partner.

$$\Pr[r^k = 1 \mid \{k \text{ is B-type}\},\quad \mu_k^1 \neq k] \quad = \quad 1 \qquad (2.2.3)$$

$$\Pr[r^k = 1 \mid \{k \text{ is S-type}\},\quad \mu_k^1 \neq k,\quad D_k, C_{\mu_k^1} \in a^1] \quad = \quad 1 \qquad (2.2.4)$$

In contrast to some reputation models, in our case a player's reputation is not publicly available — only some players hear about it. Let $s_i(k) \in \{\{1\}, \emptyset\}$ be the signal that player i sends about player k, where $s_i(k) = 1$ signifies k's bad reputation.[17] We assume that a player emits a signal about another player if and only if that player has a bad reputation and was his period 1 partner. That is, only the true victim of a deviation will emit a signal. This assumption requires a certain degree of truth-telling because we assume that players do not initiate false bad reputations about other players.

Assumption 2 $s_i(k) = 1 \iff r^k = 1 \text{ and } \mu_k^1 = i \quad \forall i, k \in N$

After the signal is emitted, it may be received by other players: let $\rho_j(k) \in \{\{1\}, \emptyset\}$ be the signal that j receives about k. The probability $p_{ij}(k)$ that a signal emitted by i about k will reach j is as follows.

$$\Pr\left[\rho_j(k) = 1 \mid s_i(k)\right] = \begin{cases} p_{ij}(k) & \text{if } s_i(k) = 1 \\ 0 & \text{if } s_i(k) = \emptyset \end{cases} \qquad \forall i, j, k \in N \qquad (2.2.5)$$

Due to Assumption 2, note that $p_{ij}(k) = 0$ if $k \in \{i, j\}$, because a player cannot

[17] Samuelson and Mailath (2006) show how this separating equilibrium with a bad type is one alternative for reputational games; the other is a pooling equilibrium with a 'good type'. See Appendix 2.A for this alternative in our model. Spence (1973) and Breza and Chandrasekhar (2015) use a specification of the reputation model where both good and bad signals are emitted.

emit a signal about himself — only his victim can.

2.2.5 The repeated game

Players are risk-neutral von Neumann-Morgenstern expected utility maximisers, with expected utility function $v_i(\cdot)$. Given players' common discount factor $\delta \in]0,1]$, payoffs in the repeated game with strategy profile b and realised matches μ are given by

$$u_i(b, \mu) = U_i(a_i^1, a_{\mu_i^1}^1, \gamma_{i\mu_i^1}) + \delta U_i(a_i^3, a_{\mu_i^3}^3, \gamma_{i\mu_i^3}) \tag{2.2.6}$$

Let $b_i = (a_i^1, a_i^3)$ be player i's pure strategy in the repeated game where $b_i \in B_i = \{\{a_i^1, a_i^3\} \mid a_i^1 : \mu_i^1 \to A_i, \quad a_i^3 : \{\mu_i^3, h_i^3\} \to A_i\}$. Let the pure strategy space be $B = \prod_{i \in N} B_i$ and $b = (b_i)_{i \in N}$ be a pure strategy profile of the repeated game.

Let $\rho_j = (\rho_j(k))_{k \in N}$ be player j's 'network signal'. Player i's history (information set) is empty in period 1, and is given by $h_i^3 = (\mu_i^1, r^i, r^{\mu_i^1}, \rho_i)$ at the beginning of period 3. That is, he knows his own and his period 1 partner's reputations, and could also receive a network signal about any other player. But he has not observed his period 1 match's type, only their reputation. The repeated game is defined as the tuple:

$$F \equiv (N, (B_i)_{i \in N}, (u_i)_{i \in N}, \Xi, \phi, \mathbf{M}, \mathbf{\Gamma}, [p_{ij}(k)]_{i,j,k \in N})$$

2.2.6 Timing of the game

In summary, the order of the game is as follows:

Period 1

- Players are randomly matched for period 1: μ^1 is chosen
- Players choose actions a^1 and receive payoffs
- Players' reputations are updated, given their partner's payoffs. Each player i observes his own and his partner's reputations $r^i, r^{\mu_i^1}$ with certainty
- For any player i with $r^i = 1$, a signal is emitted by his partner, $s_{\mu_i^1}(i) = 1$

Period 2

- Information travels between players according to the probabilities $[p_{ij}(k)]_{i,j,k \in N}$

period 3

- Players observe a network signal ρ_i
- Players are randomly matched for period 3: μ^3 is chosen
- Players choose actions a^3 and receive payoffs

2.3 Equilibrium

We would like to construct an equilibrium with cooperation; in particular, where cooperation in period 1 can be supported by community enforcement in period 3.

We construct this equilibrium as follows.[18] In period 1, players either cooperate against a cooperating partner or defect against a defecting partner. In period 3, almost all players defect. Those who do not defect choose exit, which happens if and only if they know that their period 3 partner deviated from the equilibrium strategy in period 1. So if a player deviates by defecting against a cooperating partner in period 1, he knows that any player he is matched with in period 3 — if they find out about his deviation — will choose exit against him, not defection. According to Table 2.1, mutual defection gives a strictly positive payoff and exit gives a zero payoff to both players, so a player could lose out on positive period 3 payoffs if he deviates in period 1. This expected loss — this punishment — can sustain cooperation.

At this equilibrium, actions may not be symmetric — in period 1 some players may be cooperating while others may not — but all players use the same decision rule for their action choice, a rule that is based on the expected probability of punishment. We are particularly interested in how many players cooperate at this equilibrium, and which ones they are with respect to their network position.

Proposition 3. *A perfect Bayesian equilibrium in the repeated game F for all S-type players $i \in N$ is given by the following equilibrium strategy:*

Period 1 *Player i cooperates with his partner j if and only if his expected losses from deviation are above a threshold value, that is $L_i^j \geq L_{ij}^*$, and also that an equivalent threshold value is met for his partner so that $L_j^i \geq L_{ji}^*$. Otherwise,*

[18] We expect there to be many equilibria of this game; we do not attempt to characterise them here. Like Dixit (2003b), we focus only on our equilibrium of interest. For example, there is also an equilibrium where no information is passed, since players are indifferent to passing information in the final round of information transmission, and so an equilibrium strategy including information transmission is only weakly preferred.

he defects.

Period 2 *Player i passes on signals about other players $k \neq i$, but does not pass on signals about himself.*

Period 3 *Player i exits against his partner k if and only if he believes that he is a B-type: either having heard a signal about him or having matched with him in the previous period. Otherwise, he defects.*

Proof. See Propositions 4,5 and 6, and Remark 2.3.1. ■

If there were no B-types, this equilibrium would be subgame perfect Nash equilibrium (Benoit and Krishna, 1985). The existence of the B-types implies that the punishment is renegotiation-proof (Benoit and Krishna, 1993) and allows us to pin down expectations off of the equilibrium path.[19] The equilibrium concept is perfect Bayesian equilibrium, because players update their beliefs about their partner's type according to the signal(s) they receive and Bayes' rule.

To construct this equilibrium, we proceed by backward induction and examine only the payoffs and decisions of the S-type players. To simplify the notation in this section, let player i's partner in period 1 be player $j = \mu_i^1$, and his partner in period 3 be player $k = \mu_i^3$.

[19]This is because in equilibrium, B-types will deviate and any players who hear about it will update their beliefs on those players' types. Without the B-types, players would not expect anyone to deviate. In this case, if they do hear about a deviation, their beliefs are not clearly specified. The B-types also mean that punishment is renegotiation-proof because the preferred action against a B-type player is always exit. If players face instead a known S-type who they are supposed to punish with an exit action, it would be in both players' interests to forgo punishment and mutually defect.

2.3.1 Period 3

In period 3 each player knows his own reputation and that of his period 1 partner, r^i and r^j; and his network signal ρ_i, but he does not know who anyone else matched with or what transpired in those matches. There are two general possibilities for player i's history in period 3. If he met another S-type in period 1, he has history $(r^i = 0, r^j = 0)$ since neither of them deviated. Alternatively i met a B-type in period 1 and has history $(r^i = 0, r^j = 1)$, since he received a negative payoff at the hands of his partner.

In the conjectured equilibrium, if a player he hears a signal about another player, he believes him to be a B-type with probability 1. This means that we can combine (2.2.4) and Assumption 2 to give

$$\Pr[\{k \text{ is B-type}\} \mid \rho_i(k) = 1] = 1 \tag{2.3.1}$$

Remark 2.3.1. *The S-type player's equilibrium strategy for period 3 is that he exits if and only if he believes for sure that his partner is a B-type: either having heard a signal about him or having matched with him in the previous period. Otherwise, he defects.*

Proof. Clearly, from Table 2.1, if i believes k is a B-type, his only rational action is to choose exit. If i has not heard a signal, the probability[20] that his partner is a B-type is ϕ. By Assumption 1, this possibility of meeting a B-type is low enough that expected payoffs from choosing defection are positive. ■

[20]In fact, player i's updated subjective expected probability of each partner k being a B-type could be lower than ϕ, and depends on the network probabilities of information transmission. See Appendix 2.C.

Given this equilibrium strategy, we can now identify the payoffs in period 3.

Proposition 4. *Let $V_i^j(k)$ be the period 3 expected payoffs in the repeated game F for an S-type player i who met player j, in period 1, did not deviate, and then meets player $k \neq j, i$ in period 3. Period 3 payoffs are*

$$V_i^j(k) \quad = \quad \gamma_{ik} \left[\sigma(1-\phi) - \beta\phi(1 - Q_i^j(k)) \right] \tag{2.3.2}$$

Where $Q_i^j(k)$ is the conditional probability that an S-type i, having met another S-type j in period 1, hears a signal about player k, if k is a B-type. If i meets the same player $j \neq i$ in both periods 1 and 3, payoffs are $V_i^j(j) = \sigma$ if j was an S-type and zero if j was a B-type (since i knows this type of his period 2 partner). Finally, $V_i^j(i) = 0$ when i sits out.

Proof. for $V_i^j(k)$ with $k \notin \{i, j\}$, there are two possibilities: either k is a B-type, or he is an S-type. With probability $1 - \phi$, k is an S-type, and payoffs are $\gamma_{ik}\sigma$ as the equilibrium strategy requires both players to defect. With probability ϕ, k is a B-type, and player i's strategy depends on whether or not he has heard a signal about him. Let this probability of a signal being received be given by $Q_i^j(k)$. If i has heard, he will choose exit with payoff 0. If he has not heard, he will choose defection with payoff $-\beta$. ∎

Remark 2.3.2. *The conditional probability $Q_i^j(k)$ is given by $\frac{\sum_{h \neq i,j} p_{hi}(k) m_{hk}}{\sum_{h \neq i,j} m_{hk}}$.*

Proposition 4 shows how players benefit from being ***trusting***: if there is a higher probability that they receive a signal about a bad reputation, their expected payoffs are increased, because they are more likely to have heard if their partner is a B-type and hence choose exit against him and protect themselves.

This is the period of the game where payoffs depend on a player's ability to detect *types*. Here payoffs are increased when a player is more likely to receive signals.

2.3.2 Period 2

The possibility of obstruction provides an action space for the players in period 2 of our cooperation game. In our model, all signals are true and are distinguished only by their subject — the identity of the player whose bad reputation is being transmitted. There are T rounds (or sub-periods) of information transmission within period 2. Players need to decide whether to pass on signals about each of the players, in each of the rounds of information transmission. This means that the action space for each player in period 2 consists of $\prod_{k\in N,\tau\leq T}\{$pass on signals about player k in round τ, do not pass on signals about player k in round $\tau\}$. Recall from the previous Section that the S-type players' equilibrium strategy in period 3 is {exit if a signal has been received about your partner, otherwise defect}. We can now observe the following.

Proposition 5. *When players' equilibrium strategies in period 3 are those given in Remark 2.3.1, and signals flow in period 2 through the network by obstructed diffusion given in Definition 5 of Chapter 1, a player who is the subject of a signal strictly prefers to conceal it, and players who are not the subject of a signal weakly prefer to pass it on.*

Proof. See Appendix. ∎

This means that we can identify the network links through which a signal about a player can travel in period 2 — it is all the network links except those which include the player himself. We can now identify the probabilities that true signals emitted about deviations in the first round will travel through the network by obstructed diffusion to the other players, given the information structure $\mathbf{\Omega}$. These are the obstructed word-of-mouth probabilities given in Definition 6 in Chapter 1.

While we have observed that the B-type will not pass on information about himself, a player cannot infer anything about his neighbour's type just because he does not receive any signal from him. This is due to the stochastic nature of information transmission. [21]

2.3.3 Period 1

Next we examine conditions under which we can expect cooperation in period 1. Consider the case where player i expects his period 1 partner j to cooperate.[22] If i defects, j will get a negative payoff and send a signal about it, and any S-type player who receives that signal will exit if they are matched with i in period 3.

Let $V_i^j = \sum_{k\neq i} V_i^j(k)m_{ik} = \sum_{k\neq i,j} V_i^j(k)m_{ik} + \sigma m_{ij}$ (recall that $V_i^j(j) = \sigma$ because j is an S-type). Let $V_i = \sum_{j\neq i} V_i^j m_{ij}$ be i's ex ante expected payoffs over all possible period 1 partners j. Now we can write player i's overall expected payoffs from either cooperating or defecting when his S-type partner

[21]However, a player can use the structure of the network to update his beliefs about the likelihood of his period-3 partner being a B-type, given that he did not hear a signal about him. This depends on the network links between his partner's possible period 1 matches and himself and is given in Appendix 2.C. These beliefs would not change his behaviour due to Assumption 1: he would only choose exit against a certain B-type.

[22]Note that meeting a B-type does not impact a player's incentive for cooperation, because he cannot impose a negative payoff on a B-type and hence cannot get a bad reputation from defecting against him when he should have cooperated. Hence we can exclude the possibility of meeting a B-type in period 1 from our study of the equilibrium incentives for cooperation. In this case his payoffs are $\beta\gamma_{ij} + \sum_{k\neq i,j} V_i^j(k)m_{ik}$, which enter the expression for overall payoffs given later in Proposition 9.

j cooperates.[23]

$$
\begin{aligned}
v_i(C_i^1, C_j^1) &= \gamma_{ij} + \delta V_i^j &(2.3.3)\\
v_i(D_i^1, C_j^1) &= \alpha\gamma_{ij} + \delta V_i^j - \delta\sigma(1-\phi)\sum_{k\neq i}\gamma_{ik}p_{jk}(i)m_{ik} &(2.3.4)
\end{aligned}
$$

As shown in (2.3.4), expected losses that i incurs if he defects against j who cooperates are $\sigma(1-\phi)\sum_{k\neq i}\gamma_{ik}p_{jk}(i)m_{ik}$, which is broken down as follows.

- The payoff from mutual defection in period 3 is σ (the 'reward' for cooperation);
- He will only be punished if his future partner is an S-type, which happens with probability $(1-\phi)$;
- Punishment occurs if a signal emitted by j about i reaches his potential future partners $k \in N \setminus i$, the probability of which is given by $p_{jk}(i)$. This is weighted by $\gamma_{ik}m_{ik}$, the probability i matches with k, and the payoffs if he does;
- This is summed over all $k \neq i$ because if i matches with himself, payoffs are zero. Meanwhile if he matches with j again, payoffs are also zero as j knows for sure about his deviation because $p_{jj}(i) = 1$.

Cooperation requires $v_i(C_i^1, C_j^1) \geq v_i(D_i^1, C_j^1)$ and we can rearrange (2.3.3) and (2.3.4) to find that cooperation for i, when matched with a partner j who he expects to cooperate, requires expected losses L_i^j from deviation to be above

[23] By Assumption 1, players will not exit against an unknown player in period 1: they either cooperate or defect.

a threshold L^*_{ij}, where

$$L^j_i \equiv \sigma(1-\phi)\sum_{k\neq i}\gamma_{ik}p_{jk}(i)m_{ik} \geq \frac{\gamma_{ij}(\alpha-1)}{\delta} \equiv L^*_{ij} \tag{2.3.5}$$

We can say that if $L^j_i \geq L^*_{ij}$, then player i is ***trusted*** by player j. This is because player i has an incentive such that if he expects player j to cooperate, player i will also cooperate.

We can observe that a player's propensity to cooperate is weakly *increasing in his own losses*. Higher expected losses from a deviation are to a player's advantage because they give him an incentive to be honest. If losses are high enough, he is more likely to be trusted, and therefore more likely to take part in cooperation, with higher payoffs.

In particular, i's expected losses from defecting against j are strictly increasing in the probabilities that j can inform other players — i's potential future matches — about a deviation by i. Player i is more willing to cooperate with j, if j is better able to inform other players about i's bad behaviour. So it is better for incentive-based trust if players can talk about each other.[24] The extent of which they can depends on the network structure and in fact, because of obstruction, on the network structure that remains when each node is removed. This is because each player is unable to commit to passing a message about their deviation, and has to rely on others to do so, incentivising him with their threat of gossip.

Secondly, the probabilities of information transmission are weighted by the matching probabilities. This is because a player not only cares whether his deviation would be detected — he cares if it is detected by the players he is likely

[24] Larson (2017) finds a similar result in a different setting.

to match with, because only a matched partner can carry out the punishment. Expected losses would be quite low if a potential victim who was only able to inform players who were unlikely to match with a deviator in period 3.

This period of the model is where incentives play the key role. When incentives matter, a player wants others to be able to communicate about him, encouraging him to cooperate. This is in contrast to period 3, which focuses on types, where a player wants to be able to communicate about others.

2.3.3.1 Cooperation

We have noted that if a player's expected losses are less than L_{ij}^*, he would defect. But if one player in a pair is tempted to defect, knowing this, their partner will defect too, even if their losses would otherwise be high enough to deter a deviation. So we need *both* partners in a pair to have high enough expected losses for cooperation to occur: they must both be trusted by each other. Otherwise, they both defect, coordinating on a Nash equilibrium of the one-shot game and both avoiding the bad reputation.[25]

Proposition 6. *A pair of players will cooperate in period 1 if and only if the expected losses of both players in the pair are above a threshold level, L_{ij}^* for player i matched with player j, $\forall i, j \in N$. Otherwise, they will both defect.*

Proof. If $L_i^j \geq L_{ij}^*$ and i expects j to cooperate, then i will also cooperate due to (2.3.5). Similarly if $L_j^i \geq L_{ji}^*$, and j expects i to cooperate, j will also cooperate. In contrast, if $L_i^j \geq L_{ij}^*$ but i expects j to defect (which he would if $L_j^i < L_{ji}^*$),

[25]As described earlier, we could apply this model to a one-sided trust game rather than a two-sided cooperation (prisoners' dilemma) game, and from that build a directed trust network that shows which players would trust each other in the one-sided game.

then i will also defect because $\sigma > -\beta$. Equally if $L_i^j \geq L_{ij}^*$ but $L_i^j < L_{ij}^*$, both players defect. They also both defect if $L_i^j < L_{ij}^*$ and $L_j^i < L_{ji}^*$. Hence both players in a pair will cooperate if and only if both players have losses above the relevant threshold. ■

2.3.4 Payoffs and centrality

We can now show how equilibrium payoffs depend on centrality measures, which we can find from the network of information transmission.

Proposition 7. *For uniform random matching and* $\gamma_{ij} = 1 \quad \forall j \neq i \quad \forall i \in N$, *period 3 expected payoffs are increasing in obstructed centrality* $P_i(\mathbf{\Omega})$, *given in Definition 10 of Chapter 1. That is,* $\frac{\partial V_i}{\partial P_i} > 0 \quad \forall i, j \in N$. *For general matching, period 3 expected payoffs are increasing in a weighted version of obstructed centrality.*

Proof. See Appendix ■

Players' *obstructed centrality* shows how network position affects the average probability that players receive signals via the network. This shows, on average, how *trusting* they are — that is, how easily they can receive signals from the network about other players.

Proposition 8. *In period 1, whether players are trusted or not is linked to their obstructing centrality defined in Chapter 1 as* $O_k = \frac{1}{(n-1)^2} \sum_{i \neq i} \sum_{j \neq i} p_{ij}(k, \mathbf{\Omega})$.

Proof. We can rearrange (2.3.5) to give the following threshold requirement

$$\sum_{k \neq i} \frac{\gamma_{ik}}{\gamma_{ij}} p_{jk}(i) m_{ik} \geq \frac{(\alpha - 1)}{\delta\sigma(1 - \phi)} \tag{2.3.6}$$

Taking the average expected loss over all potential period 1 partners j, we have

$$\frac{1}{1-m_{ii}}\sum_{j\neq i}\sum_{k\neq i}\frac{\gamma_{ik}}{\gamma_{ij}}p_{jk}(i)m_{ik}m_{ij}\geq\frac{(\alpha-1)}{\delta\sigma(1-\phi)} \tag{2.3.7}$$

Which we can observe is a weighted version of obstructing centrality. ■

This shows us that the extent to which a player is *trusted* depends on their *obstructing centrality* — that is, how easily other players can communicate about them, when they cannot commit to pass information on about themselves. These two results also highlight the symmetric case — when payoffs and matching probabilities are the same for all players, and so $\gamma_{ij} = 1 \quad \forall i, j$ with uniform random matching. With symmetry, the two aspects of trust and the payoffs in each period are directly linked to the centrality measures given in Chapter 1.

2.4 Welfare and the cooperation network

These equilibrium conditions mean that for any parameters of the model, we can find out which players can cooperate with each other, and which ones cannot. We list these cooperative pairs as the *cooperation network*, $\mathbf{G}^c$, which is endogenous to the information network $\mathbf{G}$, the interaction network $\mathbf{M}$ and the other parameters. Individual payoffs in period 1 depend on the number of cooperative relationships each player has: that is, their degree (number of neighbours) in the cooperation network. Hence welfare depends on the size of the trading world: that is, the number of edges in $\mathbf{G}^c$.

Proposition 9. *The cooperation network $G^c = [g^c_{ij}]_{ij}$ is given by*

$$g^c_{ij} = g^c_{ji} = \begin{cases} 1, & \text{if } L^j_i \geq L^*_{ij} \text{ and } L^i_j \geq L^*_{ji} \\ 0, & \text{otherwise} \end{cases}$$

and overall expected payoffs in the repeated game F are given by

$$v_i = \sum_{j \neq i} \left[\gamma_{ij} \left[(1-\phi)(g^c_{ij} + (1-g^c_{ij})\sigma) - \phi\beta \right] + \delta V^j_i - \phi\delta\sigma m_{ij} \right] m_{ij}$$

Proof. This follows from (2.3.2) and (2.3.3). ■

Figure 2.1 shows some example cooperation networks for a group of eight players, and the information networks they depend on.[26] We can now use the previous results to show the effect of the probabilities of information transmission on cooperation.

Proposition 10. *Trust, cooperation and welfare are weakly increasing in the probabilities of information transmission.*

Proof. See Appendix. ■

The observation that more information supports cooperation and welfare is intuitive, and supports Kandori's (1992) assertion that information about a player's reputation can sustain cooperation within a community. It also

[26]Parameters for all examples are: $\alpha = 1.15, \delta = 0.95, \beta = 1.3, \sigma = 0.7$. We set $T = 6$, for the example of a weekly market day, which would have 6 intervening days during which information flows through the information network before the next market day. Banerjee et al. (2013) set T using the number of visits to a village made by data collectors, an average of 6.6.

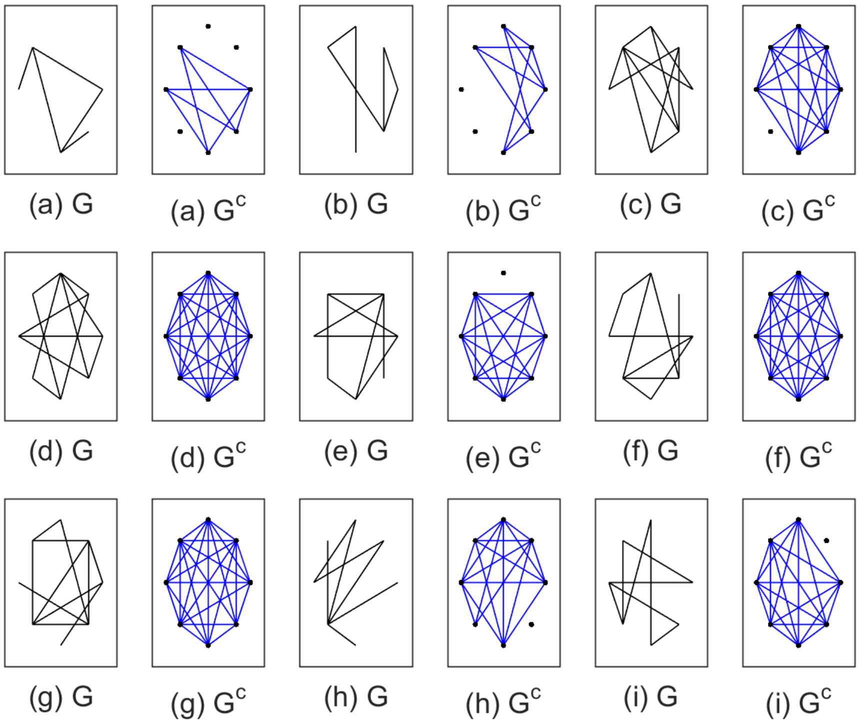


Figure 2.1: Random networks of eight nodes: $\mathbf{G}$ and $\mathbf{G}^c$

echoes results from imperfect private monitoring in infinitely repeated games by Sekiguchi (1997) and Bhaskar and Obara (2002) where, providing that monitoring is sufficiently accurate, the symmetric efficient payoff can be approximated. Experimental evidence such as that by Gallo and Yan (2015) also finds that information plays an important role in supporting cooperation.

We showed in Chapter 1 how the probabilities of information transmission are weakly increasing in the three aspects of the information structure $\mathbf{\Omega}$. Combining this with Proposition 10, we can observe that cooperation and welfare are weakly increasing in the three aspects of $\mathbf{\Omega}$. We can also note that the cooperation network defined in Proposition 9, which lists the cooperative pairs in the game

F, is endogenous to the information network $\mathbf{G}$, the matching probabilities $\mathbf{M}$ and the other parameters of the model. We say that a community of N players has *full cooperation* if and only if $L_i^j \geq L_{ij}^* \forall i, j \in N$.

2.4.1 Example: the size of the trading world

Dixit (2003b) arranged a continuum of players on a circle such that, as the distance between players around the circle increases, there is a decreasing probability of information transmission and matching, and increasing payoffs. Using this model, Dixit identified the 'size of the trading world', the arc of the circle within which players can cooperate with each other. Let us illustrate our approach and the result that increasing information can support higher cooperation and welfare in networks by using this example.

We arrange players in groups, and then place the groups in a circular shape within the information network. The four groups labelled A, B, C and D, each of four players, and the connections between the groups, are shown in the top left of Figure 2.2. These groups form an equitable partition (Allouch, 2017; Powers and Sulaiman, 1982) in that each member of a group has the same number of links to other groups. This structure leads to a greater probability that group members receive signals from each other than from players in other groups, who are located further away in the information network. We also assign increasing payoffs to matches with players in more distant groups, so that $\gamma_{AA} = 0.9$, $\gamma_{AB} = 1$, and $\gamma_{AD} = 1.1$, and this is symmetric for each group.

To use Dixit's model, we need decreasing matching probabilities with further-away groups — players should be most likely to meet their own group, then the

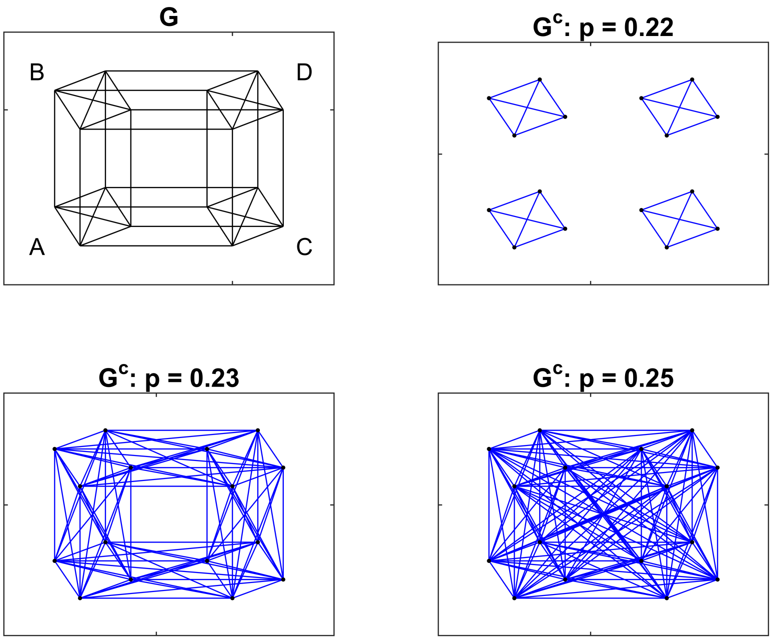


Figure 2.2: 'The size of the trading world' for groups in a network

neighbouring group, and least likely to meet the furthest group. To find matching probabilities that fit this bill, we use a simple approach where the probability of two players meeting is inversely proportional to their social distance, up to a maximum distance T_m, beyond which it is negligible but positive.[27] Let m^1 be the matching probabilities where $T_m = 1$ and players only have a non-negligible probability of meeting their direct neighbours, m^2 is the case when $T_m = 2$, etc. m^U gives uniform random matching where $m_{ij} = \frac{1}{n-1}$ $\forall\, j \neq i$ and $m_{ii} = 0$.

In this example we use matching probabilities m^3 which, since the diameter of the information network is 3, means that all players have a non-negligible

[27]See the Appendix on local matching for details. There exist alternative approaches that provide perfect matching and other desirable attributes, and we focus on this version only because it is easily applicable to the examples we have chosen.

probability of meeting. This structure gives us matching probabilities of: 0.105 for direct neighbours; 0.053 for players at social distance 2; and 0.035 for players at social distance 3.

Having translated Dixit's model into a network setting, we can see that our 'cooperation network' is analogous to Dixit's 'trading world'. The top right of Figure 2.2 shows the cooperation network at $p = 0.22$. Cooperative links are only present within groups: at this low level of p, players can only cooperate with members of their own group. The lower two graphics in Figure 2.2 show that as p increases and information is more likely to flow along the links of the information network, players can cooperate with members of their neighbouring groups, and then with groups further away. Welfare increases as there are more links in the cooperation network: that is, the size of the trading world increases. This illustrates the effect of increasing information transmission on the level of cooperation in this example.

2.4.2 Example: the dilution of social capital

A key observation from the expression for expected losses in (2.3.5) was that cooperation is more likely if a potential victim can tell the deviator's future partners about his behaviour. And while the information network determines who the victim can communicate with, it is the matching probabilities that determine whether or not those recipients of the signal are likely to be matched with the deviator in future. Therefore the joint configuration of the matching probabilities and information network are very important for the pattern of cooperation in our repeated game.

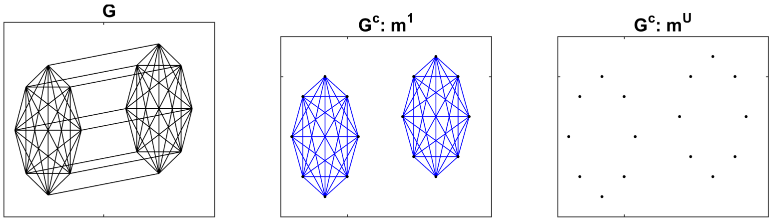


Figure 2.3: 'The dilution of social capital' for groups in a network

Figure 2.3 shows an example where, for a given information network, changes in matching probabilities affect the overall level of cooperation. The information network is shown in the top left, with two groups A and B of eight players each, who are completely connected within the group, and where each player has one connection in the other group. As before there are higher payoffs for playing the other group: $\gamma_{AA} = 1$, $\gamma_{AB} = 1.1$; now $p = 0.12$.

The cooperation network under m^1 is shown in the top right of Figure 2.3, while the cases with m^2 and m^U are shown in the lower left. This shows how the cooperation network with m^1 just connects players within each group: so when players are only likely to meet their direct neighbours in the information network, they can cooperate within their own group but not with the other group. With m^2 and m^U, there is an increased likelihood that players can meet those in the other group, and we get the somewhat surprising result that compared to m^1, this leads to the breakdown of cooperation *within* each group. This is because there is a much lower probability of a signal being received between groups, than within groups. If a player meets someone from his own group in period 1, it is not likely that they will be able to inform someone from the other group about any deviation. So if there is a good chance that a player is matched outside his group in period 3, the probability that he will be punished for a deviation against someone in his own group is lower, and he will be tempted to deviate against them. This tallies with the case of 'dilution of social capital' described by Meagher (2006), where the entry of additional groups into the informal sector of garment and shoe production in Aba in South-Eastern Nigeria reduced cooperation *within* the groups that were already occupying the sector.

2.4.2.1 Cooperation and social distance

In these examples we have shown two cases where players only cooperate within their groups and are not able to trust or cooperate with players in other groups. This echoes experimental work finding decreasing cooperation with increased social distance by Chandrasekhar et al. (2014) and Riyanto and Yeo (2014), among others. Our model identifies two mechanisms which could lead to this outcome: if, as social distance increases, either or both of the matching probabilities or the probabilities of information transmission decrease.

2.5 Trust and obstruction

Having shown that a network which supports greater information transmission can support more cooperation, next we analyse the effect of different network positions on individual levels of trust, cooperation and payoffs. We identified two aspects of trust — players are trusted if other players can send and receive signals about them, whereas they are trusting if they are likely to receive signals about other players. Now we examine how network positions are linked with these two aspects of trust.

2.5.1 Trust and visibility

Following from our observations in the previous Section, it is clear that if player i emits a signal about player k, the probabilities that the signal is received by other players are determined by player i's position in $\mathbf{G}_{-k}$, the network omitting k. From Definition 6, we can see that obstructed word-of-mouth probabilities are zero if the social distance between players is greater than T, the maximum number of links that a signal can travel. This means that i's signal about k could only be received by players at a social distance of T or less from i in $\mathbf{G}_{-k}$. Players can *communicate* if there is a positive probability that a signal sent by one of them will be received by the other. As defined in Chapter 1, a player k is *visible* if $p_{ij}(k, \mathbf{\Omega}) > 0 \ \ \forall\ i, j \in N \setminus k$: that is, k is visible if everyone can communicate about him. By Remark 1.4.1 in Chapter 1, if the network is 2-connected and $\max_{k \in N}\{D_{\mathbf{G}_{-k}}\} \leq T$, all players are visible. If, in addition $p = 1$, then $p_{ij}(k, \mathbf{\Omega}) = 1\ \forall\ i, j, k$, which is equivalent to perfect information.

Remark 2.5.1. *Given previous observations that* $V_i^j = \sum_k V_i^j(k) m_{ik}$ *and* $L_i^j = \sum_k p_{jk}(i) V_i^j(k) m_{ik}$*, a player's potential losses from a deviation can reach his total period 3 payoff if and only if he is visible. In particular,*

- *Player i is visible* $\iff$ $L_i^j \leq V_i^j\ \forall j$.
- *If player i is not visible such that $D_{\boldsymbol{G}_{-i}} > T$ but $\boldsymbol{G}_{-i}$ is connected, then for j, k such that $d_{jk}(\boldsymbol{G}_{-i}) > T$ we have that $L_i^j < V_i^j$ and $L_i^k < V_i^k$.*
- *If $\boldsymbol{G}_{-i}$ is not connected then $L_i^j < V_i^j\ \forall j$.*

Only a visible player has a positive probability that everyone could find out if he deviates against any of his matches, so only a visible player can risk the maximum losses from a deviation in every match. If a player is not visible, then he cannot lose his total period 3 payoffs from a deviation in some (or all) of his matches. This lack of visibility reduces his losses and hence his likelihood of cooperation.

The link between player i's visibility and the connectedness of $\mathbf{G}_{-i}$ echoes the importance of 2-connectedness for cooperation that is highlighted by Renault and Tomala (1998) and Wolitzky (2014), because 2-connectedness is clearly a necessary condition for visibility. Like Kinateder (2008), we also find that the diameter of the network is important for cooperation, although in our case — because of obstruction — it is the diameter of the network that remains when a player is removed that matters. In fact, a sufficient condition for player i's visibility is that $D_{\mathbf{G}_{-i}} \leq T$.

2.5.2 Example: star network

We can illustrate the importance of 2-connectedness with an example information network: the star network, shown in the top left of Figure 2.1 with eight players. The star network is not 2-connected with respect to the centre because without him, all other players — the periphery — are singletons. On the other hand, the network is 2-connected with respect to the periphery players because they would not disconnect the network by their absence. Since the star network is not 2-connected with respect to all nodes, it is not 2-connected.

The cooperation network in the case of uniform random matching is shown in the lower left of Figure 2.1 ($p = 0.5$). Perhaps surprisingly, we observe that the player in the centre of the star cannot cooperate with any other player, while

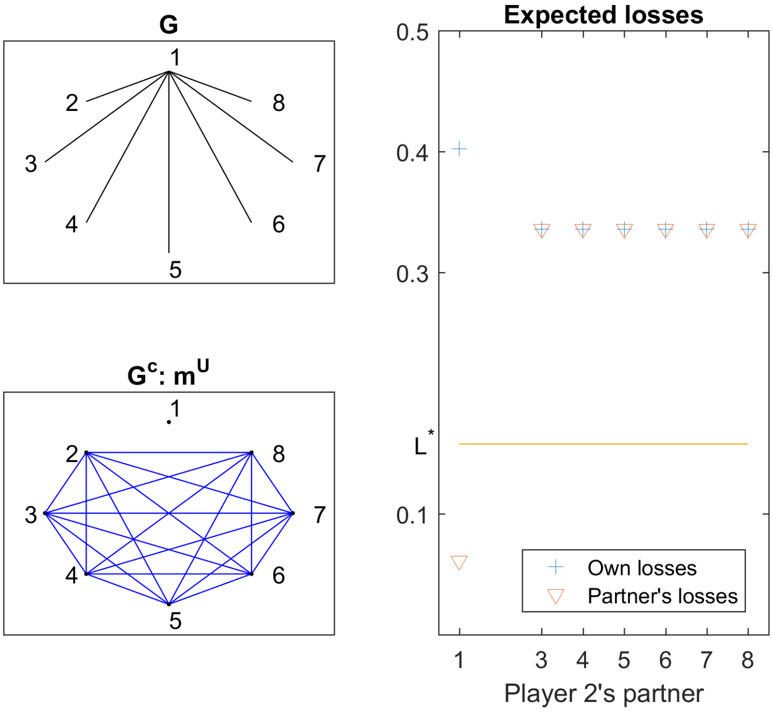


Figure 2.1: Star network with eight players

the players on the periphery can all cooperate with each other. To find out why, we can look at the losses in each partnership, shown in the chart on the right of Figure 2.1. The solid red line denotes the threshold losses L^* that must be attained by both players in a match to ensure cooperation [28]. The crosses show player 2's losses from defecting in each of his partnerships, and the triangles show his partners' losses.

We can see that player 1, in the centre of the star, can trust player 2 on the periphery. This is because 2's losses from a deviation against 1 are high, since a signal emitted by 1 about 2's deviation only has to travel one link to be received by the other periphery players — 2's potential future partners. But 2 cannot trust 1 in return, because 1's losses when matched with 2 are below the threshold line L^*. This is because if 1 were to deviate against 2, player 1 would obstruct any signal 2 would send about it — and without 1, player 2 is a singleton and so could not tell anyone. So 2 expects 1 to defect and therefore will also defect, and cooperation breaks down between them. This structure recalls the *gatekeeping* and *end network* effects highlighted by Lippert and Spagnolo (2011), because player 1 acts like a 'gatekeeper' of the information network with respect to the periphery players. In fact, the periphery nodes could cooperate with the centre if they had additional links to each other. This recalls Myerson's (2008) model of an autocrat whose support depends on the ability of his 'courtiers' to observe his behaviour towards each of them, ensuring fairness.

On the other hand, as shown in the chart, periphery players have relatively high losses when matched with each other, and these are symmetric, so they can all cooperate with each other. This is because player 1 in the centre of the star

[28]This threshold is the same for all players since $\gamma_{ij} = 1 \quad \forall j \neq i$ in this example.

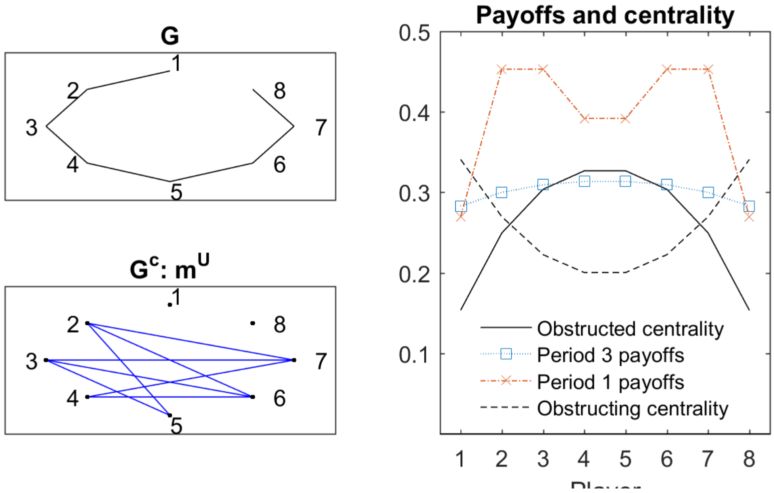


Figure 2.2: Line network with eight players

provides a walk of length two between all the periphery players, so a signal is very likely to pass between them if any of them deviate, leading to high losses and therefore more trust. The centre of the star misses out on cooperation himself, but supports cooperation by the other players, by ensuring they can communicate with each other.

For a given number of players in a connected information network that is informative and a tree, Remark 1.4.3 in Chapter 1 showed that the star network has the most visible players of any tree configuration. This means that the star network has the most partnerships with maximum losses, of any tree network.

2.5.3 Example: line network

We can illustrate the links between payoffs and the different centrality measures using the line network of eight players, shown in the top left of Figure 2.2. The cooperation network with uniform random matching is shown in the bottom left, and payoffs and obstructed centrality are shown on the right. Obstructed centrality increases for those players located nearer to the centre of the line. As expected from Proposition 7, period 3 payoffs rise monotonically with obstructed centrality, because players who are more likely to receive information from the network are more trusting.

On the other hand, there is a non-monotonic relationship between *period 1 payoffs* and obstructed centrality. The chart shows that players 2, 3, 6 and 7 have the highest cooperation levels in equilibrium, but only moderate levels of obstructed centrality. Looking at the cooperation network, we can see the same pattern: players 1 and 8 are not able to cooperate with anyone, and players 4 and 5 have fewer links in the cooperation network than players 2, 3, 6 and 7. This non-monotonic relationship between centrality ranking and cooperation is similar to a concept known as *middle-status conformity*, identified by sociologists Phillips and Zuckerman (2001), where those with a 'middle' level of status or ranking are most likely to conform to society's norms. In another setting, Butler et al. (2009) also find a non-monotonic relationship between payoffs and trust.

2.5.3.1 A counterfactual without obstruction

We can use word-of-mouth probabilities — without obstruction — as a useful counterfactual to investigate the effect of obstruction on cooperation in this network. These are given in Definition 2 in Chapter 1 and allow us to construct counterfactual expected losses without obstruction, that is $LW_i^j = \sum_k w_{jk}(\mathbf{\Omega})V_i^j(k)m_{ik}$ compared to $L_i^j = \sum_k p_{jk}(i, \mathbf{\Omega})V_i^j(k)m_{ik}$. These gives us a counterfactual cooperation network, showing which players would cooperate, were it not for obstruction. This hypothetical network (not shown) includes additional cooperative links between players: 4 with 2, 3 and 5; and 5 with 4, 6 and 7.

This means that the losses without obstruction — LW_i^j — for the partners of players 1 and 8 are still too low to deter defection; we say that 1 and 8 have *poor network positions* in an *absolute* sense because they are too 'tempting' for anyone to cooperate with, even if they did not obstruct their signals: they can trust no-one. On the other hand, we find that 4 and 5 have three more cooperative links in the counterfactual network; in fact the non-monotonic relationship between cooperation and obstructed centrality disappears when we remove the effect of obstruction. We say that 4 and 5 have *poor relative network positions* because they would have cooperated, were it not for their obstruction of signals. Players with poor network positions in either sense reduce cooperation levels and hence welfare.

2.6 Conclusion

This chapter investigates the extent of cooperation in a finitely repeated game in a network setting. We apply Dixit (2003b)'s continuous model to a network: a discrete community of players who occupy its nodes. The players are randomly matched in pairs in the first and last periods, and play the stage game of a modified prisoners' dilemma. From the fixed information network, the model allows us to generate an endogenous network of potentially cooperative relationships. From this we can characterise how levels of cooperation depend on the structure of the information network. Individual players' payoffs are linked to whether their network positions mean they are *trusting* and/or *trusted*. Players are trusting if they are likely to receive information from the network, while they are trusted if others can pass signals about them. A pair of players can only cooperate if they are both trusted by each other. Using the word-of-mouth probabilities for information transmission developed in Chapter 1, and find that cooperation and welfare both increase with these probabilities.

We find that players with higher obstructed centrality (constructed from the probabilities of information transmission) receive more information from the network and hence are more trusting. But there can be a non-monotonic relationship between centrality and the extent to which players are trusted, leading to cooperation patterns with middle-status conformity. This is interesting because one might expect the most central player to have the highest payoffs, while we find that a player's central position may actually reduce his capacity to be trusted by others. This is because players cannot commit to pass on a signal about their own bad reputation. Knowing this, players who rely on a central player for information transmission may not trust him, because they know he will obstruct any signal that they send to warn others about his bad reputation. This highlights the importance of 2-connectedness and cycles for cooperation and welfare, because these structures can prevent players from completely obstructing signals about their reputations travelling between other players, ensuring that they are *visible*. Since the non-monotonic relationship between cooperation and centrality disappears in our counterfactual example of a line network without obstruction, we conjecture (though we have no formal proof) that general results may exist linking middle-status conformity to line networks, or other acyclic networks.

The possible link between middle-status conformity and acyclic networks may also be of empirical interest. There is some experimental evidence that players with high centrality may be less 'reciprocal' in trust games (Riyanto and Yeo, 2014; Barr et al., 2009). Obstruction may also imply that acyclic networks are less likely to be observed in communities that use this kind of community enforcement mechanism. Where acyclic networks are present, we may find that central individuals seek other ways to dampen the negative effect of obstruction on their capacity to cooperate. For example, they may enlist their own neighbours (not just the neighbours of their potential victim) as *witnesses* to observe their actions, increasing their potential losses and making them more trustworthy. Secondly, obstructive, bridging players may *specialise* in information transmission: even though they cannot pass signals about themselves and are hence not trusted, they could share in the benefits of cooperation if transfers from other cooperating parties can be arranged. Finally, *local matching* may have a mitigating effect because if someone is more likely to meet the same player again, he will have higher losses from deviating against them, even if that player cannot communicate with others due to obstruction.

Some interesting extensions suggest themselves. The model looks only at the case of a bad reputation, and there may be interesting effects when a 'good label' rather than a bad one is emitted, or both, as in Spence (1973) and Breza and Chandrasekhar (2015). It could also be interesting to introduce some stochasticity in order to investigate the effect of risk on cooperative relationships, as observed by Baker (1984). Lastly, it may be possible to use the model to investigate the interaction of formal and informal enforcement regimes, as examined by Kranton (1996), Dhillon and Rigolini (2011) and Dixit (2003a).

Appendix

2.A A good reputation

In our framework, an alternative specification with a good signal would work as follows.

Period 3 For punishment to work, there would be a parametric assumption that a player would only be rewarded for cooperation in the first round if a good signal about them was received by their final-round partner. This means that if a player received a good signal about their partner in the final round they would they defect, and otherwise they would punish (exit). Hence Assumption 1 could no longer hold. And it would imply that some S-type players would be punished in equilibrium, because a signal was not received by their partner - in contrast to our model with a bad signal, where only the B-types are punished.

Period 2 With a good signal, this would mean everyone had an incentive to pass on a signal about themselves and about other players. This is because the reward for cooperation (mutual defection) would only be possible if both partners received a good signal about each other. This means there would be no obstruction.

Period 1 In the first round, incentives to be honest would depend on the probability that a good signal emitted by a player's partner would reach his future partners, in the same way as the current model - but without obstruction. However, because Assumption 1 would not hold, players who could not cooperate would coordinate on the Nash equilibrium of mutual exit, rather than mutual defection.

This is an interesting thought experiment because it suggests that there may be different welfare impacts of either good or bad signals in different networks and with different parameters. On one hand, cooperation might be higher with a good signal because there is no obstruction and so greater probabilities of information transmission. But on the other hand, welfare might be lower because some S-types would be wrongly punished in period 3, and because non-cooperating partners in period 1 could not defect, only exit.

2.B Proofs

2.B.1 Proof of Remark 2.3.2

$Q_i^j(k)$ is the conditional probability that, if player k is a B-type, an S-type player i who matched with another S-type player j in period 1 will hear a signal about player k:

$$Q_i^j(k) \equiv \Pr\left[\rho_i(k) = 1 \mid \{k \text{ is B-type}\}, \quad r^i = r^j = 0, \quad k, \mu_k^1 \notin \{i, j\}\right]$$

By the law of total expectations, $Q_i^j(k)$ is given by:

$$\begin{aligned}
&\Pr\left[\rho_i(k)=1 \mid \{k \text{ is B-type}\},\ \ r^i=r^j=0,\ \ k,\mu_k^1\notin\{i,j\}\right]\\
&=\sum_{h\in N}\left[\Pr\left[\rho_i(k)=1 \mid \{k \text{ is B-type}\},\ r^i=r^j=0,\ \ k,\mu_k^1\notin\{i,j\},\ \mu_k^1=h\right]\Pr\left[\mu_k^1=h \mid k,\mu_k^1\notin\{i,j\}\right]\right]
\end{aligned}$$

The second term is

$$\begin{aligned}
\Pr\left[\mu_k^1=h \mid k,\mu_k^1\notin\{i,j\}\right] &= \frac{\Pr\left[(\mu_k^1=h)\cap(k,\mu_k^1\notin\{i,j\})\right]}{\Pr\left[k,\mu_k^1\notin\{i,j\}\right]}\\
&= \begin{cases} \frac{m_{hk}}{\sum_{h\neq i,j} m_{hk}} & \forall\ h\notin\{i,j\}\\ 0 & \forall\ h\in\{i,j\}\end{cases}
\end{aligned}$$

The first term is

$$\Pr\left[\rho_i(k)=1 \mid \{k \text{ is B-type}\},\ r^i=r^j=0,\ k,\mu_k^1\notin\{i,j\},\ \mu_k^1=h\right]=\begin{cases}0 & \forall\ k\in\{i,j\}\\ 0 & \forall\ h\in\{i,j\}\\ p_{hi}(k) & \text{otherwise}\end{cases}$$

Therefore $Q_i^j(i)=Q_i^j(j)=0$, and otherwise we have that:

$$Q_i^j(k)=\frac{\sum_{h\neq i,j} p_{hi}(k)m_{hk}}{\sum_{h\neq i,j} m_{hk}} \qquad \forall\ k\notin\{i,j\} \tag{2.B.1}$$

2.B.2 Proof of Proposition 5

Assume $\gamma_{ij} = 1 \ \forall i, j$ WLG. In Definition 5 we assumed that the probability of information travelling along each walk is independent, implying that players only recall signals they receive in the last round of information transmission, and forget signals received and passed on in earlier rounds. There are four cases.

An S-type player and a signal about himself: This S-type player has deviated in period 1. He will have strictly lower payoffs if his future partner is an S-type and has heard about his deviation, because they will exit against him instead of defecting. His period 3 payoffs are unchanged if he meets a B-type (because a B-type would not 'punish him'. He could match with anyone in period 3 (because $m_{ij} > 0 \ \forall \ j \neq i$) and because he does not know which of these players are S-types and which are B-types, he does not want any of them to find out about his deviation. Therefore he will strictly prefer to conceal a signal about himself from all other players. This holds for all rounds of information transmission.

An S-type player and a signal about someone else: S-type players are not expecting another S-type to deviate, so a signal about another player would tell them that he is a B-type. An S-type player who has heard a signal about a B-type player, and meets him in period 3, has payoff 0. If he has not heard the signal, his payoff on meeting a B-type is $-\beta$. If he meets an S-type, his payoffs are unchanged by hearing a signal about a B-type. As he could meet any of the players in period 3, an S-type player strictly prefers to receive a signal about a B-type.

However it is not the case that passing on a signal always increases the probability that it is received. Sometimes it may have no effect. For example, a player passing on a signal in round T cannot increase his probability of receiving a signal, since it will not have time to return to him, as there are no more rounds of information transmission. So he is indifferent between the actions of either passing on a signal or not in round T. However, if players do not pass on the signal in round T, they cannot increase their probability of receiving it by passing it on in round $T-1$, and so information transmission could quickly unravel. Specifically, a player can strictly increase his probability of receiving a signal by passing it on, if and only if other players pass it on in the following rounds, and he is part of a walk that returns to his network position in a number of links which is a factor of the number of information transmission rounds remaining. We can observe that the strategy for *all* S-type players to pass on signals about other players in *all* rounds is weakly preferred.

A B-type player and a signal about himself: A B-type's payoffs are $x/2$ if his future partner has heard the signal and x otherwise, so he strictly prefers to conceal a signal about himself.

A B-type player and a signal about someone else: In this case, a B-type's payoffs in period 3 are not affected by whether he has heard a signal about another B-type or not. So he is indifferent between passing signals about other players and not passing them. Therefore the strategy to pass on signals about all other players in all rounds is weakly preferred. This means that in our model a B-type player has the same strategy for passing on signals as does an S-type. This seems reasonable: a B-type player may not wish to draw attention to himself by not passing on signals about other players, when it would not decrease his payoffs to do so.

2.B.3 Proof of Proposition 7

For **uniform random matching** and when $\gamma_{ij} = 1 \;\; \forall j \neq i \in N$ $(\gamma_{ii} = 0)$, we want to show that period 3 payoffs are increasing in obstructed centrality, which is given in Definition 10 as

$$P_i(\mathbf{\Omega}) \;\; = \;\; \frac{1}{(n-1)^2} \sum_{k \neq i} \sum_{j \neq i} p_{ij}(k, \mathbf{\Omega})$$

Ex ante, i could meet any player k in period 3, and any player j in period 1. So we want to find V_i, the average expected period 3 payoffs over any k and any j, and check that it is (weakly) increasing in P_i.

$$\begin{aligned} V_i &= \frac{1}{(n-1)^2} \sum_{k \neq i} \sum_{j \neq i} V_i^j(k) \\ &= \frac{1}{(n-1)^2} \sum_{k \neq i} \sum_{j \neq i} \left[\sigma(1-\phi) - \beta\phi(1 - Q_i^j(k))\right] \\ &= \sigma(1-\phi) - \beta\phi + \beta\phi \frac{1}{(n-1)^2} \sum_{k \neq i} \sum_{j \neq i} Q_i^j(k) \end{aligned} \tag{2.B.2}$$

Next, rewriting (2.B.1) with uniform random matching gives the following expression

$$Q_i^j(l) = \frac{1}{n-2} \sum_{h \neq i,j} p_{hi}(l) \tag{2.B.3}$$

Let $P_i(k) = \frac{1}{n-1} \sum_{h \neq i} p_{hi}(k)$, noting that $\frac{1}{n-1} \sum_{k \neq i} P_i(k) = P_i$. We can substitute this into the expression for $Q_i^j(k)$ (2.B.1) to give

$$Q_i^j(k) \;\; = \;\; \frac{1}{n-2} \left[\sum_h p_{hi}(l) - p_{ji}(k) \right] = \frac{1}{n-2} \left[(n-1) P_i(k) - p_{ji}(k) \right]$$

Next we take the averages required

$$\begin{aligned}
\frac{1}{(n-1)^2}\sum_{k\neq i}\sum_{j\neq i} Q_i^j(k) &= \frac{1}{(n-1)^2}\sum_{k\neq i}\sum_{j\neq i}\frac{1}{n-2}\left[(n-1)P_i(k)-p_{ji}(k)\right] \\
&= \frac{1}{(n-1)(n-2)}\sum_{k\neq i}\left[(n-1)P_i(k)-\frac{1}{(n-1)}\sum_{j\neq i}p_{ji}(k)\right] \\
&= \frac{1}{(n-2)}\left[(n-1)P_i-\frac{1}{(n-1)^2}\sum_{k\neq i}\sum_{j\neq i}p_{ji}(k)\right] \\
&= \frac{1}{(n-2)}\left[(n-1)P_i-P_i\right]=P_i
\end{aligned}$$

And therefore from (2.B.2) we have that $\frac{\partial V_i}{\partial P_i}=\beta\phi>0$, as required.

In the case of **general matching**, (2.3.2) shows that the only way that network probabilities affect payoffs is via $Q_i^j(k)$. Clearly $\frac{\partial V_i^j(k)}{\partial Q_i^j(k)}=\phi\beta>0$ as required. Let $\sum_{k\neq i}\gamma_{ik}m_{ik}=\gamma$, which would imply that $\frac{1}{n-1}\sum_{k\neq i}\gamma_{ik}=\gamma$ in the uniform matching case.

$$\begin{aligned}
V_i &= \sum_{k\neq i}\sum_{j\neq i}V_i^j(k)m_{ij}m_{ik} \\
&= \sum_{k\neq i}\sum_{j\neq i}\gamma_{ik}\left[\sigma(1-\phi)-\beta\phi(1-Q_i^j(k))\right]m_{ij}m_{ik} \\
&= \sum_{k\neq i}\gamma_{ik}\sum_{j\neq i}\left[\sigma(1-\phi)-\beta\phi\right]m_{ij}m_{ik}+\beta\phi\sum_{k\neq i}\sum_{j\neq i}Q_i^j(k)m_{ij}\gamma_{ik}m_{ik} \\
&= \gamma(1-m_{ii})[\sigma(1-\phi)-\beta\phi]+\beta\phi\sum_{k\neq i}\sum_{j\neq i}Q_i^j(k)m_{ij}\gamma_{ik}m_{ik}
\end{aligned}$$

Let $\sum_{h\neq i}p_{hi}(k)m_{hk}=P_i^m(k)$. Now we have that

$$Q_i^j(k)=\frac{\sum_{h\neq i,j}p_{hi}(k)m_{hk}}{\sum_{h\neq i,j}m_{hk}}=\frac{P_i^m(k)-p_{ji}(k)m_{jk}}{1-m_{ik}-m_{jk}} \tag{2.B.4}$$

Now we need to find $\sum_{k \neq i} \sum_{j \neq i} Q_i^j(k) m_{ij} \gamma_{ik} m_{ik} =$

$$\begin{aligned}
&= \sum_{k \neq i} \sum_{j \neq i} \frac{P_i^m(k) - p_{ji}(k) m_{jk}}{1 - m_{ik} - m_{jk}} m_{ij} \gamma_{ik} m_{ik} \\
&= \sum_{k \neq i} \sum_{j \neq i} \left[\frac{P_i^m(k)}{1 - m_{ik} - m_{jk}} - \frac{p_{ji}(k) m_{jk}}{1 - m_{ik} - m_{jk}} \right] m_{ij} \gamma_{ik} m_{ik} \\
&= \sum_{k \neq i} \left[\sum_{j \neq i} \frac{P_i^m(k)}{1 - m_{ik} - m_{jk}} m_{ij} - \sum_{j \neq i} \frac{p_{ji}(k) m_{jk}}{1 - m_{ik} - m_{jk}} m_{ij} \right] \gamma_{ik} m_{ik} \\
&= \sum_{k \neq i} \left[P_i^m(k) \sum_{j \neq i} \frac{m_{ij}}{1 - m_{ik} - m_{jk}} - \sum_{j \neq i} \frac{p_{ji}(k) m_{jk} m_{ij}}{1 - m_{ik} - m_{jk}} \right] \gamma_{ik} m_{ik}
\end{aligned}$$

Clearly, the second term is another weighted version of $P_i(k)$, but using different weights to $P_i^m(k)$. Averaging both these terms over k means that we have a weighted version of P_i, which we call P_i^X. We have that $\frac{\partial V_i}{\partial P_i^X} = \beta\phi > 0$ as required.

2.B.4 Proof of Proposition 10

The expression for each player's utility given in Proposition 9 is strictly increasing in both the number of cooperative partnerships, and in period 3 payoffs. From Proposition 6 the number of cooperative partnerships is weakly increasing in the losses in each partnership, and from (2.3.5) losses are strictly increasing in information transmission probabilities. From Proposition 7, period 3 payoffs are increasing in information transmission probabilities. Putting these together, overall utility is weakly increasing in the probabilities of information transmission. So more information, in the sense of greater probabilities of information transmission, weakly increases the levels of cooperation and welfare in this repeated game.

2.C Updated subjective probabilities

If i meets k in period 3 and has not heard any signal, there are two possibilities: either k is an S-type; or k is a B-type but i has not heard about it. Player i will still defect against the unknown player k due to Assumption 1. Now $\phi_i^j(k)$ is his updated belief that k is a B-type player, given that he has heard no signal about him, that is:

$$\phi_i^j(k) \quad \equiv \quad \Pr[\{k \text{ is B-type}\} \mid \rho_i(k) = \emptyset, \quad r^i = r^j = 0, \quad k, \mu_k^1 \notin \{i,j\}]$$

Let $Q_i^j(k)$ be the conditional probability that, if k is a B-type, an S-type player i who matched with another S-type player j in period 1 will hear a signal about player k (see next subsection).

$$Q_i^j(k) \quad \equiv \quad \Pr\left[\rho_i(k) = 1 \mid \{k \text{ is B-type}\}, \quad r^i = r^j = 0, \quad k, \mu_k^1 \notin \{i,j\}\right]$$

To find $\phi_i^j(k)$, we need the following expressions:

$$\begin{aligned}
\Pr\left[\rho_i(k) = \emptyset \mid \{k \text{ is S-type}\}, \quad r^i = r^j = 0, \quad k, \mu_k^1 \notin \{i,j\}\right] &= 1 \\
\Pr\left[\rho_i(k) = \emptyset \cap \{k \text{ is S-type}\} \mid r^i = r^j = 0, \quad k, \mu_k^1 \notin \{i,j\}\right] &= 1 - \phi \\
\Pr\left[\rho_i(k) = \emptyset \mid \{k \text{ is B-type}\}, \quad r^i = r^j = 0, \quad k, \mu_k^1 \notin \{i,j\}\right] &= 1 - Q_i^j(k) \\
\Pr\left[\rho_i(k) = \emptyset \cap \{k \text{ is B-type}\} \mid r^i = r^j = 0, \quad k, \mu_k^1 \notin \{i,j\}\right] &= \phi(1 - Q_i^j(k))
\end{aligned}$$

Using the final equation from this list and Bayes' rule we have that

$$\begin{aligned}\phi_i^j(k) &= \Pr[\{k \text{ is B-type}\} \mid \rho_i(k) = \emptyset, \quad r^i = r^j = 0, \quad k, \mu_k^1 \notin \{i, j\}] \\ &= \frac{\phi(1 - Q_i^j(k))}{\phi(1 - Q_i^j(k)) + 1 - \phi} = \frac{\phi(1 - Q_i^j(k))}{1 - \phi Q_i^j(k)}\end{aligned} \tag{2.C.1}$$

2.D Local matching based on social distance

As discussed in Section 2.2, we might expect that matching probabilities would decrease as social distance increases, which we call *local matching*. Here we propose a simple form for this function. For this purpose, we assume there is a parameter, T_m, which gives the maximum social distance over which players have a non-negligible probability of matching. Since our model requires a positive probability for any pair to match, beyond T_m we assume the matching probability is negligible but positive.

Our function takes the form $m_{ij} = m^{T_m}(d_{ij}(\mathbf{G})) = \frac{1}{\lambda d_{ij}(\mathbf{G})}$ for some parameter λ, so that matching probabilities are inversely proportional to their social distance.

As set out in Subsection 2.2.2, any matching probabilities in our model must meet the following conditions: $m_{ij} = m_{ji} \quad \forall i, j \in N$; $m_{ij} > 0 \ \forall \ j \neq i \in N$; $\sum_j m_{ij} = 1$ and $m_{ii} \geq 0 \ \ \forall i \in N$. It now remains to identify λ in order to specify our function $m(\cdot)$.

Since we have that $\sum_j m_{ij} = 1$, assume for a moment that there is no sitting out and $m_{ii} = 0 \ \ \forall i \in N$, since $\frac{1}{d_{ii}(\mathbf{G})}$ is not defined. Then we can define λ_i as follows

$$1 = \sum_{j \neq i \text{ s.t. } d_{ij}(\mathbf{G}) \leq T_m} m_{ij} = \sum_{j \neq i \text{ s.t. } d_{ij}(\mathbf{G}) \leq T_m} \frac{1}{\lambda_i d_{ij}(\mathbf{G})}$$

Which rearranges to

$$\lambda_i = \sum_{j \neq i \text{ s.t.} d_{ij}(\mathbf{G}) \leq T_m} \frac{1}{d_{ij}(\mathbf{G})}$$

.

Then let $\lambda = \max_{i \in N}\{\lambda_i\}$ and use this to calculate the pairwise matching probabilities m_{ij}. For any i such that $\sum_{j \neq i} m_{ij} < 1$, we set $m_{ii} = 1 - \sum_{j \neq i} m_{ij}$ so that $\sum_j m_{ij} = 1 \ \ \forall i$, as required.

This means that m^1 gives the matching probabilities where $T_m = 1$ and players only have a non-negligible probability of meeting their direct neighbours. Now $\lambda_i = |\mathcal{N}_i|$ and λ is the maximum degree of the network. We have that $m_{ij} = 1/\lambda \quad \forall i \neq j$, which means that $m_{ii} = 1 - \frac{|\mathcal{N}_i|}{\lambda} \quad \forall i$. Players with the maximum degree have $m_{ii} = 0$ and they never sit out. Let m^2 be the case when $T_m = 2$, m^3 be when $T_m = 3$, etc. Meanwhile m^U gives uniform random matching where $m_{ij} = \frac{1}{n-1} \ \ \forall\ j \neq i$ and $m_{ii} = 0$.

Chapter 3

Redistribution in networks

3.1 Introduction

Research on the private provision of public goods has been, from the first, focused on welfare implications. Important results by Malinvaud (1972), Warr (1983) and Bergstrom, Blume and Varian (1986) (henceforth BBV) show firstly, that the public good is under-provided relative to the efficient level; and secondly, that income redistribution that leaves the set of contributors unchanged is 'neutral'. Neutrality means that contributors adjust their contributions to exactly offset the transfer, meaning that there is no change in provision levels or welfare from such a policy.

Public goods are often 'local' in the sense that consumers only benefit from the provision of their direct neighbours. Hence the network context, where local influences are heterogeneous among consumers, is a natural setting to examine private provision of public goods. In a key contribution, Bramoullé and Kranton (2007) show that when neighbours' actions are perfect strategic substitutes, specialised Nash equilibria correspond to the maximal independent sets of the network. Bramoullé, Kranton and d'Amours (2014) investigated the whole range of strategic substitution and identified a threshold of impact related to the lowest eigenvalue of the network. Below the threshold, the uniqueness and stability of a Nash equilibrium hold. Beyond it, multiple Nash equilibria will in general exist, and stability holds only for corner equilibria. Allouch (2015) extends this model to the non-linear case, with a condition on the normality of the public good which follows BBV's approach.[29]

[29] Ballester, Calvó-Armengol and Zenou (2006) first showed that consumers' equilibrium were proportional to their 'Bonacich' centrality (Bonacich, 1987), a measure which gives the number of walks throughout the network that begin from each consumer. Other recent and relevant

Allouch (2015) also shows that neutrality no longer holds for incomplete networks, opening the door to policy interventions that could improve welfare. Redistribution of endowments is the benchmark policy choice, as shown in the Second Welfare Theorem. Unlike the competitive equilibrium, where efficiency is always assured and welfare is only affected in a normative sense through improvements to equity, with private provision, neither equilibrium (before or after redistribution) is efficient. Allouch (2017) first investigated the benchmark policy of income redistribution between contributors, focusing on preferences that yield affine Engel curves[30] and using a standard utilitarian approach. These were based on income redistribution involving 'relatively small' budget-balanced transfers between players which leave the set of contributors unchanged. The findings show that transfers to consumers with the lowest Bonacich centrality increased welfare, by simultaneously reducing aggregate public good provision and increasing aggregate consumption. Low Bonacich centrality consumers who are, in the case of strategic substitutes, the most central consumers in the network, are actually those who could free-ride the most by virtue of their network position. When they provide relatively more public goods, spillovers increase. So those consumers whose network position means they have the lowest propensity to contribute, are exactly those who should be induced to increase their provision.

contributions to the network literature include those by Galeotti et al. (2010); Ghiglino and Goyal (2010); Elliott and Golub (2015); Acemoglu et al. (2016); Kinateder and Merlino (2017); Bourlès et al. (2017) and López-Pintado (2017).

[30]Known also as Gorman (1961) of which Cobb-Douglas is a special case.

3.1.1 Policy reform

This chapter explores the pattern of welfare impacts due to income redistribution with general preferences and a weighted welfare social function. For complete networks, neutrality leads to an invariance of private consumption — as well as welfare — from income redistribution. Beyond complete networks, neutrality does not hold generally, so this type of redistribution does have welfare implications. This insight can be used to illuminate the optimal direction of policy reform in the tradition of Dixit (1975), Guesnerie (1977), Weymark (1981) and Ahmad and Stern (1984). In our case policy reform consists of infinitesimally and relatively small budget-balanced transfers between players which leave the set of contributors unchanged.

This chapter offers two main contributions. Firstly, we provide a new perspective on the neutrality result by showing that it corresponds to a change of direction in the policy impact. To do so, we show that a transfer affects each consumer only insofar as it affects the consumer's neighbourhood. That is, it is the aggregate transfer to the consumer's neighbourhood, rather than the individual transfer to the consumer, that affects consumption. As a consequence, we identify the -1 eigenvalue, as not only the condition for neutral transfers, but also the switch point of the impact of a transfer on each consumer's neighbourhood, and therefore as key to the outcome of income redistribution.

Secondly, we characterise two mutually exclusive cases — either there is a Pareto-improving income redistribution, or if not, we can identify the implicit welfare weights of the initial private provision equilibrium. As a consequence, our policy reform analysis leads to a full characterisation of the welfare impact of infinitesimally and relatively small income redistribution between contributors to local public goods.

Finally we illustrate these scenarios with two core-periphery networks and establish rather surprising results: although consumers' provision levels may respond positively when they receive transfers, the direction of consumption changes may not be in the same direction as the transfer. Whether transfers and consumption/utility move in the same direction or not depends on the architecture of the network. This is because a consumer's utility indirectly depends on their *'social wealth'*, which includes not only an individual's own income but also the total public good provision of his neighbours. We identify cases where the recipient of the welfare-improving transfer is made individually worse off: a type of *'transfer paradox'* (Leontief, 1936; Samuelson, 1952; Yano, 1983; Balasko, 2014; Rasmusen and Kang, 2016). We identify examples of both strong and weak transfer paradoxes, and find a particular case of networks where there is a Pareto improvement that does not depend on preferences but only on the structure of the network.

The chapter is structured as follows. Section 3.2 sets out the general model, and Section 3.3 describes results linked to neutrality. Section 3.4 describes the social welfare implications of income redistribution, Section 3.5 provides some examples of the results, and Section 3.6 concludes.

3.2 The general model

We consider a society comprising n consumers who occupy the nodes of a fixed network $\mathbf{g}$ of social interactions. Let $\mathbf{G} = [g_{ij}]$ denote the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} = 1$ indicates that consumer $i \neq j$ are neighbours and $g_{ij} = 0$ otherwise. The adjacency matrix of the network, $\mathbf{G}$, is symmetric with non-negative entries and therefore has a complete set of real eigenvalues (not necessarily distinct), denoted by $\lambda_{\max}(\mathbf{G}) = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = \lambda_{\min}(\mathbf{G})$, where $\lambda_{\max}(\mathbf{G})$ is the largest eigenvalue and $\lambda_{\min}(\mathbf{G})$ is the lowest eigenvalue of $\mathbf{G}$. By the Perron–Frobenius Theorem, it holds that $\lambda_{\max}(\mathbf{G}) \geq -\lambda_{\min}(\mathbf{G}) > 0$.

Consumer i's neighbours in the network $\mathbf{g}$ are given by $\mathcal{N}_i$. The preferences of each consumer i are represented by a twice continuously differentiable, strictly increasing, and strictly quasi-concave utility function $u_i(x_i, q_i + Q_{-i})$, where x_i is consumer i's private good consumption, q_i is consumer i's public good provision, and $Q_{-i} = \sum_{j \in \mathcal{N}_i} q_j$ is the sum of public good provisions of consumer i's neighbours in the society. Furthermore, the public good can be produced from the private good via a unit-linear production technology. That is, any non-negative quantity of the private good can be converted into the same quantity of the public good. For simplicity, the prices of the private good and the public good can be normalised to $\mathbf{p} = (p_x, p_Q) = (1, 1)$. Each consumer i faces the utility maximisation problem

$$\max_{x_i, q_i} u_i(x_i, q_i + Q_{-i}) \tag{3.2.1}$$

$$\text{s.t.} \quad x_i + q_i = w_i \text{ and } q_i \geq 0,$$

where w_i is his income (exogenously fixed). The utility maximisation problem can be represented equivalently as

$$\max_{x_i, Q_i} u_i(x_i, Q_i) \tag{3.2.2}$$

$$\text{s.t.} \quad x_i + Q_i = w_i + Q_{-i} \text{ and } Q_i \geq Q_{-i},$$

where consumer i chooses his (local) public good consumption, $Q_i = q_i + Q_{-i}$. Let γ_i be the Engel curve of consumer i.[31] Then consumer i's local public good demand depends on $w_i + Q_{-i}$, each player's 'social wealth' (Becker, 1974):

$$Q_i = \max\{\gamma_i(w_i + Q_{-i}), Q_{-i}\},$$

or, equivalently,

$$q_i = Q_i - Q_{-i} = \max\{\gamma_i(w_i + Q_{-i}) - Q_{-i}, 0\}. \tag{3.2.3}$$

Definition 12. Network normality. (Allouch, 2015) For each consumer $i = 1, \ldots, n$, the Engel curve γ_i is differentiable and it holds that $1 + \frac{1}{\lambda_{\min}(\mathbf{G})} < \gamma_i'(\cdot) < 1$.

Theorem 3.2.1. *(Allouch, 2015) Assume network normality. Then there exists a unique Nash equilibrium in the private provision of public goods on networks.*

[31] We could also find the demand curve, which would include prices in the function, but here prices are unchanged at 1.

3.2.1 Income redistribution in general networks

This section investigates the impact of a social planner's intervention on the private provision of public goods. The social planner aims to achieve socially optimal outcomes by drawing on income redistribution as a policy instrument. Income redistribution takes the form of lump-sum transfers, which are traditionally viewed as a benchmark for other policy instruments. Like Warr (1983) and BBV we focus our analysis on income redistributions that leave the set of contributors unchanged, referring to them as 'relatively small'.

In general, there are compelling reasons for presuming that not all consumers will be contributing to public goods. In the following, for simplicity of notations, we will focus our analysis on just one component of contributors by passing to the subnetwork induced by the component.[32]

Let $\mathbf{q}^* = (q_1^*, \ldots, q_n^*)$ be the Nash equilibrium associated with $\mathbf{w} = (w_1, \ldots, w_n)$ and let $\mathbf{t} = (t_1, \ldots, t_n)$ denote a (budget-neutral) income transfer, that is,

$$\sum_{i=1}^{n} t_i = 0.$$

Where transfers could be a tax ($t_i < 0$) or a subsidy ($t_i \geq 0$). Let $\mathbf{q}^t = (q_1^t, \ldots, q_n^t)$ be the Nash equilibrium after an income transfer $\mathbf{t}$, that is, the Nash equilibrium corresponding to the income distribution $\mathbf{w} + \mathbf{t} = (w_1 + t_1, \ldots, w_n + t_n)^T$. We say that a transfer $\mathbf{t}$ is 'neutral' if for each i

$$(x_i^{\mathbf{t}}, Q_i^{\mathbf{t}}) = (x_i^*, Q_i^*).$$

[32]Note that if we pass our analysis to several components of contributors, while we can fully characterise the provision of public goods, we can no longer consider the consumption or welfare of non-contributors.

The question of neutrality has been to a large extent settled for *pure public goods* by the neutrality result of Warr (1983) and BBV. Their neutrality result shows that contributors exactly offset their public good provision by the value of the transfer so that for each consumer i it holds that $q_i^{\mathbf{t}} - q_i^* = t_i$, leading to an unchanged private good consumption:

$$x_i^{\mathbf{t}} = w_i + t_i - q_i^{\mathbf{t}} = w_i - q_i^* = x_i^*$$

and an unchanged public good consumption:

$$Q^{\mathbf{t}} = \sum_{i=1}^{n} q_i^{\mathbf{t}} = \sum_{i=1}^{n} q_i^* + \sum_{i=1}^{n} t_i = Q^*.$$

We examine the case of general networks where, rather than pure public goods, players only benefit from their neighbours' public good provision. Pure public goods are equivalent to the special case of a complete network where everyone is connected to everyone else.

Proposition 11. *Assume network normality and that all consumers are contributors. Then, for any relatively small transfer* $\mathbf{t}$*, the change in consumers' private and public good consumption after the transfer are given by:*

$$\mathbf{Q^t} - \mathbf{Q}^* = (\mathbf{A}^{-1} - \mathbf{I})[\mathbf{x^t} - \mathbf{x}^*] = (\mathbf{A}^{-1} - \mathbf{I})(\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\ \mathbf{t}$$

Proof. See Appendix. ∎

Proposition 11 shows that a transfer impacts each consumer's consumption of either the public or private good only insofar as it impacts his neighbourhood,

$i \cup \mathcal{N}_i$. Let us denote $(\mathbf{I}+\mathbf{G})\mathbf{t}$ as the *'neighbourhood transfer'*: the aggregate tax/transfer in the neighbourhood of each consumer i. The neighbourhood transfer determines how transfers impact consumers in terms of their consumption of either the public or private good. In particular, it follows that

Corollary 2. *A transfer* $\mathbf{t}$ *is neutral if and only if*

$$(\mathbf{I}+\mathbf{G})\ \mathbf{t} = 0.$$

Corollary 2 shows that a transfer $\mathbf{t}$ is neutral if and only if $\mathbf{t}$ is an eigenvector[33] with a corresponding eigenvalue of -1. This is because in this case $(\mathbf{I}+\mathbf{G})\mathbf{t} = (1+\lambda_i)\mathbf{t} = 0$ and hence the neighbourhood transfer is null for each consumer.

More generally, it follows from Proposition 11 that if the transfer is an eigenvector corresponding to the eigenvalue of λ_i, then the neighbourhood transfer follows the same direction as the individual transfer if $\lambda_i > -1$, and is in the opposite direction if $\lambda_i < -1$. Therefore, the neighbourhood transfer changes from one direction to another depending on the eigenvalue λ_i, and the point at which the direction switches is the -1 eigenvalue. The point of policy neutrality is also a change of direction of policy impact.

Observe that we cannot say whether the eventual impact on consumption of public and private goods will be positive or negative, since the direction of the impact will also depend on the matrix $(\mathbf{I}+\mathbf{AG})^{-1}\mathbf{A}$. In the simple case of Cobb-Douglas preferences where $u_i(x_i, Q_i) = x_i^a Q_i^{1-a}$, we have the following result.

[33] As $\mathbf{G}$ is a symmetric square matrix, then its eigenvectors are orthogonal and provide a basis for the space $\mathbb{R}^n$. This means that the vectors of possible transfers $\mathbf{t}$ can be associated with any of the eigenvectors of $\mathbf{G}$.

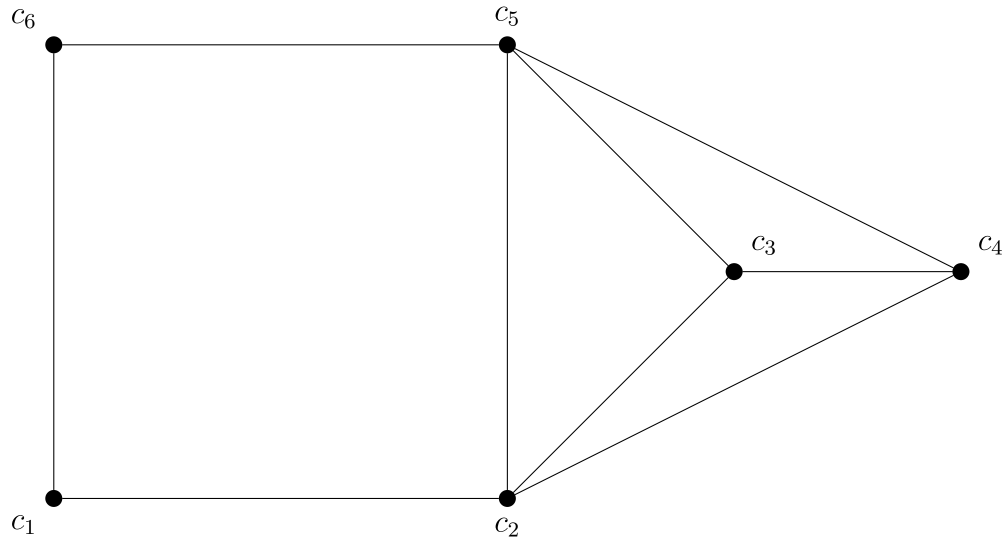


Figure 3.1: A network $\mathbf{g}$ with six nodes

Corollary 3. *Assume Cobb-Douglas preferences. If the transfer* $\mathbf{t}$ *is an eigenvector of the adjacency matrix* $\mathbf{G}$ *associated with eigenvalue* λ_i*, then it holds that*

$$\mathbf{Q^t} - \mathbf{Q}^* = \frac{1-a}{a}\,(\mathbf{x^t} - \mathbf{x}^*) = (1-a)\,(\mathbf{I} + a\mathbf{G})^{-1}(\mathbf{I} + \mathbf{G})\ \mathbf{t} = (1-a)\frac{1+\lambda_i}{1+a\lambda_i}\ \mathbf{t}.$$

Having assumed network normality throughout the chapter, we note that this is equivalent to $a \in]0, -\frac{1}{\lambda_{\min}(\mathbf{G})}[$ in the C-D case, which ensures that the denominator is positive. This means that if the transfer $\mathbf{t}$ is an eigenvector of the adjacency matrix with a corresponding eigenvalue λ_i, then the impact of the transfer on each consumer is in the same direction of the transfer if $\lambda_i > -1$ and is in the opposite direction if $\lambda_i < -1$.

3.2.2 Example

We can illustrate this result with an example network, shown in Figure 3.1. Some of the eigenvalues and eigenvectors of the network are below.

$$\lambda_6 = -2, \quad \lambda_4 = -1, \quad \lambda_3 = 0$$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

If the budget-balanced transfers $\mathbf{t}$ have directions $\mathbf{t} = (1, 1, 0, 0, -1, -1)$ corresponding to the eigenvalue $\lambda_3 = 0$, we can find the neighbourhood transfer by applying $(\mathbf{I} + \mathbf{G})\mathbf{t} = (1 + \lambda_3)\mathbf{t} = \mathbf{t}$: it is the same. On the other hand if $\mathbf{t} = (0, 0, 1, -1, 0, 0)$, corresponding to the eigenvalue $\lambda_6 = -2$, then when we apply $(\mathbf{I} + \mathbf{G})\mathbf{t} = (1 + \lambda_6)\mathbf{t} = -\mathbf{t}$, we find that the neighbourhood transfer is exactly opposite to the individual transfers. Finally the effect of $\mathbf{t} = (0, 0, 1, -1, 0, 0)$ corresponding to the eigenvalue $\lambda_4 = -1$ is given by $(\mathbf{I} + \mathbf{G})\mathbf{t} = (1 + \lambda_4)\mathbf{t} = \mathbf{0}$: neutrality.

3.3 Neutrality space

Next we investigate how the network structure affects policy reform in networks.

3.3.1 A spectral characterisation

Consider first the linear subspace of budget-balanced transfers, the **transfer space** $\mathcal{T}$, which has dimension $n-1$:

$$\mathcal{T} = \{\mathbf{t} \in \mathbb{R}^n \mid \mathbf{1} \cdot \mathbf{t} = 0\} = \mathbf{1}^{\perp}. \tag{3.3.1}$$

As we noted in Corollary 2, a vector of neutral transfers are the eigenvectors associated with the -1 eigenvalue. Since transfers are budget-balanced, that is, orthogonal to the vector of all ones, $\mathbf{1}$, this means we can characterise the space of neutral transfers as follows:

Definition 13. The **neutrality space** $\mathcal{K}$ denotes the space of neutral transfers.

$$\mathcal{K} = \mathrm{Eig}(-1) \cap \mathbf{1}^{\perp}.$$

Roughly speaking, the neutrality space represents the policy constraints faced by the social planner due to the network structure. In particular, the dimension of the neutrality space tells us how many dimensions of potential transfers are neutral. The link between the budget balance requirement and the vector $\mathbf{1}$ highlights the role of main eigenvalues in our analysis. A main eigenvalue is an eigenvalue which has an associated eigenvector not orthogonal to the vector $\mathbf{1}$ (Cvetkovic, 1970), which are useful in our setting because the balanced budget means that the transfer space is orthogonal to $\mathbf{1}$. The distinct main eigenvalues of $\mathbf{G}$ form the main part of the spectrum, denoted by $\mathcal{M}$ (Harary and Schwenk, 1979).[34]

[34] By the Perron–Frobenius Theorem, the maximum eigenvalue of $\mathbf{G}$ has an associated

Proposition 12. *Let $\psi(\mathbf{G})$ be the multiplicity of the -1 eigenvalue in the spectrum of* $\mathbf{G}$*. Then we have that*

- $-1 \in \mathcal{M}$*:* $dim(\mathcal{K}) = \psi(\mathbf{G}) - 1$
- $-1 \notin \mathcal{M}$*:* $dim(\mathcal{K}) = \psi(\mathbf{G})$

Proof. Following from the discussion above, if -1 is a (distinct) main eigenvalue, its eigenvector will not appear in the neutrality space. Therefore $\dim(\mathcal{K})$ has the dimension of the eigenspace of the -1 eigenvalue(s) that is orthogonal to $\mathbf{1}$. ∎

3.3.2 A structural characterisation

In the following we examine neutrality from a network architecture standpoint.

Definition 14 (Neighbourhood-homogenous[35] subset)**.** A subset of consumers S is 'neighbourhood homogenous' if for any i, j in S it holds that

$$i \cup \mathcal{N}_i = j \cup \mathcal{N}_j.$$

For a network with S connected consumers who have the same neighbourhood, the neighbourhood-homogenous subset has size $|S|$. Clearly, transfers between such consumers will be neutral since members of S are not only fully connected, but also indistinguishable in terms of their network position from the consumers in $N \setminus S$. In the matrix $\mathbf{I} + \mathbf{G}$, the $|S|$ rows and columns referring to the neighbourhood-homogenous nodes are identical. This means that any binary transfer that redistributes between two members of S is an eigenvector

eigenvector with all its entries positive and, therefore, is a main eigenvalue.

[35]For examples of neighbourhood-homogenous subsets, see Allouch (2015).

corresponding to the eigenvalue -1 and hence neutral. If $|S| = 2$, there is one neutral transfer available: between the pair; if $|S| = 3$ there are two pairwise transfers from which any transfer vector involving members of S can be constructed; and so on. So the dimension of neutral transfers that corresponds to the neighbourhood-homogenous subset S is $|S| - 1$.

This means we have that $|S| - 1 \leq \dim(\mathcal{K})$, showing us another way to find that all transfers in a complete network are neutral. In this case, $|S| = n$ and so $n - 1 \leq \dim(\mathcal{K})$. Since the maximum dimension of the neutrality space ($\mathcal{K}$) is $n - 1$ (the dimension of the transfer space), all transfers must be neutral in complete networks.

3.3.3 Other neutral transfers

Of course, there could be other neutral transfers in the network as well as those between members of a neighbourhood-homogenous subset. We have already shown it suffices for a transfer to be an eigenvector corresponding to the -1 eigenvalue to be neutral, or equivalently, that the sum of transfers within each neighbourhood is zero. An example of a neutral transfer without neighbourhood-homogeneity is given in Figure 3.1. As described previously, neutral transfers correspond to the eigenvector of the -1 eigenvalue; the eigensystem of this network is

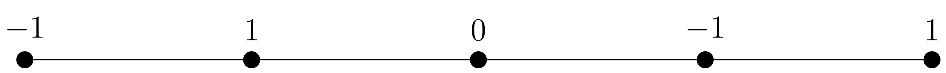


Figure 3.1: Neutral transfer but no subset is neighbourhood-homogenous

$$\lambda_5 = -\sqrt{3}, \quad \lambda_4 = -1 \quad \lambda_3 = 0, \quad \lambda_2 = 1, \quad \lambda_1 = \sqrt{3}$$

$$\begin{pmatrix} 1 \\ -\sqrt{3} \\ 2 \\ -\sqrt{3} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \sqrt{3} \\ 2 \\ \sqrt{3} \\ 1 \end{pmatrix}$$

Here we can observe that the transfer shown in Figure 3.1 is neutral, since it is associated with the -1 eigenvalue. However, this transfer is not within a subset of nodes that is neighbourhood-homogenous.[36]

Observe that a key policy implication of investigating neutrality in general networks is that two seemingly unrelated transfers are policy equivalent (that is, they lead to the same change in terms of private and public goods consumption) if and only if their difference is a neutral transfer.[37] For instance, the two transfers in Figure 3.2 are policy equivalent, that is, they lead to the same change in consumption, since their difference is the neutral transfer given in Figure 3.1.

3.4 Welfare impact of income redistribution

Having found that income redistribution can affect consumption, let us further examine the potential welfare impact. Let v_i^* denote the indirect utility

[36] Actually, there is no neighbourhood-homogenous subset in this network.

[37] Previous analysis in the case of pure public goods has mostly exploited the neighbourhood-homogenous property of complete networks. That is, a redistribution between a subset of contributors that leaves the set of contributors unchanged is neutral.

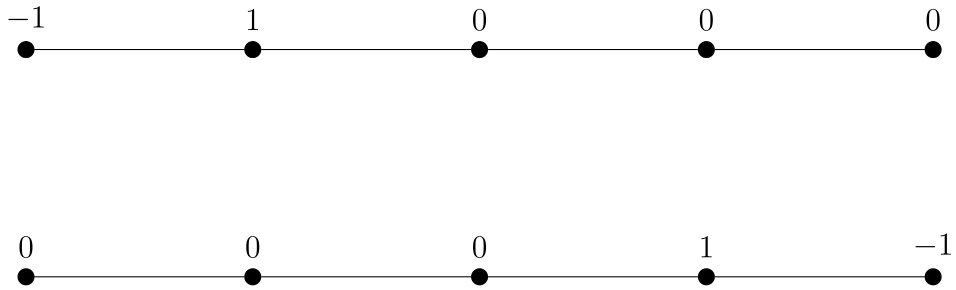


Figure 3.2: Policy equivalent transfers

function of the consumer i corresponding to the unique Nash equilibrium at the initial equilibrium, when the income distribution is $\mathbf{w} = (w_1, \ldots, w_n)^T$. To be able to approximate welfare changes, we will focus our analysis on transfers that are infinitesimal. The vector of indirect utilities of all consumers at the initial equilibrium is $\mathbf{v}^* = (u_1(x_1^*, Q_1^*)). \ldots, u_n(x_n^*, Q_n^*))$ and $\mathbf{v}^{\mathbf{t}} = (u_1(x_1^{\mathbf{t}}, Q_1^{\mathbf{t}})). \ldots, u_n(x_n^{\mathbf{t}}, Q_n^{\mathbf{t}}))$ after transfer.

Proposition 13. *The change in consumers' utilities following an infinitesimal transfer* $\mathbf{t}$ *is*

$$\Delta \boldsymbol{v}(\boldsymbol{t}) = \mathbf{v}^{\mathbf{t}} - \mathbf{v}^* \approx \mathbf{B}_{\mathbf{v}} \ (\mathbf{I} + \mathbf{G}) \ \mathbf{t}, \tag{3.4.1}$$

where $\mathbf{B}_{\mathbf{v}}$ *is an invertible matrix.*

Proof. See Appendix. ■

Let the *impact space* $\mathcal{V}$ be the space of utility changes that can be achieved with budget-neutral transfers

$$\mathcal{V} = \{\Delta \mathbf{v}(\mathbf{t}) \in \mathbb{R}^n \mid \Delta \mathbf{v}(\mathbf{t}) = \mathbf{B}_{\mathbf{v}}(\mathbf{I} + \mathbf{G})\mathbf{t} \text{ for a } \ \mathbf{t} \in \mathcal{T}\}. \tag{3.4.2}$$

Proposition 14. *The dimension of the impact space is the dimension of the* *transfer space reduced by the dimension of the neutrality space:* $\dim(\mathcal{V}) = n - 1 - \dim(\mathcal{K})$.

Proof. We know that $\dim(\mathcal{V}) \leq n-1$ since $\mathcal{V}$ is a transformation of $\mathcal{T}$, which has dimension $n-1$ (due to budget balance). Next, we see that the extent to which the dimension of $\mathcal{V}$ is further reduced with respect to $\mathcal{T}$ is equal to the dimension of the neutrality space $\mathcal{K}$, since transfers in this space are neutral and, hence, can have no effect on utility. ∎

Proposition 14 shows that even when neutrality does not hold, the network architecture still determines, and actually reduces, the dimension of the impact space.

Corollary 4. *If a subset S is neighbourhood-homogenous then* $\dim(\mathcal{V}) \leq n - |S|$.

Proof. As discussed above we have that $|S| - 1 \leq \dim(\mathcal{K})$. Combining this with Proposition 14 gives the result. ∎

We can also observe that if the public good is pure then $\dim(\mathcal{V}) = 0$. This is because $|S| = n$ in a complete network, the case for pure public goods. As we mentioned before, we get the same result via the spectral characterisation.

In order to investigate the welfare impact of income distribution, we define a weighted utilitarian social welfare function, with general welfare weights given by $\mathbf{r} \in \mathbb{R}^n_{++}$.

$$\mathcal{S}W^*_{\mathbf{r}} \stackrel{\text{def}}{=} \sum_i r_i v^*_i = \mathbf{r} \cdot \mathbf{v}^* \tag{3.4.3}$$

And the overall welfare effect of a set of transfers $\mathbf{t}$ is given by

$$\Delta \mathcal{S}W_{\mathbf{r}_M}(\mathbf{t}) = \mathcal{S}W_{\mathbf{r}}^{\mathbf{t}} - \mathcal{S}W_{\mathbf{r}}^{*} \quad = \quad \mathbf{r} \cdot \Delta \mathbf{v}(\mathbf{t}) \tag{3.4.4}$$

3.4.1 Pareto-improving reform

Next we investigate whether a Pareto-improving income redistribution exists.

Proposition 15. *There are two mutually exclusive possibilities, (a) and (b).*

(a) There exists a weakly Pareto-improving infinitesimal transfer $\mathbf{t}_0$*, that is,*

$$\Delta \mathbf{v}(\mathbf{t}_0) \in \mathbb{R}_+^n \quad \textit{for} \quad \mathbf{t} \in \mathcal{T}$$

(b) There exist positive social welfare weights which render any policy change welfare-neutral, that is, weights $\mathbf{r}_M \in \mathbb{R}_{++}^n$ *such that*

$$\Delta \mathcal{S}W_{\mathbf{r}_M}(\mathbf{t}) = \mathbf{r}_M \cdot \Delta \mathbf{v}(\mathbf{t}) = 0, \quad \textit{for all} \quad \mathbf{t} \in \mathcal{T}$$

Proof. Corollary $3'$ of Ben-Israel (1964) states that, if L and $L^{\perp}$ are complementary orthogonal subspaces in $\mathbb{R}^n$, then the following two statements are equivalent:

$$L \cap \mathbb{R}_+^n = \{0\} \text{ and } L^{\perp} \cap \mathbb{R}_{++}^n \neq \varnothing$$

Therefore, given $\mathcal{V}$ is a linear subspace, there are two mutually exclusive possibilities, (a) and (b):

$$\begin{aligned} (a) \quad & \mathcal{V} \cap \mathbb{R}_+^n \neq \{0\} \iff \mathcal{V}^{\perp} \cap \mathbb{R}_{++}^n = \varnothing \\ (b) \quad & \mathcal{V} \cap \mathbb{R}_+^n = \{0\} \iff \mathcal{V}^{\perp} \cap \mathbb{R}_{++}^n \neq \varnothing \end{aligned}$$

If (a) then there exist weakly Pareto-improving transfers in the subspace $\mathcal{V}$.

If (b) then weakly Pareto-improving transfers do not exist, and $\mathcal{V}^{\perp}$ is the linear space of strictly positive welfare weights $\mathbf{r}$ that are orthogonal to $\Delta\mathbf{v}(\mathbf{t})$, giving $\Delta\mathcal{S}W_{\mathbf{r}}(\mathbf{t}) = \mathbf{r}\cdot\Delta\mathbf{v}(\mathbf{t}) = 0$ and thereby rendering any policy change neutral in social welfare terms. ■

Proposition 15 shows that either a strict Pareto-improving income redistribution can be found, or else the initial private provision equilibrium is an optimum (amongst the private provision equilibria achieved by infinitesimal income distributions). In the case of no possible Pareto improvement, the social welfare weights, also known as *Motzkin weights*,[38] represent the implicit welfare weights at the initial equilibrium.

The intuition behind the proof of Proposition 15 is as follows: observe that the impact space $\mathcal{V}$ determines whether or not a Pareto improvement is possible with infinitesimal transfers in any network. If $\mathcal{V}$ intersects the non-negative orthant then a Pareto improvement is possible, giving the list of non-negative utility changes that result from the transfer. If $\mathcal{V}$ does not intersect the non-negative orthant, then a Pareto improvement is not possible. At the same time $\mathcal{V}^{\perp}$ will intersect the positive orthant, giving the positive implicit welfare weights of the initial equilibrium.

Note that this terminology is not the same as in the previous discussion. Here we use the term 'welfare-neutral' to distinguish it from the concept of 'neutrality' that we have been discussing so far. Welfare-neutrality is an aggregate concept,

[38] The name of *Motzkin weights* originates from the application of Motzkin's Theorem of the Alternative to find Pareto-improving income redistributions for which income redistribution is welfare-neutral. These weights also provide the *inverse optimum* (Ahmad and Stern, 1984; Dixit, 1975).

where there could be changes to individual consumption but these are offset by other consumers' consumption in the opposite direction, leaving aggregate welfare unchanged. On the other hand neutrality means that consumption is unchanged. Neutrality is sufficient but not necessary for welfare-neutrality to hold.

3.5 Welfare in example networks

We now examine some example networks, using the simple case of Cobb-Douglas preferences where $u_i(x_i, Q_i) = x_i^a Q_i^{1-a}$. As we will see, in view of the homogeneity of preferences, the feasibility of welfare-improving reform will depend greatly on the network structure.

Proposition 16. *In the Cobb-Douglas case, the change in consumers' utility from an income redistribution* $\mathbf{t}$ *is as follows, where* $\alpha = a^a(1-a)^{(1-a)}$

$$\mathbf{v}^{\mathbf{t}} - \mathbf{v}^* = \alpha(\mathbf{I} + a\mathbf{G})^{-1}(\mathbf{I} + \mathbf{G})\mathbf{t}. \tag{3.5.1}$$

The overall welfare effect of transfers $\mathbf{t}$ *in the C-D case is given by*

$$\begin{aligned} \Delta \mathcal{S}W_{\mathbf{r}}(\mathbf{t}) &= \mathbf{r} \cdot \Delta \mathbf{v}(\mathbf{t}) & (3.5.2) \\ &= \alpha \mathbf{r}^T(\mathbf{I} + a\mathbf{G})^{-1}(\mathbf{I} + \mathbf{G})\mathbf{t}. & (3.5.3) \end{aligned}$$

Proof. See Appendix. ■

Given this expression, we can now easily find either the vector of non-negative utility changes in $\mathcal{V}$ that characterise a Pareto improvement due to transfers, or the Motzkin weights (defined earlier) in $\mathcal{V}^{\perp}$, the normal to the impact space. In particular,

Corollary 5. *With Cobb-Douglas preferences, if* -1 *is a main eigenvalue then there exists a Pareto improvement.*

Proof. See Appendix. ■

Next we look at some example networks; of particular interest are core-periphery networks, defined by Borgatti and Everett (2000). In core-periphery networks there are two groups of consumers, the centre C and the periphery P. Nodes in C are completely connected to each other, and may be connected to nodes in P, while nodes in P are only connected to nodes in C and not connected to each other. Networks of this type have many applications such as information-sharing, where consumers in the core share the cost of information collection, which is then accessed by the periphery (Galeotti and Goyal, 2010). Other examples could be spatial or geographical, for example central and suburban residences in a town or city, or financial, such as liquidity networks (Elliott et al., 2014).

3.5.1 Star network

The simplest core-periphery network has only one central consumer: the star network. The star network with two periphery consumers is shown in Figure 3.1

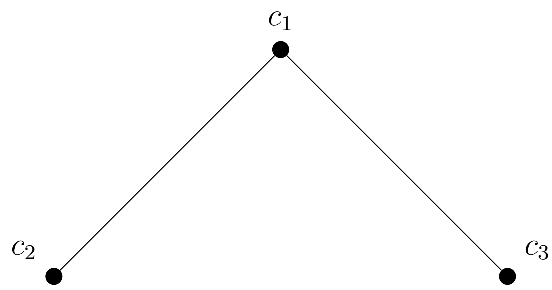


Figure 3.1: The star network of three nodes

and the eigensystem is

$$\lambda_3 = -\sqrt{2}, \quad \lambda_2 = 0, \quad \lambda_1 = \sqrt{2}$$
$$\begin{pmatrix} -\sqrt{2} \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{2} \\ 1 \\ 1 \end{pmatrix}$$

We can observe immediately that there are no neutral transfers in this network, since -1 is not an eigenvalue. Let us designate the two periphery players as 2 and 3 and the central player as 1, with transfers of $\mathbf{t} = (t_1, t_2, t_3)$. Since these must be budget-balanced, let $t_1 = -t_2 - t_3$. Then we can write the utility changes of a budget-balanced transfer regime as

$$\begin{bmatrix} \Delta v_1(\mathbf{t}) \\ \Delta v_2(\mathbf{t}) \\ \Delta v_3(\mathbf{t}) \end{bmatrix} = \frac{1}{1-2a^2} \begin{bmatrix} a(t_2 + t_3) \\ -a^2 t_2 - (1-a^2)t_3 \\ -a^2 t_3 - (1-a^2)t_2 \end{bmatrix} \tag{3.5.4}$$

We can make two observations from (3.5.4). Firstly, if transfers to the periphery players are positive, this *increases* the utility of the central player and decreases the utility of the periphery. For each consumer, transfers to themselves have a negative effect on their utility (for player 1, this effect can be written as $-at_1$). This is called a *transfer paradox* because the effect on utility is opposite to the direction of transfer. In fact, this is a *strong transfer paradox* because the opposite direction of change occurs for both players — those who give and receive transfers.

The intuition for the transfer paradox is as follows. As shown previously, the change in utility of each agent is an increasing function of their social wealth, $w_i + Q_{-i}$. So there are two ways that utility is affected by transfers. First, at the new equilibrium after transfers to the centre, wealth w_i changes by t_i and so the centre has more private wealth and the periphery has less. Second, neighbourhood public good provision Q_{-i} is also different at the new equilibrium. The periphery agents contribute less public good at the new equilibrium, while the centre increases his provision. This means that the neighbourhood provision of the periphery $Q_{-2} = Q_{-3} = q_1$ increases, and that of the centre $Q_{-1} = q_2 + q_3$ decreases. So for both periphery and centre, the two effects of the transfers on social wealth $w_i + Q_{-i}$ move in different directions. In the case of the star, the net effect on the central agent's social wealth is negative, and on the periphery agents is positive: hence there is a strong transfer paradox.

Secondly, (3.5.4) shows us that there is no Pareto improvement available in the star network, since we can observe that it is not possible that all three entries in the vector of utilities could be positive, for any budget-balanced transfers. This means that we can find the Motzkin weights as the normal to the impact space: these are $(r_1^M, r_2^M, r_3^M) = (1, a, a)$.

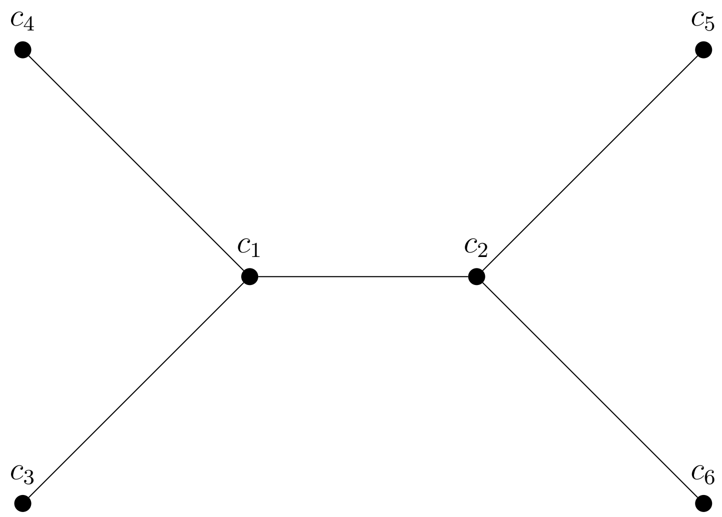


Figure 3.2: The balanced network of six nodes

3.5.2 Balanced network

Our second example is the larger core-periphery network shown in Figure 3.2. The eigensystem is

$$\begin{array}{cccccc} \lambda_6 = -2, & \lambda_5 = -1, & \lambda_4 = 0 & \lambda_3 = 0, & \lambda_2 = 1, & \lambda_1 = 2 \\ \begin{pmatrix} -2 \\ 2 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, & \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}, & \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, & \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \end{array}$$

We observe from the eigensystem that although there is a -1 eigenvalue, this does not allow the possibility of neutral transfers because the associated eigenvector is not budget neutral.[39] So there are no neutral transfers in this network. The neutrality space has dimension zero, and that of the impact space is its maximum at $n-1=5$.

As there are no neutral transfers, this means that any budget-balanced transfers will have an impact on individual utilities. There are two possibilities. If all the utility changes are non-negative this would be a Pareto improvement. Necessarily, the aggregate effect on welfare would be non-negative. On the other hand, if some of the utility changes are negative and some positive, then the overall impact could be welfare-neutral if the utility changes are weighted by the Motzkin weights, in which case the positive and negative impacts offset each other.

We are interested in whether a Pareto improvement exists or not, and we can find an example of a Pareto improvement in the simple case that follows. First we designate players 1 and 2 as the central nodes and $3, 4, 5$ and 6 as the periphery nodes. Let transfers to the centre be $t_c = t_1 = t_2$ and transfers to the periphery be $t_p = t_3 = t_4 = t_5 = t_6$. Budget balance requires that $t_c = -2t_p$. Then the utility changes of the central nodes and periphery nodes, $\Delta v_c(\mathbf{t})$ and $\Delta v_p(\mathbf{t})$ respectively, are as follows.

$$\begin{bmatrix} \Delta v_c(\mathbf{t}) \\ \Delta v_p(\mathbf{t}) \end{bmatrix} = \frac{t_c}{1+2a} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \tag{3.5.5}$$

We can observe from (3.5.5) that in this network, there is always a Pareto improvement when transfers to the centre are positive. In particular, we cannot

[39] Observe that -1 is a main eigenvalue as the associated eigenvector is not orthogonal to $\mathbf{1}$.

increase the welfare of the periphery without increasing that of the centre at double the rate. We term the central players in this network 'welfare winners' because they will always gain from any attempt to increase aggregate welfare. In fact, this is an example of a *weak transfer paradox* because some consumers — here the periphery — gains in utility terms even though they are losing in income terms by paying a transfer to the centre.

It is interesting to note that the 'balanced' network in Figure 3.2 can be formed by joining together the centre nodes of two of the star networks shown in Figure 3.1. In this case, the centre is no longer worse off at the new equilibrium - we have a Pareto improvement and a weak rather than strong transfer paradox. This is because of the impact of the transfers on social wealth. With the balanced network, the centre's neighbourhood provision is $Q_{-c} = q_c + 2q_p$, whereas in the star network it is $Q_{-c} = 2q_p$. The fact that the centre nodes in the balanced network can benefit from each others' increased public good provision means that their utility increases in the new equilibrium after transfers, leading to a Pareto improvement.

It is interesting to note that the direction of utility changes are not affected by preferences via the parameter a. And since there are Pareto-improving transfers for this network, the Motzkin weights do not exist.

3.6 Conclusion

We investigate the welfare impact of lump-sum income redistribution on individuals who privately provide public goods in networks. We fully characterise the possible directions of optimal policy reform by using general welfare weights. As usual, understanding the architecture of the network has important implications for policy reform in networks. We find that neutrality is associated with the eigenvalue -1 and leads to a change of sign of policy. Unlike Allouch (2015) and Allouch (2017) which focus on network architectures that are not amenable to a neutrality (-like) results in the tradition of BBV, our analysis shows that even when neutrality fails to hold, the impact of income redistribution is still hampered by the architecture. We find that income redistributions in networks with local public goods can be Pareto-improving, and if not, it identifies implicit social welfare weights of the initial equilibrium, invoking Motzkin's Theorem of the Alternative. Examples of core-periphery networks illustrate some features of the proposed transfer regime, including the possibility of a transfer paradox and welfare winners.

Appendix

3.A Proofs

3.A.1 Proof of Proposition 11

From Allouch (2015) we have that [40]

$$\mathbf{q^t} - \mathbf{q}^* = (\mathbf{I} + \mathbf{AG})^{-1}(\mathbf{I} - \mathbf{A})\mathbf{t} \tag{3.A.5}$$

[40]The proof in Allouch (2015) is as follows. From (3.2.3), it follows that for each consumer $i \in C$

$$\begin{aligned} q_i^{\mathbf{t}} - q_i^* &= (\gamma_i(w_i + t_i + Q_{-i}^{\mathbf{t}}) - Q_{-i}^{\mathbf{t}}) - (\gamma_i(w_i + Q_{-i}^*) - Q_{-i}^*) && (3.A.1)\\ &= \gamma_i(w_i + t_i + Q_{-i}^{\mathbf{t}}) - \gamma_i(w_i + Q_{-i}^*) - (Q_{-i}^{\mathbf{t}} - Q_{-i}^*) && (3.A.2) \end{aligned}$$

From the mean value theorem, we have that $(b-a)f'(c) = f(b) - f(a)$ for $a < b$ and $c \in]a, b[$, when f is continuous and differentiable. So it follows that for each $i \in C$ such that $t_i + Q_{-i}^{\mathbf{t}} \neq Q_{-i}^*$, there exists a real number $\beta_i \in](w_i + Q_{-i}^*), (w_i + t_i + Q_{-i}^{\mathbf{t}})[$ such that

$$\gamma_i'(\beta_i)(t_i + Q_{-i}^{\mathbf{t}} - Q_{-i}^*) = \gamma_i(w_i + t_i + Q_{-i}^{\mathbf{t}}) - \gamma_i(w_i + Q_{-i}^*)$$

On the other hand if for $i \in C$ we have $t_i + Q_{-i}^{\mathbf{t}} = Q_{-i}^*$, this means that $\gamma_i(w_i + t_i + Q_{-i}^{\mathbf{t}}) - \gamma_i(w_i + Q_{-i}^*) = 0$. Let $\beta_i = w_i + Q_{-i}^*$; then for each consumer $i \in C$ it holds that

$$\begin{aligned} q_i^{\mathbf{t}} - q_i^* &= \gamma_i'(\beta_i)(t_i + Q_{-i}^{\mathbf{t}} - Q_{-i}^*) - (Q_{-i}^{\mathbf{t}} - Q_{-i}^*) && (3.A.3)\\ q_i^{\mathbf{t}} - q_i^* &+ (1 - \gamma_i'(\beta_i))(Q_{-i}^{\mathbf{t}} - Q_{-i}^*) = \gamma_i'(\beta_i)t_i && (3.A.4) \end{aligned}$$

Consequently, it holds that

$$(\mathbf{I} + \mathbf{AG})(\mathbf{q^t} - \mathbf{q}^*) = (\mathbf{I} - \mathbf{A})\mathbf{t},$$

where $\mathbf{A} = \mathrm{diag}(1 - \gamma_i'(\beta_i))_{i \in C}$. Applying Lemma 1 (Allouch, 2015) for $\mathbf{B} = \mathbf{A}$ and $\mathbf{U} = \mathbf{I}$, it follows that $\mathbf{I} + \mathbf{AG}$ is invertible since it has positive eigenvalues. Hence, $\mathbf{q^t} - \mathbf{q}^* = (\mathbf{I} + \mathbf{AG})^{-1}(\mathbf{I} - \mathbf{A})\mathbf{t}$.

where $\mathbf{A} = \text{diag}(1 - \gamma_i'(\beta_i))_{i \in N}$ for some β_i and $\mathbf{t} = (t_i)_{i \in N}$. We can now calculate the changes in private and public good consumption from (3.A.5). We know that $\mathbf{x^t} - \mathbf{x}^* + \mathbf{q^t} - \mathbf{q}^* = \mathbf{t}$, so

$$\begin{aligned}
\mathbf{x^t} - \mathbf{x}^* &= \mathbf{t} - (\mathbf{q^t} - \mathbf{q}^*) \\
&= \mathbf{t} - (\mathbf{I} + \mathbf{AG})^{-1}(\mathbf{I} - \mathbf{A})\mathbf{t} \\
&= [\mathbf{I} - (\mathbf{I} + \mathbf{AG})^{-1}(\mathbf{I} - \mathbf{A})]\mathbf{t} \\
&= [(\mathbf{I} + \mathbf{AG})^{-1}(\mathbf{I} + \mathbf{AG}) - (\mathbf{I} + \mathbf{AG})^{-1}(\mathbf{I} - \mathbf{A})]\mathbf{t} \\
&= (\mathbf{I} + \mathbf{AG})^{-1}[(\mathbf{I} + \mathbf{AG}) - (\mathbf{I} - \mathbf{A})]\mathbf{t} \\
&= (\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\mathbf{t} \qquad (3.A.6)
\end{aligned}$$

Meanwhile we also know that $\mathbf{Q^t} - \mathbf{Q}^* = (\mathbf{I} + \mathbf{G})(\mathbf{q^t} - \mathbf{q}^*)$ so we have that

$$\begin{aligned}
\mathbf{Q^t} - \mathbf{Q}^* &= (\mathbf{I} + \mathbf{G})(\mathbf{q^t} - \mathbf{q}^*) = (\mathbf{I} + \mathbf{G})[\mathbf{t} - (\mathbf{x^t} - \mathbf{x}^*)] \\
&= (\mathbf{I} + \mathbf{G})[\mathbf{t} - (\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\mathbf{t}] \\
&= (\mathbf{I} + \mathbf{G})\mathbf{t} - (\mathbf{I} + \mathbf{G})(\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\mathbf{t} \\
&= [\mathbf{I} - (\mathbf{I} + \mathbf{G})(\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}](\mathbf{I} + \mathbf{G})\mathbf{t} \\
&= [\mathbf{A}^{-1}(\mathbf{I} + \mathbf{AG}) - (\mathbf{I} + \mathbf{G})](\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\mathbf{t} \\
&= [\mathbf{A}^{-1} - \mathbf{I}](\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\mathbf{t} \quad \text{as required.}
\end{aligned}$$

3.A.2 Proof of Proposition 13

First, from the proof of Proposition 11, we have that

$$\mathbf{Q}_i^{\mathbf{t}} - \mathbf{Q}_i^* = [\mathbf{A}^{-1} - \mathbf{I}](\mathbf{x}_i^{\mathbf{t}} - \mathbf{x}_i^*)$$

which implies that, $\forall i$

$$Q_i^{\mathbf{t}} - Q_i^* = \left[\frac{1}{1-\gamma_i'(\cdot)} - 1\right](x_i^{\mathbf{t}} - x_i^*) \tag{3.A.7}$$

Next, we can use (3.A.7) and a Taylor approximation such that:

$$\begin{aligned} u_i(x_i^{\mathbf{t}}, Q_i^{\mathbf{t}}) - u_i(x_i^*, Q_i^*) &\approx \frac{\partial u_i}{\partial x_i}(x_i^{\mathbf{t}} - x_i^*) + \frac{\partial u_i}{\partial Q_i}(Q_i^{\mathbf{t}} - Q_i^*) && (3.A.8)\\ &= \left[\frac{\partial u_i}{\partial x_i} + \left[\frac{1}{1-\gamma_i'(\cdot)} - 1\right]\frac{\partial u_i}{\partial Q_i}\right](x_i^{\mathbf{t}} - x_i^*) && (3.A.9)\end{aligned}$$

Since we normalised prices to $(1,1)$, and assumed an interior solution where all players are contributors, it follows that $\frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial Q_i}$ $\forall i$ and so we have that

$$u_i(x_i^{\mathbf{t}}, Q_i^{\mathbf{t}}) - u_i(x_i^*, Q_i^*) \approx \frac{\partial u_i}{\partial x_i}\frac{1}{1-\gamma_i'(\cdot)}(x_i^{\mathbf{t}} - x_i^*) \tag{3.A.10}$$

Let $\nabla \mathbf{u} = \text{diag}(\frac{\partial u_i}{\partial x_i})$. Then

$$\mathbf{v}^{\mathbf{t}} - \mathbf{v}^* \approx \nabla\mathbf{u}\ \mathbf{A}^{-1}(\mathbf{I} + \mathbf{A}\mathbf{G})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\mathbf{t}.$$

So that $\mathbf{B_v} = \nabla\mathbf{u}\ \mathbf{A}^{-1}(\mathbf{I} + \mathbf{A}\mathbf{G})^{-1}\mathbf{A}$, which is invertible as required.

3.A.3 Proof of Proposition 16

Following from (3.A.5) in the case of C-D preferences, the change in consumers' public good provision due to an infinitesimal transfer $\mathbf{t}$ is given by

$$\mathbf{q}^{\mathbf{t}} - \mathbf{q}^* = (1-a)(\mathbf{I} + a\mathbf{G})^{-1}\mathbf{t}$$

Let $\mathbf{Q}_-$ be the vector of Q_{-i}; now we can observe that

$$\mathbf{Q}_-^{\mathbf{t}} - \mathbf{Q}_-^* = (1-a)\mathbf{G}(\mathbf{I}+a\mathbf{G})^{-1}\mathbf{t}$$

In the Cobb-Douglas case, where $\alpha = a^a(1-a)^{(1-a)}$ the indirect utility function is

$$v_i^* = \alpha(w_i + Q_{-i}^*) \tag{3.A.11}$$

So the list of changes in utility from lump-sum transfers $\mathbf{t}$ is

$$\begin{aligned}
\mathbf{v}^{\mathbf{t}} - \mathbf{v}^* &= [u_i(x_i^{\mathbf{t}}, q_i^{\mathbf{t}} + Q_{-i}^{\mathbf{t}}) - u_i(x_i^*, q_i^* + Q_{-i}^*)]_{i\in N} \\
&= [\alpha(w_i + t_i + Q_{-i}^{\mathbf{t}}) - \alpha(w_i + Q_{-i}^*)]_{i\in N} \\
&= \alpha[t_i + Q_{-i}^{\mathbf{t}} - Q_{-i}^*]_{i\in N} \\
&= \alpha[\mathbf{G}(\mathbf{q}^{\mathbf{t}} - \mathbf{q}^*) + \mathbf{I}\mathbf{t}] \\
&= \alpha[(1-a)\mathbf{G}(\mathbf{I}+a\mathbf{G})^{-1}\mathbf{t} + \mathbf{I}\mathbf{t}] \\
&= \alpha\left[\frac{(1-a)}{a}[a\mathbf{G} + \mathbf{I} - \mathbf{I}](\mathbf{I}+a\mathbf{G})^{-1}\mathbf{t} + \mathbf{I}\mathbf{t}\right] \\
&= \alpha\left[\frac{(1-a)}{a}[\mathbf{I} - (\mathbf{I}+a\mathbf{G})^{-1}]\mathbf{t} + \ \mathbf{I}\mathbf{t}\right] \\
&= \alpha\left[\frac{1}{a}\mathbf{I}\mathbf{t} - \frac{(1-a)}{a}(\mathbf{I}+a\mathbf{G})^{-1}]\mathbf{t}\right] \\
&= \frac{\alpha}{a}\left[\mathbf{I} - (1-a)(\mathbf{I}+a\mathbf{G})^{-1}\right]\mathbf{t} \\
&= \frac{\alpha}{a}(\mathbf{I}+a\mathbf{G})^{-1}[(\mathbf{I}+a\mathbf{G}) - (1-a)\mathbf{I}]\mathbf{t} \\
&= \alpha(\mathbf{I}+a\mathbf{G})^{-1}(\mathbf{I}+\mathbf{G})\mathbf{t}.
\end{aligned}$$

3.A.4 Proof of Corollary 5

Observe that since -1 is a main eigenvalue there exists a transfer $\mathbf{t}_0 = \kappa \mathbf{u_1} + \beta \mathbf{u_l}$, where $\mathbf{u_1}$ is the eigenvector corresponding to λ_{max} and $\mathbf{u_l}$ is the eigenvector corresponding to -1, and $\kappa, \beta \neq 0$. This is not the case if -1 is a non-main eigenvalue, because then $\mathbf{u_l}$ would be orthogonal to $\mathbf{1}$, and so adding the all-positive $\mathbf{u_1}$ to it would not lead to budget-balanced transfers. On the other hand, if $\mathbf{u_l}$ is not orthogonal to $\mathbf{1}$, since $\mathbf{u_1}$ and $\mathbf{u_l}$ are orthogonal, they span a hyperplane which must intersect the $n-1$ space of $\mathbf{1}^{\perp}$ at some point, giving the budget-balanced transfer for some κ, β. From 3.5.1 it follows that

$$\begin{aligned}
\mathbf{v^t} - \mathbf{v}^* &= \alpha(\mathbf{I} + a\mathbf{G})^{-1}(\mathbf{I} + \mathbf{G})\mathbf{t}_0 && (3.A.12)\\
&= \alpha(\mathbf{I} + a\mathbf{G})^{-1}(\mathbf{I} + \mathbf{G})(\kappa \mathbf{u_1} + \beta \mathbf{u_l}) && (3.A.13)\\
&= \alpha(\mathbf{I} + a\mathbf{G})^{-1}(\mathbf{I} + \mathbf{G})\kappa \mathbf{u_1} + \alpha(\mathbf{I} + a\mathbf{G})^{-1}(\mathbf{I} + \mathbf{G})\beta \mathbf{u_l} && (3.A.14)\\
&= \alpha(1 + \lambda_1)/(1 + a\lambda_1)\kappa \mathbf{u_1} && (3.A.15)
\end{aligned}$$

So in this case the impact of the transfer is results in a Pareto improvement.

Bibliography

Acemoglu, Daron, Kostas Bimpikis, and Asuman Ozdaglar (2014) "Dynamics of information exchange in endogenous social networks," *Theoretical Economics*, Vol. 9, pp. 41–97. 13

Acemoglu, Daron, Azarakhsh Malekian, and Asu Ozdaglar (2016) "Network security and contagion," *Journal of Economic Theory*, Vol. 166, pp. 536–585. 110

Ahmad, Ehtisham and Nicholas Stern (1984) "The theory of reform and Indian indirect taxes," *Journal of Public Economics*, Vol. 25, pp. 259–298. 111, 127

Ahn, Illtae and Matti Suominen (2001) "Word-Of-Mouth Communication and Community Enforcement," *International Economic Review*, Vol. 42, pp. 399–415. 13, 59, 61

Ali, S Nageeb and David A Miller (2013) "Enforcing cooperation in networked societies," *Unpublished manuscript, University of California at San Diego*. 60, 61

Ali, S. Nageeb and David A. Miller (2016) "Ostracism and Forgiveness," *American Economic Review*, Vol. 106, pp. 2329–2348. 61

Allouch, Nizar (2015) "On the private provision of public goods on networks," *Journal of Economic Theory*, Vol. 157, pp. 527–552. 109, 114, 121, 137

——— (2017) "The Cost of Segregation in Social Networks," *Games and Economic Behavior*, Vol. Forthcoming. 30, 83, 110

Ambrus, Attila, Arun G. Chandrasekhar, and Matt Elliott (2014) "Social Investments, Informal Risk Sharing, and Inequality," Working Paper 20669, National Bureau of Economic Research, DOI: 10.3386/w20669. 19

Annen, Kurt (2003) "Social capital, inclusive networks, and economic performance," *Journal of Economic Behavior & Organization*, Vol. 50, pp. 449–463. 58

——— (2011) "Lies and slander: truth-telling in repeated matching games with private monitoring," *Social Choice and Welfare*, Vol. 37, pp. 269–285. 59

Bailey, Norman T. J. (1957) *Mathematical Theory of Epidemics*: Arnold. 43

Baker, Wayne E. (1984) "The social structure of a national securities market," *American Journal of Sociology*, pp. 775–811. 96

Balasko, Yves (2014) "The transfer problem: A complete characterization," *Theoretical Economics*, Vol. 9, pp. 435–444. 112

Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou (2006) "Who's Who in Networks. Wanted: The Key Player," *Econometrica*, Vol. 74, pp. 1403–1417. 109

Balmaceda, Felipe and Juan Escobar (2013) "Trust in Cohesive Communities,"Technical report, Diego Portales University. 59

Banerjee, Abhijit, Arun G. Chandrasekhar, Esther Duflo, and Matthew O. Jackson (2013) "The Diffusion of Microfinance," *Science*, Vol. 341, p. 1236498. 13, 14, 17, 19, 20, 43, 44, 48, 59, 81

——— (2014) "Gossip: Identifying central individuals in a social network,"Technical report, NBER. 43, 59

Barr, Abigail, Jean Ensminger, and Jeffrey Johnson (2009) "Social networks and trust in cross-cultural economic experiments," in *Whom Can We Trust? How Groups, Networks, and Institutions Make Trust Possible*, pp. 65–90. 96

Becker, Gary S. (1974) "A Theory of Social Interactions," Working Paper 42, National Bureau of Economic Research. 114

Ben-Israel, Adi (1964) "Notes on linear inequalities, I: The intersection of the nonnegative orthant with complementary orthogonal subspaces," *Journal of Mathematical Analysis and Applications*, Vol. 9, pp. 303–314. 126

Benoit, Jean-Pierre and Vijay Krishna (1985) "Finitely Repeated Games," *Econometrica*, Vol. 53, pp. 905–922. 61, 64, 71

——— (1993) "Renegotiation in Finitely Repeated Games," *Econometrica*, Vol. 61, p. 303. 61, 71

Bergstrom, Ted, Lawrence Blume, and Hal Varian (1986) "On the private provision of public goods," *Journal of Public Economics*, Vol. 29, pp. 25–49. 109

Bhaskar, V. and Ichiro Obara (2002) "Belief-Based Equilibria in the Repeated Prisoners' Dilemma with Private Monitoring," *Journal of Economic Theory*, Vol. 102, pp. 40–69. 48, 60, 82

Bloch, Francis, Gabrielle Demange, and Rachel Kranton (2014) "Rumors and Social Networks," PSE Working Paper halshs-00966234, HAL. 59

Bloch, Francis, Garance Genicot, and Debraj Ray (2008) "Informal insurance in social networks," *Journal of Economic Theory*, Vol. 143, pp. 36–58. 59, 60

Bonacich, Phillip (1987) "Power and Centrality: A Family of Measures," *American Journal of Sociology*, Vol. 92, pp. 1170–1182. 30, 43, 109

Borgatti, Stephen P. (2005) "Centrality and network flow," *Social Networks*, Vol. 27, pp. 55–71. 13, 59

Borgatti, Stephen P. and Martin G. Everett (2000) "Models of core/periphery structures," *Social networks*, Vol. 21, pp. 375–395. 129

Bourlès, Renaud, Yann Bramoullé, and Eduardo Perez-Richet (2017) "Altruism in Networks," *Econometrica*, Vol. 85, pp. 675–689. 110

Bramoullé, Yann and Rachel Kranton (2007) "Public goods in networks," *Journal of Economic Theory*, Vol. 135, pp. 478–494. 109

Bramoullé, Yann, Rachel Kranton, and Martin D'Amours (2014) "Strategic Interaction and Networks," *American Economic Review*, Vol. 104, pp. 898–930. 109

Breza, Emily and Arun G. Chandrasekhar (2015) "Social Networks, Reputation and Commitment: Evidence from a Savings Monitors Experiment," SSRN Scholarly Paper ID 2607359, Social Science Research Network, Rochester, NY. 14, 44, 48, 49, 58, 67, 96

Butler, Jeffrey, Paola Giuliano, and Luigi Guiso (2009) "The right amount of trust,"Technical report, NBER. 93

Chandrasekhar, Arun G., Cynthia Kinnan, and Horacio Larreguy (2014) "Social Networks as Contract Enforcement: Evidence from a Lab Experiment in the Field," Working Paper 20259, NBER. 87

Chen, Bo (2010) "A belief-based approach to the repeated prisoners' dilemma with asymmetric private monitoring," *Journal of Economic Theory*, Vol. 145, pp. 402–420. 48, 60

Coleman, James S (1988) "Social capital in the creation of human capital," *American journal of sociology*, pp. S95–S120. 41, 55

Crawford, Vincent P. and Joel Sobel (1982) "Strategic Information Transmission," *Econometrica*, Vol. 50, pp. 1431–1451. 13

Currarini, Sergio, Matthew O. Jackson, and Paolo Pin (2009) "An Economic Model of Friendship: Homophily, Minorities,andSegregation," *Econometrica*, Vol. 77, pp. 1003–1045. 25

Cvetkovic, Dragos (1970) "The generating function for variations with restrictions and paths of the graph and self-complementary graphs," *Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat.*, pp. 27–34. 120

Degroot, Morris H. (1974) "Reaching a Consensus," *Journal of the American Statistical Association*, Vol. 69, pp. 118–121. 13, 60

Dhillon, Amrita and Jamele Rigolini (2011) "Development and the interaction of enforcement institutions," *Journal of Public Economics*, Vol. 95, pp. 79–87. 96

Dixit, Avinash (1975) "Welfare effects of tax and price changes," *Journal of Public Economics*, Vol. 4, pp. 103–123. 111, 127

——— (2003a) "On modes of economic governance," *Econometrica*, Vol. 71, pp. 449–481. 96

——— (2003b) "Trade expansion and contract enforcement," *Journal of Political Economy*, Vol. 111, pp. 1293–1317. 58, 59, 60, 61, 66, 70, 83, 94

Elliott, Matthew and Benjamin Golub (2015) "A Network Approach to Public Goods," SSRN Scholarly Paper ID 2436683, Social Science Research Network, Rochester, NY. 110

Elliott, Matthew, Benjamin Golub, and Matthew O. Jackson (2014) "Financial Networks and Contagion," *The American Economic Review*, Vol. 104, pp. 3115–3153. 129

Ellison, Glenn (1994) "Cooperation in the prisoner's dilemma with anonymous random matching," *The Review of Economic Studies*, Vol. 61, pp. 567–588. 61

Fafchamps, Marcel and Julien Labonne (2016) "Family Networks and Distributive Politics,"Technical report, CEPR Discussion Papers. 14, 48, 49

Fainmesser, Itay P (2012) "Community Structure and Market Outcomes: A Repeated Games-in-Networks Approach," *American Economic Journal: Microeconomics*, Vol. 4, pp. 32–69. 60

Fainmesser, Itay P. and David A. Goldberg (2012) "Cooperation in partly observable networked markets,"Technical report, Working paper, July. 60

Farrell, Joseph and Eric Maskin (1989) "Renegotiation in repeated games," *Games and Economic Behavior*, Vol. 1, pp. 327–360. 61

Freeman, Linton (1977) "A Set of Measures of Centrality Based on Betweenness," *Sociometry*, Vol. 40, pp. 35–41. 46

Fuente, Angel de la (2000) *Mathematical Methods and Models for Economists*, Cambridge, UK ; New York, NY: Cambridge University Press. 18

Fukuyama, Francis (1996) *Trust: The Social Virtues and the Creation of Prosperity*, New York, NY: The Free Press, 1st edition. 55

——— (2001) "Social capital, civil society and development," *Third World Quarterly*, Vol. 22, pp. 7–20. 58

Galeotti, Andrea, Christian Ghiglino, and Francesco Squintani (2013) "Strategic information transmission networks," *Journal of Economic Theory*, Vol. 148, pp. 1751–1769. 13

Galeotti, Andrea and Sanjeev Goyal (2010) "The Law of the Few," *American Economic Review*, Vol. 100, pp. 1468–1492. 129

Galeotti, Andrea, Sanjeev Goyal, Matthew O. Jackson, Fernando Vega-Redondo, and Leeat Yariv (2010) "Network Games," *Review of Economic Studies*, Vol. 77, pp. 218–244. 110

Gallo, Edoardo (2014) "Social learning by chit-chat," *Journal of Economic Theory*, Vol. 153, pp. 313–343. 60

Gallo, Edoardo and Chang Yan (2015) "The effects of reputational and social knowledge on cooperation," *Proceedings of the National Academy of Sciences*, Vol. 112, pp. 3647–3652. 82

Ghiglino, Christian and Sanjeev Goyal (2010) "Keeping up with the Neighbors: Social Interaction in a Market Economy," *Journal of the European Economic Association*, Vol. 8, pp. 90–119. 110

Godsil, Chris (1993) *Algebraic Combinatorics*: CRC Press. 26

Golub, Benjamin and Matthew O. Jackson (2012) "How Homophily Affects the Speed of Learning and Best-Response Dynamics," *The Quarterly Journal of Economics*, Vol. 127, pp. 1287–1338. 13, 25, 60

Gorman, W. M. (1961) "On a Class of Preference Fields," *Metroeconomica*, Vol. 13, pp. 53–56. 110

Goyal, Sanjeev (2007) *Connections: An Introduction to the Economics of Networks*, Princeton, N.J.: Princeton University Press. 58

——— (2016) "Networks in Economics." 13

Greif, Avner (1993) "Contract Enforceability and Economic Institutions in Early Trade: The Maghribi Traders' Coalition," *The American Economic Review*, Vol. 83, pp. pp. 525–548. 56

Guesnerie, Roger (1977) "On the direction of tax reform," *Journal of Public Economics*, Vol. 7, pp. 179–202. 111

Hagenbach, Jeanne and Frédéric Koessler (2010) "Strategic communication networks," *The Review of Economic Studies*, Vol. 77, pp. 1072–1099. 13

Harary, Frank and Allen J. Schwenk (1979) "The spectral approach to determining the number of walks in a graph.," *Pacific Journal of Mathematics*, Vol. 80, pp. 443–449. 120

Jackson, Matthew O. (2008) *Social and Economic Networks*, Princeton, NJ: Princeton University Press. 58

Jackson, Matthew O., Tomas Rodriguez-Barraquer, and Xu Tan (2012) "Social capital and social quilts: Network patterns of favor exchange," *The American Economic Review*, Vol. 102, pp. 1857–1897. 41, 61

Kandori, Michihiro (1992) "Social norms and community enforcement," *The Review of Economic Studies*, Vol. 59, pp. 63–80. 56, 59, 60, 61, 64, 66, 81

——— (2002) "Introduction to Repeated Games with Private Monitoring," *Journal of Economic Theory*, Vol. 102, pp. 1–15. 48, 60

Karlan, Dean, Markus Mobius, Tanya Rosenblat, and Adam Szeidl (2009) "Trust and Social Collateral," *The Quarterly Journal of Economics*, Vol. 124, pp. 1307–1361. 58

Kermack, William and Anderson McKendrick (1927) "A contribution to the mathematical theory of epidemics," *Proceedings of the Royal Society of London A: mathematical, physical and engineering sciences*, Vol. 115. 43

Kinateder, Markus (2008) "Repeated Games Played in a Network," SSRN Scholarly Paper ID 1115733, Social Science Research Network, Rochester, NY. 59, 89

Kinateder, Markus and Luca Paolo Merlino (2017) "Public Goods in Endogenous Networks," *American Economic Journal: Microeconomics*, Vol. Forthcoming. 110

Knack, Stephen and Philip Keefer (1997) "Does social capital have an economic payoff? A cross-country investigation," *The Quarterly journal of economics*, pp. 1251–1288. 55

Kranton, Rachel E. (1996) "Reciprocal Exchange: A Self-Sustaining System," *The American Economic Review*, Vol. 86, pp. 830–851. 96

Laclau, Marie (2012) "A folk theorem for repeated games played on a network," *Games and Economic Behavior*, Vol. 76, pp. 711–737. 60

——— (2014) "Communication in repeated network games with imperfect monitoring," *Games and Economic Behavior*, Vol. 87, pp. 136–160. 59

Larson, Jennifer M. (2017) "Networks and Interethnic Cooperation," *The Journal of Politics*, Vol. 79, pp. 546–559. 60, 77

Larson, Jennifer Mary (2014) "Cheating Because They Can: Social Networks and Norm Violators," Ph.D. dissertation, Harvard. 60

Leeson, Peter T. (2005) "Endogenizing fractionalization," *Journal of Institutional Economics*, Vol. 1, pp. 75–98. 55

Leontief, Wassily (1936) "Note on the pure theory of capital transfer," *Explorations in economics: notes and essays contributed in honor of FW Taussig*, pp. 84–91. 112

Levy, Gilat and Ronny Razin (2014) "A simple Bayesian heuristic for social learning and groupthink." 13

Lippert, Steffen and Giancarlo Spagnolo (2011) "Networks of relations and word-of-mouth communication," *Games and Economic Behavior*, Vol. 72, pp. 202–217. 13, 59, 60, 91

López-Pintado, Dunia (2017) "Influence networks and public goods," *SERIEs*, Vol. 8, pp. 97–112. 110

Malinvaud, Edmond (1972) *Lectures on Microeconomic Theory*: North-Holland. 109

Meagher, Kate (2006) "Social capital, social liabilities, and political capital: Social networks and informal manufacturing in Nigeria," *African Affairs*, Vol. 105, pp. 553–582. 87

Moore, Cristopher and Mark E.J. Newman (2000) "Epidemics and percolation in small-world networks," *Physical Review E*, Vol. 61, p. 5678. 43

Mueller-Frank, Manuel and Claudia Neri (2015) "Quasi-Bayesian updating in social networks," *Available at SSRN*. 13

Myerson, Roger B. (2008) "The Autocrat's Credibility Problem and Foundations of the Constitutional State," *American Political Science Review*, Vol. 102. 91

Nava, Francesco (2016) "Repeated Games and Networks," in Yann Bramoullé, Andrea Galeotti, and Brian Rogers eds. *The Oxford Handbook of the Economics of Networks*: OUP USA. 13, 36, 58

Nava, Francesco and Michele Piccione (2014) "Efficiency in repeated games with local interaction and uncertain local monitoring," *Theoretical Economics*, Vol. 9, pp. 279–312. 58

Noh, Jae Dong and Heiko Rieger (2004) "Random Walks on Complex Networks," *Physical Review Letters*, Vol. 92. 43

Ostrom, Elinor (1990) *Governing the Commons: The Evolution of Institutions for Collective Action*, Cambridge ; New York: Cambridge University Press. 55, 60

Phillips, Damon J. and Ezra Zuckerman (2001) "Middle Status Conformity: Theoretical Restatement and Empirical Demonstration in Two Markets," *American Journal of Sociology*, Vol. 107, pp. 379–429. 57, 93

Piraveenan, Mahendra, Mikhail Prokopenko, and Liaquat Hossain (2013) "Percolation Centrality: Quantifying Graph-Theoretic Impact of Nodes during Percolation in Networks," *PLoS ONE*, Vol. 8, p. e53095. 43

Powers, David L. and Mohammad M. Sulaiman (1982) "The walk partition and colorations of a graph," *Linear Algebra and its Applications*, Vol. 48, pp. 145–159. 26, 83

Putnam, Robert D., Robert Leonardi, and Raffaella Y. Nanetti (1994) *Making Democracy Work: Civic Traditions in Modern Italy*, Princeton, NJ: Princeton University Press, new ed edition edition. 55

Rasmusen, Eric and Minwook Kang (2016) "The Transfer Paradox in International Trade," August. 112

Raub, Werner and Jeroen Weesie (1990) "Reputation and efficiency in social interactions: An example of network effects," *American Journal of Sociology*, pp. 626–654. 59, 61

Renault, Jerome and Tristan Tomala (1998) "Repeated proximity games," *International Journal of Game Theory*, Vol. 27, pp. 539–559. 59, 89

Riyanto, Yohanes E. and Jonathan Yeo (2014) "Directed Trust and Reciprocity in a Real-Life Social Network: An Experimental Investigation." 87, 96

Samuelson, Larry and George J. Mailath (2006) *Repeated Games and Reputations: Long-Run Relationships*, Oxford ; New York: OUP USA. 61, 65, 67

Samuelson, Paul A. (1952) "The Transfer Problem and Transport Costs: The Terms of Trade When Impediments are Absent," *The Economic Journal*, Vol. 62, pp. 278–304. 112

Sekiguchi, Tadashi (1997) "Efficiency in Repeated Prisoner's Dilemma with Private Monitoring," *Journal of Economic Theory*, Vol. 76, pp. 345–361. 48, 60, 82

Sobel, Joel (2002) "Can we trust social capital?" *Journal of economic literature*, Vol. 40, pp. 139–154. 60

Spence, Michael (1973) "Job Market Signaling," *The Quarterly Journal of Economics*, Vol. 87, pp. 355–374. 67, 96

Stephenson, Karen and Marvin Zelen (1989) "Rethinking centrality: Methods and examples," *Social Networks*, Vol. 11, pp. 1–37. 43

Teytelboym, Alex, Yongwhan Lim, and Asu Ozdaglar (2015) "A simple model of cascades in networks." 43

Tilly, Charles (2004) "Trust and Rule," *Theory and Society*, Vol. 33, pp. 1–30. 55

Warr, Peter G. (1983) "The private provision of a public good is independent of the distribution of income," *Economics Letters*, Vol. 13, pp. 207–211. 109, 115, 116

Weymark, John A. (1981) "Undominated directions of tax reform," *Journal of Public Economics*, Vol. 16, pp. 343–369. 111

Wolitzky, Alexander (2013) "Cooperation with network monitoring," *The Review of Economic Studies*, Vol. 80, pp. 395–427. 58

——— (2014) "Communication with tokens in repeated games on networks," *Theoretical Economics*. 13, 36, 59, 89

Yano, Makoto (1983) "Welfare aspects of the transfer problem," *Journal of International Economics*, Vol. 15, pp. 277–289. 112