Essays on term structure models

A thesis submitted in partial fulfillment of the requirements of the degree of Doctor of Philosophy (Ph.D.) in Economics

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to my beloved dad

Declaration

I wish to declare

I, Sarah Mouabbi, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications:
Parts of Chapter 4 were undertaken as joint work with Professor Andrea Carriero and Elisabetta Vangelista.

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Abstract

Estimating risk premia has been at the forefront of the financial economics' literature due to their informational content. Risk premia are of particular interest to academics, policymakers and practitioners given the information they disclose on expected asset returns for a given level of risk, their contribution in asset pricing and their ability to disentangle the different sources of risk. However, risk premia are unobserved and their estimates strongly differ from one study to another, as they are highly sensitive to the specification of the underlying model, sparking hence a strong interest in their analysis. The aim of the thesis is to estimate risk premia in a dynamic term structure model setting. The first part of the thesis comprises of an overview of a particular class of dynamic term structure models, namely affine term structure models. The overview will include important concepts and definitions. The second part of the thesis uses a risk-averse formulation of the uncovered interest rate parity to determine exchange rates through interest rate differentials, and ultimately extract currency risk premia. The method proposed consists of developing an affine Arbitrage-Free class of dynamic Nelson-Siegel term structure models (AFNS) with stochastic volatility to obtain the domestic and foreign discount rate variations, which in turn are used to derive a representation of exchange rate depreciations and risk premia. The third part of the thesis studies both the nominal and real UK term structure of interest rates using a Gaussian dynamic term structure model, which imposes the non-negativity of nominal short maturity rates. Estimates of the term premia, inflation risk premia and market-implied inflation expectations are provided.

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Chapter 1

Introduction

1.1 Motivation

One basic issue facing economists is the appraisal of risk premia, which is the compensation offered to investors for being exposed to a specific risk. The intricacy of risk premia is rooted in the fact that they are unobserved, whilst their appeal is nurtured by their informational content. The relevance of risk premia stems from their ability to convey expected asset returns for a given level of risk, their capacity to disentangle the different sources of risk and their contribution in asset pricing.

This thesis consists in utilizing dynamic term structure models, particularly affine term structure models of interest rates, to monetary finance applications, with the aim to extract risk premia estimates.

In a first instance, key definitions of notions that are recurrently used throughout the thesis are introduced. This outline is followed by an account of term structure models which are typically used to fit the time series and cross-sectional dynamics of yields. The affine class of term structure models has been distinctly popular in the literature and hence this segment of the thesis further elaborates on this specific class. The popularity of affine models is partially justified by their flexibility and hence their ease in developing extensions that can be used in monetary and financial applications. This chapter of the thesis sets the scene for the remainder chapters which extend affine term structure models to the application of currency risk premia and inflation risk premia and expectations.

In a second instance, this thesis studies links between the term structure of interest rates and exchange rates and extracts currency risk premia. An established strand of the economic literature has long formed strong relations between interest rates and exchange rates. This chapter delves into this stand of the literature, particularly on the concept of the uncovered interest rate parity which stipulates that high interest rate countries typically observe a depreciation in their currency. However, empirical evidence has so far rejected the use of the uncovered interest rate parity to determine exchange rates. Its failure is often accredited to the existence of a time-varying risk premium. The third chapter uses a risk-averse formulation of the uncovered interest rate parity to determine exchange rates through interest rate differentials, and ultimately extract currency risk premia. The strategy, starts by fitting domestic and foreign yields using a parsimonious arbitrage-free term structure model. Stochastic discount factors are consequently estimated to extract the depreciation of exchange rates. Finally, currency risk premia are endogenously determined. One of the main contributions of the model is that it constitutes the first Nelson-Siegel model for the determination of exchange rate depreciations and currency risk premia. The framework built is an extension of Christensen, Lopez, and Rudebusch (2010a)s AFNS model with stochastic volatility for the extraction of exchange rates expected depreciations and risk premia. Additionally, the model is theoretically robust and empirically successful and is able to alleviate the global optimum problem encountered in canonically affine term structure models by utilizing the Nelson-Siegel interpolation and no-arbitrage restrictions. Empirical findings suggest that estimated currency risk premia are able to account for the forward premium puzzle.

In the fourth chapter of my thesis, the effect of recent UK monetary policies, that have pushed nominal yields to near zero levels, are analyzed. Monetary policy actions in the United Kingdom, amongst other countries, have followed unconventional strategies in recent months, in an attempt to stimulate the economy. This segment studies the consequences of suppressing short-term interest rates near the zero lower bound on term structure modeling and quantifies the effects of the Bank of England's injection of money on inflation. By allowing nominal short-term interest rates to fall to the zero lower bound, the Bank of England practically disjoints the behavior of nominal and real rates. In particular, due to the existence of currency, nominal yields are to remain around the zero lower bound, whilst real yields are permitted to go below zero. These developments urge us to question the use of standard affine Gaussian dynamic term structure models as they face the risk of violating the inherent non-negativity assumption of nominal yields. It becomes hence of crucial importance to refine these prominent models and equip them with the ability to restrain nominal yields from being negative, whilst not restricting the behavior of real yields. Acknowledging Black (1995)'s astute use of shadow short rates and Krippner (2012)'s tractable estimation method, Christensen and Rudebusch (2013) developed a shadow-rate Arbitrage-Free Nelson Siegel (AFNS) term structure model which imposes the non-negativity of interest rates. An AFNS model is hence used to jointly estimate both the nominal and real UK term structure of interest rates, whilst imposing the now crucial non-negative property of nominal rates. Having addressed the delicate complications these monetary policy actions have on the modeling aspect of the paper, we now proceed to their economic implications. After the withdrawal of the European Exchange Rate Mechanism in 1992, inflation targeting has become one of the core objectives of the Bank of England's monetary policies. Thus, anchoring particularly long term inflation expectations is of primordial importance to the credibility of the Monetary Policy Committee. The question that now arises is how do we measure inflation? Amongst a multitude of indices and methods available, one measure that is widespread in central banks is the use of breakeven inflation (BEI) rates which consist of the mere difference between nominal and real yields. BEI rates are often used in lieu of surveys and forecasts, however, their use is far more intricate as a component for the risk premium of inflation typically contaminates the BEI rates as measures of inflation expectations. The benefits of using a no-arbitrage model come now into play by enabling the disentanglement of inflation risk premia from BEI rates, thus providing us with estimates of inflation expectations. This project is the outcome of a fruitful collaboration with the Debt Management Office, HM Treasury which kindly supplied me with the data set.

1.2 Outline

This thesis is structured as follows. Chapter 2 elaborates on the general concepts under which term structure models are based on. It provides key definitions that serve as a complement to the remaining chapters of the thesis. It introduces a specific class of term structure models, known as affine models, and examines their flexibility in being extended into macro-finance frameworks. Potential interesting applications and extensions of affine term structure models are presented, illustrating thus their strong pliability.

Chapter 3 builds a bilateral framework that jointly prices the term structure of interest rates of two countries and extracts the risk premia of the exchange-rate in question. This framework is based on a risk-averse take of the uncovered interest rate parity. The model is applied on a specific currency pair, namely the GBP/USD, which historically is known to fail to account for the forward premium puzzle.

Chapter 4 constructs a joint AFNS model for nominal and real yields which imposes the non-negativity assumption of nominal rates. Estimates of term premia and breakeven inflation rates are provided and further decomposed into two components, namely into inflation risk premia and inflation expectations. The zero lower bound assumption is found to be necessary to reflect the countercyclicality in the model-implied nominal term premia.

Chapter 5 concludes this thesis and discusses future projects.

Chapter 2

Affine term structure models

2.1 Introduction

Monetary policy, forecasting and derivative pricing are a handful of the many reasons that have sparked an interest in bond yields. Most modern economies utilize the term structure of interest rates to conduct monetary policy. Particularly, the short end of the yield curve is exploited to drive changes in the medium and long end of the curve. Focus is drawn to medium and long term yields due to their inherent association with borrowing costs and consequently their tight link to the economy's aggregate demand. Current yield curves bear informational content on future curves and economic activity, rendering them a potent tool for forecasting. Additionally, the valuation of complex financial instruments is often determined through interest rate models. However, despite the fact that bond prices are typically observed, bond yields need to be extrapolated by these bond prices and as a consequence, the estimation of term structure models of interest rates has spawned a wide literature due to its importance to policymakers, academics and practitioners.

Bonds, unlike other financial assets and macroeconomic variables, enjoy the peculiarity of having many observed yields associated with different maturities, at every given point in time, thus rendering both their time series and cross-sectional properties of interest. An analysis ignoring cross-sectional restrictions is possible, when focusing on a particular segment of the yield curve. However, the incorporation of cross-sectional restrictions comes with its own benefits. First and foremost, the imposition of no-arbitrage restrictions allows the extraction of risk premia by alleviating the difficulty that usually arises, that is, the inability to disentangle risk premia from expectations. Accounting for no-arbitrage introduces an additional probability measure to the physical one, known as the risk-neutral measure. By computing the difference between those two measures, one is capable to obtain estimates of the risk premium. It is important to note that the assumption of no-arbitrage is well grounded given the highly liquid nature of bond markets. In addition, these restrictions further enhance the consistency of yields across time and maturities and improve out-of-sample forecasts by reducing the number of parameters to be estimated within the model.

Having addressed the importance of working on a set of yields that vary across time and maturities, multivariate models are perceived as the appealing paradigm to capture yield dynamics. A natural response is to consider an unrestricted vector autoregression model. However, the latter is paired with the disadvantage of losing degrees of freedom due to the high-dimensionality of the model. At this point, the advantages of cross-sectional restrictions enter into play by allowing a low-dimensional factor structure to approximate the high-dimensional system. A factor structure appears to be sufficient to be able to replicate all possible shapes of the yield curve. Specifically, yield curves take different forms across time, from U-shaped curves, all the way to flat, upward or downward sloping curves. Nonetheless, typical stylized facts of yield curves include the notion that yields ought to increase with maturity, thus rendering upward sloping curves more characteristic. This fact enhances the liquidity preference theory, which stipulates that a time-varying term premium is required on long term yields to compensate for their relative lack of liquidity. Yields are also known to be highly persistent, as indicated by their strong autocorrelations. An additional trait of the yield curve is the fact that its short end is typically more volatile than its long end. This last stylized fact becomes of particular interest in today's economy, with unconventional monetary policy strategies driving short yields near their zero lower bound. By anchoring the short end of the curve, the volatility has been seen to pick up in the long end of the curve and inversely decrease in the short end. These very stylized facts aid in imposing the restrictions necessary to achieve the factor structure.

Reaching a consensus that a low-dimensional factor structure has the ability to summarize a complex and high-dimensional structure, the econometrician is now faced with a wide choice of factor structures. At this stage, it is important to note that it is widely accepted, in the literature, that three factors are typically considered sufficient (see Litterman and Scheinkman (1991), Ang and Piazzesi (2003)). The choice of factor structures can be synthesized in the following list of alternative models: principal components, interpolation methods and term structure models. In this chapter, arguments are made in support of the latter alternative, as it not only encompasses consistency of yield dynamics through the imposition of no-arbitrage, but it further allows the dissociation of risk premia from expectations' estimates. This chapter, thus, resumes in clarifying the ties yield curve models may have with financial and economic variables, including exchange rates, inflation and growth. This segment builds the necessary grounds for the following chapters, which apply affine term structure models of interest rates to monetary finance applications, with the aim to extract risk premia estimates.

This chapter benefits from the work of Piazzesi (2010) and Diebold and Rudebusch (2012), and is constructed as follows. In the second section, the basic concepts surrounding bond yields and prices are tackled. In the third section, affine term structure models are introduced. The fourth section includes a brief account of the recent developments within this literature, known as macro-finance models. The fifth section provides conclusive remarks.

2.2 Bond prices and yields

This section establishes the main definitions revolving around term structure modeling. It is important to note that term structure models focus on specific bonds, namely zero-coupon bonds. Those pay no coupons and only pay a single payoff at maturity, known as the face value of the bond, which for simplicity is assumed to amount to 1 unit of currency. Zero-coupon bonds are characterized by being purchased at discount and by the fact that they are considered as default free. Let $P_t(\tau)$ denote the price of a bond at time t that matures in τ periods and $y_t(\tau)$ denote the yield to maturity, compounded continuously. The following relationship holds.

$$P_t\left(\tau\right) = exp\left[-\tau y_t(\tau)\right] \tag{2.2.1}$$

Yields to maturity, also known as zero coupon yields, are thus naturally implied by zero coupon bond prices as follows.

$$y_t\left(\tau\right) = -\frac{logP_t\left(\tau\right)}{\tau} \tag{2.2.2}$$

Yields can also be expressed as an average of forward rates, which are the increment observed in the yield for prolonging the maturity by one additional period. The relationship of zero coupon yields and forward rates, in continuous time, is given below.

$$y_t\left(\tau\right) = \frac{1}{\tau}\int_0^\tau f_u du \tag{2.2.3}$$

In addition, by combining equations 2.2.2 and 2.2.3, the forward rate curve can be extracted by using the formula below.

$$f_t(\tau) = -\frac{P_t'(\tau)}{P_t(\tau)} \tag{2.2.4}$$

where $P_t'(\tau)$ designates the first derivative of the bond price $P_t(\tau)$. It is interesting to note that out of the three variables in question, $P(\tau)$, $y(\tau)$ and $f(\tau)$, only one of them suffices to derive the remaining two.

As previously mentioned, bond yields are not observed and need to be extracted by transforming observed bond prices. Many approaches have been taken across the years. One of them consists of the use of spline methods, including polynomial splines and exponential splines, to name a few. These were deemed dated due to their incapacity to ensure positive forward rates. Fama and Bliss (1987) elaborate on this flaw by deriving the yield curve using forward rates. This very method is typically used to obtain what are known as unsmoothed Fama-Bliss forward rates. The preponderance of central banks often use interpolation methods, such as the Nelson-Siegel or Svensson method, on those unsmoothed yields, in order to smoothen them. Factor models have become increasingly popular in the estimation of term structure models as they reduce the dimensionality of the problem whilst enabling the replication of all possible shapes of the yield curve. The most widespread factor designs in term structure modeling are broadly segregated into three families. The first factor structure stems from a principal component analysis, which by construction imposes factors to remain orthogonal whilst factor loadings are left unconstrained. A second structure involves interpolation methods that fit empirical yield curves. Unlike the previous method, factors remain unconstrained and factor loadings are the ones that inherit a particular empirical structure. The third category is known as the no-arbitrage dynamic term structure model. This method imposes restrictions on both factors and loadings. The most important trait of this structure revolves around the imposition of no-arbitrage restrictions on the factor loadings. Although this last class of models is very broad, the most noteworthy subclass is known to be affine term structure models. The next section comprises of a brief account of affine models.

2.3 Affine term structure models

The pricing of bonds necessitates an equivalent probability measure to the physical one $\mathbb{P}$, known as the risk-neutral probability measure, denoted by $\mathbb{Q}$. The very introduction of a second probability measure allows the imposition of the absence of arbitrage opportunities, which according to Almeida and Vicente (2008), enhances estimation and forecasting efficiency as well as solidifies the consistency of the model. Assuming no-arbitrage, a bond, that pays a payoff $\Pi(T)$ at time T, is priced under the physical measure $\mathbb{P}$ using a pricing kernel M(t). The current price $\Pi(t)$ is thus the expectation of the discounted future cash flows, as seen below, where $E_t^{\mathbb{P}}$ denotes the expectation at time t under the physical measure.

$$\Pi(t) = E_t^{\mathbb{P}} \left[\frac{M(T)}{M(t)} \Pi(T) \right] \tag{2.3.5}$$

Assuming the kernel dynamics given in equation 2.3.6, where $\Gamma(t)$ and W(t) represent, respectively, the price of risk and a standard Brownian motion, the two measures, $\mathbb{P}$ and $\mathbb{Q}$, are linked through the Radon-Nikodym derivative given in equation 2.3.7.

$$\frac{dM(t)}{M(t)} = -r(t)dt - \Gamma(t)'dW(t) \tag{2.3.6}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = exp\left[-\frac{1}{2}\int_t^T \Gamma(s)'\Gamma(s)ds - \int_t^T \Gamma(s)dWs\right] \tag{2.3.7}$$

It follows that equation 2.3.5 is transformed as shown below.

$$\Pi(t) = E_t^{\mathbb{Q}}\left[exp\left(-\int_t^T r_u du\, \Pi(T)\right)\right] \tag{2.3.8}$$

Let T denote the maturity of a zero-coupon bond that pays one unit of currency at maturity and $\tau = T - t$ designate the time to maturity. The instantaneous rate, denoted by r_t, is given by the limit of yields $y_t(\tau)$ as time t tends to T and the bond price is given as follows.

$$P_t\left(\tau\right) = E_t^{\mathbb{Q}}\left[exp\left(-\int_t^T r_u du\right)\right] \tag{2.3.9}$$

It is clearly reflected in equation 2.3.9 that there are two key components to modeling the yield curve, those being the existence of an equivalent measure $\mathbb{Q}$ to the physical measure $\mathbb{P}$ and the dynamics of the instantaneous rate r_t under $\mathbb{Q}$. In affine term structure models the dynamics of the instantaneous rate r_t under $\mathbb{Q}$ ought to be an affine function of the state variable X_t, which itself is an affine diffusion under the risk-neutral probability measure. The state dynamics follow an affine diffusion process, provided below:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW(t) \tag{2.3.10}$$

where the drift $\mu(X_t)$ and the variance-covariance matrix $\sigma(X_t)\sigma(X_t)'$ are affine in X_t. The drift of the state dynamics takes the following form, $\mu(X_t) = \kappa(\theta - X_t)$, where κ is the mean reversion matrix and θ represents the unconditional mean. As for the diffusion of the process, it takes the following form, $\sigma(X_t) = \Sigma s(X_t)$, where $s(X_t)$ is equal to the identity matrix in Gaussian affine models and is a diagonal matrix, of the form $s_{ii}(X_t) = \sqrt{s_{0,ii} + s'_{1,ii}X_t}$, in the stochastic volatility class of models. More is said on the latter models, given chapter 3 focuses on an exchange rate application of an affine term structure model with stochastic volatility.

Bond prices thus inherit an exponentially affine representation, which is the solution of a system of ordinary differential equations (ODE). These ODE have a closed-form solution when the model is Gaussian and are solved numerically when the model encompasses stochastic volatility.

It is important to note that Gaussian affine models do not preclude interest rates from being negative. This issue is not of particular interest when interest rates are at a safe distance of the zero lower bound. However, with recent economic developments, interest rates have plummeted to unprecedented levels, sparking thus the need to impose the non-negativity of interest rates. Three different classes of models have been developed to accommodate this situation: shadow rate models, Cox-Ingersoll-Ross models and quadratic models. Quadratic models as in Ahn, Dittmar, and Gallant (2002) and CIR models are, nonetheless, unable to conform to prolonged periods of zero or near zero interest rates. Conversely, shadow rate models are able to cope with extended periods of near zero rates by rendering instantaneous rates non-linear. Chapter 4 elaborates on the particularity of estimating rates in the vicinity of zero and builds an inflation application of both a Gaussian affine term structure model and a shadow model.

It is worth noting that affine models, despite their advantages in precluding arbitrage opportunities and obtaining known expressions for term premia, come at the disadvantage of being hard to estimate and interpret. More specifically, common issues that arise are the inability to interpret intuitively the latent factors and the global optimum problem.

2.4 Macro-finance extensions

The two previous sections have established that term structure models are of importance to model the dynamics of yields across both their cross-section and time series and are particularly interesting tools due to their simplicity in extending them to more complex and complete frameworks. It has long been instilled that the state of the economy has an impact on financial variables. A clear example of macroeconomic variables feeding financial variables is the effect of the level of inflation on the future bank rate, which eventually translates to all yields in the market. Nonetheless, it is becoming increasingly apparent that the health of financial and banking institutions can have an effect on economic variables. The advent of the recent financial crisis has thus strengthened the relation between financial and economic variables, rendering macro-finance models of great importance. This section analyzes the recent developments in the use of term structure models of interest rates to macroeconomic and financial applications.

The most natural account of a macroeconomic model is the Taylor rule, which accounts for fluctuations in short rates by using the output gap and inflation gap which are the dispersion of actual levels of output and inflation, respectively, from their target values. Ang, Dong, and Piazzesi (2005) estimate a Taylor rule and are able to draw the monetary policy shocks by imposing cross-sectional restrictions. An interesting attempt of a macro model is made by Aruoba and Diebold (2010), who model the yield curve using level, slope and curvature factors as well as observable macroeconomic variables, amongst which are monetary policy tools, inflation and real activity. Pooter, Ravazzolo, and van Dijk (2010) have a similar approach by analyzing the effect of the inclusion of macroeconomic variables on the forecasting of the term structure of interest rates. Reported results suggest that accounting for macroeconomic informational content improves the forecasting of yields.

On the finance end of the spectrum, Campbell and Taksler (2003) examine the interrelation between the expected excess returns on bonds and equity and find that changes in these expected excess returns, real yields and risk levels bear a predictable component. Similarly, Lettau and Wachter (2011) expand upon this idea by jointly pricing the term structure of interest rates, the risk-return levels of stocks and the returns on the aggregate market.

A recently popular extension of the term structure literature consists in shedding some light on the following twofold research questions. Does the yield curve span yields' volatility, or is volatility unspanned? Those inquisitions have been triggered by a very common phenomenon in the term structure literature, that is the inability of models to jointly capture the first and second moment of yields. Andersen and Benzoni (2007) examine whether bonds do span the yield volatility and find arguments against this hypothesis. Their conclusion was supported by the fact that yield volatility factors were uncorrelated to the yields' cross-section. According to Joslin (2013), volatility is said to be unspanned if bonds are unable to hedge the volatility risk. On this front, it is found that current unspanned stochastic volatility models cannot capture the cross-section of bond volatility. Moreover, Coroneo, Giannone, and Modugno (2012) assess whether macroeconomic content has a predictive ability on the yield curve and on excess bond returns. The use of macroeconomic variables is extended to both the obtention of yield curve factors and the identification of the sources of risk which are not hedged by bonds. Therefore, spanned and unspanned stochastic volatility is a potentially prolific strand of the term structure literature which necessitates further investigation and requires further advances in the years to come.

Interesting extensions to term structure models can be found in the two types of vector autoregression (VAR) models that follow. The first consists in studying term structure models in a global scale, in the spirit of Diebold, Li, and Yue (2008) that fits the yield curve of multiple countries by featuring global and country-specific factors. Similarly, Chudik and Pesaran (2014) introduce the Global VAR model (GVAR). This paper studies the joint forecasting of financial and macroeconomic variables at an international level. Advances in the literature are expected to be made on the selection and number of global factors and individual factors. Additional consideration ought to be made on the existence of regional factor structures. An alternative is to use a Bayesian VAR (BVAR) à la Carriero (2011). This paper, with the help of artificial data, uses a term structure model as a prior. This approach allows the loose imposition of no-arbitrage conditions whilst further alleviating the dimensionality problem and accounting for possible model misspecifications.

2.5 Conclusion

This chapter provides a brief and concise outline of term structure models, covering basic concepts and introducing several advances within this literature. The general idea that transcends within the chapter is the complexity involved in estimating the term structure of interest rates as well as their potency in extracting information regarding macroeconomic and financial variables. The two following chapters will utilize term structure models in order to extract risk premia. Specifically, chapter 3 emphasizes on the link between term structure models and currencies whilst chapter 4 concentrates on the strong relationship between the yield curve and inflation. Both chapters emphasize on the affine class of term structure models and more specifically on a Nelson-Siegel affine term structure model which further imposes no-arbitrage conditions to ensure the consistency of yield dynamics.

Chapter 3

An arbitrage-free Nelson-Siegel term structure model for the determination of currency risk premia

3.1 Introduction

Exchange rate fluctuations have substantial implications for the pricing and allocation of assets. Characterized by seemingly weak links to fundamentals and by a volatile nature, exchange rates still remain at the forefront of a multitude of papers. These stylized facts, better known as the exchange rate determination and excess volatility puzzles, render the modeling of exchange rate movements and the caption of their volatility increasingly intricate.

A significant strand of the exchange rate literature has long been devoted to tying exchange rates to interest rates through the so called covered and uncovered interest rate parities. Under the validity of perfect asset substitutability and capital mobility, the principle of these two parities revolves around the premise of no-arbitrage, whereby low interest rate countries ought to be compensated by an appreciated currency in order to maintain the indifference of the global investor. Despite the highly intuitive nature of these theoretical equilibrium relations, severe deviations from postulated equilibrium levels have, on multiple occasions, been recorded through empirical tests. The observed divergences are expressed by the susceptibility of low interest rate countries to currency depreciations and are typically known as the forward premium puzzle.

A plethora of studies has been dedicated to justifying these deviations. What seems to be the most convincing interpretation so far is the one proposed by Fama (1984), advocating the presence of a time-varying risk premium. The latter represents the compensation to the investor for being exposed to exchange rate risk. Fama (1984) stipulates that currency risk premia ought to have a greater variance than expected exchange rate variations and that both variables need to be negatively correlated in order to explain the puzzle. Following this noteworthy account, many papers have attempted to model a currency risk premium using statistical methods and conventional asset pricing methods, including consumption based asset pricing theory, equilibrium models, but with arguably limited success (see, for example, Frankel and Engel (1984), Domowitz and Hakkio (1985), Mark (1988), Bekaert (1996) and Lustig and Verdelhan (2011)).

Though unobserved, currency risk premia have the potential to enhance asset allocation and risk management decisions. This explains why attempts to estimate currency risk premia are persistently found in the literature. The purpose of this study is to examine whether a newly established framework for the term structure of interest rates, the Arbitrage Free Nelson Siegel term structure model (AFNS) with stochastic volatility, introduced by Christensen, Lopez, and Rudebusch (2010a), can be further extended to jointly price both the term structure of interest rates of two countries and exchange rate depreciations. Once the exchange rate depreciation is estimated through the Bilateral Arbitrage-Free Nelson-Siegel model with stochastic volatility (BAFNS), no-arbitrage conditions allow for the endogenous extraction of the risk premium.

The above-mentioned approach of exploiting existing affine term structure models in order to derive risk premia has previously been employed in several different contexts. In an influential study by Backus (2001), the issue of whether the popular affine term structure model by Duffie and Kan (1996) is capable of capturing the forward premium anomaly is considered. Similarly, Sarno, Schneider, and Wagner (2012) derive a multi-currency term structure model that gives rise to the foreign exchange risk premium, the properties of which are examined. Graveline (2006), examines the forward premium anomaly using an arbitrage-free model, including options prices. Similar methods and applications can be found in Brandt and Santa-Clara (2001), where excess volatility in an incomplete market setting is examined, in Ahn (2004), Inci and Lu (2004) and Anderson, Hammond, and Ramezani (2010), who compare the different implications of local and global factors, and Brennan and Xia (2006), where the volatility of pricing kernels is tied to exchange rate volatility and risk premia. More recently, term structure models have been used to obtain equity premia (see Brennan, Wang, and Xia (2004) and Lettau and Wachter (2011)) and underpin inflation expectations and risk premia (see Christensen, Lopez, and Rudebusch (2010b) and Chernov and Mueller (2012)).

Although Sarno, Schneider, and Wagner (2012)'s analysis appears to be the most complete and well-rounded piece of work to date, it suffers from a cumbersome Bayesian estimation procedure requiring the use of priors, which might, therefore, impact the estimation results. Moreover, an additional step is further required stemming from the necessity to use rotations in order to interpret the latent factors. In this chapter, attention is drawn towards employing the AFNS model due to the favorable properties it agglomerates. In particular, this model encompasses sound theoretical grounds through no-arbitrage restrictions, whilst also preserving robust estimation procedures with the imposition of the Dynamic Nelson Siegel (DNS) structure. Specifically, the imposition of the DNS structure provides a level, slope and curvature interpretation to the latent factors without performing any rotation. Additionally, the flexibility of the AFNS model allows to extend its use beyond simple estimation and makes it appealing for forecasting exercises. Furthermore, the AFNS is found to be successful not only in the blunt determination of the term structure of interest rates but also in more synthesized problems such as the estimation of inflation expectations; hence motivating the use of this specific model to estimating currency risk premia. This chapter further shifts its focus towards analyzing the impact of the different assumptions set on the diffusion of the process (ie. Gaussian or with stochastic volatility) on the properties acquired by the estimates of the model, namely, the yields, exchange rate variations and currency risk premia.

A six-factor AFNS model with stochastic volatility is estimated to jointly underpin the term structure of two countries, whilst exchange rate depreciations and risk premia are derived endogenously. For robustness purposes, a Gaussian multilateral AFNS model with twenty one factors (three factors for each country included) is examined in appendix 3.B of this chapter. Results suggest that the Gaussian AFNS model provides a better fit for interest rates and allows for a joint multi-currency estimation rather than restricting the model to a bilateral estimation. On the other hand, the volatility of exchange rate differentials is better captured using the AFNS model with stochastic volatility rather than the Gaussian version of the model. Additionally, the risk premium generated from the bilateral AFNS model with stochastic volatility respects the above mentioned Fama conditions, hence offering a legitimate explanation for the forward bias puzzle without resorting to departures from rational expectations. The main drivers of exchange rate depreciations and risk premia are found to be the two curvature factors whilst currency risk premia display a countercyclical nature. Finally, Graveline (2006) argues that the use of options helps in fitting the volatility of exchange rates. In this regard, this chapter shows that it is possible to reasonably capture the volatility of exchange rate depreciations and risk premia without the inclusion of options in the model. More specifically, this result extends to first and second conditional moments.

The remainder of the chapter is structured as follows. The second section consists of a selective overview of the uncovered interest rate parity, the existing AFNS model with stochastic volatility, and pricing kernels as the connecting link of interest rates to exchange rates. In the third section, the BAFNS model is derived with the aim of extrapolating both exchange rate depreciations and risk premia. The fourth section comprises of an empirical study of the performance of the BAFNS model in determining exchange rate changes and extracting risk premia. This section also specifies the estimation procedure followed and its substantial benefits. The fifth section provides conclusive remarks.

3.2 Exchange rates and interest rates at a glance

This segment aims to motivate the sections that follow by building a review of the link between interest rates and exchange rates as well as the affine term structures model that is utilized to derive exchange rate variations.

3.2.1 The uncovered interest rate parity

Let $y^D(t,T)$ and $y^F(t,T)$ denote the zero coupon bond yields with maturity T at time t, of the domestic and foreign countries, and s_t and $f_{t,T}$ denote the logarithm of the spot and T-forward exchange rate, respectively. For the remainder of the chapter, the United States is considered as the domestic country. The United Kingdom represents the foreign country in the main analysis of the chapter, whilst additional foreign countries, including Australia, Canada, Switzerland, Japan and Sweden are examined in appendix 3.B. All exchange rates are denominated in U.S. dollars, and hence represent the price of one unit of foreign currency in US dollars.

The covered interest rate parity stipulates that, under rational expectations and risk-neutrality, the expected exchange rate depreciation equals the difference between the forward and spot exchange rates. By the same token, the uncovered interest rate parity builds an exact relationship between the expected exchange rate depreciation and the domestic and foreign interest rate differential. The two relationships are shown in the equations below,

$$\mathbb{E}^{\mathbb{P}}\left[\Delta s_{t,T}|\mathcal{F}_t\right] = f_{t,T} - s_t \tag{3.2.1}$$

$$\mathbb{E}^{\mathbb{P}}\left[\Delta s_{t,T}|\mathcal{F}_t\right] = y^D(t,T) - y^F(t,T) \tag{3.2.2}$$

where $\mathbb{E}^{\mathbb{P}}$ is the expectation under the data generating probability measure, $\mathcal{F}_t$ is the filtration and $\Delta s_{t,T} = s_T - s_t$. Drawing from equation (3.2.1), the forward exchange rate ought to be an unbiased predictor of the future spot exchange rate. Using the traditional Fama regressions given below, the validity of the forward rate unbiasedness hypothesis is confirmed if $\alpha_i = 0$, $\beta_i = 1$ and $\xi_{i;t,T}$ displays no serial correlation, for $i = 1, 2$.

$$\Delta s_{t,T} = \alpha_1 + \beta_1(f_{t,T} - s_t) + \xi_{1;t,T} \tag{3.2.3}$$

$$\Delta s_{t,T} = \alpha_2 + \beta_2\left[y^D(t,T) - y^F(t,T))\right] + \xi_{2;t,T} \tag{3.2.4}$$

The preponderance of empirical results have, however, disputed the claim of the hypothesis, hence raising theories for the existence of a time-varying risk premium, amongst others. Conceptually, the existence of a risk premium signifies a departure from risk-neutrality given it represents a compensation, to the investor, for being exposed to currency risk as well as interest rate risk. A risk-averse interpretation of the uncovered interest rate parity is given below,

$$\Delta s_{t,T} = \left[y^D(t,T) - y^F(t,T)\right] - \rho_{t,T} + \zeta_{t,T} \tag{3.2.5}$$

with $\rho_{t,T}$ representing the risk premium, which varies with time t and maturity T and $\zeta_{t,T}$ being the regression residual. The risk premium component bears a negative sign due to the fact that exchange rates are denominated in US dollars (ie. the domestic currency). A negative exchange rate depreciation signals an appreciated US currency, hence implying a higher purchasing power and risk premium. Fama (1984) stipulates that there are two necessary conditions the risk premium needs to feature in order to ensure its ability to explain the departures from the levels dictated by the uncovered interest rate parity. These conditions are stated below.

$$\mathbb{V}^{\mathbb{P}}\left[\rho_{t,T}\right] > \mathbb{V}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}{}_t\left(\Delta s_{t,T}\right)\right] \tag{3.2.6}$$

$$\mathbb{C}ov^{\mathbb{P}}\left[\rho_{t,T}, \mathbb{E}^{\mathbb{P}}{}_t\left(\Delta s_{t,T}\right)\right] < 0 \tag{3.2.7}$$

where $\mathbb{V}^{\mathbb{P}}$ and $\mathbb{C}ov^{\mathbb{P}}$ represent the variance and covariance under the physical measure, respectively.

Specifically, omitting the risk premium typically generates a negative slope of the Fama regression in equation (3.2.4). Fama (1984) shows that if the risk premium admits these two conditions then the negative bias of the slope is corrected, advocating, hence, in favor of the risk premium hypothesis as a reasonable correction to the risk neutral uncovered interest rate parity.

3.2.2 The arbitrage-free Nelson-Siegel model with stochastic volatility

In this segment, the model, used to fit the term structure of interest rates of the domestic and foreign countries, is presented in its simplest, unilateral form.

One of the most prominent models, empirically, for the term structure of interest rates is the one developed by Nelson and Siegel (1987). The popularity of this model mainly stems from its stable estimation and its flexibility in fitting both the cross section and time series properties of interest rates. Diebold and Li (2006) have extended it to a dynamic factor model where latent factors bear the level, slope and curvature interpretation, whilst, Koopman, Mallee, and Van der Wel (2010) have allowed for time-varying parameters and a non-Gaussian setting. Although empirically these models have been highly praised for their performance, they have sustained an equal amount of criticism for their lack of theoretical grounding.

Conversely, affine term structure models imposing no-arbitrage restrictions, such as the canonical model by Duffie and Kan (1996), have been found challenging in their estimation due to the difficulty in pinning down the global optimum, (see Joslin, Singleton, and Zhu (2011a) and Duffee and Stanton (2012)), as well as in their empirical success (see Duffee (2002). Christensen, Diebold, and Rudebusch (2011) develop an affine Arbitrage-Free class of dynamic Nelson-Siegel term structure models which combine the benefits of the two strands of models above whilst simultaneously alleviating their disadvantages. However, due to the Gaussian nature of the model, it is highly unlikely to be able to capture the volatility displayed by exchange rates. A stochastic version of the AFNS model is hence adopted following Christensen, Lopez, and Rudebusch (2010a).

The details of the three-factor AFNS model with stochastic volatility generated by all three factors ($AFNS_3$) are provided below. Let $X_t = (L_t, S_t, C_t)'$ denote the latent state variables, which can be interpreted as level, slope and curvature factors. In addition, assume the state vector X_t follows a Cox-Ingersoll-Ross process under the risk neutral $\mathbb{Q}$ measure. $\kappa^{\mathbb{Q}}$ is the mean-reversion matrix, $\theta^{\mathbb{Q}}$ the unconditional mean vector and $W_t^{X,\mathbb{Q}}$ denotes a three dimensional Wiener process.

$$dX_t = \kappa^{\mathbb{Q}} \left[\theta^{\mathbb{Q}} - X_t\right] dt + \Sigma diag[\sqrt{X_t}] dW_t^{X,\mathbb{Q}} \tag{3.2.8}$$

Christensen, Diebold, and Rudebusch (2011) show that with no loss of generality, $\theta^{\mathbb{Q}}$ can be set to zero. The system of stochastic differential equations, under the risk neutral probability measure, is hence re-written as follows.

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = -\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} \sqrt{L_t} & 0 & 0 \\ 0 & \sqrt{S_t} & 0 \\ 0 & 0 & \sqrt{C_t} \end{pmatrix} \begin{pmatrix} dW_t^{L,\mathbb{Q}} \\ dW_t^{S,\mathbb{Q}} \\ dW_t^{C,\mathbb{Q}} \end{pmatrix} \tag{3.2.9}$$

where λ is the mean-reversion parameter and $\epsilon = 10^{-6}$ to have a near unit root behavior for the level factor. In particular, the level factor typically displays a unit root, implying that the first element of the mean-reversion matrix ought to be equal to zero. However, the breach of Gaussianity would prevent the use of the Kalman filter. Setting this element equal to ϵ, a very small yet non-zero number, allows to preserve a near unit root feature whilst still allowing the use of the Kalman filter.

As demonstrated by Ang and Piazzesi (2003), nominal zero-coupon bond prices are exponentially affine functions of the state variables,

$$P\left(t,T\right) = E_t^{\mathbb{Q}}\left[exp\left(-\int_t^T r_u du\right)\right] = exp\left(A\left(t,T\right) + B\left(t,T\right)' X_t\right) \tag{3.2.10}$$

where r_t denotes the instantaneous risk-free rate and $(A\left(t,T\right))$ and $(B\left(t,T\right))$ are, respectively, the intercept and slope of the affine expression.

Consequently, the representation of zero-coupon yields with maturity T at time t is given by an affine function of the state variables, as shown below,

$$y(t,T) = -\frac{1}{T-t}logP\left(t,T\right) = -\frac{A\left(t,T\right)}{T-t} - \frac{B\left(t,T\right)'}{T-t}X_t \tag{3.2.11}$$

with $A\left(t,T\right)$ and $B\left(t,T\right)$ being the unique solutions to a system of Riccati equations. $A\left(t,T\right)$ is known as the adjustment term, which is added to maintain no-arbitrage conditions, whilst the factor loadings $B\left(t,T\right)$, retain the interpretation of level, slope and curvature, although they no longer match the exact form of the Nelson-Siegel factor loadings. The Riccati differential equations are listed below.

$$\begin{cases} \frac{B_1(t,T)}{dt}\left(t,T\right) = 1 + \epsilon B_1(t,T) - \frac{1}{2}\sigma_{11}^2 B_1^2(t,T) \\ \frac{B_2(t,T)}{dt}\left(t,T\right) = 1 + \lambda B_2(t,T) - \frac{1}{2}\sigma_{22}^2 B_2^2(t,T) \\ \frac{B_3(t,T)}{dt}\left(t,T\right) = -\lambda B_2(t,T) + \lambda B_3(t,T) - \frac{1}{2}\sigma_{33}^2 B_3^2(t,T) \\ \frac{A(t,T)}{dt}\left(t,T\right) \ = -B(t,T)'\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}} \end{cases} \tag{3.2.12}$$

The instantaneous risk-free rate is an affine function of the state variables given by the sum of the level and slope factors, as stated in equation (3.2.13). This representation is justified by the fact that the level factor affects yields of all maturities, including the short rate, while the slope factor typically influences yields of short maturities. The curvature factor is unnecessary in the spectrum of the short rate since it typically influences yields of medium horizons.

$$r_t = L_t + S_t \tag{3.2.13}$$

The AFNS model with stochastic volatility is a continuous-time model and Girsanov's theorem ensures the change from the data generated process measure, also known as the physical measure, to the risk-neutral measure as such, $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Gamma_t dt$, where Γ_t is the market price of risk and under the extended affine risk premium specification defined in Cheridito, Filipovic, and Kimmel (2007), it takes the form below:

$$\begin{aligned}\Gamma_t = &\begin{pmatrix} \sqrt{L_t} & 0 & 0 \\ 0 & \sqrt{S_t} & 0 \\ 0 & 0 & \sqrt{C_t} \end{pmatrix} \begin{pmatrix} \gamma_{1,1} \\ \gamma_{1,2} \\ \gamma_{1,3} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{S_t}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{C_t}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \gamma_{2,21} & 0 & \gamma_{2,23} \\ \gamma_{2,31} & \gamma_{2,32} & 0 \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{S_t}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{C_t}} \end{pmatrix} \begin{pmatrix} 0 \\ \gamma_{3,2} \\ \gamma_{3,3} \end{pmatrix}\end{aligned} \tag{3.2.14}$$

The extended specification for the market price of risk encompasses the essentially affine risk premium specification provided by Duffee (2002), which itself is a generalization of the completely affine formulation of the canonical model by Dai and Singleton (2000). Subtracting $\Sigma diag[\sqrt{X_t}]\Gamma_t dt$ from the risk-neutral dynamics and substituting the Brownian motion under the risk-neutral measure with its physical counterpart allows the extraction of the latent state variables $X_t = (L_t, S_t, C_t)'$ under the physical measure. The dynamics are given by the following stochastic differential equation:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathbb{P}} & 0 & 0 \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} & \kappa_{23}^{\mathbb{P}} \\ \kappa_{31}^{\mathbb{P}} & \kappa_{32}^{\mathbb{P}} & \kappa_{33}^{\mathbb{P}} \end{pmatrix} \left[\begin{pmatrix} \theta_t^{L,\mathbb{P}} \\ \theta_t^{S,\mathbb{P}} \\ \theta_t^{C,\mathbb{P}} \end{pmatrix} - \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} \right] dt$$
$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} \sqrt{L_t} & 0 & 0 \\ 0 & \sqrt{S_t} & 0 \\ 0 & 0 & \sqrt{C_t} \end{pmatrix} \begin{pmatrix} dW_t^{L,\mathbb{P}} \\ dW_t^{S,\mathbb{P}} \\ dW_t^{C,\mathbb{P}} \end{pmatrix} \tag{3.2.15}$$

It is important to note that Feller conditions need to be satisfied in order to prevent states from hitting the zero-bound, as it would induce the states to remain at zero. These conditions are:

$$\begin{cases} \kappa_{21}^{\mathbb{P}}\theta_1^{\mathbb{P}} + \kappa_{22}^{\mathbb{P}}\theta_2^{\mathbb{P}} + \kappa_{23}^{\mathbb{P}}\theta_3^{\mathbb{P}} > \frac{1}{2}\sigma_{22}^2 \\ \lambda\theta_2^{\mathbb{Q}} - \lambda\theta_3^{\mathbb{Q}} > \frac{1}{2}\sigma_{22}^2 \\ \kappa_{31}^{\mathbb{P}}\theta_1^{\mathbb{P}} + \kappa_{32}^{\mathbb{P}}\theta_2^{\mathbb{P}} + \kappa_{33}^{\mathbb{P}}\theta_3^{\mathbb{P}} > \frac{1}{2}\sigma_{33}^2 \\ \lambda\theta_3^{\mathbb{Q}} > \frac{1}{2}\sigma_{33}^2 \end{cases} \tag{3.2.16}$$

There are additional admissibility restrictions that also need to be respected in order to ensure that the Nelson-Siegel factor loadings are being as feasibly approximated as possible, as well as for the model to remain free from arbitrage opportunities. These are:

$$\begin{cases} \epsilon\theta_1^{\mathbb{Q}} = \kappa_{11}^{\mathbb{P}}\theta_1^{\mathbb{P}} \\ \epsilon\theta_1^{\mathbb{Q}} > 0, \kappa_{11}^{\mathbb{P}}\theta_1^{\mathbb{P}} > 0 \\ \kappa_{21}^{\mathbb{P}} \leq 0, \kappa_{23}^{\mathbb{P}} \leq 0, \kappa_{31}^{\mathbb{P}} \leq 0, \kappa_{32}^{\mathbb{P}} \leq 0 \\ \theta_3^{\mathbb{Q}} = \frac{\lambda\theta_2^{\mathbb{Q}} - \frac{1}{2}\sigma_{22}^2}{\lambda} - \epsilon \end{cases} \tag{3.2.17}$$

Further, to ensure stationarity, the eigenvalues of $\kappa^{\mathbb{P}}$ have to be strictly positive. Finally, the latent factor L_t is interpreted as a level factor, which theoretically has a unit root. However, a unit root in the diffusion process induces complications in the estimation procedure. An adequate compromise is to settle for a near unit root behavior. Hence, in order to prevent the latent factor from displaying a unit root, additional restrictions are imposed on the relevant parameters. More specifically, $\kappa_{11}^{\mathbb{P}}$ and $\theta_1^{\mathbb{P}}$ are set to be strictly positive and $\kappa_{11}^{\mathbb{Q}} = \epsilon = 10^{-6}$, thus ensuring a near unit root behavior.

3.2.3 Stochastic discount factors

Let P_t^D and P_t^F denote the domestic and foreign price at time t of a future payment P_T^D and P_T^F, respectively.

$$P_t^D = \mathbb{E}^{\mathbb{P}}\left[\frac{M_T^D}{M_t^D}P_T^D\right] \tag{3.2.18}$$

$$P_t^F = \mathbb{E}^{\mathbb{P}}\left[\frac{M_T^F}{M_t^F}P_T^F\right] \tag{3.2.19}$$

where M^D and M^F are the domestic and foreign stochastic discount factors. Stochastic discount factors, also known as pricing kernels, establish the existence of a risk neutral probability measure and dictate the price of state-dependent claims. According to Graveline (2006), there exists a unique minimum variance stochastic discount factor with the following dynamics.

$$\frac{dM_t^D}{M_t^D} = -r_t^D dt - \Gamma_t^{D'} dW_t^{\mathbb{P}} \tag{3.2.20}$$

$$\frac{dM_t^F}{M_t^F} = -r_t^F dt - \Gamma_t^{F'} dW_t^{\mathbb{P}} \tag{3.2.21}$$

r_t^D and r_t^F denote the instantaneous domestic and foreign risk-free rate, respectively, and $W_t^{\mathbb{P}}$ represents a Wiener process. The diffusions of the pricing kernels, Γ_t^D and Γ_t^F, are the domestic and foreign prices of risk. The benefits of adopting a no-arbitrage setting come into play by enforcing a relationship between domestic and foreign bond prices and more importantly by setting a direct link relating interest rates to exchange rates, as shown below.

$$\frac{M_T^F}{M_t^F} \equiv \frac{S_T}{S_t}\frac{M_T^D}{M_t^D} \tag{3.2.22}$$

The above relationship states that one of the three random variables can be replicated using the remaining two variables. Hence, one of the stochastic processes can be determined endogenously, assuming that the remaining two dynamics are known. As in Backus (2001), the two pricing kernels are used to endogenously extract the exchange rate dynamics. This strategy allows the preservation of symmetry between the theoretical frameworks of the two countries. In particular, this chapter aims to extract information from the term structures of interest rates in order to explain exchange rate movements. Thus, the two term structures are modeled using exactly the same theoretical model for consistency purposes.

3.3 Theoretical framework: a dynamic bilateral asset pricing model

This section builds a bilateral extension for the AFNS model with stochastic volatility generated by all factors included in the model. The endogenous representations of the exchange rate depreciation, expected exchange rate return and currency risk premia are then derived.

3.3.1 The bilateral arbitrage-free Nelson-Siegel model with stochastic volatility

Extrapolating from the $AFNS_3$ to encompass two countries requires six factors. Let $X_t^J = \left(L_t^D, S_t^D, C_t^D, L_t^F, S_t^F, C_t^F\right)'$ denote the state vector for the joint model, including the level, slope and curvature factors for the domestic and foreign countries. One core advantage in using an extension of the AFNS stems from the fact that no additional rotation is necessary to interpret the latent factors. Under the risk-neutral measure, the state variable $X_t^J = \left(X_t^D, X_t^F\right)'$ solves the following stochastic differential equation.

$$dX_t^J = -\begin{pmatrix} \kappa^{D,\mathbb{Q}} & 0 \\ 0 & \kappa^{F,\mathbb{Q}} \end{pmatrix} \begin{pmatrix} X_t^D \\ X_t^F \end{pmatrix} dt + \begin{pmatrix} \Sigma^D & 0 \\ 0 & \Sigma^F \end{pmatrix} \begin{pmatrix} diag\sqrt{X_t^D} & 0 \\ 0 & diag\sqrt{X_t^F} \end{pmatrix} \begin{pmatrix} dW_t^{D,\mathbb{Q}} \\ dW_t^{F,\mathbb{Q}} \end{pmatrix} \tag{3.3.23}$$

where $W_t^{D,\mathbb{Q}}$ and $W_t^{F,\mathbb{Q}}$ are three dimensional Brownian motions and $\kappa^{D,\mathbb{Q}}$, $\kappa^{F,\mathbb{Q}}$, Σ^D and Σ^F are defined as follows.

$$\kappa^{D,\mathbb{Q}} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \lambda^D & -\lambda^D \\ 0 & 0 & \lambda^D \end{pmatrix} ; \; \kappa^{F,\mathbb{Q}} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \lambda^F & -\lambda^F \\ 0 & 0 & \lambda^F \end{pmatrix} ; \; \Sigma^D = diag \begin{pmatrix} \sigma_{11}^D \\ \sigma_{22}^D \\ \sigma_{33}^D \end{pmatrix} ; \; \Sigma^F = diag \begin{pmatrix} \sigma_{44}^F \\ \sigma_{55}^F \\ \sigma_{66}^F \end{pmatrix} \tag{3.3.24}$$

It is important to note that the off-diagonal elements of the mean-reversion matrix, in equation (3.3.23), are set to zero in order to preserve an independence between the latent factors in the domestic and foreign economy. Specifically, using the pairwise approach for the analysis of more than two countries, say $n+1$ countries including the domestic economy, induces the domestic economy to have n sets of estimates, one for each pair of currencies; generating hence a consistency problem. Keeping domestic and foreign latent factors independent alleviates this issue and preserves the consistency of the model in a bilateral setting. However, in a multilateral setting, consistency can be achieved in two ways, either by using a joint pricing for the $n+1$ term structures of interest rates, or by conducting the estimation for each country on an individual basis.

The instantaneous risk-free rates for the domestic and foreign countries are affine functions of the state variables and are given below.

$$r_t^D = L_t^D + S_t^D \tag{3.3.25}$$

$$r_t^F = L_t^F + S_t^F \tag{3.3.26}$$

Additionally, let $y(t,T)$ be the column vector of dimension 2Nx1, composed of the concatenation of N-maturities of domestic and foreign yields. The representations of domestic and foreign zero-coupon yields with maturity T at time t are given by an affine function of the state variables, as shown below,

$$y(t,T) = \begin{bmatrix} y^D(t,T) \\ y^F(t,T) \end{bmatrix} = -\begin{pmatrix} \frac{A^D(t,T)}{T-t} \\ \frac{A^F(t,T)}{T-t} \end{pmatrix} - \begin{pmatrix} \frac{B^D(t,T)'}{T-t} & 0 \\ 0 & \frac{B^F(t,T)'}{T-t} \end{pmatrix} X_t^J \tag{3.3.27}$$

where $A^D(t,T)$, $A^F(t,T)$, $B^D(t,T)$ and $B^F(t,T)$ are the unique solutions to a system of Riccati equations which are a natural extension to the system in equation (3.2.12). The intercept terms are the no-arbitrage adjustment terms and the factor loadings capture the level, slope and curvature interpretations.

Suppose a diffusion process of the form $dx_t = \mu(x_t)dt + \sigma(x_t)dW_t$ with $\mu^{\mathbb{P}}(x_t)$ and $\mu^{\mathbb{Q}}(x_t)$ denoting the drift terms of the state diffusion process under the physical and risk neutral probability measures, respectively. The price of risk is defined as follows.

$$\Gamma_t(x_t) = (\sigma(x_t))^{-1}\left[\mu^{\mathbb{P}}(x_t) - \mu^{\mathbb{Q}}(x_t)\right] \tag{3.3.28}$$

The dynamics of the state vector X_t^J, under the physical probability measure $\mathbb{P}$, are consequently drawn and given by the following stochastic differential equation, with $W_t^{J,\mathbb{P}}$ being a six dimensional Wiener process.

$$dX_t^J = \kappa^{J,\mathbb{P}}\left[\theta^{J,\mathbb{P}} - X_t^J\right]dt + \Sigma^J diag[\sqrt{X_t^J}]dW_t^{J,\mathbb{P}} \tag{3.3.29}$$

with $\kappa^{J,\mathbb{P}}$ being set to a diagonal matrix for simplicity and $W_t^{J,\mathbb{P}}$ being a six dimensional Brownian motion. The square matrix $\kappa^{J,\mathbb{P}}$ and vectors $\theta^{J,\mathbb{P}}$ and X_t^J are all six-dimensional.

3.3.2 Deriving exchange rate depreciations

In order to derive the exchange rate differences, a formulation for the domestic and foreign pricing kernels is necessary. Denote by M^D and M^F the domestic and foreign stochastic discount factors with the following dynamics,

$$\frac{dM_t^D}{M_t^D} = -r_t^D dt - \Gamma_t^D (X_t^J)' dW_t^{\mathbb{P}} \tag{3.3.30}$$

$$\frac{dM_t^F}{M_t^F} = -r_t^F dt - \Gamma_t^F (X_t^J)' dW_t^{\mathbb{P}} \tag{3.3.31}$$

$$= -r_t^F dt - \left(\Gamma_t^D (X_t^J)' - \gamma^* \Sigma^J \sqrt{X_t^J} \right) dW_t^{\mathbb{P}} \tag{3.3.32}$$

with $\gamma^* = (0,0,0,1,1,1)$ and $W_t^{\mathbb{P}}$ being a six dimensional Wiener process. It is interesting to note that the foreign stochastic discount factor has two representations given by equations (3.3.31) and (3.3.32). The latter is the one used in the extraction of the depreciation of exchange rates due to its ability to create correlations amongst the domestic and foreign economies. In a more general setting, with n currency pairs, the domestic risk factors act as global risk factors for the international economy.

Using equation (3.2.22), the dynamics of the exchange rate S_t are derived. Moreover, using Ito's lemma, the dynamics of the logarithm of the exchange rate, denoted by s_t are also retrieved. It is interesting to note that the dynamics of the exchange rate are no longer affine in the state variable.

$$\frac{dS_t}{S_t} = \left(r_t^D - r_t^F + \gamma^* \Sigma^J \sqrt{X_t^J} \Gamma_t^D (X_t^J) \right) dt + \gamma^* \Sigma^J \sqrt{X_t^J} dW_t^{\mathbb{P}} \tag{3.3.33}$$

$$ds_t = \left(r_t^D - r_t^F + \gamma^* \Sigma^J \sqrt{X_t^J} \Gamma_t^D (X_t^J) - \frac{1}{2} \gamma^* \Sigma^J X_t^J \Sigma^{J'} \gamma^{*'} \right) dt + \gamma^* \Sigma^J \sqrt{X_t^J} dW_t^{\mathbb{P}} \tag{3.3.34}$$

A clear parallelism is derived between the two equations above and equation (3.2.5), keeping in mind that $r_t^D - r_t^F$ is the short rate differential and $\gamma^* \Sigma^J \sqrt{X_t^J} dW_t^{\mathbb{P}}$ is the disturbance term.

3.3.3 Extracting currency risk premia

Having established the endogenous relationship of the variation in the logarithm of exchange rates implied by the model, the extraction of the risk premium is fairly straight-forward. Using equation (3.3.33), the drift is now composed of two components, the interest rate differentials and a second component, which englobes the risk premium, as shown below.

$$rp_t = -\gamma^* \Sigma^J \sqrt{X_t^J} \Gamma_t^D (X_t^J) \tag{3.3.35}$$

The risk premium is hence obtained by differencing the expectations of the exchange rate depreciation under the risk-neutral and physical probability measures. An equivalent representation can be derived using the dynamics of the logarithm of the exchange rate.

It is further possible to obtain a representation of the continuously compounded expected return of exchange rates by taking the expectation, under the physical measure, of equation (3.3.33).

$$\mathbb{E}^{\mathbb{P}} \left[S_t^{ret} | \mathcal{F}_t \right] = r_t^D - r_t^F + \gamma^* \Sigma^J \sqrt{X_t^J} \Gamma_t^D (X_t^J) \tag{3.3.36}$$

The expected return of exchange rates assumes rational expectations and sets the expectations of $\gamma^* \Sigma^J \sqrt{X_t^J} dW_t^{\mathbb{P}}$, under the data generating process measure, equal to zero.

3.4 Empirical analysis

This section is devoted to the empirical estimation of the bilateral AFNS with stochastic volatility on domestic and foreign zero-coupon yields. In a first instance, the characteristics of the data set are studied, sequentially, the estimation method is described and finally, all empirical results are presented.

3.4.1 Data description

The data set consists of monthly nominal yields for the United Kingdom and the United States, spanning from September 1989 to October 2008 and includes a set of nine maturities for each country, namely 3, 6, 12, 18, 24, 30, 36, 42 and 48 months. The time period includes the abandonment of the European Exchange Rate Mechanism in September 2002 by the UK as well as the beginning of the recent financial crisis caused by the burst of the housing bubble in the US market. The data set's timespan is specifically selected to coincide with the timespan of the data set included in Sarno, Schneider, and Wagner (2012), given it is the most recent paper in this strand of the literature, thus facilitating comparison of results. However, the use of short and medium term maturities is perfectly warranted as most violations of the uncovered interest rate parity are reported to occur in the short run, whilst empirical evidence supports claims of the parity holding in the long run.

The data set is kindly made available by Jonathan Wright and can be found on the following link -http://econ.jhu.edu/directory/jonathan-wright-.

Additionally, the monthly GBP/USD spot exchange rate is obtained through Datastream, and is denominated in US dollars; the same timespan applies, commencing in September 1989 and ending in October 2008.

Table 3.1 displays the descriptive statistics, namely the mean, standard deviation, skewness, kurtosis and first lag autocorrelation, of the level of interest rates for the US and the UK as well as the level of the exchange rate and logarithm of the exchange rate. The UK yields are characterized by a positive skew and excess kurtosis, especially at short and medium term maturities. All variables have a high first autocorrelation, close to unity, indicating highly persistent behaviors.

Throughout the chapter, differentials of variables are used. Panel A of Table 3.2 presents the descriptive statistics for the variables' differentials. Those are defined as the difference between domestic and foreign rates for yields at all maturities, and a first lag difference for exchange rates and the logarithm of exchange rates. Both exchange rate differentials display strong excess kurtosis. The results for the Fama regression in equation (3.2.4) are reported in Panel B of Table 3.2. The findings confirm the empirical results found in the majority of the literature, whereby the intercept of the regressions is statistically insignificant, while the slope coefficient rejects the null hypothesis of unity at all conventional significance levels. Additionally, the R squared coefficient displays a very weak goodness of fit. These results motivate the methodology of incorporating a time-varying risk premium.

It is common practice to use three factors to fit the term structure of interest rates of a single country. Additionally, following convention, the level factor affects yields at all maturities, the slope factor influences short-term yields, whilst the curvature factor is of importance for medium-term maturities. The maturities used in this empirical section span from 3 to 48 months, hence justifying the use of three factors per economy. However, before proceeding to the estimation procedure, a preliminary study is conducted to best specify the model. A principal component analysis (PCA) is used to determine how many pricing factors are required to explain the cross-sectional variation of domestic and foreign yields. The loadings for the six first principal components for the entire set of maturities are reported on Table 3.3. The PCA results validate our use of 6 latent factors given the first six components explain 99.98% of the cross-sectional yield variation.

3.4.2 Estimation procedure: Kalman filtering

The model, so far presented, naturally adopts a state space representation, with equations (3.4.37) and (3.4.38) below being the transition and measurement equations, respectively. The state-space representation is given below, in its discretized form, with $X_t^J = \left(X_t^D, X_t^F\right)'$ and $y(t,T) = \left(y(t,T)^D, y(t,T)^F\right)'$,

$$X_T^J = \left[I - exp(-\kappa^{\mathbb{P}}(T-t))\right]\theta^{\mathbb{P}} + exp(-\kappa^{\mathbb{P}}(T-t))X_t^J + \eta_t \tag{3.4.37}$$

$$y(t,T) = -\frac{A(t,T)}{T-t} - \frac{B(t,T)'}{T-t}X_t^J + \epsilon_t \tag{3.4.38}$$

where the measurement errors η_t and ϵ_t are assumed to be orthogonal and ϵ_t is i.i.d.

The bilateral AFNS model with spanned volatility theoretically ought to be estimated through an extended Kalman filter, due to the non-Gaussian nature of the state variables. However, it is widely accepted in the literature that the state variables can be treated as if they were Gaussian. Amongst many other references, Fisher and Gilles (1996) and Christensen, Lopez, and Rudebusch (2010a) have used the Kalman filter and the two first moments to approximate the probability distribution function of the non-Gaussian state variables. The estimation procedure, hence, generates quasi-maximum likelihood estimates due to the approximation applied.

The moments conditions are displayed below.

$$\mathbb{E}^{\mathbb{P}}\left[X_T^J|\mathcal{F}_t\right] = \left[I - exp(-\kappa^{\mathbb{P}}(T-t))\right]\theta^{\mathbb{P}} + exp(-\kappa^{\mathbb{P}}(T-t))X_t^J \tag{3.4.39}$$

$$\mathbb{V}^{\mathbb{P}}\left[X_T^J|\mathcal{F}_t\right] = \int_t^T exp(-\kappa^{\mathbb{P}}(T-s))\Sigma\sqrt{\mathbb{E}^{\mathbb{P}}\left[X_s^J|\mathcal{F}_t\right]}\sqrt{\mathbb{E}^{\mathbb{P}}\left[X_s^J|\mathcal{F}_t\right]}'\Sigma' exp(-\kappa^{\mathbb{P}'}(T-s))ds \tag{3.4.40}$$

The initial conditions for the Kalman filter are set to the unconditional mean and covariance matrix, given in equation (3.4.41) and (3.4.42).

$$\hat{X_0^J} = \theta^{\mathbb{P}} \tag{3.4.41}$$

$$\hat{\Sigma_0} = \int_0^\infty exp(-\kappa^{\mathbb{P}}s)\Sigma\sqrt{\theta^{\mathbb{P}}}\sqrt{\theta^{\mathbb{P}}}'\Sigma' exp(-\kappa^{\mathbb{P}'}s)ds \tag{3.4.42}$$

The conditional and unconditional covariance matrix in equation (3.4.42) are estimated using the analytical solutions provided in Jacobs and Karoui (2009).

Finally, to estimate the logarithmic exchange depreciation implied by the model, a discretization of equation (3.3.34) is used,

$$\Delta s_{t+\omega} = \left[r_t^D - r_t^F + \gamma^*\Sigma^J\sqrt{X_t^J}\Gamma_t^D(X_t^J) - \frac{1}{2}\gamma^*\Sigma^J X_t^J \Sigma^{J'}\gamma^{*'}\right]\omega + \gamma^*\Sigma^J\sqrt{X_t^J}\Delta W_{t+\omega}^{\mathbb{P}} \tag{3.4.43}$$

where $\Delta W_{t+\omega}^{\mathbb{P}}$ is approximated by the following expression.

$$\Delta W_{t+\omega}^{\mathbb{P}} \approx \left[\Sigma^J\sqrt{X_t^J}\right]^{-1}\left[\Delta X_{t+\omega}^J - \left(\kappa^{\mathbb{P}}\left(\theta^{\mathbb{P}} - X_t^J\right)\right)\omega\right] \tag{3.4.44}$$

The above expression is derived by re-arranging a discretized version of the state dynamics.

3.4.3 Empirical findings

The estimates for the six factor bilateral AFNS model with stochastic volatility are provided in Table 3.4. These results are found using solely the US and UK nominal yields with maturities 3, 6, 12, 18, 24, 30, 36, 42 and 48 months. The specification for the mean reversion matrix $\kappa^{\mathbb{P}}$ is set to a diagonal matrix. The results indicate that the first and fourth factor do display near unit root behaviors. This result is clearer when the discretized states are considered. The estimates for the unconditional mean $\theta^{\mathbb{P}}$ and diffusion matrix Σ are also displayed. The two mean reversion parameters, under the risk neutral probability measure, λ^D and λ^F are comparable to the ones found in the literature. Additionally, the log-likelihood value obtained by the quasi-maximum likelihood estimation is reasonably high compared to the ones found in this strand of the literature.

Table 3.5 elaborates on the fit of the six factor bilateral AFNS model. Both the mean and root mean squared error (RMSE) are provided. It is clearly visible that the short maturities are extremely hard to fit. The shorter the maturity of the first yield in the sample, the higher the ability of extracting the appropriate cross-section of the yields. The fact that the shortest maturity used is 3 months, could explain the difficulty in fitting the short yields appropriately. Using swap and libor rates to bootstrap short rates has the potential to improve significantly the fit of short term yields, however, this exercise is left for future research. On the other hand, the fit of yields is successful especially in medium term maturities. Attention is drawn to the appendix section, specifically appendix 3.B, which contains a robustness check using a multilateral Gaussian AFNS model. The results for the US, using the bilateral AFNS model with stochastic volatility, are comparable to those found in the multilateral Gaussian AFNS, while the fit for the UK is visibly poorer. This can be explained by the particularity of the UK term structure of interest rates which has succumbed an inversion of the yield curve.

Table 3.6 allows to compare the findings of the model's implied logarithmic exchange rate depreciations with the actual variation in log exchange rates. The means of the two variables are significantly similar, while the standard deviation of the model implied depreciation is lower, which is a major improvement to the findings under the Gaussian AFNS model. The mean and standard deviation of the implied risk premium and expected exchange rate return are also reported. The risk premium is comparable to the ones found in similar studies.

Moreover, Table 3.7 indicates that the correlation found between the actual and estimated exchange rate depreciations is equal to 16.03%. This finding might be misinterpreted as a poor fit, however it is important to note that comparable bilateral studies have found correlations well below 10% and on some occasions correlations slightly below 0%, thus indicating an improvement in the fit (see Sarno, Schneider, and Wagner (2012)). Additionally, the implied risk premium does validate the two Fama conditions, hence providing empirical support to Fama (1984)'s claim and indicating that the model does offer a correction to the uncovered interest rate parity by incorporating a time-dependent risk premium.

In addition, Figure 3.1 displays the actual and estimated exchange rate depreciations' time series. It is noticeable that the mean is successfully captured, and the variance is closely matched. It is also clear that interest rate differentials are not the only drivers of exchange rate changes. The consideration of unspanned volatility and a macro-finance approach to the model are two interesting extensions of the current study which are left for future investigation.

Figure 3.2 plots the estimated expected exchange rate return and exchange rate risk premia. In recent literature, claims increasingly stipulate that currency risk premia are countercyclical. This graph also supports theories of countercyclicality, the risk premium thus tracking expected returns. Assuming the foreign country has a lower interest rate than the domestic country, the risk premium tends to be positive given an appreciation of the domestic currency is denoted by a decrease of the exchange rate. Vice-versa, a foreign country with historically higher interest rates than the domestic country will mostly display a negative currency risk premium in order to reflect the appreciation of the foreign currency which is coupled with an increase of the exchange rate. The more the domestic country is considered risky vis-a-vis the foreign country, the larger the magnitude of the risk premium. Hence, the higher the liquidity constraints and economic uncertainty, the more likely the risk premium is to increase, thus reinforcing arguments of flight-to-liquidity and flight-to-quality. Moreover, the expected return on the pound fluctuates between -1.07% and 8.75%; whilst its mean and standard deviation are equal to 3.06% and 2.08%, respectively. The estimation provides similar results with Graveline (2006)'s findings using options prices. In particular, Graveline (2006) did a comparative study between two models, with and without options. He concluded that models that do not use option prices usually display a lot of variability. The findings in this chapter show that option prices are not necessary to retrieve expected return on currencies that have a low variance.

Figure 3.3 provides a graphical representation of the contribution of each of the six risk factors to the risk premium. Interestingly, this figure corroborates Graveline

(2006)'s results by displaying the same low variances in risk premium contributions for risk factors that have a greater impact on exchange rates. Hence, the domestic and foreign curvature factors appear to be the key drivers of both exchange rate depreciations and currency risk premia, whilst they also appear to be the most persistent factors.

Finally, Figure 3.4 plots the contribution of a carry trade risk factor to the currency risk premium. The carry trade factor, in this case, is represented by the short interest rate differential. Using equation (3.3.25) and (3.3.26), the carry trade risk factor is easily derived by summing the fourth and fifth risk factors (ie. level and slope of the foreign economy, which in this case is the UK) and subtracting the first and second risk factors (ie. level and slope of the domestic economy, in this case the US). The contribution of the carry trade factor to currency risk premia, on average, is equal to -1.60%, whilst the integrity of the currency risk premium is on average equal to -5.66%. It is clear that the carry trade factor is a driver of currency risk premia, as demonstrated by Lustig, Roussanov, and Verdelhan (2010). However, the carry trade factor is found not to contain all the information of currency risk premia in its integrity, hence rendering the two curvature factors particularly important. Moreover, the carry trade risk factor's contribution to currency risk premia mainly contains exchange rate risk in short maturities, whilst it is contaminated by an additional component for interest rate risk in long maturities. A recent study by Lustig, Stathopoulos, and Verdelhan (2013) indicates that carry trade risk premia are indicative of temporary shocks and hence their term structure tends to be downward sloping. This finding is confirmed by the persistence of curvature factors which do not feature within carry trade factors. On the other hand, currency risk premia at long horizons seem to be driven by the permanent component of stochastic discount factors.

3.5 Conclusion

In conclusion, in this chapter, a bilateral AFNS model with stochastic volatility for the joint pricing of the term structure of interest rates for both the domestic and foreign countries that is further able to derive exchange rate variations is developed. The model proposed benefits from the Nelson-Siegel factor loadings yielding a robust and tractable estimation procedure. The no-arbitrage restrictions enhance the theoretical grounds whilst simultaneously allowing the extraction of currency risk premia.

This chapter compares the effect of the different assumptions set on the diffusion of the process (ie. Gaussian or with stochastic volatility) on the properties adopted by the estimates of the yields, exchange rate variations and currency risk premia. To summarize, the use of a stochastic volatility version rather than a Gaussian take of the AFNS model comes with the detriment of having an inferior fit for the yields. However, the very inclusion of stochastic volatility endows the model with the capacity to capture to some extent the volatility of exchange rate depreciations and successfully derive a risk premium that respects the two Fama conditions. The model's implied risk premium provides, thus, an adaptation of the uncovered interest rate parity that alleviates the recorded puzzle in the literature whilst solely assuming a departure from risk neutrality. On the other hand, a Gaussian AFNS model allows a better fit for the yields, whilst the variance of exchange rate fluctuations is not fully captured. It is interesting to note that the Gaussian AFNS is easily extended to a multi-currency model which not only benefits from an elegant estimation procedure, but also takes advantage of the fact that currency portfolios tend to be more predictable than individual exchange rates.

Finally, the extension of the stochastic volatility AFNS model to a multi-currency framework is left for future research.

Appendix

3.A Appendix A: Bilateral AFNS model with stochastic volatility

Table 3.1: Descriptive statistics of the level of interest rates and exchange rates

Maturity	Mean	Standard Deviation	Skewness	Kurtosis	Autocorrelation
Panel A: Domestic Country -United States-					
3 months	0.0405	0.0178	-0.1129	2.4204	0.9716
6 months	0.0434	0.0184	-0.1388	2.3578	0.9761
12 months	0.0449	0.0183	-0.1729	2.3694	0.9735
18 months	0.0464	0.0179	-0.1604	2.4065	0.9710
24 months	0.0476	0.0174	-0.1286	2.4320	0.9689
30 months	0.0488	0.0170	-0.0874	2.4381	0.9675
36 months	0.0498	0.0165	-0.0415	2.4270	0.9667
42 months	0.0508	0.0162	0.0061	2.4036	0.9664
48 months	0.0517	0.0158	0.0533	2.3731	0.9666
Panel B: Foreign Country -United Kingdom-					
Maturity	Mean	Standard Deviation	Skewness	Kurtosis	Autocorrelation
3 months	0.0666	0.0289	1.7187	5.1836	0.9790
6 months	0.0632	0.0270	1.7349	5.3003	0.9759
12 months	0.0624	0.0244	1.5979	5.0414	0.9711
18 months	0.0626	0.0233	1.4388	4.5814	0.9708
24 months	0.0630	0.0226	1.2987	4.1433	0.9717
30 months	0.0634	0.0222	1.1902	3.7899	0.9730
36 months	0.0637	0.0219	1.1094	3.5196	0.9745
42 months	0.0640	0.0217	1.0496	3.3161	0.9758
48 months	0.0642	0.0216	1.0046	3.1617	0.9771
Panel C: Exchange rate and logarithm of the exchange rate					
Maturity	Mean	Standard Deviation	Skewness	Kurtosis	Autocorrelation
S_t	1.6778	0.1685	0.5156	2.2601	0.9602
s_t	0.5126	0.0987	0.3673	2.1500	0.9608

NOTE: The descriptive statistics for the level of domestic and foreign yields at all the maturity set and the exchange rate and logarithmic exchange rate are given. The data comprises of monthly nominal zero coupon bond yields for the US and the UK and the GBP/USD exchange rate denominated in US dollars, from September 1989 to October 2008.

Table 3.2: Stylized facts of interest rates and exchange rates differentials

Panel A: Descriptive Statistics on variable differentials					
Variable	Mean	Standard Deviation	Skewness	Kurtosis	Autocorrelation
$y_3^D - y_3^F$	-0.0260	0.0214	-1.0106	3.0320	0.9747
$y_6^D - y_6^F$	-0.0198	0.0194	-0.9846	3.1179	0.9796
$y_{12}^D - y_{12}^F$	-0.0175	0.0165	-0.8513	2.9694	0.9726
$y_{18}^D - y_{18}^F$	-0.0163	0.0148	-0.8185	3.0088	0.9695
$y_{24}^D - y_{24}^F$	-0.0154	0.0135	-0.7860	3.0647	0.9668
$y_{30}^D - y_{30}^F$	-0.0146	0.0126	-0.7493	3.1037	0.9644
$y_{36}^D - y_{36}^F$	-0.0139	0.0118	-0.7096	3.1137	0.9623
$y_{42}^D - y_{42}^F$	-0.0132	0.0113	-0.6698	3.0963	0.9606
$y_{48}^D - y_{48}^F$	-0.0134	0.0113	-0.5359	2.9016	0.9627
$S_t - S_{t-1}$	0.0009	0.0466	-1.3608	8.5506	0.1267
$s_t - s_{t-1}$	0.0005	0.0270	-1.1966	7.6103	0.1219

Panel B: Fama Regression					
Variable		α	β	$t[\beta = 1]$	R^2
		-0.0003	-0.0313	-12.2090	0.0006
		(0.0028)	(0.0845)		

NOTE: The descriptive statistics for the differentials of domestic and foreign yields at all the maturity set and the exchange rate and logarithmic exchange rate are given in Panel A. The results of the Fama regression are provided in Panel B. The numbers in parenthesis are the standard errors of the estimates. The data comprises of monthly nominal zero coupon bond yields for the US and the UK and the GBP/USD exchange rate denominated in US dollars, from September 1989 to October 2008.

Table 3.3: First three principal components in nominal yields

Maturity	First PC	Second PC	Third PC	Fourth PC	Fifth PC	Sixth PC
y_3^D	0.1801	0.3175	0.2843	0.4628	0.0661	0.6147
y_6^D	0.1862	0.3433	0.2584	0.3217	0.1315	-0.2112
y_{12}^D	0.1895	0.3340	0.1353	0.1071	0.0349	-0.3551
y_{18}^D	0.1895	0.3101	0.0381	-0.0449	-0.0320	-0.2756
y_{24}^D	0.1880	0.2817	-0.0379	-0.1512	-0.0599	-0.1437
y_{30}^D	0.1858	0.2527	-0.0979	-0.2266	-0.0584	-0.0110
y_{36}^D	0.1833	0.2249	-0.1458	-0.2810	-0.0379	0.1066
y_{42}^D	0.1806	0.1988	-0.1844	-0.3212	-0.0059	0.2053
y_{48}^D	0.1778	0.1747	-0.2159	-0.3511	0.0323	0.2860
y_3^F	0.3251	-0.3049	0.4696	-0.3411	0.5749	-0.1615
y_6^F	0.3106	-0.2494	0.3475	-0.1346	-0.1926	0.3827
y_{12}^F	0.2863	-0.1885	0.1653	0.0071	-0.4419	-0.0901
y_{18}^F	0.2760	-0.1594	0.0107	0.0729	-0.3792	-0.1258
y_{24}^F	0.2679	-0.1442	-0.1083	0.1183	-0.2355	-0.1100
y_{30}^F	0.2617	-0.1370	-0.1977	0.1507	-0.0735	-0.0743
y_{36}^F	0.2569	-0.1342	-0.2656	0.1750	0.0854	-0.0297
y_{42}^F	0.2533	-0.1339	-0.3178	0.1943	0.2339	0.0175
y_{48}^F	0.2504	-0.1350	-0.3588	0.2104	0.3702	0.0637
% explained	87.83	97.46	99.61	99.83	99.94	99.98

NOTE: The loadings of the yields of the set of maturities on the first six principal components are given. The percentage of all bond yields' cross-sectional variation accounted for by each component is displayed on the final row. The data comprises of monthly zero coupon bonds from September 1989 to October 2008 for the United States and the United Kingdom.

Table 3.4: 6 factor BAFNS estimates for domestic and foreign rates

$\kappa^{\mathbb{P}}_{i,i}$	θ^{P}	$\Sigma_{i,i}$
0.1000	0.0099	0.0163
(0.000032)	(0.000034)	(0.000120)
0.1996	0.0243	0.0583
(0.000032)	(0.000058)	(0.001229)
0.4997	0.0328	0.0325
(0.000032)	(0.000077)	(0.000216)
0.0991	0.0100	0.0319
(0.000048)	(0.000068)	(0.008248)
0.1985	0.0396	0.0841
(0.000077)	(0.000901)	(0.030020)
0.4995	0.0264	0.0442
(0.000041)	(0.000625)	(0.024991)

NOTE: The estimated parameters of the $\kappa^{\mathbb{P}}$ matrix, $\theta^{\mathbb{P}}$ vector, and diagonal diffusion matrix $\Sigma_{i,i}$ are given for the six-factor bilateral AFNS model for domestic and foreign yields. The estimated value of λ^{D} is 0.4974 with standard deviation of 0.000045 and λ^{F} is 0.4965 with standard deviation of 0.000156. The numbers in parentheses are the standard deviations of the estimated parameters. The log likelihood is equal to 10290.1208.

Table 3.5: Measures of fit for the bilateral AFNS model

Maturity in months	Mean(in bp)	RMSE(in bp)
y_3^D	17.1459	36.9274
y_6^D	-1.9856	16.1301
y_{12}^D	-0.7299	5.5918
y_{18}^D	-0.2130	1.6230
y_{24}^D	0.0060	0.0382
y_{30}^D	-0.0479	1.0435
y_{36}^D	-0.4106	2.3444
y_{42}^D	-1.0902	3.9866
y_{48}^D	-2.0643	5.8888
y_3^F	-14.7128	47.0974
y_6^F	9.6541	18.0715
y_{12}^F	6.5452	8.6486
y_{18}^F	0.6465	4.6115
y_{24}^F	-1.1010	1.3787
y_{30}^F	0.9950	3.3367
y_{36}^F	6.0909	6.8855
y_{42}^F	13.3451	9.7477
y_{48}^F	22.0650	16.1391

NOTE: The mean and RMSE of fitted errors of the six-factor bilateral AFNS model with stochastic volatility for domestic and foreign yields are given. All values are measured in basis points. The nominal yields span from September 1989 to October 2008.

Table 3.6: Model implied findings

Variable	Δs_{t+1}	$\Delta \hat{s}_{t+1}$	$\hat{rp}_t$	$\mathbb{E}^{\mathbb{P}}\left[\hat{S_t^{ret}}\vert\mathcal{F}_t\right]$
Mean	0.0005	0.0006	-0.0566	0.0306
Standard deviation	0.0270	0.0172	0.0224	0.0203

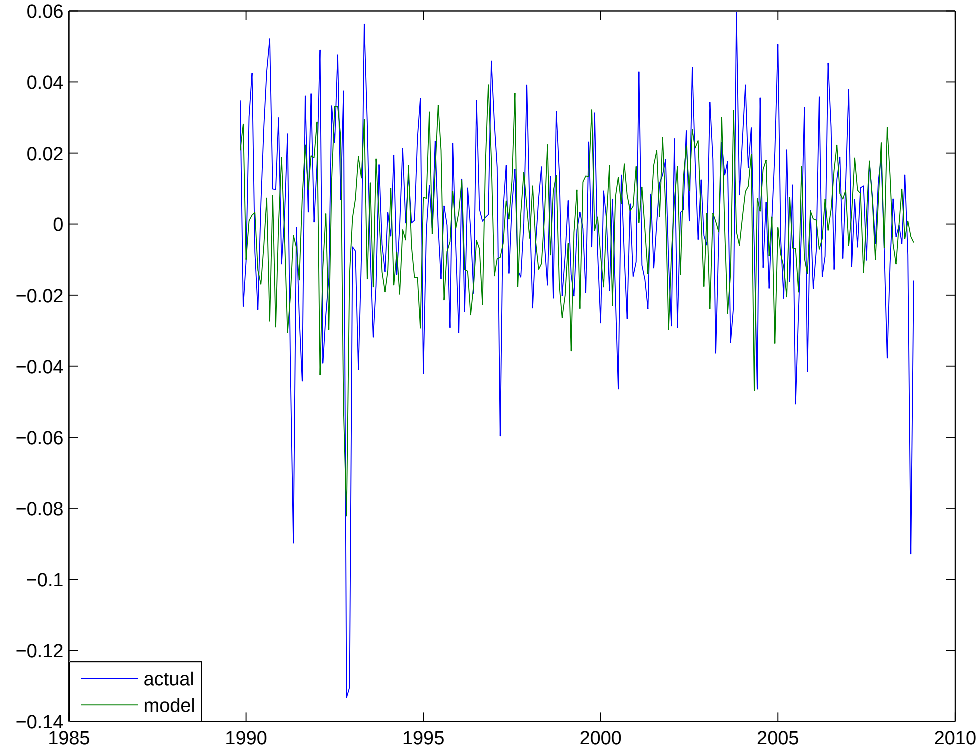
NOTE: The mean and standard deviation of the implied exchange rate depreciation, risk premium and exchange rate expected return are provided. The actual depreciation exchange rate mean and standard deviation are also included to facilitate the comparison with the estimates. The exchange rates span from September 1989 to October 2008.

Table 3.7: Analysis of the model implied exchange rate depreciation and risk premium

Panel A: model implied exchange rate depreciation	
$corr(\Delta s_{t+1}, \Delta \hat{s_{t+1}})$	0.1603
Panel B: Fama conditions	
$VR = \frac{r\hat{p}_t}{\Delta \hat{s_{t+1}}}$	1.7017
$corr(\Delta \hat{s_{t+1}}, r\hat{p}_t)$	-0.0718

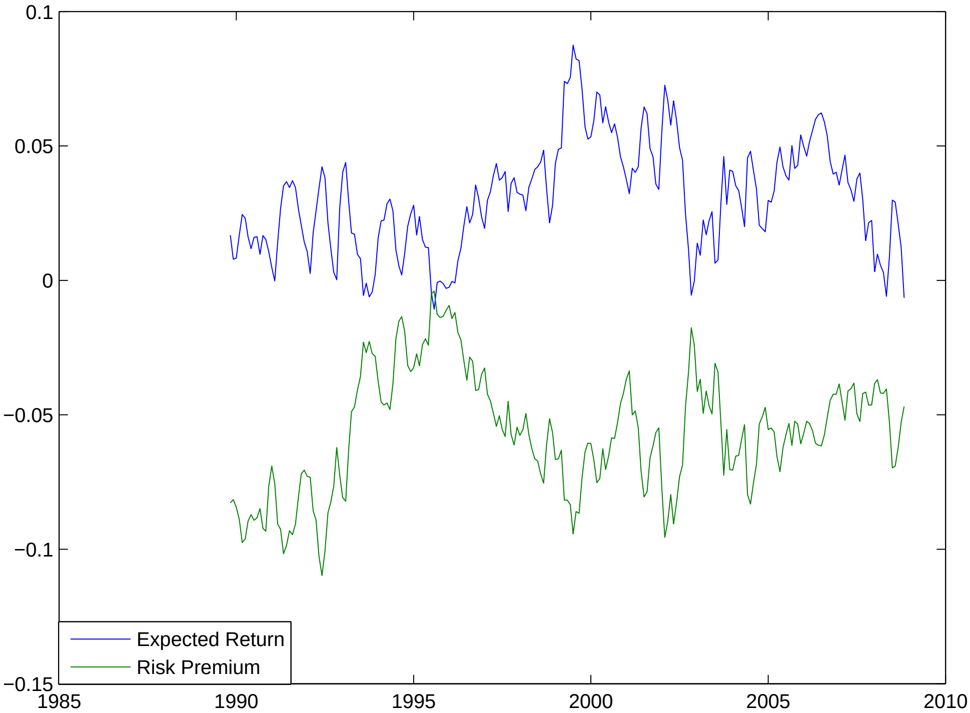
NOTE: Panel A displays the correlation between the actual and model implied exchange rate depreciations. In panel B, the variance ratio of the implied risk premium and actual exchange rate depreciations are provided. The correlation of the implied risk premium and actual exchange rate depreciations are also displayed. If the variance ratio figure is above 1 and the correlation is below 0 then the Fama conditions are verified. The exchange rates span from September 1989 to October 2008.

Figure 3.1: Actual and model implied log exchange rate depreciations



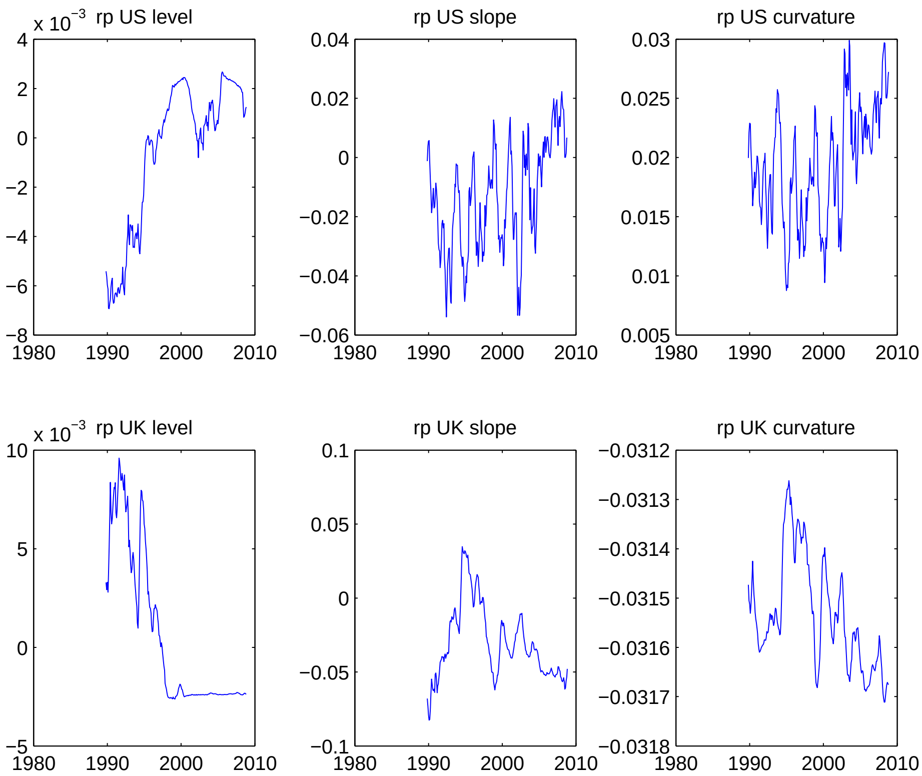
NOTE: Comparison of the actual and model implied log GBP/USD exchange depreciations across time. The exchange rates span from September 1989 to October 2008.

Figure 3.2: Expected exchange rate return and exchange rate risk premium



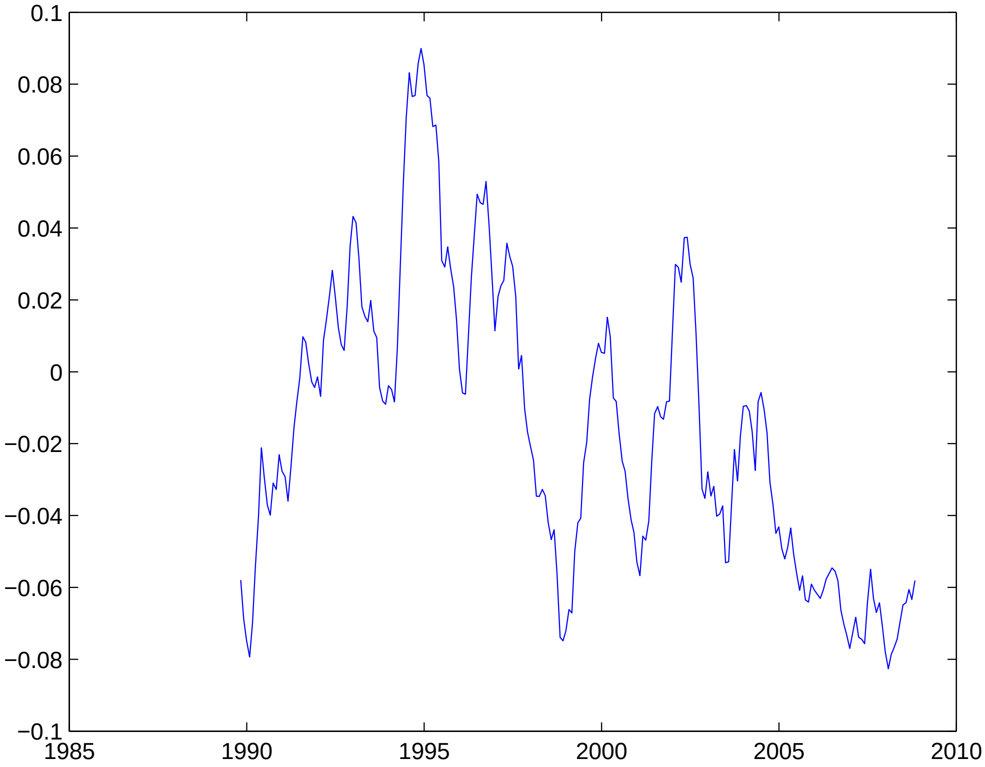
NOTE: Comparison of the expected exchange rate return and exchange rate risk premium across time, with exchange rates spanning from September 1989 to October 2008.

Figure 3.3: Contribution of risk factors to risk premium



NOTE: Comparison of the contribution of each risk factor to the risk premium. The six risk factors considered are namely the domestic and foreign level, slope and curvature factors.

Figure 3.4: Contribution of the carry trade factor to risk premium



NOTE: Contribution of a carry trade risk factor to the risk premium. The carry trade factor is computed by summing the foreign-UK level and slope factors and deducting the domestic-US level and slope factors.

3.B Appendix B: Multilateral Gaussian AFNS model

This appendix segment is dedicated to conducting a robustness check with a different specification for the model. The empirical exercise, for this section, consists of an analysis of the Gaussian AFNS model extended to a multi-currency setting. Specifically, the United States is preserved as the domestic country and six more countries, including the United Kingdom, are treated as foreign countries. The model investigated includes twenty one latent factors; three factors for each country in the sample.

The data set consists of monthly nominal yields for the United States, the United Kingdom, Australia, Canada, Switzerland, Japan and Sweden spanning from January 1995 to May 2009 and includes a set of six maturities for each country, namely 3, 6, 12, 24, 36 and 48 months. The yields are available in Jonathan Wright's homepage.

Moreover, the monthly GBP/USD, AUD/USD, CAD/USD, CHF/USD, JPY/USD and SEK/USD spot exchange rates are obtained through Datastream, using a denomination in US dollars. The same timespan applies, commencing in January 1995 and ending in May 2009. The data set is comprised of a balanced panel and is truncated vis a vis to the empirical analysis' data set due to unavailability of data.

It is important to note that the model is Gaussian, which allows the uncontested use of the Kalman filter to obtain the maximum likelihood estimates.

Table 3.8 reports the fit of the yields for all seven countries across the entire set of maturities. The mean and Root Mean Squared Error (RMSE) indicate that with the exception of the three month yield for the US and the UK, all remaining yields are strikingly well captured.

Figures 3.5 to 3.10 display the comparison between the actual and the model implied logarithmic exchange rate depreciations for all the six pairs of currencies. The mean of the exchange rate depreciations seems to be appropriately captured, however their variance is clearly underestimated. The correlation between the two time series above mentioned tend to be significantly lower than in the setting of the bilateral AFNS model.

As a final note, the fit of the yields is superior under the Gaussian multilateral AFNS rather than the bilateral AFNS with stochastic volatility. However, there seems to be an obvious trade-off between fitting yields and capturing the exchange rate depreciation properties. As Sarno, Schneider, and Wagner (2012) suggest, selecting between two extensions of a given model, in this case between the bilateral AFNS with stochastic volatility and the multilateral Gaussian AFNS model, will depend entirely on the purpose of the exercise, hence by whether the objective of the analysis is to fit yields or exchange rates.

Table 3.8: Measures of fit for the multilateral AFNS model

Maturity in months	Mean(in bp)	RMSE(in bp)
Panel A: Fit for domestic yields - US		
y_3^D	23.2560	35.2626
y_6^D	0.5516	7.3573
y_{12}^D	-0.0003	0.0086
y_{24}^D	-0.3060	1.1311
y_{36}^D	0.0000	0.0000
y_{48}^D	-0.9072	2.1180
Panel B: Fit for foreign yields - UK		
y_3^F	-30.9604	46.0448
y_6^F	-3.2131	8.6486
y_{12}^F	0.1757	0.6775
y_{24}^F	0.0587	0.3522
y_{36}^F	-0.0001	0.0003
y_{48}^F	0.3937	0.9682
Panel C: Fit for foreign yields - Australia		
y_3^F	-0.1564	5.3435
y_6^F	0.0000	0.0000
y_{12}^F	-0.0984	2.2626
y_{24}^F	0.0000	0.0000
y_{36}^F	0.0000	0.0000
y_{48}^F	-0.5561	2.2206
Panel D: Fit for foreign yields - Canada		
y_3^F	-1.2376	7.6264
y_6^F	0.0000	0.0000
y_{12}^F	0.0033	3.1487
y_{24}^F	-0.0643	0.1456
y_{36}^F	0.5218	0.8229
y_{48}^F	-1.2093	2.1356

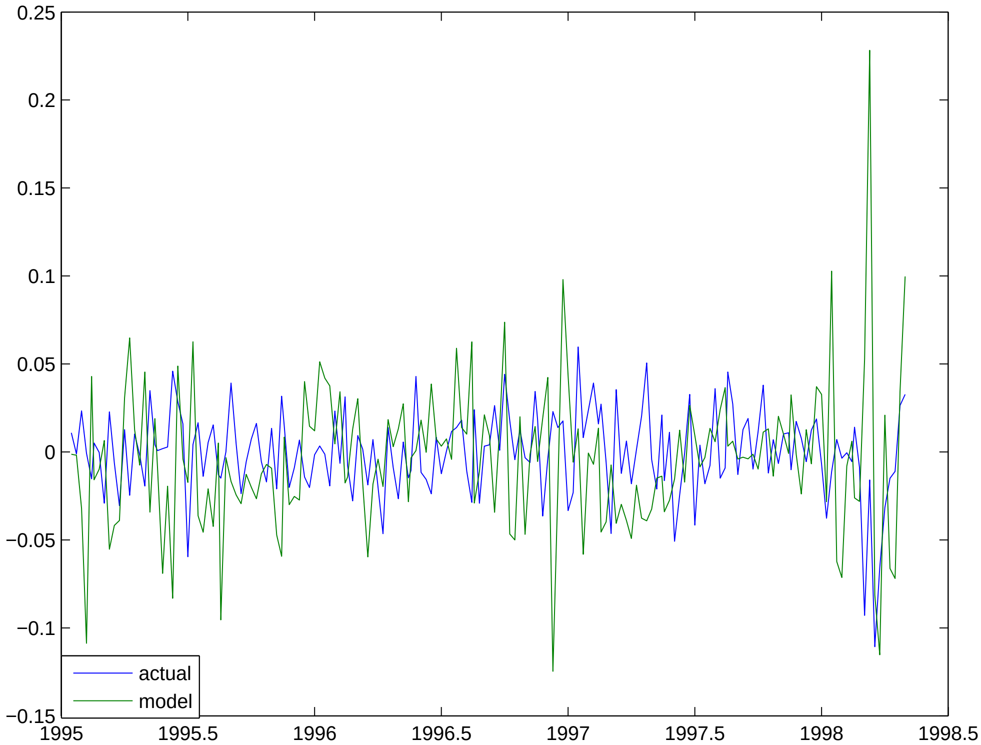
NOTE: The mean and RMSE of fitted errors of the multilateral Gaussian AFNS model for domestic and foreign yields are given. Panel A displays the fit for the US (domestic) yields, panel B for the UK (foreign), panel C for Australia (foreign), panel D for Canada (foreign), panel E for Switzerland (foreign), panel F for Japan (foreign) and panel G for Sweden (foreign). All values are measured in basis points. The nominal yields span from from September 1989 to October 2008.

Table 3.8 Continued: Measures of fit for the multilateral AFNS model

Maturity in months	Mean(in bp)	RMSE(in bp)
Panel E: Fit for foreign yields - Switzerland		
y_3^F	6.6244	13.1451
y_6^F	0.0531	0.6131
y_{12}^F	-2.8266	5.4623
y_{24}^F	-0.7319	2.5425
y_{36}^F	0.1603	0.2882
y_{48}^F	-0.8433	1.9154
Panel F: Fit for foreign yields - Japan		
y_3^F	0.2589	0.5446
y_6^F	0.0000	0.0000
y_{12}^F	0.0000	0.0001
y_{24}^F	0.4940	0.8120
y_{36}^F	0.0000	0.0000
y_{48}^F	-2.1471	3.0299
Panel G: Fit for foreign yields - Sweden		
y_3^F	-0.9590	5.9960
y_6^F	0.0003	0.0006
y_{12}^F	-0.4138	2.1575
y_{24}^F	-0.0026	0.0082
y_{36}^F	0.2150	0.4063
y_{48}^F	-0.9196	1.8548

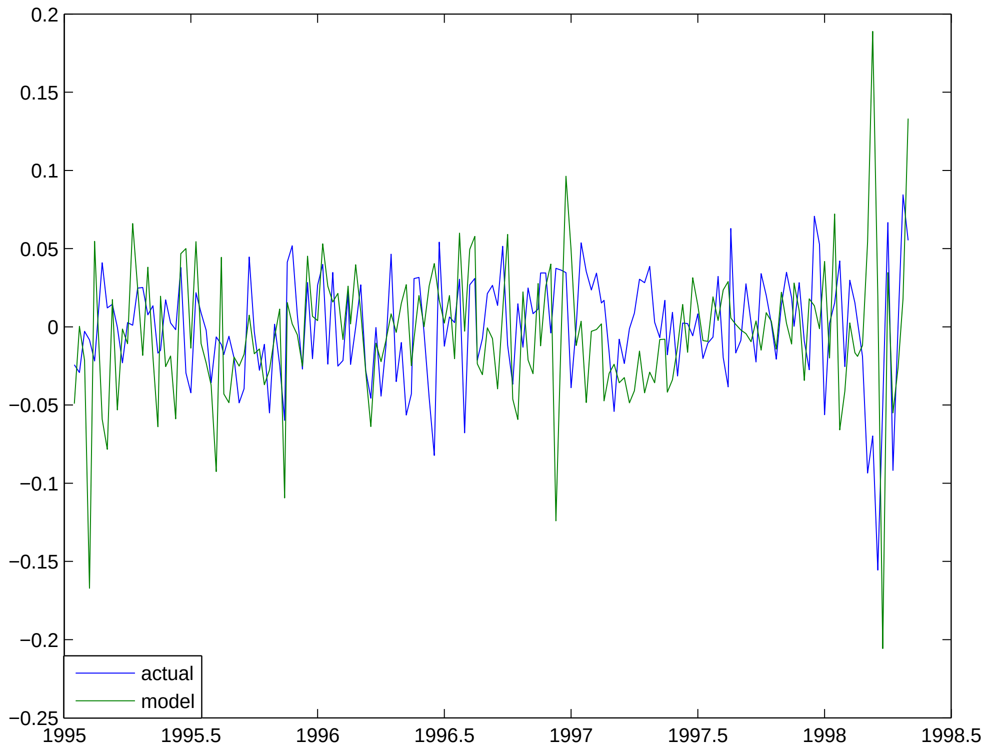
NOTE: The mean and RMSE of fitted errors of the multilateral Gaussian AFNS model for domestic and foreign yields are given. Panel A displays the fit for the US (domestic) yields, panel B for the UK (foreign), panel C for Australia (foreign), panel D for Canada (foreign), panel E for Switzerland (foreign), panel F for Japan (foreign) and panel G for Sweden (foreign). All values are measured in basis points. The nominal yields span from from September 1989 to October 2008.

Figure 3.5: Actual and model implied log exchange rate depreciations for the GBP/USD



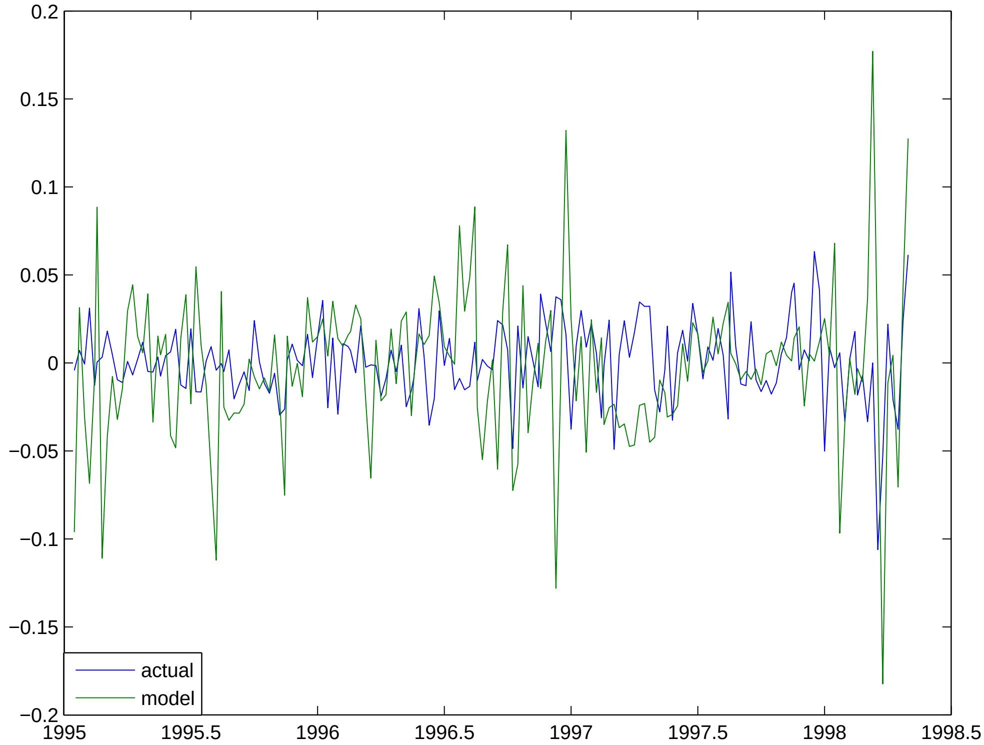
NOTE: Comparison of the actual and model implied log GBP/USD exchange depreciations across time.

Figure 3.6: Actual and model implied log exchange rate depreciations for the AUD/USD



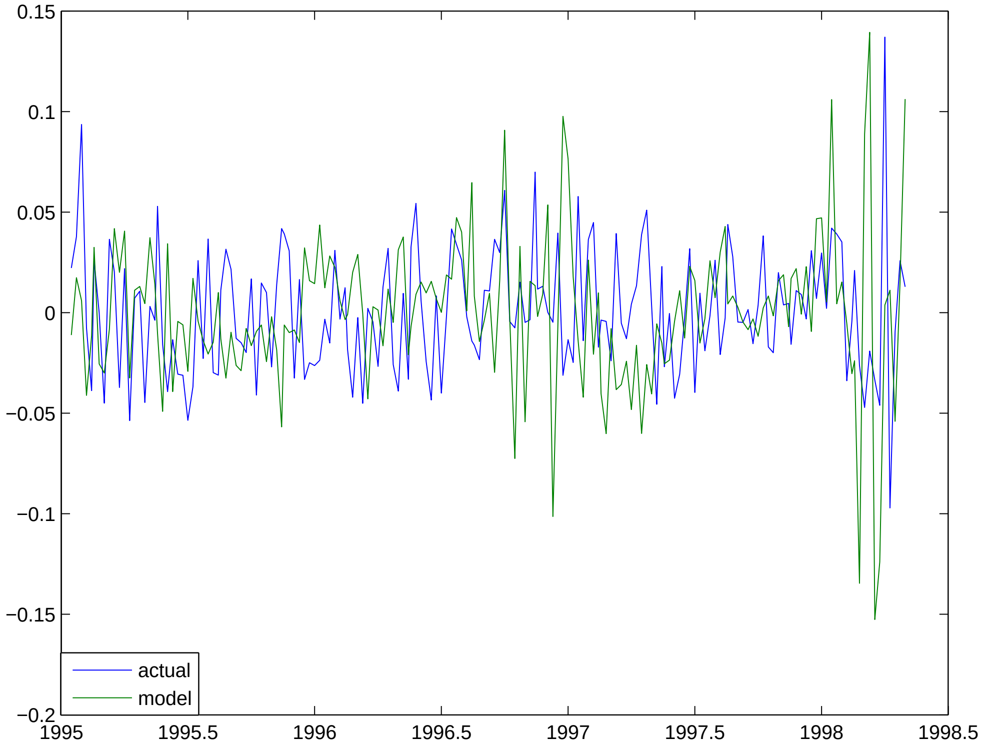
NOTE: Comparison of the actual and model implied log AUD/USD exchange depreciations across time.

Figure 3.7: Actual and model implied log exchange rate depreciations for the CAD/USD



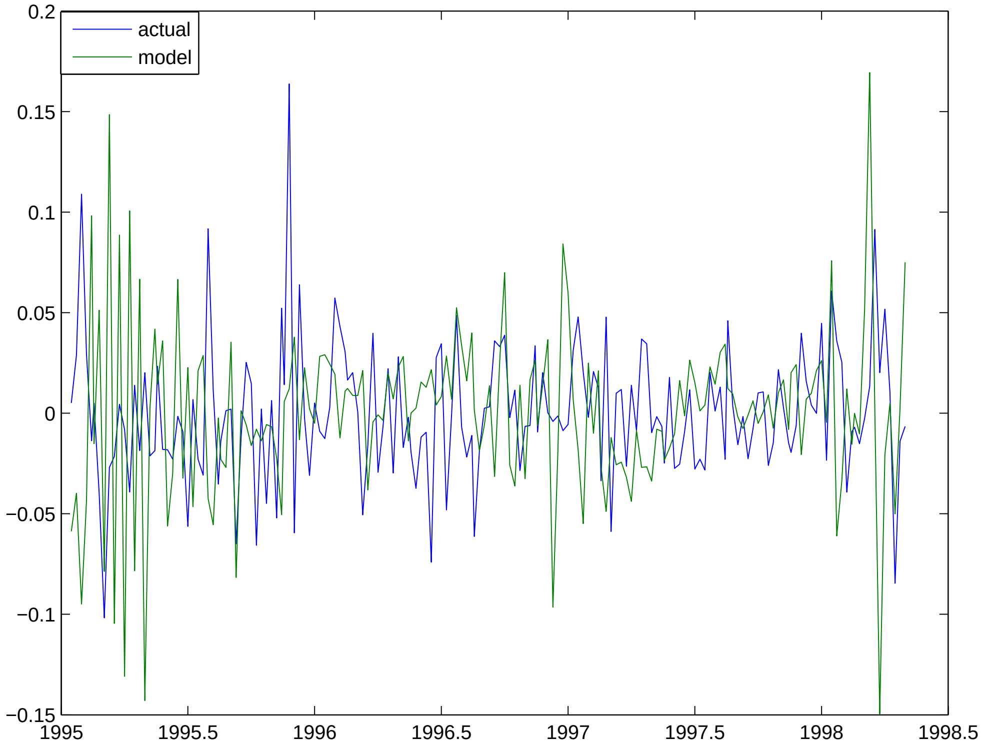
NOTE: Comparison of the actual and model implied log CAD/USD exchange depreciations across time.

Figure 3.8: Actual and model implied log exchange rate depreciations for the CHF/USD



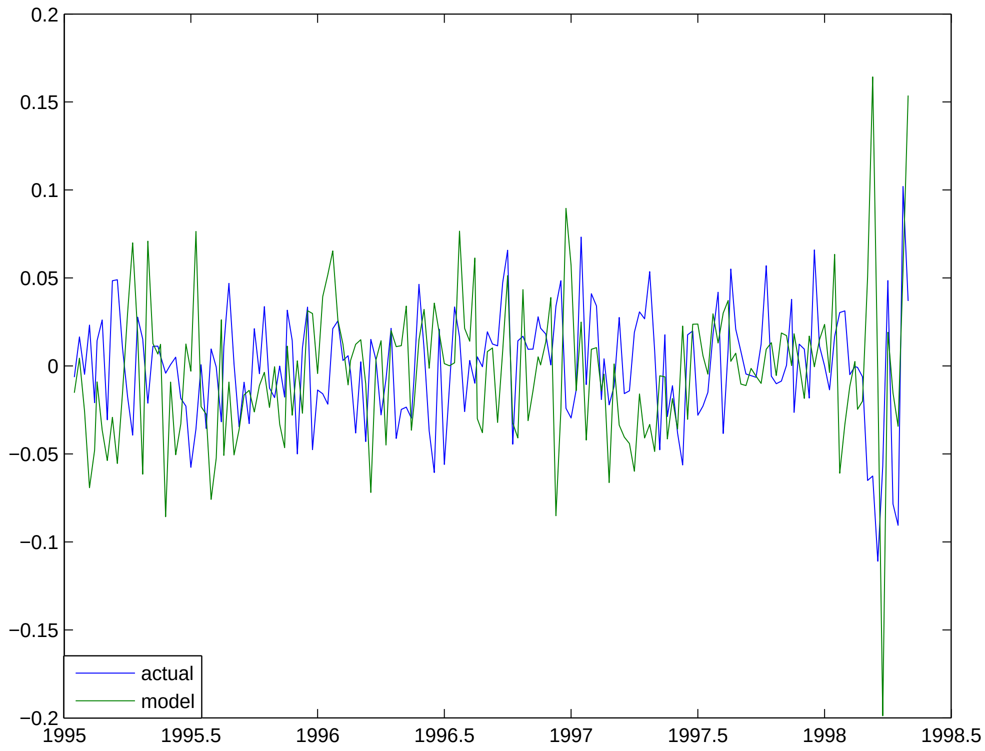
NOTE: Comparison of the actual and model implied log CHF/USD exchange depreciations across time.

Figure 3.9: Actual and model implied log exchange rate depreciations for the JPY/USD



NOTE: Comparison of the actual and model implied log JPY/USD exchange depreciations across time.

Figure 3.10: Actual and model implied log exchange rate depreciations for the SEK/USD



NOTE: Comparison of the actual and model implied log SEK/USD exchange depreciations across time.

Chapter 4

The UK term structure at the zero lower bound

4.1 Introduction

In March 2009, the Monetary Policy Committee announced a cut of the policy rate to 0.5%, from a level of 4.5% six months earlier. As a response to the recent financial crisis, that not only involved the UK but further extended to a global phenomenon, this decision was accompanied by an economic stimulus amounting to a running total of £375bn. Other countries, including the US and more recently Japan, also followed unconventional monetary policy strategies. Since 2009 short nominal yields in the UK gilt and Treasury bill markets reached historically low levels. Negative nominal yields remain a possibility in periods of crisis, when bondholders require an insurance to safe-guard their investments, although these measures ought to be temporary.

In this chapter, we study the consequences that pushing the short term interest rates near the zero lower bound have on agents expectations, and therefore on term premia and inflation premia. With short nominal interest rates close to zero, the yield curve is anchored at the short end, agents expectations reflect the belief that the policy rate would not be further reduced, and the volatility of short term rates falls.

These considerations lead to question the use of standard affine Gaussian dynamic term structure models as the expectations implied by these models might be violating the inherent non-negativity assumption of nominal yields. As a result, these models can generate, on the one hand, implausible nominal risk premia at short maturities (as seen in Kim and Singleton (2012)), and on the other hand, imprecise future long term expected inflation projections. Thus, it becomes of crucial importance to refine these models and equip them with the ability to restrain nominal yields from being negative, whilst also reflecting a clear relationship between the level and volatility of yields. The model should not restrict the behavior of real yields, which remain free to take any sign.

In this chapter we address this issue by using the model recently proposed by Christensen and Rudebusch (2013), which builds on Black (1995)'s and Krippner (2012)'s shadow rate framework. The model is a shadow-rate Arbitrage-Free Nelson Siegel (AFNS) term structure model which imposes the non-negativity of interest rates. Unlike Kim and Singleton (2012)'s model, this particular representation has the benefit of being capable of encompassing more than two factors, concurrently preserving the simplicity of standard Gaussian models. Additionally, the factor loadings, borrowed from Nelson and Siegel (1987)'s model, facilitate the tractability of the no-arbitrage model and offer a reasonable interpretation of level, slope and curvature to the factors. As far as future inflation projections are concerned, the benefits of using a no-arbitrage model come into play by enabling the disentanglement of inflation risk premia from Break-Even Inflation (BEI) rates, thus providing estimates of pure inflation expectations.

In recent years, there have been a considerable number of papers examining inflation expectations and risk premia using affine models. Amongst them, Chernov and Mueller (2012) develop a no-arbitrage affine model that uses survey based forecasts in addition to US nominal Treasury yields. Papers that similarly utilize survey expectations and the use of macroeconomic variables include Chun (2011) and Grishchenko and Huang (2012). D'Amico, Kim, and Wei (2010) use a no-arbitrage affine model on a combination of US nominal and real rates combined with survey based inflation data and forecasts. Additional studies on BEI rates are Chen, Liu, and Cheng (2005) and Hordahl and Tristani (2010). However, limited literature is available for UK yields, despite the fact that the UK linker market is one of the most liquid ones and the UK Debt Management Office - an Executive Agency of HM Treasury - is committed to maintain this liquidity with regular issuance of inflation-linked bonds. Joyce, Lildholdt, and Sorensen (2010) study UK inflation using affine models. Specifically, they obtain inflation projections up to 2009, thus before unconventional monetary policies were put in place. The paper most affiliated with our study, is the study by Christensen, Lopez, and Rudebusch (2010b). They use a joint AFNS model for nominal and real yields to extract US inflation expectations. Our study mainly differs in our use of the property of the zero lower bound in the fitting of nominal yields as well as our choice in the use of UK data. Unlike the four-factor model by Christensen, Lopez, and Rudebusch (2010b), we use a five-factor model to jointly fit the term structure of nominal and real yields, due to the peculiarity of the shape of the UK yield curve.

Inflation can be measured through indices. Since 2004, the UK's main inflation index is the CPI (Consumer Price Index) and its target is set at 2%. Before 2004, the main reference index for monetary policy objectives was the RPIX (Retail Price Index excluding mortgage interest), introduced in 1992, for the first time. At the time, the targeting was expected to anchor long term inflation expectations and further promote financial stability. UK inflation-linked government bonds ('gilts')[1] have always been indexed to the Retail Price Index (RPI). These bonds set a pre-agreed coupon which adjusts through time, co-moving with the RPI. Furthermore, the principal payment is also aligned with changes in the RPI. Hence these instruments combined with conventional bonds, can be used as a means to gain insight into inflation expectations, defined as BEI. BEI rates are often used in

[1] Refer to http://www.dmo.gov.uk/index.aspx?page=Research/research.

lieu of surveys and forecasts, however, their use is far more intricate as they contain risk premia for inflation uncertainty[2] that contaminate the BEI rates as measures of pure inflation expectations.

From a debt management policy perspective, risk premia are crucial as they determine the debt servicing cost (interest) of an issuer. Inflation risk premia, in particular, determine the relative cost effectiveness of issuing a conventional bond as opposed to an inflation-linked bond. For a given bond maturity, the risk premium represents the additional expected cost to the issuer over that bond's life relative to a short maturity (6 months) issuance strategy that rolls over. As premia tend to grow with maturity, it can also be perceived as the cost of buying protection against refinancing risk.

Proceeding to the structure of the chapter, in the second section we estimate individual models, particularly, an AFNS model enforcing non-negativity for nominal yields and a standard AFNS model for real yields. In the third section we estimate a joint term structure model of nominal and real curves using an AFNS model that restricts solely nominal yields in a positive domain. No-arbitrage conditions allow us to further decompose BEI rates into two components, inflation risk premia and expectations, which can be found in section four. We provide concluding remarks in the fifth and final section.

[2]Assuming a good liquidity in both conventional and inflation-linked markets.

4.2 Empirical affine models for nominal and real yields

The two individual estimations of the Gaussian affine models on nominal and real yields that follow, are essential in the construction of the joint model. More particularly, the choice of the number and selection of the factors highly relies on their results.

4.2.1 Shadow-rate AFNS model for nominal yields

We devote this section to the estimation of a shadow-rate AFNS model on nominal zero-coupon UK yields. The data set consists of continuously-compounded monthly nominal yields spanning from October 1986 to December 2011 and includes a set of seven maturities, namely 6, 12, 24, 36, 60, 84 and 120 months[3]. Interestingly, the time period incorporates two main changes in monetary policy practices in the UK, the introduction of inflation targeting in 1992 and the introduction of 'Quantitative Easing' in March 2009.

Before proceeding to the estimation, we need to go through two preliminary stages to best specify our model. In the first instance, we conduct a principal component analysis (PCA) to determine how many pricing factors are required to explain the cross-sectional variation of nominal yields. In the second instance, we use a general-to-specific method in order to impose the relevant restrictions to our model. It is important to note that we apply this method on a standard AFNS model that does not enforce the zero lower bound due to the fact that the shadow-

[3]The UK DMO issues bonds that have maturities of up to around 55 years. The aim of this study is to only analyze rate dynamics from short to medium horizons.

rate AFNS model is computationally more involved rendering it unfeasible for such a strategy.

Table 4.1 displays the loadings from the principal component analysis for the set of maturities and the percentage of variation of yields that is being captured by each component. We notice that the first component is characteristic of a level factor due to its homogeneity, the second component incorporates a sign switch between shorter and longer maturities hence displaying a slope feature and finally the third component, being parabolic, has the behavior of a curvature factor. Additionally, the first three components explain 99.99% of the cross-sectional yield variation. The PCA results validate our use of three factors bearing the interpretation of level, slope and curvature. These models have been used extensively in the literature, (refer to Diebold and Li (2006) and Koopman, Mallee, and Van der Wel (2010)).

We now proceed in adopting a three factor AFNS model, following Christensen, Diebold, and Rudebusch (2011). The latent state variables given by $X_t^N = \left(L_t^N, S_t^N, C_t^N\right)'$ solve the following system of stochastic differential equations under the risk-neutral $\mathbb{Q}$ measure, where λ^N is the mean reversion parameter, $W_t^{\mathbb{Q}}$ denotes a three dimensional Wiener process and the diffusion is diagonal.

$$\begin{pmatrix} dL_t^N \\ dS_t^N \\ dC_t^N \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda^N & -\lambda^N \\ 0 & 0 & \lambda^N \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t^N \\ C_t^N \end{pmatrix} dt + \begin{pmatrix} \sigma_{11,N} & 0 & 0 \\ 0 & \sigma_{22,N} & 0 \\ 0 & 0 & \sigma_{33,N} \end{pmatrix} \begin{pmatrix} dW_t^{L^N,\mathbb{Q}} \\ dW_t^{S^N,\mathbb{Q}} \\ dW_t^{C^N,\mathbb{Q}} \end{pmatrix} \tag{4.2.1}$$

The instantaneous risk-free rate is an affine function of the state variables and is specifically defined as the sum of the level and slope factors:

$$r_t^N = L_t^N + S_t^N \tag{4.2.2}$$

As demonstrated by Ang and Piazzesi (2003), nominal zero-coupon bond prices are exponentially affine functions of the state variables. As an immediate consequence, the representation of nominal zero-coupon yields with maturity T at time t is given by an affine function of the state variables, as shown below. $A^N(t,T)$ and $B^N(t,T)$ are the unique solutions to a system of Riccati equations, where $A^N(t,T)$ is known as the adjustment term (see Christensen, Diebold, and Rudebusch (2011) for the derivation) and $B^N(t,T)$ matches the Nelson-Siegel factor loadings.

$$\begin{aligned} y^N(t,T) &= -\frac{A^N(t,T)}{T-t} - \frac{B^N(t,T)'}{T-t} X_t^N \\ &= L_t^N + \left(\frac{1-e^{-\lambda^N(T-t)}}{\lambda^N(T-t)}\right) S_t^N + \left(\frac{1-e^{-\lambda^N(T-t)}}{\lambda^N(T-t)} - e^{-\lambda^N(T-t)}\right) C_t^N - \frac{A^N(t,T)}{T-t} \end{aligned} \tag{4.2.3}$$

The AFNS model is formulated in continuous time and Girsanov's theorem ensures the change from the physical to the risk-neutral measure, as such, $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Gamma_t^N dt$, where Γ_t^N is the market price of risk and under essentially affine risk premium specifications (see Duffee (2002) and Cheridito, Filipovic, and Kimmel (2007)), it takes the form below, with γ_0^N being a three-dimensional vector and γ_1^N a 3x3 matrix:

$$\Gamma_t^N = \gamma_0^N + \gamma_1^N X_t^N \tag{4.2.4}$$

Having all the tools necessary, we can now extract the latent state variables $X_t^N = \left(L_t^N, S_t^N, C_t^N\right)'$ under the physical measure. The key parameters are $\kappa^{N,\mathbb{P}}$ and $\theta^{N,\mathbb{P}}$ which are unrestricted and σ^N which has a diagonal structure. The dynamics are given by the following stochastic differential equation:

$$dX_t^N = \kappa^{N,\mathbb{P}}(t)\left[\theta^{N,\mathbb{P}}(t) - X_t^N\right]dt + \sigma^N dW_t^{X^N,\mathbb{P}} \tag{4.2.5}$$

It is at this point that the general-to-specific strategy comes into play, as we implement it to find the best specification for the $\kappa^{N,\mathbb{P}}$ matrix. The procedure goes as follows, we estimate an unrestricted AFNS and set the least significant element of $\kappa^{N,\mathbb{P}}$ to zero. We repeat this process until we are left with a diagonal $\kappa^{N,\mathbb{P}}$. Two criteria, the Akaike Information Criterion (AIC) and Bayes Information Criterion (BIC), are provided on Table 4.2, and we will rule our decision by minimizing the AIC. The preferred specification is thus given by specification (6) which is consistent with Christensen and Rudebusch (2012)'s findings.

Having found the preferred specification, we move on to the implementation of the shadow-rate AFNS which restricts nominal yields in the positive domain. The most striking difference will stem from the introduction of a shadow-rate which will have the same dynamics as the instantaneous risk-free rate under the standard AFNS, whilst the new dynamics for the instantaneous rate will consist of the maximum between the shadow-rate and zero. The latent shadow-rates and instantaneous rates are respectively defined as:

$$s_t^N = L_t^N + S_t^N \tag{4.2.6}$$

$$\underline{r}_t^N = max\left\{0, s_t^N\right\} \tag{4.2.7}$$

As in the standard AFNS, the state dynamics under the risk-neutral $\mathbb{Q}$ measure and the DGP $\mathbb{P}$ measure are given by equation (4.2.1) and (4.2.5), respectively.

We will now use a few important concepts borrowed from the bond option price literature. There has been a considerable amount of bond option pricing papers over the years (see Jamshidian (1989), Chen and Scott (1995) and Singleton and Umantsev (2002)). More recently, Krippner (2012) developed a shadow-rate framework in which a representation for the Zero Lower Bound (ZLB) instantaneous forward rate is provided. This representation is valid for all Gaussian models, including the AFNS, and depends on the instantaneous forward shadow-rates as well as an additional component which is a function of the conditional variance of a European call. In the case of the shadow-rate AFNS, analytical solutions for the instantaneous forward shadow-rates and the conditional variance are provided by Christensen and Rudebusch (2013). Their results can be found in appendix 4.B. Let us now denote by $\underline{y}^N(t,T)$, the Zero Lower Bound (ZLB) zero-coupon bond yields. We use appendix 4.B to derive $\underline{y}^N(t,T)$, by setting the vector $(X_1, X_2, X_3)'$ equal to $(L_t^N, S_t^N, C_t^N)'$ and the variables $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ equal to $(\sigma_{11,N}, \sigma_{22,N}, \sigma_{33,N})$.

$$\underline{y}^N(t,T) = \frac{1}{T-t}\int_t^T \left[f(t,s)\Phi\left(\frac{f(t,s)}{\omega(t,s)}\right) + \omega(t,s)\frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}\left[\frac{f(t,s)}{\omega(t,s)}\right]^2\right)\right] ds \tag{4.2.8}$$

It is important to note at this stage that $\underline{y}(t,T)$ is no longer a linear function of the state variables, unlike in the standard AFNS model. This non-linearity is translated in the estimation procedure, whereby a conventional Kalman Filter cannot be used and should be replaced by an Extended Kalman Filter (see appendix 4.C).

As in the standard AFNS case, the change of measure $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Gamma_t^N dt$ combined with the essentially affine specification of risk $\Gamma_t^N = \gamma_0^N + \gamma_1^N X_t^N$ allow us to have the preferred specification's representation of the state dynamics under the physical measure:

$$\begin{pmatrix} dL_t^N \\ dS_t^N \\ dC_t^N \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{N,\mathbb{P}} & 0 & 0 \\ 0 & \kappa_{22}^{N,\mathbb{P}} & \kappa_{23}^{N,\mathbb{P}} \\ 0 & 0 & \kappa_{33}^{N,\mathbb{P}} \end{pmatrix} \left[\begin{pmatrix} \theta_t^{L^N,\mathbb{P}} \\ \theta_t^{S^N,\mathbb{P}} \\ \theta_t^{C^N,\mathbb{P}} \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t^N \\ C_t^N \end{pmatrix} \right] dt + \begin{pmatrix} \sigma_{11,N} & 0 & 0 \\ 0 & \sigma_{22,N} & 0 \\ 0 & 0 & \sigma_{33,N} \end{pmatrix} \begin{pmatrix} dW_t^{L^N,\mathbb{P}} \\ dW_t^{S^N,\mathbb{P}} \\ dW_t^{C^N,\mathbb{P}} \end{pmatrix} \tag{4.2.9}$$

The results of the estimated parameters can be found in Table 4.3, whilst the in-sample fit results, in Table 4.4, report a good fit for all maturities, particularly for medium-term tenors.

Following Campbell and Shiller (1991), bond risk premia have been at the forefront of many studies, including Dai and Singleton (2002), Duffee (2002) and Cochrane and Piazzesi (2005). We provide estimates of the term premia across maturities, with and without the ZLB assumption, in Figures 4.1 and 4.2 respectively. The most striking difference is that term premia change sign after 2008, coinciding with the start of the crisis. With the ZLB specification, term premia now display a countercyclical nature, after 2009, hence providing a better representation than under the previous setting that does not impose the non-negativity assumption. Figure 4.3 plots the one year tenor forward and expected forward rates, in December 2011, along with the shadow rate. It is clear that the omission of the ZLB assumption can generate negative nominal short yields. As noted earlier, market demand can drive short maturity yields to negative territories, especially if bonds are perceived by investors as a ‘safe haven’. However, a prolonged period of negative short nominal rates, or equivalently, a negative policy rate, might not be reasonable for monetary policy objectives and would result in price tensions in market dynamics. Here, we note that shadow rates can turn significantly negative when modeled using the standard linear Gaussian AFNS mapping. What is observed in reality is that short rates are rather anchored at zero, hence capping the theoretical price of a zero coupon bond at 100 (see Krippner (2012)). If short rates were to go negative (Gaussian assumption), the price of a theoretical zero coupon bond ('shadow bond') would float anywhere above par. In essence, with the use of the properties of bond option pricing, it is now possible to uncover the non-linear relationship between prices, yields, and volatilities, and to price convexity effects in short maturity rates. This relationship becomes evident when rates are at the zero lower bound and the option is in/at the money [4]. The new shadow path is considerably more negative at shorter maturities suggesting term premia were previously estimated unreasonably low, especially at short maturities. Noteworthy is the fact that we identify three distinct sub-periods within our sample (1985-1998, 1998-2008 and 2008-2011) which correspond to different cycles of the economy. Figure 4.4 includes a decomposition of nominal yields into two components: the so called risk-neutral yields and the term premia. The term premia are given by:

$$TP^N(t,T) = \underline{y}^N(t,T) - \frac{1}{T-t}\int_t^T \mathbb{E}_t^{\mathbb{P}}\left[\underline{r}_s^N\right] ds \tag{4.2.10}$$

[4] Moneyness is the difference between strike price and future expected price. If the option is significantly in the money, the shadow bond price is well above par.

4.2.2 Empirical AFNS model for real yields

We now proceed to the estimation of a standard AFNS model for real zero-coupon UK bond yields. The data set consists of continuously-compounded monthly yields spanning from October 1986 to December 2011 and includes a set of seven fixed maturities: 60, 72, 84, 90, 96, 108 and 120 months. It is important to note that we have chosen longer maturities for real yields, in comparison to nominal yields, although we reserve to review this assumption in future work.

Table 4.5 displays the results of a principal component analysis on the set of real yields. It is clear that the first principal component that bears attributes of a level factor, explains a greater cross-sectional variation in real yields, in contrast to the case of nominal yields. One could argue that 2 factors suffice in the modeling of this set of real yields given they explain 99.99% of the variation. However, we take a closer look at the third component and notice that the typical U-shaped behaviour of a curvature factor persists. Moreover, our ultimate goal lies in estimating long term inflation expectations and it is common knowledge that the curvature factor is of high importance to longer maturity yields. Hence these two arguments fully justify our choice of using a three-factor AFNS model to fit real yields. More importantly, it is crucial to identify that the second component bears a positive sign for shorter maturities and a negative sign for longer maturities, indicating the UK real yield curve has been inverted.

We denote by $X_t^R = \left(L_t^R, S_t^R, C_t^R\right)'$, the latent state variables. Under the risk-neutral measure $\mathbb{Q}$, where λ^R is the mean reversion parameter, $W_t^{\mathbb{Q}}$ denotes a three dimensional Wiener process and the diffusion is diagonal, the state dynamics are given by the following system of stochastic differential equations:

$$\begin{pmatrix} dL_t^R \\ dS_t^R \\ dC_t^R \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda^R & -\lambda^R \\ 0 & 0 & \lambda^R \end{pmatrix} \begin{pmatrix} L_t^R \\ S_t^R \\ C_t^R \end{pmatrix} dt + \begin{pmatrix} \sigma_{11,R} & 0 & 0 \\ 0 & \sigma_{22,R} & 0 \\ 0 & 0 & \sigma_{33,R} \end{pmatrix} \begin{pmatrix} dW_t^{L^R,\mathbb{Q}} \\ dW_t^{S^R,\mathbb{Q}} \\ dW_t^{C^R,\mathbb{Q}} \end{pmatrix} \quad (4.2.11)$$

The instantaneous risk-free real rate is an affine function of the state variables and is defined as the sum of the level and slope factors:

$$r_t^R = L_t^R + S_t^R \quad (4.2.12)$$

Real zero-coupon bond yields have the following structure, where $A^R(t,T)$ is the adjustment term and $B^R(t,T)$ are the Nelson Siegel loadings:

$$\begin{aligned} y^R(t,T) &= -\frac{A^R(t,T)}{T-t} - \frac{B^R(t,T)'}{T-t} X_t^R \\ &= L_t^R + \left(\frac{1-e^{-\lambda^R(T-t)}}{\lambda^R(T-t)} \right) S_t^R + \left(\frac{1-e^{-\lambda^R(T-t)}}{\lambda^R(T-t)} - e^{-\lambda^R(T-t)} \right) C_t^R - \frac{A^R(t,T)}{T-t} \end{aligned} \quad (4.2.13)$$

Exactly as in the nominal case, the market price of risk takes an essentially affine specification seen below:

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Gamma_t^R dt \quad (4.2.14)$$

$$\Gamma_t^R = \gamma_0^R + \gamma_1^R X_t^R \quad (4.2.15)$$

We can now apply the change of measure to obtain the latent state variables $X_t^R = \left(L_t^R, S_t^R, C_t^R\right)'$ under the physical measure. The key parameters are $\kappa^{R,\mathbb{P}}$ and

$\theta^{R,\mathbb{P}}$ which are unrestricted and σ^R which has a diagonal structure.

$$dX_t^R = \kappa^{R,\mathbb{P}}(t)\left[\theta^{R,\mathbb{P}}(t) - X_t^R\right]dt + \sigma^R dW_t^{X^R,\mathbb{P}} \tag{4.2.16}$$

Given we use a three-factor AFNS model to fit real yields which, at first glance, do not seem to necessitate so many factors, it is very likely that some parameters may not be statistically significant. To accommodate for this possibility, we use a general-to-specific method, as before, to find the optimal specification of the $\kappa^{R,\mathbb{P}}$ matrix. The results reported on Table 4.6, indicate that the diagonal specification (7) is the one that minimizes both information criteria, and consequently is our preferred specification. The dynamics are given by the following stochastic differential equation:

$$\begin{pmatrix} dL_t^R \\ dS_t^R \\ dC_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{R,\mathbb{P}} & 0 & 0 \\ 0 & \kappa_{22}^{R,\mathbb{P}} & 0 \\ 0 & 0 & \kappa_{33}^{R,\mathbb{P}} \end{pmatrix} \left[\begin{pmatrix} \theta_t^{L^R,\mathbb{P}} \\ \theta_t^{S^R,\mathbb{P}} \\ \theta_t^{C^R,\mathbb{P}} \end{pmatrix} - \begin{pmatrix} L_t^R \\ S_t^R \\ C_t^R \end{pmatrix} \right] dt + \begin{pmatrix} \sigma_{11,R} & 0 & 0 \\ 0 & \sigma_{22,R} & 0 \\ 0 & 0 & \sigma_{33,R} \end{pmatrix} \begin{pmatrix} dW_t^{L^R,\mathbb{P}} \\ dW_t^{S^R,\mathbb{P}} \\ dW_t^{C^R,\mathbb{P}} \end{pmatrix} \tag{4.2.17}$$

The parameter estimates and in-sample fit can be found on Tables 4.7 and 4.8, respectively. We notice that the Root Mean Squared Error (RMSE), for maturities 84, 90 and 96 months, is negligible, confirming the findings of Chen and Scott (1995), supporting that some yields are measured without error.

4.3 Empirical joint shadow-rate AFNS model for nominal and real yields

In this section, we estimate a joint AFNS model for nominal and real yields. We impose the non-negativity assumption solely on nominal yields without restricting real yields. Nonetheless, before proceeding to our joint partial shadow-rate AFNS model, we need to establish the number of factors to be considered, as well as the interpretation we wish to give to these factors, in other words, level, slope or curvature. To do so, we first perform a principal component analysis displayed in Table 4.9. We consider a data set combining exactly the two panels studied in the previous section. Therefore, the data consists of continuously-compounded monthly nominal and real yields spanning from October 1986 to December 2011 and includes a set of seven maturities for nominal yields, namely, 6, 12, 24, 36, 60, 84 and 120 months, and an additional set of seven maturities for real yields: 60, 72, 84, 90, 96, 108 and 120 months. At first glance, we can see that the use of six factors would be somewhat of a stretch. By the same token, the use of three factors seems, a priori, far too restrictive to be able to fit the term structure of nominal and real yields appropriately. We now face the dilemma between using four or five factors. On the one hand, our nominal yields' data set includes short, medium and long term maturities, which implies the need for a level, slope and curvature factor. On the other hand, real yields comprise solely of medium and long term maturities, which ultimately give a greater weight to the level and curvature factors. One could hence argue that an appropriate model could have a level, slope and curvature for nominal yields, a curvature for real yields and finally a common level and slope factor, as it is the case in Christensen, Lopez, and Rudebusch (2010b). However, this model would be unfeasible as it would violate the no-arbitrage assumption imposed on the AFNS model in order to retrieve the Nelson-Siegel factor loadings (see Christensen, Diebold, and Rudebusch (2009)). The assumption of no-arbitrage is of primordial significance given we require both the risk-neutral and physical measure in order to retrieve inflation risk premia. In addition, we find that, empirically, the correlation between long nominal and real yields, representing the level, has been historically very stable over time and that nominal yields moved very much in line with real yields, thus supporting the specification of using one single level factor to explain both nominal and real rates. We find that nominal and real rates' slopes, especially at 5 and 10-year maturities, also display a historically stable correlation, however, this pattern changes after 2008. This coincides with the timing of the sudden decrease in nominal rates and the significant increase in the steepness of the nominal curve, resulting in the sharp increase in BEI at 5 and 10-year maturities. In practice, if we were to use a single slope factor, we would misestimate the short real rate consequently also affecting inflation expectations after 2008. We therefore choose to use a five factor model which consists of an extension of the Svensson model. This model has the capacity to capture the inversion of real yields, by allowing their slope to vary independently from the slope of nominal yields. The five first principal components explain 99.99% of the cross-sectional variation of nominal and real yields, therefore the choice of five factors is pefectly reasonable. We are hence left with a single interpretation for our factors, whereby the first three factors represent the level, slope and curvature of nominal yields, whilst the fourth and fifth factors represent the slope and curvature of real yields, respectively. By deduction, the level factor will be common across the two sets of yields. We denote by α^R the weight of real yields on the level of nominal yields.

As in the nominal case, before enforcing the zero lower-bound on nominal yields, we need to first find the preferred specification of our mean reversion matrix $\kappa^{J,\mathbb{P}}$. Using the so-called preferred specification is of great importance due to the sensitivity of results to different specifications (see Joslin, Priebsch, and Singleton (2013), Joslin, Singleton, and Zhu (2011b) and Christensen and Rudebusch (2013)). The issue of sensitivity is of greater importance when considering the estimation of risk premia, given they rely heavily on the estimation of $\kappa^{J,\mathbb{P}}$. Once again, however, it is unfeasible to conduct a general-to-specific strategy on a shadow-rate AFNS due to the computational burden of the model. We hence proceed in conducting such a strategy on a standard joint AFNS model, and its preferred specification is subsequently used in a joint partial shadow-rate AFNS.

We first consider the structure of our standard joint AFNS. The joint latent state vector is given by $X_t^J = \left(L_t, S_t^N, C_t^N, S_t^R, C_t^R\right)'$ and solves the following stochastic differential equations under the risk-neutral measure $\mathbb{Q}$:

$$\begin{pmatrix} dL_t \\ dS_t^N \\ dC_t^N \\ dS_t^R \\ dC_t^R \end{pmatrix} = - \begin{pmatrix} \epsilon & 0 & 0 & 0 & 0 \\ 0 & \lambda^N & -\lambda^N & 0 & 0 \\ 0 & 0 & \lambda^N & 0 & 0 \\ 0 & 0 & 0 & \lambda^R & -\lambda^R \\ 0 & 0 & 0 & 0 & \lambda^R \end{pmatrix} \begin{pmatrix} L_t \\ S_t^N \\ C_t^N \\ S_t^R \\ C_t^R \end{pmatrix} dt + \begin{pmatrix} \sigma_{11,J} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22,J} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33,J} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{44,J} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{55,J} \end{pmatrix} \begin{pmatrix} dW_t^{L,\mathbb{Q}} \\ dW_t^{S^N,\mathbb{Q}} \\ dW_t^{C^N,\mathbb{Q}} \\ dW_t^{S^R,\mathbb{Q}} \\ dW_t^{C^R,\mathbb{Q}} \end{pmatrix} \tag{4.3.18}$$

where $\lambda^N$and λ^R are scalars that represent the speed of mean-reversion for nominal and real yields respectively, $dW_t^{\mathbb{Q}}$ is a five-dimensional Wiener process and the diffusion matrix is diagonal. We assume the nominal and real instantaneous risk-free rates are defined respectively as follows

$$r_t^N = L_t + S_t^N \tag{4.3.19}$$

$$r_t^R = \alpha^R L_t + S_t^R \tag{4.3.20}$$

Nominal and real yields are respectively given by the two following equations, containing the typical adjustment term of AFNS models and the Nelson-Siegel factor loadings. We note that real yields have a weighted Nelson-Siegel loading for the level factor, given this factor is common to nominal and real yields.

$$y^N(t,T) = L_t + \frac{1-e^{-\lambda^N \tau}}{\lambda^N \tau} S_t^N + \left(\frac{1-e^{-\lambda^N \tau}}{\lambda^N \tau} - e^{-\lambda^N \tau} \right) C_t^N - \frac{A^N(\tau)}{\tau} \quad (4.3.21)$$

$$y^R(t,T) = \alpha^R L_t + \left(\frac{1-e^{-\lambda^R \tau}}{\lambda^R \tau} \right) S_t^R + \left(\frac{1-e^{-\lambda^R \tau}}{\lambda^R \tau} - e^{-\lambda^R \tau} \right) C_t^R - \frac{A^R(\tau)}{\tau} \quad (4.3.22)$$

Γ_t is the market price of risk and under essentially affine risk premium specifications it takes the following affine form:

$$\Gamma_t = \gamma_0^J + \gamma_1^J X_t^J \quad (4.3.23)$$

By applying the change of measure, we extract the latent state variable vector $X_t^J = \left(L_t, S_t^N, C_t^N, S_t^R, C_t^R\right)'$ which solves the stochastic differential equations below under the physical measure:

$$dX_t^J = \kappa^{J,\mathbb{P}}(t) \left[\theta^{J,\mathbb{P}}(t) - X_t^J\right] dt + \sigma^J dW_t^{X^J,\mathbb{P}} \quad (4.3.24)$$

We can now implement a general-to-specific method to find the best specification for the $\kappa^{J,\mathbb{P}}$ matrix. We first start by estimating an unrestricted AFNS model and continue by setting the least significant element of $\kappa^{J,\mathbb{P}}$ to zero. This process is repeated until we are left with a diagonal $\kappa^{J,\mathbb{P}}$. For each step, the log-likelihood, AIC and BIC are reported on Table 4.10. We aim to minimize the information criteria, in this case the decision rule of the AIC and BIC does not coincide. For the sake of consistency, we will minimize the AIC, as we previously did in the nominal yields' section. We therefore designate specification (17) as our preferred specification.

Having found our preferred specification, we proceed to the implementation of the partial shadow-rate AFNS model which restricts nominal yields in the positive domain whilst simultaneously keeping real yields unrestricted. The instantaneous risk-free nominal and real rates are given respectively by:

$$\underline{r}_t^N = max\left\{0, L_t + S_t^N\right\} \tag{4.3.25}$$

$$r_t^R = \alpha^R L_t + S_t^R \tag{4.3.26}$$

We note that the nominal instantaneous risk-free rate is the maximum between zero and the nominal shadow-rate, whilst the real instantaneous risk-free rate coincides with the real shadow-rate. Let us now denote by $\underline{y}^N(t,T)$ and $y^R(t,T)$, the ZLB nominal zero-coupon bond yields and the real zero coupon yields, respectively. We use appendix 4.B to derive $\underline{y}^N(t,T)$, by setting the vector $(X_1, X_2, X_3)'$ equal to $(L_t, S_t^N, C_t^N)'$ and the variables $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ equal to $(\sigma_{11,J}, \sigma_{22,J}, \sigma_{33,J})$. Their representations are given below:

$$\underline{y}^N(t,T) = \frac{1}{T-t}\int_t^T \left[f^N(t,s)\Phi\left(\frac{f^N(t,s)}{\omega^N(t,s)}\right) + \omega^N(t,s)\frac{1}{\sqrt{2\pi}}exp\left(-\frac{1}{2}\left[\frac{f^N(t,s)}{\omega_N(t,s)}\right]^2\right)\right] ds \tag{4.3.27}$$

$$y^R(t,T) = \alpha^R L_t + \left(\frac{1-e^{-\lambda^R\tau}}{\lambda^R\tau}\right) S_t^R + \left(\frac{1-e^{-\lambda^R\tau}}{\lambda^R\tau} - e^{-\lambda^R\tau}\right) C_t^R - \frac{A^R(\tau)}{\tau} \tag{4.3.28}$$

Our model naturally takes a state-space representation. It is crucial to observe that nominal yields are non-linear functions of the state vector and real yields are affine function of the latent state variables. As a consequence, to accommodate for the non-linearity, the estimation procedure requires the use of an extended Kalman Filter (see appendix 4.C).

The market price of risk under the essentially affine risk premium specifications takes the form:

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Gamma_t^J dt \tag{4.3.29}$$

$$\Gamma_t^J = \gamma_0^J + \gamma_1^J X_t^J \tag{4.3.30}$$

The latent state variable $X_t^J = \left(L_t, S_t^N, C_t^N, S_t^R, C_t^R\right)'$ solves the following stochastic differential equation under the physical measure, for our preferred specification:

$$\begin{pmatrix} dL_t \\ dS_t^N \\ dC_t^N \\ dS_t^R \\ dC_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{J,\mathbb{P}} & 0 & 0 & 0 & 0 \\ 0 & \kappa_{22}^{J,\mathbb{P}} & 0 & 0 & \kappa_{25}^{J,\mathbb{P}} \\ 0 & 0 & \kappa_{33}^{J,\mathbb{P}} & \kappa_{34}^{J,\mathbb{P}} & \kappa_{35}^{J,\mathbb{P}} \\ 0 & 0 & 0 & \kappa_{44}^{J,\mathbb{P}} & 0 \\ 0 & 0 & 0 & \kappa_{54}^{J,\mathbb{P}} & \kappa_{55}^{J,\mathbb{P}} \end{pmatrix} \left[\begin{pmatrix} \theta_t^L \\ \theta_t^{S^N} \\ \theta_t^{C^N} \\ \theta_t^{S^R} \\ \theta_t^{C^R} \end{pmatrix} - \begin{pmatrix} L_t \\ S_t^N \\ C_t^N \\ S_t^R \\ C_t^R \end{pmatrix} \right] dt$$
$$+ \, diag \begin{pmatrix} \sigma_{11,J} \\ \sigma_{22,J} \\ \sigma_{33,J} \\ \sigma_{44,J} \\ \sigma_{55,J} \end{pmatrix} \begin{pmatrix} dW_t^{L,\mathbb{P}} \\ dW_t^{S^N,\mathbb{P}} \\ dW_t^{C^N,\mathbb{P}} \\ dW_t^{S^R,\mathbb{P}} \\ dW_t^{C^R,\mathbb{P}} \end{pmatrix} \tag{4.3.31}$$

The estimated parameters comprising the equation above are reported in Table 4.11 and the in-sample fit is displayed in Table 4.12. The findings under the joint model are consistent with the individual models' results. The fit of both nominal and real yields is very satisfactory and further allows us to explore, in the next section, inflation expectations and risk premia.

The ZLB term premia of nominal yields is given by:

$$TP^N(t,T) = \underline{y}^N(t,T) - \frac{1}{T-t}\int_t^T \mathbb{E}_t^{\mathbb{P}}\left[\underline{r}_s^N\right] ds \tag{4.3.32}$$

Figures 4.5 and 4.6 provide plots of the nominal term premia by maturity, and the decomposed nominal yields, respectively. The shape of the term premia is comparable to our findings in the individual nominal models. However the magnitude strongly differs due to the sensitivity risk premia have to the structure and composition of the mean reversion matrix $\kappa^{J,\mathbb{P}}$. Specifically, term premia up to 1992 appear to be very different, leaving room for further investigation. In addition, the decomposition of nominal yields is relatively similar under both joint and individual nominal models. We remark clearly that the risk premia are on average fluctuating around zero in both cases, and even more so in the individual nominal model.

4.4 Inflation expectations and risk premia

We now pursue our analysis by decomposing BEI rates into inflation risk premia and expectations. The no-arbitrage condition so far imposed on all AFNS models gains further importance in this section as it is precisely the existence of a risk-neutral and physical measure that eventually allows us to proceed in this decomposition. We denote by $\frac{dM_t^N}{M_t^N}$ and $\frac{dM_t^R}{M_t^R}$, the nominal and real pricing kernel dynamics, respectively, and provide their expressions below:

$$\frac{dM_t^N}{M_t^N} = -\underline{r}_s^N dt - \Gamma_t^{J'} dW_t^{J,\mathbb{P}} \tag{4.4.33}$$

$$\frac{dM_t^R}{M_t^R} = -r_t^R dt - \Gamma_t^{J'} dW_t^{J,\mathbb{P}} \tag{4.4.34}$$

By manipulating the two stochastic discount factors above, (see Christensen, Lopez, and Rudebusch (2010b) for further details), one can extract the following system of equations:

$$BEI(t,T) \equiv \underline{y}_t^N(t,T) - y_t^R(t,T) \tag{4.4.35}$$

$$= \pi_t^e(t,T) + \phi_t(t,T) \tag{4.4.36}$$

$$\pi_t^e(t,T) = -\frac{1}{T-t} ln \left\{ \mathbb{E}_t^{\mathbb{P}} \left[exp \left(- \int_t^T (\underline{r}_u^N - r_u^R) du \right) \right] \right\} \tag{4.4.37}$$

where $\pi_t^e(t,T)$ and $\phi_t(t,T)$ denote respectively the inflation expectations and inflation risk premia for maturity T, estimated at time t. Moreover, the solution to the expression is obtained through numerical procedures.

In Figure 4.7, we display the inflation expectations for maturities of 5 and 10 years. We note that since 1992, inflation expectations have decreased, possibly as a result of investors' confidence in the new monetary policy framework that was reinforced in 1998. There is a tendency for 5-year spot inflation projections to be above the current inflation target[5], while at a 10-year horizon, inflation projections systematically undershoot target inflation after 1994. In 2008, inflation expectations decreased significantly, perhaps overly so, relatively to the magnitude of deflationary shock observed in CPI inflation thereafter. Historically, this occurred in conjunction

[5]We took into account that inflation expectations are RPI based.

with large volatility in the inflation-linked bond market, which witnessed reduced liquidity. At that time, inflation-linked gilt asset swap spreads sharply widened to historical highs. As a result, it is possible our estimation has been affected by this event and that inflation expectations and risk premia require an adjustment for liquidity premia, especially at longer horizons[6]. Linkers are typically less liquid than conventional bonds of similar maturity. We tested the drop in 2008 against alternative data sources, including inflation survey forecast data[7]. Our results confirm the fall in 2008 is likely to be the product of a distortion in market prices. Subsequently to the sharp drop, expectations have picked up and have reached, once again, post-1990 average levels. In Figure 4.8, we plot the inflation premia at 5 and 10- year maturities. Inflation risk premia dropped following the introduction of inflation targeting, conveying a period of lower uncertainty. In March 2009, inflation risk premia strongly increased[8]. Since then, inflation premia have decreased, as investors might be placing less weight on future inflation uncertainty. In Figure 4.9, we display actual and model-implied BEI rates. Finally, in Figure 4.10, we decompose BEI rates into pure inflation expectations and inflation risk premia.

[6]If future inflation expectations are underestimated, inflation risk premia tend to be overestimated in our model.

[7]From Consensus Economics.

[8]As previously noted, this is likely to be due to a pricing distortion in the linker market.

4.5 Conclusion

In conclusion, we estimate a joint shadow-rate AFNS model that is able to impose the zero lower bound restriction on nominal yields whilst allowing real yields to fall below zero. The model proposed benefits from the Nelson Siegel factor loadings which induce a robust estimation procedure and tractability. The no-arbitrage restrictions enhance the theoretical grounds whilst simultaneously allowing the decomposition of BEI rates into inflation expectations and risk premia. Our model successfully fits both nominal and real yields as well as BEI rates.

We find that imposing the zero lower bound has corrected the risk premia projections of nominal rates that would otherwise appear too low after 2009. Inflation premia are larger in longer maturity nominal yields. Our results show that the bond market has on average priced long term inflation in line with its target after the early 1990s, which suggests monetary policy credibility. The shadow rate projections show the 'standard reaction function of the central bank', independently of the vicinity of rates to the zero lower bound. Under this scenario, current monetary policy looks restrictive.

Finally, countercyclicality of risk premia paired with the fact that they increase with maturity suggest that in times of a recession - below trend growth -, issuing more short maturity bonds and rolling them over is likely to be more cost effective over the long horizon than issuing long maturity bonds. On the other hand, when the economy is in expansion, it could become more favorable to issue longer maturity bonds, as the premium paid to investors, relative to short maturity bonds, is lower, and the hedging of refinancing risk is cheaper on a relative scale.

Appendix

4.A Appendix A: Tables and Figures

Table 4.1: First three principal components in nominal yields

Maturity	First PC	Second PC	Third PC
6 months	0.4254	-0.4838	0.5294
12 months	0.4134	-0.3685	0.0913
24 months	0.3952	-0.1724	-0.3401
36 months	0.3806	-0.0042	-0.4878
60 months	0.3591	0.2580	-0.3268
84 months	0.3423	0.4339	0.0465
120 months	0.3177	0.5879	0.4988
% explained	97.28	2.55	0.17

NOTE: We provide the loadings of the yields of the set of maturities on the first three principal components. The percentage of all nominal bond yields' cross-sectional variation accounted for by each component is displayed on the final row. The data comprises of monthly nominal zero coupon bonds from October 1986 to December 2011.

Table 4.2: Evaluation of alternative specifications of the 3 factor AFNS model - Nominal -

Alternative specifications	logL	k	p-value	AIC	BIC
(1) Unrestricted $\kappa^{\mathbb{P}}$	12620.8009	23		-25195.6019	-25109.0747
(2) $\kappa^{\mathbb{P}}_{32} = 0$	12620.6907	22	0.6386	-25197.3813	-25114.6162
(3) $\kappa^{\mathbb{P}}_{32} = \kappa^{\mathbb{P}}_{31} = 0$	12620.6858	21	0.9952	-25199.3716	-25120.3685
(4) $\kappa^{\mathbb{P}}_{32} = \kappa^{\mathbb{P}}_{31} = \kappa^{\mathbb{P}}_{12} = 0$	12620.3134	20	0.8626	-25200.6268	-25125.3858
(5) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{21} = 0$	12620.2646	19	0.9988	-25202.5292	-25131.0503
(6) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{13} = 0$	12620.2040	18	0.9997	**-25204.4080**	-25136.6910
(7) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{23} = 0$	12618.3373	17	0.7127	-25202.6745	**-25138.7197**

NOTE: We estimate and evaluate seven alternative specifications of the individual standard AFNS model on nominal yields. For each specification, we record its log-likelihood (LogL), number of parameters (k) and the p-value of a likelihood ratio test of the hypothesis that a specification with (k-i) parameters is different from the one with (k-i+1) parameters. The information criteria (AIC and BIC) are reported and we display their minimum in bold.

Table 4.3: 3 factor Shadow-rate AFNS estimates for nominal rates

$\kappa^{\mathbb{P}}_t$	$\kappa^{\mathbb{P}}_{\cdot,1}$	$\kappa^{\mathbb{P}}_{\cdot,2}$	$\kappa^{\mathbb{P}}_{\cdot,3}$	θ^P	$\sigma^N_{i,i}$
$\kappa^{\mathbb{P}}_{1,\cdot}$	0.0455	0.0000	0.0000	0.0594	0.0155
	(0.0350)			(0.0203)	(0.0005)
$\kappa^{\mathbb{P}}_{2,\cdot}$	0.0000	0.2691	-0.1893	-0.0111	0.0193
		(0.0353)	(0.0506)	(0.0230)	(0.0008)
$\kappa^{\mathbb{P}}_{3,\cdot}$	0.0000	0.0000	0.3644	-0.0130	0.0313
			(0.0324)	(0.0237)	(0.0016)

NOTE: The estimated parameters of the $\kappa^{N,\mathbb{P}}$ matrix, $\theta^{N,\mathbb{P}}$ vector, and diagonal diffusion matrix $\sigma^N_{i,i}$ are given for our preferred individual three-factor shadow-rate AFNS model for nominal yields. The estimated value of λ^N is 0.4760 with standard deviation of 0.0154. The numbers in parentheses are the standard deviations of the estimated parameters.

Table 4.4: Measures of fit for the 3 factor Shadow-rate AFNS model for nominal yields

Maturity in months	Mean(in bp)	RMSE(in bp)
6	-0.4162	6.3991
12	0.1495	0.8540
24	0.1356	1.8846
36	0.2227	1.1613
60	0.3416	2.8460
84	0.3176	2.3382
120	-0.0822	9.7383

NOTE: The mean and RMSE of fitted errors of the preferred individual three-factor shadow-rate AFNS model for nominal yields are given. All values are measured in basis points. The nominal yields span from October 1986 to December 2011.

Table 4.5: First three principal components in real yields

Maturity	First PC	Second PC	Third PC
60 months	0.3853	0.6832	0.5069
72 months	0.3799	0.3391	-0.1831
84 months	0.3770	0.0557	-0.3971
90 months	0.3763	-0.0653	-0.3644
96 months	0.3758	-0.1738	-0.2597
108 months	0.3755	-0.3574	0.1048
120 months	0.3757	-0.5030	0.5836
% explained	98.46	1.52	0.01

NOTE: We provide the loadings of the yields of the set of maturities on the first three principal components. The percentage of all real bond yields' cross-sectional variation accounted for by each component is displayed on the final row. The data comprises of monthly real zero coupon bonds from October 1986 to December 2011.

Table 4.6: Evaluation of alternative specifications of the 3 factor AFNS model - Real -

Alternative specifications	logL	k	p-value	AIC	BIC
(1) Unrestricted $\kappa^{\mathbb{P}}$	15784.2651	23		-31522.5303	-31437.1144
(2) $\kappa^{\mathbb{P}}_{32} = 0$	15784.2650	22	0.9877	-31524.5300	-31442.8279
(3) $\kappa^{\mathbb{P}}_{32} = \kappa^{\mathbb{P}}_{31} = 0$	15784.2649	21	0.9999	-31526.5298	-31448.5414
(4) $\kappa^{\mathbb{P}}_{32} = \kappa^{\mathbb{P}}_{31} = \kappa^{\mathbb{P}}_{21} = 0$	15784.2641	20	1.0000	-31528.5283	-31454.2536
(5) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{23} = 0$	15784.2630	19	1.0000	-31530.5259	-31459.9650
(6) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{13} = 0$	15784.2288	18	0.9999	**-31532.4576**	**-31465.6104**
(7) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{12} = 0$	15740.6851	17	0.0000	-31447.3702	-31384.2367

NOTE: We estimate and evaluate seven alternative specifications of the individual standard AFNS model on real yields. For each specification, we record its log-likelihood (LogL), number of parameters (k) and the p-value of a likelihood ratio test of the hypothesis that a specification with k-i parameters is different from the one with k-i+1 parameters. The information criteria (AIC and BIC) are reported and we display their minimum in bold.

Table 4.7: 3 factor AFNS estimates for real rates

$\kappa_t^{\mathbb{P}}$	$\kappa_{\cdot,1}^{\mathbb{P}}$	$\kappa_{\cdot,2}^{\mathbb{P}}$	$\kappa_{\cdot,3}^{\mathbb{P}}$	θ^{P}	$\sigma_{i,i}^{R}$
$\kappa_{1,\cdot}^{\mathbb{P}}$	0.0979	0.0027	0.0000	0.0054	0.0053
	(0.0316)	(0.0317)		(0.0315)	(0.0002)
$\kappa_{2,\cdot}^{\mathbb{P}}$	0.0000	0.1001	0.0000	0.0000	0.0458
		(0.0316)		(0.0316)	(0.0282)
$\kappa_{3,\cdot}^{\mathbb{P}}$	0.0000	0.0000	0.1001	-0.0001	0.0578
			(0.0316)	(0.0316)	(0.0292)

NOTE: The estimated parameters of the $\kappa^{R,\mathbb{P}}$ matrix, $\theta^{R,\mathbb{P}}$ vector, and diagonal diffusion matrix $\sigma_{i,i}^{R}$ are given for our preferred individual three-factor standard AFNS model for real yields. The estimated value of λ^R is 0.7108 with standard deviation of 0.0303. The numbers in parentheses are the standard deviations of the estimated parameters.

Table 4.8: Measures of fit for the 3 factor AFNS model for real yields

Maturity in months	Mean (in bp)	RMSE (in bp)
60	0.4839	1.8435
72	-0.0190	0.3016
84	-0.0001	0.0002
90	0.0174	0.0620
96	-0.0001	0.0004
108	-0.1589	0.5680
120	-0.4290	1.6344

NOTE: The mean and RMSE of fitted errors of the preferred individual three-factor standard AFNS model for real yields are given. All values are measured in basis points. The real yields span from October 1986 to December 2011.

Table 4.9: First six principal components in nominal and real yields

Maturity	First PC	Second PC	Third PC	Fourth PC	Fifth PC	Sixth PC
Nominal yields						
6 months	0.3945	0.4250	-0.2763	0.4725	0.2821	-0.3863
12 months	0.3841	0.3304	-0.2186	0.1014	-0.0294	0.3039
24 months	0.3681	0.1868	-0.1008	-0.2689	-0.2551	0.4024
36 months	0.3551	0.0791	0.0176	-0.4072	-0.2765	0.0541
60 months	0.3355	-0.0524	0.2419	-0.3228	-0.1074	-0.4768
84 months	0.3198	-0.1148	0.4212	-0.0675	0.1188	-0.3572
120 months	0.2968	-0.1675	0.5828	0.2762	0.3816	0.4820
Real yields						
60 months	0.1354	-0.3054	-0.3665	-0.3340	0.5170	0.0098
72 months	0.1355	-0.3049	-0.2701	-0.1194	0.2453	0.0285
84 months	0.1364	-0.3023	-0.1905	0.0467	0.0220	0.0254
90 months	0.1370	-0.3004	-0.1562	0.1135	-0.0726	0.0170
96 months	0.1378	-0.2981	-0.1253	0.1708	-0.1568	0.0053
108 months	0.1393	-0.2929	-0.0722	0.2607	-0.2961	-0.0256
120 months	0.1409	-0.2873	-0.0293	0.3246	-0.4016	-0.0613
% explained	94.76	3.08	1.97	0.14	0.03	0.01

NOTE: We provide the loadings of the yields of the set of maturities on the first three principal components. The percentage of all nominal and real bond yields' cross-sectional variation accounted for by each component is displayed on the final row. The data comprises of monthly nominal and real zero coupon bonds from October 1986 to December 2011.

Table 4.10: Evalutation of alternative specifications of the 5 factor joint AFNS model

Alternative specifications	logL	k	p-value	AIC	BIC
(1) Unrestricted $\kappa^{\mathbb{P}}$	26534.3876	52		-52964.7753	-52771.6611
(2) $\kappa^{\mathbb{P}}_{32} = 0$	26534.3876	51	1.0000	-52966.7753	-52777.3749
(3) $\kappa^{\mathbb{P}}_{32} = \kappa^{\mathbb{P}}_{53} = 0$	26534.3868	50	0.9992	-52968.7736	-52783.0870
(4) $\kappa^{\mathbb{P}}_{32} = \kappa^{\mathbb{P}}_{53} = \kappa^{\mathbb{P}}_{41} = 0$	26534.3825	49	0.9998	-52970.7650	-52788.7921
(5) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{45} = 0$	26534.3798	48	1.0000	-52972.7596	-52794.5004
(6) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{21} = 0$	26534.3787	47	1.0000	-52974.7574	-52800.2120
(7) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{12} = 0$	26534.3729	46	1.0000	-52976.7459	-52805.9142
(8) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{15} = 0$	26534.3083	45	1.0000	-52978.6166	-52811.4986
(9) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{31} = 0$	26534.3079	44	1.0000	-52980.6158	-52817.2115
(10) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{51} = 0$	26534.2901	43	1.0000	-52982.5802	-52822.8897
(11) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{23} = 0$	26534.2888	42	1.0000	-52984.5777	-52828.6009
(12) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{24} = 0$	26534.2784	41	1.0000	-52986.5569	-52834.2938
(13) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{43} = 0$	26534.2271	40	1.0000	-52988.4543	-52839.9050
(14) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{52} = 0$	26534.1479	39	1.0000	-52990.2957	-52845.4601
(15) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{13} = 0$	26534.1478	38	1.0000	-52992.2957	-52851.1738
(16) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{14} = 0$	26534.1476	37	1.0000	-52994.2953	-52856.8872
(17) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{42} = 0$	26534.1475	36	1.0000	**-52996.2951**	-52862.6007
(18) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{54} = 0$	26528.5899	35	0.8505	-52987.1798	-52857.1992
(19) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{25} = 0$	26528.5811	34	1.0000	-52989.1623	-52862.8953
(20) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{35} = 0$	26528.5326	33	1.0000	-52991.0653	**-52868.5121**
(21) $\kappa^{\mathbb{P}}_{32} = ... = \kappa^{\mathbb{P}}_{34} = 0$	26490.6458	32	0.0000	-52917.2917	-52798.4522

NOTE: We estimate and evaluate thirteen alternative specifications of the joint standard AFNS model on nominal and real yields. For each specification, we record its log-likelihood (LogL), number of parameters (k) and the p-value of a likelihood ratio test of the hypothesis that a specification with (k-i) parameters is different from the one with (k-i+1) parameters. The information criteria (AIC and BIC) are reported and we display their minimum in bold.

Table 4.11: 5 factor joint Shadow-rate AFNS estimates

$\kappa_t^{\mathbb{P}}$	$\kappa_{.,1}^{\mathbb{P}}$	$\kappa_{.,2}^{\mathbb{P}}$	$\kappa_{.,3}^{\mathbb{P}}$	$\kappa_{.,4}^{\mathbb{P}}$	$\kappa_{.,5}^{\mathbb{P}}$	θ^P	$\sigma_{i,i}^J$
$\kappa_{1,.}^{\mathbb{P}}$	0.0120	0.0000	0.0000	0.0000	0.0000	0.0265	0.0149
	(0.0314)					(0.0022)	(0.0005)
$\kappa_{2,.}^{\mathbb{P}}$	0.0000	0.1011	0.0000	0.0000	0.0823	-0.0047	0.0297
		(0.0316)			(0.0127)	(0.0014)	(0.0010)
$\kappa_{3,.}^{\mathbb{P}}$	0.0000	0.0000	0.0898	0.1442	-0.1847	0.0196	0.0278
			(0.0314)	(0.0283)	(0.0256)	(0.0033)	(0.0011)
$\kappa_{4,.}^{\mathbb{P}}$	0.0000	0.0000	0.0000	0.0158	0.0000	-0.0002	-0.0162
				(0.0277)		(0.0307)	(0.0012)
$\kappa_{5,.}^{\mathbb{P}}$	0.0000	0.0000	0.0000	-0.0534	0.1561	0.0130	0.0210
				(0.0263)	(0.0215)	(0.0298)	(0.0007)

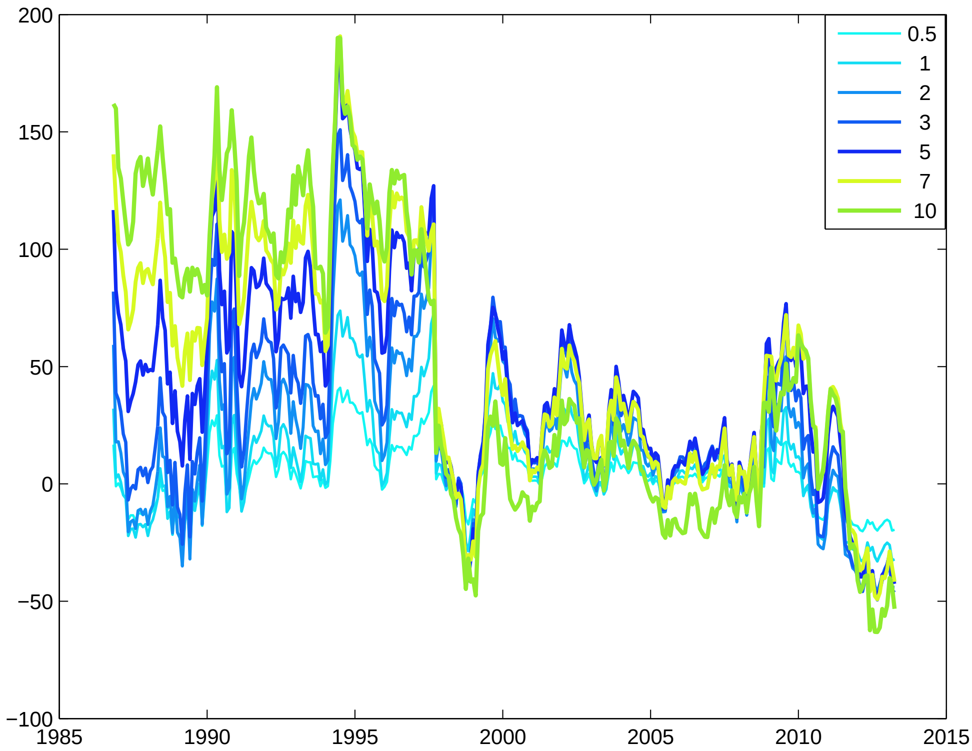
NOTE: The estimated parameters of the $\kappa^{J,\mathbb{P}}$ matrix, $\theta^{J,\mathbb{P}}$ vector, and diagonal diffusion matrix $\sigma_{i,i}^J$ are given for our preferred joint five-factor shadow-rate AFNS model for nominal and real yields. The estimated value of λ^N is 0.5311 with standard deviation of 0.0187 and the estimated value of λ^R is 0.1765 with standard deviation of 0.0095. The estimated value of α^R is 0.5538 with standard deviation of 0.0355. The numbers in parentheses are the standard deviations of the estimated parameters.

Table 4.12: Measures of fit for the 5 factor joint Shadow-rate AFNS model

Maturity in months	Mean(in bp)	RMSE(in bp)
Nominal yield		
6	-0.0545	7.5306
12	0.3977	3.9062
24	0.4343	5.3254
36	0.5764	6.6117
60	0.7707	9.0720
84	0.8617	10.1789
120	0.8483	15.1684
Real yield		
60	-1.8197	6.6548
72	-0.5877	2.0162
84	-0.0040	0.0133
90	0.0655	0.2045
96	0.0000	0.0001
108	-0.4821	1.3792
120	-1.3252	3.6122

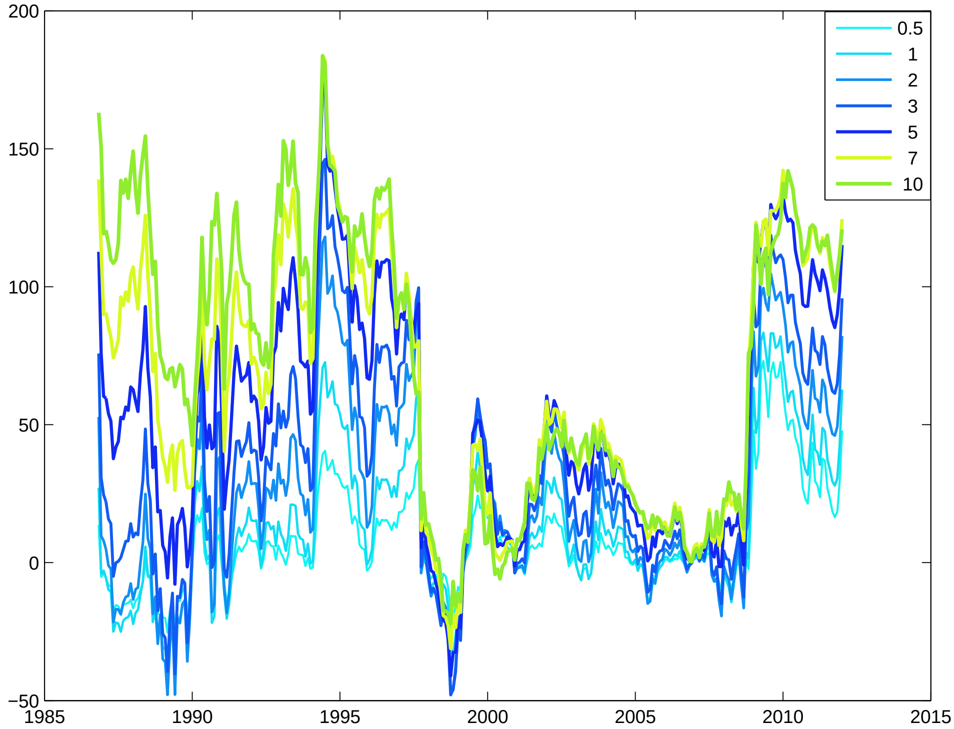
NOTE: The mean and RMSE of fitted errors of the preferred joint shadow-rate AFNS model for nominal and real yields are given. All values are measured in basis points. The nominal and real yields span from October 1986 to December 2011.

Figure 4.1: Nominal risk premia by maturity - AFNS - (in bp)



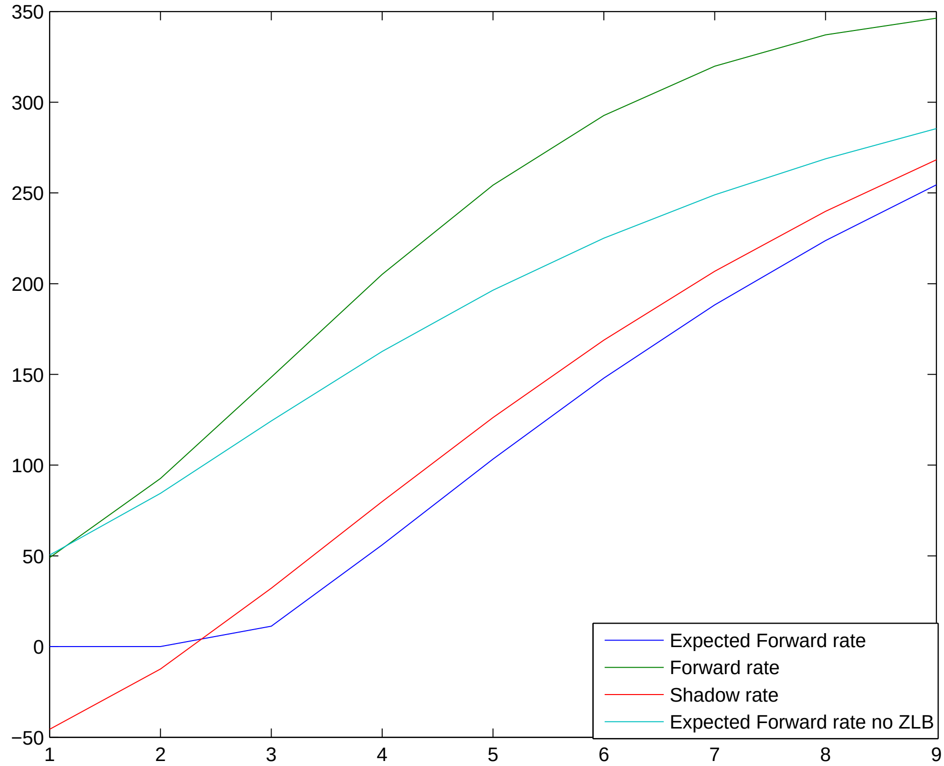
NOTE: Term premia of nominal yields at all maturities, measured in basis points, estimated with the preferred individual three-factor AFNS.

Figure 4.2: Nominal risk premia by maturity - Shadow-rate AFNS - (in bp)



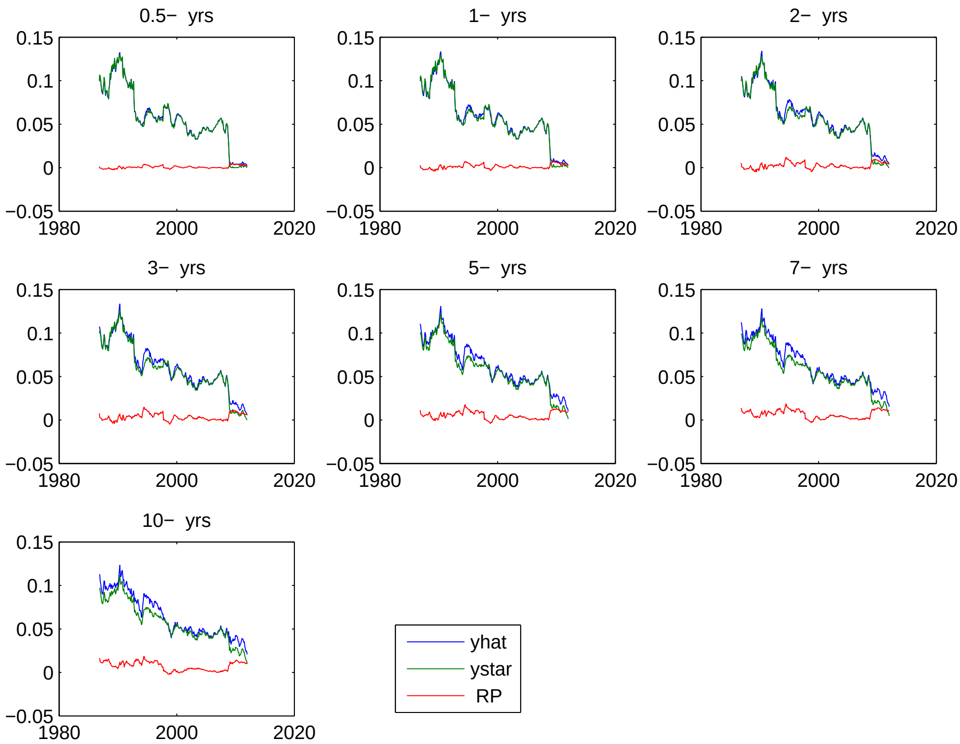
NOTE: Term premia of nominal yields at all maturities, measured in basis points, estimated with the preferred individual three-factor shadow-rate AFNS.

Figure 4.3: Forward and expected forward curves (in bp)



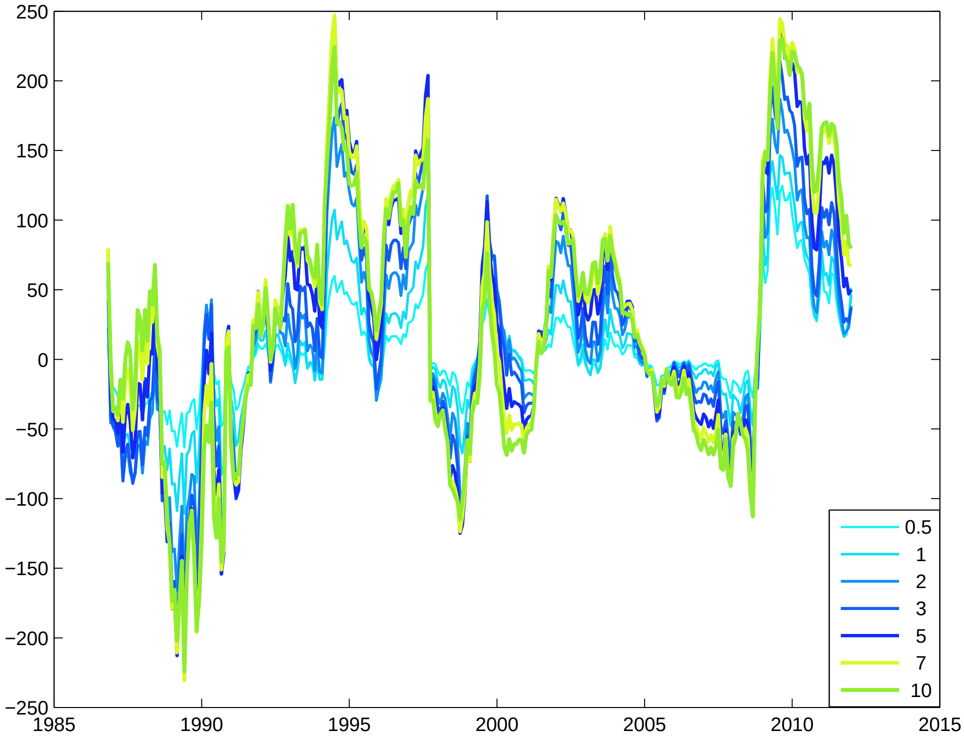
NOTE: The one-year forward curve of nominal yields, the expected one-year forward curves of nominal yields stemming from the two models (namely, the preferred individual three-factor AFNS and shadow-rate AFNS), as well as the shadow rate. All curves are extracted for December 2011 and are measured in basis points.

Figure 4.4: Model-implied nominal yield decomposition



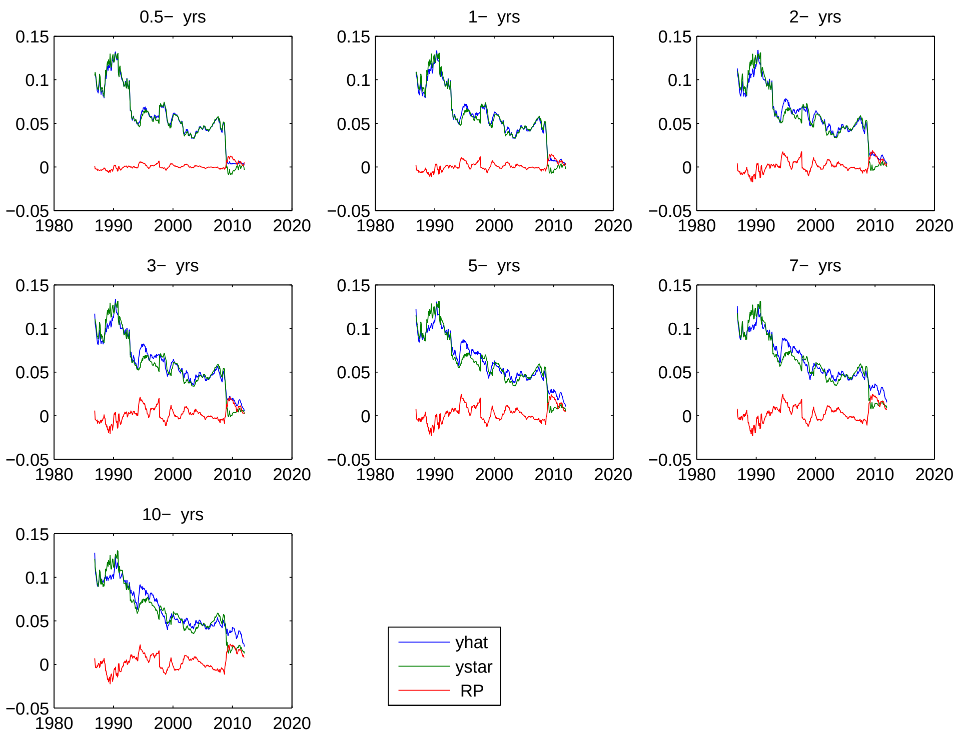
NOTE: Nominal yields (yhat), estimated with the preferred individual three-factor shadow-rate AFNS, decomposed into risk-neutral (ystar) and term premia (RP) components, by maturity.

Figure 4.5: Nominal risk premia by maturity (in bp)



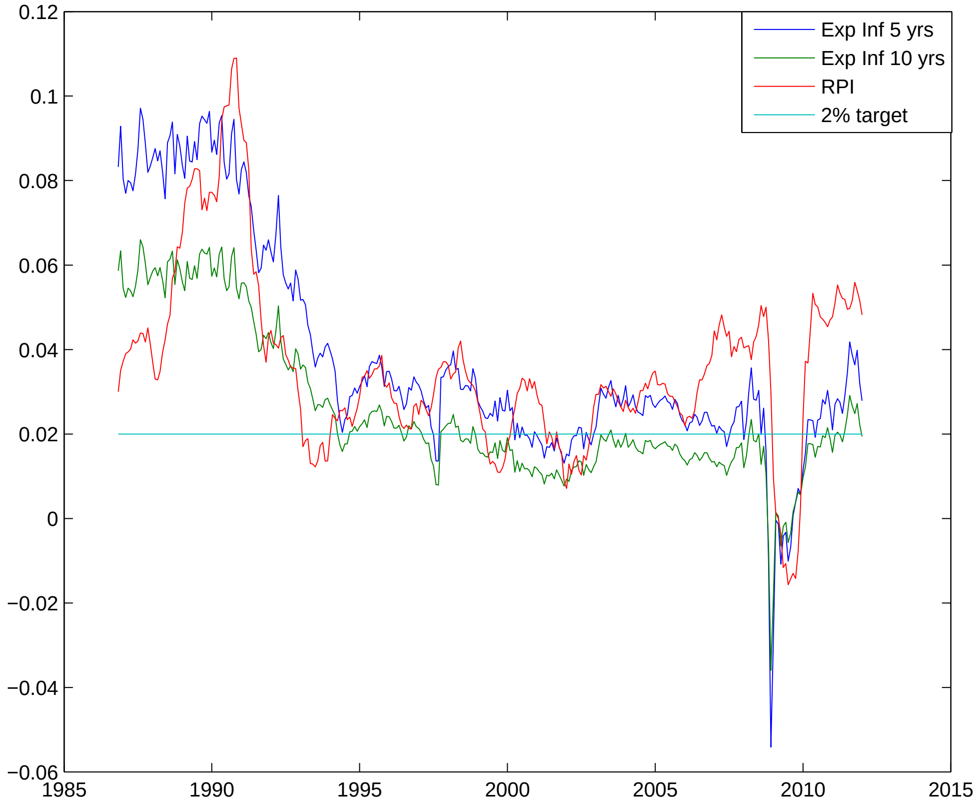
NOTE: Term premia of nominal yields at all maturities, measured in basis points, estimated with the preferred joint shadow-rate AFNS.

Figure 4.6: Model-implied nominal yield decomposition



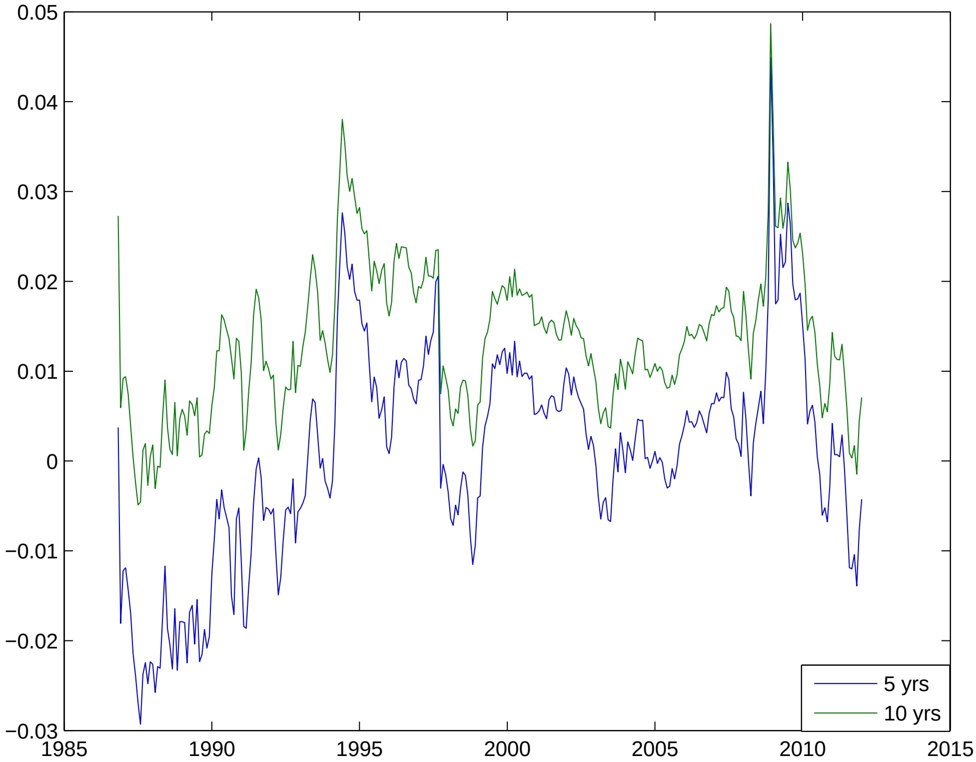
NOTE: Nominal yields (yhat), estimated with the preferred joint shadow-rate AFNS, decomposed into risk-neutral (ystar) and term premia (RP) components, by maturity

Figure 4.7: Inflation expectations by maturity and RPI



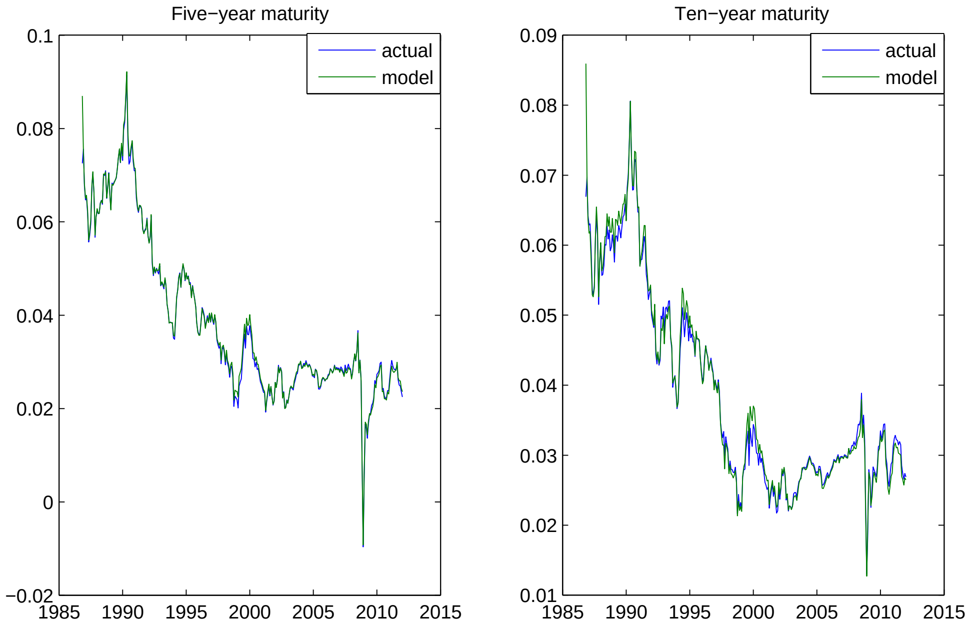
NOTE: The 5- and 10- year expected inflation rates, implied from the preferred joint shadow-rate AFNS model, and historical RPI and CPI inflation target.

Figure 4.8: Inflation premia by maturity



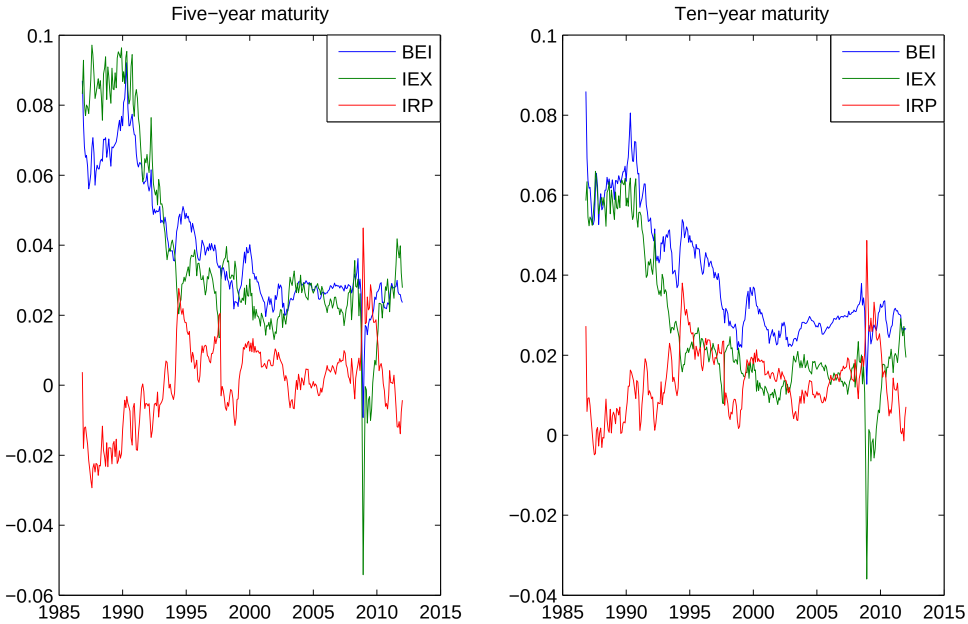
NOTE: The 5- and 10- year inflation risk premia, implied from the preferred joint shadow-rate AFNS model.

Figure 4.9: BEI rates: actual vs. model-implied



NOTE: The 5- and 10- year BEI rates, implied from the preferred joint AFNS model.

Figure 4.10: Model-implied decomposed BEI rates



NOTE: The 5- and 10- year BEI rates, implied from the preferred joint AFNS model, decomposed into inflation expectation (IEX) and risk premia (IRP) components.

4.B Appendix B: Shadow-rate AFNS model à la Krippner

The instantaneous shadow forward rates are obtained by deriving the logarithmic bond prices P(t,T) with respect to the maturity T, as follows:

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T} lnP(t,T) \\ &= X_1 + e^{-\lambda(T-t)}X_2 + \lambda(T-t)e^{-\lambda(T-t)}X_3 + A^f(t,T) \end{aligned} \tag{4.B.38}$$

where $A^f(t,T)$ is obtained below:

$$\begin{aligned} A^f(t,T) &= -\frac{\partial A(t,T)}{\partial T} \\ &= -\frac{1}{2}\sigma_{11}^2(T-t)^2 - \frac{1}{2}\sigma_{22}^2\left(\frac{1-e^{-\lambda(T-t)}}{\lambda}\right)^2 \\ &- \frac{1}{2}\sigma_{33}^2\left((T-t)e^{-\lambda(T-t)} - \frac{1-e^{-\lambda(T-t)}}{\lambda}\right)^2 \end{aligned} \tag{4.B.39}$$

Let us now denote by $\underline{f}(t,T)$, the Zero Lower Bound (ZLB) instantaneous forward rate. Setting $\Phi(.)$ to be the standard normal cumulative probability, we obtain a representation for $\underline{f}(t,T)$:

$$\underline{f}(t,T) = f(t,T)\Phi\left(\frac{f(t,T)}{\omega(t,T)}\right) + \omega(t,T)\frac{1}{\sqrt{2\pi}}exp\left(-\frac{1}{2}\left[\frac{f(t,T)}{\omega(t,T)}\right]^2\right) \tag{4.B.40}$$

where $\omega(t,T)$ is defined below as a transformation of the conditional variance of a European call option.

We denote by $v(t,T,T+\epsilon)$ the conditional variance of a European call option maturing at time T, contingent on the zero-coupon bond with maturity $T+\epsilon$.

$$
\begin{aligned}
v(t,T,T+\epsilon) &= \sigma_{11}^2\epsilon^2(T-t) + \sigma_{22}^2\left(\frac{1-e^{-\lambda\epsilon}}{\lambda}\right)^2 \frac{1-e^{-2\lambda(T-t)}}{2\lambda} \\
&+ \sigma_{33}^2\Bigg[\left(\frac{1-e^{-\lambda\epsilon}}{\lambda}\right)^2 \frac{1-e^{-2\lambda(T-t)}}{2\lambda} \\
&+ e^{-2\lambda\epsilon}\left[\frac{\epsilon^2-(T-t+\epsilon)^2e^{-2\lambda(T-t)}}{2\lambda} + \frac{\epsilon-(T-t+\epsilon)e^{-2\lambda(T-t)}}{2\lambda^2} + \frac{1-e^{-2\lambda(T-t)}}{4\lambda^3}\right] \\
&- \frac{1}{2\lambda}(T-t)^2e^{-2\lambda(T-t)} - \frac{1}{2\lambda^2}(T-t)e^{-2\lambda(T-t)} + \frac{1-e^{-2\lambda(T-t)}}{4\lambda^3} \\
&- \frac{\left(1-e^{-\lambda\epsilon}\right)e^{-\lambda\epsilon}}{\lambda^2}\left[\epsilon-(T-t+\epsilon)e^{-2\lambda(T-t)} + \frac{1-e^{-2\lambda(T-t)}}{2\lambda}\right] \\
&+ \frac{\left(1-e^{-\lambda\epsilon}\right)}{\lambda^2}\left[\frac{1-e^{-2\lambda(T-t)}}{2\lambda} - (T-t)e^{-2\lambda(T-t)}\right] \\
&+ \frac{\epsilon e^{-\lambda\epsilon}}{\lambda}\left[(T-t)e^{-2\lambda(T-t)} - \frac{1-e^{-2\lambda(T-t)}}{2\lambda}\right] \\
&+ \frac{\epsilon e^{-\lambda\epsilon}}{\lambda}\left[(T-t)^2e^{-2\lambda(T-t)} + \frac{1}{\lambda}(T-t)e^{-2\lambda(T-t)} - \frac{1-e^{-2\lambda(T-t)}}{2\lambda^2}\right]\Bigg]
\end{aligned}
\tag{4.B.41}
$$

The conditional variance is further transformed to obtain a representation of $\omega(t,T)^2$:

$$
\begin{aligned}
\omega(t,T)^2 &= \frac{1}{2}\lim_{\epsilon\to 0}\frac{\partial^2 v(t,T,T+\epsilon)}{\partial\epsilon^2} \\
&= \sigma_{11}^2(T-t) + \sigma_{22}^2\left(\frac{1-e^{-2\lambda(T-t)}}{2\lambda}\right) \\
&+ \sigma_{33}^2\left[\frac{1-e^{-2\lambda(T-t)}}{4\lambda} - \frac{1}{2}(T-t)e^{-2\lambda(T-t)} - \frac{1}{2}\lambda(T-t)^2e^{-2\lambda(T-t)}\right]
\end{aligned}
\tag{4.B.42}
$$

4.C Appendix C: Extended Kalman filter

The estimation of a shadow rate term structure model resembles the one of a Gaussian model in many ways. Specifically, the state equation of the state-space representation remains intact and the sole change in the algorithm stems from the non-linearity in the space equation. Therefore, rather than using a Kalman filter routine, an Extended Kalman filter is used, whereby the algorithm remains identical in all the steps that relate to the state equation, and the only change that occurs is to perform a Taylor expansion in order to approximate the space equation and linearize it.

First, let us disclose the details pertaining to the state equation, which are identical to the standard Kalman filter. Below is the transition equation in its discretized form.

$$X_T = \left[I - exp(-\kappa^{\mathbb{P}}(T-t))\right]\theta^{\mathbb{P}} + exp(-\kappa^{\mathbb{P}}(T-t))X_t + \eta_t \tag{4.C.43}$$

The standard moments conditions are displayed below.

$$\mathbb{E}^{\mathbb{P}}\left[X_T|\mathcal{F}_t\right] = \left[I - exp(-\kappa^{\mathbb{P}}(T-t))\right]\theta^{\mathbb{P}} + exp(-\kappa^{\mathbb{P}}(T-t))X_t \tag{4.C.44}$$

$$\mathbb{V}^{\mathbb{P}}\left[X_T|\mathcal{F}_t\right] = \int_t^T exp(-\kappa^{\mathbb{P}}(T-s))\Sigma\Sigma' exp(-\kappa^{\mathbb{P}'}(T-s))ds \tag{4.C.45}$$

The initial conditions for the Extended Kalman filter are set to the unconditional mean and covariance matrix, given in equation (4.C.46) and (4.C.47), as in the standard case.

$$\hat{X}_0 = \theta^{\mathbb{P}} \tag{4.C.46}$$

$$\hat{\Sigma}_0 = \int_0^{\infty} exp(-\kappa^{\mathbb{P}} s)\Sigma\Sigma' exp(-\kappa^{\mathbb{P}'} s)ds \tag{4.C.47}$$

Now, proceeding to the differences that stem from the non-linearity of the measurement equation. Denote by ψ the parameters of the model and assume the error terms η_t and ϵ_t are orthogonal and ϵ_t is i.i.d. The space equation can be written as follows, where the function k is non-linear.

$$y_t = k(X_t; \psi) + \epsilon_t \tag{4.C.48}$$

This equation is now linearized using a first-order Taylor expansion as shown below. The approximation is performed around the optimal guess of X_t within the prediction step of the algorithm, given by $X_{t|t-1}$.

$$k(X_t; \psi) \approx k(X_{t|t-1}; \psi) + \frac{\partial k(X_t; \psi)}{\partial X_t}|_{X_t = X_{t|t-1}}(X_t - X_{t|t-1}) \tag{4.C.49}$$

The space equation takes the following form.

$$y_t = \mathbb{A}_t(\psi) + \mathbb{B}_t(\psi)X_t + \epsilon_t \tag{4.C.50}$$

where $\mathbb{A}_t(\psi)$ and $\mathbb{B}_t(\psi)$ are provided below.

$$\mathbb{A}_t(\psi) = k(X_{t|t-1};\psi) - \frac{\partial k(X_t;\psi)}{\partial X_t}|_{X_t=X_{t|t-1}} X_{t|t-1} \tag{4.C.51}$$

$$\mathbb{B}_t(\psi) = \frac{\partial k(X_t;\psi)}{\partial X_t}|_{X_t=X_{t|t-1}} \tag{4.C.52}$$

Chapter 5

Conclusion

Throughout this thesis, an outline of affine term structure models is provided. This particular class of term structure models has been made very popular in recent years due to its ability to capture the dynamics of yields both across their time series and cross-section and its ease in imposing the absence of arbitrage, allowing in turn the obtention of adaptable risk premia specifications. Affine term structure models have the advantage of allowing various extensions, in a wide range, to their basic primary setup, asserting their importance in the literature. However, difficulties do arise in their estimation and in the interpretation of the latent factors used. This thesis addresses both problems by utilizing a specific structure to the factor loadings, known as the Nelson-Siegel method. The estimation of this term structure model not only circumvents the global optimum issues but further provides some interpretation to the factors, given the level, slope and curvature factors of the Nelson-Siegel interpolation are not only intuitive in their nature, but also have reliable macroeconomic links.

The present thesis introduces and employs dynamic term structure models to macroeconomic and financial research questions. More precisely, this study initially pertains to financial markets by establishing a tie between interest rates and exchange rates. The study follows by concerning itself with macroeconomic objectives, by exploiting the relationship between yields and inflation.

In a first instance, this study exploits a theoretical relationship between interest rates and exchange rates, namely the uncovered interest rate parity, with the aim to extract currency risk premia through a bilateral affine term structure model with stochastic volatility. The method proposed consists of developing an affine Arbitrage-Free class of dynamic Nelson-Siegel term structure models (AFNS) with stochastic volatility to obtain the domestic and foreign discount rate variations, which in turn are used to derive a representation of exchange rate depreciations. The manipulation of no-arbitrage restrictions allows to endogenously capture currency risk premia. The estimation exercise comprises of a state-space analysis using the Kalman filter. The imposition of the Dynamic Nelson-Siegel (DNS) structure allows for a tractable and robust estimation, offering significant computational benefits, whilst no-arbitrage restrictions enforce the model with theoretically appealing properties. Empirical findings suggest that estimated currency risk premia are able to account for the forward premium puzzle.

In a second instance, inflation expectations and inflation risk premia are derived using a shadow rate class of term structure models. In response to the recent financial crisis, the Bank of England reduced short term interest rates to 0.5%. With such low short term rates, traditional term structure models are likely to be inappropriate for estimating inflation expectations and risk premia, because expectations based on such models might implicitly violate the zero lower bound condition. In this segment both the nominal and real UK term structure of interest rates are studied, using the dynamic term structure model introduced by Christensen and Rudebusch (2013), which imposes the non-negativity of nominal short maturity rates. Estimates of the term premia, inflation risk premia and market-implied inflation expectations are provided. Findings indicate that the zero lower bound specification is necessary to reflect countercyclicality in nominal term premia projections and that medium and long term inflation expectations have been contained within narrower bounds since the early 1990s, suggesting monetary policy credibility after the introduction of inflation targeting.

For my future research projects, I wish to draw from the analysis and discussion of this thesis and elaborate further on this strand of the literature, this time emphasizing on the joint effect of monetary economics and finance on asset prices, financial markets and monetary policy. Two main perspectives emerge within my research agenda. A potential project, that inclines more towards macroeconomic concepts, consists in building an extension of the two above-mentioned models by constructing a Taylor rule type of model which would further extend to include growth. Furthermore, an alternative suggests to further exploit the interaction between macroeconomic and financial data to explore a gap in the literature. Specifically, the study includes providing an economic interpretation to the latent factors, used in the state-space representation, by venturing towards macro-finance models and high frequency data. This analysis is built on the prior belief that assets are affected by macroeconomic conditions but simultaneously suffer from microstructure phenomena.

Notwithstanding the extensions listed above, it is crucial to note that the most important message to draw from this thesis is that the literature on risk premia is still at its infancy due to the striking complexity involved in estimating an unobservable variable which nonetheless contains a very rich informational content. In turn, in future research, I wish to investigate the sensitivity of the price of risk, and consequently of risk premia, to different specifications in the mean reversion matrix of the states' dynamics. The aim is to determine a preferred specification for dynamic term structure models using a Bayesian shrinkage estimation approach.

To conclude, this thesis builds a spherical account of the versatility of affine models by implementing them to distinct monetary finance applications. Several of the pending issues in the literature are addressed and the grounds for future interesting questions are paved.

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