Stationary integrated ARCH(∞) and AR(∞) processes with finite variance

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Abstract

We prove the long standing conjecture of Ding and Granger (1996) about the existence of a stationary Long Memory ARCH model with finite fourth moment. This result follows from the necessary and sufficient conditions for the existence of covariance stationary integrated AR(∞), ARCH(∞) and FIGARCH models obtained in the present paper. We also prove that such processes always have long memory.

Keywords: IAR, IARCH, LM-ARCH, FIGARCH, long memory.
JEL classification: C15; C22

1 Introduction

A non-negative random process $\{\tau_k\} = \{\tau_k,\, k \in \mathbb{Z}\}$ is said to satisfy an ARCH(∞) equation if there exists a sequence of nonnegative i.i.d. random variables $\{\varepsilon_k\}$ with unit mean $E\varepsilon_0 = 1$, a nonnegative number $\omega \ge 0$ and a deterministic sequence $b_j \ge 0$, $j = 1, 2, \dots$, such that

$$\tau_k \;=\; \varepsilon_k\big(\omega + \sum_{j=1}^{\infty} b_j \tau_{k-j}\big), \qquad k \in \mathbb{Z}. \tag{1.1}$$

In this paper we assume that the process $\{\tau_k\}$ described by equations (1.1) is *causal*, i.e., for any k, τ_k can be represented as a measurable function $f(\varepsilon_k, \varepsilon_{k-1}, \dots)$ of the present and past values $\varepsilon_s, s \le k$. The last property implies that a stationary $\{\tau_k\}$ process is ergodic, and ε_k is independent of $\tau_s, s < k$. Therefore,

$$E[\tau_k|\tau_s, s < k] = h_k, \quad h_k = \omega + \textstyle\sum_{j=1}^{\infty} b_j \tau_{k-j}.$$

A typical example of τ_k and ε_k in financial econometrics is that of squared returns and squared innovations, viz., $\tau_k = r_k^2$, $\varepsilon_k = z_k^2$, where the return process $\{r_k\}$ satisfies the

ARCH equations

$$r_k = z_k h_k^{1/2}, \quad h_k = \omega + \sum_{j=1}^{\infty} b_j r_{k-j}^2 \qquad k \in \mathbb{Z}, \tag{1.2}$$

$\{z_k\}$ is a standardized i.i.d. $(0,1)$-noise and h_k is volatility. The class of ARCH(∞) processes (1.1) includes the parametric stationary ARCH and GARCH models of Engle (1982) and Bollerslev (1986).

The ARCH(∞) process was introduced by Robinson (1991) and later studied in Giraitis, Kokoszka and Leipus (2000a), Giraitis and Surgailis (2002), Kazakevičius and Leipus (2002) and elsewhere. In contrast to a standard stationary GARCH(p,q) process whose autocorrelations decay exponentially, the ARCH(∞) process may have autocovariances $\mathrm{cov}(\tau_k, \tau_0)$ decaying to zero at a rate $k^{-\gamma}$ with $\gamma > 1$ arbitrarily close to 1. However, a finite variance stationary solution to the ARCH equations in (1.1) with $\omega > 0$, if it exists, has *short memory* or *absolutely summable autocovariance function*, see Giraitis and Surgailis (2002). The existence of such a solution necessarily implies $\sum_{j=1}^{\infty} b_j < 1$ by $E\tau_k = \omega + (\sum_{j=1}^{\infty} b_j) E\tau_k > (\sum_{j=1}^{\infty} b_j) E\tau_k$, excluding stationary Integrated ARCH models with $\sum_{j=1}^{\infty} b_j = 1$. Because of the well-known empirical phenomenon of long memory of squared returns, the latter finding may be considered as a limitation of ARCH modeling. Subsequently, it initiated and justified the study of other ARCH-type models for which the long memory property can be rigorously established, see Giraitis, Robinson and Surgailis (2000b), Giraitis, Leipus and Surgailis (2009).

A particular case of the IARCH model is the well-known FIGARCH equation

$$\tau_k \;=\; \varepsilon_k \big\{ \omega + \big(1 - (1-L)^d\big) \tau_k \big\} \;=\; \varepsilon_k \big(\omega + \sum_{j=1}^{\infty} b_j \tau_{k-j} \big), \quad k \in \mathbb{Z}, \tag{1.3}$$

where $0 < d < 1/2$ is the fractional differencing parameter, L is the backshift operator and the coefficients b_j are determined by the generating function $B(z) = \sum_{j=1}^{\infty} b_j z^j = 1 - (1-z)^d$. Here, $b_j > 0$, $\sum_{j=1}^{\infty} b_j = 1$ and $b_j = O(j^{-1-d})$ decay hyperbolically with $j \to \infty$. The FIGARCH equation was introduced by Baillie, Bollerslev, and Mikkelsen (1996) to capture the long memory effect in volatility. Independently of the last paper, Ding and Granger (1996, Eq. (4.24)) introduced the LM(d)-ARCH model

$$r_k^2 = z_k^2 h_k, \quad h_k = \mu(1-\theta) + \theta(1 - (1-L)^d) r_k^2, \qquad k \in \mathbb{Z}, \tag{1.4}$$

where $\theta \in [0,1]$, $\mu > 0$ and r_k, z_k are related to τ_k, ε_k as in (1.2). A similar long memory model for absolute returns was proposed by Granger and Ding (1995). Ding and Granger (1996) derived (1.4) via contemporaneous aggregation of a large number of GARCH(1,1) processes with random Beta distributed coefficients. Ding and Granger (1996) note that in the integrated case $\theta = 1$, (1.4) coincides with the special case $\omega = 0$ of the FIGARCH model in

(1.3). Ding and Granger (1996, pp. 206-207) argue that a stationary solution of (1.4) with finite fourth moment has long memory, in the sense that

$$\text{(1.5)} \qquad \operatorname{corr}(r_0^2, r_k^2) \sim \frac{\Gamma(1-d)}{\Gamma(d)} k^{-1+2d}.$$

The results in Baillie *et al.* (1996, pp. 10-11) imply a similar long memory behavior of the FIGARCH model[1].

However, the existence of a stationary solution of the LM(d)-ARCH equation in (1.4) with finite fourth moment was not rigorously established and the validity of (1.5) remained open. See Davidson (2004), Giraitis *et al.* (2000a), Kazakevičius and Leipus (2003), Mikosch and Stărică (2000, 2003) for a discussion of controversies surrounding the FIGARCH and the LM(d)-ARCH models.

The present paper solves the long standing conjecture (1.5) of Ding and Granger (1996). We prove that the necessary and sufficient condition for the existence of a covariance stationary solution of the FIGARCH equation in (1.3) with $\omega = 0$ is

$$\text{(1.6)} \qquad E\varepsilon_0^2 < \frac{\Gamma(1-2d)}{\Gamma(1-2d) - \Gamma^2(1-d)},$$

and therefore conditions (1.6) and $\theta = 1$ being necessary and sufficient for (1.5)[2]. See Corollary 2.2 below.

The above result is a particular case of more general result concerning integrated ARCH(∞), or IARCH(∞) equation with zero intercept:

$$\text{(1.7)} \qquad \tau_k = \varepsilon_k \Big(\sum_{j=1}^{\infty} b_j \tau_{k-j} \Big), \quad k \in \mathbb{Z}, \quad \text{with} \quad \sum_{j=1}^{\infty} b_j = 1.$$

Note that for $\sum_{j=1}^{\infty} b_j < 1$, equation (1.7) has only the trivial stationary solution $\tau_k = 0$ with finite mean, which follows from $E\tau = (\sum_{j=1}^{\infty} b_j) E\tau$ by taking expectations. Our main result is Theorem 2.1 saying that a covariance stationary solution of the IARCH equation in (1.7) with $b_j \geq 0$ exists if and only if

$$\text{(1.8)} \qquad \|g\|^2 = \sum_{j=0}^{\infty} g_j^2 < (1+\sigma^2)/\sigma^2,$$

where $\sigma^2 = \operatorname{var}(\varepsilon_0)$ and g_j are determined from the power expansion

$$\text{(1.9)} \qquad \textstyle\sum_{j=0}^{\infty} g_j L^j = (1 - B(L))^{-1}, \qquad \text{where} \qquad B(L) = \sum_{j=1}^{\infty} b_j L^j.$$

[1]In Baillie, Bollerslev, and Mikkelsen (1996) on page 10, bottom line and on page 11, line 6, d should be replaced by $-d$.

[2]Condition (1.6) for the existence of a stationary solution of the FIGARCH equation in (1.3) with $\omega = 0$ was independently obtained in the unpublished paper by Koulikov (2003) who used a similar approach for constructing the solution. However, the proof in Koulikov (2003, Theorem 2) uses erroneous assumption (9) which contradicts the IARCH condition $\sum_{j=1}^{\infty} b_j = 1$.

Condition (1.8) rules out integrated GARCH(p, q) as well as any IARCH(∞) models with sufficiently fast decaying lags which are known to admit a stationary solution with infinite variance, see Kazakevičius and Leipus (2003), Douc, Roueff, and Soulier (2008), Robinson and Zaffaroni (2006). It turns out that covariance stationary solutions of (1.7) always have long memory, in the sense that the covariance function is nonsummable and the spectral density is infinite at the origin, see Corollary 2.1.

The main idea of constructing a stationary L_2-solution τ_k of the IARCH equation (1.7) with mean $\mu = E\tau_k > 0$ is the reduction of equation (1.7) to the linear Integrated AR (IAR) equation for the centered process $Y_k = \tau_k - \mu$:

$$Y_k = \sum_{j=1}^{\infty} b_j Y_{k-j} + M_k, \qquad k \in \mathbb{Z} \tag{1.10}$$

with a conditionally heteroskedastic martingale difference noise $\{M_k\}$ defined as

$$M_k = \zeta_k \big(\mu\sigma + \sigma \sum_{j=1}^{\infty} b_j Y_{k-j}\big), \tag{1.11}$$

where $\zeta_k = (\varepsilon_k - 1)/\sigma$, $\sigma^2 = \text{var}(\varepsilon_1) < \infty$. In turn, from (1.10) and (1.11) the process $\{M_k\}$ can be defined as a stationary solution of the LARCH (Linear ARCH) equation (2.5) with standardized zero mean i.i.d. innovations $\{\zeta_k\}$ discussed in Giraitis *et al.* (2000b, 2004), given by convergent Volterra series in (2.6). Then, a causal L_2-solution $\{Y_k\}$ can be obtained by inverting the linear IAR equation in (1.10).

The last question is studied in Section 3 where we establish sufficient and necessary conditions for the existence of a covariance stationary solution of the linear Integrated AR(∞) equation generalizing (1.10):

$$x_k - \sum_{j=1}^{\infty} b_j x_{k-j} = \xi_k \qquad k \in \mathbb{Z}, \tag{1.12}$$

where $b_j \ge 0$, $\sum_{j=1}^{\infty} b_k = 1$ and $\{\xi_k\}$ is a stationary short memory process, in particular, white noise. Theorem 3.1 says that covariance stationary solutions of (1.12) always have long memory, which originates from integration property $\sum_{j=1}^{\infty} b_j = 1$ with infinite number of $b_j \ge 0$. This result is in deep contrast with the well-known fact that integrated AR$(p), p < \infty$ processes are non-stationary and need to be differenced to achieve stationarity.

The paper is structured as follows. Section 2 discusses stationary L_2-solutions of the ARCH(∞) (1.1) and bilinear (1.10)-(1.11) equations and their mutual relationship. It contains Theorem 2.1 together with several corollaries. Section 3 discusses solvability and second-order properties of IAR(∞) equation (1.12). All proofs are relegated to Sections 4 and 5.

In the sequel, we set $\Pi = [-\pi, \pi]$, and write $a_n \sim b_n$ if $a_n/b_n \to 1$. All (in)equalities involving random variables in this paper are supposed to hold almost surely. C stands for a generic positive constant whose precise value is not required and which may assume different values at various locations.

2 Stationary solutions of FIGARCH, IARCH and ARCH equations

In this section we discuss the existence of a stationary L_2-solution of ARCH(∞) equation (1.1) in the integrated case $\sum_{j=1}^{\infty} b_j = 1$. We first explain the idea of solving the ARCH(∞) equation (1.1) with a non-negative i.i.d. noise $\{\varepsilon_k\}$ by reducing it to a bilinear equation with a zero mean i.i.d. noise $\{\zeta_k\}$ used in Giraitis and Surgailis (2002). Recall the definition of ARCH(∞) model in (1.1). Specifically, for a stationary ARCH(∞) process τ_k in (1.1) with mean $E\tau_k = \mu$, we set

$$Y_k = \tau_k - \mu.$$

Let $\theta = \sum_{j=1}^{\infty} b_j$. We focus on two cases, a) $\omega > 0$ and $0 < \theta < 1$, and b) $\omega = 0$ and $\theta = 1$. As noted above, case $\omega = 0$ and $\theta < 1$ is uninteresting and excluded from the subsequent discussion since it leads to the unique trivial solution $\tau_k = 0$. By taking expectations, equation (1.1) implies $E\tau_k = \omega + \theta E\tau_k$, or $\mu = E\tau_k = \omega/(1-\theta)$ in case a), while in case b), it does not contradict a free choice of $\mu > 0$. Motivated by these facts, put

$$\mu = \begin{cases} \omega/(1-\theta) & \text{if } \theta < 1, \text{ and } \omega > 0, \\ \text{any positive number } \mu > 0 & \text{if } \theta = 1 \text{ and } \omega = 0. \end{cases} \tag{2.1}$$

Assume $\sigma^2 = \mathrm{var}(\varepsilon_1) < \infty$ and let $\{\zeta_k = (\varepsilon_k - 1)/\sigma,\ k \in \mathbb{Z}\}$ be the centered i.i.d. noise (recall that ε_k in (1.1) are standardized: $E\varepsilon_k = 1$). With this notation, the ARCH equation of (1.1) can be written as the bilinear equation

$$Y_k = \sum_{j=1}^{\infty} b_j Y_{k-j} + \zeta_k \Big(\mu\sigma + \sigma \sum_{j=1}^{\infty} b_j Y_{k-j}\Big), \tag{2.2}$$

see also Giraitis and Surgailis (2002). As noted in Giraitis *et al.* (2000b), Giraitis and Surgailis (2002), (2.2) is a different class from bilinear equations discussed in Granger and Andersen (1978), Subba Rao (1981) due to the presence of cross terms $\zeta_k Y_{k-j}$. Set

$$M_k = Y_k - \textstyle\sum_{j=1}^{\infty} b_j Y_{k-j} = (1 - B(L))Y_k.$$

Then $Y_k = (1 - B(L))^{-1} M_k = G(L) M_k = \sum_{j=0}^{\infty} g_j M_{k-j}$ and

$$\sigma \textstyle\sum_{j=1}^{\infty} b_j Y_{k-j} = \sigma B(L)(1 - B(L))^{-1} M_k = H(L) M_k = \sum_{j=1}^{\infty} h_j M_{k-j},$$

where coefficients g_j, h_j of the generating functions $G(z), H(z)$ are defined by

$$(2.3)\qquad G(z) = \frac{1}{1-B(z)} = \sum_{j=0}^{\infty} g_j z^j, \qquad H(z) = \frac{\sigma B(z)}{1-B(z)} = \sum_{j=1}^{\infty} h_j z^j, \qquad |z| < 1.$$

Notice that $h_j = \sigma g_j$ $(j \ge 1)$, $g_0 = 1$, $h_0 = 0$ follows from equality $H(z) = \sigma(G(z)-1)$ which in turn follows from (2.3). Hence (2.2) can be written as the system of two equations:

$$(2.4)\qquad (a)\ Y_k = \sum_{j=1}^{\infty} b_j Y_{k-j} + M_k, \qquad (b)\ M_k = \zeta_k\big(\mu\sigma + \sum_{j=1}^{\infty} h_j M_{k-j}\big).$$

Note that equation (2.4)(b) does not contain the Y_k and coincides with the so-called LARCH model studied in Giraitis *et al.* (2000b, 2004) and elsewhere. Also observe that $\{M_k\}$ is a martingale difference sequence which may be written as

$$(2.5)\qquad M_k = \zeta_k v_k, \qquad v_k = \mu\sigma + \sum_{j=1}^{\infty} h_j M_{k-j},$$

where v_k may be interpreted as volatility. A stationary solution $\{M_k\}$ of equation (2.5) is constructed in terms of causal Volterra series in i.i.d. innovations $\zeta_s, s \le k$:

$$(2.6)\qquad M_k = \mu\sigma\zeta_k\Big(1 + \sum_{m=1}^{\infty} \sum_{s_m < \cdots < s_1 < k} h_{k-s_1} h_{s_1-s_2} \cdots h_{s_{m-1}-s_m} \zeta_{s_1} \cdots \zeta_{s_m}\Big),$$

see Giraitis *et al.* (2000b, 2004). The series in (2.6) converges in L_2 if and only if

$$(2.7)\qquad \sigma_M^2 = EM_k^2 = (\mu\sigma)^2\Big(1 + \sum_{m=1}^{\infty} \sum_{s_m < \cdots < s_1 < k} h_{k-s_1}^2 h_{s_1-s_2}^2 \cdots h_{s_{m-1}-s_m}^2\Big)$$
$$= (\mu\sigma^2)\big(1 + \sum_{m=1}^{\infty} \|h\|^{2m}\big) = (\mu\sigma^2)(1 - \|h\|^2)^{-1} < \infty,$$

or $\|h\|^2 = \sum_{j=1}^{\infty} h_j^2 < 1$, which is equivalent to $\|g\|^2 = \sum_{j=0}^{\infty} g_j^2 < (1+\sigma^2)/\sigma^2$. After solving equation (2.4)(b), equation (2.4)(a) in the integrated case $\theta = \sum_{j=1}^{\infty} b_j = 1$ represents a particular case of the IAR(∞) model with causal uncorrelated noise $\{M_k\}$ discussed in Theorem 3.1 below. Accordingly, the stationary solution of the bilinear equation (2.2) and hence the ARCH equation (1.1) can be obtained by inverting (2.4) (a), viz.,

$$(2.8)\qquad Y_k = (1 - B(L))^{-1} M_k = \sum_{j=0}^{\infty} g_j M_{k-j}$$
$$= \mu\sigma\Big(\textstyle\sum_{m=1}^{\infty} \sum_{-\infty < s_m < \cdots < s_1 \le k} g_{k-s_1} h_{s_1-s_2} \ldots \cdots h_{s_{m-1}-s_m} \zeta_{s_1} \cdots \zeta_{s_m}\Big),$$

as a solution of the AR(∞) equation with martingale difference innovations M_{k-j} determined by equation (2.4) (b), or (2.5), see Proposition 5.1(iii).

In what follows, the term 'causal' indicates a stationary process $\{y_k\}$ written as a measurable function of the present and past values $\zeta_s, s \le k$ or, equivalently, $\varepsilon_s, s \le k$. By an L_2-solution of equations (1.1), (2.2), (2.4) we mean a random process with finite second moment such that all series in these equations converge in mean square and the corresponding equations hold for each $k \in \mathbb{Z}$.

The main result of this paper is the following theorem which establishes sufficient and necessary conditions for the existence of a causal L_2-solution $\{\tau_k\}$ of the ARCH(∞) equation (1.1) and $\{(Y_k, M_k)\}$ of bilinear equations in (2.2), (2.4). Denote the transfer function

$$A(x) = (1 - B(\mathrm{e}^{\mathrm{i}x}))^{-1}, \quad B(\mathrm{e}^{\mathrm{i}x}) = \textstyle\sum_{j=1}^{\infty} b_j \mathrm{e}^{\mathrm{i}jx}, \qquad x \in \Pi, \tag{2.9}$$

and set $\|g\|^2 = \sum_{j=0}^{\infty} g_j^2$ and $\|A\|^2 = \int_{\Pi} |A(x)|^2 \mathrm{d}x$.

Theorem 2.1 (a) *ARCH equation (1.1) has a nontrivial causal L_2-solution $\{\tau_k\}$ if and only if*

$$\|g\|^2 < (1 + \sigma^2)/\sigma^2. \tag{2.10}$$

Condition (2.10) is equivalent to

$$\|A\|^2 < 2\pi(1 + \sigma^2)/\sigma^2. \tag{2.11}$$

(b) *Let* (2.10) *or* (2.11) *be satisfied, and let Y_k be defined as in* (2.8), (2.6).

(i) *If $\omega > 0$, $0 < \theta < 1$ then ARCH equation (1.1) has a unique causal L_2-solution $\{\tau_k = \mu + Y_k\}$, where $\mu = \omega/(1-\theta) = E\tau_k$.*

(ii) *If $\omega = 0$, $\theta = 1$ then for each $\mu > 0$, $\{\tau_k = \mu + Y_k\}$ is a unique causal L_2-solution of (1.1) with mean $E\tau_k = \mu$.*

Theorem 2.1 is new in the integrated case $\theta = 1$ only, since for $\theta < 1$ it follows from Giraitis and Surgailis (2002). Case $\theta < 1$ is included above for comparison. While for $\theta < 1$ the solution is unique, for $\theta = 1$ the IARCH equation (1.1) has an infinite number of causal L_2-solutions parametrized by $E\tau_k = \mu$. Since the g_j are expressed through the b_j via multiple infinite series, see (3.2), direct verification of condition (2.10) may be hard. On the other hand, condition (2.11) in some cases can be verified rather easily if the transfer function $A(x)$ is explicitly known, as in the case of the FIGARCH model.

The following corollary establishes the long memory property of the stationary IARCH model.

Corollary 2.1 *The IARCH equation (1.7) has a non-trivial stationary causal L_2-solution if and only if $\sigma^2 = \mathrm{var}(\varepsilon_1)$ and b_j satisfy condition (2.11). In the latter case,*

(i) *For each* $\mu > 0$, *the process* $\{\tau_k = \mu + Y_k\}$ *with* Y_k *defined in* (2.8), (2.6) *is a unique causal* L_2*-solution of (1.7) with mean* $E\tau_k = \mu$.

(ii) *The covariance function of the solution* $\{\tau_k = \mu + Y_k\}$ *is given by*

$$\mathrm{cov}(\tau_0, \tau_k) = \sigma_M^2 \sum_{j=0}^{\infty} g_j g_{k+j}. \tag{2.12}$$

(iii) *The covariance function in* (2.12) *is nonnegative,* $\mathrm{cov}(\tau_0, \tau_k) \ge 0$, *and nonsummable:* $\sum_{k\in\mathbb{Z}} \mathrm{cov}(\tau_0, \tau_k) = \infty$. *Moreover,* $\{\tau_k\}$ *has spectral density* $f(x) = \mu^2(\sigma_M^2/2\pi)|1-B(\mathrm{e}^{-\mathrm{i}x})|^{-2}$, $x \in \Pi$ *that is unbounded at the zero frequency:* $f(x) \to \infty$ *as* $x \to 0$.

The above corollary together with Lemma 6.1 (iii) imply that the IARCH model in (1.7) with $\omega = 0$ does not have a stationary solution with finite variance if the b_j tend to zero fast enough, e.g. exponentially or decay at rate $b_j = O(j^{-\gamma})$ for some $\gamma \ge 3/2$. In contrast, the sufficient conditions for the existence of a stationary IARCH process with non-zero intercept $\omega > 0$ and infinite mean $E\tau_k = \infty$ obtained in Kazakevičius and Leipus (2003) require an exponential decay of b_j as $j \to \infty$.

The next corollary details the case of the FIGARCH equation in (1.3) with zero intercept $\omega = 0$. It establishes existence of stationary long memory FIGARCH processes $\{\tau_k\}$ and shows that their covariance function $\mathrm{cov}(\tau_k, \tau_0)$ decays to zero hyperbolically slowly as in (2.13).

Corollary 2.2 *For the FIGARCH model in (1.3) with* $\omega = 0$ *and* $d \in (0, 1/2)$, *condition (2.10) is equivalent to* (1.6). *Under this condition, the statements of Corollary 2.1 hold. Moreover, as* $k \to \infty$, *the covariance and spectral density of the FIGARCH process* $\{\tau_k\}$ *with* $E\tau_k = \mu$ *satisfy*

$$\mathrm{cov}(\tau_0, \tau_k) \sim \mu^2 c_\gamma k^{-1+2d}, \tag{2.13}$$

where $c_\gamma = \sigma_M^2 \Gamma(1-2d)/\{\Gamma(d)\Gamma(1-d)\}$, $\sigma_M^2 = \sigma^2/\big(1 + \sigma^2 - \sigma^2(\Gamma(1-2d)/\Gamma^2(1-d))\big)$ *and*

$$f(x) = (\sigma_M^2/2\pi)|1 - \mathrm{e}^{\mathrm{i}x}|^{-2d} \sim (\sigma_M^2/2\pi)|x|^{-2d}, \quad x \to 0. \tag{2.14}$$

For comparison, Corollary 2.3 below recovers the results on the existence of a stationary finite variance solution of ARCH(∞) equation with $\theta = \sum_{j=1}^{\infty} b_j < 1$, obtained in Giraitis and Surgailis (2002). As noted above, the existence of such solution in this case necessarily implies $E\tau_k = \mu = \omega/(1-\theta)$. In sharp contrast to a finite variance stationary IARCH process, which can have only long memory, see Corollary 2.1, stationary finite variance process ARCH process with $\theta < 1$ always has short memory.

Corollary 2.3 *ARCH(∞) equation in (1.1) with* $\omega > 0$ *and* $\theta = \sum_{j=1}^{\infty} b_j < 1$ *has a unique stationary causal* L_2*-solution* τ_k *if and only if condition (2.10) is satisfied. The above solution* *is given by* $\{\tau_k = \mu + Y_k\}$, *with* $\mu = \omega/(1-\theta)$ *and* Y_k *defined in* (2.8), (2.6). *It has mean* $E\tau_k = \mu = \omega/(1-\theta)$ *and non-negative covariance function given in (2.12). Moreover,*

$$\sum_{k=0}^{\infty} \mathrm{cov}(\tau_0, \tau_k) < \infty, \quad \sum_{k=0}^{\infty} g_k < \infty.$$

The following Corollary 2.4 discusses the weak convergence in the Skorohod space $D[0,1]$, denoted by $\to_{D[0,1]}$, of the partial sums process of $\{\tau_k\}$. Part (i) of this corollary is known, see Giraitis *et al.* (2007, 2000a). Below, $\{B(t), t \in [0,1]\}$ denotes standard Brownian motion with variance $EB^2(t) = t$ and $\{B_{d+1/2}(t), t \in [0,1]\}$ a fractional Brownian motion with variance $EB^2_{d+1/2}(t) = t^{2d+1}$, $d \in (0, 1/2)$.

Corollary 2.4 *Suppose that (2.10) holds.*
(i) *Let* $\omega > 0$, $\theta < 1$ *and* $\{\tau_k\}$ *be the ARCH(∞) process as in Corollary 2.3. Then*

$$n^{-1/2} \sum_{k=1}^{[nt]} (\tau_k - E\tau_k) \to_{D[0,1]} s^2 B(t), \qquad s^2 = \sum_{k\in\mathbb{Z}} \mathrm{cov}(\tau_0, \tau_k).$$

(ii) *Let* $\{\tau_k\}$ *be the FIGARCH process as in Corollary 2.2. Then*

$$n^{-1/2-d} \sum_{k=1}^{[nt]} (\tau_k - E\tau_k) \to_{D[0,1]} s_d B_{d+1/2}(t), \qquad s_d^2 = \mu^2 c_\gamma / (d(1+2d)).$$

We are able to give a final answer to the conjecture (1.5) of Ding and Granger (1996), which assumes the existence of a stationary solution $\{r_t\}$ of the LM(d)-ARCH model in (1.4) with $Er_t^4 < \infty$, for arbitrary parameters $\theta \in (0,1], 0 < d < 1/2$, and $\mu > 0$. Although this conjecture is proved for $\theta = 1$ only, the fact that it is invalid for all $0 < \theta < 1$ is also new since previously the failure of (1.5) was shown for $\theta < 1/\sqrt{Ez_0^4} < 1$ alone, see Giraitis *et al.* (2000a, pp. 15).

Corollary 2.5 *The conjecture* (1.5) *of Ding and Granger (1996) about the LM(d)-ARCH model in* (1.4) *is true if and only if* $\theta = 1$ *and* $Ez_0^4 = E\varepsilon_0^2$ *satisfies condition* (1.6).

3 Stationary Integrated AR(∞) process: origins of long memory

As explained in the Introduction, our construction of a stationary solution of the IARCH model relies on solving the IAR equation (1.10) with martingale difference innovations $\{M_k\}$. Particularly, we want to know which conditions on the filter b_j guarantee that the IAR equation has a stationary solution and when this solution has covariance long memory, in the sense that its covariance function is nonsummable?

It turns out that the two questions are closely related, in the sense that the existence of a stationary solution of the IAR equation implies the long memory property of its solution. This question is of independent interest apart from ARCH models since it indicates a general mechanism for generating a long memory process different from fractional differencing or explicit ARFIMA (p,d,q) modeling commonly used in the time series literature, see e.g. Brockwell and Davis (1991), Giraitis, Koul, and Surgailis (2012). Being a technical tool for generating parametric long memory time series, fractional filtering/differencing cannot fully explain the phenomenon and the mechanism by which long memory is induced, which has sometimes led to controversies justifying the use of long memory processes and explaining their generating process. See Lieberman and Phillips (2008) for an illustrative analysis of how long memory may arise in realized volatility.

In this section we discuss stationary solution of Integrated AR(∞) equation:

$$x_k - \sum_{j=1}^{\infty} b_j x_{k-j} = \xi_k, \qquad k \in \mathbb{Z}, \tag{3.1}$$

where b_j are non-negative, $\sum_{j=1}^{\infty} b_j = 1$, and $\{\xi_k\}$ is a white noise (a stationary sequence of uncorrelated random variables with zero mean and finite variance $\sigma_\xi^2 = E\xi_1^2 < \infty$). Everywhere in this section by stationarity we mean weak sense or covariance stationarity since no other properties of random variables with exception of the two finite first moments will be used.

Definition 3.1 *We say that a random process $\{x_k\}$ is a L_2-solution of (3.1) if $Ex_k^2 < \infty$ for each $k \in \mathbb{Z}$, the series $\sum_{j=1}^{\infty} b_j x_{k-j}$ converges in mean square and (3.1) holds.*

The above definition is very general and does not assume causality or even ergodicity of $\{x_k\}$ since any constant r.v. $x \equiv x_k$, $Ex^2 < \infty$ is a L_2-solution of the homogeneous equation $x_k - \sum_{j=1}^{\infty} b_j x_{k-j} = 0$. As for the IARCH equation (1.7), a (stationary) L_2-solution $\{x_k\}$ of (3.1), if it exists, is not unique: for any real μ, $\{x_k + \mu\}$ is also a L_2-solution of (3.1).

A causal solution of (3.1) can be constructed by inverting the filter $1 - B(z)$ with the inverse filter coefficients g_j, $j \geq 0$ defined as in (2.3) by using the power expansion of the analytic function $G(z) = (1 - B(z))^{-1} = \sum_{j=0}^{\infty} g_j z^j$ on the unit disc $\{|z| < 1\}$. The resulting coefficients are nonnegative and given by

$$g_j = \sum_{m=1}^{j} \sum_{0 < s_{m-1} < \cdots < s_1 < j} b_{j-s_1} b_{s_1 - s_2} \cdots b_{s_{m-2} - s_{m-1}} b_{s_{m-1}}, \quad j \geq 1, \qquad g_0 = 1, \tag{3.2}$$

which follows from equality $(1 - B(z))^{-1} = \sum_{m=0}^{\infty} B^m(z)$. Assuming that $\|g\| = (\sum_{j=0}^{\infty} g_j^2)^{1/2} < \infty$, we can define a stationary L_2-solution of (3.1) as

$$\widetilde{x}_k = \sum_{j=0}^{\infty} g_j \xi_{k-j}, \qquad k \in \mathbb{Z}. \tag{3.3}$$

As shown below in Lemma 6.2, if the transfer function $A(x) = (1 - B(\mathrm{e}^{\mathrm{i}x}))^{-1}$ is L_2-integrable: $\|A\| = (\int_\Pi |A(x)|^2 \mathrm{d}x)^{1/2} < \infty$, the Fourier coefficients of $A(x)$ agree with g_j in (3.2):

$$g_j \;=\; (2\pi)^{-1} \int_\Pi A(x) \mathrm{e}^{-\mathrm{i}xj} \mathrm{d}x \quad \text{and} \quad A(x) = \sum_{j=0}^\infty g_j \mathrm{e}^{\mathrm{i}xj}. \tag{3.4}$$

Notice that equalities (3.4) are not obvious since the g_js are defined by the power expansion of $G(z)$ in the open disc $|z| < 1$ while the definition of $A(x)$ requires only $B(\mathrm{e}^{\mathrm{i}x}) \neq 1$ a.e.

The next theorem establishes the equivalence of conditions $\|g\| < \infty$ and $\|A\| < \infty$ and the representations (3.2) and (3.4). It also obtains conditions for the existence and uniqueness of a stationary L_2-solution of (3.1) and its long memory property.

Theorem 3.1 (i) *Assumption $\|g\| < \infty$ is necessary and sufficient for the existence of a stationary L_2-solution $\{x_k\}$ of (3.1).*

(ii) *If $\|g\| < \infty$ then with $\tilde{x}_k$ as in (3.3) for each real μ,*

$$x_k = \mu + \tilde{x}_k, \quad k \in \mathbb{Z} \tag{3.5}$$

is a stationary L_2-solution of (3.1) with $Ex_k = \mu$. The above solution is unique in the class of all stationary linear processes $x_k = \mu + \sum_{j\in\mathbb{Z}} c_j \xi_{k-j}$ with $\sum_{j\in\mathbb{Z}} c_j^2 < \infty$.

(iii) *The solution x_k in (3.5) has a non-negative and non-summable covariance function:*

$$\mathrm{cov}(x_0, x_k) = \sigma_\xi^2 \sum_{j=0}^\infty g_j g_{k+j} \geq 0, \qquad \sum_{k\in\mathbb{Z}} \mathrm{cov}(x_0, x_k) = \infty \tag{3.6}$$

and unbounded spectral density $f(x) = \frac{\sigma_\xi^2}{2\pi} |1 - B(\mathrm{e}^{\mathrm{i}x})|^{-2}$ with $\lim_{x\to 0} f(x) = \infty$.

(iv) $\|g\| < \infty$ *implies* $\|A\| < \infty$ *and* (3.4). *Conversely,* $\|A\| < \infty$ *implies* $\|g\| < \infty$.

A surprising consequence of Theorem 3.1 is the fact that a stationary solution (3.3) of (3.1) does not exist if the b_js vanish for j large enough. The validity of this conclusion is not obvious from the representation of g_j in (3.2) but follows easily from (3.4). Indeed, since $|A(x)|^{-1} = |1 - B(\mathrm{e}^{\mathrm{i}x})| = |\sum_{j=0}^\infty b_j(1 - \mathrm{e}^{\mathrm{i}jx})| \leq |x| \sum_{j=1}^\infty j|b_j| \leq C|x|$ this implies $\int_\Pi |A(x)|^2 \mathrm{d}x \geq C^{-2} \int_\Pi x^{-2} \mathrm{d}x = \infty$ and $\|g\| = \infty$ according to (3.4). The above argument combined with Lemma 6.1 (iii) is formalized in the following corollary.

Corollary 3.1 The IAR(∞) equation in (3.1) does not have a stationary L_2-solution if the b_js decay as $j^{-3/2}$ or faster. In particular, the latter holds if $b_j = 0, j > j_0$ for some $j_0 \geq 1$, or $b_j = O(\mathrm{e}^{-cj})$ for $j \geq 1, c > 0$.

Remark 3.1 The exponent 3/2 in the critical decay rate condition $b_j = O(j^{-3/2})$ in the above corollary which does not permit a stationary solution cannot be replaced by a smaller one. Indeed, the ARFIMA$(0, d, 0)$ equation with $b_j \sim j^{-d-1}/\Gamma(-d)$, $d + 1 < 3/2$ does have a stationary L_2-solution, see Example 3.1 below.

The requirement of Theorem 3.1 that the r.h.s. $\{\xi_k\}$ in the IAR equation (3.1) is a white noise is restrictive and can be relaxed. Theorem 3.2 extends Theorem 3.1 to the case when $\{\xi_k\}$ is a short memory process as described below.

Theorem 3.2 *Let $\{\xi_k\}$ be a stationary process with zero mean, finite variance and a spectral density f_ξ which is bounded away from 0 and ∞:*

$$(3.7)\qquad c_1 \le f_\xi(x) \le c_2, \quad \forall\, x \in \Pi, \qquad \exists\, 0 < c_1 < c_2 < \infty.$$

Then statements (i) and (ii) of Theorem 3.1 remain valid while statement (iii) has to be modified as follows:
(iii') The solution $\{x_k\}$ in (3.5) has unbounded spectral density $f(x) = |1 - B(\mathrm{e}^{\mathrm{i}x})|^{-2} f_\xi(x)$, $\lim_{x\to 0} f(x) = \infty$ and a non-summable autocovariance: $\sum_{k\in\mathbb{Z}} |\mathrm{cov}(x_0, x_k)| = \infty$.

Apparently, the class of stationary IAR(∞) processes with long memory satisfying the conditions of Theorems 3.1 or 3.2 is quite large. Since condition $\theta = \sum_{j=1}^{\infty} b_j = 1$ does not assume any particular form of b_j, it seems that the spectral density of an IAR(∞) process need not grow regularly as a power function $|x|^{-\alpha}$, $0 < \alpha < 1$ at $x = 0$ and, similarly, the covariance function need not decay regularly with the lag as $k^{-1+\alpha}$. The latter properties are key features of fractionally integrated ARFIMA models, see e.g. Hosking (1981), also (Giraitis *et al.* (2012), Chapter 7).

Example 3.1 The ARFIMA$(0, d, 0)$ model is defined as a stationary solution of the equation $(1 - L)^d x_k = \xi_k$, $0 < d < 1/2$, where $\{\xi_k\}$ is an uncorrelated white noise with $E\xi_k = 0$, $E\xi_k^2 = \sigma_\xi^2$. It can be written as the IAR(∞) equation in (3.1) with b_j generated by $B(z) = 1 - (1 - z)^d = \sum_{j=1}^{\infty} b_j z^j$. The transfer function $A(x) = (1 - B(\mathrm{e}^{-\mathrm{i}x}))^{-1}$ satisfies $|A(x)| = |1 - \mathrm{e}^{-\mathrm{i}x}|^{-d} \sim |x|^{-d}$ as $x \to 0$ and is square integrable. The coefficients b_j and g_j of the generating functions $B(z)$ and $G(z) = (1 - B(z))^{-1} = (1 - z)^{-d}$ are given by

$$(3.8)\qquad b_j = -\frac{\Gamma(j - d)}{\Gamma(j + 1)\Gamma(-d)}, \quad g_j = \frac{\Gamma(j + d)}{\Gamma(j + 1)\Gamma(d)}, \quad j \ge 1, \quad g_0 = 1.$$

They have properties $b_j > 0$, $g_j > 0$, $\theta = \sum_{j=1}^{\infty} b_j = 1$ and

$$(3.9)\qquad b_j \sim -j^{-d-1}/\Gamma(-d), \quad g_j \sim j^{d-1}/\Gamma(d), \qquad j \to \infty$$

so that $\|g\| < \infty$. Relations (3.9) imply that the covariance $\gamma_k = \mathrm{cov}(x_0, x_k) = \sigma_\xi^2 \sum_{j=0}^{\infty} g_j g_{k+j}$ decays hyperbolically, viz.

$$(3.10)\qquad \gamma_k \sim c_\gamma k^{-1+2d}, \qquad c_\gamma = \frac{\sigma_\xi^2 \Gamma(1 - 2d)}{\Gamma(d)\Gamma(1 - d)}.$$

Example 3.2 A nonparametric (depending on infinite number of parameters) class of IAR processes $x_k = \sum_{j=1}^{\infty} b_j x_{k-j} + \xi_k$ generalizing the previous example is defined by equation (3.1) with uncorrelated noise $\{\xi_k\}$ and coefficients b_j generated by the operator

$$B(L) \;=\; \big(1-(1-L)^d\big)P(L) = \textstyle\sum_{j=1}^{\infty} b_j L^j, \quad 0 < d < 1/2. \tag{3.11}$$

Here, $P(z) = \sum_{j=0}^{\infty} p_j z^j$ is a generating function with coefficients satisfying

$$p_j \ge 0, \quad p_1 > 0, \quad \textstyle\sum_{j=0}^{\infty} p_j = 1 \quad \text{and} \quad \sum_{j=1}^{\infty} j p_j < \infty. \tag{3.12}$$

Then $b_j = \sum_{k=0}^{j-1} p_k b_{j-k}^0$ where b_j^0 are the coefficients of the expansion $1-(1-z)^d = \sum_{j=1}^{\infty} b_j^0 z^j$, see (3.8). Hence, the b_j in (3.11) are non-negative and sum up to 1.

Let us show that $|A(x)|^2 = |1 - B(\mathrm{e}^{\mathrm{i}x})|^{-2}$ is integrable. Since $b_1 = p_0 b_1^0 > 0$, by Lemma 6.1(ii) $|A(x)|$ is bounded on $[\epsilon, \pi]$ for any $\epsilon > 0$. Therefore, it suffices to show that $|A(x)|^2$ is integrable at $x = 0$. Rewrite $1 - B(\mathrm{e}^{\mathrm{i}x}) = 1 - \big(1-(1-\mathrm{e}^{\mathrm{i}x})^d\big)P(\mathrm{e}^{\mathrm{i}x}) = (1-\mathrm{e}^{\mathrm{i}x})^d h(x)$, where

$$h(x) = P(\mathrm{e}^{\mathrm{i}x}) - \big(P(\mathrm{e}^{\mathrm{i}x}) - 1\big)(1-\mathrm{e}^{\mathrm{i}x})^{-d}. \tag{3.13}$$

From (3.12) we have $|P(\mathrm{e}^{\mathrm{i}x}) - 1| = \sum_{j=1}^{\infty} |\mathrm{e}^{\mathrm{i}jx} - 1| p_j \le |x| \sum_{j=1}^{\infty} j p_j = O(|x|) = o(|(1-\mathrm{e}^{\mathrm{i}x})^d|)$ and therefore $\lim_{x\to 0} h(x) = h(0) = P(1) = 1$. Hence, $|A(x)|^2 \sim |x|^{-2d}$, $x \to 0$, proving the integrability of $|A(x)|^2$ for $d \in (0, 1/2)$. The corresponding stationary solution $\{x_k\}$ of (3.1) with uncorrelated noise $\{\xi_k\}$ has the spectral density

$$f(x) = (\sigma_\xi^2/2\pi)|1 - B(\mathrm{e}^{-\mathrm{i}x})|^{-2} = (\sigma_\xi^2/2\pi)|1-\mathrm{e}^{-\mathrm{i}x}|^{-2d}|h(x)|^{-2}, \quad x \in \Pi \tag{3.14}$$

with h defined at (3.13). It satisfies $f(x) \sim (\sigma_\xi^2/2\pi)|x|^{-2d}, x \to 0$, and is a continuous bounded function on intervals $[\epsilon, \pi], \epsilon > 0$. Moreover, using (3.14), (3.13), (3.12) and Lemma 2.3.1 of Giraitis *et al.* (2012), one can show that the asymptotics of the covariance function $\mathrm{cov}(x_0, x_k) \sim c_\gamma k^{-1+2d}, k \to \infty$ with c_γ given in (3.10) is the same as for ARFIMA$(0, d, 0)$ model. Hence, the p_j or $P(L)$ in (3.11) essentially affects the short memory dynamics and do not affect the long-run behavior of the corresponding IAR process.

Example 3.3 (The IAR(q, d) model). We introduce the parametric class IAR(q, d) consisting of IAR(∞) processes (3.1) with $B(L)$ as in (3.11) and $P(L)$ a polynomial of degree q satisfying (3.12). It is convenient to parameterize such polynomials as

$$P(z) = \frac{1 + r_1 z + \cdots + r_q z^q}{1 + r_1 + \cdots + r_q}, \quad r_1 \ge 0, \ldots, r_q \ge 0. \tag{3.15}$$

Thus, $p_i = r_i/(1+\cdots+r_q)$, $1 \le i \le q$, $p_0 = 1/(1+\cdots+r_q)$ satisfy (3.12) so that IAR(q, d) is a particular case and shares the same long memory properties as IAR in Example 3.2. Note that IAR$(0, d)$ model coincides with ARFIMA$(0, d, 0)$. Apart from this case, it seems that IAR(q, d) are different from ARFIMA(p, d, q) models. For example, the model $(1-B(L))x_k = \xi_k$ with $B(z) = (1-(1-z)^d)(1+rz)/(1+r)$ with $P(z) = (1+rz)/(1+r)$ generates a different covariance structure than ARFIMA$(1, d, 0)$ model $(1-L)^d(1+rL)x_k = \xi_k$.

4 Conclusion

Ding and Granger (1996) proposed the Long Memory ARCH model to capture hyperbolic decay of sample autocorrelations of speculative squared returns. The LM ARCH model is closely related to the FIGARCH model which was independently introduced in Baillie *et a.* (1996). However the existence of covariance stationary solutions of these models was not established and, thus, the possibility of long memory in the ARCH setting was doubtful. The present paper resolves this doubt and associated controversy by showing that FIGARCH and IARCH(∞) equations with zero intercept may have a non-trivial covariance stationary solution with long memory. It also obtains necessary and sufficient conditions for the existence of stationary integrated AR(∞) processes with finite variance and proves that such processes always have long memory. The paper provides a complete answer to the long standing conjecture of Ding and Granger (1996) about the existence of the Long Memory ARCH model.

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5 Appendix A: Proofs of Theorem 2.1 and Corollaries 2.1-2.4

The following proposition used to prove Theorem 2.1 establishes the relation between solutions τ_k of (1.1), and (Y_k, M_k) of (2.2) and (2.4) with ε_k and ζ_k related by $\varepsilon_k = \sigma\zeta_k + 1$, and $\omega = \mu(1-\theta)$. For Y_k in (2.2), we define the 'noise' as $M_k = \zeta_k(\mu\sigma + \sigma\sum_{j=1}^{\infty} b_j Y_{k-j})$. For M_k in (2.4), the volatility process v_k is defined in (2.5).

Proposition 5.1 *Let $0 < \mu < \infty$ and $\theta \in (0, 1]$.*

(i) *If $\{\tau_k\}$ is a causal L_2-solution of (1.1) then $\{Y_k = \tau_k - \mu\}$ is a causal L_2-solution of (2.2) such that $Y_k \geq -\mu$.*

(ii) *If $\{Y_k\}$ is a causal L_2-solution of (2.2) such that $Y_k \geq -\mu$, then $\{\tau_k = Y_k + \mu\}$ is a causal L_2-solution of equation (1.1).*

(iii) *$\{Y_k\}$ is a causal L_2-solution of bilinear equation (2.2) if and only if $\{Y_k, M_k\}$ is a causal L_2-solution of equation (2.4). Moreover, $\{Y_k \geq -\mu\}$ is equivalent to $\{v_k \geq 0\}$ with v_k as in (2.5).*

Proof. The equivalence of (i) and (ii) is immediate. It remains to prove (iii). Let $\{Y_k\}$ be a causal L_2-solution of (2.2). Set $M_k = \zeta_k(\mu\sigma + \sigma\sum_{j=1}^\infty b_j Y_{k-j})$ and denote $v_k = \mu\sigma + \sum_{j=1}^\infty h_j M_{k-j}$. Let us prove that $\{Y_k, M_k\}$ is a causal L_2-solution of (2.4). This follows from (2.2) and equality

$$v_k = \mu\sigma + \sigma\sum_{j=1}^\infty b_j Y_{k-j} \tag{5.1}$$

which is verified below. From the definition of M_k and (2.2) it follows that the Y_k satisfy the IAR equation $Y_k - \sum_{j=1}^\infty b_j Y_{t-j} = M_k$ where $\{M_k\}$ is a causal uncorrelated process with finite variance. Therefore by Theorem 3.1 we have $Y_k = \sum_{j=0}^\infty g_j M_{k-j}$ which implies that $\sigma\sum_{j=1}^\infty b_j Y_{k-j} = \sigma\sum_{j=1}^\infty b_j \sum_{i=0}^\infty g_i M_{k-j-i} = \sum_{j=1}^\infty h_j M_{k-j}$ in view of the definition of h_j in (2.3), proving (5.1) and the fact that $\{Y_k, M_k\}$ is a causal L_2-solution of (2.4). Moreover, $Y_{k-j} \geq -\mu$ and (5.1) imply $v_k \geq \mu\sigma + \sigma(\sum_{j=1}^\infty b_j)(-\mu) = \mu\sigma(1-\theta) \geq 0$.

Conversely, assume that $\{Y_k, M_k\}$ is a causal L_2-solution of (2.4). Then causality of L_2-solution of (2.2) follows from (5.1) which in turn follows from Theorem 3.1 using exactly the same argument as above. Finally, from $v_k \geq 0$, (5.1), (2.2) and $\zeta_k \geq -1/\sigma$ we obtain

$$\begin{aligned} Y_k &= \sum_{j=0}^\infty b_j Y_{k-j} + \zeta_k v_k \geq \sum_{j=0}^\infty b_j Y_{k-j} - (1/\sigma)v_k \\ &= \sum_{j=0}^\infty b_j Y_{k-j} - (1/\sigma)(\mu\sigma + \sigma\sum_{j=0}^\infty b_j Y_{k-j}) = -\mu, \end{aligned}$$

proving part (iii) and the proposition. □

Proof of Theorem 2.1. (a) The equivalence of (2.10) and (2.11) follows from the equivalence of $\|g\| < \infty$ and $\|A\| < \infty$, see Lemma 6.2, and Parseval's identity $\|g\| = 2\pi\|A\|$. Let us prove the necessity of condition (2.10), or $\|h\| < 1$, for the existence of a stationary solution. Assume that $\{\tau_k\}$ is an L_2-solution of ARCH equation (1.1). Then by Proposition 5.1 (i), the last fact implies that for $\mu > 0$, $\{Y_k = \tau_k - \mu, M_k = \zeta_k(\mu\sigma + \sigma\sum_{j=1}^\infty b_j Y_{k-j})\}$ is an L_2-solution of the bilinear equation (2.4). Consequently, $\sigma_M^2 = EM_k^2 = E\big(\mu\sigma + \sum_{j=1}^\infty h_j M_{k-j}\big)^2 = (\mu\sigma)^2 + (\sum_{j=1}^\infty h_j^2)\sigma_M^2 = (\mu\sigma)^2 + \|h\|^2\sigma_M^2$, yielding $\|h\|^2 < 1$, or (2.10), since $\|h\|^2 = \sigma^2(\|g\|^2 - 1)$.

Conversely, let us show that $\|h\| < 1$ implies the existence of L_2-solution $\{\tau_k\}$ of (1.1) with $E\tau_k = \mu$ given by $\tau_k = Y_k + \mu$ and Y_k defined in (2.8), (2.6). As shown in (2.7), $\|h\| < 1$

guarantees that $\{Y_k\}$ is an L_2-solution of (2.2). Therefore by Proposition 5.1 (ii), it suffices to prove that

$$Y_k \geq -\mu. \tag{5.2}$$

To show (5.2), we approximate Y_k by

$$Y_{k,p} = (\mu\sigma)\sum_{m=1}^{\infty}\Big(\sum_{p<s_m<\cdots<s_1\le k} g_{k-s_1}h_{s_1-s_2}\cdots h_{s_{m-1}-s_m}\zeta_{s_1}\cdots\zeta_{s_m}\Big),$$

where $-p \geq 1$ is a large integer. Observe that for $k > p$ the $Y_{k,p}$ satisfy equation (2.2), viz.,

$$Y_{k,p} = \zeta_k\Big(\mu\sigma + \sigma\sum_{j=1}^{\infty} b_j Y_{k-j,p}\Big) + \sum_{j=1}^{\infty} b_j Y_{k-j,p}, \qquad \text{for } k > p, \tag{5.3}$$

while $Y_{k,p} = 0$ for $k \leq p$. Moreover, by orthogonality of Volterra series, $E(Y_k - Y_{k,p})^2 = (\mu\sigma)^2 \sum_{m=1}^{\infty} J_{k,p}^{(m)}$, where

$$J_{k,p}^{(m)} = \sum_{s_m<\cdots<s_1\le k,\, s_m\le p} g^2_{k-s_1}h^2_{s_1-s_2}\cdots h^2_{s_{m-1}-s_m}.$$

Notice that $J_{k,p}^{(m)} \leq \|g\|^2\|h\|^{2(m-1)}$, where $\|h\| < 1$. Hence $\sum_{m=1}^{\infty} J_{k,p}^{(m)}$ is dominated by a converging series. Moreover, for each $m \geq 1$, $J_{k,p}^{(m)} \to 0$ as $p \to -\infty$. Hence, $\lim_{p\to-\infty} E(Y_k - Y_{k,p})^2 = 0$ for any $k \in \mathbb{Z}$ by the dominated convergence theorem. Therefore, (5.2) follows if we show that for any $p \in \mathbb{Z}$,

$$Y_{k,p} \geq -\mu, \qquad k \in \mathbb{Z}. \tag{5.4}$$

To prove (5.4), we use induction on k. Clearly, (5.4) holds for $k \leq p$ because by definition $Y_{k,p} = 0 > -\mu$ for $k \leq p$. Also, (5.4) holds for $k = p+1$ since $Y_{p+1,p} = (\mu\sigma)\zeta_{p+1} \geq -\mu$ because $(\mu\sigma)\zeta_j = (\mu\sigma)(\varepsilon_j - 1)/\sigma \geq -\mu$, for $j \in \mathbb{Z}$. Let $k > p+1$. Assume by induction that $Y_{s,p} \geq -\mu$ for all $s < k$. Then, by (5.3) and the inductive assumption,

$$Y_{k,p} = \zeta_k(\mu\sigma) + (\zeta_k\sigma + 1)\Big(\sum_{j=1}^{\infty} b_j Y_{k-j,p}\Big) \geq \zeta_k(\mu\sigma) + (\zeta_k\sigma+1)\Big(\sum_{j=1}^{\infty} b_j\Big)(-\mu) = -\mu.$$

This proves the induction step $k-1 \to k$ and (5.4), (5.2), too, thereby proving part (a) of the theorem.

(b) Claim (i) is shown in Giraitis and Surgailis (2002), Thm. 3.1. Let us prove (ii). By part (a), it suffices to prove the uniqueness of the solution $\{\tau_k\}$. Let $\{\tau'_k\}, \{\tau''_k\}$ be two causal L_2-solutions of (1.1) with $E\tau'_k = E\tau''_k$. Then $\tau'_k - \tau''_k = Y_k$ is a causal L_2-solution of $Y_k = \sum_{j=1}^{\infty} b_j Y_{k-j} + M_k$, where $M_k = \zeta_k\sigma\sum_{j=1}^{\infty} b_j Y_{k-j}$. By causality, the stationary process $Y_k = f(\varepsilon_k, \varepsilon_{k-1}, \dots)$ is a function of lagged i.i.d. variables. Hence, $\{Y_k\}$ is a regular process with $EY_k^2 < \infty$, having a spectral density, see Ibragimov and Linnik (1971), Thm. 17.1.2. Moreover, $M_k = \zeta_k \sum_{j=1}^\infty h_j M_{k-j}$, see (2.4) (b), where $\{M_k\}$ is a covariance stationary white noise and $\sum_{j=1}^\infty h_j^2 = \|h\|^2 < 1$, $E\zeta_k^2 = 1$. Then $EM_k^2 = \sum_{j=1}^\infty h_j^2 EM_{k-j}^2 = \|h\|^2 EM_k^2$ implies $EM_k^2 = 0$ and hence $M_k = 0$. Therefore $\{Y_k\}$ has spectral density and is a stationary solution of the homogeneous equation $Y_k - \sum_{j=1}^\infty b_j Y_{k-j} = 0$. As shown in the proof of Theorem 3.1 (ii) below, such an equation has a unique solution $Y_k = 0$, proving the uniqueness of $\{\tau_k\}$. Theorem 2.1 is proved. □

Proof of Corollary 2.1. All claims with exception of (iii) follow from Theorem 2.1, and the claim (iii) follows from Theorem 3.1(iii). □

Proof of Corollary 2.2. Note $\sigma^2 = E\varepsilon_0^2 - 1$. We have $\|A\|^2 = \int_\Pi |1 - \mathrm{e}^{\mathrm{i}x}|^{-2d} \mathrm{d}x = 2\pi\Gamma(1-2d)/\Gamma^2(1-d)$ yielding the equivalence of (2.11) and (1.6). The remaining claims follow from Corollary 2.1 and the fact (3.10) in Example 3.1. □

Proof of Corollary 2.3. All statements with exception of the last claim follow from Theorem 2.1. To show it, note that $g_j \geq 0$ in (3.2) satisfy $\sum_{j=0}^\infty g_j \leq \sum_{m=0}^\infty \theta^m < \infty$ since $\theta < 1$. □

Proof of Corollary 2.4. It suffices to show part (ii) only. Since by (2.8) $Y_k = \tau_k - E\tau_k = \sum_{j=0}^\infty g_j M_{j-k}$ is a moving average in stationary ergodic martingale differences $\{M_s\}$ of (2.6) with coefficients g_j given in (3.8) and satisfying (3.9), the convergence in (ii) follows from Theorem 3.1 in Abadir et al. (2014) or Theorem 6.2 in Giraitis and Surgailis (2002). □

Proof of Corollary 2.5. Let $\theta = 1$. Then the LM(d)-ARCH model in (1.4) coincides with the FIGARCH model in (1.3) with $\omega = 0$ and the statement follows from Corollary 2.2. Next, let $\theta < 1$. Then (1.4) can be written as ARCH(∞) equation in (1.1) with $\omega = \mu(1-\theta) > 0$. According to Corollary 2.3, the squared process $\{r_k^2 = \tau_k\}$ has short memory and summable autocovariance $\sum_{k=0}^\infty \mathrm{cov}(r_0^2, r_k^2) < \infty$ which contradicts (1.5). □

6 Appendix B: Proofs of Theorems 3.1 and 3.2

The proof of Theorem 3.1 uses auxiliary Lemmas 6.1 and 6.2. The proofs of these lemmas are postponed till the end of this section. Denote $J_b = \{j \geq 1 : b_j > 0\}$, and assume J_b has at least two elements. Denote by $\gcd(J_b)$ the greatest common divisor of $j \in J_b$. For example, if $b_1 > 0$ then $\gcd(J_b) = 1$, and if $b_{2j} > 0$, $b_{2j-1} = 0$, $j = 1, 2, \ldots$ then $\gcd(J_b) = 2$.

Lemma 6.1 *Let $\theta = 1$.*

(i) The function $1 - B(\mathrm{e}^{\mathrm{i}x})$, $x \in \Pi$ has only finite number of zeroes on Π, including $x = 0$.

(ii) The point $x = 0$ is the unique zero of $1 - B(\mathrm{e}^{\mathrm{i}x})$ if and only if $\gcd(J_b) = 1$.

(iii) If $b_k = O(k^{-\gamma})$, $k \to \infty$, for some $\gamma \geq 3/2$ then $\|A\| = \infty$.

Lemma 6.2 *Let $\theta \le 1$.*

(i) If $\|g\| < \infty$ then $\|A\| < \infty$ and (3.4) *hold.*

(ii) If $\|A\| < \infty$ then $\|g\| < \infty$.

Proof of Theorem 3.1. All statements in (iv) follow from Lemma 6.2.

(i) If $\|g\| < \infty$, then by (iv), $\|A\| < \infty$ and (3.4) holds. Evidently, this implies that (3.3) is a stationary solution of (3.1).

Conversely, if a stationary solution $\{x_k\}$ of (3.1) exists, it suffices to prove that $\|A\| < \infty$ which by (iv) implies $\|g\| < \infty$. Let $x_k = \int_\Pi \mathrm{e}^{\mathrm{i}ky} Z_x(\mathrm{d}y)$ be the spectral representation of $\{x_k\}$ and $F_x(\mathrm{d}y) = E|Z_x(\mathrm{d}y)|^2$ be its spectral measure. (We do not assume *a priori* that $\{x_k\}$ has a spectral density.) Denote by $\xi_k = \int_\Pi \mathrm{e}^{\mathrm{i}ky} Z_\xi(\mathrm{d}y)$ the spectral representation of the noise $\{\xi_k\}$ and by $F_\xi(\mathrm{d}y) = E|Z_\xi(\mathrm{d}y)|^2 = (\sigma_\xi^2/2\pi)\mathrm{d}y$ its spectral measure. Since the series $B(\mathrm{e}^{\mathrm{i}y}) = \sum_{j=1}^\infty b_j \mathrm{e}^{\mathrm{i}jy}$ converges uniformly in Π to a bounded function, so $x_k - \sum_{j=1}^\infty b_j x_{k-j} = \int_\Pi (1 - B(\mathrm{e}^{\mathrm{i}y})) Z_x(\mathrm{d}y) = \xi_k = \int_\Pi \mathrm{e}^{\mathrm{i}ky} Z_\xi(\mathrm{d}y)$ leading to

$$|1 - B(\mathrm{e}^{\mathrm{i}y})|^2 F_x(\mathrm{d}y) = F_\xi(\mathrm{d}y) = (\sigma_\xi^2/2\pi)\mathrm{d}y, \quad y \in \Pi. \tag{6.1}$$

By Lemma 6.1(i), $1 - B(\mathrm{e}^{-\mathrm{i}y})$ has a finite number of zeros $y_1, \dots, y_m \in \Pi$. Since F_x is non-decreasing, (6.1) implies that $F_x(\mathrm{d}y)$ coincides with $f(y)\mathrm{d}y$, $f(y) = (\sigma_\xi^2/2\pi)|A(y)|^2$ except for possible jumps at the points $y_1, \dots, y_m$, i.e. $F_x(\mathrm{d}y) = f(y)\mathrm{d}y + \sum_{i=1}^m c_i \delta_{y_i}$, where $c_i \ge 0$ are some non-negative constants. Therefore,

$$\infty > Ex_k^2 = \int_\Pi F_x(\mathrm{d}y) \ge \int_\Pi f(y)\mathrm{d}y = (\sigma_\xi^2/2\pi) \int_\Pi |A(y)|^2 \mathrm{d}y,$$

proving $\|A\| < \infty$.

(ii) Since $\{\tilde{x}_k\}$ in (3.3) is a zero mean L_2-solution of equation (3.1), see the proof of (i) above, it remains to show the uniqueness of solution $x_k = \mu + \tilde{x}_k$ of (3.1) with the stated properties. Let $\{x'_k\}, \{x''_k\}$ be two stationary L_2-solutions of (3.1) with $Ex'_k = Ex''_k$ and let $y_k = x'_k - x''_k$. Moreover, by the assumption in (ii), y_k has the form $y_k = \sum_{j\in\mathbb{Z}} c_j \xi_{k-j}$ with $\sum_{j\in\mathbb{Z}} c_j^2 < \infty$. The above facts imply that $\{y_k\}$ is a L_2-solution of the homogeneous equation $y_k - \sum_{j=1}^\infty b_j y_{k-j} = 0$ and a stationary process with absolutely continuous spectral measure $F_y(\mathrm{d}x) = f_y(x)\mathrm{d}x$, $f_y(x) = (\sigma_\xi^2/2\pi)|\sum_{j\in\mathbb{Z}} c_j \mathrm{e}^{\mathrm{i}jx}|^2$ and a spectral representation $y_k = \int_\Pi \mathrm{e}^{\mathrm{i}kx} Z_y(\mathrm{d}x)$. Since $\sum_{j=1}^\infty \mathrm{e}^{\mathrm{i}jx} b_j$ converges uniformly on Π, hence also in $L_2(F_y)$, it follows that $y_k - \sum_{j=1}^\infty b_j y_{k-j} = \int_\Pi (1 - B(\mathrm{e}^{\mathrm{i}x})) Z_y(\mathrm{d}x) = 0$ and $\int_\Pi |1 - B(\mathrm{e}^{\mathrm{i}x})|^2 F_y(\mathrm{d}x) = 0$. Together with Lemma 6.1(i) this implies that $f_y(x) = 0$ a.e. on Π and hence $F_y = 0$ and $y_k = 0$, proving part (ii).

(iii) As noted above, solution $\tilde{x}_k$ in (3.3) has spectral density $f(x) = (\sigma_\xi^2/2\pi)|1 - B(\mathrm{e}^{\mathrm{i}x})|^{-2}$. Relation $\lim_{x\to 0} f(x) = \infty$ follows from $B(1) = 1$, continuity of $B(\mathrm{e}^{-\mathrm{i}x})$, and the fact $|B(x)| < 1$ for $0 < x < x_0$ for some $x_0 > 0$ which holds by Lemma 6.1 (i). The divergence $\sum_{k\in\mathbb{Z}} |\mathrm{cov}(x_0, x_k)| = \infty$ is immediate from the previous fact. Finally, the first claim in (3.6) is a consequence of the moving average representation (3.3) and positivity of g_j. Theorem 3.1 is proved. □

Proof of Theorem 3.2. The proof follows using the same arguments as in the proof of Theorem 3.1. □

Proof of Lemma 6.1. (i) First observe that $B(\mathrm{e}^{\mathrm{i}x}) = \theta = 1$ holds for $x = 0$. Suppose that $x \in (0, 2\pi)$ is such that $1 = B(\mathrm{e}^{\mathrm{i}x})$. Then $B(\mathrm{e}^{\mathrm{i}x})$ is a real number: $B(\mathrm{e}^{\mathrm{i}x}) = \sum_{j=1}^{\infty} b_j \cos(jx)$ and then $1 = \sum_{j=1}^{\infty} b_j \cos(jx) \le \sum_{j=1}^{\infty} b_j = 1$ is possible if and only if $1 = \cos(jx) = \mathrm{e}^{\mathrm{i}jx}$ for all $j \in J_b$, or

$$x/2\pi \in \bigcap_{j\in J_b} I_j, \quad \text{where} \quad I_j = \{\frac{1}{j}, \frac{2}{j}, \dots, \frac{j-1}{j}\}. \tag{6.2}$$

Clearly, since each I_j is a finite set, the intersection in (6.2) is a finite set, too, proving (i).
(ii) Let $\gcd(J_b) = 1$. Then $\gcd(j_1, j_2) = 1$ for $j_1, j_2 \in J_b, j_1 \ne j_2$. It suffices to show that $I_{j_1} \cap I_{j_2} = \emptyset$. Indeed, assume *ad absurdum* that $I_{j_1} \cap I_{j_2} \ne \emptyset$, then $k_2 = k_1 j_2/j_1$ for some integers $1 \le k_1 < j_1$, $1 \le k_2 < j_2$ by definition of I_j in (6.2). Since j_1 and j_2 are coprimes, this means that j_1 is a divisor of k_1, or $k_1 \in \{j_1, 2j_1, \dots\}$, which contradicts $k_1 < j_1$.

Let $p = \gcd(J_b) \ge 2$. Then for any $j \in J_b$, $j = j'p$ with $1 \le j' < j$. Thus, $j/p \in \{1, 2, \dots, j-1\}$, implying $1/p \in I_j$ for all $j \in J_b$ and $1/p \in \bigcap_{j\in J_b} I_j$. Particularly, $x = 2\pi/p \ne 0$ is a zero of $1 - B(\mathrm{e}^{\mathrm{i}x})$.
(iii) It suffices to show that $b_j \le Cj^{-\gamma} \le Cj^{-3/2}$, $\sum_{j=1}^{\infty} b_j = 1$ imply $|1 - B(\mathrm{e}^{\mathrm{i}x})| \le C_1|x|^{1/2}$, $C_1 = \max(2, 16C)$ as this in turn implies $\int_\Pi |1 - B(\mathrm{e}^{\mathrm{i}x})|^{-2}\mathrm{d}x \ge C_1^{-2} \int_\Pi \mathrm{d}x/|x| = \infty$. Clearly, $|1 - B(\mathrm{e}^{\mathrm{i}x})| \le 2 \le C_1|x|^{1/2}$ for $|x| \ge 1$. On the other hand for $|x| \le 1$ using $|1-\mathrm{e}^{\mathrm{i}jx}| \le 2\min(j|x|, 1)$ we obtain $|1-B(\mathrm{e}^{\mathrm{i}x})| = |\sum_{j=1}^{\infty} b_j(1-\mathrm{e}^{\mathrm{i}jx})| \le C|x|\sum_{1\le j<1/|x|} j^{-1/2} + C\sum_{j\ge 1/|x|} j^{-3/2}$, where $\sum_{1\le j<1/|x|} j^{-1/2} \le 16|x|^{-1/2}$ and $\sum_{j\ge 1/|x|} j^{-3/2} \le 16|x|^{1/2}$. This proves (iii) and the lemma, too. □

Proof of Lemma 6.2. (i) Suppose that $\|g\| < \infty$. Set $k_r(x) = \sum_{j=0}^{\infty} g_j r^j \mathrm{e}^{\mathrm{i}jx}$, $0 < r < 1$. Then $\{k_r\}$ is a Cauchy sequence in $L_2(\Pi)$:

$$\|k_r - k_{r'}\|^2 = \int_\Pi |\sum_{j=0}^{\infty} g_j(r^j - r'^j)\mathrm{e}^{\mathrm{i}jx}|^2\mathrm{d}x = 2\pi\sum_{j=0}^{\infty} g_j^2|r^j - r'^j|^2 \to 0 \quad \text{as } r, r' \uparrow 1.$$

Moreover, $k_r(x) \to A(x) = (1 - B(\mathrm{e}^{\mathrm{i}x}))^{-1}$ a.e. in Π as $r \uparrow 1$, since $k_r(x) = G(r\mathrm{e}^{\mathrm{i}x}) = (1 - B(r\mathrm{e}^{\mathrm{i}x}))^{-1}$ for $0 < r < 1$ and $1 - B(\mathrm{e}^{\mathrm{i}x}) \ne 0$ a.e. in Π (see Lemma 6.1 (i)). Therefore, $\|k_r - A\| \to 0$ as $r \uparrow 1$ and $\|A\| < \infty$, see Rudin (1978, 3.12 Thm. 3.12). Since $k_1 \in L_2(\Pi)$ and $\|k_r - k_1\|^2 \to 0$ as $r \uparrow 1$, then $A = k_1$ in $L_2(\Pi)$ which proves (3.4).
(ii) Let $\|A\| < \infty$. Then the functions $h_k(x) = \mathrm{e}^{-\mathrm{i}kx}/(1 - B(\mathrm{e}^{\mathrm{i}x})) = \mathrm{e}^{-\mathrm{i}kx}A(x)$, $x \in \Pi$, $k \in \mathbb{Z}$ belong to the Hilbert space $L_2(\Pi)$ with the norm $\|h\| = (\int_\Pi |h(x)|^2\mathrm{d}x)^{1/2}$. So, $\|h_k\| = \|A\| < \infty$. Then $h_k(x) - \sum_{j=1}^{\infty} b_j h_{k-j}(x) = \mathrm{e}^{\mathrm{i}kx}$, where the series converges in $L_2(\Pi)$. By Lemma

6.1 (iii), $\|A\| < \infty$ implies that $b_j > 0$ for infinite number of j. For a large $p \ge 1$ denote $b'_j = b_j I(j \le p)$ and $B'(\mathrm{e}^{\mathrm{i}x}) = \sum_{j=1}^{\infty} b'_j \mathrm{e}^{\mathrm{i}jx}$. Then

$$h_k(x) - \sum_{j=1}^{p} b_j h_{k-j}(x) = \mathrm{e}^{\mathrm{i}kx} + u_k(x), \quad \text{where} \quad u_k(x) = \sum_{j=p+1}^{\infty} b_j h_{k-j}(x).$$

Since $\sum_{j=1}^{p} b_j h_{k-j}(x) = h_k(x) B'(\mathrm{e}^{\mathrm{i}x})$ and $\sum_{j=1}^{p} b_j = \sum_{j=1}^{\infty} b'_j < 1$ we obtain

$$h_k(x) = \frac{\mathrm{e}^{\mathrm{i}kx} + u_k(x)}{1 - B'(\mathrm{e}^{\mathrm{i}x})} = \xi'_k(x) + u'_k(x), \quad \text{where} \quad \xi'_k(x) = \frac{\mathrm{e}^{\mathrm{i}kx}}{1 - B'(\mathrm{e}^{\mathrm{i}x})}, \ u'_k(x) = \frac{u_k(x)}{1 - B'(\mathrm{e}^{\mathrm{i}x})}.$$

We claim that under assumption $\|g\| = \infty$, as $p \to \infty$

$$\|u'_k\| \le \|h_k\| = \|A\| < \infty \quad \text{and} \quad \|\xi'_k\| \to \infty. \tag{6.3}$$

On the other hand, $\|\xi'_k\| = \|h_k - u'_k\| \le \|h_k\| + \|u'_k\| \le 2\|h_k\| < C < \infty$ which leads to a contradiction, implying $\|g\| < \infty$.

To prove (6.3), note that from the definition of $u'_k(x)$ and $1 \le |1 - \sum_{j=1}^{p} b_j \mathrm{e}^{-\mathrm{i}jx}| + |\sum_{j=1}^{p} b_j \mathrm{e}^{-\mathrm{i}jx}| \le |1 - \sum_{j=1}^{p} b_j \mathrm{e}^{-\mathrm{i}jx}| + \sum_{j=1}^{p} b_j$ we obtain

$$\begin{aligned} \|u'_k\|^2 &= \int_{\Pi} \Big| \frac{\sum_{j=p+1}^{\infty} b_j \mathrm{e}^{-\mathrm{i}jx}}{(1 - B(\mathrm{e}^{\mathrm{i}x}))(1 - \sum_{j=1}^{p} b_j \mathrm{e}^{-\mathrm{i}jx})} \Big|^2 \mathrm{d}x \\ &\le \int_{\Pi} \frac{\mathrm{d}x}{|1 - B(\mathrm{e}^{\mathrm{i}x})|^2} \Big(\frac{\sum_{j=p+1}^{\infty} b_j}{1 - \sum_{j=1}^{p} b_j} \Big)^2 \le \int_{\Pi} \frac{\mathrm{d}x}{|1 - B(\mathrm{e}^{\mathrm{i}x})|^2} = \|h_k\|^2, \end{aligned}$$

proving the first relation in (6.3). The second claim in (6.3) follows from

$$\|\xi'_k\|^2 = \int_{\Pi} \frac{\mathrm{d}x}{|1 - B'(\mathrm{e}^{\mathrm{i}x})|^2} = \int_{\Pi} |G'(\mathrm{e}^{\mathrm{i}x})|^2 \mathrm{d}x = 2\pi \sum_{j=0}^{\infty} (g'_j)^2$$

where g'_j are power coefficients of the analytic function $G'(z) = (1 - B'(z))^{-1} = \sum_{j=0}^{\infty} g'_j z^j$, $|z| < 1$ as given by (3.2) with b_j replaced by b'_j. Note $0 \le g'_j \to g_j$ monotonically as $p \to \infty$ and therefore $\sum_{j=0}^{\infty} (g'_j)^2 \to \|g\|^2 = \infty$. This completes the proof of (ii) and the lemma. $\square$