



# Extended solutions for the biadjoint scalar field



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## ABSTRACT

Biadjoint scalar field theories are increasingly important in the study of scattering amplitudes in various string and field theories. Recently, some first exact nonperturbative solutions of biadjoint scalar theory were presented, with a pure power-like form corresponding to isolated monopole-like objects located at the origin of space. In this paper, we find a novel family of extended solutions, involving non-trivial form factors that partially screen the divergent field at the origin. All previous solutions emerge as special cases.

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## 1. Introduction

The study of (quantum) field theories remains a highly active research area, given that such theories describe the four fundamental forces in nature. It is widely believed that these forces may emerge as the low energy limit of an underlying framework, which may not necessarily be a field theory (e.g. it may be a string theory). It is thus interesting to elucidate common structures between theories, and also to investigate how different types of theory are related to each other. In this context, so-called *biadjoint scalar theory* has recently been increasingly studied, whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \partial^\mu \Phi^{aa'} \partial_\mu \Phi^{aa'} + \frac{y}{3} f^{abc} \tilde{f}^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'}. \quad (1)$$

Here the (un)primed indices are adjoint indices associated with two different Lie groups, with structure constants  $f^{abc}$  and  $\tilde{f}^{a'b'c'}$  respectively, and  $y$  is a coupling constant. Although this theory does not seem to be directly physically applicable by itself, there is increasing evidence that it underlies the dynamics in more relevant theories. For example, perturbative scattering amplitudes in this theory can be used as building blocks for amplitudes in non-abelian gauge theories [1,2]. The latter are related to amplitudes in gravity theories by the *double copy* of refs. [3–5], thus providing a ladder of theories related by taking the field  $\Phi^{aa'}$  of eq. (1), and replacing either of its adjoint indices with Lorentz (spacetime) indices. This same procedure can be applied to exact classical solutions as well as perturbative amplitudes [6–13], where again the

biadjoint theory plays a crucial role. Perturbative radiative solutions have also been considered recently [14–16], as well as amplitudes in curved space [17]. A natural framework for unifying the description of amplitudes in biadjoint, gauge and gravity theories is the CHY equations of refs. [18–21], which have been shown to emerge from a string theory in ambitwistor space [22–27], itself a limit of conventional string theory [28]. Loop level aspects of biadjoint theories in this formalism have been recently explored in ref. [29]. For related studies, see also refs. [30–37].

Given the wide range of instances in which the biadjoint scalar theory appears, it is clearly worth studying this theory in its own right. All of the above examples of its use involve perturbative solutions of the equation of motion, which from eq. (1) is found to be

$$\partial_\mu \partial^\mu \Phi^{aa'} - y f^{abc} \tilde{f}^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0. \quad (2)$$

Instead, one may consider exact nonlinear solutions, which may involve inverse powers of the coupling  $y$ . A number of solutions of this type were recently presented in ref. [38], consisting of singular point-like disturbances located at the origin. The first of these has spherical symmetry, and is applicable when the two Lie groups are identified:

$$\Phi^{ab} = -\frac{2\delta^{ab}}{yT_A r^2}. \quad (3)$$

Here  $r$  is the radial space coordinate,<sup>1</sup> and we define

$$f^{abc} f^{dbc} = T_A \delta^{ad}. \quad (4)$$

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<sup>1</sup> We use the metric (+, −, −, −) throughout.

Further solutions are possible if one takes the common group to be  $SU(2)$ , which allows mixing between spacetime and adjoint indices, analogous to known non-perturbative solutions in non-abelian gauge theories [39–44]. In particular, ref. [38] presented a one-parameter family of solutions with an axial form in the internal space:

$$\Phi^{ab} = \frac{1}{yr^2} \left[ -k \left( \delta^{ab} - \frac{x^a x^b}{r^2} \right) \pm \sqrt{2k - k^2} \frac{\epsilon^{abd} x^d}{r} \right], \quad (5)$$

where  $0 \leq k \leq 2$  if  $\Phi^{ab} \in \mathbb{R}$ . Like eq. (3), this has a pure power-like behaviour in  $r$ . Indeed, this is the only possibility if an axial form is imposed [38], such that it tempting to think that the spectrum of nonperturbative solutions in biadjoint scalar theory is much simpler than that of non-abelian gauge theories. The aim of this paper, however, is to show that extended solutions do in fact exist. That is, it is possible to dress the power-like divergence in  $r$  with a non-trivial form factor, such that all previously found solutions emerge as special cases. The new solutions are still singular at the origin (a consequence of Derrick's theorem for scalar field theories [45], which prohibits finite energy). However, one solution in particular will exhibit an interesting screening behaviour, such that the strength of the divergence in energy is ameliorated. Our results will be useful in future studies of biadjoint scalar theory, including the issue of whether or not double copy-like relationships can be generalised beyond the perturbative sector.

The structure of the paper is as follows. In section 2, we present extended solutions for the case in which both Lie groups in the biadjoint scalar Lagrangian are the same. In section 3, we apply similar techniques to construct an extended solution for the case in which both Lie groups are  $SU(2)$ . We discuss our results and conclude in section 4.

## 2. Extended solutions for common Lie groups

In this section, we consider the Lagrangian of eq. (1), but where both sets of structure constants are associated with the same Lie group. We may then make a similar ansatz to that made in ref. [38], namely

$$\Phi^{ab} = \delta^{ab} S(r), \quad S(r) = \frac{\bar{S}(r)}{y T_A}. \quad (6)$$

Substituting this in eq. (2) yields a second-order non-linear differential equation for  $\bar{S}(r)$ :

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\bar{S}(r)}{dr} \right) + \bar{S}^2(r) = 0. \quad (7)$$

We now look for extended solutions in which the  $r^{-2}$  divergence of the field at the origin, of eq. (3), is dressed by a finite form factor. To this end, we may write

$$K(r) = 1 + r^2 \bar{S}(r), \quad (8)$$

where  $K(r)$  is finite for all  $r$ . Equation (7) then becomes

$$r^2 K''(r) - 2rK'(r) + K^2(r) - 1 = 0. \quad (9)$$

We can study this further by defining

$$r = e^{-\xi} \quad (10)$$

so that eq. (9) becomes

$$\frac{\partial^2 K}{\partial \xi^2} + 3 \frac{\partial K}{\partial \xi} - 1 + K^2 = 0. \quad (11)$$

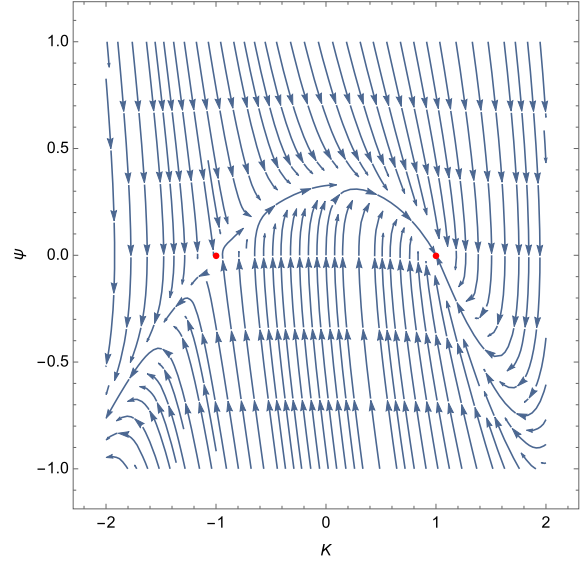


Fig. 1. The integral curves of the vector field defined by (14). The stationary points  $(\pm 1, 0)$  are marked in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

We may transform this into a more recognisable form by setting

$$\frac{\partial K}{\partial \xi} = -3w(K), \quad (12)$$

so that eq. (11) implies

$$w(K)w'(K) - w(K) + \frac{-1 + K^2}{9} = 0. \quad (13)$$

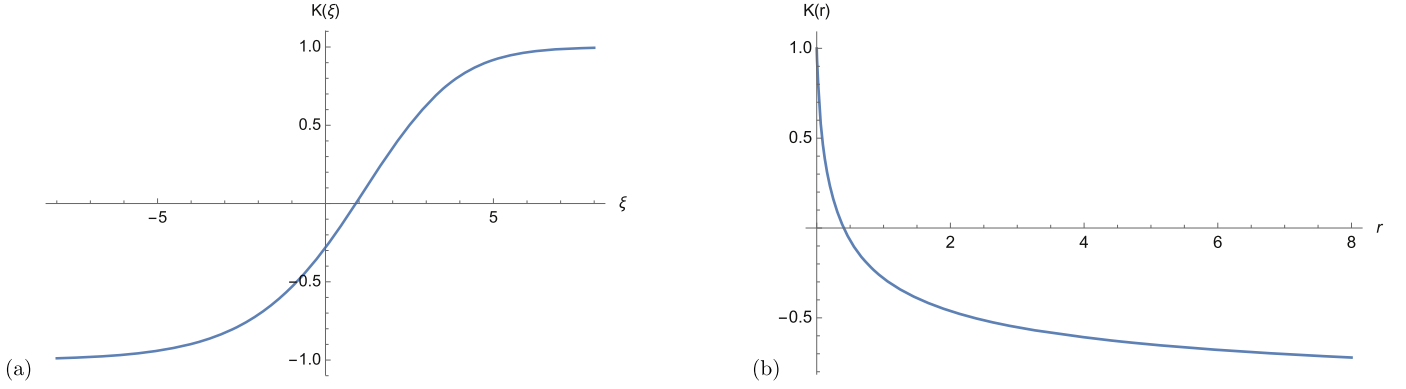
This is an Abel equation of the second kind. Unlike some equations of this type, there appears to be no analytic solution in terms of known functions (e.g. it is not listed in the well-known compendium of ref. [46]). Instead, we may revert back to eq. (11) and analyse it using a method similar to that used by Wu and Yang [44] to construct nonperturbative solutions in pure Yang–Mills theory. First, one may turn eq. (11) into two coupled first-order differential equations as follows:

$$\left( \frac{\partial K}{\partial \xi}, \frac{\partial \psi}{\partial \xi} \right) = \left( \psi, -3\psi + 1 - K^2 \right). \quad (14)$$

This defines a vector field in the  $(K, \psi)$  plane, where the curves that are tangent to this vector field are the solutions we desire. We show a plot of these curves in Fig. 1, and solutions for  $K(\xi)$  that are finite for all values of  $r$  correspond to curves which are bounded in the plane. By inspection, and noting that the vector field is zero at  $(K, \psi) = (\pm 1, 0)$ , there are three possibilities:

1.  $K(r) = 1$ : this corresponds to  $\bar{S}(r) = 0$ , and hence the trivial solution  $\Phi^{ab} = 0$ .
2.  $K(r) = -1$ : this yields  $\bar{S}(r) = -2/r^2$ , which is the solution already found in ref. [38], and reported here in eq. (3).
3.  $K(r) \rightarrow \pm 1$  as  $\xi \rightarrow \pm\infty$  respectively. This is a non-trivial solution corresponding to the single curve that flows from  $(-1, 0)$  to  $(+1, 0)$  in Fig. 1. It has not been previously obtained.

Let us analyse the new solution yet further. In fact, this is a one-parameter family of solutions: if  $(K(\xi), \psi(\xi))$  solve eq. (14), then  $(K(\xi - \xi_0), \psi(\xi - \xi_0))$ , for arbitrary constant  $\xi_0$ , are also solutions. Translating back to the spatial coordinate  $r$ , this implies that if  $K(r)$  solves eq. (11), so does the rescaled function  $K(r/\lambda)$ , for general constant  $\lambda$ .



**Fig. 2.** (a) Numerical solution of (11) with asymptotic behaviour given by (17). We have chosen  $A = 1$ ; other choices of  $A$  will lead to the same graph but translated along the  $\xi$ -axis. (b) The behaviour of  $K$  as a function of  $r$ .

Although a complete analytic solution for  $K(\xi)$  is not possible, we can examine its behaviour in asymptotic limits. For  $\xi \rightarrow -\infty$  one may write

$$K(\xi) = -1 + f(\xi).$$

Upon substituting this into eq. (11) and neglecting terms quadratic and higher in  $f(\xi)$ , one finds the general solution

$$f(\xi) = Ae^{\left(\frac{\sqrt{17}}{2}-\frac{3}{2}\right)\xi} + Be^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)\xi}. \quad (15)$$

The boundary condition  $f(\xi) \rightarrow 0$  as  $\xi \rightarrow -\infty$  then requires  $B = 0$ . Similarly, for  $\xi \rightarrow +\infty$  we may substitute

$$K(\xi) = 1 + g(\xi)$$

in eq. (11) and ignore terms quadratic and higher in  $g(\xi)$ , leading to the general solution

$$g(\xi) = A'e^{-\xi} + B'e^{-2\xi}, \quad (16)$$

where the first term gives the dominant behaviour. We conclude that

$$K(\xi) \simeq \begin{cases} -1 + Ae^{\left(\frac{\sqrt{17}}{2}-\frac{3}{2}\right)\xi}, & \xi \rightarrow -\infty \\ +1 + A'e^{-\xi}, & \xi \rightarrow +\infty \end{cases}, \quad (17)$$

or, translating back to the function  $\bar{S}(r)$ ,

$$\bar{S}(r) \simeq \begin{cases} \frac{1}{r^2} \left[ -2 + Ar^{-\frac{\sqrt{17}}{2} + \frac{3}{2}} \right], & r \rightarrow \infty \\ \frac{A'}{r}, & r \rightarrow 0. \end{cases} \quad (18)$$

The two parameters  $A$  and  $A'$  are not independent. As discussed above, all solutions for  $K(r)$  constitute a rescaling of a canonical solution. The scaling is fixed by choosing either  $A$  or  $A'$ , such that the other parameter also becomes fixed. For the particular choice  $A = 1$ , we show a numerical solution for  $K(\xi)$  in Fig. 2.

The simple power-like solution of eq. (3) is reminiscent of a Coulomb solution in gauge theory, with a correspondingly divergent field at the origin.<sup>2</sup> The new solution found here consists of dressing the previous solution with a form factor, such that the divergence at  $r \rightarrow 0$  is partially screened. That is, the  $r^{-2}$  behaviour is softened to  $r^{-1}$ , and by choosing the rescaling parameter  $\lambda$  in  $K(r/\lambda)$ , this screening can be made as smooth or abrupt as desired. The softening of the divergence in the field is also reflected

in the energy of the solution. From eq. (1), the Hamiltonian density of the biadjoint scalar theory is found to be [38]

$$\mathcal{H} = \frac{1}{2} \left[ (\dot{\Phi}^{aa'})^2 + \nabla \Phi^{aa'} \cdot \nabla \Phi^{aa'} \right] - \frac{y}{3} f^{abc} \tilde{f}^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'}, \quad (19)$$

which for the ansatz of eq. (6) evaluates to

$$\mathcal{H} = \frac{\mathcal{N}}{y^2 T_A^2} \left( \frac{1}{2} \bar{S}'(r)^2 - \frac{1}{3} \bar{S}(r)^3 \right), \quad (20)$$

with  $\mathcal{N}$  the dimension of the (common) Lie group. The energy is divergent due to the singularity at the origin, but can be evaluated by implementing a short-distance cutoff  $r_0$ , corresponding to a charge radius:

$$E = 4\pi \int_{r_0}^{\infty} dr r^2 \mathcal{H}. \quad (21)$$

For the solution with the asymptotic behaviour of eq. (18), this gives (as  $r_0 \rightarrow 0$ )

$$E \simeq \frac{\mathcal{N}}{y^2 T_A^2} \frac{2\pi (A')^2}{r_0}, \quad (22)$$

which is a much softer divergence than the energy associated with eq. (3), that behaves as  $E \sim r_0^{-3}$  [38].

### 3. Extended solutions for a common group of SU(2)

Having succeeded in constructing an extended solution for the case in which both gauge groups are identical, let us see if further such solutions are possible upon restricting each gauge group to be SU(2), for which one may make a similar ansatz to that used in ref. [38]

$$\Phi^{ab} = \frac{1}{r^2} \left( A(r)\delta^{ab} + B(r)\frac{x^a x^b}{r^2} + C(r)\epsilon^{abd}\frac{x^d}{r} \right). \quad (23)$$

Introducing  $\bar{A} = A/y$  etc., eq. (2) then implies the coupled non-linear differential equations

$$r^2 \bar{A}'' - 2r \bar{A}' + 2\bar{A}\bar{B} + 2\bar{A}^2 + 2\bar{A} + 2\bar{B} = 0; \quad (24)$$

$$r^2 \bar{B}'' - 2r \bar{B}' - 2\bar{A}\bar{B} - 4\bar{B} + 2\bar{C}^2 = 0; \quad (25)$$

$$r^2 \bar{C}'' - 2r \bar{C}' + 2\bar{A}\bar{C} + 2\bar{B}\bar{C} = 0. \quad (26)$$

<sup>2</sup> The fact that the power associated with the divergence is different to the Coulomb solution can be traced to the fact that the coupling constant  $y$  is dimensionful [38].

To simplify this we make a further ansatz, namely we write

$$\bar{A}(r) = c_1 f(r) + c_2 \quad (27)$$

$$\bar{B}(r) = c_3 f(r) + c_4 \quad (28)$$

$$\bar{C}(r) = c_5 f(r) + c_6, \quad (29)$$

and then we look for special values of the constants  $c_1, \dots, c_6$  so that the three equations (24)–(26) reduce to a single differential equation for  $f(r)$ . We thus find two new types of solution that were not presented previously in ref. [38].

### 3.1. Multiple power-like solution

The first new solution is obtained upon choosing

$$\bar{A}(r) = -1, \quad \bar{C}(r) = 0, \quad (30)$$

in which case  $\bar{B}(r)$  satisfies the linear homogeneous differential equation

$$r^2 \bar{B}''(r) - 2r \bar{B}'(r) - 2\bar{B}(r) = 0, \quad (31)$$

whose general solution is

$$\bar{B}(r) = r^{3/2} \left( b_1 r^{\frac{\sqrt{17}}{2}} + b_2 r^{-\frac{\sqrt{17}}{2}} \right), \quad (32)$$

where  $b_1$  and  $b_2$  are arbitrary constants. The full solution for  $\Phi^{ab}$  is then

$$\begin{aligned} \Phi^{ab} &= \frac{1}{yr^2} \left[ -\delta^{ab} + \left( b_1 r^{\frac{3}{2} + \frac{\sqrt{17}}{2}} + b_2 r^{\frac{3}{2} - \frac{\sqrt{17}}{2}} \right) \frac{x^a x^b}{r^2} \right] \\ &\simeq \frac{1}{yr^2} \left[ -\delta^{ab} + \left( b_1 r^{3.562} + b_2 r^{-0.562} \right) \frac{x^a x^b}{r^2} \right]. \end{aligned} \quad (33)$$

Here the second term in the square brackets has a part which diverges more rapidly at the origin than the  $r^{-2}$  behaviour of the solutions of ref. [38], and a term which is softer. If one requires a finite energy upon integrating to infinity, however, one must set  $b_1 = 0$ . The energy, from eq. (21), is then found to be

$$E = 4\pi \left( \frac{1}{4} (1 + \sqrt{17}) b_2^2 r_0^{-\sqrt{17}} - 2b_2 r_0^{-\frac{3}{2} - \frac{\sqrt{17}}{2}} + \frac{8}{3r_0^3} \right), \quad (34)$$

where again  $r_0$  is a small-distance cutoff. For small  $r_0$ , the energy diverges as  $E \sim r_0^{-\sqrt{17}} \simeq r_0^{-4.123}$  (assuming  $b_2 \neq 0$ ). Interestingly, the solution of eq. (33) involves the same power of  $r^{\frac{3}{2} - \frac{\sqrt{17}}{2}}$  as appears in the asymptotic behaviour for the extended solution of the previous section, eq. (18).

### 3.2. Extended solutions

We have also found extended solutions of eqs. (24)–(26), that do not have a pure power-like form. For these solutions, one has

$$\bar{A}(r) = -1 + \frac{1}{2}(c^2 - 1)\bar{B}(r), \quad \bar{C}(r) = c\bar{B}(r), \quad (35)$$

where  $c$  is a constant, and  $B(r)$  satisfies

$$r^2 \bar{B}''(r) - 2r \bar{B}'(r) + (c^2 + 1)\bar{B}(r) - 2\bar{B}(r) = 0. \quad (36)$$

Upon writing

$$\bar{B}(r) = \frac{1}{c^2 + 1} (K(r) + 1), \quad (37)$$

one finds that  $K(r)$  satisfies eq. (9). Thus, an extended solution for  $\Phi^{ab}$  is given by

$$\Phi^{ab} = \frac{1}{yr^2} \left[ -\delta^{ab} + \frac{K(r) + 1}{c^2 + 1} \left( \frac{c^2 - 1}{2} \delta^{ab} + \frac{x^a x^b}{r^2} + c \epsilon^{abd} \frac{x^d}{r} \right) \right]. \quad (38)$$

In section 2 we found three solutions for  $K(r)$ , which lead to the following cases:

1.  $K(r) = 1$ : in this case, eq. (38) reduces to

$$\Phi^{ab} = \frac{1}{yr^2} \left[ -\frac{2}{c^2 + 1} \left( \delta^{ab} - \frac{x^a x^b}{r^2} \right) + \frac{2c}{c^2 + 1} \epsilon^{abd} \frac{x^d}{r} \right]. \quad (39)$$

By replacing

$$c \rightarrow \pm \sqrt{\frac{2-k}{k}}, \quad (40)$$

we see that eq. (39) is the same as the solution of eq. (5), that was already presented in ref. [38].

2.  $K(r) = -1$ : in this case, eq. (38) reduces to

$$\Phi^{ab} = -\frac{1}{yr^2} \delta^{ab},$$

which is the same as eq. (3) for the special case in which both gauge groups are  $SU(2)$ , so that  $T_A = 2$ .

3. The most general solution has  $K(r)$  given by the function of Fig. 2, with asymptotic limits given by eq. (18). This solution is new.

The extended solution no longer has the pure-axial property of eq. (5), consistent with the fact noted in ref. [38] that extended axial solutions cannot exist. The energy of the solution in case 3 above, in terms of the usual short-distance cutoff  $r_0$ , is found to have the leading behaviour (as  $r_0 \rightarrow 0$ )

$$E \simeq \frac{32\pi}{y^2} \frac{1}{(1 + c^2)r_0^3}. \quad (41)$$

Note that the extended solution of section 2 (where again  $T_A = 2$ ) can also be obtained from eq. (38), by choosing  $c \rightarrow \infty$ :

$$\Phi^{ab} \xrightarrow{c \rightarrow \infty} \frac{K(r) - 1}{2yr^2} \delta^{ab}. \quad (42)$$

Finally, we note that a complex solution is also possible. By choosing  $c^2 = -1$  in eq. (36), this reduces to eq. (31), whose general solution is given by eq. (32). One then obtains

$$\Phi^{ab} = \frac{1}{yr^2} \left[ -\delta^{ab} + b_2 r^{\frac{3}{2} - \frac{\sqrt{17}}{2}} \left( -\delta^{ab} + \frac{x^a x^b}{r^2} \pm i \epsilon^{abd} \frac{x^d}{r} \right) \right], \quad (43)$$

where we have again set  $b_1 \rightarrow 0$  to ensure that the energy is bounded at spatial infinity.

## 4. Conclusion

Biadjoint scalar field theory occurs in a number of contexts, and plays an intriguing role in determining the dynamics of perturbative scattering amplitudes in gauge and gravity theories. The nonperturbative properties of this theory remain relatively unexplored, and in this paper we have presented a number of new solutions involving inverse powers of the coupling constant. Unlike the first solutions presented in ref. [38], the results of the present paper have an extended structure, indicating that the spectrum of nonperturbative solutions of biadjoint scalar theory is much richer

than has previously been suggested. Interestingly, the solutions presented here include those in which the strong divergent behaviour of the field at the origin is partially screened.

All known examples of the double copy between gauge and gravity theories (or the zeroth copy between gauge and biadjoint scalar theories) involve positive powers of the coupling constant, rather than negative powers as in this paper. It is thus not at all clear how the double copy can be extended to the nonperturbative sector. Nevertheless, our hope is that by collecting a catalogue of such solutions, one may be able to formulate a suitable copy procedure, which would then dramatically extend the remit of the double copy and related ideas. The simplest nonperturbative copy procedure would be expected between biadjoint and gauge theories, which motivates the present study. Furthermore, the fact that very little is so far known about biadjoint theories, despite their cropping up in a number of different contexts, makes these solutions of interest in themselves.

There are a number of avenues for further work. Firstly, it may be possible to find more solutions of the biadjoint scalar theory, including non-static solutions. Secondly, one may imagine coupling the biadjoint scalar to a gauge field, as occurs in some applications of the double copy. Finally, the question of whether the nonperturbative solutions found here and in ref. [38] can themselves be copied to gauge theory or gravity deserves further attention.

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