

THE OUTERCOARSENESS OF THE n -CUBE

ALEX FINK AND RICHARD K. GUY

ABSTRACT. Guy and Nowakowski showed that the outercoarseness of the n -cube was, for sufficiently large n , at least 0.96 of its maximum possible value, $n \cdot 2^{n-4}$. Here we give some exact results, including that the maximum is attained for all $n \geq 24$. We construct explicit partitions of the edges of the cube attaining this maximum in which each part is a *tepee*, namely, the three-cube with a vertex and a non-incident edge deleted. Its vertices and those of the cube are given binary labels, which we often write in octal (base 8) or hexadecimal (base 16) notation.

1. INTRODUCTION

The *coarseness* of a graph is the maximum possible number of components in an edge-partition of the graph with each part a non-planar graph. Interest in coarseness, a one of several quantities one might associate to a graph to measure its deviation from planarity, dates from Erdős's introduction of the concept. One application is to printing of circuits: if one can print conducting material onto insulating sheets, with contact between the sheets at and only at a set of points serving as the vertices, then the coarseness of a graph is the number of sheets required to realise it [6]. Bounds or exact values have been computed for the coarseness of several frequently studied families of graphs, among them the n -dimensional cube Q_n [5, 4].

Similarly, the *outercoarseness* $\xi_o(G)$ of a graph G is the maximum number of parts in an edge partition with each part non-outerplanar. An *outerplanar graph* is a planar graph for which there is an imbedding in the plane with all its vertices on the boundary of a single cell. Bounds for the outercoarseness of the cube Q_n were first announced without proof in [1], with the proofs appearing in the follow-up article [2]. The upper bound given in these works was

$$(1.1) \quad \xi_o(Q_n) \leq n \cdot 2^{n-4},$$

a consequence of a simple edge-counting argument, while the lower bound was $(0.96n - 1.15)2^{n-4} < \xi_o(Q_n)$ for n not equal to 2, 5, 6 or 9.

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The present paper closes the gap between the above bounds for all $n \geq 24$, as well as certain lesser values of n , by exhibiting edge-partitions that show that the upper bound (1.1) is in fact attainable. We also give improved bounds in some smaller cases.

Halin [3] has given the analog, for outerplanar graphs, of Kuratowski's theorem for planar graphs:

Theorem 1.1 (Halin). *A graph is outerplanar just if it does not contain a subgraph homeomorphic to the complete graph K_4 or to the complete bipartite graph $K_{2,3}$.*

When we refer to a product $G \times H$ of two graphs we mean the *Cartesian product*, in which two vertices (v, w) and (v', w') are joined by an edge if and only if either $v = v'$ and w is adjacent to w' in H , or v is adjacent to v' in G and $w = w'$.

For example, the n -cube, Q_n , is the n -fold Cartesian product of $Q_1 = K_2$, and is therefore a graph on 2^n vertices. If we give the names 0 and 1 to the two vertices of Q_1 , the vertices of Q_n are thereby labelled with the n -digit binary numbers. The vertex labels form a n -dimensional vector space over \mathbb{F}_2 , and the symbols “+” and “span” will refer to operations inside this vector space. The operation + is also known as *bitwise xor*, or *nim-sum*. After this introduction, we will begin to write the binary labels in octal or hexadecimal notation. Although this labelling may be novel, it is purely a space-saving measure, compressing three or four coordinates of the vertex label to a single symbol.

The n -cube is regular with valence n . The $n \cdot 2^{n-1}$ edges connect pairs of vertices whose labels differ in just one binary digit. Of the edges there are 2^{n-1} in each of n different *directions*, or having n different *colors*, according to which of the n Cartesian factors Q_1 give rise to them. The n -cube is bipartite: the parts are the set of *odious* vertices and the set of *evil* vertices, the digit-sums of whose labels are respectively odd and even, and which are respectively represented by small black and white circles in Figure 1.

The smallest non-outerplanar subgraphs of Q_n are shown in Figure 1, where (a), (b), (c) are homeomorphs of $K_{2,3}$ and (d) is a homeomorph of K_4 . The vertex labels are in binary. Only (a) has as few as 8 edges, and it is easy to see that no such graph can have fewer edges. So (a) will be the graph of our choice, which we call a *tepee*.

This immediately gives us the upper bound (1.1),

$$\xi_o(Q_n) \leq n \cdot 2^{n-4}.$$

We set some terminology for the anatomy of a tepee. Let $e_0 = 00 \dots 01$, $e_1 = 00 \dots 10$, \dots , $e_{n-1} = 10 \dots 00$ be the standard basis of the \mathbb{F}_2 -vector space whose elements are the vertices of Q_n , so that the neighbours of a vertex v are all vertices of form $v + e_i$ with $0 \leq i \leq n - 1$. Then a *tepee in*

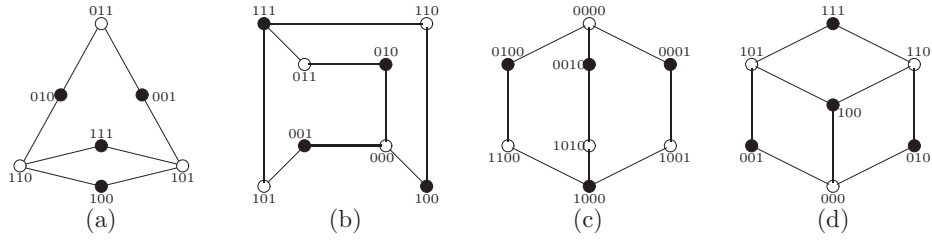


FIGURE 1. Non-outerplanar subgraphs of Q_n

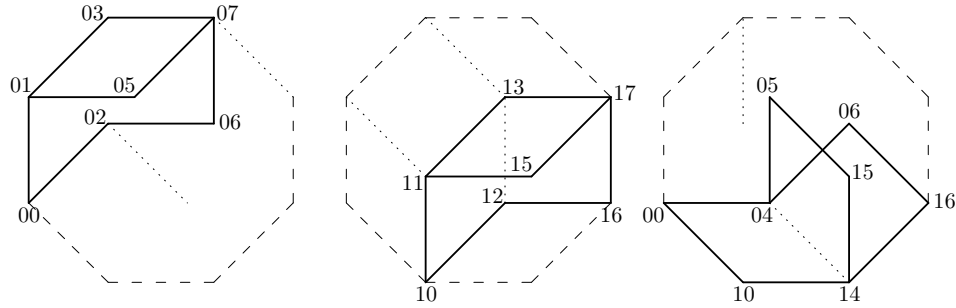
orientation ijk contains the edges

$$\begin{aligned}
 & (v, v + e_i), & & (v, v + e_j), \\
 & (v + e_i + e_k, v + e_i), & & (v + e_j + e_k, v + e_j), \\
 & (v + e_i + e_k, v + e_k), & & (v + e_j + e_k, v + e_k), \\
 (1.2) \quad & (v + e_i + e_k, v + e_i + e_j + e_k), & & (v + e_j + e_k, v + e_i + e_j + e_k)
 \end{aligned}$$

for some vertex v , which we will call the *tip* of the tepee. The tepee's trivalent vertices are $v + e_i + e_k$ and $v + e_j + e_k$, and its remaining vertices are bivalent. A tepee subgraph of a cube is completely specified by its tip and orientation, and this specification is unique up to exchange of i and j . For example, the tepee in Figure 1(a) has tip 011, and orientation 012 (or equally 102). When checking the orientations, bear in mind that we have labelled the bits *from right to left*, e.g., vertex 011 means $0e_2 + 1e_1 + 1e_0$!

2. SMALL VALUES OF n

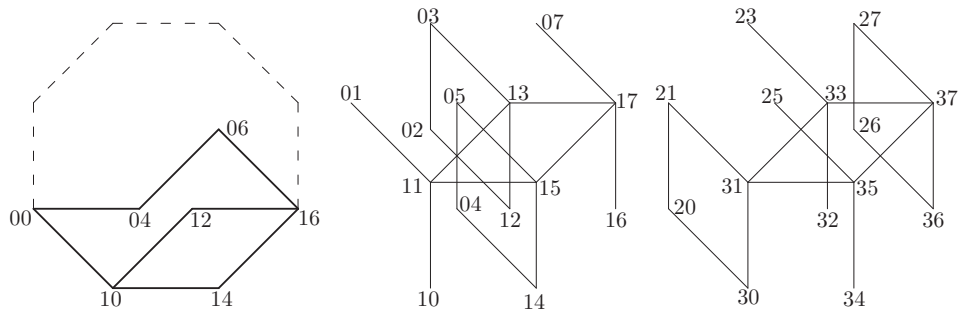
The bound (1.1) can't be attained for small n . For $n = 1$ and 2 , Q_n is outerplanar, so that $\xi_o(Q_1) = \xi_o(Q_2) = 0$. On the other hand Q_3 is not outerplanar, but has only 12 edges, so that $\xi_o(Q_3) = 1$. Although Q_4 has 32 edges, the bound of 4 can't be attained since the graph of Figure 1(a) has two trivalent vertices, so that some of the edges of the 4-valent graph Q_4 can't be used. That $\xi_o(Q_4) = 3$ may be seen from Figure 2 whose vertex labels are written in octal. It shows the decomposition of Q_4 into two tepees in orientation 120 with tips 02 and 12, and a copy of Figure 1(c): there are 7 unused edges, indicated by dotted lines.

FIGURE 2. A decomposition of Q_4

3. THE 5-CUBE

We can see that $\xi_o(Q_5) < 10$ because the 20 3-valent vertices of 10 tepees can occupy only 20 of the 5-valent vertices of Q_5 , leaving 12 odd-valent vertices, so that at least 6 edges must be wasted.

To see that $\xi_o(Q_5) = 9$, we will write vertex labels in octal. We regard Q_5 as the product $Q_4 \times Q_1$ with the edges coming from the Q_4 factor having directions 0, 1, 2 and 3, and from the Q_1 having direction 4. This gives two subgraphs Q_4 of Q_5 , one of which has vertices 00 through 17, the other 20 through 37. Into the subgraph on vertices 00 through 17, we place the tepee shown in Figure 3, at left, together with the one shown in the left of Figure 2. The 16 unused edges are shown in the second part of Figure 3.

FIGURE 3. Two decompositions of Q_4

We also place two tepees in the Q_4 subgraph on vertices 20 through 37, namely the images of the first two under the isomorphism between the two Q_4 taking binary $0dcba$ to $0dbca + 00010$. The unused edges in this Q_4 form the configuration in the third part of Figure 3.

The second and third parts of Figure 3 are connected by sixteen edges in direction 4. Four of these, namely octal $(0x, 2x)$, $x = 0, 2, 4, 6$, are wasted.

Four of them, $(1y, 3y)$, $y = 1, 3, 5, 7$, together with the edges between their endpoints in the two Q_4 , form a Q_3 , which is non-outerplanar. The four pairs, $(1x, 3x)$ and $(0y, 2y)$, $y = x+1$, together with the edges in the centre and right of Figure 3 in directions 0 and 3 form tepees. With the tepees from the two Q_4 , we have $1 + 4 + (2 \times 2) = 9$ non-outerplanar graphs and

$$\xi_o(Q_5) = 9.$$

4. THE 6-CUBE

If Q_6 had an exact packing with 24 tepees, every vertex of Q_6 would have to be occupied either by three bivalent or two trivalent tepee vertices. Thus the tepees would be arranged into rings, where neighboring tepees in each ring share a trivalent vertex.

A program was run to find packings with rings of tepees, which found that the maximal packing of this sort uses 16 tepees. This establishes $\xi_o(Q_6) < 24$. Here is a packing of 16 tepees in four rings of four, with vertex labels given in two-digit octal, and each ring described by one column:

orient ⁿ	tip	orient ⁿ	tip	orient ⁿ	tip	orient ⁿ	tip
012	06	012	53	012	17	012	42
345	50	345	05	345	41	345	14
012	35	012	60	012	24	012	71
345	63	345	36	345	72	345	27

More compactly, this arrangement uses 8 copies of the half-ring consisting of one tepee in orientation 012 with tip 06 and one in orientation 345 with tip 50, translating it by the elements of $\text{span}\{11, 22, 44\}$.

We also wrote a second, unclever, program which placed tepees unrestrictedly. It quickly found our current best known solutions with 21 tepees; it seemed to plateau there, though we never let it run to exhaustion. The table below exhibits one of its packings of 21 tepees in Q_6 . The source code of both of these programs, with a brief explanation of their output formats, can be found at <http://www.maths.qmul.ac.uk/~fink/outercoarseness/Q6.html>.

orient ⁿ	tip	orient ⁿ	tip	orient ⁿ	tip
012	05	145	44	123	33
345	50	053	44	150	74
342	11	054	47	231	44
054	41	150	50	123	55
032	12	124	30	024	50
054	42	450	17	123	72
345	53	231	24	123	73

The resulting bounds are

$$21 \leq \xi_o(Q_6) \leq 23.$$

5. THE 7-CUBE

We can show that $48 \leq \xi_o(Q_7) \leq 54$.

Taking the upper bound first, every non-outerplanar subgraph of a cube has at least 7 vertices with valence ≥ 2 ; observe that this is true of the minimal such graphs in Figure 1. Suppose that t non-outerplanar graphs are packed in Q_7 . Each vertex of Q_7 can only accommodate three vertices of valence ≥ 2 , proving $t \leq \lfloor 3 \cdot 2^7 / 7 \rfloor = 54$.

The lower bound is attained by the following packing of 48 tepees. Writing in octal, place 32 tepees with their tips two each at the sixteen vertices of $\text{span}\{011, 022, 044, 100\}$, one in orientation 012 and one in orientation 345 for each tip location. Place the remaining sixteen as follows:

- four in orientation 036 with tips $003 + \text{span}\{022, 044\}$;
- four in orientation 146 with tips $003 + \text{span}\{011, 044\}$;
- four in orientation 256 with tips $003 + \text{span}\{011, 022\}$;
- and four in orientation 256 with tips $\text{span}\{011, 022\}$.

6. VALUES OF n ATTAINING THE UPPER BOUND

Here we construct exact packings demonstrating that $\xi_o(Q_n)$ attains its maximum value $n \cdot 2^{n-4}$ for all $n \geq 8$ other than $n = 9, 10, 11, 14, 15, 19, 23$, which we cannot address at present. These constructions are based on the following elements:

- (1) Exact tepee packings of Q_8 , Q_{12} , $Q_4 \times K_{1,2}$ and $Q_8 \times K_{1,2}$. By taking products of these we obtain all graphs $Q_{4k} \times (Q_4 \times K_{1,2})^m$ for integers $k, m \geq 0$, aside from Q_4 .
- (2) A partial arrangement of tepees in Q_5 which, together with packings of G and $G \times K_{1,2}$, provides a packing of $G \times Q_5$ for any graph G . In view of item 1, iteration of this construction yields an exact tepee packing of every cube Q_{4k+9m} for integers $k, m \geq 0$, aside from Q_4 and Q_9 .

We provide the constructions in items 1 and 2 above in sections 6.1 and 6.2 respectively.

6.1. Cubes and cubes times $K_{1,2}$.

6.1.1. *The 8-cube.* That $\xi_o(Q_8) = 128$ is shown by the following elegant packing of 128 tepees. In Figure 4 the vertex labels are 2-digit hexadecimal numbers. Each diagram represents four tepees, and is to be replicated eight times. The upper two by adding the span of $\{\text{ff}, 66, 0\text{c}\}$ over \mathbb{F}_2 (i.e., $00, \text{ff}, 66, 99, 0\text{c}, \text{f3}, 6\text{a}, 95$) and the lower two by adding the span of $\{\text{ff}, 66, 03\}$ (i.e., $00, \text{ff}, 66, 99, 03, \text{fc}, 65, 9\text{a}$). The reader may verify that these exactly cover the $8 \cdot 2^7$ edges of Q_8 .

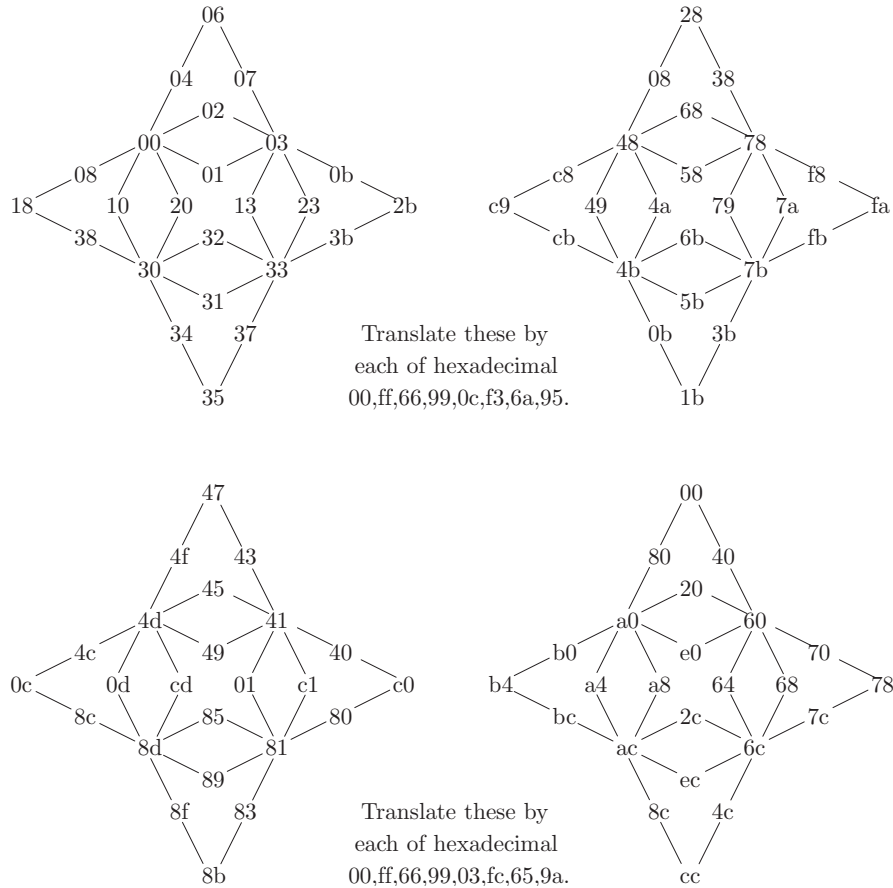


FIGURE 4. Q_8 is the union of 128 tepees

6.1.2. *The 12-cube.* We exhibit an exact packing of Q_{12} establishing that $\xi_o(Q_{12}) = 3072$.

There is a graph homomorphism p from Q_{12} to Q_6 sending the vertex $(x, y) \in \mathbb{F}_2^6 \times \mathbb{F}_2^6$ to $x + y \in \mathbb{F}_2^6$. This homomorphism induces a two-to-one map from the twelve edge directions of Q_{12} to the six of Q_6 , following which we grant the directions of Q_{12} the names $0x, \dots, 5x$ and $0y, \dots, 5y$.

We produce our packing of Q_{12} as follows: we provide an arrangement of tepees in Q_6 , and in each tepee, tag each class of edges in a single direction with the symbol x or y . Altogether each edge of Q_6 will be covered by two tepee edges, one tagged x and one tagged y . This arrangement on Q_6 can be pulled back across the homomorphism p to the requisite packing of Q_{12} , as suggested by the labels: a tepee with tip v and orientation abc , whose edges in directions a, b, c are tagged respectively $\alpha, \beta, \gamma \in \{x, y\}$, pulls back to 64 tepees with tips at each of the preimages of v , all in orientation $\alpha\alpha\beta\beta c\gamma$.

We present our arrangement on Q_6 using two-digit octal numbers for the vertices. It has 48 tepees; together these have the requisite $64 \cdot 48 = 3072$

preimages in Q_{12} . At each of the vertices 00, 11, 22, 33, 44, 55, 66, 77 are placed the tips of six tepees.

Tip location	Orientations
00	0y1x2x 1y2x3x 2y3x4x 3y4x5x 4y5x0x 5y0x1x
11	0x1x2y 1y2y3y 2x3y4x 3x4x5y 4y5y0y 5x0y1x
22	0x1y2x 1x2x3y 2y3y4y 3x4y5x 4x5x0y 5y0y1y
33	0y1y2y 1x2y3x 2x3x4y 3y4y5y 4x5y0x 5x0x1y
44	0y1y2y 1x2y3x 2x3x4y 3y4y5y 4x5y0x 5x0x1y
55	0x1y2x 1x2x3y 2y3y4y 3x4y5x 4x5x0y 5y0y1y
66	0x1x2y 1y2y3y 2x3y4x 3x4x5y 4y5y0y 5x0y1x
77	0y1x2x 1y2x3x 2y3x4x 3y4x5x 4y5x0x 5y0x1x

6.1.3. *The Cartesian product $Q_4 \times K_{1,2}$.* Let Q_4 sit in directions 0 through 3, and label its vertices in hexadecimal; and let the directions of the two edges of $K_{1,2}$ be 4 and 5. All of the tepees in the packing have their tips at the middle vertex of $K_{1,2}$, so we will only give their position in Q_4 . They have

orientation 014	and	tips 0, 5, a, f;
orientation 015	and	tips 3, 6, 9, c;
orientation 234	and	tips 1, 4, b, e;
orientation 235	and	tips 2, 7, 8, d.

6.1.4. *The Cartesian product $Q_8 \times K_{1,2}$.* We first look at a single Q_8 , and label its vertices in hexadecimal. The capital letters $ABCDEFGH$ will denote the directions or colors of edges; edges in these directions respectively have ends with difference (or, equivalently, sum) 01 02 04 08 10 20 40 80.

Consider two rings of four tepees, with trivalent vertices at 00 03 0c 0f and 30 33 3c 3f, respectively, where the third color of the tepees whose sets of trivalent vertices are $\{00, 03\}$, $\{0c, 0f\}$, $\{30, 3c\}$, and $\{33, 3f\}$ is E , and the third color of the tepees whose sets of trivalent vertices are $\{00, 0c\}$, $\{03, 0f\}$, $\{30, 33\}$, and $\{3c, 3f\}$ is F . The 64 edges of these tepees consist entirely of squares in the pairs of colors AB , CD , and EF . They include the sixteen squares with pairs of opposite vertices

$$\begin{aligned} &\{00, 03\}, \{0c, 0f\}, \{10, 13\}, \{1c, 1f\}, \{30, 33\}, \{3c, 3f\}, \{00, 30\}, \{03, 33\} \\ &\{00, 0c\}, \{03, 0f\}, \{20, 2c\}, \{23, 2f\}, \{30, 3c\}, \{33, 3f\}, \{0c, 3c\}, \{0f, 3f\}. \end{aligned}$$

We place eight such groups of eight tepees in Q_8 . Four of them are in the orientation displayed above, displaced by 00, 55, aa, and ff respectively. The other four are the images of the displayed tepee under the symmetries of Q_8 that exchange directions A with G , B with H , C with E , and D with F , and map 00 to 01, 54, ab, and fe respectively.

These 64 tepees occupy half of the edges in Q_8 . In any Q_4 with colors $ABCD$ or $EF GH$, exactly four of the squares with colors AB or CD or EF or GH have been completely filled, and the other four such squares are completely empty. Furthermore, in a Q_4 of colors $ABCD$, each evil vertex

lies on just one of the unfilled squares, whereas in a Q_4 of colors $EF GH$, each odious vertex lies on just one of the unfilled squares.

In our whole graph $Q_8 \times K_{1,2}$, we will place this configuration of 64 tepees in each of the three Q_8 . This leaves us to fill the remaining half of the edges in each Q_8 as well as all of the edges of the original 256 copies of $K_{1,2}$.

Two tepees can be placed in $Q_2 \times K_{1,2}$ so as to leave just two opposite subgraphs $K_{1,2}$ unfilled. We pack the remainder of the graph with these, oriented so that they fill a square in each Q_8 and the two $K_{1,2}$ corresponding to opposite vertices of each square.

In particular, we place one of these pairs of tepees in each of the $4 \cdot 32 = 128$ unfilled squares of colors AB or CD or EF or GH in the three subgraphs Q_8 . This fills every remaining edge of these subgraphs. We place a copy of $K_{1,2}$ at each evil vertex of a square of colors AB or CD , and at each odious vertex of a square of colors EF or GH . By our earlier observations, there's always just one square available for each vertex of the Q_8 , so each $K_{1,2}$ is filled just once. This completes the packing. In all we have used $3 \cdot 64 + 128 \cdot 2 = 448$ tepees, as required to pack $3 \cdot 8 \cdot 128 + 256 \cdot 2 = 3584$ edges.

6.2. A partial Q_5 . We will pack Q_5 with eight copies of the graph U shown in Figure 5 obtained from Q_3 by deleting two opposite edges. The graph U accommodates a tepee in any of four ways with $K_{1,2}$ remaining.

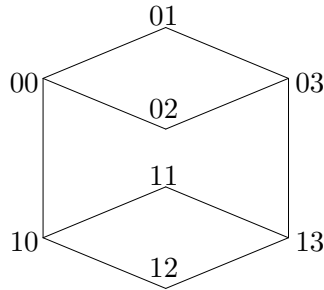


FIGURE 5. The graph U

View Q_4 as $Q_2 \times Q_2$, and select a pair of opposite vertices of each Q_2 in such a way that every vertex is selected once (for instance, select the evil vertices in squares in one pair of directions and the odious vertices in squares in the other pair of directions). Now, pack Q_5 with two parallel copies of this Q_4 , and join each square in one Q_4 to its counterpart in the other by edges between the selected vertices. This yields a packing of Q_5 with eight copies of U .

If we label the vertices of Q_5 with hexadecimal labels, then the trivalent vertices of the eight copies of U can be taken to fall at

$$\begin{aligned} &\{00, 03\}, \quad \{05, 06\}, \quad \{0a, 09\}, \quad \{0f, 0c\} \\ &\{11, 1d\}, \quad \{14, 18\}, \quad \{1b, 17\}, \quad \{1e, 12\} \end{aligned}$$

(these are pairs of trivalent vertices at opposite corners of a square) and the sums (over \mathbb{F}_2) of 10 with each of these. Call these copies U_0, \dots, U_7 .

It is possible to select a tepee from each U in such a way that the remaining edges form eight disconnected copies of $K_{1,2}$, for instance by removing from each U the two edges incident to 12,17,18,1d,09,0c,03,06 in U_0, \dots, U_7 respectively.

Therefore let G be a graph such that G and $G \times K_{1,2}$ have exact packings. In the graph $G \times Q_5$, each of the copies of Q_5 can be packed with eight copies of U , with the unused edges positioned so as to make up eight copies each of G and $G \times K_{1,2}$, which have exact packings themselves. Altogether this yields an exact packing of $G \times Q_5$.

As we saw in item 2 of the discussion opening Section 6, this construction results in packings of all of the cube graphs Q_{4k+9m} for integers $k, m \geq 0$ aside from Q_4 and Q_9 .

6.2.1. *Example: the 13-cube.* View Q_{13} as $Q_8 \times Q_5$. After partially filling the copies of Q_5 as in Section 6.2, we are left with eight copies each of Q_8 and $Q_8 \times K_{1,2}$, whose packings we have presented in Sections 6.1.1 and 6.1.4. Altogether this uses

$$\begin{aligned} & 2^8 \cdot 8 \text{ tepees in the } 2^8 \text{ partially filled subgraphs } Q_5; \\ & 2^3 \cdot 2^7 \text{ tepees in the } 2^3 Q_8 \text{ corresponding to the full vertices} \\ & \text{of the } Q_5; \\ & 2^3 \cdot 3 \cdot 2^3 \cdot 2^3 \text{ tepees in the } 2^3 \cdot 3 \text{ half-filled } Q_8 \text{ within the } 2^3 \\ & Q_8 \times K_{1,2}; \\ & 2^3 \cdot 2^7 \cdot 2 \text{ tepees in the rest of the } Q_8 \times K_{1,2}; \end{aligned}$$

which totals $13 \cdot 2^9$.

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QUEEN MARY, UNIVERSITY OF LONDON, LONDON, ENGLAND
E-mail address, Alex Fink: a.fink@qmul.ac.uk

THE UNIVERSITY OF CALGARY, CALGARY ALBERTA, CANADA
E-mail address, Richard K. Guy: rkg@cpsc.ucalgary.ca