Fluctuations in the two-dimensional one-component plasma and associated fourth-order phase transition

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We study the distribution of the mean radial displacement of charges of a 2D one-component plasma in the thermodynamic limit \( N \to \infty \) at finite temperature \( \beta > 0 \). We compute explicitly the large deviation functions showing the emergence of a fourth-order phase transition as a consequence of a change of topology in the plasma distribution. This weak phase transition occurs exactly at the ground state of the plasma. These results have been compared with the integrable case (finite \( N \)) of plasma parameter \( \beta q^2 = 2 \). In this case the problem can be mapped to the stationary properties of 2D Dyson Brownian particles and to a non-Hermitian matrix model.

In this paper we present a study on the large deviations (rare events) of 2D-OCP systems. We will consider a rather simple observable, namely the mean radial displacement \( \Delta_N \) at finite temperature \( \beta > 0 \) (the angle brackets denote the canonical average). As a byproduct, at \( \beta = 2 \) these results provide large deviations functions for the Ginibre ensemble and 2D Dyson Brownian particles.

Results on the probability of rare events are popular in the one-dimensional case, in particular in the literature on Hermitian random matrices [18–25]. In most of the models analysed the 2D-OCP is confined on the line and undergoes a phase transition, usually of third-order if the plasma is constrained by a hard wall (for an interesting review see [26]). Large deviations have been much less explored in two dimensions and few results are available [27–30]. Here we show explicitly that the statistical distribution of the mean displacement \( \Delta_N \) in the plane unveils a fourth-order phase transition of the 2D-OCP related to a change of topology in the equilibrium distribution of the plasma. This phenomenon is absent in one dimension. We point out that the extraordinary weak singularity of \( P_{N,\beta}(x) \) (or \( \tilde{P}_{N,\beta}(s) \)) occurs at the average value \( x = \langle \Delta_N \rangle \) (at \( s = 0 \) in the Laplace domain), i.e. at the ground state of the 2D plasma. Therefore this phase transition may be detected by the anomalous behaviour of the high order cumulants of \( \Delta_N \).

Results. The general theory of large deviations of log-gases [29, 30] ensures that the Laplace transform \( \tilde{P}_{N,\beta}(s) \) of \( \Delta_N \) has a thermodynamic limit

\[
J(s) = - \lim_{N \to \infty} \frac{1}{\beta N^2} \log \tilde{P}_{N,\beta}(s)
\]
order phase transition the large deviation function is not analytic. The satisfies the normalization condition sequences of (5). The cumulant GF is strictly concave and of the equilibrium distribution x ~. From left to right s = 1, 0, −0.5. Bottom: The corresponding radial distributions \[ \int_0^{2\pi} d\mu^* \].

for all s ∈ ℝ. Such a limit is the scaled cumulant generating function (cumulant GF) of Δ_N. Then, by the Gärtner-Ellis theorem [31, 32], the probability law of Δ_N satisfies a large deviation principle [48] with speed βN^2, i.e. \( \mathcal{P}_{N,\beta}(x) \approx e^{-\beta N^2 \Psi(x)} \), where the rate function \( \Psi(x) \) is the Legendre-Fenchel transform of the cumulant GF (3):

\[
\Psi(x) = - \lim_{N \to \infty} \frac{1}{\beta N^2} \log \mathcal{P}_{N,\beta}(x) = \inf_s [J(s) - s x].
\] (4)

The challenge here is to compute explicitly the large deviation functions J(s) and \( \Psi(x) \). We will show that the cumulative GF has the following expression independent of β:

\[
J(s) = \frac{1}{2} \arcsinh \left( \frac{s}{2} \right) \frac{s^2}{4} + \frac{s}{48} \left[ (s^2 + 10) \sqrt{s^2 + 4} - |s|^3 \right].
\] (5)

This is the central result of the paper. Before turning to the derivation, we make few observations and we discuss the consequences of (5). The cumulative GF is strictly concave and satisfies the normalization condition J(0) = 0. Notice that the large deviation function is not analytic. The fourth derivative of J(s) is discontinuous at s = 0 with a finite jump \( J^{(4)}(0^+) - J^{(4)}(0^-) = 1 \); this is a consequence of a fourth-order phase transition in the equilibrium distribution of the 2D-OCP. Phase transitions of 2D-OCP constrained on a line, i.e. of eigenvalues of Hermitian random matrices, are well understood. In two dimensions, however, they are more gentle than for Hermitian matrices, where the singularities are of third order. This behaviour corresponds to the physical intuition that the logarithmic repulsion has stronger effects in 1D than in 2D gases. Here the singularity emerges in the thermodynamic limit as a consequence of a change in topology (disk-annulus) of the support of the equilibrium distribution of the charges. This mechanism is absent in one dimension. We note that the phase transition would not occur in the absence the logarithmic interaction, since without it (2) is the Hamiltonian of a system of free particles, which cannot undergo a phase transition — this case is discussed in more detail in the last section. As a final remark, we stress that the phase transition occurs at the ground state \( s = 0 \) and hence is not related to the atypical fluctuations (this is another departure from the one-dimensional scenario for continuous statistics).

At leading order in N, the first three cumulants of Δ_N can be obtained by differentiation of J(s) at s = 0

\[
\kappa_1(\Delta_N) = \frac{2}{\beta}, \quad \kappa_2(\Delta_N) = \frac{1}{\beta N^2}, \quad \kappa_3(\Delta_N) = -\frac{1}{2\beta N^4}.
\] (6)

At β = 2 the first two cumulants (average and variance) agree with asymptotic formulas for the Ginibre matrix ensemble [33]. The nonzero third cumulant measures the deviation from the Gaussian behaviour at leading order in N. Higher cumulants cannot be obtained by differentiation of J(s) (we will elaborate more on this point later). Since J(s) is everywhere differentiable, we can apply the Gärtner-Ellis theorem and recover the rate function \( \Psi(x) \) using (4). The limiting behaviour of \( \mathcal{P}_{N,\beta}(x) \approx e^{-\beta N^2 \Psi(x)} \) may be summarized as follows

\[
\mathcal{P}_{N,\beta}(x) \approx \begin{cases} 
\left( \frac{2x}{\beta} \right)^{\beta N^2/2} e^{-(2/3)\beta N^2(x^2-4/9)} & \text{for } x \ll 1 \\
\exp \left[-\beta N^2 (x - x_0)^2 \right] & \text{for } x \approx \frac{2}{\beta} \\
\left( \frac{2x}{\beta} \right)^{\beta N^2/2} e^{-(1/2)\beta N^2(x^2-4/9)} & \text{for } x \gg 1.
\end{cases}
\] (7)

The rate function \( \Psi(x) \) is plotted in the top panel of Fig.1. From the Gaussian form around \( x \approx 2/3 \), one easily reads off the values for mean and variance: \( \langle \Delta_N \rangle = 2/3 \) and var(Δ_N) = 1/(2βN^2), according to (6). On the other hand, the rate function is not quadratic and this implies the polynomial and sub-Gaussian far tails of \( \mathcal{P}_{N,\beta}(x) \) for \( x \to 0 \) and \( x \to \infty \), respectively, in (7).

We stress that the exact computation of J(s) is essential to unveil the phase transition. Standard computations usually concern with the quadratic approximation about the ground state of the 2D-OCP (Gaussian approximation of J(s) about \( s = 0 \)). However the fourth-order singularity at \( s = 0 \) does not emerge at the level of quadratic (i.e. \( O(s^2) \)) effects. 

\textbf{Derivation.} In order to compute the cumulant GF (3), we cast the Laplace transform of Δ_N as a ratio of two partition functions \( \mathcal{Z}_{N,\beta}(s) = \frac{\mathcal{Z}_{N,\beta}(s)}{\mathcal{Z}_{N,\beta}(0)} \), where \( \mathcal{Z}_{N,\beta}(s) = \int e^{-s x} \mathcal{P}_{N,\beta}(x) dx \).
\[ \int e^{-\beta H(r_1, \ldots, r_N; s)} \] is the partition function of the 2D-OCP with modified energy

\[ H(r_1, \ldots, r_N; s) = -\frac{1}{2} \sum_{i \neq j} \log r_{ij} + N \sum_k V_s(r_k) \]  

(8)

\[ V_s(r) = \frac{1}{2} r^2 + sr. \]  

(9)

Clearly, at zero temperature \( \beta \to \infty \) the only contribution to \( \tilde{Z}_{N, \beta}(s) \) comes the configurations of the system that minimize the energy \( H(r_1, \ldots, r_N; s) \). At finite temperature \( \beta < \infty \) in the large \( N \) limit, the picture is not very different in the sense that the ‘typical’ configurations of the system are still the minimizers of the energy and one can characterize the probability of ‘atypical’ configurations. For large \( N \), we have \( H(r_1, \ldots, r_N; s) = N^2 (H_s[\mu_N] + o(1)) \) where \( \mu_N = N^{-1} \sum \delta_{\bar{r}_i} \) is the normalized distribution of the charges and \( H_s[\cdot] \) is the mean-field energy functional:

\[ H_s[\mu] = -\frac{1}{2} \int_{\mathbb{R}^2} d\mu(\bar{r}) \int_{\mathbb{R}^2} d\mu(\bar{r}') |\bar{r} - \bar{r}'| + \int_{\mathbb{R}^2} d\mu(\bar{r}) V_s(|\bar{r}|). \]  

(10)

These considerations justify a saddle-point evaluation of the large deviation function \( J(s) \) as excess free energy

\[ J(s) = -\lim_{N \to \infty} \frac{1}{\beta N^2} \log \left[ \frac{\int e^{-\beta H(r_1, \ldots, r_N; s)}}{\int e^{-\beta H(r_1, \ldots, r_N; 0)}} \right] = H_s[\mu_s^*] - H_0[\mu_0^*], \]  

(11)

where \( \mu_s^* \) (resp. \( \mu_0^* \)) is the minimizer of the energy functional \( H_s[\cdot] \) (resp. \( H_0[\cdot] \)):

\[ H_s[\mu_s^*] = \min \left\{ H_s[\mu] : \mu \geq 0, \int_{\mathbb{R}^2} d\mu(\bar{r}) = 1 \right\}. \]  

(12)

Hence, the problem of computing \( J(s) \) reduces to find the equilibrium configuration \( \mu_s^* \) of the charges of plasma in the large \( N \) limit. Find the configuration of the 2D-OCP that minimizes a mean-field energy functional like (10) is one of the central objectives in potential theory [34]. Fortunately, minimization problems such as (12) can be solved in closed form with a judicious use of classical electrostatics. We first observe that the problem is radially symmetric and the radial potential \( V_s \) is convex. Therefore, the equilibrium measure inherits the radial symmetry of the energy \( d\mu_s^*(\bar{r}) = (2\pi)^{-1} d\mu_s^*(r) d\varphi \) and is supported in general on the annulus

\[ \Sigma(s) = \left\{ \bar{r} \in \mathbb{R}^2 : r_0(s) \leq r \leq R_0(s) \right\}, \]  

(13)

where we have switched to polar coordinates \( \bar{r} = (r, \varphi) \). At equilibrium, the Gauss law on a generic domain of the plane holds. For a centered disk of radius \( r \), the Gauss law reads

\[ V_s^0(r) r = 2\pi \int_0^r d\mu_s^*(r') r'. \]  

(14)

Using the positivity and the normalization of \( \mu_s^* \), it is easy to find the conditions \( V_s^0(r_0) = 0 \) and \( V_s^0(R_0) R_0 = 1 \) on the inner and outer radii of the support \( \Sigma(s) \). From (14), the equilibrium distribution of the charge has the explicit form

\[ d\mu_s^* (\bar{r}) = \left( \frac{2|\bar{r}| + s}{2\pi |\bar{r}|} \right) 1_{\bar{r} \in \Sigma(s)} d\bar{r}, \]  

(15)

\[ r_0(s) = \max \{ 0, -s \}, \quad R_0(s) = \frac{1}{2} \sqrt{s^2 + 4 - s}. \]  

(16)

We see that the optimal distribution of the plasma experiences a change of topology driven by \( s \): \( \mu_s^* \) is supported on a disk for \( s \geq 0 \) and on an annulus for \( s < 0 \) (see Fig. 1). The fourth-order singularity of \( J(s) \) at \( s = 0 \) witnesses this sudden disk-to-annulus change of topology. In particular, for \( s = 0 \) we get the uniform measure on the unit disk:

\[ d\mu_0^*(\bar{r}) = \frac{1}{\pi} 1_{|\bar{r}| \leq 1} d\bar{r}. \]  

This corresponds to the circular law [7, 35–37], one of the most celebrated results on the empirical density of the eigenvalues of non-Hermitian random matrices. For a generic value \( s \in \mathbb{R} \), the distribution \( \mu_s^* \) is the typical configuration of the 2D-OCP with a fixed value of \( \Delta_N \) given by

\[ x(s) = \int d\mu_s^*(\bar{r}) |\bar{r}| = \left( \frac{s^2 + 4}{12} \right)^{3/2} - 6s - |s|^3. \]  

(17)

The energy of the equilibrium configuration at \( s = 0 \) (i.e. at the ground state) is \( H_0[\mu_s^*] = 3/8 \), a value known as \( H \)-stability bound of the 2D-OCP [38, 39]. For a generic value of \( s \), the minimal energy attained at \( \mu_s^* \) is

\[ H_s[\mu_s^*] = \frac{1}{2} \left[ \int_{\mathbb{R}^2} d\mu_s^*(r) V_s(|\bar{r}|) + V_s(R_0) - \log R_0 \right]. \]  

(18)

Evaluating (18), we get the cumulant GF \( J(s) \) in (5) as excess energy (11).

Since \( J(s) \) is strictly concave and differentiable, using the properties of the Legendre-Fenchel transform, we can write the thermodynamical relation \( J(s) - \Psi(x) = sx \), where the conjugate variables \( x \) and \( s \) are related by

\[ x(s) = J'(s) \quad \text{and} \quad s(x) = -\Psi'(x), \]  

(19)

with \( x(s) \) given in (17). These relations, supplemented with the conditions \( J(0) = \Psi(x(0)) = 0 \), provide the following simple expression for the rate function \( \Psi(x) \):

\[ \Psi(x) = -\int_{\frac{s}{2}}^x s(x') dx', \]  

(20)

where \( s(x) \) is the (real non increasing) inverse of (17) and we have used \( x(0) = 2/3 \). One may verify that the cumulant GF (5) can be computed also from (19). The asymptotic analysis of \( \psi(x) \) in (20) provides the limiting behaviour summarized in (7).

The integrable case \( \beta = 2 \). As already mentioned, the value \( \beta = 2 \) is special. At this value of the plasma parameter the 2D-OCP canonical measure has a determinantal structure and the partition function is integrable at finite \( N \). This fact has been largely exploited in random matrix theory. For \( \beta = 2 \), the Laplace transform of radially symmetric linear statistics
has been computed exactly by Forrester [33]. By specializing his result to \( \Delta_N \) we obtain the following finite \( N \) formula at (crucially) \( \beta = 2 \):

\[
\hat{P}_{N,\beta=2}(s) = \prod_{\ell=1}^{N} \left[ \frac{\int_{0}^{\infty} dt e^{-(\ell+2s\sqrt{\ell}N) t^{-1}}}{\int_{0}^{\infty} dt e^{-\ell t^{-1}}} \right].
\]  

(21)

In Fig. 2 we show that our large \( N \) result (5) agrees with this finite \( N \) result even for moderate values of \( N \). We stress however that the large deviation formula is valid for any value of \( \beta > 0 \). Moreover, the route presented here shows explicitly that, even at \( \beta = 2 \), the function \(-1/(\beta N^2) \log \hat{P}_{N,\beta}(s)\) analytic at finite \( N \) (see (21)) develops a non-analyticity as a result of the thermodynamic limit. To understand better the emergence of the singularity we have performed another numerical experiment. For \( \beta = 2 \) (the unordered!) collection of displacements \( \{r_k\} \) of the charges is distributed as a collection of independent random variables \( \{N^{-1/2}\xi_k\} \) where \( \xi_k \geq 0 \) has density \( f_k(x) = (2/\Gamma(k))x^{2k-1}\exp(-x^2) \) and moments \( \langle \xi_k^m \rangle = \Gamma(k + m/2)/\Gamma(k) \) [40, 41]. The cumulants \( \kappa_m \) are linear functionals for independent variables and hence we have \( \kappa_m(\Delta_N) = N^{-m/2} \sum_{k=1}^{N} \kappa_m(\xi_k) \). We have evaluated numerically this sum for increasing values of \( N \) and compared the output with our \( N \gg 1 \) results on the cumulants (6) (see Fig. 2). For \( m = 1, 2, 3 \), the sum asymptotically approaches (6), while the fourth cumulant \( \kappa_4(\Delta_N) \) shows a bizarre behaviour for large \( N \). For \( \beta = 2 \), this is a further confirmation of the weak singularity of the large deviations functions.

**Role of the logarithmic interaction.** In order to convey the key ideas of this paper, we conclude with a brief digression to clarify the role of the 2D Coulomb interaction in the mechanism of the phase transition. The computation of \( J(s) \) led to study the minimisation problem (12). For \( s \geq 0 \) the effective external potential \( V_s(r) \) in (9) is convex and the distribution of charges \( \mu^*_s \) is supported on a disk. When \( s \) is negative the convexity of \( V_s(r) \) is broken and the plasma distribution concentrates on an annulus. One might think that this change in the potential is sufficient to modify the topology of the charge distribution and to induce a phase transition even in absence of interaction between the charges. This is incorrect, as we will demonstrate.

Removing the logarithmic interaction from (2) turns the plasma into a non-interacting gas with single-particle energy \( H(\{r_k\}) = (1/2) \sum r_k^2 \) at inverse temperature \( \beta > 0 \) [49]. Since the particles are independent, the single-particle distribution is Gaussian (the system is no more confined in a bounded region) and \( \Delta_N = N^{-1} \sum r_k \) is a sum of independent and identically distributed displacements. Hence, an elementary computation shows that \( \langle \Delta_N \rangle = \sqrt{\pi/2} \beta, \text{var}(\Delta_N) = (4 - \pi)/(2\beta N) \) and that the Laplace transform is \( \hat{P}_{N,\beta}(s) = \langle e^{-\beta s N \Delta_N} \rangle = [1 - \sqrt{2\pi \beta s} \exp(-s^2/2) \Phi(-s)]^N \), where \( \Phi(x) \) is the standard Gaussian cumulative function. (Note the different scaling in \( N \) and \( \beta \)-dependence in the above expressions compared with the 2D-OCP.) Therefore, in absence of interaction the free energy of the system is analytic in \( s \): non-interacting particles do not undergo phase transitions.

**Conclusions.** Many of the ideas presented here can be generalized to other problems. In particular, the method to compute large deviations formulas for 2D-OCP is applicable to a large class of radially symmetric observables \( A_N = \sum f(|r_k|^2) \) with \( f(r) \) continuous. The problem of discontinuous statistics, like the charge fluctuations (i.e. the index problem in random matrices), is delicate and needs more refined tools (see [42] for recent developments).

We detect a novel and unusual type of weak phase transition due to a change of topology in the plasma distribution. Therefore this work naturally prompts the following question: is this fourth-order phase transition on the plane as ubiquitous and universal as the third-order transition for charged particles on the line? We mention that similar disk-annulus phase transitions have been detected for non-Hermitian non-Gaussian random matrices [43, 44]. However, non-Gaussian ensembles do not have a 2D-OCP interpretation and it is not clear how to relate these works with our findings.

An interesting question is to look at the finite \( N \) corrections of the large deviations principle (3)-(4). At microscopic scales it has been shown [1, 17, 39] that, among all lattice configurations, the charges of the plasma prefer to arrange themselves in a triangular one (the very same of the Abrikosov lattice [45] of vortices in the Ginzburg-Landau model). Showing that the triangular lattice is the global minimizer of the 2D-OCP energy requires a careful analysis beyond the mean-field limit. Such a
conjecture is still unsolved even in the integrable case $\beta = 2$.

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[48] For more references on the terminology see [46, 47].

[49] Here we drop off the $N$ scaling in front of the harmonic potential in order to keep the particles of the gas $r_k = O(1)$. 