

QUANTUM BIANCHI IDENTITIES VIA DG CATEGORIES

EDWIN J. BEGGS & SHAHN MAJID

ABSTRACT. We use DG categories to derive analogues of the Bianchi identities for the curvature of a connection in noncommutative differential geometry. We also revisit the Chern-Connes pairing but following the line of Chern's original derivation. We show that a related DG category of extendable bimodule connections is a monoidal tensor category and in the metric compatible case obtain an analogue of a classical antisymmetry of the Riemann tensor. The monoidal structure implies the existence of a cup product on noncommutative sheaf cohomology. Another application shows that the curvature of a line module reduces to a 2-form on the base algebra. We illustrate the theory on the q -sphere, the permutation group S_3 and the bicrossproduct quantum spacetime $[r, t] = \lambda r$.

1. INTRODUCTION

Noncommutative differential geometry allows geometric ideas to extend to situations where the 'coordinate algebra' A is noncommutative[12]. The formulation of vector bundles as projective modules E , connections as maps $\nabla : E \rightarrow \Omega^1 \otimes_A E$ obeying a left Leibniz-type rule and principal bundles with quantum group fibre to which such E are sections of associated bundles[10], are all well known and there is a large modern literature. We follow here a constructive approach in which the structures of differential geometry are built up layer by layer starting with the 1-forms Ω^1 of which a recent review is [18]. As well as left connections we will also need 'bimodule connections', being left connections for which there exists a bimodule map $\sigma : E \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A E$ with respect to which a right Leibniz rule also holds[14, 26]. This approach allows for the formulation of metric compatibility and is an approach particularly to noncommutative Riemannian geometry[4, 5, 7].

In this paper we take a close look at the curvature of a connection in order to address two fundamental issues that remain open within the constructive approach. Namely, we establish the two Bianchi identities and in the metric bimodule connection case the antisymmetry with respect to the metric of the Riemann tensor. The first of the Bianchi identities is elementary and follows directly from definition of curvature, but the 2nd Bianchi identity and further symmetry or the curvature are less immediate and arise in formulation in the context of a certain DG category ${}_A\mathcal{G}$ for noncommutative differential geometry, which in classical geometry would simply be an implicit structure behind the symmetry rules for permuting indices for covariant derivatives of tensors, but which in noncommutative geometry we need to define more explicitly in order to understand the right formulation of such

Key words and phrases. Quantum Riemannian geometry, noncommutative geometry, Chern-Connes pairing, Bianchi identity, tensor category, quantum group, q -sphere, quantum gravity, quantum spacetime.

symmetries. As DG categories are now central to many approaches to algebraic geometry, their natural appearance in noncommutative differential geometry is not unexpected. For an excellent reference on DG categories see [16]. From the physics side, a deeper understanding of the full Bianchi identities and the symmetries of the Riemann tensor is likely a necessary step towards the right concept of Einstein tensor in noncommutative geometries. The latter is an open problem which is plausibly key to applications of noncommutative geometry to model Planck scale or quantum gravity effects. Such effects are expected to result in noncommutative or ‘quantum’ spacetime of which there are now numerous models; see [5, 23] for two models that include noncommutative Riemannian curvature.

We also revisit the well-known Chern-Connes pairing between K-theory and cyclic cohomology [12] in order more fully to connect the Chern and Connes approaches. At the time of Chern’s work [11], cyclic homology and cohomology had yet to be invented and instead of a pairing with K -theory, Chern’s proof relied on showing that the trace of a power of the curvature was independent of the connection chosen on a particular vector bundle. With the machinery of DG categories we now follow Chern’s original derivation followed by Connes’ idea of n -cycles for the final pairing. The answers obtained are exactly the same as those given by Connes’ theory, on substituting in the Grassmann connection defined in terms of a matrix idempotent, but our input data and derivation of the formula are different. Of course, Connes’ pairing with K -theory can make sense even when there is no sensible differential structure to calculate with, whereas our construction is explicitly differential geometric.

An outline of the paper is as follows. Section 2 has some preliminary definitions working over a possibly non-commutative unital algebra A over a field \mathbb{k} . Differential forms of different degrees are specified together as a differential graded algebra (Ω, d) generated by A, Ω^1 . We also define line bundles (‘line modules’) and the Fröhlich map. DG categories are defined at the start of Section 3 but the key idea is that in classical differential geometry, natural maps between vector bundles with connections do not always commute with the connections; one also needs also to be able to covariantly differentiate bundle maps. (Classically, a bundle map is simply a tensor with appropriate up and down indices, and is covariantly differentiated by standard rules.) Section 3 then constructs the DG category ${}_A\mathcal{G}$ where objects are bundles with connection but morphisms include maps $E \rightarrow \Omega^n \otimes_A F$ for each n . Theorem 3.2 proves the 2nd Bianchi identity and Corollary 3.4 revisits the 1st Bianchi identity in our formulation. Section 4 contains the Chern construction with connection-independence proven in Theorem 4.3. Section 5.1 switches to bimodule connections and introduces a notion of extendability that we need. Section 5 introduces the bimodule version ${}_A\mathcal{G}_A$ of bundles with bimodule connections, this time with tensor product over A i.e. we show that this is a monoidal category, which we do in Section 5.3. Corollary 5.8 is our Riemann curvature antisymmetry result when the connection preserves the quantum metric $g \in \Omega^1 \otimes_A \Omega^1$ (i.e., we prove a quantum version of the remaining classical symmetry of the Riemann tensor). A second application, Proposition 5.9, is to line bundles where we show that the curvature canonically defines a 2-form on the base of the bundle just as in classical electromagnetism. A third application, Corollary 5.10, uses a

quantum metric on a module to take the trace of the curvature of a connection to give a noncommutative de Rham class in the sense of the cohomology of (Ω, d) .

The paper concludes in Section 6 with some detailed examples including the quantum cotangent bundle on the group S_3 of permutations of three elements, a standard 2-dimensional quantum spacetime model and the standard q -sphere from the q -Hopf fibration with its associated q -monopole bundles E_n and q -Riemannian geometry. These are basic ‘test examples’ of noncommutative differential geometry in our approach [4, 7, 5, 20, 18, 21]. For examples like the q -sphere, extendability does not impose further restrictions thanks to our basic Lemma 5.3. Thus all the charge n q -monopole connections and the canonical torsion-free metric compatible or ‘quantum-Levi Civita’ connection ∇_{Ω^1} in [20] are extendable. The quantum antisymmetry holds and we compute the quantum metric trace of the curvature, which classically vanishes but now has a $q^4 - 1$ factor. The other two examples are at the other extreme in the sense that we show in these cases that extendability of a connection on Ω^1 is linked to flatness under certain homogeneity or symmetry conditions. For the bicrossproduct model spacetime we find, in particular, a unique metric compatible extendable connection among the homogenous type studied in [5], which is flat but with torsion. For S_3 with its 2-cycles calculus we find exactly four metric compatible extendable connections among the ad-invariant connections studied in [4]. Such flat connections classically are data for the quantisation of the differential structure in Poisson-Riemannian geometry[6] and it is remarkable that quantum versions arise now form our algebraic criterion of extendability. We also exhibit the full moduli of torsion-free and cotorsion-free or ‘weak quantum Levi-Civita’ connections and of metric compatible with-torsion connections, illustrating the Bianchi identities on these.

2. PRELIMINARIES

We recall that a differential graded algebra (DGA) (Ω, d) means graded components Ω^n for $n \in \mathbb{N}$, differentials $d : \Omega^n \rightarrow \Omega^{n+1}$ and an associative product $\wedge : \Omega^n \otimes \Omega^m \rightarrow \Omega^{n+m}$ obeying

$$d(\xi \wedge \eta) = (d\xi) \wedge \eta + (-1)^n \xi \wedge (d\eta) , \quad d^2 = 0$$

for $\xi \in \Omega^n$ and $\eta \in \Omega^m$. We suppose that $A = \Omega^0$ is a unital algebra, that Ω is generated by A under d and \wedge , and then call (Ω, d, \wedge) a differential calculus or ‘exterior algebra’ for A . We shall work over the general field \mathbb{k} where possible. Also well known (for example in the works of Quillen in the 1970s) is the notion left connection (or covariant derivative) $\nabla_E : E \rightarrow \Omega^1 \otimes_A E$ on a left A -module E as a linear map obeying the left Leibniz rule,

$$\nabla_E(a.e) = da \otimes e + a.\nabla_E(e)$$

for all $a \in A$ and $e \in E$. This can be extended to

$$(1) \quad \nabla_E^{[n]} : \Omega^n \otimes_A E \rightarrow \Omega^{n+1} \otimes_A E , \quad \xi \otimes e \mapsto d\xi \otimes e + (-1)^n \xi \wedge \nabla_E e$$

for $n \geq 0$, the $n = 0$ case giving ∇_E . These extensions are used in [2] to define a sheaf cohomology, and to write the curvature R_E of (E, ∇_E) as the left module map

$$R_E = \nabla_E^{[1]} \nabla_E^{[0]} : E \rightarrow \Omega^2 \otimes_A E.$$

We also need right connections, which we write as $\tilde{\nabla}_F : F \rightarrow F \otimes_A \Omega^1$ for a right A -module F , and which obey the right Leibniz rule

$$\tilde{\nabla}_F(f.a) = f \otimes da + (\tilde{\nabla}_F f)a$$

for $a \in A$ and $f \in F$.

In the special case where we have a left connection ∇_{Ω^1} on the bimodule Ω^1 , we have the further notion of ‘torsion’

$$T = \wedge \nabla_{\Omega^1} - d : \Omega^1 \rightarrow \Omega^2.$$

We may also be interested in a right connection $\tilde{\nabla}_{\Omega^1}$ in which case the formula for the torsion is $\tilde{T} = \wedge \tilde{\nabla}_{\Omega^1} + d : \Omega^1 \rightarrow \Omega^2$. A brief check shows that T is a left module map and that \tilde{T} is a right module map. A modest first result in this case, which we state for left connections (there is a similar formula for right ones) is the following.

Lemma 2.1 (1st Bianchi identity). *For a left connection on Ω^1 ,*

$$\wedge R_{\Omega^1} = dT - (\text{id} \wedge T)\nabla_{\Omega^1}.$$

Proof. We adopt the shorthand $\nabla_{\Omega^1}\xi = \xi^1 \otimes \xi^2$ (summation understood) then

$$\begin{aligned} \wedge R_{\Omega^1}(\xi) &= (d\xi^1) \wedge \xi^2 - (\xi^1 \wedge \xi^{21}) \wedge \xi^{22} \\ &= d(\xi^1 \wedge \xi^2) + \xi^1 \wedge (d\xi^2 - \xi^{21} \wedge \xi^{22}) = d^2\xi + dT(\xi) - \xi^1 \wedge T(\xi^2) \end{aligned}$$

for all $\xi \in \Omega^1$. We use $d^2 = 0$ and we used associativity of the wedge product. \square

For quantum Riemannian geometry in the constructive approach one also has a metric $g \in \Omega^1 \otimes_A \Omega^1$ and in this context we define the ‘cotorsion’ of a left connection as

$$\text{co}T = (d \otimes \text{id} - \text{id} \wedge \nabla_{\Omega^1})g$$

of which the vanishing is a weak notion of metric-compatibility[19, 4, 18]. For full metric compatibility we will need more structure as we describe later, in Section 5.1.

Next we recall that a left A -module E has dual the right module $E^b = {}_A\text{Hom}(E, A)$ (the left module maps from E to A) with evaluation a bimodule map $\text{ev} : E \otimes E^b \rightarrow A$. By definition, a left finitely generated projective (left fgp for short) A -module E has a dual basis $e^i \in E$ and $e_i \in E^b$ (for a finite number of indices i) so that (sum understood)

$$(2) \quad (\text{ev} \otimes \text{id})(\text{id} \otimes e_i \otimes e^i) = \text{id} : E \rightarrow E .$$

In the case where E is also a bimodule we can define a coevaluation bimodule map $\text{coev} : A \rightarrow E^b \otimes_A E$ so that $\text{coev}(1) = e_i \otimes e^i$ (sum understood). There are formulae for the change of dual basis which read

$$(3) \quad e^i = c_j(e^i).c^j, \quad e_i = c_j.e_i(c^j), \quad c^j = e_i(c^j).e^i, \quad c_j = e_i.c_j(e^i).$$

for an alternative dual basis $c^j \in E$ and $c_j \in E^b$. We shall need a result on dual connections on E^b from [4]: For the left fgp module E with left connection ∇_E , there is a unique right connection $\tilde{\nabla}_{E^b} : E^b \rightarrow E^b \otimes_A \Omega^1$ given by

$$\tilde{\nabla}_{E^b}(\alpha) = e_j \otimes d \text{ev}(e^j \otimes \alpha) - e_j \otimes (\text{id} \otimes \text{ev}_E)(\nabla_E(e^j) \otimes \alpha).$$

so that

$$d \circ \text{ev} = (\text{id} \otimes \text{ev})(\nabla_E \otimes \text{id}) + (\text{ev} \otimes \text{id})(\text{id} \otimes \tilde{\nabla}_{E^b}) : E \otimes E^b \rightarrow \Omega^1.$$

Now we apply the covariant derivatives to the dual basis to get

$$\begin{aligned} & (\tilde{\nabla}_{E^b} \otimes \text{id} + \text{id} \otimes \nabla_E)(e_i \otimes e^i) \\ &= e_j \otimes d \text{ev}(e^j \otimes e_i) \otimes e^i - e_j \otimes (\text{id} \otimes \text{ev}_E)(\nabla_E(e^j) \otimes e_i) \otimes e^i + e_i \otimes \nabla_E(e^i) \\ &= e_j \otimes d \text{ev}(e^j \otimes e_i) \otimes e^i \end{aligned}$$

where we have used (2). Now setting $P_{ji} = \text{ev}(e^j \otimes e_i)$ we have

$$(\tilde{\nabla}_{E^b} \otimes \text{id} + \text{id} \otimes \nabla_E)(e_i \otimes e^i) = e_j \otimes dP_{ji} \otimes e^i = e_q \otimes P_{qj} \cdot dP_{ji} \cdot P_{ik} \otimes e^k$$

and applying d to the matrix product $P^2 = P$ we get $P \cdot dP \cdot P = 0$, so

$$(4) \quad (\tilde{\nabla}_{E^b} \otimes \text{id} + \text{id} \otimes \nabla_E)(e_i \otimes e^i) = 0.$$

We also recall the definition of a line module, which is the direct noncommutative generalisation of a line bundle, over a unital algebra A . This means a left fgp A -bimodule L for which the bimodule maps of coevaluation $\text{coev} : A \rightarrow L^b \otimes_A L$ and evaluation $\text{ev} : L \otimes_A L^b \rightarrow A$ are isomorphisms. Such modules were originally introduced as Morita contexts in algebraic K -theory (see [1]), and later as geometric objects in [3]. A standard result we will need later (from e.g. [1]) is:

Lemma 2.2. *Let L be a line A -module. If $T : L \rightarrow L \otimes_A F$ is a left A -module map for some left A -module F then there is a unique $f \in F$ such that $T(e) = e \otimes f$. If $S : L \rightarrow E \otimes_A L$ is a right A -module map for some right A -module E then there is a unique $g \in E$ so that $S(e) = g \otimes e$.*

It is shown in [15] that associated to a line module L is a unique unital algebra automorphism Φ_L of the centre $Z(A)$ (we will call it the Fröhlich map) such that

$$z \cdot e = e \cdot \Phi_L(z)$$

for all $e \in L$ and $z \in Z(A)$. This map depends only on the isomorphism class of the bimodule and obeys $\Phi_{L \otimes M} = \Phi_M \circ \Phi_L$.

3. DG CATEGORY ${}_A\mathcal{G}$ AND THE 2ND BIANCHI IDENTITY

For any field \mathbb{k} , we define (see [25]) a \mathbb{k} -category to be an additive category with a ring homomorphism from \mathbb{k} to the natural transformations from the identity functor to itself. An example of a \mathbb{k} -category is the category of vector spaces and linear maps over \mathbb{k} , where \mathbb{k} acts by multiplication on each object. Now we define a DG category (see [16]):

Definition 3.1. *A differential graded category (or DG category) is a \mathbb{k} -category where each $\text{Mor}(X, Y)$ is a cochain complex (in a weak sense where we do not require the differential squared to be zero) of \mathbb{k} -vector spaces, and composition $\circ : \text{Mor}(Y, Z) \otimes \text{Mor}(X, Y) \rightarrow \text{Mor}(X, Z)$ is a map of cochain complexes.*

In what follows we use an order of products which is natural for modules and connections, even if it is not standard for DG categories. Given two left A -modules with connection (E, ∇_E) and (F, ∇_F) a left module map $\phi : E \rightarrow F$ is said to intertwine the connections if

$$\nabla_F \circ \phi = (\text{id} \otimes \phi) \circ \nabla_E : E \rightarrow \Omega^1 \otimes_A F.$$

If we generalise the maps which we consider to all left module maps then it is useful to define the derivative of $\phi : E \rightarrow F$ to be

$$\mathbb{W}(\phi) = \nabla_F \circ \phi - (\text{id} \otimes \phi) \circ \nabla_E : E \rightarrow \Omega^1 \otimes_A F$$

so $\mathbb{W}(\phi) = 0$ precisely when ϕ intertwines the connections. Note that $\mathbb{W}(\phi)$ is a left module map when ϕ is. It is convenient to form a category with objects left modules with connections, and to grade the morphisms from E to F by \mathbb{N} so that $\phi : E \rightarrow F$ is in $\text{Mor}_0(E, F)$ and $\mathbb{W}(\phi)$ is in $\text{Mor}_1(E, F)$. Our basic example of a DG category for noncommutative geometry will be based on the following category with graded morphisms.

Theorem 3.2. *For an algebra with differential calculus (A, Ω, d, \wedge) we let ${}_A\mathcal{G}$ be the category with morphisms graded by \mathbb{N} (including $0 \in \mathbb{N}$) as given by the table:*

Name	Objects	$\text{Mor}_n((E, \nabla_E), (F, \nabla_F))$	derivative
${}_A\mathcal{G}$	(E, ∇_E) left modules \mathcal{E} left connections	$\psi : E \rightarrow \Omega^n \otimes_A F$ left module maps	$\mathbb{W} : \text{Mor}_n(E, F) \rightarrow \text{Mor}_{n+1}(E, F)$ $\mathbb{W}(\psi) = \nabla_F^{[n]} \circ \psi - (\text{id} \wedge \psi) \nabla_E$

Composition of morphisms is given by

$$\phi \circ \psi = (\text{id} \wedge \phi)\psi : E \rightarrow \Omega^{n+m} \otimes_A G$$

for $\phi \in \text{Mor}_m((F, \nabla_F), (G, \nabla_G))$. Then ${}_A\mathcal{G}$ is a DG category, with

$$\mathbb{W}(\phi \circ \psi) = \phi \circ \mathbb{W}(\psi) + (-1)^n \mathbb{W}(\phi) \circ \psi.$$

Moreover,

- (1) (2nd Bianchi) The curvature $R_E \in \text{Mor}_2((E, \nabla_E), (E, \nabla_E))$ has $\mathbb{W}(R_E) = 0$;
- (2) For all morphisms $\psi : E \rightarrow F$, $\mathbb{W}(\mathbb{W}(\psi)) = R_F \circ \psi - \psi \circ R_E$.

Proof. We first show that $\mathbb{W}(\psi)$ is a left module map. For $a \in A$, $e \in E$, and $\psi(e) = \xi \otimes f$ (summation implicit),

$$\begin{aligned} \nabla_F^{[n]} \circ \psi(a.e) - (\text{id} \wedge \psi) \nabla_E(a.e) &= \nabla_F^{[n]}(a.\xi \otimes f) - (\text{id} \wedge \psi)(da \otimes e + a.\nabla_E(e)) \\ &= da \wedge \xi \otimes f + a.\nabla_F^{[n]}(\xi \otimes f) - da \wedge \psi(e) + a.\nabla_E(e) = a.(\nabla_F^{[n]} \circ \psi(e) - (\text{id} \wedge \psi) \nabla_E(e)). \end{aligned}$$

Next setting $\phi(f) = \kappa \otimes g$ (summation implicit),

$$\begin{aligned} \nabla_G^{[n+m]}(\text{id} \wedge \phi)\psi(e) &= \nabla_G^{[n+m]}(\xi \wedge \phi(f)) = \nabla_G^{[n+m]}(\xi \wedge \kappa \otimes g) \\ &= d(\xi \wedge \kappa) \otimes g + (-1)^{n+m} \xi \wedge \kappa \wedge \nabla_G g = d\xi \wedge \phi(f) + (-1)^n \xi \wedge \nabla_G^{[m]}(\phi(f)) \\ &= (\text{id} \wedge \phi) \nabla_F^{[n]} \psi(e) + (-1)^n \xi \wedge \mathbb{W}(\phi) f \\ &= (\text{id} \wedge \phi) \mathbb{W}(\psi) e + (\text{id} \wedge (\text{id} \wedge \phi) \psi) \nabla_E e + (-1)^n \xi \wedge \mathbb{W}(\phi) f. \end{aligned}$$

For (1),

$$\begin{aligned}\mathbb{W}(R_E) &= \nabla_E^{[2]} \circ R_E - (\text{id} \wedge R_E) \nabla_E \\ &= \nabla_E^{[2]} \circ \nabla_E^{[1]} \circ \nabla_E - (\text{id} \wedge (\nabla_E^{[1]} \circ \nabla_E)) \nabla_E.\end{aligned}$$

Now set $\nabla_E e = \xi \otimes h$ and $\nabla_E h = \eta \otimes f$ (summation implicit),

$$\begin{aligned}\mathbb{W}(R_E)(e) &= \nabla_E^{[2]}(\nabla_E^{[1]}(\xi \otimes h)) - \xi \wedge (\nabla_E^{[1]} \circ \nabla_E)h \\ &= \nabla_E^{[2]}(d\xi \otimes h - \xi \wedge \nabla_E h) - \xi \wedge (\nabla_E^{[1]} \circ \nabla_E)h \\ &= -d(\xi \wedge \eta) \otimes f + d\xi \otimes \nabla_E h - \xi \wedge \eta \wedge \nabla_E f - \xi \wedge d\eta \otimes f + \xi \wedge \eta \wedge \nabla f = 0.\end{aligned}$$

To prove (2) we begin by showing that for all $n \geq 0$,

$$\nabla_E^{[n+1]} \circ \nabla_E^{[n]} = \text{id} \wedge R_E : \Omega^n \otimes_A E \rightarrow \Omega^{n+2} \otimes_A E.$$

Putting $\nabla_E e = \xi \otimes h$ again for convenience, and for all $\omega \in \Omega^n$,

$$\begin{aligned}\nabla_E^{[n+1]}(\nabla_E^{[n]}(\omega \otimes e)) &= \nabla_E^{[n+1]}(d\omega \otimes e + (-1)^n \omega \wedge \nabla_E e) \\ &= (-1)^{n+1} d\omega \wedge \nabla_E e + (-1)^n d\omega \wedge \xi \otimes e + \omega \wedge d\xi \otimes h - \omega \wedge \xi \wedge \nabla_E h \\ &= \omega \wedge (d\xi \otimes h - \xi \wedge \nabla_E h) = \omega \wedge R_E(e).\end{aligned}$$

To complete the proof of (2),

$$\begin{aligned}\mathbb{W}(\mathbb{W}(\psi)) &= \nabla_F^{[n+1]} \circ \mathbb{W}(\psi) - (\text{id} \wedge \mathbb{W}(\psi)) \nabla_E \\ &= \nabla_F^{[n+1]} \circ \nabla_F^{[n]} \circ \psi - \nabla_F^{[n+1]} \circ (\text{id} \wedge \psi) \nabla_E - (\text{id} \wedge (\nabla_F^{[n]} \circ \psi)) \nabla_E \\ &\quad + (\text{id} \wedge \psi)(\text{id} \wedge \nabla_E) \nabla_E \\ &= (\text{id} \wedge R_F) \psi - \nabla_F^{[n+1]} \circ (\text{id} \wedge \psi) \nabla_E - (\text{id} \wedge (\nabla_F^{[n]} \circ \psi)) \nabla_E \\ &\quad - (\text{id} \wedge \psi) R_E + (\text{id} \wedge \psi)(d \wedge \text{id}) \nabla_E.\end{aligned}$$

and if $\nabla_E e = \xi \otimes h$ and $\psi(h) = \eta \otimes f$ then

$$\begin{aligned}\nabla_F^{[n+1]} \circ (\text{id} \wedge \psi) \nabla_E(e) &= \nabla_F^{[n+1]}(\xi \wedge \eta \otimes f) \\ &= d\xi \wedge \eta \otimes f - \xi \wedge d\eta \otimes f - (-1)^n \xi \wedge \eta \wedge \nabla_F f, \\ (\text{id} \wedge (\nabla_F^{[n]} \circ \psi)) \nabla_E(e) &= \xi \wedge \nabla_F^{[n]}(\eta \otimes f) \\ &= \xi \wedge d\eta \otimes f + (-1)^n \xi \wedge \eta \wedge \nabla_F f. \quad \square\end{aligned}$$

In this result (1) becomes the usual 2nd Bianchi identity in the classical case when this is viewed in Ω^3 , while (2) tells that the curvatures form an obstruction to the cochain complex on each morphism space having differential \mathbb{W} that squares to zero. This means that each $(\text{Mor}(E, E), \mathbb{W}, R_E)$ in ${}_A\mathcal{G}$ forms a curved DGA as described in [8]. Now we show how some almost tautological morphisms in ${}_A\mathcal{G}$ can have a nice geometric interpretation, including a version of the first Bianchi identity.

Example 3.3. *Let A be an algebra with differential structure and consider (A, d) as a bimodule with connection. Suppose that Ω^1 has a left connection ∇ and consider the morphism $\tau \in \text{Mor}_1(\Omega^1, A)$ given by $\xi \mapsto \xi \otimes_A 1$. Then $(\mathbb{W}\tau)(\xi) = d\xi \otimes 1 - (\text{id} \wedge \tau) \nabla \xi = (d\xi - \wedge \nabla \xi) \otimes 1$ showing that $\mathbb{W}\tau = -T_\nabla \in \text{Mor}_2(\Omega^1, A)$ where T_∇ is the torsion.*

There is a corollary of Theorem 3.2 which will give us a more conceptual way of thinking about the 1st Bianchi identity. We note that classically the vector fields are dual to the 1-forms, and this continues to provide a good definition of vector field in noncommutative geometry (as opposed to the idea of identifying vector fields with derivations). Of course in noncommutative geometry we have to choose a side.

Corollary 3.4 (1st Bianchi identity revisited). *Suppose that Ω^1 is right finitely generated projective with dual \mathfrak{X}^R (the right vector fields) and suppose \mathfrak{X}^R has a left connection with $(\mathfrak{X}^R, \nabla_{\mathfrak{X}}) \in {}_A\mathcal{G}$. Then for the coevaluation map $\text{coev} : A \rightarrow \Omega^1 \otimes_A \mathfrak{X}^R$,*

$$\mathbb{W}(\mathbb{W}(\text{coev})) = R_{\mathfrak{X}} \circ \text{coev} \in \text{Mor}_3(A, \mathfrak{X}^R).$$

Proof. Under our assumptions, $\text{coev} : A \rightarrow \Omega^1 \otimes_A \mathfrak{X}^R$ is in $\text{Mor}_1(A, \mathfrak{X}^R)$ and from Theorem 3.2 we have the 1st Bianchi identity as stated. \square

At first sight this looks nothing like the first Bianchi identity but remember that if we set E to be the left fgp module \mathfrak{X}^R then its dual E^b is Ω^1 so we have a corresponding right connection $\tilde{\nabla} = \tilde{\nabla}_{\Omega^1}$ on Ω^1 such that (4) becomes

$$(5) \quad (\tilde{\nabla}_{\Omega^1} \otimes \text{id} + \text{id} \otimes \nabla_{\mathfrak{X}})\text{coev}(1) = 0 \in \Omega^1 \otimes_A \Omega^1 \otimes_A \mathfrak{X}^R.$$

Corollary 3.5. *If the torsion \tilde{T} of $\tilde{\nabla}$ is a bimodule map then the 1st Bianchi identity in Corollary 3.4 becomes*

$$\mathbb{W}((\tilde{T} \otimes \text{id})\text{coev}) = (\text{id} \wedge R_{\mathfrak{X}})\text{coev} : A \rightarrow \Omega^3 \otimes_A \mathfrak{X}^R.$$

Proof. Applying $\wedge \otimes \text{id}$ to (5) gives

$$((\wedge \tilde{\nabla}_{\Omega^1}) \otimes \text{id} + \text{id} \wedge \nabla_{\mathfrak{X}})\text{coev}(1) = 0 \in \Omega^2 \otimes_A \mathfrak{X}^R$$

which we rewrite as

$$(\text{d} \otimes \text{id} + (\wedge \tilde{\nabla}_{\Omega^1}) \otimes \text{id})\text{coev}(1) = (\text{d} \otimes \text{id} - \text{id} \wedge \nabla_{\mathfrak{X}})\text{coev}(1).$$

Hence if $\tilde{T} = \text{d} + (\wedge \tilde{\nabla}_{\Omega^1}) : \Omega^1 \rightarrow \Omega^2$, i.e. the (right handed) torsion of $\tilde{\nabla}$, is a left (and hence bi-)module map then

$$\mathbb{W}(\text{coev}) = (\tilde{T} \otimes \text{id})\text{coev} \in \text{Mor}_2(A, \mathfrak{X}^R)$$

and we then use Corollary 3.4 to obtain the stated result. \square

The form version of this is the right handed version of Lemma 2.1, for example $\wedge \tilde{R} = 0$ in the case of zero torsion, but the above gives the Bianchi identity more conventionally expressed in terms of the curvature on vector fields. In the classical case with zero torsion and using the usual conventions for the Riemann tensor, our result in Corollary 3.5 reads

$$R^a{}_{bcd} dx^b \wedge dx^c \wedge dx^d \otimes \frac{\partial}{\partial x^a} = 0$$

which is the familiar classical first Bianchi identity in the absence of torsion.

4. CHARACTERISTIC CLASSES AND INVARIANCE

Chern defined characteristic classes of a vector bundle in terms of the trace of powers of the curvature [11]. To follow this construction literally in noncommutative geometry would require a formulation of trace, which is problematic without additional structures such as braiding. However, we can use a rather more limited version of trace.

Lemma 4.1. *Let F be an A -bimodule, E a left finitely generated projective module and a linear map $\phi : F \rightarrow \mathbb{k}$ have the trace property $\phi(a.f) = \phi(f.a)$ for all $a \in A$, $f \in F$. Then a left module map $\theta : E \rightarrow F \otimes_A E$ has a well defined trace*

$$\mathrm{Tr}_\phi(\theta) = \phi((\mathrm{id} \otimes \mathrm{ev})(\theta e^i \otimes e_i)) \in \mathbb{k}.$$

independently of the choice of dual basis e^i of E and e_i of E^\flat .

Proof. We need to show that the formula does not depend on the choice of dual basis. If we take a different dual basis c_j, c^j of E^\flat, E then

$$\begin{aligned} \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(c^j) \otimes c_j)) &= \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(e_i(c^j).e^i) \otimes e_k.c_j(e^k))) \\ &= \phi(e_i(c^j).(\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes e_k.c_j(e^k))) = \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes e_k.c_j(e^k) e_i(c^j))) \\ &= \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes e_k.e_i(c_j(e^k) c^j))) = \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes e_k.e_i(e^k))) \\ &= \phi((\mathrm{id} \otimes \mathrm{ev})(\theta(e^i) \otimes e_i)) \end{aligned}$$

on using the change of basis formulae in (3) and summing over i, j, k . \square

The linear maps we use here will be n -cycles in the sense of [12], since we want to recover the standard Chern-Connes pairing. Given a differential calculus Ω on A , Connes defines an n -cycle as a linear map $\int : \Omega^n \rightarrow \mathbb{k}$ satisfying the conditions

$$(6) \quad \int d\xi = 0, \quad \int \omega \wedge \rho = (-1)^{|\omega||\rho|} \int \rho \wedge \omega$$

for all $\xi \in \Omega^{n-1}$ and all forms ω, ρ with sum of degrees $|\omega| + |\rho| = n$. We now combine n -cycles with the derivatives of morphisms in ${}_A\mathcal{G}$.

Proposition 4.2. *Let E be a finitely generated projective left A -module and \int an $n+1$ -cycle on Ω . If $(E, \nabla_E) \in {}_A\mathcal{G}$ then any $\theta \in \mathrm{Mor}_n(E, E)$ has $\mathrm{Tr}_\int(\mathbb{V}(\theta)) = 0$.*

Proof. Take a dual basis $e^i \in E$ and $e_i \in E^\flat$. Using the dual connection,

$$\begin{aligned} \mathrm{Tr}_\int(\mathbb{V}(\theta)) &= \int (\mathrm{id} \otimes \mathrm{ev}_E)(\mathbb{V}(\theta)(e^i) \otimes e_i) \\ &= \int (\mathrm{id} \otimes \mathrm{ev}_E)((d \otimes \mathrm{id} + (-1)^n (\mathrm{id} \wedge \nabla_E))\theta(e^i) \otimes e_i - (\mathrm{id} \wedge \theta)\nabla_E(e^i) \otimes e_i) \\ &= \int d(\mathrm{id} \otimes \mathrm{ev}_E)(\theta(e^i) \otimes e_i) - (-1)^n \int (\mathrm{id} \wedge \mathrm{ev}_E \wedge \mathrm{id})(\theta(e^i) \otimes \tilde{\nabla}_{E^\flat}(e_i)) \\ &\quad - \int (\mathrm{id} \otimes \mathrm{ev}_E)((\mathrm{id} \wedge \theta)\nabla_E(e^i) \otimes e_i). \end{aligned}$$

Using that \int is an $n + 1$ cycle and the shorthand $\tilde{\nabla}_{E^b}(e_i) = e_j \otimes \eta_{ji}$ we have

$$(-1)^n \int (\text{id} \wedge \text{ev}_E \wedge \text{id})(\theta(e^i) \otimes \tilde{\nabla}_{E^b}(e_i)) = \int (\text{id} \otimes \text{ev}_E)(\eta_{ji} \wedge \theta(e^i) \otimes e_j)$$

while using the explicit formula for η_{ji} in the dual connection and the shorthand $\nabla_E(e^j) = \kappa \otimes f$ (summation implicit),

$$\begin{aligned} \eta_{ji} \wedge \theta(e^i) &= d \text{ev}(e^j \otimes e_i) \wedge \theta(e^i) - (\text{id} \otimes \text{ev}_E)(\nabla_E(e^j) \otimes e_i) \wedge \theta(e^i) \\ &= d \text{ev}(e^j \otimes e_i) \wedge \theta(e^i) - \kappa \text{ev}_E(f \otimes e_i) \wedge \theta(e^i) \\ &= d \text{ev}(e^j \otimes e_i) \wedge \theta(e^i) - \kappa \wedge \theta(\text{ev}_E(f \otimes e_i) e^i) \\ &= d \text{ev}(e^j \otimes e_i) \wedge \theta(e^i) - (\text{id} \wedge \theta) \nabla_E(e^j). \end{aligned}$$

Substituting our these expressions and setting $P_{ji} = \text{ev}(e^j \otimes e_i)$ so that $e^i = P_{ik} e^k$ and $e_j = e_m P_{mj}$, we arrive at

$$\begin{aligned} \text{Tr}_f(\mathbb{W}(\theta)) &= - \int (\text{id} \otimes \text{ev}_E)(d \text{ev}(e^j \otimes e_i) \wedge \theta(e^i) \otimes e_j) \\ &= - \int (\text{id} \otimes \text{ev}_E)(P_{mj} \cdot dP_{ji} \cdot P_{ik} \wedge \theta(e^k) \otimes e_m). \end{aligned}$$

But applying d to $P^2 = P$ tells us that $P \cdot dP \cdot P = 0$. \square

To apply these results to characteristic classes, we use powers of the curvature, and for $(E, \nabla_E) \in {}_A\mathcal{G}$ it is natural to take these as $R_E^2 = R_E \circ R_E$, $R_E^3 = R_E \circ R_E \circ R_E$ etc. Now from Theorem 3.2 we have $\mathbb{W}(R_E^n) = 0$. The next result is a noncommutative version of a standard classical result at the heart of Chern's construction, that the cohomology class of the trace of the curvature is independent of the connection. We do not construct characteristic classes themselves but only their value against a cycle.

Theorem 4.3. *Suppose that E is a finitely generated projective left A -module, and that \int is a $2n$ -cycle on the differential calculus Ω . Then for any left covariant derivative ∇_E and any n , the value of $\text{Tr}_f(R_E^n)$ is independent of the choice of ∇_E .*

Proof. The set of left connections on E is an affine space, so any two left connections can be connected by a straight line path. Take such a path, with connection ∇_E^t parameterised by $t \in \mathbb{R}$, and define $\dot{\nabla}_E : E \rightarrow \Omega^1 A \otimes_A E$ by

$$\dot{\nabla}_E(e) = \frac{d\nabla_E^t(e)}{dt}.$$

(As we are moving along a straight line interpolating two connections, $\dot{\nabla}_E$ does not depend on t .) For all t , we have $\nabla_E^t(a \cdot e) = da \otimes e + a \cdot \nabla_E^t(e)$ from which it follows that $\dot{\nabla}_E$ is a left module map. Differentiating $R_E^t = (\nabla_E^t)^{[1]} \circ \nabla_E^t$ (which is a quadratic function of t), we obtain

$$\dot{R}_E^t = -(\text{id} \wedge \dot{\nabla}_E) \nabla_E^t + (\nabla_E^t)^{[1]} \dot{\nabla}_E = \mathbb{W}^t(\dot{\nabla}_E).$$

Now omitting the explicit t dependence for clarity, by Theorem 3.2,

$$\begin{aligned} \frac{d}{dt} R_E^n &= \dot{R}_E \circ R_E^{n-1} + R_E \circ \dot{R}_E \circ R_E^{n-2} + \cdots + R_E^{n-1} \circ \dot{R}_E \\ &= \mathbb{W}(\dot{\nabla}_E) \circ R_E^{n-1} + R_E \circ \mathbb{W}(\dot{\nabla}_E) \circ R_E^{n-2} + \cdots + R_E^{n-1} \circ \mathbb{W}(\dot{\nabla}_E) \\ &= \mathbb{W}(\dot{\nabla}_E \circ R_E^{n-1} + R_E \circ \dot{\nabla}_E \circ R_E^{n-2} + \cdots + R_E^{n-1} \circ \dot{\nabla}_E). \end{aligned}$$

Then $\frac{d}{dt} \text{Tr}_f(R_E^n) = 0$ by Proposition 4.2. \square

We still need to know that left connections actually exist. This is known for any projective module, although here we recall just the finitely generated case given in terms of projection matrices. (By the Cuntz-Quillen theorem [13] a module is projective if and only if it has a connection for the *universal* calculus.) Thus, if E is an fgp left A -module and $e^i \in E$ and $e_i \in E^b$ a choice of dual bases, there is a well-known *Grassmann connection* with left covariant derivative $\nabla_E : E \rightarrow \Omega^1 \otimes_A E$ and curvature

$$\nabla_E e^i = \sum_i dP_{ij} \cdot P_{jk} \otimes e^k, \quad R_E e^i = - \sum_{j,k,m} dP_{ij} \wedge dP_{jk} \cdot P_{km} \otimes e^m,$$

where $P_{ij} = \text{ev}(e^i \otimes e_j) = e_j(e^i)$. Inserting this expression for R_E into Theorem 4.3, we clearly recover the usual formula for invariants associated to projection matrices in [12]. Of course, this was exactly the motivation behind this Chern-Connes pairing formula in the first place; the difference is that we do now follow Connes' construction via cyclic theory but stick to a noncommutative differential version of the original Chern one. In special cases with more structure, such as Corollary 5.10 below, we can obtain actual cohomology classes more like in Chern's theory, rather than evaluating them against a cycle.

5. THE DG CATEGORY ${}_A\mathcal{G}_A$

To proceed further, we will need our connections to be bimodule connections as these allow tensor products, and therefore a decent notion of metric compatibility with respect to a metric $g \in \Omega^1 \otimes_A \Omega^1$. We start with a technical extendability property, then define the DG category ${}_A\mathcal{G}_A$, and then prove it is monoidal under tensor products of bimodules over A . We assume throughout that A is equipped with a differential calculus (Ω, d, \wedge) .

5.1. Bimodule connections and extendability. For bimodules over a noncommutative algebra it turns out that asking for the left and right Leibniz rules to hold for a connection simultaneously is too strong a condition, and instead we use the following definition going back to [14, 26]. As mentioned in the introduction, a left *bimodule connection* (∇_E, σ_E) on a bimodule E is a left connection $\nabla_E : E \rightarrow \Omega^1 \otimes_A E$ such that there exists a bimodule map $\sigma_E : E \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A E$ obeying, for $e \in E$ and $a \in A$,

$$\nabla_E(e.a) = \sigma_E(e \otimes da) + (\nabla_E e).a.$$

Note that if it exists σ is uniquely determined so this is a property that some left connection have rather than additional data. The most important feature of bimodule connections is that we can tensor product them. Thus, if (E, ∇_E, σ_E) is a (left) bimodule connection on

the bimodule E and that (F, ∇_F) is a left connection on the left module F . Then we can define a tensor product connection on $E \otimes_A F$ by [9].

$$(7) \quad \nabla_{E \otimes F} = \nabla_E \otimes \text{id} + (\sigma_E \otimes \text{id})(\text{id} \otimes \nabla_F) : E \otimes_A F \rightarrow \Omega^1 \otimes_A E \otimes_A F.$$

Further if F is a bimodule with (F, ∇_F, σ_F) a (left) bimodule connection then we have a tensor product bimodule connection $(E \otimes_A F, \nabla_{E \otimes F}, \sigma_{E \otimes F})$, where

$$\sigma_{E \otimes F} = (\sigma_E \otimes \text{id})(\text{id} \otimes \sigma_F) : E \otimes_A F \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A E \otimes_A F.$$

This is key to applications in noncommutative Riemannian geometry[4, 5] as it means that a bimodule connection on Ω^1 acts on the metric tensor $g \in \Omega^1 \otimes_A \Omega^1$ so that we can write $\nabla g = 0$ (otherwise for general left connections one has only the weaker notion of cotorsion free). It also means that we have the monoidal categories ${}_A\mathcal{E}_A$, ${}_A\mathcal{I}_A$ as in [4] where the former has objects which are bimodules with bimodule connection and the latter is the same with σ invertible. Morphisms were defined as bimodule maps which intertwine the connections.

Before generalising ${}_A\mathcal{E}_A$, we need to specify an additional possible property of bimodule covariant derivatives, which is the ability to swap the order of n -forms and modules in the same way that the definition of a bimodule connection incorporates swapping 1-forms and modules.

Definition 5.1. *An A -bimodule with left bimodule connection (E, ∇_E, σ_E) is called extendable if $\sigma_E : E \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A E$ extends for all $n \geq 1$ to $\sigma_E : E \otimes_A \Omega^n \rightarrow \Omega^n \otimes_A E$ such that*

$$(\wedge \otimes \text{id})(\text{id} \otimes \sigma_E)(\sigma_E \otimes \text{id}) = \sigma_E(\text{id} \otimes \wedge) : E \otimes_A \Omega^n \otimes_A \Omega^m \rightarrow \Omega^{n+m} \otimes_A E.$$

for all $m \geq 1$. We include the case $m, n = 0$ with $\sigma_E(e \otimes 1) = 1 \otimes e$.

As Ω is generated by A, dA the extended σ_E (should they exist) are uniquely determined by σ_E on the 1-forms. There is a useful property which holds when both extendability and curvature being a right module map hold:

Lemma 5.2. *Suppose that (E, ∇_E, σ_E) is an extendable left A -bimodule connection. Then the curvature R_E is a right module map if and only if*

$$\nabla_E^{[n]} \sigma_E = (\text{id} \wedge \sigma_E)(\nabla_E \otimes \text{id}) + \sigma_E(\text{id} \otimes d) : E \otimes_A \Omega^n \rightarrow \Omega^{n+1} \otimes_A E.$$

holds for $n = 1$. In this case it also holds for all $n \geq 0$ and we also have

$$(\text{id} \wedge R_E) \sigma_E = (\text{id} \wedge \sigma_E)(R_E \otimes \text{id}) : E \otimes_A \Omega^n \rightarrow \Omega^{n+2} \otimes_A E .$$

Proof. For $a \in A$ and $e \in E$ and writing $\nabla e = \eta \otimes f$ (summation implicit), we have $\nabla_E(e.a) = \sigma_E(e \otimes da) + \eta \otimes f.a$ and

$$\begin{aligned} R_E(e.a) &= (d \otimes \text{id} - \text{id} \wedge \nabla_E) \sigma_E(e \otimes da) + d\eta \otimes f.a - \eta \wedge \nabla_E(f.a) \\ &= (d \otimes \text{id} - \text{id} \wedge \nabla_E) \sigma_E(e \otimes da) + R_E(e).a - \eta \wedge \sigma_E(f \otimes da) . \end{aligned}$$

Thus R_E is a right module map if and only if

$$(8) \quad (d \otimes \text{id} - \text{id} \wedge \nabla_E) \sigma_E(e \otimes da) = (\text{id} \wedge \sigma_E)(\nabla_E e \otimes da), \quad \forall e \in E, a \in A.$$

In this case, applying this equation to $e.c \otimes da$ for $c \in A$ instead of $e \otimes da$ gives

$$\begin{aligned} (d \otimes \text{id} - \text{id} \wedge \nabla_E) \sigma_E(e.c \otimes da) &= (\text{id} \wedge \sigma_E)(\nabla_E(e.c) \otimes da) \\ &= (\text{id} \wedge \sigma_E)(\nabla_E e \otimes c.da) + (\text{id} \wedge \sigma_E)(\sigma_E(e \otimes dc) \otimes da). \end{aligned}$$

Using extendability this is

$$(d \otimes \text{id} - \text{id} \wedge \nabla_E) \sigma_E(e \otimes c.da) = (\text{id} \wedge \sigma_E)(\nabla_E e \otimes c.da) + \sigma_E(e \otimes dc \wedge da),$$

or for $\xi = c.da \in \Omega^1$,

$$(d \otimes \text{id} - \text{id} \wedge \nabla_E) \sigma_E(e \otimes \xi) = (\text{id} \wedge \sigma_E)(\nabla_E e \otimes \xi) + \sigma_E(e \otimes d\xi).$$

From this $n = 1$ case we prove the general equality by induction on n : Suppose that it is true for n , and consider it for $n + 1$. It is enough to check it on products $\xi \wedge \eta$, for $\xi \in \Omega^1$ and $\eta \in \Omega^n$,

$$(9) \quad \begin{aligned} \nabla_E^{[n+1]} \sigma_E(e \otimes \xi \wedge \eta) &= \nabla_E^{[n+1]} (\wedge \otimes \text{id})(\text{id} \otimes \sigma_E)(\sigma_E(e \otimes \xi) \otimes \eta) \\ &= (d \wedge \sigma_E - \text{id} \wedge \nabla_E^{[n]} \sigma_E)(\sigma_E(e \otimes \xi) \otimes \eta) \end{aligned}$$

by using extendability. Now by the $n = 1$ case and more extendability,

$$\begin{aligned} (d \otimes \text{id}) \sigma_E(e \otimes \xi) &= (\text{id} \wedge \nabla_E) \sigma_E(e \otimes \xi) + (\text{id} \wedge \sigma_E)(\nabla_E e \otimes \xi) + \sigma_E(e \otimes d\xi), \\ (d \wedge \sigma_E)(\sigma_E(e \otimes \xi) \otimes \eta) &= (\text{id} \wedge \sigma_E)((\text{id} \wedge \nabla_E) \sigma_E(e \otimes \xi) \otimes \eta) \\ &\quad + (\text{id} \wedge \sigma_E)((\text{id} \wedge \sigma_E)(\nabla_E e \otimes \xi) \otimes \eta) + (\text{id} \wedge \sigma_E)(\sigma_E(e \otimes d\xi) \otimes \eta) \\ &= (\text{id} \wedge \sigma_E)((\text{id} \wedge \nabla_E) \sigma_E(e \otimes \xi) \otimes \eta) + (\text{id} \wedge \sigma_E)(\nabla_E e \otimes \xi \wedge \eta) + \sigma_E(e \otimes d\xi \wedge \eta). \end{aligned}$$

Also the inductive hypothesis for n and more extendability gives

$$\begin{aligned} &(\text{id} \wedge \nabla_E^{[n]} \sigma_E)(\sigma_E(e \otimes \xi) \otimes \eta) \\ &= (\text{id} \wedge ((\text{id} \wedge \sigma_E)(\nabla_E \otimes \text{id}) + \sigma_E(\text{id} \otimes d)))(\sigma_E(e \otimes \xi) \otimes \eta) \\ &= (\text{id} \wedge \text{id} \wedge \sigma_E)(\text{id} \otimes \nabla_E \otimes \text{id})(\sigma_E(e \otimes \xi) \otimes \eta) + (\text{id} \wedge \sigma_E)(\sigma_E \otimes \text{id})(e \otimes \xi \otimes d\eta) \\ &= (\text{id} \wedge \sigma_E)((\text{id} \wedge \nabla_E) \sigma_E(e \otimes \xi) \otimes \eta) + \sigma_E(e \otimes \xi \wedge d\eta). \end{aligned}$$

Combining these in (9) yields the required result,

$$\nabla_E^{[n+1]} \sigma_E(e \otimes \xi \wedge \eta) = (\text{id} \wedge \sigma_E)(\nabla_E e \otimes \xi \wedge \eta) + \sigma_E(e \otimes d\xi \wedge \eta) - \sigma_E(e \otimes \xi \wedge d\eta).$$

For the converse, the $n = 1$ case of the equation here implies that R_E is a right module map by condition (8) since $d^2 = 0$. Also in this case

$$\begin{aligned}
 (\text{id} \wedge R_E)\sigma_E &= (\text{id} \wedge \nabla_E^{[1]})(\text{id} \otimes \nabla_E)\sigma_E \\
 &= (\text{id} \wedge \nabla_E^{[1]})(d \otimes \text{id})\sigma_E - (\text{id} \wedge \nabla_E^{[1]})\nabla_E^{[1]}\sigma_E = \nabla_E^{[2]}\nabla_E^{[1]}\sigma_E \\
 &= \nabla_E^{[2]}(\text{id} \wedge \sigma_E)(\nabla_E \otimes \text{id}) + \nabla_E^{[2]}\sigma_E(\text{id} \otimes d) \\
 &= \nabla_E^{[2]}(\text{id} \wedge \sigma_E)(\nabla_E \otimes \text{id}) + (\text{id} \wedge \sigma_E)(\nabla_E \otimes d) + \sigma_E(\text{id} \otimes d)(\text{id} \otimes d) \\
 &= (d \wedge \sigma_E)(\nabla_E \otimes \text{id}) - (\text{id} \wedge (\text{id} \wedge \sigma_E)(\nabla_E \otimes \text{id}))(\nabla_E \otimes \text{id}) \\
 &\quad - (\text{id} \wedge \sigma_E(\text{id} \otimes d))(\nabla_E \otimes \text{id}) + (\text{id} \wedge \sigma_E)(\nabla_E \otimes d) \\
 &= (d \wedge \sigma_E)(\nabla_E \otimes \text{id}) - (\text{id} \wedge (\text{id} \wedge \sigma_E)(\nabla_E \otimes \text{id}))(\nabla_E \otimes \text{id}) \\
 &= (\text{id} \wedge \sigma_E)(\nabla_E^{[1]}\nabla_E \otimes \text{id}) = (\text{id} \wedge \sigma_E)(R_E \otimes \text{id}).
 \end{aligned}$$

Expressing the stated condition in diagrammatic form

is the easiest way to extended to other n by induction, a tedious but straightforward exercise in associativity of wedge products. \square

We give a class of examples where extendability of the bimodule connections is automatic once the curvature is assumed to be a bimodule map. Recall that the maximal prolongation differential calculus extends a given first order differential calculus to all orders by imposing quadratic relations which are precisely $da \wedge db + dr \wedge ds = 0 \in \Omega^2$ for every relation $a.db = dr.s \in \Omega^1$ (summation on both sides implicit, for some $a, b, r, s \in A$).

Lemma 5.3. *Suppose that A is given the maximal prolongation differential calculus for some first order differential calculus and that (E, ∇_E, σ_E) is a left bimodule connection whose curvature R_E is a right module map. Then (E, ∇_E, σ_E) is an extendable bimodule connection.*

Proof. Suppose that $a.db = dr.s \in \Omega^1$ (summation implicit, for some $a, b, r, s \in A$). We also set $\sigma_E(e \otimes dr) = \xi \otimes f$ (summation understood) as a shorthand. Then

$$\begin{aligned}
 (\text{id} \wedge \sigma_E)(\sigma_E(e \otimes dr) \otimes ds) &= \xi \wedge (\nabla_E(f.s) - (\nabla_E f).s) \\
 &= d\xi \otimes f.s - (d \otimes \text{id} - \text{id} \wedge \nabla_E)(\xi \otimes f.s) - \xi \wedge (\nabla_E f).s \\
 &= (d \otimes \text{id} - \text{id} \wedge \nabla_E)(\xi \otimes f).s - (d \otimes \text{id} - \text{id} \wedge \nabla_E)(\xi \otimes f.s) \\
 &= (d \otimes \text{id} - \text{id} \wedge \nabla_E)\sigma_E(e \otimes dr).s - (d \otimes \text{id} - \text{id} \wedge \nabla_E)\sigma_E(e \otimes dr.s) \\
 &= (d \otimes \text{id} - \text{id} \wedge \nabla_E)\sigma_E(e \otimes dr).s - (d \otimes \text{id} - \text{id} \wedge \nabla_E)\sigma_E(e.a \otimes db).
 \end{aligned}$$

If we use the expression for $R_E(e.a) - R_E(e).a$ from the proof of Lemma 5.2 then

$$\begin{aligned}
 &(\text{id} \wedge \sigma_E)(\sigma_E(e \otimes dr) \otimes ds) \\
 &= (\text{id} \wedge \sigma_E)(\nabla_E e \otimes dr.s - \nabla_E(e.a) \otimes db) + R_E(er)s - R_E(e)rs - R_E(eab) + R_E(ea)b \\
 &= (\text{id} \wedge \sigma_E)(\nabla_E e \otimes a.db - \nabla_E(e.a) \otimes db) + R_E(er)s - R_E(e)rs - R_E(eab) + R_E(ea)b \\
 &= (\text{id} \wedge \sigma_E)((\nabla_E(e).a - \nabla_E(e.a)) \otimes db) + R_E(er)s - R_E(e)rs - R_E(eab) + R_E(ea)b \\
 &= -(\text{id} \wedge \sigma_E)(\sigma_E(e \otimes da) \otimes db) + R_E(er)s - R_E(e)rs - R_E(eab) + R_E(ea)b.
 \end{aligned}$$

Hence

$$(\text{id} \wedge \sigma_E)(\sigma_E \otimes \text{id})(e \otimes (da \otimes db + dr \otimes ds)) = R_E(er)s - R_E(e)rs - R_E(eab) + R_E(ea)b.$$

In particular, if R_E is a right module map then the LHS vanishes for all e and all a, b, r, s (sum of such) obeying the condition stated. The relations for the wedge product are given by $da \otimes db + dr \otimes ds$ whenever $a.db = dr.s \in \Omega^1$ (summation implicit). Hence if R_E is a right module map, for $(\text{id} \otimes \sigma_E)(\sigma_E \otimes \text{id}) : E \otimes_A \Omega^1 \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1 \otimes_A E$ then we find

$$(10) \quad (\text{id} \otimes \sigma_E)(\sigma_E \otimes \text{id})(E \otimes_A \ker \wedge) \subset \ker \wedge \otimes_A E.$$

As Ω^2 is $\Omega^1 \otimes_A \Omega^1$ quotiented by $\ker \wedge$, we find that $(\text{id} \otimes \sigma_E)(\sigma_E \otimes \text{id})$ induces a map $\sigma_E : E \otimes_A \Omega^2 \rightarrow \Omega^2 \otimes_A E$ so we have proven extendability in degree 2. The higher extendability in the case of the maximal prolongation is then automatic. Thus for Ω^3 consider the map

$$(\text{id}^{\otimes 2} \otimes \sigma_E)(\text{id} \otimes \sigma_E \otimes \text{id})(\sigma_E \otimes \text{id}^{\otimes 2}) : E \otimes_A \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1 \otimes_A E$$

By (10), this sends $E \otimes_A \ker \wedge \otimes_A \Omega^1$ to $\ker \wedge \otimes_A E \otimes_A \Omega^1$ and sends $E \otimes_A \Omega^1 \otimes_A \ker \wedge$ to $\Omega^1 \otimes_A E \otimes_A \ker \wedge$. Then by definition of maximal prolongation there is a well defined map from $E \otimes_A \Omega^3$ to $\Omega^3 \otimes_A E$. Similar for higher degree. \square

We can use extendability to give a simple statement about the curvature of a tensor product:

Lemma 5.4. *For a left A -module F and an A -bimodule E , suppose that (E, ∇_E, σ_E) is an extendable bimodule connection whose curvature is a right module map and that (F, ∇_F) is a left connection. Then*

$$R_{E \otimes F} = R_E \otimes \text{id} + (\sigma_E \otimes \text{id})(\text{id} \otimes R_F) : E \otimes_A F \rightarrow \Omega^2 \otimes_A E \otimes_A F$$

Proof. Using Lemma 5.2:

$$\begin{aligned}
 \nabla_{E \otimes F}^{[1]} \nabla_{E \otimes F} &= \nabla_{E \otimes F}^{[1]} (\nabla_E \otimes \text{id} + (\sigma_E \otimes \text{id})(\text{id} \otimes \nabla_F)) \\
 &= \nabla_E^{[1]} \nabla_E \otimes \text{id} - (\text{id} \wedge \sigma_E \otimes \text{id})(\nabla_E \otimes \nabla_F) \\
 &\quad + (\nabla_E^{[1]} \sigma_E \otimes \text{id})(\text{id} \otimes \nabla_F) - (\text{id} \wedge \sigma_E \otimes \text{id})(\sigma_E \otimes \nabla_F)(\text{id} \otimes \nabla_F) \\
 &= \nabla_E^{[1]} \nabla_E \otimes \text{id} + (\sigma_E \otimes \text{id})(\text{id} \otimes d \otimes \text{id})(\text{id} \otimes \nabla_F) \\
 &\quad - (\text{id} \wedge \sigma_E \otimes \text{id})(\sigma_E \otimes \nabla_F)(\text{id} \otimes \nabla_F) \\
 &= \nabla_E^{[1]} \nabla_E \otimes \text{id} + (\sigma_E \otimes \text{id})(\text{id} \otimes d \otimes \text{id})(\text{id} \otimes \nabla_F) \\
 &\quad - (\sigma_E \otimes \text{id})(\text{id} \otimes (\text{id} \wedge \nabla_F) \nabla_F)
 \end{aligned}$$

which gives the required formula \square

5.2. Construction of ${}_A\mathcal{G}_A$. Consider two A -bimodules with left bimodule connections (E, ∇_E, σ_E) and (F, ∇_F, σ_F) , and suppose that $\phi : E \rightarrow F$ is a bimodule map. We know that $\mathbb{W}(\phi) : E \rightarrow \Omega^1 \otimes_A F$ is a left module map, and we now check the right action:

$$\begin{aligned}
 \mathbb{W}(\phi)(e.a) &= \nabla_F \phi(e.a) - (\text{id} \otimes \phi) \nabla_E(e.a) = \nabla_F((\phi e).a) - (\text{id} \otimes \phi) \nabla_E(e.a) \\
 &= (\nabla_F \phi e).a - ((\text{id} \otimes \phi) \nabla_E e).a + \sigma_F(\phi(e) \otimes da) - (\text{id} \otimes \phi) \sigma_E(e \otimes da) \\
 (11) \quad &= (\mathbb{W}(\phi)(e)).a + \sigma_F(\phi e \otimes da) - (\text{id} \otimes \phi) \sigma_E(e \otimes da)
 \end{aligned}$$

for $a \in A$ and $e \in E$. So $\mathbb{W}(\phi)$ is a bimodule map if and only if $(\text{id} \otimes \phi) \sigma_E = \sigma_F(\phi \otimes \text{id})$. If we have 0-morphisms being bimodule maps it is reasonable to expect that their derivatives, the 1-morphisms, should also be bimodule maps. Thus we assume the additional condition $(\text{id} \otimes \phi) \sigma_E = \sigma_F(\phi \otimes \text{id})$ for 0-morphisms in our construction of a bimodule DG category. The required generalisation for morphisms $\psi : E \rightarrow \Omega^n \otimes_A F$ of grade n is bimodule maps obeying the condition

$$(12) \quad (\text{id} \wedge \psi) \sigma_E = (-1)^n (\text{id} \wedge \sigma_F)(\phi \otimes \text{id}) : E \otimes_A \Omega^1 \rightarrow \Omega^{n+1} \otimes_A F$$

and Lemma 5.6 will say that $\mathbb{W}(\psi)$ is then an $n+1$ -morphism. This gives the morphisms in our bimodule category ${}_A\mathcal{G}_A$. Remembering how important the curvature in the theory is, it is reasonable to restrict to objects whose curvature is a bimodule map. Also as we wish to take tensor products, it is reasonable to restrict to extendable bimodule connections as then we have the result of Lemma 5.4 on the curvature of tensor products. (In fact the technical result Lemma 5.2 will prove useful in numerous places.) We summarise our proposed DG category by the following table:

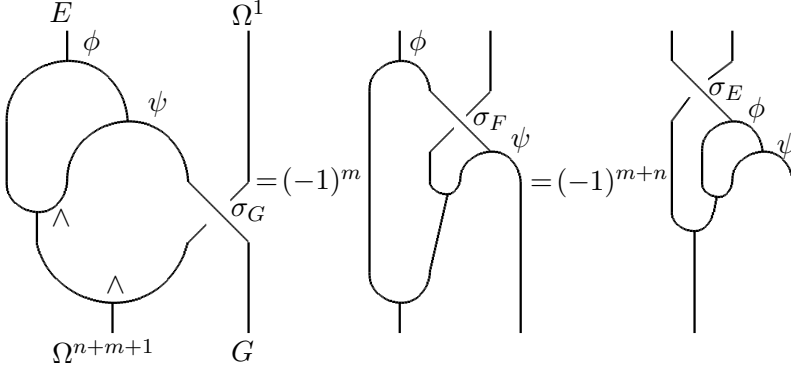
Name	Objects	$\text{Mor}_n((E, \nabla_E, \sigma_E), (F, \nabla_F, \sigma_F))$
${}_A\mathcal{G}_A$	(E, ∇_E, σ_E) extendable left $A - A$ -bimodule connections with R_E a bimodule map	$\phi : E \rightarrow \Omega^n \otimes_A F$ a bimodule map obeying (12)

It is useful to note that by extendability and a use of induction a bimodule map ϕ obeying (12) also obeys the following for all $n, m \geq 0$:

$$(13) \quad (\text{id} \wedge \phi)\sigma_E = (-1)^{nm}(\text{id} \wedge \sigma_F)(\phi \otimes \text{id}) : E \otimes_A \Omega^m \rightarrow \Omega^{n+m} \otimes_A F$$

Theorem 5.5. *If we use the composition and derivative of morphisms as for ${}_A\mathcal{G}$, then ${}_A\mathcal{G}_A$ as summarised above is a DG category and the conclusions of Theorem 3.2 hold but now for ${}_A\mathcal{G}_A$.*

Proof. We begin by checking that ${}_A\mathcal{G}_A$ is a category, i.e. that we can compose morphisms. This is shown using diagrammatic notation to check that (12) holds for the composition of $\phi \in \text{Mor}_n(E, F)$ and $\psi \in \text{Mor}_m(F, G)$:



It is clear that $\phi \circ \psi$ is a bimodule map if ϕ, ψ are. Lemma 5.2 showed that the curvature is a morphism and Lemma 5.6 will show that the differential of a morphism is a morphism. The rest follows just as for ${}_A\mathcal{G}$. \square

To conclude the proof of Theorem 5.5 we need the following:

Lemma 5.6. *If $\phi \in \text{Mor}_n(E, F)$ in ${}_A\mathcal{G}_A$, then $\mathbb{V}(\phi) \in \text{Mor}_{n+1}(E, F)$.*

Proof. First we check that $\mathbb{V}(\phi)$ is a right module map. For all $e \in E$ and $a \in A$ and setting $\phi e = \xi \otimes f$ (summation understood),

$$\begin{aligned} \mathbb{V}(\phi)(e.a) &= \nabla_F^{[n]}(\phi(e.a)) - (\text{id} \wedge \phi)\nabla_E(e.a) \\ &= \nabla_F^{[n]}(\xi \otimes f.a) - (\text{id} \wedge \phi)(\nabla_E e).a - (\text{id} \wedge \phi)\sigma_E(e \otimes da) \\ &= d\xi \otimes f.a + (-1)^n \xi \wedge \nabla_F(f.a) - (\text{id} \wedge \phi)(\nabla_E e).a - (\text{id} \wedge \phi)\sigma_E(e \otimes da) \\ &= \mathbb{V}(\phi)(e).a + (-1)^n \xi \wedge \sigma_F(f \otimes da) - (\text{id} \wedge \phi)\sigma_E(e \otimes da) = \mathbb{V}(\phi)(e).a \end{aligned}$$

where the last equality is the assumption (12). Now check that $\mathbb{V}(\phi)$ itself obeys the condition (12) in degree $n+1$. Here

$$(\text{id} \wedge \mathbb{V}(\phi))\sigma_E = (\text{id} \wedge \nabla_F^{[n]}\phi)\sigma_E - (\text{id} \wedge \phi)(\text{id} \wedge \nabla_E)\sigma_E : E \otimes_A \Omega^1 \rightarrow \Omega^{n+2} \otimes_A F.$$

From the definition of $\nabla_F^{[n+1]}$,

$$\nabla_F^{[n+1]}(\text{id} \wedge \phi)\sigma_E = (d \wedge \phi)\sigma_E - (\text{id} \wedge \nabla_F^{[n]}\phi)\sigma_E,$$

so using Lemma 5.2,

$$\begin{aligned}
 (\text{id} \wedge \mathbb{W}(\phi))\sigma_E &= (d \wedge \phi)\sigma_E - \nabla_F^{[n+1]}(\text{id} \wedge \phi)\sigma_E - (\text{id} \wedge \phi)(\text{id} \wedge \nabla_E)\sigma_E \\
 &= (\text{id} \wedge \phi)\nabla_E^{[1]}\sigma_E - (-1)^n \nabla_F^{[n+1]}(\text{id} \wedge \sigma_F)(\phi \otimes \text{id}) \\
 &= (\text{id} \wedge \phi)(\text{id} \wedge \sigma_E)(\nabla_E \otimes \text{id}) + (\text{id} \wedge \phi)\sigma_E(\text{id} \otimes d) - (-1)^n \nabla_F^{[n+1]}(\text{id} \wedge \sigma_F)(\phi \otimes \text{id}) \\
 &= (\text{id} \wedge (\text{id} \wedge \phi)\sigma_E)(\nabla_E \otimes \text{id}) + (\text{id} \wedge \phi)\sigma_E(\text{id} \otimes d) - (-1)^n \nabla_F^{[n+1]}(\text{id} \wedge \sigma_F)(\phi \otimes \text{id}).
 \end{aligned}$$

Taking some care over the signs and using (13),

$$\begin{aligned}
 (-1)^n (\text{id} \wedge \mathbb{W}(\phi))\sigma_E &= (\text{id} \wedge (\text{id} \wedge \sigma_F)(\phi \otimes \text{id}))(\nabla_E \otimes \text{id}) \\
 &\quad + (-1)^n (\text{id} \wedge \sigma_F)(\phi \otimes \text{id})(\text{id} \otimes d) - \nabla_F^{[n+1]}(\text{id} \wedge \sigma_F)(\phi \otimes \text{id}) \\
 &= (\text{id} \wedge \sigma_F)(\text{id} \wedge \phi \otimes \text{id})(\nabla_E \otimes \text{id}) + (-1)^n (\text{id} \wedge \sigma_F)(\phi \otimes d) \\
 &\quad - (d \wedge \sigma_F)(\phi \otimes \text{id}) - (-1)^n (\text{id} \wedge \nabla_F^{[1]}\sigma_F)(\phi \otimes \text{id}) \\
 &= (\text{id} \wedge \sigma_F)(\text{id} \wedge \phi \otimes \text{id})(\nabla_E \otimes \text{id}) + (-1)^n (\text{id} \wedge \sigma_F)(\phi \otimes d) \\
 &\quad - (\text{id} \wedge \sigma_F)((d \otimes \text{id})\phi \otimes \text{id}) - (-1)^n (\text{id} \wedge (\text{id} \wedge \sigma_F)(\nabla_F \otimes \text{id}))(\phi \otimes \text{id}) \\
 &\quad - (-1)^n (\text{id} \wedge \sigma_F(\text{id} \otimes d))(\phi \otimes \text{id}) \\
 &= (\text{id} \wedge \sigma_F)((\text{id} \wedge \phi \otimes \text{id})(\nabla_E \otimes \text{id}) - ((d \otimes \text{id})\phi \otimes \text{id}) - (-1)^n (\text{id} \wedge \nabla_F \otimes \text{id}))(\phi \otimes \text{id}) \\
 &= - (\text{id} \wedge \sigma_F)(\mathbb{W}(\phi) \otimes \text{id}) . \quad \square
 \end{aligned}$$

5.3. Monoidal structure on ${}_A\mathcal{G}_A$.

Theorem 5.7. *There is an associative functor \boxtimes from ${}_A\mathcal{G}_A \times {}_A\mathcal{G}_A$ to ${}_A\mathcal{G}_A$, with product of objects being the usual tensor product of bimodules with bimodule connection. The tensor product of morphisms $\phi \in \text{Mor}_n(E, G)$ and $\psi \in \text{Mor}_m(F, H)$ in ${}_A\mathcal{G}_A$ is*

$$\phi \boxtimes \psi = (\text{id} \wedge \sigma_F \otimes \text{id})(\phi \otimes \psi) : E \otimes_A F \rightarrow \Omega^{n+m} \otimes_A G \otimes_A H.$$

There is a signed rule for composition of tensor products:

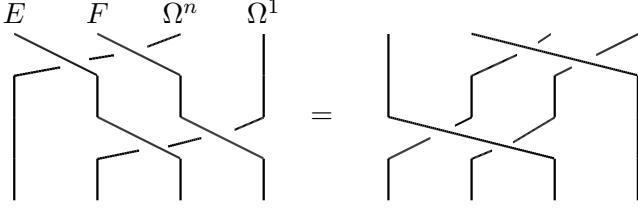
$$(\phi \boxtimes \kappa) \circ (\psi \boxtimes \tau) = (-1)^{|\psi||\kappa|} (\phi \circ \psi) \boxtimes (\kappa \circ \tau).$$

For the differential of morphisms,

$$\mathbb{W}(\phi \boxtimes \psi) = \mathbb{W}(\phi) \boxtimes \psi + (-1)^n \phi \boxtimes \mathbb{W}(\psi) ,$$

and for curvatures $R_{E \otimes F} = R_E \boxtimes \text{id} + \text{id} \boxtimes R_F$.

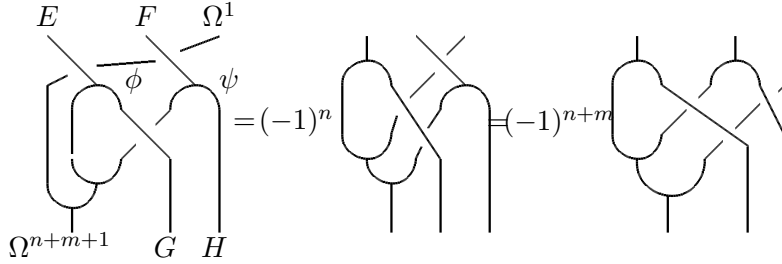
Proof. First show that given objects (E, ∇_E, σ_E) and (F, ∇_F, σ_F) then $(E \otimes_A F, \nabla_{E \otimes F}, \sigma_{E \otimes F})$ is also an object. The extendable connection condition for $E \otimes_A F$ follows fairly immediately from rearranging the crossings for $E \otimes_A F$ (on the LHS of the following picture) in terms of the crossings for E and the crossings for F on the RHS:



Next we check the displayed condition in Lemma 5.2 for $R_{E \otimes F}$ to be a bimodule map (with $n = 1$ there):

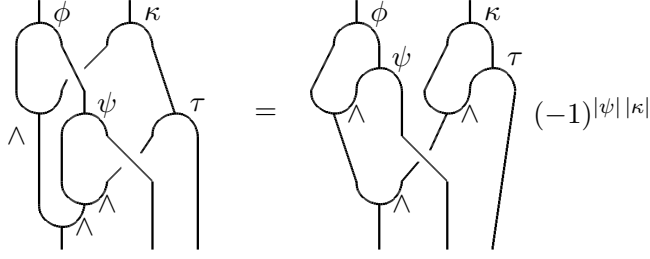
$$\begin{aligned}
 \nabla_{E \otimes F}^{[1]} \sigma_{E \otimes F} &= (d \otimes \text{id} \otimes \text{id}) \sigma_{E \otimes F} - (\text{id} \wedge \nabla_{E \otimes F}) \sigma_{E \otimes F} \\
 &= (d \otimes \text{id} \otimes \text{id}) \sigma_{E \otimes F} - (\text{id} \wedge \nabla_E \otimes \text{id}) \sigma_{E \otimes F} - (\text{id} \wedge \sigma_E \otimes \text{id})(\text{id} \otimes \text{id} \otimes \nabla_F) \sigma_{E \otimes F} \\
 &= (\text{id} \wedge \sigma_E \otimes \text{id})(\nabla_E \otimes \text{id} \otimes \text{id})(\text{id} \otimes \sigma_F) + (\sigma_E \otimes \text{id})(\text{id} \otimes d \otimes \text{id})(\text{id} \otimes \sigma_F) \\
 &\quad - ((\text{id} \wedge \sigma_E)(\sigma_E \otimes \text{id}) \otimes \text{id})(\text{id} \otimes \text{id} \otimes \nabla_F)(\text{id} \otimes \sigma_F) \\
 &= (\text{id} \wedge \sigma_{E \otimes F})(\nabla_E \otimes \text{id} \otimes \text{id}) + (\sigma_E \otimes \text{id})(\text{id} \otimes d \otimes \text{id})(\text{id} \otimes \sigma_F) \\
 &\quad - (\sigma_E \otimes \text{id})(\text{id} \otimes \text{id} \wedge \nabla_F)(\text{id} \otimes \sigma_F) \\
 &= (\text{id} \wedge \sigma_{E \otimes F})(\nabla_E \otimes \text{id} \otimes \text{id}) + (\sigma_E \otimes \text{id})(\text{id} \otimes \nabla_F^{[1]} \sigma_F) \\
 &= (\text{id} \wedge \sigma_{E \otimes F})(\nabla_E \otimes \text{id} \otimes \text{id}) + (\sigma_E \otimes \text{id})(\text{id} \otimes (\text{id} \wedge \sigma_F)(\nabla_F \otimes \text{id})) \\
 &\quad + (\sigma_E \otimes \text{id})(\text{id} \otimes \sigma_F(\text{id} \otimes d)) \\
 &= (\text{id} \wedge \sigma_{E \otimes F})(\nabla_{E \otimes F} \otimes \text{id}) + \sigma_{E \otimes F}(\text{id} \otimes \text{id} \otimes d).
 \end{aligned}$$

For morphisms the formula for $\phi \boxtimes \psi$ shows that it is a bimodule map, and we also check that it satisfies the σ condition to be a morphism, which is best proved diagrammatically:



At each stage we have used the morphism property for one of ϕ and ψ , and also associativity of \wedge product. The diagram for the property used for a single morphism appears at the end of the proof of Lemma 5.2, in the special case of the curvature (as $(-1)^2 = 1$ the sign does not appear there).

We also have to check associativity of \boxtimes and the signed rule for compatibility with composition, both of which are routine exercises using the definitions and associativity of the wedge product (and best done diagrammatically). The more complicated case is the composition of tensor products, which is represented by the following diagram, where we use (13) and extendability:



Now we verify the formula for the derivative of a tensor product of morphisms: Suppose ${}_A\mathcal{G}_A$, for $\phi \in \text{Mor}_n(E, G)$ and $\psi \in \text{Mor}_m(F, H)$, Then

$$\begin{aligned} \nabla_{G \otimes H}^{[n+m]} \circ (\phi \boxtimes \psi) &= (d \wedge \sigma_G \otimes \text{id})(\phi \otimes \psi) + (-1)^n (\text{id} \wedge \nabla_G^{[m]} \sigma_G \otimes \text{id})(\phi \otimes \psi) \\ &\quad + (-1)^{n+m} (\text{id} \wedge \sigma_G \otimes \text{id})(\phi \otimes (\text{id} \wedge \nabla_H) \psi) \\ &= (d \wedge \sigma_G \otimes \text{id})(\phi \otimes \psi) + (-1)^n (\text{id} \wedge (\text{id} \wedge \sigma_G)(\nabla_G \otimes \text{id}) \otimes \text{id})(\phi \otimes \psi) \\ &\quad + (-1)^n (\text{id} \wedge \sigma_G (\text{id} \otimes d) \otimes \text{id})(\phi \otimes \psi) \\ &\quad + (-1)^{n+m} (\text{id} \wedge \sigma_G \otimes \text{id})(\phi \otimes (\text{id} \wedge \nabla_H) \psi) \\ &= (\text{id} \wedge \sigma_G \otimes \text{id})(\nabla_G^{[n]} \phi \otimes \psi + (-1)^n \phi \otimes \nabla_H^{[m]} \psi), \end{aligned}$$

while

$$\begin{aligned} (\text{id} \wedge (\phi \boxtimes \psi)) \nabla_{E \otimes F} &= (\text{id} \wedge (\phi \boxtimes \psi)) (\nabla_E \otimes \text{id}) + (\text{id} \wedge (\phi \boxtimes \psi)) (\sigma_E \otimes \text{id})(\text{id} \otimes \nabla_F) \\ &= (\text{id} \wedge \sigma_G \otimes \text{id}) ((\text{id} \wedge \phi) \nabla_E \otimes \psi) + (\text{id} \wedge (\text{id} \wedge \sigma_G \otimes \text{id})) (\phi \otimes \psi) (\sigma_E \otimes \text{id})(\text{id} \otimes \nabla_F) \\ &= (\text{id} \wedge \sigma_G \otimes \text{id}) ((\text{id} \wedge \phi) \nabla_E \otimes \psi) + (-1)^n (\text{id} \wedge \sigma_G \otimes \text{id})(\phi \otimes (\text{id} \wedge \psi) \nabla_F). \end{aligned}$$

Subtracting these gives $\mathbb{W}(\phi \boxtimes \psi)$. \square

We give three applications of this machinery. The first application is find another of the symmetries of the Riemann curvature, this time involving a metric, written as $g \in \Omega^1 \otimes_A \Omega^1$.

Corollary 5.8. *Suppose that $(\Omega^1, \nabla_{\Omega^1}, \sigma_{\Omega^1}) \in {}_A\mathcal{G}_A$ and that $g \in \Omega^1 \otimes_A \Omega^1$ is preserved in the sense $\nabla_{\Omega^1 \otimes \Omega^1} g = 0$. Then*

$$(R_{\Omega^1} \otimes \text{id} + (\sigma_{\Omega^1} \otimes \text{id})(\text{id} \otimes R_{\Omega^1}))g = 0.$$

(Riemann antisymmetry identity.)

Proof. Here $\nabla_{\Omega^1 \otimes \Omega^1} g = 0$ is equivalent to $\mathbb{W}(g) = 0$ when g is viewed as a morphism in ${}_A\mathcal{G}$. This in turn implies $\mathbb{W}(\mathbb{W}(g)) = 0$ which is equivalent to $R_{\Omega^1 \otimes \Omega^1} g = 0$ by Theorem 3.2. Since $(\Omega^1, \nabla_{\Omega^1}, \sigma_{\Omega^1}) \in {}_A\mathcal{G}_A$ we can now use Theorem 5.7 to write $R_{\Omega^1 \otimes \Omega^1} = R_{\Omega^1} \boxtimes \text{id} + \text{id} \boxtimes R_{\Omega^1}$ which gives the answer. \square

To find the classical identity corresponding to this we use

$$R_{\Omega^1}(dx^i) = -\frac{1}{2} R^i_{knm} dx^n \wedge dx^m \otimes dx^k.$$

and then

$$R_{\Omega^1 \otimes \Omega^1}(g_{ij} dx^i \otimes dx^j) = -\frac{1}{2} R_{iknm} dx^n \wedge dx^m \otimes dx^k \otimes dx^i - \frac{1}{2} R_{iknm} dx^n \wedge dx^m \otimes dx^i \otimes dx^k$$

so we get the classical symmetry $R_{abcd} = -R_{bacd}$ of the Riemann curvature of the Levi-Civita connection.

We now turn to a second application, which utilises the centre $Z_A(E)$ of a bimodule E , whose elements commute with every $a \in A$.

Proposition 5.9. *Let A be an algebra with exterior algebra Ω and (L, ∇_L, σ_L) an extendable bimodule connection on a line module. There is an algebra map $\hat{\Phi}_L : Z_A(\Omega) \rightarrow Z_A(\Omega)$ defined by $\hat{\Phi}_L(\xi) \otimes e = \sigma_L(e \otimes \xi)$ and if R_L is a bimodule map (so $(L, \nabla_L, \sigma_L) \in {}_A\mathcal{G}_A$) then there is a unique $\omega_L \in Z_A(\Omega^2)$ such that $R_L(e) = \omega_L \otimes e$. If (M, ∇_M, σ_M) is another such line module then so is their tensor product and $\omega_{L \otimes_A M}$ is given by $\omega_L + \hat{\Phi}_L(\omega_M)$.*

Proof. Here $\hat{\Phi}_L$ extends the (inverse of) the Fröhlich map to higher degree. If $\xi \in Z_A(\Omega^n)$ then the map $e \mapsto \sigma_L(e \otimes \xi)$ is a right module map from L to $\Omega^n \otimes_A L$, so by Lemma 2.2 it is given by $e \mapsto \hat{\Phi}_L(\xi) \otimes e$ for some $\hat{\Phi}_L(\xi)$. Uniqueness also tells us that $\hat{\Phi}_{L \otimes_A M} = \hat{\Phi}_L \circ \hat{\Phi}_M$. Now suppose that R_L is a bimodule map. The existence and uniqueness of $\omega_L \in \Omega^2$ follows from applying Lemma 2.2 in the right A -module map case to R_L . Taking this form of R_L and knowing that it is a left module map tells us that ω_L is central. For the curvature of the tensor product we use Theorem 5.7 and our observation about $\hat{\Phi}_{L \otimes M}$. \square

Recall that in classical electromagnetism the curvature is often simply quoted as a 2-form rather than an operator. The above generalise this to line modules with connection in ${}_A\mathcal{G}_A$, giving the curvature as a 2-form ω_L . A third application is to take traces of operators on objects $E \in {}_A\mathcal{G}_A$ equipped with a quantum metric. We are interested particularly in the ‘trace’ of powers of the curvature.

Corollary 5.10. *Let $E \in {}_A\mathcal{G}_A$ and suppose that $g : A \rightarrow E \otimes_A E$ and its inverse $(,) : E \otimes_A E \rightarrow A$ are morphisms in ${}_A\mathcal{G}_A$ and that $\mathbb{W}(g) = 0$. Then we have an element of de Rham cohomology*

$$[(\text{id} \otimes (,)) (R_E^n \otimes \text{id}) g(1)] \in H_{\text{dR}}^{2n}(A).$$

Proof. In fact we only need to know that g is a bimodule map and $\mathbb{W}(g) = 0$ to know that it is a morphism. Here $\mathbb{W}(g) = 0$ is equivalent to $\nabla_{E \otimes E}(g) = 0$ for the corresponding $g \in E \otimes_A E$. From the equation $((,) \otimes \text{id})(\text{id} \otimes g) = \text{id} : E \rightarrow E$ we deduce

$$(\mathbb{W}((,)) \boxtimes \text{id}) \circ (\text{id} \boxtimes g) + ((,) \boxtimes \text{id}) \circ (\text{id} \boxtimes \mathbb{W}(g)) = \mathbb{W}(\text{id}) = 0$$

so $\mathbb{W}((,)) = 0$ also holds. Now by Theorem 3.2 and Theorem 5.7, for all $n \geq 0$

$$\mathbb{W}((\text{id} \otimes (,)) (R_E^n \otimes \text{id}) g) = 0$$

which implies the result. \square

In the $n = 0$ case we call this trace the ‘metric dimension’ $\underline{\dim}_E = (,)(g)$ of E since classically the metric dimension will just recover the fibre dimension. For example, if $\Omega^1 \in {}_A\mathcal{G}_A$ and ∇_{Ω^1} is metric compatible as in Corollary 5.8 then the above implies that

the quantum dimension is in the kernel of d and hence constant if the differential calculus is connected. In the case of the line module L with connection in Proposition 5.9 preserving a metric as just discussed we get the class of ω_L^n times the metric dimension in $H_{dR}^{2n}(A)$.

6. EXAMPLES

We collect some examples at the end of the paper in order not to distract from the general results. Since our results are very general there are plenty more examples, although for the Chern theory we have the same limitations as in cyclic cohomology due to the lack of cycles of the relevant dimension (eg for the q -sphere we would need twisted cycles as in [17] which requires more analysis for our results). We also discover that in some noncommutative models, extendability is linked to zero curvature.

6.1. Connections on $\Omega^1(S_3)$. Finite groups G provide nice examples where the differentials are noncommutative even though coordinate algebra $A = \mathbb{k}(G)$ is commutative. A bicovariant calculus here is fixed by $\mathcal{C} \subseteq G \setminus \{e\}$ stable under conjugation and has left invariant basis elements e_a for $a \in \mathcal{C}$ and relations and exterior derivative

$$e_a \cdot f = R_a(f)e_a, \quad df = \sum_{a \in \mathcal{C}} (R_a(f) - f)e_a$$

where $f \in \mathbb{k}(G)$ and $R_a(f)(x) = f(xa)$. The canonical ‘Woronowicz construction’[27] exterior algebra in this case is defined via the crossed module braiding $\Psi(e_a \otimes e_b) = e_{aba^{-1}} \otimes e_a$ and in degree 2 consists in setting $\ker(\text{id} - \Psi)$ to zero. The above is immediate from a general theory of Hopf algebras in [27], while a recent nontrivial result [22] is that the noncommutative de Rham cohomology for the calculus is $H^1(G) = \mathbb{k}\theta$ with $\theta = \sum_a e_a$, when \mathbb{k} has characteristic zero and \mathcal{C} as a quandle is ‘locally skew’ (a concept that includes all Weyl groups of complex semisimple Lie algebras with \mathcal{C} given by reflections). We refer to [22] for details. We use this setting to illustrate our above results on $G = S_3$, the permutation group of 3 objects, and $\mathcal{C} = \{u, v, w\}$ the set of 2-cycles (or transpositions) in S_3 where $u^2 = v^2 = e$ and $w = uvu = vuv$. The quantum metric here is the Euclidean one $g = e_u \otimes e_u + e_v \otimes e_v + e_w \otimes e_w$.

(i) Then the canonical exterior algebra $\Omega(S_3)$ in for this calculus is well-known to have relations

$$\begin{aligned} e_u \wedge e_v + e_v \wedge e_w + e_w \wedge e_u &= 0, & e_v \wedge e_u + e_u \wedge e_w + e_w \wedge e_v &= 0, & e_u^2 = e_v^2 = e_w^2 &= 0 \\ de_u + e_v \wedge e_w + e_w \wedge e_v &= 0, & de_v + e_w \wedge e_u + e_u \wedge e_w &= 0, & de_w + e_u \wedge e_v + e_v \wedge e_u &= 0 \end{aligned}$$

which imply dimensions $1 : 3 : 4 : 3 : 1$ and in particular a unique up to scale ‘top form’

$$\text{Vol} = e_u \wedge e_v \wedge e_u \wedge e_w = e_v \wedge e_u \wedge e_v \wedge e_w = -e_w \wedge e_u \wedge e_v \wedge e_u = -e_w \wedge e_v \wedge e_u \wedge e_v$$

and equal to the two cyclic permutations $u \rightarrow v \rightarrow w \rightarrow u$.

We now observe that this exterior algebra has a well-defined 4-cycle defined by $\int(f\text{Vol}) = \int f = \sum_{g \in S_3} f(g)$. First, from the above versions of Vol we see that swapping $e_u \wedge e_v \wedge e_u$ with e_w gives a minus sign and similarly for other identities for (6) for ω, ρ 1- and 3-forms with constant coefficients. Here $e_u \wedge e_v \wedge e_w = e_w \wedge e_v \wedge e_u = -e_w \wedge e_u \wedge e_v = -e_u \wedge e_w \wedge e_u$ and its two cyclic permutations are a convenient basis of 3-forms. Similarly in degree 2 we

have basis $e_u \wedge e_v, e_v \wedge e_u, e_v \wedge e_w, e_w \wedge e_v$ and these mutually commute so that (6) holds on basic 2-forms as well. Also note that Vol is zero unless the total degree given by the product of the basis labels is e as it is known that Vol here is central (these basic facts were used recently in [21]). Then for the general case (where we sum over labels belonging to our declared bases)

$$\begin{aligned} \int (\omega \wedge \rho) &= \int \omega_a e_a \wedge \rho_{bcd} e_b \wedge e_c \wedge e_d = \int \omega_a R_a(\rho_{bcd}) e_a \wedge e_b \wedge e_c \wedge e_d \\ &= - \int R_{bcd}(\omega_a) \rho_{bcd} e_b \wedge e_c \wedge e_d \wedge e_a = - \int \rho_{bcd} e_b \wedge e_c \wedge e_d \wedge \omega_a e_a = - \int \rho \wedge \eta \end{aligned}$$

for degrees 1,3 (and similarly for degrees 3,1), and

$$\begin{aligned} \int (\omega \wedge \rho) &= \int \omega_{ab} e_a \wedge e_b \wedge \rho_{cd} e_c \wedge e_d = \int \omega_{ab} R_{ab}(\rho_{cd}) e_a \wedge e_b \wedge e_c \wedge e_d \\ &= \int R_{cd}(\omega_{ab}) \rho_{cd} e_c \wedge e_d \wedge e_a \wedge e_b = \int \rho_{cd} e_c \wedge e_d \wedge \omega_{ab} e_a \wedge e_b = \int \rho \wedge \eta \end{aligned}$$

for degree 2,2. We used of course that \sum over S_3 is translation invariant. We have $\int d\omega = \int [\theta, \omega] = 0$ (where we use the graded-commutator) since the calculus is inner by $\theta = e_u + e_v + e_w$.

(ii) Next we look at connections on Ω^1 . We focus on left-invariant connections, which means of the form

$$\nabla_{\Omega^1} e_a = - \sum_{b,c \in \mathcal{C}} \Gamma^a_{bc} e_b \otimes e_c$$

where $\Gamma^a_{bc} \in \mathbb{k}$. For simplicity, we further restrict to the case where ∇_{Ω^1} is also right-invariant, which reduces to $\Gamma^{gag^{-1}}_{gbg^{-1},gcg^{-1}} = \Gamma^a_{b,c}$ for all $a, b, c \in \mathcal{C}$ and $g \in G$ in the case of general group. In our case, as conjugation induces all possible permutations of the set \mathcal{C} , we deduce that there are only 5 possible different values of the Christoffel symbols Γ^a_{bc} , namely of the form $\Gamma^x_{xx}, \Gamma^x_{yz}, \Gamma^x_{yx}, \Gamma^x_{xy}$ and Γ^y_{xx} (where x, y, z are all different). We set

$$(14) \quad \Gamma^x_{xx} = a - 1, \quad \Gamma^x_{yz} = c, \quad \Gamma^x_{yx} = d - 1, \quad \Gamma^x_{xy} = e, \quad \Gamma^y_{xx} = b$$

with parameters $a, b, c, d, e \in \mathbb{k}$ (these should not be confused with generic elements of \mathcal{C} which we won't need at this point) and since we use computer algebra we assume \mathbb{k} has characteristic zero. The curvature in our context is

$$R_{\Omega^1}(e_a) = - (\Gamma^a_{bc} d e_b + \Gamma^a_{bs} \Gamma^s_{rc} e_b \wedge e_r) \otimes e_c.$$

which in terms of the parameters works out as the curvature given by the following (and permutations of this),

$$\begin{aligned} R_{\Omega^1}(e_u) &= e_w \wedge e_u \otimes ((-b^2 - ad + d^2 + ce)e_u + (d - c)(b - e)e_v + (b^2 + ac - c^2 - de)e_w) \\ &\quad + e_u \wedge e_v \otimes ((bc - ad + d^2 - e^2)e_u + (-ab + bd + cd - ae - be + ce)e_v + (b - e)(b + d + e)e_w) \\ &\quad + e_u \wedge e_w \otimes ((bc - ad + d^2 - e^2)e_u + (b - e)(b + d + e)e_v + (-ab + bd + cd - ae - be + ce)e_w) \\ &\quad + e_v \wedge e_u \otimes ((-b^2 - ad + d^2 + ce)e_u + (b^2 + ac - c^2 - de)e_v + (d - c)(b - e)e_w) . \end{aligned}$$

Using the relations and monomial equalities in (i) one can check that the first Bianchi identity holds in the form

$$\wedge R_{\Omega^1}(e_u) = 0$$

and similarly for e_v, e_w as it must in the absence of torsion by Lemma 2.1. The 2nd Bianchi identity

$$(d \otimes \text{id} + \text{id} \wedge \nabla_{\Omega^1})R_{\Omega^1} = (\text{id} \wedge R_{\Omega^1})\nabla_{\Omega^1}$$

in Theorem 3.2 must also hold and can be verified. A short computation gives

$$(\text{id} \otimes (,))(R_{\Omega^1} \otimes \text{id})g = \text{Tr}R_{\Omega^1} = 0$$

where the metric trace and usual trace coincide because of the trivial Euclidean form of the metric. The same is true for the trace of $R_{\Omega^1}^2$ but we check it in the following more manageable case.

(iii) Geometrically speaking, we are interested in ad-invariant connections that are torsion free, which is $c - 1 = e = d - 1$, and cotorsion-free, which is $c - 1 = b = d - 1$. These are as close as one can come to ‘Levi-Civita’ in this example with respect to the Euclidean metric $g = \sum_a e_a \otimes e_a$. There is a two-parameter moduli space of such connections and we write the parameters as $\lambda = c - a - 2$ and $\mu = -c$. Then one can compute that

$$\begin{aligned} R_{\Omega^1}(e_u) = & -de_u \otimes e_u - de_v \otimes e_w - de_w \otimes e_v \\ & - (3 + \lambda)(\mu de_u \otimes e_u + (2(1 + \mu)e_u \wedge e_v - \mu e_v \wedge e_u) \otimes e_v) \\ & - (3 + \lambda)(2(1 + \mu)e_u \wedge e_w - \mu e_w \wedge e_u) \otimes e_w \end{aligned}$$

with the same under $e_u \rightarrow e_v \rightarrow e_w \rightarrow e_u$. We write the curvature in the shorthand form

$$R_{\Omega^1}(e_u) = \alpha(e_v \wedge e_w + e_w \wedge e_v) \otimes e_u + (\beta e_u \wedge e_v + \gamma e_v \wedge e_u) \otimes e_v + (\beta e_u \wedge e_w + \gamma e_w \wedge e_u) \otimes e_w$$

for coefficients $\alpha = (3 + \lambda)\mu + 1, \beta = -1 - 2(3 + \lambda)(1 + \mu), \gamma = (3 + \lambda)\mu - 1$ and the same under $u \rightarrow v \rightarrow w \rightarrow u$. Then trace of $R_{\Omega^1}^2$ is given by the $\otimes e_a$ coefficient of $R_{\Omega^1}^2(e_a) = (\wedge \otimes \text{id})(\text{id} \otimes R_{\Omega^1})R_{\Omega^1}(e_a)$, summed over $a = u, v, w$. Here

$$\begin{aligned} R_{\Omega^1}^2(e_u)|_{\otimes e_u} = & \alpha^2(e_v \wedge e_w + e_w \wedge e_v)^2 + (\beta e_u \wedge e_v + \gamma e_v \wedge e_u) \wedge (\beta e_v \wedge e_u + \gamma e_u \wedge e_v) \\ & + (\beta e_u \wedge e_w + \gamma e_w \wedge e_u) \wedge (\beta e_w \wedge e_u + \gamma e_u \wedge e_w) \end{aligned}$$

where in the first term we apply R_{Ω^1} to the $\otimes e_u$ term of $R_{\Omega^1}(e_u)$, for second term we apply it to the $\otimes e_v$ term of $R_{\Omega^1}(e_u)$ and for third term we apply it to the $\otimes e_w$ terms of $R_{\Omega^1}(e_u)$, and in all cases we pick off the $\otimes e_u$ term of this second application of R_{Ω^1} . Now using the relations of the exterior algebra we obtain

$$R_{\Omega^1}^2(e_u)|_{\otimes e_u} = \alpha^2((e_v \wedge e_w)^2 + (e_w \wedge e_v)^2) + \beta\gamma((e_u \wedge e_v)^2 + (e_v \wedge e_u)^2 + (e_u \wedge e_w)^2 + (e_w \wedge e_u)^2) = 0$$

since all of these squares vanish using the commutation relations (for example, $e_u \wedge e_v \wedge e_u \wedge e_v = -e_u \wedge (e_u \wedge e_w + e_w \wedge e_v) \wedge e_v = 0$). Similarly for the e_v, e_w terms of the trace. Hence $\int \text{Tr}R_{\Omega^1}^2 = 0$, at least for this class of connections, as each term in the trace vanishes. Here integration just sums over the group but all our coefficients are constant so this just multiplies by the order of the group.

(iv) We next ask when $(\Omega^1, \nabla_{\Omega^1})$ is an objects of ${}_A\mathcal{G}_A$ for $A = C(S_3)$ with the same calculus. From the bimodule commutation relations we have

$$(\Gamma^a_{bc}de_b + \Gamma^a_{bs}\Gamma^s_{rc}e_b \wedge e_r) f = R_{ac^{-1}}(f) (\Gamma^a_{bc}de_b + \Gamma^a_{bs}\Gamma^s_{rc}e_b \wedge e_r)$$

for any f and the $\otimes e_c$ component of $R_{\Omega^1}(e_a)$ for a connection in setting above. In view of this, a calculation for our full 5-parameter moduli space of bicovariant connections gives the following conditions on the parameters for the curvature being a right- and hence bi-module map:

- (1) $b = c = d = e = 0$,
- (2) $a = -b(b+c)/c, c \neq 0, d = c, e = -b - c$,
- (3) $d = a, e = c = b$,
- (4) $a = c, e = d = b$,
- (5) $a = b + c - b^2/c, c \neq 0, d = c, e = b$.

of which one can check using the formula for R_{∇} in part (ii) that (1)-(4) are just the cases of zero curvature and the only possibly nonzero case of the curvature a bimodule map is (5), where

$$R_{\nabla}(e_u) = \frac{(c-e)^2(c+2e)}{c}(e_u \wedge e_v \otimes e_v + e_u \wedge e_w \wedge e_u).$$

Here $e = c, -c/2$ fall back to an instance of cases (4),(2) respectively.

Moreover, it is known [4] that all left connections on Ω^1 as in (ii) are bimodule ones and using the tensor product basis $\{e_I\}$ in the lexicographical order $e_u \otimes e_u, e_u \otimes e_v, e_u \otimes e_w, e_v \otimes e_u$ to $e_w \otimes e_w$ that

$$\sigma = \begin{pmatrix} a & 0 & 0 & 0 & b & 0 & 0 & 0 & b \\ 0 & e & 0 & 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & e & d & 0 & 0 & 0 & c & 0 \\ 0 & 0 & c & e & 0 & 0 & 0 & d & 0 \\ b & 0 & 0 & 0 & a & 0 & 0 & 0 & b \\ 0 & d & 0 & 0 & 0 & e & c & 0 & 0 \\ 0 & c & 0 & 0 & 0 & d & e & 0 & 0 \\ 0 & 0 & d & c & 0 & 0 & 0 & e & 0 \\ b & 0 & 0 & 0 & b & 0 & 0 & 0 & a \end{pmatrix}$$

where $\sigma_{\Omega^1}(e_I) = \sum_J \sigma_I^J e_J$ and $\{\sigma_I^J\}$ is the matrix as shown (for example $\sigma_{\Omega^1}(e_u \otimes e_v) = e e_u \otimes e_v + c e_v \otimes e_w + d e_w \otimes e_u$ according to the second row).

We also ask which of our 5-parameter moduli of connections is extendable. Here the extended $\sigma_{\Omega^1} : \Omega^1 \otimes_A \Omega^2 \rightarrow \Omega^2 \otimes_A \Omega^1$, if it exists, must obey

$$\sigma_{\Omega^1}(\omega \otimes \eta \wedge \zeta) = (\wedge \otimes \text{id})(\text{id} \otimes \sigma_{\Omega^1})(\sigma_{\Omega^1} \otimes \text{id})(\omega \otimes \eta \otimes \zeta)$$

and if so then it further extends to higher forms to give an extendable bimodule connection in the sense of Definition 5.1 since the relations of the exterior algebra in the present

example are quadratic. We ask when this is well defined. For example

$$\begin{aligned} 0 &= \sigma_{\Omega^1}(e_u \otimes e_u \wedge e_u) = (\wedge \otimes \text{id})(\text{id} \otimes \sigma_{\Omega^1})(ae_u \otimes e_u + be_v \otimes e_v + be_w \otimes e_w) \otimes e_u \\ &= ae_u \wedge (be_v \otimes e_v + be_w \otimes e_w) + be_v \wedge (ce_u \otimes e_w + de_w \otimes e_v) + be_w \wedge (ce_u \otimes e_v + de_v \otimes e_w) \end{aligned}$$

which given the quadratic relations of the algebra requires $ba = bd = bc$. Similarly $0 = \sigma_{\Omega^1}(e_u \otimes e_v \wedge e_v)$ gives us $ea = ed = ec$ and $cd = eb$. Proceeding in the same way, $\sigma_{\Omega^1}(e_u \otimes ())$ applied to the two 3-term relations give us

$$e^2 + ad + ec = b^2 + ad + bc = d^2 + bc + ce, \quad e^2 + ac + ed = b^2 + ac + bd = c^2 + bd + de$$

Taken together, these nine equations have six 1-parameter moduli of solutions as follows:

- (1) $b = c = d = e = 0$
- (2) $a = c = d, b = q^{-1}a, e = qa$ where q is any cube root of 1 (three cases)
- (3) $a = d, b = c = e = 0$
- (4) $a = c, b = d = e = 0$.

We have done this by hand but a general approach is to ask which σ_{Ω^1} obey the mixed braid relation

$$(15) \quad (\text{id} \otimes \sigma_{\Omega^1})(\sigma_{\Omega^1} \otimes \text{id})(\text{id} \otimes \Psi) = (\Psi \otimes \text{id})(\text{id} \otimes \sigma_{\Omega^1})(\sigma_{\Omega^1} \otimes \text{id}),$$

where $\Psi(e_u \otimes e_v) = e_w \otimes e_u$ etc. is the crossed-module braiding that defines the calculus quadratic relations in our case. Such σ_{Ω^1} extend to the standard exterior algebra at least to degree 2 where the relations are given by $\ker(\text{id} - \Psi)$. In the present example, we obtain all extendable σ_{Ω^1} , i.e. the same moduli (1)-(4) as found directly. On the other hand, one can easily see that these cases are all instances of the corresponding moduli of $R_{\Omega^1} = 0$ connections already computed:

Lemma 6.1. *All six 1-parameter moduli of ad-invariant extendable connections make Ω^1 a flat object of $C(S_3)\mathcal{G}_{C(S_3)}$.*

Exactly two points among all six moduli are cotorsion-free, namely

$$a = c = d = b + 1 = \frac{1}{1 - q^{-1}}, \quad e = qd$$

where $q \neq 1$ is a cube root of unity, but neither of these is torsion-free. Exactly two points among all six moduli are torsion free, namely

$$a = c = d = e + 1 = \frac{1}{1 - q}, \quad b = q^{-1}d$$

where $q \neq 1$ is a cube root of unity, but neither of these is cotorsion-free.

(iv) We can also ask about full metric compatibility. For the full 5-parameter moduli of connections, the equations of $\nabla_{\Omega^1 \otimes \Omega^1} g = 0$ of metric compatibility come down to

$$a^2 + 2be = 1, \quad bc + ed + cd = 0, \quad be + c^2 + d^2 = 1, \quad ab + ae + be = 0, \quad ec + bd + cd = 0.$$

This has a discrete moduli of solutions as follows.

- (a): $a = \pm 1, b = d = e = 0, c = \epsilon a$ where $\epsilon = \pm 1$.
- (b): $a = \pm 1, b = c = e = 0, d = \epsilon a$ where $\epsilon = \pm 1$.

(c): $a = c = d = \pm 1/\sqrt{3}$, $b = q^{-1}a$, $e = qa$ where $q \neq 1$ is a cube root of 1.

(d): $a = \pm i$, $b = za$, $c = d = 0$, $e = -z^{-1}a$ where $z^2 - z - 1 = 0$ (for the golden ratio).

(e): $a = \pm 1/3$, $b = e = -2a$, $c = xa$, $d = \frac{2xa}{x-2}$ where $(x-1)(x+2)(x^2 - 5x + 10) = 0$.

None of these are torsion free, so there are no strict quantum Levi-Civita connections as fore-warned already in part (ii).

We also see that the metric preserving bimodule connections which are extendable are the four cases (a),(b) both with $\epsilon = 1$, and (c) with the two values of q . These are flat as we know from the above. The other curvatures are respectively,

$$(a) \quad \epsilon = -1 : \quad R_{\Omega^1}(e_u) = -2(e_w \wedge e_u \otimes e_w + e_v \wedge e_u \otimes e_v)$$

$$(b) \quad \epsilon = -1 : \quad R_{\Omega^1}(e_u) = -2(e_v \wedge e_w + e_w \wedge e_u) \otimes e_u$$

$$(d) \quad R_{\Omega^1}(e_u) = z^2(e_w \wedge e_u \otimes (e_u - e_w) + e_v \wedge e_u \otimes (e_u - e_v)) \\ + \frac{1}{z^2}e_u \wedge (e_v \otimes (e_u + (1 - z^4)e_w) + e_w \otimes e_u + (1 - z^4)e_v)$$

$$(e) \quad R_{\Omega^1}(e_u) = \frac{(x-4)(x-1)(x+2)}{9(x-2)^2} \\ (2(e_v \wedge e_w + e_w \wedge e_v) \otimes e_u + (2-x)(e_w \wedge e_u \otimes e_w + e_v \wedge e_u \otimes e_v)).$$

6.2. Bicrossproduct model quantum spacetime revisited. Here we look at the 2D noncommutative model with curvature found in [5] for the quantum spacetime A with differential algebra

$$[r, t] = \lambda r, \quad [r, dt] = \lambda dr, \quad [t, dt] = \lambda dt, \quad [r, dr] = 0 = [t, dr]$$

with dt, dr anticommuting, which admits a unique form of central quantum metric

$$g = ((1 + b\lambda^2)dr - \lambda bv) \otimes dr + bv \otimes v, \quad v = rdt - tdr.$$

Here dr, v are central and $dr \wedge v + v \wedge dr = 0$, $(dr)^2 = 0$ but $v^2 = \lambda \text{Vol}$, where $\text{Vol} = v \wedge dr$ is of top degree. It is already known [5] that there are two moduli of $*$ -preserving metric compatible bimodule connections for this model when b is real and λ imaginary, namely a line and a conic, with a unique torsion free ‘quantum Levi-Civita connection’ in each component. Only one of these, on the conic, has a classical limit as $\lambda \rightarrow 0$ so there is a unique quantum Levi-Civita connection with classical limit.

(i) Here we will look at bimodule connections more generally but still of the homogeneous form studied in [5], where

$$\nabla_{\Omega^1} dr = \frac{1}{r}(\alpha v \otimes v + \beta v \otimes dr + \gamma dr \otimes v + \delta dr \otimes dr)$$

$$\nabla_{\Omega^1} v = \frac{1}{r}(\alpha' v \otimes v + \beta' v \otimes dr + \gamma' dr \otimes v + \delta' dr \otimes dr)$$

for constants $\alpha, \dots, \delta, \alpha', \dots, \delta'$. These have braiding $\sigma_{\Omega^1}(dr \otimes dr) = dr \otimes dr$, $\sigma_{\Omega^1}(v \otimes dr) = dr \otimes v$ and

$$\sigma_{\Omega^1}(dr \otimes v) = \lambda \alpha v \otimes v + (1 + \lambda \beta)v \otimes dr + \lambda \gamma dr \otimes v + \lambda \delta dr \otimes dr$$

$$\sigma_{\Omega^1}(v \otimes v) = (1 + \lambda\alpha')v \otimes v + \lambda\beta'v \otimes dr + \lambda\gamma'dr \otimes v + \lambda\delta'dr \otimes dr,$$

torsion [5]

$$T(dr) = \frac{1}{r}(\lambda\alpha + \beta - \gamma)\text{Vol}, \quad T(v) = \frac{1}{r}(\lambda\alpha' + \beta' - \gamma' + 2)\text{Vol}$$

and curvature of the form

$$R_{\Omega^1}(dr) = -\frac{1}{r^2}\text{Vol} \otimes (c_1v + c_2dr), \quad R_{\Omega^1}(v) = -\frac{1}{r^2}\text{Vol} \otimes (c_3v + c_4dr)$$

for some constant coefficients c_i given explicitly in [5] in terms of our eight parameters. Both the torsion and curvature are bimodule maps. The Bianchi identities are also automatic as $\Omega^3 = 0$. The quantum dimension from [5] and now the quantum metric trace of the curvature using

$$(R_{\Omega^1} \otimes \text{id})(g) = -\frac{(1 + b\lambda^2)}{r^2}\text{Vol} \otimes (c_1v + c_2dr) \otimes dr - \frac{b}{r^2}\text{Vol} \otimes (c_3v + c_4dr) \otimes (v - \lambda dr)$$

are

$$\underline{\dim}_{\Omega^1} = \frac{2 + b\lambda^2}{1 + b\lambda^2}, \quad (\text{id} \otimes (,))(R_{\Omega^1} \otimes \text{id})(g) = -\frac{1}{r^2}\text{Vol} \left(\lambda c_1 + c_2 + \frac{c_3 - \lambda c_4}{1 + b\lambda^2} \right)$$

which deforms the usual trace. Note that in general in Section 3 we combined a usual trace and a cycle to give a well defined Tr_f when taken together. The metric trace, by contrast, is always well-defined but in general will depend on the metric in the noncommutative case.

(ii) We check when we have an extendable connection. In fact the result is surprising:

Lemma 6.2. *Bimodule connections of the above homogeneous form are extendable if and only they are flat. In this case*

$$\sigma_{\Omega^1}(dr \otimes \text{Vol}) = \text{Vol} \otimes ((1 + \lambda\beta)dr + \lambda\alpha v), \quad \sigma_{\Omega^1}(v \otimes \text{Vol}) = \text{Vol} \otimes (\lambda\beta'dr + (1 + \lambda\alpha')v).$$

Proof. We set

$$\sigma_{\Omega^1}(dr \otimes v \wedge dr) = (\wedge \otimes \text{id})(\text{id} \otimes \sigma_{\Omega^1})(\sigma_{\Omega^1} \otimes \text{id})(dr \otimes v \otimes dr)$$

etc, and find the values stated, while $\sigma_{\Omega^1}(() \otimes (dr)^2) = 0$, $\sigma_{\Omega^1}(() \otimes (dr) \wedge v) = -\sigma_{\Omega^1}(() \otimes \text{Vol})$ when computed in the same way. The only extendability constraints then come from $\sigma_{\Omega^1}(() \otimes v^2) = \lambda\sigma_{\Omega^1}(() \otimes \text{Vol})$ which gives 4 equations

$$(\beta - \alpha')\gamma + \alpha(1 - \delta + \gamma' + \lambda\alpha' + \lambda\beta) = 0, \quad \beta + \alpha\delta' - \gamma\beta' + \lambda\beta^2 + \lambda\alpha\beta' = 0$$

$$\alpha' - \alpha\delta' + \gamma\beta' + \lambda\alpha'^2 + \lambda\alpha\beta' = 0, \quad (\alpha' - \beta)\delta' + \beta'(1 + \delta - \gamma' + \lambda\alpha' + \lambda\beta) = 0$$

for extendability. Remarkably, the four expressions here being set to zero are exactly the coefficients c_1, c_2, c_3, c_4 for the curvature listed in [5]. \square

Using Mathematica, we find two 4-parameter moduli of bimodule connections of the homogeneous form that are flat/extendable and non-singular in the classical limit:

(1) We take $\alpha, \beta, \gamma, \delta$ as the parameters and

$$\alpha' = -\beta, \quad \beta' = -\frac{\beta^2}{\alpha}, \quad \gamma' = \delta - 1 - \frac{2\beta\gamma}{\alpha}, \quad \delta' = -\frac{\beta(\alpha + \beta\gamma)}{\alpha^2}$$

(2) We take $\gamma, \gamma', \delta, \delta'$ as the parameters and $\alpha = \alpha' = \beta = \beta' = 0$.

So these are all the connections of our restricted form that make $(\Omega^1, \nabla_{\Omega^1}, \sigma_{\Omega^1})$ objects of ${}_A\mathcal{G}_A$ and are non-singular as $\lambda \rightarrow 0$, and all necessarily flat.

(iii) Next we ask which of our two 4-parameter moduli of nonsingular extendable connections in (ii) are metric compatible. In fact the family (1) have no intersection with metric compatibility for generic or nonsingular b . The family (2) has a 1-parameter sub-family of metric-compatible connections, namely $\alpha = \alpha' = \beta = \beta' = 0$ and

$$\gamma' = -\delta = \frac{\gamma\lambda}{2}, \quad \delta' = -\frac{\gamma(1+b\lambda^2)}{b}$$

where we take γ to be the free variable. Their torsion is always nonzero,

$$T(dr) = -\gamma r^{-1} \text{Vol}, \quad T(v) = 2r^{-1} \text{Vol}$$

and the only $*$ -preserving connection in this 1-parameter family is the one with $\gamma = 0$. This unique point has $\nabla_{\Omega^1}(dr) = \nabla_{\Omega^1}(v) = 0$ and hence exactly quantises the flat metric-compatible Poisson-compatible connection which underlies the quantisation of the differential calculus and metric in Poisson-Riemannian geometry[6]. This is the quantum connection ∇_Q constructed in the latter paper at semiclassical order. In particular, the unique $*$ -preserving non-singular quantum Levi-Civita connection in [5] cannot be extendable by our above analysis as it is not flat.

(iii) As is often the case in noncommutative geometry, one may need something weaker than metric compatibility, namely cotorsion-free, and in our case extendable connections. Here of type (1) we have two cases

$$(1a): \quad \gamma = -\frac{\alpha(2\alpha b\lambda^2 + \alpha + \beta b\lambda + b)}{b(\alpha\lambda + \beta)}, \quad \delta = -\frac{\alpha(2\beta b\lambda^2 + \beta + 2b\lambda) + \beta b(\beta\lambda + 3)}{b(\alpha\lambda + \beta)}$$

where α, β are the parameters, and

$$(1b): \quad \beta = -\alpha\lambda, \quad \delta = \frac{\alpha}{b}(1 + b\lambda^2) - \gamma\lambda - 1$$

where α, γ are the parameters. Of type (2) we have another two-parameter family, namely $\alpha = \alpha' = \beta = \beta' = 0$ and

$$\gamma' = \gamma\lambda - 2, \quad \delta' = (\delta + 2)\lambda$$

where γ, δ are the parameters. Thus, we have three 2-parameter moduli of extendable cotorsion free connections, all of them necessarily flat.

If we now also impose torsion-free then we have a unique torsion free cotorsion-free or ‘weak quantum Levi-Civita’ extendable connection of type (1b) with

$$\alpha = \frac{4b}{1 + b\lambda^2}, \quad \gamma = 0.$$

In the classical limit this gives us a natural connection

$$\nabla dr = \frac{1}{r}(4bv \otimes v + 3dr \otimes dr), \quad \nabla v = \frac{2}{r}dr \otimes v$$

on our curved manifold which is torsion free and weakly metric compatible in the sense $(\wedge \otimes \text{id})\nabla g = 0$. Unlike the Levi-Civita connection, this one is extendable on quantisation.

6.3. Extended connections on the q -sphere. We study $A = \mathbb{C}_q[S^2]$ the standard q -sphere appearing in the base of the q -Hopf fibration cf[10, 20] as the degree 0 component of $\mathbb{C}_q[SL_2]$ for a \mathbb{Z} -grading corresponding to a diagonal coaction of $\mathbb{C}[t, t^{-1}]$. Here the standard quantum group generators a, c have grade 1 and b, d grade -1 while the q -sphere generators are $z = cd = q^{-1}dc, z^* = -qab = -ba, x = -q^{-1}bc$ with the inherited relations

$$zz^* = q^2x(1 - q^2x), \quad z^*z = x(1 - x), \quad zx = q^2xz, \quad z^*x = q^{-1}xz^*.$$

The quantum principal bundle here has associated quantum vector bundles or projective modules E_n for each integer n realised as the grade $-n$ component of $\mathbb{C}_q[SL_2]$. These are each bimodules by multiplication in the quantum group and have an induced charge n q -monopole connection ∇_{E_n} with curvature $R_{E_n}(e) = q^3[n]_{q^2}\text{Vol} \otimes e$, see [20, 18], where we used $[n]_{q^2} = (1 - q^{2n})/(1 - q^2)$. This is a bimodule map as Vol is central. It is also possible to show that ∇_{E_n} is a bimodule connection. This was already explained for charge ± 1 in [7] in our geometric realisation of the q -Dirac operator but the same argument applies also for all n . Then Lemma 5.3 tells us that ∇_{E_n} is extendable with

$$\sigma_{E_n}(f \otimes \text{Vol}) = q^{2n}\text{Vol} \otimes f$$

for $f \in E_n$. In this way $(E_n, \nabla_{E_n}, \sigma_{E_n})$ are objects of ${}_{\mathbb{C}_q[S^2]}\mathcal{G}_{\mathbb{C}_q[S^2]}$. Clearly the curvature 2-form according to Proposition 5.9 is $\omega_{E_n} = q^3[n]_{q^2}\text{Vol}$.

To give details of the braiding we will focus on $E_1 = \mathbb{C}_q[SL_2]_{-1}$ with module generators b, d and the Grassmann connection for a standard choice of projector matrix, which equivalently constructs the q -monopole. This can be written as cf. [18],

$$\nabla_{E_1}f = d(fa) \otimes d - q^{-1}d(fc) \otimes b$$

for all $f \in E_1$. This is a bimodule connection with

$$\begin{aligned} \sigma_{E_1}(b \otimes dz) &= \nabla_{E_1}(bz) - (\nabla_{E_1}b)z = d(bza) \otimes d - q^{-1}d(bzc) \otimes b - d(ba) \otimes dz + q^{-1}d(bc) \otimes bz \\ &= d(cdab) \otimes d - d(bc) \otimes b - q(d(ba))z \otimes d + q^{-2}(d(bc))z \otimes b \\ &= -q^{-1}d(zz^*) \otimes d + qd(xz) \otimes b + q(dz^*)z \otimes d - q^{-1}(dx)z \otimes b \\ &= (qd(xz) - q^{-1}(dx)z) \otimes b + (q(dz^*)z - q^{-1}d(zz^*)) \otimes d \\ &= qxdz \otimes b + q((q^2 - 1)xdx - z^*dz) \otimes d \end{aligned}$$

where we move z from the right using the commutation relations in $\mathbb{C}_q[SL_2]$ and through \otimes_A . We then identify resulting expressions in terms of the q -sphere generators, use the module relations

$$(x - 1)b = z^*d, \quad zb = -q^2xd$$

and the q -sphere relations for the final form. In the same way one can find

$$\begin{aligned}\sigma_{E_1}(b \otimes dz^*) &= -qd(z^*)^2 \otimes d + q((dz^*)(q^2x - 1) + q^{-2}d(xz^*) - (dx)z^*) \otimes b \\ &= q(xdz^* - (q^2 - 1)z^*dx) \otimes b - qz^*dz^* \otimes d \\ \sigma_{E_1}(b \otimes dx) &= xdx \otimes b - q^2z^*dx \otimes d.\end{aligned}$$

These σ_{E_1} are necessarily compatible with the sphere projector relation

$$qz^*dz + q^{-1}zdz^* + q^2((2)_q x - q^{-1})dx = 0$$

where $(2)_q = q + q^{-1}$, which is a nice check of the formulae using the holomorphic calculus in [20]. This gives us the charge 1 q -monopole more explicitly as an object of ${}_{\mathbb{C}_q[S^2]}\mathcal{G}_{\mathbb{C}_q[S^2]}$. The curvature 2-form according to Proposition 5.9 comes out as $\omega_{E_1} = q^3\text{Vol}$.

We also conclude that $\Omega^1 \cong E_2 \oplus E_{-2}$ from [20] together with the induced connection is an object of ${}_{\mathbb{C}_q[S^2]}\mathcal{G}_{\mathbb{C}_q[S^2]}$. This connection ∇_{Ω^1} is torsion-free and metric-compatible or ‘quantum Levi-Civita’ for the metric

$$g = qdz^* \otimes dz + q^{-1}dz \otimes dz^* + q^2(2)_q dx \otimes dx$$

as essentially shown in [20]. That work did not consider ∇_{Ω^1} as a bimodule connection and hence only showed the weaker cotorsion freeness, but it is a bimodule connection and full metric compatibility can be checked after constructing the braiding σ by the method in [7]. We can therefore apply Corollary 5.8 and Corollary 5.10. Here[20]

$$R_{\Omega^1}(\partial f) = q^4(2)_q \text{Vol} \otimes \partial f, \quad R_{\Omega^1}(\bar{\partial} f) = -(2)_q \text{Vol} \otimes \bar{\partial} f$$

for $f \in \mathbb{C}_q[S^2]$ so that

$$(R_{\Omega^1} \otimes \text{id} + (\sigma_{\Omega^1} \otimes \text{id})(\text{id} \otimes R_{\Omega^1}))g = 0$$

as one can also see directly given that g has only mixed terms in the decomposition into holomorphic and holomorphic parts. For example, if the first tensor factor of a part of g is holomorphic, we have $q^4(2)_q$ from the first term. R_{Ω^1} in the second term is $-(2)_q$ and σ_{Ω^1} acting on the holomorphic factor $\otimes \text{Vol}$ gives q^4 as this part is in E_2 . This illustrates the antisymmetry of the Riemann tensor in Corollary 5.8. There are no non-trivial Bianchi identities to illustrate here as $\Omega^3 = 0$.

For the metric traces in Corollary 5.10, we find similarly find

$$\underline{\dim}_{\Omega^1} = (2)_{q^2}, \quad (\text{id} \otimes (,))(R_{\Omega^1} \otimes \text{id})g = (q^4 - 1)(2)_q(2)_{q^2} \text{Vol}$$

where $1, \text{Vol}$ are the generators of H_{dR} . We see that the metric traces q -deform their classical values.

REFERENCES

- [1] H. Bass, Algebraic K -theory, W.A. Benjamin Inc., New York & Amsterdam, 1968.
- [2] E.J. Beggs & T. Brzeziński, The Serre spectral sequence of a noncommutative fibration for de Rham cohomology, Acta Math., 195 (2005) 155–196
- [3] E.J. Beggs & T. Brzeziński, Line bundles and the Thom construction in noncommutative geometry, J. Noncomm. Geom., 8 (2014) 61–105

- [4] E.J. Beggs & S. Majid, \ast -compatible connections in noncommutative Riemannian geometry, *J. Geom. Phys.* 61 (2011) 95–124
- [5] E.J. Beggs & S. Majid, Gravity induced from quantum spacetime, *Class. Quant. Grav.* 31 (2014) 035020 (39pp)
- [6] E.J. Beggs & S. Majid, Poisson-Riemannian geometry, *J. Geom. Phys.* 114 (2017) 450–491
- [7] E.J. Beggs & S. Majid, Spectral triples from bimodule connections and Chern connections, *J. Non-comm. Geom.*, 11 (2017) 669–701
- [8] J. Block, Duality and Equivalence of Module Categories in Noncommutative Geometry, CRM Proceedings and Lecture Notes, Vol. 50, 2010
- [9] K. Bressler, F. Müller-Hoissen, A. Dimakis & A. Sitarz, Noncommutative geometry of finite groups, *J. of Physics A*, 29 (1996) 2705–2735
- [10] T. Brzezinski & S. Majid, Quantum group gauge theory on classical spaces, *Phys. Lett. B.* 298 (1993) 339–343
- [11] S.S. Chern, Characteristic classes of Hermitian Manifolds, *Annals of Mathematics*, 47 (1946) 85–121
- [12] A. Connes, *Noncommutative Geometry*, Academic Press (1994).
- [13] J. Cuntz & D. Quillen, Algebra extensions and nonsingularity, *Jour. Amer. Math. Soc.* 8 (1995) 251–289
- [14] M. Dubois-Violette & P.W. Michor, Connections on central bimodules in noncommutative differential geometry, *J. Geom. Phys.* 20 (1996) 218 –232
- [15] A. Fröhlich, The Picard group of noncommutative rings, in particular of orders, *Trans. Amer. Math. Soc.* 180 (1973) 1–45
- [16] B. Keller, On differential graded categories, *ICM proceedings*, Vol. 2, 2006
- [17] J. Kustermans, G. Murphy & L. Tuset, Differential calculi over quantum groups and twisted cyclic cocycles, *J. Geom. Phys.* 44 (2003) 570–594
- [18] S. Majid, Noncommutative differential geometry, in *LTCC Lecture Notes Series: Analysis and Mathematical Physics*, eds. S. Bullet, T. Fearn and F. Smith, World Sci. (2017) 139–176
- [19] S. Majid, Quantum and braided group Riemannian geometry, *J. Geom. Phys.* 30 (1999) 113–146
- [20] S. Majid, Noncommutative Riemannian and spin geometry of the standard q -Sphere, *Comm. Math. Phys.*, 256 (2005) 255–285.
- [21] S. Majid, Hodge star as braided Fourier transform, *Alg. Reprn. Theory*, 20 (2017) 695–733
- [22] S. Majid & K. Rietsch, Lie theory and coverings of finite groups, *J. Algebra*, 389 (2013) 137–150
- [23] S. Majid & W.-Q. Tao, Cosmological constant from quantum spacetime, *Phys. Rev. D* 91 (2015) 124028 (12pp)
- [24] S. Majid & W.-Q. Tao, Generalised noncommutative geometry on finite groups and Hopf quivers, 41pp. arXiv:1612.09258 (math.QA)
- [25] B. Mitchell, Rings with Several Objects, *Adv. Math.* 8 (1972) 1–161
- [26] J. Mourad, Linear connections in noncommutative geometry, *Class. Quant. Grav.* 12 (1995) 965 – 974
- [27] S.L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups). *Comm. Math. Phys.* 122 (1989) 125–170

EJB: DEPT OF MATHEMATICS, SWANSEA UNIVERSITY, SINGLETON PARC, SWANSEA SA2 8PP, SM:
 SCHOOL OF MATHEMATICAL SCIENCES, QUEEN MARY, UNIVERSITY OF LONDON, MILE END RD, LONDON
 E1 4NS