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Applications of conformal methods to the analysis of global properties of solutions to the Einstein field equations

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Submitted in partial fulfilment of the requirements for the degree of
Doctor of Philosophy

Declaration

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Details of collaboration and publications
The content of this thesis is based in the following papers:

- Gasperín E. and Valiente Kroon J.A., "Spinorial wave equations and stability of the Milne spacetime," *Classical and Quantum Gravity* **32** (Sept., 2015) 185021, `arXiv:1407.3317 [gr-qc]`.

- Gasperín E. and Valiente Kroon J.A., "Perturbations of the asymptotic region of the Schwarzschild–de sitter spacetime," *Annales Henri Poincaré* (2017) 1–73. `http://dx.doi.org/10.1007/s00023-016-0544-z`.

- Gasperín E. and Valiente Kroon J.A., "Zero rest-mass fields and the Newman-Penrose constants on flat space," *ArXiv e-prints* (Aug., 2016) , `arXiv:1608.05716 [gr-qc]`.

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Abstract

Although the study of the initial value problem in General Relativity started in the decade of 1950 with the work of Fourès-Bruhat, addressing the problem of global non-linear stability of solutions to the Einstein field equations is in general a hard problem. The first non-linear global stability result in General Relativity was obtained for the de-Sitter spacetime by means of the so-called conformal Einstein field equations introduced by H. Friedrich in the decade of 1980. The latter constitutes the main conceptual and technical tool for the results discussed in this thesis. In Chapter 1 the physical and geometrical motivation for these equations is discussed. In Chapter 2 the conformal Einstein equations are presented and first order hyperbolic reduction strategies are discussed. Chapter 3 contains the first result of this work; a second order hyperbolic reduction of the spinorial formulation of the conformal Einstein field equations. Chapter 4 makes use of the latter equations to give a discussion of the non-linear stability of the Milne universe. Chapter 5 is devoted to the analysis of perturbations of the Schwarzschild-de Sitter spacetime via suitably posed asymptotic initial value problems. Chapter 6 gives a partial generalisation of the results of Chapter 5. Finally a result relating the Newman-Penrose constants at future and past null infinity for spin-1 and spin-2 fields propagating on Minkowski spacetime close to spatial infinity is discussed in Chapter 7 exploiting the framework of the cylinder at spatial infinity. Collectively, these results show how the conformal Einstein field equations and more generally conformal methods can be employed for analysing perturbations of spacetimes of interest and extract information about their conformal structure.

Acknowledgements

Foremost I would like to thank Dr Juan A. Valiente Kroon for his guidance during my PhD studies. I am grateful for his patience during long discussions about the conformal Einstein field equations, mentoring and advice. During this period I have had the opportunity to see the Mathematical Relativity Group at Queen Mary University of London grow and evolve. In this process I have met people that undoubtedly had a positive impact in my research and my personal point of view about science. In this group I would like to especially thank Dr Shabnam Beheshti for her academic advice and support. On a more personal level, to my mentors: Juan and Shabnam, my scientific and personal admiration. I have benefited from interaction with the Cosmology Group and the Geometry and Analysis Group at Queen Mary University of London. In this regard, I would like to thank Dr Timothy Clifton and Dr Reto Buzano. My friends and office-mates (Andrea Salas, Jarrod Williams, Adem Hursit, José Luis Obregón, Susana Tecante, Manik Nava, Paulina Flores, Liam Williams, Adrián Ortiz, Sol Gil, Diego Carranza and Michael Cole) played also an important role during this period of my life both academically and personally. I have profited from interaction and discussion with Dr Christian Lübbe and Dr Alfonso García-Parrado. I want to thank Dr José Luis Jaramillo for his support and advice. I would like to thank Dr Xavier Hernández and Dr Sergio Mendoza, thanks for the encouragement to pursuit my goals! I would like to use this opportunity to thank my examiners Prof. Paul Tod and Dr Gustav Holzegel in advance for reading this thesis.

I gratefully acknowledge financial support from Consejo Nacional de Ciencia y Tecnología (CONACyT scholarship 494039/218141) and Secretaría de Educación Pública (Beca complemento SEP, 2013, 2014, 2015 & 2016). I also received financial support to fund conference travels from Institute of Physics IoP (Barber Trust Fund and Research Student Conference Fund), Queen Mary University of London (Eileen Colyer Prize), University of Potsdam (Andrejewski Days: 100 Years of General Relativity), and the Erwin Schrödinger Institute for Mathematics and Physics, (Geometric transport equations in General Relativity). I also want to thank the Gravitational Physics Group at University of Vienna for a space provided during a one-month academic visit.

A mi familia, los presentes y los que ya partieron, Alma Gpe., Arsenia, Alicia, Magali, Marissa, Pedro E., Olga y César A. gracias por su apoyo en mis proyectos y su interminable cariño. Asimismo, agradezco a los Sres. Ángeles y Benito por su confianza en mi. Finalmente quiero agradecer a Mónica por su apoyo durante estos años, por haber estado siempre ahí, en los buenos y los malos momentos. Gracias por nunca dejar de creer en mi.

Contents

1 Introduction

1.1 General Relativity

The General theory of Relativity is the most successful theory of gravity. Since its birth in 1916, starting with the prediction of the redshift effect, the bending of light rays and explaining the perihelion of Mercury, General Relativity has passed numerous observational tests. More recently, in February 2016, the detection of Gravitational waves originating from a pair of merging black holes announced by the LIGO Scientific Collaboration confirms the validity of General Relativity not only in the regime of weak gravitational fields but situations where both strong and dynamical gravitational fields interact. In contrast with Newtonian gravity where the gravitational field is encoded in one single scalar, in General Relativity the gravitational field is encoded in 10 quantities corresponding to the components of a Lorentzian metric. This issue in conjunction with the fact that the equations governing the gravitational field in the theory —the Einstein field equations— are a coupled system of nonlinear second order partial differential equations for the metric components makes very difficult to obtain information about the behaviour of the spacetime in realistic scenarios such as the one described above. Despite the complexity of the Einstein field equations it was very rapidly realised that under suitable assumptions representing idealised physical scenarios one can find explicit solutions to the Einstein field equations. The paradigmatic example is the Schwarzschild solution representing the exterior gravitational field of a spherically symmetric and static configuration of matter. The study of exact solutions showed unexpected features of the theory such as the existence of black holes and singularities. At first instance some of these features may look as merely artefacts of the high symmetry which one would expect to disappear once one is confronted with more realistic situations i.e., less symmetric and dynamical. Nevertheless, singularities in General Relativity are ubiquitous and, as shown by R. Penrose and S. Hawking in their *singularity theorems*, if a *trapped surface* (a 2-surface for which the null expansions $\theta_{\pm}$ are negative) exists and suitable energy conditions are satisfied the solutions to the Einstein field equations will be geodesically incomplete —see [1, 2]. In some cases the latter can be interpreted as an indication of the presence of a curvature singularity. Whether or not this is generically the case is still a research question —see for instance Conjecture 17.2 in [3]. Additionally, the study of exact solutions shows that in several situations, when present, the curvature singularity is hidden by a event horizon so that it is causally disconnected from the rest of the spacetime, i.e., singularities could not be seen from distant observers. The latter constitutes the so-called *weak cosmic censorship conjecture*. Despite the fact that great insight is gained from the study of exact solutions they do not exhaust all the space of solutions to the Einstein field equations. Consequently, if a deeper analysis of the generic properties of the solutions of the Einstein field equations is in order, one requires a more systematic approach to explore the space of solutions of the theory. Moreover, to understand the relevance and physical significance of these explicit solutions it is necessary to analyse their stability. In other words, to investigate the general behaviour of perturbed solutions to the Einstein field equations which are in some sense close to a reference solution. The latter could help to distinguish those properties and structures which are preserved in more realistic situations from those which only arise in idealised scenarios. The most systematic way to approach this problem is through suitable posed initial value problems.

1.2 The Cauchy problem in General Relativity

The initial value problem in General Relativity started with the seminal work of Fourès-Bruhat [4] in which it was shown that, if the gauge is fixed appropriately, the equations governing General Relativity split into constraint and evolution equations. To see this in more detail recall that the Einstein field equations in vacuum with vanishing cosmological constant, $\tilde{R}_{\mu\nu} = 0$, in local coordinates (x^μ) read

$$\tilde{R}_{\mu\nu} = -\frac{1}{2}\tilde{g}^{\lambda\rho}\partial_\lambda\partial_\rho\tilde{g}_{\mu\nu} + \tilde{\nabla}_{(\mu}\tilde{\Gamma}_{\nu)} + \tilde{g}_{\lambda\rho}\tilde{g}^{\sigma\tau}\tilde{\Gamma}_\sigma{}^\lambda{}_\mu\tilde{\Gamma}_\tau{}^\rho{}_\nu + 2\tilde{\Gamma}_\lambda{}^\sigma{}_\rho\tilde{g}^{\lambda\tau}\tilde{g}_{\sigma(\mu}\tilde{\Gamma}_{\nu)}{}^\rho{}_\tau = 0,$$

where

$$\tilde{\Gamma}_\mu{}^\nu{}_\lambda = \frac{1}{2}\tilde{g}^{\nu\rho}(\partial_\mu\tilde{g}_{\rho\lambda} + \partial_\lambda\tilde{g}_{\mu\rho} - \partial_\rho\tilde{g}_{\mu\lambda})$$

and $\tilde{\Gamma}_\mu \equiv \tilde{g}^{\alpha\beta}\tilde{\Gamma}_\alpha{}^\mu{}_\beta$ are the so-called *contracted Christoffel symbols*. Define $\tilde{H}_{\mu\nu}$ via

$$\tilde{H}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{S}_{\mu\nu},$$

where

$$\tilde{S}_{\mu\nu} \equiv \partial_\mu\tilde{\Gamma}_\nu + \partial_\nu\tilde{\Gamma}_\mu.$$

A computation shows that the contracted Christoffel symbols are associated to the choice of coordinates since

$$\tilde{\square}x^\mu = \tilde{\Gamma}^\mu,$$

where $\tilde{\square} \equiv \tilde{\nabla}^\nu\tilde{\nabla}_\nu$. Notice that the principal part of $\tilde{H}_{\mu\nu}$ coincides with that of $\tilde{\square}\tilde{g}_{\mu\nu}$. With these definitions introduced and observations made, one of the main ideas in the work of Fourès-Bruhat is to consider a Cauchy problem for the *reduced Einstein field equations*

$$\tilde{H}_{\mu\nu} = 0, \tag{1.1}$$

which constitute a system of wave equations for the metric components $\tilde{g}_{\mu\nu}$. Let $\tilde{\mathcal{S}}$ denote a spacelike hypersurface with normal $\tilde{n}^\mu$. The equation (1.1) is then supplemented with data

$$\tilde{g}_{\mu\nu}|_{\tilde{\mathcal{S}}} = \tilde{h}_{\mu\nu}, \qquad \tilde{n}^\alpha \partial_\alpha \tilde{g}_{\mu\nu}|_{\tilde{\mathcal{S}}} = 2\tilde{K}_{\mu\nu},$$

satisfying

$$\tilde{\Gamma}_\mu|_{\tilde{\mathcal{S}}} = 0, \qquad \tilde{n}^\nu \partial_\nu \tilde{\Gamma}_\mu|_{\tilde{\mathcal{S}}} = 0, \tag{1.2}$$

where $\tilde{h}_{\mu\nu}$ is a 3-dimensional Riemannian metric and $\tilde{K}_{\mu\nu}$ is a symmetric tensor on $\tilde{\mathcal{S}}$. A calculation using the contracted second Bianchi identity shows that $\tilde{\Gamma}_\mu$ satisfy a system of homogeneous linear wave equations, hence, the conditions (1.2) ensure that $\tilde{\Gamma}_\mu = 0$. The latter, in turn, implies that $\tilde{H}_{\mu\nu}$ and $\tilde{R}_{\mu\nu}$ coincide. The conditions (1.2) are associated to the *Hamiltonian and momentum constraints* —see [4, 5] for a comprehensive discussion. In this formulation of General Relativity, the initial data corresponds to a triple $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ where $\tilde{\mathcal{S}}$ denotes a 3-dimensional manifold, $\tilde{\boldsymbol{h}}$ a Riemannian metric and $\tilde{\boldsymbol{K}}$ a symmetric tensor. One of the most important results in this regard was proved by Choquet-Bruhat and Geroch in [6] where it was shown that associated to each triple $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ satisfying the constraint equations there exists a unique maximal globally hyperbolic development $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. In other words, $\tilde{\mathcal{S}}$ is a spacelike hypersurface of $\tilde{\mathcal{M}}$, $\tilde{\boldsymbol{h}}$ is the induced metric of $\tilde{\boldsymbol{g}}$ on $\tilde{\mathcal{S}}$ and $\tilde{\boldsymbol{K}}$ is associated with the extrinsic curvature. The adjective hyperbolic makes reference to the fact that the evolution equations obtained in this formulation of the Einstein field equations are hyperbolic. This property is fundamental from the physical point of view as it is closely related to the notion of causality and to the finite speed of propagation of signals.

Remark 1. Despite the fact that, Fourès-Bruhat's approach leads naturally to the analysis of wave equations, one can study the Cauchy problem in General Relativity via first order symmetric hyperbolic systems. To see this, assume that a local system of coordinates (x^μ) on $\tilde{\mathcal{M}}$ has been fixed such that $\tilde{\boldsymbol{n}} = \boldsymbol{\partial}_{x^0} = \boldsymbol{\partial}_0$ is the normal vector to a spacelike hypersurface $\tilde{\mathcal{S}}$. Now, consider the wave equation

$$\tilde{\Box} u = F(x, u, \partial u), \tag{1.3}$$

where u is a scalar function on $\tilde{\mathcal{M}}$, F is a smooth function of its arguments and ∂u denotes, collectively, the first derivatives of u. Introduce a new variable $w_\mu = \partial_\mu u$.

Then,

$$
\begin{aligned}
-\tilde{g}^{00}\partial_0 w_0 - 2\tilde{g}^{0\mu}\partial_\mu w_0 &= \tilde{g}^{\mu\nu}\partial_\mu w_\nu + F(x,u,w_\mu), && (1.4)\\
\tilde{g}^{\mu\nu}\partial_0 w_\mu &= \tilde{g}^{\mu\nu}\partial_\mu w_0, && (1.5)\\
\partial_0 u &= w_0, && (1.6)
\end{aligned}
$$

with initial data $(u, w_\mu)|_{\tilde{\mathcal{S}}}$ satisfying

$$
w_\mu|_{\tilde{\mathcal{S}}} = (\partial_\mu u)|_{\tilde{\mathcal{S}}}, \tag{1.7}
$$

is a symmetric hyperbolic system for (u, w_μ) —see [7] for further discussion.

1.3 Conformal methods in General Relativity

Although the study of the initial value problem in General Relativity started in the decade of 1950 with the work of Fourès-Bruhat, addressing the problem of global non-linear stability of solutions to the Einstein field equations is in general a hard problem. In fact, the first global non-linear stability result in General Relativity had to wait until the decade of 1980 when in [8] and [9] H. Friedrich proved the *semi-global non-linear stability of the Minkowski spacetime* and the *global non-linear stability of the de-Sitter spacetime*. A proof of the full non-linear stability of the Minkowski spacetime using *vector field methods* has been given in [10]. One of the essential ideas in [8] and [9] was the use of the so-called conformal Einstein field equations —introduced in [11]— to pose an initial value problem. As the name suggests, the central concept in these equations is that of a conformal transformation. Conformal transformations have a long tradition in General Relativity going back at least to the decade of 1960. R. Penrose introduced the concept of conformal rescalings for the study of *asymptotics* —the study of the behaviour of gravitational fields for large distances and late times. In his proposal one starts with a *physical* spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ where $\tilde{\mathcal{M}}$ is a 4-dimensional manifold and $\tilde{\boldsymbol{g}}$ is a Lorentzian metric which is a solution to the Einstein field equations

$$
\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda\tilde{g}_{ab} = \tilde{T}_{ab}, \tag{1.8}
$$

where $\tilde{R}_{ab}$ and $\tilde{R}$ are the Ricci tensor and Ricci scalar of $\tilde{g}_{ab}$ respectively, λ is the cosmological constant and $\tilde{T}_{ab}$ is the energy momentum tensor. Notice that, in vacuum $\tilde{T}_{ab} = 0$, the Einstein field equations (1.8) reduce to

$$
\tilde{R}_{ab} = \lambda\tilde{g}_{ab}. \tag{1.9}
$$

Then, one introduces a *unphysical* spacetime $(\mathcal{M}, \boldsymbol{g})$ into which $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is conformally embedded. Accordingly, there exists an embedding $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ such that

$$\varphi^* \boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}. \tag{1.10}$$

The scalar function Ξ is the so-called *conformal factor*. By suitably choosing Ξ the metric $\boldsymbol{g}$ may be well defined at the points where $\Xi = 0$. The set

$$\mathscr{I} \equiv \{ p \in \mathcal{M} \mid \Xi(p) = 0, \ \ \mathbf{d}\Xi(p) \neq 0 \}$$

is called the conformal boundary. The set of points where the conformal factor vanishes is at infinity from the physical spacetime perspective. More precisely, if $\tilde{s}$ and s denote, $\tilde{\boldsymbol{g}}$-affine and $\boldsymbol{g}$-affine parameters of a null geodesic $\gamma \subset \tilde{\mathcal{M}}$, then

$$\frac{d\tilde{s}}{ds} = \frac{1}{\Xi^2},$$

consequently,

$$\tilde{s} = \int \frac{1}{\Xi^2} \mathrm{d}s. \tag{1.11}$$

Since $\Xi = 0$ and $\mathbf{d}\Xi \neq 0$ on $\mathscr{I}$ then one can choose s to vanish at $\mathscr{I}$ and set $\Xi = \mathcal{O}(s^\alpha)$ with $\alpha \geq 1$ then it follows from equation (1.11) that $\tilde{s} \to \infty$ as $\Xi \to 0$ —see [12, 13] for further discussion. Thus, $\mathscr{I}$ can be identified with the collection of endpoints —on $(\mathcal{M}, \boldsymbol{g})$— of null geodesics of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$.

Definition. A *conformal extension* of a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfying the vacuum Einstein field equations (1.8) consists on a manifold $\mathcal{M}$ equipped with a metric $\boldsymbol{g}$, a smooth conformal factor Ξ and a diffeomorphism $\varphi : \tilde{\mathcal{M}} \to \mathcal{U} \subseteq \mathcal{M}$, such that:

$$\begin{aligned}
&\varphi^* \boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}} \quad \text{is well defined at} \quad \Xi = 0, \\
&\Xi > 0 \qquad \text{in} \qquad \mathcal{U}, \\
&\Xi = 0 \quad \text{and} \quad \mathbf{d}\Xi \neq 0 \quad \text{on} \quad \partial\mathcal{U}.
\end{aligned}$$

The set $\mathscr{I} \equiv \partial\mathcal{U}$ is called *conformal boundary.* Since every point in $\mathscr{I}$ can be identified with the endpoint of a null geodesics of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ the set $\mathscr{I}^+$ denoting the portion of $\mathscr{I}$ corresponding to future endpoints of null geodesics will be called *future conformal boundary.* Similarly, the set $\mathscr{I}^-$ denoting the portion of $\mathscr{I}$ corresponding to past endpoints of null geodesics will be called *past conformal boundary.*

Remark 2. Observe that the latter definition does not require that every null geodesic acquires two distinct endpoints at $\mathscr{I}$. In particular, it leaves the possibility for the existence of null geodesics that do not reach $\mathscr{I}$.

The relevance of this construction goes beyond the study of asymptotics and isolated gravitational systems since the unphysical metric $\boldsymbol{g}$ contains the same causal information as the physical metric $\tilde{\boldsymbol{g}}$. In this framework, a natural question that arises is, how do the Einstein field equations behave under a conformal transformation of the metric? A straightforward computation using the conformal transformation laws for the curvature tensors shows that, in the vacuum case with $\lambda = 0$, the Einstein field equations imply

$$R_{ab} - \frac{1}{2}Rg_{ab} = -2\Xi^{-1}(\nabla_a\nabla_b\Xi - \nabla_c\nabla^c\Xi g_{ab}) - 3\Xi^{-2}\nabla_c\nabla^c\Xi g_{ab}, \tag{1.12}$$

where R_{ab}, R and ∇_a are the Ricci tensor, Ricci scalar and Levi-Civita connection of the unphysical metric g_{ab}. From the last equation one immediately observes that the Einstein field equations are not conformally invariant. Moreover, equation (1.12) is formally singular at the conformal boundary. To have a satisfactory equation for the unphysical metric it is necessary to derive a regular version of equation (1.12). An approach to deal with this problem was given in [11] where a regular set of equations for the unphysical metric was derived. These equations are known as the *conformal Einstein field equations*. The crucial property of these equations is that they are regular at the points where $\Xi = 0$ and a solution thereof implies whenever $\Xi \neq 0$ a solution to the Einstein field equations. At its core, the conformal Einstein field equations constitute a system of differential conditions on the curvature tensors and the conformal factor.

There are two versions of these equations: the *standard conformal Einstein field equations* and the *extended conformal Einstein field equations*. In the former, these differential conditions are expressed in terms of the *Levi-Civita connection* of $\boldsymbol{g}$, while in the latter the conditions are expressed in terms of *Weyl connections*. Additionally, the standard conformal Einstein field equations can be expressed in three different formulations: the metric, frame and spinorial formulations. In the metric formulation, the unphysical metric $\boldsymbol{g}$ is part of the unknowns while in the frame version one introduces a $\boldsymbol{g}$-orthonormal frame with respect to which all the geometric quantities are expressed. The frame formulation of the equations leads naturally to a spinorial description which exhibits in a clearer way the algebraic structure of the equations. In particular, this algebraic structure can be exploited to construct alternative representations of these equations —see [14] for a discussion of a representation of the spinorial conformal Einstein field equations as wave equations. In the case of the extended conformal Einstein field equations one has frame and spinorial formulations as well. These conformally invariant representations of the Einstein field equations are not only advantageous from the purely theoretical point of view but also for applications since the conformal framework allows to recast global problems in $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ as local problems in $(\mathcal{M}, \boldsymbol{g})$.

1.4 Results obtained in this thesis

Similar to the case of the Einstein field equations, a problem that one has to face when dealing with the analysis of the conformal Einstein field equations is the issue of gauge freedom. In the classical treatment of the Cauchy problem in General Relativity —see [4]— a judicious choice of coordinates allows to reduce the equations to a system of wave equations for the metric components. Interestingly, in the original treatment of the conformal Einstein field equations the hyperbolic reduction strategies used lead to a first order system of equations —see [11, 13, 15–17]. In the case of the spinorial formulation of the standard conformal Einstein field equations the use of *gauge source functions* and the *space spinor formalism* renders a first order system of symmetric hyperbolic evolution equations. In the case of the extended conformal Einstein field equations the gauge fixing is performed exploiting a congruence of curves with special conformal properties: *conformal geodesics*. This hyperbolic reduction strategy leads to a first order system of symmetric hyperbolic equations as well. In [18] a second order hyperbolic reduction of the *metric formulation of the standard conformal Einstein field equations* has been obtained and used for the analysis of the asymptotic characteristic problem on a cone —see [19]. In Chapter 3, a second order hyperbolic reduction of the equations for the *spinorial formulation of the standard conformal Einstein field equations* is obtained. The spinorial formulation is advantageous as the algebraic structure of the equations is simpler when expressed in spinorial form and the construction of the wave equations can be done in a systematic way. In particular, the equation for the *rescaled Weyl spinor*, which can be considered as the central object in the discussion of the conformal Einstein field equations, becomes particularly simple. Additionally, the use of spinors gives access to a wider set of gauge source functions than those available in the metric formulation. As an application of the analysis in Chapter 3, a discussion of the non-linear stability of the Milne spacetime is given in Chapter 4. This spacetime is a spatially flat Friedman-Lemaître-Robertson-Walker solution to the Einstein field equations with vanishing cosmological constant —see e.g. [20]. Moreover, the Milne Universe can be seen to be a part of the Minkowski spacetime written in comoving coordinates adapted to the worldline of a particle. In this chapter, perturbations of exact initial data —for the wave equations derived in Chapter 3— corresponding to the Milne Universe are considered. Then the theory of symmetric hyperbolic systems contained in [21] is used to obtain a non-linear stability result for small perturbations of the Milne Universe.

A common feature that is exploited in the the analysis of constant curvature spacetimes by means of conformal methods (the Minkowski, de-Sitter and anti de-Sitter spacetimes) is that they can be conformally embedded in the Einstein cylinder —see [8, 9, 22]. The latter is convenient as, an explicit solution to the conformal Einstein field equations can be identified. In other words, most of the existence and stability results using the conformal Einstein field equations have been restricted to the analysis of perturbations of conformally flat spacetimes. Therefore, an interesting question is whether the conformal Einstein field equations can be exploited in the analysis of global properties of non-conformally flat spacetimes and, in particular, in the stability the analysis black hole spacetimes. On the other hand, from a physical point of view, observations have established that the universe is expanding. Therefore, spacetimes describing isolated systems embedded in a de-Sitter universe constitute a class of physically relevant spacetimes to be analysed. In view of these remarks, in Chapter 5 the Schwarzschild-de Sitter spacetime is analysed using the extended conformal Einstein field equations. The presence of a cosmological constant with a de-Sitter like value —see Section 1.6 for definitions— is of importance as it implies that the conformal boundary is spacelike. The use of conformal methods in this setting is natural as the conformal Einstein field equations allow to discuss asymptotic initial value problems: initial value problems for which the initial hypersurface corresponds to the conformal boundary. Moreover, the *conformal constraint equations* acquire a particular simple form at the conformal boundary so that the asymptotic initial data is encoded *essentially* in the *induced metric at the conformal boundary* h_{ij} and the *electric part of the rescaled Weyl tensor* d_{ij}. As discussed in detail in Chapter 5, the induced metric at the conformal boundary for the Schwarzschild-de Sitter spacetime is conformally flat. Furthermore, there exists a conformal representation in which the initial data for the rescaled Weyl tensor is regular and homogeneous so that one can integrate the extended conformal Einstein field equations along single conformal geodesics. This is not directly evident since, as discussed in detail in Chapter 5, there are conformal representations in which the initial data for the rescaled Weyl tensor becomes singular at the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ —corresponding to the region in the Penrose diagram of the Schwarzschild-de Sitter spacetime where the horizons of the spacetime appear to meet the conformal boundary. The insight gained from the analysis of the evolution of the exact asymptotic initial data corresponding to the Schwarzschild-de Sitter spacetime is used to discuss non-linear perturbations of this exact data by exploiting the theory of symmetric hyperbolic systems contained in [23]. The spacetimes constructed in this way can be regarded as perturbations of the asymptotic region of the Schwarzschild-de Sitter spacetime. Moreover, they serve as non-trivial examples of the theory of asymptotics for de Sitter-like spacetimes given in [24].

Notice that, despite the fact that the global non-linear stability of the Kerr-de Sitter spacetime has been addressed in the recent work [25], the discussion given in [25] is restricted to the black hole exterior region. In view of the domain of dependence property of solutions to the Einstein field equations, the stability of the black hole exterior can be analysed independently of the asymptotic region —see [26] for further discussion. In the asymptotic initial value problem considered in

Chapter 5, the domain of influence of the initial data is contained in the region corresponding to the asymptotic region of the Schwarzschild-de Sitter spacetime. The question whether the analysis given in Chapter 5 for the Schwarzschild-de Sitter spacetime can be generalised to the case of the Kerr-de Sitter spacetime is explored in Chapter 6. In this chapter, in particular, asymptotic initial data for the Kerr-de Sitter spacetime is obtained and a local existence result for spacetimes arising from asymptotic initial data close to that of the Kerr-de Sitter spacetime is obtained.

The singular behaviour of the asymptotic initial data for the rescaled Weyl tensor at the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ of the Schwarzschild-de Sitter spacetime is not completely unexpected. Actually, one of the main difficulties in establishing a global result for the stability of the Minkowski spacetime using conformal methods lies on the fact that the initial data for the conformal Einstein field equations are not smooth at i^0. In [11] the initial data are not prescribed on a Cauchy hypersurface but on a hyperboloid $\tilde{\mathcal{H}}$ whose conformal extension in $\mathcal{M}$ intersects $\mathscr{I}$. In the case of the problem of spatial infinity i^0, a milestone in the resolution of this problem is the construction, originally introduced in [27], of a new representation of spatial infinity known as the *cylinder at spatial infinity*. With this motivation in mind, and the fact that the analysis of conserved quantities at null infinity —the so-called Newman-Penrose constants— has gained some interest recently due to the discussion given in [28], in Chapter 7, the framework of the cylinder at spatial infinity is exploited for the analysis of the Newman-Penrose constants. More specifically, the framework of the cylinder at spatial infinity is used to clarify the correspondence between data on a spacelike hypersurface for the spin-1 and spin-2 fields —representing the Maxwell spinor and the linearised gravitational field, respectively— propagating on a Minkowski background and the value of their corresponding Newman-Penrose constants at future and past null infinity. In particular, it was shown that the electromagnetic NP constants at future and past null infinity case, are related to each other as they arise from the same terms in the initial data.

Collectively, these results show how the conformal Einstein field equations and more generally *conformal methods* can be employed for analysing perturbations of spacetimes of interest and extract information about their conformal structure.

1.5 Structure of this thesis

In Chapter 2 the conformal Einstein equations are presented and first order hyperbolic reduction strategies are discussed. Chapter 3 contains the first result of this thesis, a second order hyperbolic reduction of the spinorial formulation of the conformal Einstein field equations —see Proposition 1 and 2. Chapter 4 makes use of the latter equations to give a discussion of the non-linear stability of the Milne universe —see Main Result 1. Chapter 5 is devoted to the analysis of perturbations of the Schwarzschild-de Sitter spacetime via suitably posed *asymptotic initial value problems* —see Main Result 2. Chapter 6 provides with a generalisation of the results of Chapter 5, more specifically, an existence result for perturbations of the Kerr-de Sitter spacetime is given —see Theorem 4. Finally a result relating the Newman Penrose constants at future and past null infinity for spin-1 and spin-2 fields propagating on Minkowski spacetime close to spatial infinity is discussed in Chapter 7 exploiting the framework of the *cylinder at spatial infinity* —see Main Result 3.

1.6 Notation and Conventions

The signature convention for (Lorentzian) spacetime metrics is $(+,-,-,-)$. In these conventions the cosmological constant λ of the de Sitter spacetime takes negative values. Cosmological constants with negative (positive) values will be said to be *de Sitter-like* (*anti-de Sitter-like*). In what follows, the Latin indices ${}_a, {}_b, {}_c, \ldots$ are used as abstract tensor indices while the boldface Latin indices ${}_{\boldsymbol{a}}, {}_{\boldsymbol{b}}, {}_{\boldsymbol{c}}, \ldots$ are used as spacetime frame indices taking the values $0, \ldots, 3$. In this way, given a basis $\{\boldsymbol{e_a}\}$, a generic tensor is denoted by T_{ab} while its components in the given basis are denoted by $T_{\boldsymbol{ab}} \equiv T_{ab}\boldsymbol{e_a}{}^a\boldsymbol{e_b}{}^b$. The indices ${}_{\boldsymbol{i}}, {}_{\boldsymbol{j}}, {}_{\boldsymbol{k}}, \ldots$ are reserved to denote frame spatial indices respect to an adapted frame taking the values 1, 2, 3. Round and square brackets enclosing a group of indices (abstract or frame) will be used to denote symmetrisations and antisymmetrisations respectively, so that

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \qquad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}).$$

In addition the curly brackets will be used to denote the tracefree part of tensors, e.g.,

$$T_{\{ab\}} = T_{ab} - \frac{1}{4}T g_{ab}$$

where $T = g^{ab}T_{ab}$. Similar definitions hold for higher order tensors. For spinorial expressions the conventions and notation of Penrose & Rindler [29] will be used. In particular, ${}_A, {}_B, {}_C, \ldots$ are abstract spinorial indices while ${}_{\boldsymbol{A}}, {}_{\boldsymbol{B}}, {}_{\boldsymbol{C}}, \ldots$ will denote frame spinorial indices with respect to some specified spin dyad $\{\epsilon_{\boldsymbol{A}}{}^A\}$. The conventions for the curvature tensors will be fixed by the relation

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)v^c = R^c{}_{dab}v^d.$$

Although index notation will be preferred, for clarity some expressions will be written in index free notation, in this regard the conventions used in this thesis are the following: covectors will be denoted with bold Greek letters while vectors with bold Latin letters. Similarly a covector $\boldsymbol{\omega} \in T^*|_p(\mathcal{M})$ acting on a vector $\boldsymbol{v} \in T|_p(\mathcal{M})$ is denoted by $\langle \boldsymbol{\omega}, \boldsymbol{v} \rangle \in \mathbb{R}$. If a metric $\boldsymbol{g}$ is provided then the symbols $\flat$ and $\sharp$ (musical isomorphisms) are used to denote the action of rise or lower indices in index free notation. Namely, $\boldsymbol{v}^\flat \in T^*|_p(\mathcal{M})$ represents the index free version of $g_{ab}v^b$ and $\boldsymbol{\omega}^\sharp \in T|_p(\mathcal{M})$ represents the index free version of $g^{ab}\omega_a$. Given a map $\varphi : \mathcal{U} \to \mathcal{V}$ and $\boldsymbol{\omega} \in T^*_q\mathcal{V}$ the pull-back of this covector to $T^*_{\varphi^{-1}(q)}\mathcal{U}$ is denoted by $\varphi^*\boldsymbol{\omega}$. Similarly, given a vector $\boldsymbol{v} \in T_p\mathcal{U}$ the push-forward of this vector to $T_{\varphi(p)}\mathcal{V}$ is denoted by $\varphi_*\boldsymbol{v}$. Nevertheless, in a slight abuse of notation $\varphi(\mathcal{U})$ and $\mathcal{V}$ will be frequently identified and the map φ will be omitted. In addition, $D^+(\mathcal{A})$, $H^+(\mathcal{A})$, $J^+(\mathcal{A})$ and $I^+(\mathcal{A})$ will denote the future domain of dependence, the future Cauchy horizon, causal and chronological future of $\mathcal{A}$, respectively. The past counterparts will be denoted changing $+$ by $-$ in the above notation —see [2, 30].

2 The conformal Einstein field equations

2.1 The standard conformal Einstein field equations

As previously discussed, the are three formulations of the standard conformal Einstein field equations; the metric, frame and spinorial one. Despite being equivalent, each formulation is better suited depending on the problem that one is to analyse. The standard conformal Einstein field equations were originally introduced in [11]. In this section the conformal Einstein field equations are presented and the relation between the three different formulations briefly discussed —see [11, 31, 32] for derivations and further discussion. These formulations of the conformal Einstein field equations are well suited to analyse non-vacuum spacetimes with trace-free matter content, i.e., $\tilde{T}_a{}^a = 0$ —e.g., electrovacuum spacetimes. Nevertheless, as all the applications of these equations discussed in this thesis are restricted to vacuum spacetimes, in the following, the vacuum Einstein field equations (1.9) will be simply referred as the Einstein field equations.

2.1.1 Conformal rescalings

Two spacetimes $(\mathcal{M}, \boldsymbol{g})$ and $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ are said to be *conformally related* if $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ are related as in equation (1.10). In a slight abuse of notation $\varphi(\tilde{\mathcal{M}})$ and $\mathcal{M}\backslash\mathscr{I}$ are identified and the mapping $\varphi : \tilde{\mathcal{M}} \rightarrow \mathcal{M}$ will be omitted. Consistently one writes

$$g_{ab} = \Xi^2 \tilde{g}_{ab}.$$

For the subsequent discussion it is necessary to introduce some notation first. The *physical Schouten tensor* $\tilde{L}_{ab}$ is defined as follows

$$\tilde{L}_{ab} \equiv \frac{1}{2}\tilde{R}_{ab} - \frac{1}{12}\tilde{R}\tilde{g}_{ab}, \tag{2.1}$$

where $\tilde{R}_{ab}$ and $\tilde{R}$ represent, respectively, the Ricci tensor and Ricci scalar of $\tilde{g}_{ab}$. The *unphysical Schouten tensor* L_{ab} is defined in analogous way

$$L_{ab} \equiv \frac{1}{2}R_{ab} - \frac{1}{12}Rg_{ab}. \tag{2.2}$$

Notice that using expression (2.1) the Einstein field equations (1.9) can be rewritten in terms of the physical Schouten tensor as

$$\tilde{L}_{ab} = \frac{1}{6}\lambda\tilde{g}_{ab}. \tag{2.3}$$

Remark 3. The motivation for the introduction of the Schouten tensor will be clarified when discussing the conformal Einstein field equations in the remainder of this chapter. Despite that in the conformal Einstein field equations one could replace the Schouten tensor by the Ricci tensor and the Ricci scalar, the Schouten tensor appears naturally in the equations due to its conformal transformation properties.

2.1.2 Frame formulation of the standard conformal Einstein field equations

Let $\{\boldsymbol{e_a}\}$ denote a set of frame fields on $\mathcal{M}$ and let $\{\boldsymbol{\omega^a}\}$ be the associated coframe. Accordingly, one has that $\langle \boldsymbol{\omega^a}, \boldsymbol{e_b}\rangle = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}$. One defines the frame metric as $g_{\boldsymbol{ab}} \equiv \boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b})$ —in abstract index notation $g_{\boldsymbol{ab}} \equiv \boldsymbol{e_a}{}^a \boldsymbol{e_b}{}^b g_{ab}$. In the subsequent discussion only orthonormal frames will be considered, so that $g_{\boldsymbol{ab}} = \eta_{\boldsymbol{ab}}$, where $\eta_{\boldsymbol{ab}} = \mathrm{diag}(1,-1,-1,-1)$. The metric $\boldsymbol{g}$ is then expressed in terms of the coframe $\{\boldsymbol{\omega^a}\}$ as

$$\boldsymbol{g} = \eta_{\boldsymbol{ab}}\boldsymbol{\omega^a} \otimes \boldsymbol{\omega^b}.$$

The connection coefficients $\grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ of a connection $\grave{\boldsymbol{\nabla}}$ —which is not assumed to be the Levi-Civita connection of $\boldsymbol{g}$— with respect to the frame $\{\boldsymbol{e_a}\}$ are defined via the relation

$$\grave{\nabla}_{\boldsymbol{a}}\boldsymbol{e_b} = \grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c},$$

where $\grave{\nabla}_{\boldsymbol{a}} \equiv \boldsymbol{e_a}{}^a\grave{\nabla}_a$ denotes the covariant directional derivative in the direction of $\boldsymbol{e_a}$. The torsion $\grave{\boldsymbol{\Sigma}}$ of $\grave{\boldsymbol{\nabla}}$ can be expressed in terms of the frame $\{\boldsymbol{e_a}\}$ and the connection coefficients $\grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ via

$$\grave{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c} = [\boldsymbol{e_a}, \boldsymbol{e_b}] - (\grave{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \grave{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\boldsymbol{e_c}.$$

Since different connections will be used, all the geometrical objects derived from each connection will carry a symbol over the kernel letter to denote the connection from which they were defined. The symbols $\boldsymbol{\nabla}$ and $\tilde{\boldsymbol{\nabla}}$ will be reserved for the Levi-Civita connection of the metrics $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$. Consistent with this notation one has that $\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0$. The connection coefficients of $\boldsymbol{\nabla}$ and $\tilde{\boldsymbol{\nabla}}$ are related to each other through the expression

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} + S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}}\Upsilon_{\boldsymbol{d}}, \tag{2.4}$$

where

$$S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} \equiv \delta_{\boldsymbol{a}}{}^{\boldsymbol{c}}\delta_{\boldsymbol{b}}{}^{\boldsymbol{d}} + \delta_{\boldsymbol{b}}{}^{\boldsymbol{c}}\delta_{\boldsymbol{a}}{}^{\boldsymbol{d}} - \eta_{\boldsymbol{ab}}\eta^{\boldsymbol{cd}} \qquad \text{and} \qquad \Upsilon_{\boldsymbol{a}} \equiv \Xi^{-1}\nabla_{\boldsymbol{a}}\Xi.$$

In particular, observe that the 1-form $\boldsymbol{\Upsilon} \equiv \Upsilon_{\boldsymbol{a}}\boldsymbol{\omega}^{\boldsymbol{a}}$ is exact.

Let $R^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ denote the *geometric curvature* of $\boldsymbol{\nabla}$ —that is, the expression of the Riemann tensor of $\boldsymbol{\nabla}$ written in terms of derivatives of the connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$:

$$R^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \equiv \boldsymbol{e}_{\boldsymbol{a}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \boldsymbol{e}_{\boldsymbol{b}}(\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) + \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}}.$$

The expression of the irreducible decomposition of Riemann tensor $R^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ given by

$$\rho^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \equiv \Xi d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} + 2S_{\boldsymbol{b}[\boldsymbol{c}}{}^{\boldsymbol{af}}L_{\boldsymbol{d}]\boldsymbol{f}}. \tag{2.5}$$

will be called the *algebraic curvature.* In the last expression $L_{\boldsymbol{ab}}$ denotes the Schouten tensor of $\boldsymbol{g}$ and $d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ represents the so-called *rescaled Weyl tensor*, defined as

$$d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \equiv \Xi^{-1}C^{\boldsymbol{a}}{}_{\boldsymbol{bcd}},$$

where $C^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ is the conformally invariant Weyl tensor ($\tilde{C}^{a}{}_{bcd} = C^{a}{}_{bcd}$). Despite the fact that the definition of the rescaled Weyl tensor may look singular at the conformal boundary, it can be shown that under suitable assumptions the tensor $d^{a}{}_{bcd}$ is regular even when $\Xi = 0$ —see Remark 4. Finally, let s —the so-called *Friedrich scalar*— denote the scalar field defined as

$$s \equiv \frac{1}{4}\nabla_{\boldsymbol{a}}\nabla^{\boldsymbol{a}}\Xi + \frac{1}{24}R\Xi,$$

where R is the Ricci scalar of $\boldsymbol{g}$. Using the above definitions one can write the frame version of the conformal Einstein field equations as

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad Z_{\boldsymbol{ab}} = 0, \qquad Z_{\boldsymbol{a}} = 0, \tag{2.6a}$$

$$\Delta_{\boldsymbol{abc}} = 0, \qquad \Lambda_{\boldsymbol{abc}} = 0, \qquad Z = 0, \tag{2.6b}$$

where the so-called, zero-quantities are defined via

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} \equiv [\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}] - (\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\boldsymbol{e}_{\boldsymbol{c}}, \tag{2.7a}$$

$$\Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv R^{\boldsymbol{c}}{}_{\boldsymbol{abd}} - \rho^{\boldsymbol{c}}{}_{\boldsymbol{abd}}, \tag{2.7b}$$

$$Z_{\boldsymbol{ab}} \equiv \nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi + \Xi L_{\boldsymbol{ab}} - s\eta_{\boldsymbol{ab}}, \tag{2.7c}$$

$$Z_{\boldsymbol{a}} \equiv \nabla_{\boldsymbol{a}}s + L_{\boldsymbol{ac}}\nabla^{\boldsymbol{c}}\Xi, \tag{2.7d}$$

$$\Lambda_{bcd} \equiv \nabla_a d^a{}_{bcd}, \tag{2.7e}$$

$$\Delta_{cdb} \equiv \nabla_c L_{db} - \nabla_d L_{cb} - \nabla_a \Xi d^a{}_{bcd}, \tag{2.7f}$$

$$Z \equiv 6\Xi s - 3\nabla_a \Xi \nabla^a \Xi - \lambda. \tag{2.7g}$$

The starting point for the derivation of the conformal Einstein field equations is similar to that leading to the singular equation (1.12) of Chapter 1; one writes the conformal transformation law for the Schouten tensor and uses the Einstein field equations as written in equation (2.3) to replace the physical Schouten tensor. This equation is encoded in the zero-quantity Z_{ab}. The main conceptual difference with respect to the formally singular equation (1.12) is that the equation $Z_{ab} = 0$ is read not as an equation for the metric but as an equation for the conformal factor Ξ. The appearance of the scalar field s in the equation $Z_{ab} = 0$ requires the construction of a suitable equation for this field. Considering $\nabla^b Z_{ab} = 0$, commuting covariant derivatives and using the contracted second Bianchi identity renders such an equation. This equation has been encoded in the zero-quantity Z_a. In the frame version of the conformal Einstein field equations, the metric is not an unknown of the system, instead, equations for the frame $\boldsymbol{e}_a$ need to be incorporated. The equation for the frame is encoded in $\Sigma_a{}^c{}_b = 0$ which describes the fact that the connection $\boldsymbol{\nabla}$ is torsion-free. Consistent with this spirit, the appearance of the Schouten tensor in equation Z_a is not seen as representing second order derivatives of the metric but as an unknown which has to satisfy an equation of its own. Such equations for the curvature tensors are the content of the zero-quantities Δ_{abc}, Λ_{bcd} and $\Xi^d{}_{abc}$. In particular, equations $\Delta_{cdb} = 0$ and $\Lambda_{bcd} = 0$ encode the contracted second Bianchi identity. The equation $\Xi^a{}_{bcd} = 0$ expresses that the algebraic and geometric curvature coincide. The equation $Z = 0$ encodes the fact that λ is a constant. It is sufficient to demand that this equation holds only at one point $p \in \mathcal{M}$ since, the rest of the equations in (2.6a)-(2.6b) imply that $\nabla_a Z = 0$. Finally, observe that once the conformal factor Ξ and frame $\boldsymbol{e}_a$ are determined, one can obtain the corresponding dual coframe $\boldsymbol{\omega}^a$ —provided that $det(\eta^{ab}\boldsymbol{e}_a \otimes \boldsymbol{e}_b) \neq 0$— and the physical metric $\tilde{\boldsymbol{g}}$ can be reconstructed as $\tilde{\boldsymbol{g}} = \Xi^{-2}\eta_{ab}\boldsymbol{\omega}^a \otimes \boldsymbol{\omega}^b$.

Lemma 1. *Let*

$$\{\Xi, \boldsymbol{e}_a, s, \Gamma_a{}^c{}_b, L_{ab}, d^a{}_{bcd}\}$$

denote a solution to the frame conformal Einstein field equations with $\Gamma_a{}^c{}_b$ *satisfying the metric compatibility condition*

$$\Gamma_a{}^d{}_b \eta_{dc} + \Gamma_a{}^d{}_c \eta_{bd} = 0$$

and such that

$$\Xi \neq 0, \qquad det(\eta^{ab}\boldsymbol{e}_a \otimes \boldsymbol{e}_b) \neq 0,$$

in an open set $\mathcal{U} \subset \mathcal{M}$. *Then the metric* $\tilde{\boldsymbol{g}} = \Xi^{-2}\eta_{ab}\boldsymbol{\omega}^a \otimes \boldsymbol{\omega}^b$ *where* $\boldsymbol{\omega}^a$ *is the dual* *frame to* $\boldsymbol{e_a}$*, is a solution to the Einstein field equations* (1.9) *on* $\mathcal{U}$*.*

A detailed proof of this Lemma can be found in [11] and [13]. The proof of this Lemma exploits the geometrical significance that the conformal Einstein field equations encode. In particular, if $\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0$ then $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ correspond to the connection coefficients with respect of $\{\boldsymbol{e_a}\}$ of the Levi-Civita connection of $\boldsymbol{g} = \eta_{\boldsymbol{ab}}\boldsymbol{\omega^a} \otimes \boldsymbol{\omega^b}$. Equations $\Xi^{\boldsymbol{c}}{}_{\boldsymbol{abd}} = 0$, $\Lambda_{\boldsymbol{bcd}} = 0$ and $\Delta_{\boldsymbol{cbd}} = 0$ ensure that $L_{\boldsymbol{ab}}$ and $C^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \equiv \Xi d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ are the components of the Schouten and Weyl tensors of $\boldsymbol{\nabla}$ respect to the frame $\{\boldsymbol{e_a}\}$. Finally, equations $Z_{\boldsymbol{ab}} = 0$ and $Z_{\boldsymbol{a}} = 0$ imply that $\tilde{\boldsymbol{g}} = \Xi^{-2}\boldsymbol{g_{ab}}$ satisfy the Einstein field equations —expressed as in equation (2.3)— on $\mathcal{U}$.

2.1.3 The metric conformal Einstein field equations

The derivation of the equations (2.7c)-(2.7g) can be done in formally identical way in abstract index notation i.e., without making reference to a frame $\boldsymbol{e_a}$. In other words, in the metric formulation one considers the following zero-quantities:

$$Z_{ab} \equiv \nabla_a\nabla_b\Xi + \Xi L_{ab} - s g_{ab}, \tag{2.8a}$$

$$Z_a \equiv \nabla_a s + L_{ac}\nabla^c\Xi, \tag{2.8b}$$

$$\Lambda_{bcd} \equiv \nabla_a d^a{}_{bcd}, \tag{2.8c}$$

$$\Delta_{cdb} \equiv \nabla_c L_{db} - \nabla_d L_{cb} - \nabla_a \Xi d^a{}_{bcd}, \tag{2.8d}$$

$$Z \equiv 6\Xi s - 3\nabla_a\Xi\nabla^a\Xi - \lambda. \tag{2.8e}$$

The main conceptual difference is that in this formulation the Cartan structure equations for the frame $\boldsymbol{e_a}$ encoded in the zero-quantities (2.7a) and (2.7b) are not required. In this formulation, however, one needs to supplement the system encoded in the zero-quantities (2.8a)-(2.8e) with an equation for the unphysical metric. To do so, one considers equation (2.2) expressed in some local coordinates (x^μ). Recalling that, in local coordinates the components of the Ricci tensor can be written as second order derivatives of the metric one obtains the required equation for the unphysical metric. In concrete applications the choice of coordinates is a subtle point since not every choice would lead to an equation of recognisable form. An additional complication of this strategy is that for applications one has to analyse a system of mixed order. This approach is closer in spirit to the classical treatment of the Cauchy problem in General Relativity in [4]. Nevertheless, in view of the applications discussed in this thesis, the frame and spinorial versions of the conformal Einstein field equations will be preferred.

2.1.4 Spinorial formulation of the standard conformal Einstein field equations

A spinorial version of the extended conformal Einstein field equations (2.6a)-(2.6b) is readily obtained by suitable contraction with the *Infeld-van der Waerden symbols* $\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}$. Given the components $T_{\boldsymbol{ab}}{}^{\boldsymbol{c}}$ of a tensor $T_{ab}{}^{c}$ respect to a frame $\boldsymbol{e_a}$ field, the components of its spinorial counterpart are given by

$$T_{\boldsymbol{AA'BB'}}{}^{\boldsymbol{CC'}} \equiv T_{\boldsymbol{ab}}{}^{\boldsymbol{c}}\sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\sigma_{\boldsymbol{BB'}}{}^{\boldsymbol{b}}\sigma^{\boldsymbol{CC'}}{}_{\boldsymbol{c}},$$

where

$$\sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{0}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{1}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.9a}$$

$$\sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{2}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{3}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.9b}$$

and

$$\sigma^{\boldsymbol{AA'}}{}_{\boldsymbol{0}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^{\boldsymbol{AA'}}{}_{\boldsymbol{1}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.9c}$$

$$\sigma^{\boldsymbol{AA'}}{}_{\boldsymbol{2}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^{\boldsymbol{AA'}}{}_{\boldsymbol{3}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.9d}$$

In particular, the spinorial counterpart of the frame metric $g_{\boldsymbol{ab}} = \eta_{\boldsymbol{ab}}$ is given by $g_{\boldsymbol{AA'BB'}} \equiv \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}$. In turn, the frame $\boldsymbol{e_a}$ and coframe $\boldsymbol{\omega^a}$ imply a spinorial frame $\boldsymbol{e_{AA'}}$ and a coframe $\boldsymbol{\omega^{AA}}$ such that

$$\boldsymbol{g}(\boldsymbol{e_{AA'}}, \boldsymbol{e_{BB'}}) = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}.$$

If one denotes with the same kernel letter the unknowns of the frame version of the conformal Einstein field equations one is lead to consider the following spinorial zero-quantities:

$$\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{BB'}}\boldsymbol{e_{QQ'}} \equiv [\boldsymbol{e_{BB'}}, \boldsymbol{e_{AA'}}] - \left(\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} - \Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{AA'}}\right)\boldsymbol{e_{CC'}}, \tag{2.10a}$$

$$\Xi^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} \equiv R^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} - \rho^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}}, \tag{2.10b}$$

$$Z_{\boldsymbol{AA'BB'}} \equiv \nabla_{\boldsymbol{AA'}}\nabla_{\boldsymbol{BB'}}\Xi + \Xi L_{\boldsymbol{AA'BB'}} - s\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}, \tag{2.10c}$$

$$Z_{\boldsymbol{AA'}} \equiv \nabla_{\boldsymbol{AA'}}s + L_{\boldsymbol{AA'CC'}}\nabla^{\boldsymbol{CC'}}\Xi, \tag{2.10d}$$

$$\Delta_{\boldsymbol{CC'DD'BB'}} \equiv \nabla_{\boldsymbol{CC'}}L_{\boldsymbol{DD'BB'}} - \nabla_{\boldsymbol{DD'}}L_{\boldsymbol{CC'BB'}} - \nabla_{\boldsymbol{AA'}}\Xi d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}}, \tag{2.10e}$$

$$\Lambda_{\boldsymbol{BB'CC'DD'}} \equiv \nabla_{\boldsymbol{AA'}}d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}}, \tag{2.10f}$$

$$Z \equiv 6\Xi s - 3\nabla_{\boldsymbol{AA'}}\Xi\nabla^{\boldsymbol{AA'}}\Xi - \lambda. \tag{2.10g}$$

In terms of these zero quantities, the *spinorial formulation of the standard conformal Einstein field equations* can be succinctly expressed as

$$\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} = 0, \qquad \Xi^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} = 0, \qquad Z_{\boldsymbol{AA'BB'}} = 0, \tag{2.11a}$$

$$\Delta_{\boldsymbol{AA'BB'CC'}} = 0, \qquad \Lambda_{\boldsymbol{AA'BB'CC'}} = 0, \qquad Z_{\boldsymbol{AA'}} = 0, \qquad Z = 0. \tag{2.11b}$$

In the spinorial formulation one can exploit the symmetries of the relevant fields to obtain expressions in terms of lower valence spinors. In particular one has the following irreducible decompositions:

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} = \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}\epsilon_{\boldsymbol{C'}}{}^{\boldsymbol{B'}} + \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B'}}{}_{\boldsymbol{C'}}\epsilon_{\boldsymbol{C}}{}^{\boldsymbol{B}}, \tag{2.12a}$$

$$d_{\boldsymbol{AA'BB'CC'DD'}} = -\phi_{\boldsymbol{ABCD}}\epsilon_{\boldsymbol{A'B'}}\epsilon_{\boldsymbol{C'D'}} - \bar{\phi}_{\boldsymbol{A'B'C'D'}}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{CD}}. \tag{2.12b}$$

In the last decomposition $\phi_{\boldsymbol{ABCD}} = \phi_{(\boldsymbol{ABCD})}$ represent the components of the *rescaled Weyl spinor*. Namely, $\phi_{\boldsymbol{ABCD}} \equiv \Xi^{-1}\Psi_{\boldsymbol{ABCD}}$ where $\Psi_{\boldsymbol{ABCD}}$ is the conformally invariant *Weyl spinor*.

Remark 4. In the classical theory of asymptotics as discussed in [12, 33] it is shown that if Ψ_{ABCD} is smooth at $\mathscr{I}$ and $\tilde{T}_{ab} = \mathcal{O}(\Xi^3)$ where $\tilde{T}_{ab}$ is the *physical energy momentum tensor*, then $\Psi_{ABCD} = \mathcal{O}(\Xi)$ —see Theorem 3.5.3 in [12] and Theorem 10.3 in [13]. As mentioned previously, in the applications of the conformal Einstein field equations discussed in this thesis $\tilde{T}_{ab} = 0$.

In addition, $\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} \equiv \frac{1}{2}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{BQ'}}{}_{\boldsymbol{CQ'}}$ denote the *reduced connection coefficients*. Likewise, the geometric and algebraic curvature spinors can be decomposed as

$$R^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} = R^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}\epsilon_{\boldsymbol{D'}}{}^{\boldsymbol{C'}} + \bar{R}^{\boldsymbol{C'}}{}_{\boldsymbol{D'AA'BB'}}\epsilon_{\boldsymbol{D}}{}^{\boldsymbol{C}}, \tag{2.13a}$$

$$\rho^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} = \rho^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}\epsilon_{\boldsymbol{D'}}{}^{\boldsymbol{C'}} + \bar{\rho}^{\boldsymbol{C'}}{}_{\boldsymbol{D'AA'BB'}}\epsilon_{\boldsymbol{D}}{}^{\boldsymbol{C}}, \tag{2.13b}$$

where

$$R^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} \equiv \frac{1}{2}R^{\boldsymbol{CQ'}}{}_{\boldsymbol{DQ'AA'BB'}}, \qquad \rho_{\boldsymbol{ABCC'DD'}} \equiv \frac{1}{2}\rho_{\boldsymbol{A}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{BQ'CC'DD'}}.$$

Explicitly, in terms of the unknowns of the conformal Einstein field equations, the reduced geometric and algebraic curvature spinors are given by

$$\begin{aligned} R^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} &= \boldsymbol{e}_{\boldsymbol{AA'}}\left(\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\right) - \boldsymbol{e}_{\boldsymbol{BB'}}\left(\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\right) \\ &\quad -\Gamma_{\boldsymbol{FB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{B}} - \Gamma_{\boldsymbol{BF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{B'}} + \Gamma_{\boldsymbol{FA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{A}} \\ &\quad +\Gamma_{\boldsymbol{AF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{A'}} + \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{E}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{E}}{}_{\boldsymbol{D}} - \Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{E}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{E}}{}_{\boldsymbol{D}}, \end{aligned} \tag{2.14a}$$

$$\rho_{\boldsymbol{ABCC'DD'}} = -\Xi\phi_{\boldsymbol{ABCD}}\epsilon_{\boldsymbol{C'D'}} + L_{\boldsymbol{BC'DD'}}\epsilon_{\boldsymbol{CA}} - L_{\boldsymbol{BD'CC'}}\epsilon_{\boldsymbol{DA}}. \tag{2.14b}$$

Similarly, the zero-quantities can be decomposed as

$$\Delta_{\boldsymbol{CC'DD'BB'}} = \Delta_{\boldsymbol{CDBB'}}\epsilon_{\boldsymbol{C'D'}} + \bar{\Delta}_{\boldsymbol{C'D'BB'}}\epsilon_{\boldsymbol{CD}},$$
$$\Lambda_{\boldsymbol{BB'CC'DD'}} = \Lambda_{\boldsymbol{BB'CD}}\epsilon_{\boldsymbol{C'D'}} + \bar{\Lambda}_{\boldsymbol{B'BC'D'}}\epsilon_{\boldsymbol{CD}},$$

where

$$\Delta_{\boldsymbol{CDBB'}} \equiv \frac{1}{2}\Delta_{\boldsymbol{CQ'D}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{BB'}}, \qquad \Lambda_{\boldsymbol{BB'CD}} \equiv \frac{1}{2}\Lambda_{\boldsymbol{BB'CQ'D}}{}^{\boldsymbol{Q'}}.$$

Consequently one defines the following *reduced spinorial zero-quantities*

$$\Xi^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} \equiv R^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} - \rho^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}},$$

$$\Delta_{\boldsymbol{CDBB'}} \equiv \nabla_{(\boldsymbol{C}}{}^{\boldsymbol{Q'}}L_{\boldsymbol{D})Q'BB'} + \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\Xi\phi_{\boldsymbol{CDBQ}},$$
$$\Lambda_{\boldsymbol{BB'CD}} \equiv \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\phi_{\boldsymbol{BCDQ}},$$

With these definitions, the spinorial extended conformal Einstein field equations can be alternatively written as

$$\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} = 0, \qquad \Xi^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} = 0, \qquad Z_{\boldsymbol{AA'BB'}} = 0, \tag{2.15a}$$
$$\Delta_{\boldsymbol{CDBB'}} = 0, \qquad \Lambda_{\boldsymbol{BB'CD}} = 0, \qquad Z_{\boldsymbol{AA'}} = 0 \qquad Z = 0. \tag{2.15b}$$

The last set of equations is completely equivalent to the equations in (2.11a)-(2.11b). Moreover, since the equations (2.11a)-(2.11b) are equivalent to (2.6a)-(2.6b) an analogous result to Lemma 1 follows:

Lemma 2. *Let*

$$\{\Xi, s, \boldsymbol{e}_{\boldsymbol{AA'}}, \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{B}}, L_{\boldsymbol{AA'BB'}}, \phi_{\boldsymbol{ABCD}}\}$$

represent a solution to (2.11a)-(2.11b) *with* $\Gamma_{\boldsymbol{AA'BC}}$ *satisfying the metric compatibility condition*

$$\Gamma_{\boldsymbol{AA'BC}} = \Gamma_{\boldsymbol{AA'(BC)}}$$

and such that

$$\Xi \neq 0 \qquad and \qquad det(\epsilon^{\boldsymbol{AB}}\epsilon^{\boldsymbol{A'B'}}\boldsymbol{e}_{\boldsymbol{AA'}} \otimes \boldsymbol{e}_{\boldsymbol{BB'}}) \neq 0,$$

in an open set $\mathcal{U} \subset \mathcal{M}$. *Then the metric*

$$\tilde{\boldsymbol{g}} = \Xi^{-2}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}\boldsymbol{\omega}^{\boldsymbol{AA'}} \otimes \boldsymbol{\omega}^{\boldsymbol{BB'}}$$

where $\boldsymbol{\omega}^{\boldsymbol{AA'}}$ *is the dual coframe to* $\boldsymbol{e}_{\boldsymbol{AA'}}$, *is a solution to the Einstein field equations* (1.9) *on* $\mathcal{U}$.

2.2 The extended conformal Einstein field equations

In this section the necessary notation for discussing the extended conformal Einstein field equations will be presented. In particular, the notion of a *Weyl connection* $\hat{\boldsymbol{\nabla}}$ and the relevant transformation formulae between Weyl connections and the Levi-Civita connection will be discussed. The extended conformal Einstein field equations were originally introduced in [22] —see also [13, 27, 34–36].

2.2.1 Weyl connections

A *Weyl connection* $\hat{\boldsymbol{\nabla}}$ is a torsion-free connection satisfying the property

$$\hat{\nabla}_a g_{bc} = -2 f_a g_{bc}, \tag{2.16}$$

where f_a is an arbitrary 1-form —thus, $\hat{\boldsymbol{\nabla}}$ is not necessarily metric. Property (2.16) is preserved under the conformal rescalings (1.10) as it can be verified that $\hat{\nabla}_a \tilde{g}_{bc} = -2\tilde{f}_a \tilde{g}_{bc}$ where $\tilde{f}_a \equiv f_a + \Upsilon_a$. The connection coefficients of $\hat{\boldsymbol{\nabla}}$ are related to those of $\boldsymbol{\nabla}$ through the relation

$$\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} + S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} f_{\boldsymbol{d}}. \tag{2.17}$$

A Weyl connection is a Levi-Civita connection of some element of the conformal class $[\boldsymbol{g}]$ if and only if the 1-form f_a is exact —compare with equation (2.4). The Schouten tensors of the connections $\hat{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ are related to each other by

$$L_{ab} - \hat{L}_{ab} = \nabla_a f_b - \frac{1}{2} S_{ab}{}^{cd} f_c f_d \tag{2.18}$$

Notice that, in general, $\hat{L}_{ab} \neq \hat{L}_{(ab)}$.

2.2.2 Frame formulation of the extended conformal Einstein field equations

From now on, Weyl connections $\hat{\boldsymbol{\nabla}}$ related to a conformal metric $\boldsymbol{g}$ as in equation (2.16) will be considered. Let $\hat{R}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ denote the *geometric curvature* of $\hat{\boldsymbol{\nabla}}$ —that is, the expression of the Riemann tensor of $\hat{\boldsymbol{\nabla}}$ written in terms of derivatives of the connection coefficients $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$:

$$\hat{R}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \equiv \boldsymbol{e}_{\boldsymbol{a}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \boldsymbol{e}_{\boldsymbol{b}}(\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}}. \tag{2.19}$$

The expression for the algebraic curvature, namely the irreducible decomposition of Riemann tensor, is given by

$$\hat{\rho}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \equiv \Xi d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} + 2S_{\boldsymbol{b}[\boldsymbol{c}}{}^{\boldsymbol{af}}\hat{L}_{\boldsymbol{b}]\boldsymbol{f}}, \tag{2.20}$$

where $d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ is the rescaled Weyl tensor defined as before $d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = \Xi^{-1}C^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$. Notice that the Weyl tensor $C^{a}{}_{bcd}$ with respect to the Weyl connection $\hat{\boldsymbol{\nabla}}$ coincides with the Weyl tensor of any element of the conformal class $[\boldsymbol{g}]$. Finally, one introduces a 1-form $\boldsymbol{d}$ defined by the relation

$$d_{\boldsymbol{a}} \equiv \Xi f_{\boldsymbol{a}} + \nabla_{\boldsymbol{a}}\Xi.$$

With the above definitions one can write the *extended conformal Einstein field equations* as

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \hat{\Xi}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = 0, \qquad \hat{\Delta}_{\boldsymbol{cdb}} = 0, \qquad \hat{\Lambda}_{\boldsymbol{bcd}} = 0, \tag{2.21}$$

where

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} \equiv [\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}] - (\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\boldsymbol{e}_{\boldsymbol{c}}, \tag{2.22a}$$

$$\hat{\Xi}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \equiv \hat{R}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} - \hat{\rho}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{2.22b}$$

$$\hat{\Delta}_{\boldsymbol{cdb}} \equiv \hat{\nabla}_{\boldsymbol{c}}\hat{L}_{\boldsymbol{bd}} - \hat{\nabla}_{\boldsymbol{d}}\hat{L}_{\boldsymbol{cb}} - d_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{2.22c}$$

$$\hat{\Lambda}_{\boldsymbol{bcd}} \equiv \hat{\nabla}_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} - f_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}. \tag{2.22d}$$

The fields $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$, $\hat{\Xi}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$, $\hat{\Delta}_{\boldsymbol{cdb}}$ and $\hat{\Lambda}_{\boldsymbol{bcd}}$ encoding the extended conformal Einstein field equations will be called again *zero-quantities*. The geometric meaning of the extended conformal field equations is completely analogous to the standard conformal Einstein field equations. Nevertheless, observe that, in contrast with the formulation of the standard conformal Einstein field equations there is no differential condition for neither the 1-form $\boldsymbol{d}$ nor the conformal factor Ξ. In Section 2.3 it will be discussed how to fix these fields by adapting the gauge to a congruence of curves with special conformal properties: *conformal geodesics*. In order to relate the extended conformal Einstein equations field equations (2.21) to the Einstein field equations (1.9) one has to introduce the constraints—see Remark 5

$$\delta_{\boldsymbol{a}} = 0, \qquad \gamma_{\boldsymbol{ab}} = 0, \qquad \zeta_{\boldsymbol{ab}} = 0. \tag{2.23}$$

encoded in the *supplementary zero-quantities*

$$\delta_{\boldsymbol{a}} \equiv d_{\boldsymbol{a}} - \Xi f_{\boldsymbol{a}} - \hat{\nabla}_{\boldsymbol{a}}\Xi, \tag{2.24a}$$

$$\gamma_{\boldsymbol{ab}} \equiv \hat{L}_{\boldsymbol{ab}} + \frac{1}{6}\lambda\Xi^{-2}\eta_{\boldsymbol{ab}} - \hat{\nabla}_{\boldsymbol{a}}(\Xi^{-1}d_{\boldsymbol{b}}) - \Xi^{-2}S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}}d_{\boldsymbol{c}}d_{\boldsymbol{d}}, \tag{2.24b}$$

$$\zeta_{\boldsymbol{ab}} \equiv \hat{L}_{[\boldsymbol{ab}]} - \hat{\nabla}_{[\boldsymbol{a}}f_{\boldsymbol{b}]}. \tag{2.24c}$$

Equation (2.24a) encodes the definition of the 1-form $d_{\boldsymbol{a}}$; equation (2.24b) arises from the transformation law between the Schouten tensor $\hat{L}_{\boldsymbol{ab}}$ of $\hat{\boldsymbol{\nabla}}$ and the physical Schouten tensor $\tilde{L}_{\boldsymbol{ab}}$ determined by the Einstein field equations as expressed in equation (2.3); finally, equation (2.24c) relates the antisymmetry of the Schouten tensor $\hat{L}_{\boldsymbol{ab}}$ to derivatives of the 1-form $f_{\boldsymbol{a}}$.

Remark 5. The supplementary zero-quantities (2.24a)-(2.24c) are regarded as constraints in the sense that they are propagated by conformal evolution equations extracted from (2.22a)-(2.22d) —see Lemma 8. In other words, it is only required that (2.24a)-(2.24c) are satisfied on a spacelike hypersurface $\mathcal{S}$.

The precise relation between the extended conformal Einstein field equations and the Einstein field equations is given by the following lemma:

Lemma 3. *Let* $(\boldsymbol{e_a}, \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \hat{L}_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}})$ *denote a solution to the extended conformal Einstein field equations* (2.21) *for some choice of gauge fields* $(\Xi, d_{\boldsymbol{a}})$ *satisfying the constraint equations* (2.23)*. Assume, further, that*

$$\Xi \neq 0 \qquad \text{and} \qquad det(\eta^{\boldsymbol{ab}} \boldsymbol{e_a} \otimes \boldsymbol{e_b}) \neq 0$$

on an open subset $\mathcal{U} \subset \tilde{\mathcal{M}}$*. Then*

$$\tilde{\boldsymbol{g}} = \Xi^{-2} \eta_{\boldsymbol{ab}} \boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}$$

where $\{\boldsymbol{\omega}^{\boldsymbol{a}}\}$ *is the coframe dual to* $\{\boldsymbol{e_a}\}$ *is a solution to the Einstein field equations* (1.9) *on* $\mathcal{U}$*.*

The proof of this lemma can be found in [13, 37].

2.2.3 Spinorial formulation of the extended conformal Einstein field equations

Proceeding in a similar way as in Section 2.1.4 one can rewrite every frame expression in spinorial form by contracting with the Infeld-van der Waerden symbols. Denoting with the same kernel letter the unknowns of the extended conformal Einstein field equations one has the following zero-quantities

$$\hat{\Sigma}_{\boldsymbol{AA'BB'}} \equiv [\boldsymbol{e}_{\boldsymbol{AA'}}, \boldsymbol{e}_{\boldsymbol{BB'}}] - (\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} - \hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{AA'}}) \boldsymbol{e}_{\boldsymbol{CC'}}, \tag{2.25a}$$

$$\hat{\Xi}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} \equiv \hat{R}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} - \hat{\rho}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}}, \tag{2.25b}$$

$$\hat{\Delta}_{\boldsymbol{CC'DD'BB'}} \equiv \hat{\nabla}_{\boldsymbol{CC'}} \hat{L}_{\boldsymbol{DD'BB'}} - \hat{\nabla}_{\boldsymbol{DD'}} \hat{L}_{\boldsymbol{CC'BB'}} - d_{\boldsymbol{AA'}} d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}}, \tag{2.25c}$$

$$\hat{\Lambda}_{\boldsymbol{BB'CC'DD'}} \equiv \hat{\nabla}_{\boldsymbol{AA'}} d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}} - f_{\boldsymbol{AA'}} d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}}. \tag{2.25d}$$

The spinorial version of the extended conformal Einstein field equations are then succinctly written as

$$\hat{\Sigma}_{\boldsymbol{AA'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{BB'}}\boldsymbol{e}_{\boldsymbol{QQ'}} = 0, \qquad \hat{\Xi}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} = 0, \tag{2.26a}$$

$$\hat{\Delta}_{\boldsymbol{CC'DD'BB'}} = 0, \qquad \hat{\Lambda}_{\boldsymbol{BB'CC'DD'}} = 0. \tag{2.26b}$$

As discussed for the frame formulation, in order to relate the extended conformal Einstein field equations to the Einstein field equations one has to introduce the constrains

$$\delta_{\boldsymbol{AA'}} = 0, \qquad \gamma_{\boldsymbol{AA'BB'}} = 0, \qquad \zeta_{\boldsymbol{AA'BB'}} = 0, \tag{2.27}$$

where $\delta_{\boldsymbol{AA'}}, \gamma_{\boldsymbol{AA'BB'}}$ and $\zeta_{\boldsymbol{AA'BB'}}$ denote the spinorial counterpart of the supplementary zero-quantities given in equations (2.24a)-(2.24c).

As discussed in Section 2.1.4, one of the advantages of the spinorial formulation is that one can exploit the symmetries to express $d^{\boldsymbol{AA'}}{}_{\boldsymbol{BB'CC'DD'}}$ and $\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}}$ in terms of the lower valence spinors

$$\phi_{\boldsymbol{ABCD}}, \qquad \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}},$$

satisfying formally identical expressions of those of equations (2.12a)-(2.12b). Due to the fact that $\hat{\boldsymbol{\nabla}}$ is not metric, the reduced connection coefficients does not necessarily posses the symmetry $\Gamma_{\boldsymbol{AA'CD}} = \Gamma_{\boldsymbol{AA'(CD)}}$ which holds for the reduced connection coefficients of a Levi-Civita connection. Observe that a Weyl connection $\hat{\boldsymbol{\nabla}}$ reduces to the Levi-Civita connection of an element in the conformal class $[\boldsymbol{g}]$ if and only if the 1-form $\boldsymbol{f}$ is exact. In addition, notice that the transformation formula for the connection coefficients given in (2.17) is simpler in spinorial notation. In fact one has that

$$\hat{\Gamma}_{\boldsymbol{CC'AB}} = \Gamma_{\boldsymbol{CC'AB}} - \epsilon_{\boldsymbol{AC}} f_{\boldsymbol{BC'}}.$$

The expressions for geometric and algebraic curvature can be decomposed in a analogous way as in equation (2.13a) and (2.13b) The reduced geometric and algebraic curvature read

$$\begin{aligned}\hat{R}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} = {} & \boldsymbol{e}_{\boldsymbol{AA'}}\left(\hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\right) - \boldsymbol{e}_{\boldsymbol{BB'}}\left(\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\right)\\ & -\hat{\Gamma}_{\boldsymbol{FB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{B}} - \hat{\Gamma}_{\boldsymbol{BF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\hat{\Gamma}}_{\boldsymbol{AA'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{B'}} + \hat{\Gamma}_{\boldsymbol{FA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{A}}\\ & +\hat{\Gamma}_{\boldsymbol{AF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\hat{\Gamma}}_{\boldsymbol{BB'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{A'}} + \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{E}}\hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{E}}{}_{\boldsymbol{D}} - \hat{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{E}}\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{E}}{}_{\boldsymbol{D}},\\ \hat{\rho}_{\boldsymbol{ABCC'DD'}} = {} & -\Xi\phi_{\boldsymbol{ABCD}}\epsilon_{\boldsymbol{C'D'}} + \hat{L}_{\boldsymbol{BD'CC'}}\epsilon_{\boldsymbol{DA}} - \hat{L}_{\boldsymbol{BC'DD'}}\epsilon_{\boldsymbol{CA}}.\end{aligned}$$

From the last expressions one can notice that, in contrast with the Levi-Civita case, the contractions

$$\hat{R}^{\boldsymbol{Q}}{}_{\boldsymbol{QAA'BB'}} = \hat{\nabla}_{\boldsymbol{AA'}} f_{\boldsymbol{BB'}} - \hat{\nabla}_{\boldsymbol{BB'}} f_{\boldsymbol{AA'}}, \quad \hat{\rho}^{\boldsymbol{Q}}{}_{\boldsymbol{QCC'DD'}} = \hat{L}_{\boldsymbol{CC'DD'}} - \hat{L}_{\boldsymbol{DD'CC'}}, \tag{2.28}$$

do not necessarily vanish. Accordingly, one defines the following reduced zero-quantities

$$\hat{\Xi}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} = \hat{R}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} - \hat{\rho}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}, \tag{2.29}$$
$$\hat{\Delta}_{\boldsymbol{CDBB'}} = \hat{\nabla}_{(\boldsymbol{C}}{}^{\boldsymbol{Q'}}\hat{L}_{\boldsymbol{D})\boldsymbol{Q'BB'}} + d^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\phi_{\boldsymbol{CDBQ}}, \tag{2.30}$$
$$\hat{\Lambda}_{\boldsymbol{BB'CD}} = \hat{\nabla}^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\phi_{\boldsymbol{BCDQ}} - f^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}\phi_{\boldsymbol{BCDQ}}. \tag{2.31}$$

Henceforth, the spinorial extended conformal Einstein field equations will be alternatively written as

$$\hat{\Sigma}_{\boldsymbol{AA'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{BB'}} = 0, \quad \hat{\Xi}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} = 0, \quad \hat{\Delta}_{\boldsymbol{CDBB'}} = 0, \quad \hat{\Lambda}_{\boldsymbol{BB'CD}} = 0. \tag{2.32}$$

The last set of equations is completely equivalent to the equations (2.26a)-(2.26b). In turn, equations encoded in (2.26a)-(2.26b) are equivalent to the frame formulation of the extended conformal Einstein field equations given in equation (2.21). Therefore, a result analogous to Lemma 3 can be formulated:

Lemma 4. *Let*

$$(\boldsymbol{e}_{\boldsymbol{AA'}}, \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}, \hat{L}_{\boldsymbol{AA'BB'}}, \phi_{\boldsymbol{ABCD}})$$

denote a solution to the spinorial formulation of the extended conformal Einstein field equations (2.32) *for some choice of gauge fields* $(\Xi, d_{\boldsymbol{AA'}})$ *satisfying the constraint equations* (2.27). *Assume further that*

$$\Xi \neq 0 \qquad and \qquad det(\epsilon^{\boldsymbol{AB}}\epsilon^{\boldsymbol{A'B'}}\boldsymbol{e}_{\boldsymbol{AA'}} \otimes \boldsymbol{e}_{\boldsymbol{BB'}}) \neq 0,$$

on an open subset $\mathcal{U} \subset \tilde{\mathcal{M}}$. *Then*

$$\tilde{\boldsymbol{g}} = \Xi^{-2}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}\boldsymbol{\omega}^{\boldsymbol{AA'}} \otimes \boldsymbol{\omega}^{\boldsymbol{BB'}},$$

where $\boldsymbol{\omega}^{\boldsymbol{AA'}}$ *is the dual coframe to* $\boldsymbol{e}_{\boldsymbol{AA'}}$, *is a solution to the Einstein field equations* (1.9) *on* $\mathcal{U}$.

2.3 Conformal geodesics and conformal Gaussian systems

In this section the notion of conformal geodesics and conformal Gaussian systems is introduced. Additionally, it is discussed how to exploit the conformal geodesic equations to fix the gauge in the extended conformal Einstein field equations. Then, using the space spinor formalism, also briefly discussed in this section, it is shown how to extract a system of first order evolution equations from the extended conformal Einstein field equations. A discussion of the propagation of the constraints is also provided.

2.3.1 Conformal Geodesics

The following definitions are important for the subsequent discussion.

Definition. A conformal geodesic on spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ consist of a pair $(x(\tau), \boldsymbol{\beta}(\tau))$, where $x(\tau)$ is a curve on $\tilde{\mathcal{M}}$, $\tau \in I \subset \mathbb{R}$ with tangent $\dot{\boldsymbol{x}}$ and $\boldsymbol{\beta}$ is a 1-form defined along $x(\tau)$, satisfying the equations

$$\dot{x}^c \tilde{\nabla}_c \dot{x}^a = -\dot{x}^d \dot{x}^b S_{db}{}^{af} \beta_f \tag{2.33a}$$
$$\dot{x}^c \tilde{\nabla}_c \beta_a = \tfrac{1}{2} \dot{x}^c S_{ca}{}^{bd} \beta_b \beta_d + \tilde{L}_{ca} \dot{x}^c. \tag{2.33b}$$

where $\tilde{L}_{ab}$ denotes the Schouten tensor of $\tilde{\boldsymbol{\nabla}}$ and $S_{ab}{}^{cd}$ encodes the tensor introduced in Section 2.1.2.

Definition. A frame $\boldsymbol{e_a}$ on $\tilde{\mathcal{M}}$ is said to be *Weyl propagated* along a conformal geodesic $(x(\tau), \boldsymbol{\beta}(\tau))$ if it satisfies

$$\dot{x}^c \tilde{\nabla}_c \boldsymbol{e_a}{}^a = -S_{cd}{}^{af} \boldsymbol{e_a}{}^d \dot{x}^c \beta_f.$$

The motivation for considering curves satisfying equations (2.33a)-(2.33b) is understood when one observes their behaviour under conformal transformations and transitions to Weyl connections. Given an arbitrary 1-form $\widehat{\boldsymbol{f}}$ consider its associated Weyl connection $\widehat{\boldsymbol{\nabla}}$ i.e., such that $\widehat{\nabla}_a g_{bc} = -2\widehat{f}_a g_{bc}$. Then, defining $\widehat{\boldsymbol{\beta}} \equiv \boldsymbol{\beta} - \widehat{\boldsymbol{f}}$ the pair $(x(\tau), \widehat{\boldsymbol{\beta}}(\tau))$ will satisfy the equations

$$\dot{x}^c \widehat{\nabla}_c \dot{x}^a = -\dot{x}^d \dot{x}^b S_{db}{}^{af} \widehat{\beta}_f,$$
$$\dot{x}^c \widehat{\nabla}_c \widehat{\beta}_a = \tfrac{1}{2} \dot{x}^c S_{ca}{}^{bd} \widehat{\beta}_b \widehat{\beta}_d + \widehat{L}_{ca} \dot{x}^c,$$

where $\widehat{L}_{ab}$ is the Schouten tensor of $\widehat{\boldsymbol{\nabla}}$. Notice that if one chooses a Weyl connection $\hat{\boldsymbol{\nabla}}$ whose defining 1-form $\boldsymbol{f}$ coincides with the 1-form $\boldsymbol{\beta}$ of the $\tilde{\boldsymbol{\nabla}}$-conformal geodesic equations (2.33a)-(2.33b), then the conformal geodesic equations reduce to

$$\dot{x}^c \hat{\nabla}_c \dot{x}^a = 0, \qquad \hat{L}_{ab} \dot{x}^b = 0. \tag{2.34}$$

Similarly, the Weyl propagation of the frame becomes

$$\dot{x}^c \hat{\nabla}_c \boldsymbol{e_a}{}^a = 0. \tag{2.35}$$

The conformal geodesics equations admit more general reparametrisations than the usual affine parametrisation of metric geodesics. This is summarised in the following lemma:

Lemma 5. *The admissible reparametrisations mapping (non-null) conformal geodesics into (non-null) conformal geodesics are given by fractional transformations of the form*

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

where $a, b, c, d \in \mathbb{R}$.

The proof of this lemma can be found in [37] —see also [13, 38].

2.3.2 Conformal Gaussian systems

Besides their conformal invariance a major motivation for the introduction of conformal geodesics in the analysis of spacetimes by means of the extended conformal Einstein field equations is that they provide a geometric way for fixing the gauge fields $(\Xi, \boldsymbol{d})$ of Lemma 4. Assume that an open set $\mathcal{U} \subset \tilde{\mathcal{M}}$ of a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ can be covered by non-intersecting congruence of conformal geodesics. If one identifies the timelike leg of the tetrad $\{\boldsymbol{e_a}\}$ with the tangent to the curves, $\boldsymbol{e_0} = \dot{\boldsymbol{x}}$, then one can single out a conformal factor Θ by requiring

$$\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1, \qquad \boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}. \tag{2.36}$$

The last equation states that the parametrisation of the curve $x(\tau)$ is chosen so that the tangent vector $\dot{\boldsymbol{x}}$ is $\boldsymbol{g}$-normalised. It follows from the condition (2.36) by successive application of $\nabla_{\dot{\boldsymbol{x}}}$ —a derivative in the direction of $\dot{\boldsymbol{x}}$, e.g., $\dot{\Theta} \equiv \nabla_{\dot{\boldsymbol{x}}}\Theta$— and using the conformal geodesic equations (2.33a)-(2.33b) that the conformal factor Θ satisfies

$$\dot{\Theta} = \beta_a \dot{x}^a \Theta, \tag{2.37}$$

$$\ddot{\Theta} = \tfrac{1}{2}\Theta(\tilde{g}_{ab}\dot{x}^a\dot{x}^b)(\tilde{g}^{cd}\beta_c\beta_d) + \Theta\tilde{L}_{ab}\dot{x}^a\dot{x}^b, \tag{2.38}$$

$$\dddot{\Theta} = (\tilde{\nabla}_{\dot{\boldsymbol{x}}}(\tilde{L}_{ab}\dot{x}^a\dot{x}^b) + (\tilde{L}_{ab}\tilde{g}^{bc}\beta_c\dot{x}^a)(\tilde{g}_{pq}\dot{x}^p\dot{x}^q) + \beta_c\dot{x}^c\tilde{L}_{ab}\dot{x}^a\dot{x}^b)\Theta. \tag{2.39}$$

Moreover one can verify that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}(g_{ab}\boldsymbol{e_a}^a\boldsymbol{e_b}^b) = 0.$$

Therefore, if the frame $\{\boldsymbol{e_a}\}$ is orthogonal at one point of the conformal geodesic it will remain orthonormal along the conformal geodesic. If $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is a solution to the vacuum Einstein field equations one can use equation (2.3) to show that the right hand side of equation (2.39) vanishes. This observation is contained in the following key result:

Lemma 6. *Let* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *be a spacetime where* $\tilde{\boldsymbol{g}}$ *is a solution to the vacuum Einstein field equations* (1.9). *Moreover, let* $(x(\tau), \boldsymbol{\beta}(\tau))$ *satisfy the conformal geodesic equations* (2.33a)-(2.33b), *let* $\tau_\star \in \mathbb{R}$ *be an arbitrary constant defining the value of* τ *at* *a fiduciary point on the conformal geodesic and let* $\{\boldsymbol{e_a}\}$ *denote a Weyl propagated* $\boldsymbol{g}$*-orthonormal frame along* $x(\tau)$ *with*

$$\boldsymbol{g} \equiv \Theta^2 \tilde{\boldsymbol{g}},$$

such that

$$\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1.$$

Then the conformal factor Θ *is given, along* $x(\tau)$*, by*

$$\Theta(\tau) = \Theta_\star + \dot{\Theta}_\star(\tau - \tau_\star) + \frac{1}{2}\ddot{\Theta}_\star(\tau - \tau_\star)^2, \tag{2.40}$$

where the coefficients $\Theta_\star \equiv \Theta(\tau_\star), \dot{\Theta}_\star \equiv \dot{\Theta}(\tau_\star)$ *and* $\ddot{\Theta}_\star \equiv \ddot{\Theta}(\tau_\star)$ *are constant along the conformal geodesic and satisfy the constraints*

$$\dot{\Theta}_\star = \langle \boldsymbol{\beta}_\star, \dot{\boldsymbol{x}}_\star \rangle \Theta_\star, \qquad \Theta_\star \ddot{\Theta}_\star = \frac{1}{2} \tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}_\star, \boldsymbol{\beta}_\star) + \frac{1}{6}\lambda. \tag{2.41}$$

Moreover, along each conformal geodesic

$$\Theta \beta_{\boldsymbol{0}} = \dot{\Theta}, \qquad \Theta \beta_{\boldsymbol{i}} = \Theta_\star \beta_{\boldsymbol{i}\star},$$

where $\beta_{\boldsymbol{a}} \equiv \langle \boldsymbol{\beta}, \boldsymbol{e_a} \rangle$*.*

Finally the gauge field $\boldsymbol{d}$ can be specified via $\boldsymbol{d} \equiv \Theta \boldsymbol{\beta}$. The constraints for the initial data for Θ can then be written in terms of $\boldsymbol{d}$ as

$$\dot{\Theta}_\star = \langle \boldsymbol{d}_\star, \dot{\boldsymbol{x}}_\star \rangle, \qquad \Theta_\star \ddot{\Theta}_\star = \frac{1}{2} \boldsymbol{g}^\sharp(\boldsymbol{d}_\star, \boldsymbol{d}_\star) + \frac{1}{6}\lambda.$$

The proof of this Lemma and a further discussion of the properties of conformal geodesics can be found in [13, 22].

For spacetimes with a spacelike conformal boundary the relation between metric geodesics and conformal geodesics is particularly simple. This observation is the content of the following:

Lemma 7. *Any conformal geodesic leaving* $\mathscr{I}^+$ *(*$\mathscr{I}^-$*) orthogonally into the past (future) is up to reparametrisation a timelike future (past) complete geodesic for the physical metric* $\tilde{\boldsymbol{g}}$*. The reparametrisation required is determined by*

$$\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}\tau} = \frac{1}{\Theta(\tau)} \tag{2.42}$$

where $\tilde{\tau}$ *is the* $\tilde{\boldsymbol{g}}$*-proper time and* τ *is the* $\boldsymbol{g}$*-proper time and* $\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$*.*

The proof of this Lemma can be found in [39].

Conformal Gaussian systems

Assume, as before, that there exists a region $\mathcal{U}$ of the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ which can be covered by non-intersecting conformal geodesics. Furthermore, suppose that the tangent vector $\dot{\boldsymbol{x}}_\star \equiv \dot{\boldsymbol{x}}(\tau_\star)$ is orthogonal to some spacelike hypersurface $\tilde{\mathcal{S}} \subset \mathcal{U}$ determined locally by the condition $\tau = \tau_\star$. The conformal factor is determined by equation (2.39) and specification of the initial data $\Theta_\star$, $\dot{\Theta}_\star$ and $\ddot{\Theta}_\star$ on $\tilde{\mathcal{S}}$. As discussed in the previous section on the construction of a *conformal Gaussian system* one identifies the tangent to the conformal curve $(x(\tau), \boldsymbol{\beta}(\tau))$ with the time leg of the $\boldsymbol{g}$-orthonormal tetrad $\{\boldsymbol{e_a}\}$ —i.e. one sets $\boldsymbol{e}_0 = \dot{\boldsymbol{x}}$. This gauge choice can be specialised further by using the parameter τ along the conformal geodesics as a time coordinate so that

$$\boldsymbol{e_0} = \boldsymbol{\partial}_\tau. \tag{2.43}$$

To construct a spacetime system of coordinates consider some local (spatial) coordinates $(x^{\boldsymbol{\alpha}})$ on $\mathcal{S}$ which are extended off the initial hypersurface $\mathcal{S}$ by requiring them to remain constant along a conformal geodesic. Namely, if a conformal geodesic intersects $\mathcal{S}$ at a point p with coordinates $(x^{\boldsymbol{\alpha}}_\star)$ then the points in the conformal geodesic will have coordinates $(\tau, x^{\boldsymbol{\alpha}}_\star)$. With the above prescription $(\tau, x^{\boldsymbol{\alpha}})$ constitute a *conformal Gaussian coordinate system* on $\mathcal{U}$. This choice of gauge naturally leads to consider a 1+3 decomposition of the field equations. Another advantageous feature of considering a conformal Gaussian system is that the the conformal geodesic equations, as written in equations (2.34) and (2.35) imply the *gauge conditions*

$$\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0, \qquad \hat{L}_{\boldsymbol{0a}} = 0, \qquad f_{\boldsymbol{0}} = 0. \tag{2.44}$$

2.3.3 The $\tilde{\boldsymbol{g}}$-adapted equations

In the last section it was shown that imposing that τ corresponds to the $\boldsymbol{g}$-proper time readily selects a representative of the conformal class. However, for some applications it is more convenient to consider parametrisation of the conformal geodesics in terms of the (physical) $\tilde{\boldsymbol{g}}$-proper time $\tilde{\tau}$. To reexpress the conformal geodesic equations in terms of the physical proper time consider the parameter transformation $\tilde{\tau} = \tilde{\tau}(\tau)$ given by

$$\tilde{\tau} = \tilde{\tau}_\star + \int_{\tau_\star}^{\tau} \frac{\mathrm{d}s}{\Theta(s)}, \tag{2.45}$$

with inverse $\tau = \tau(\tilde{\tau})$. In what follows let $\tilde{x}(\tilde{\tau}) \equiv x(\tau(\tilde{\tau}))$. It can then verified that

$$\tilde{\boldsymbol{x}}' \equiv \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{\tau}} = \Theta \dot{\boldsymbol{x}}, \tag{2.46}$$

and that

$$\tilde{\boldsymbol{g}}(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{x}}') = 1$$

so that $\tilde{\tau}$ is the $\tilde{\boldsymbol{g}}$-proper time of the curve $\tilde{x}(\tilde{\tau})$. In order to write the equation for the curve $\tilde{x}(\tilde{\tau})$ in a convenient way, one considers the split

$$\boldsymbol{\beta} = \tilde{\boldsymbol{\beta}} + \varpi \dot{\boldsymbol{x}}^{\flat}, \tag{2.47}$$

where the 1-form $\tilde{\boldsymbol{\beta}}$ satisfies

$$\langle \tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}} \rangle = 0, \qquad \varpi \equiv \frac{\langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle}{\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})}, \qquad \boldsymbol{g}^{\sharp}(\boldsymbol{\beta}, \boldsymbol{\beta}) = \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle^2 + \boldsymbol{g}^{\sharp}(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}}).$$

Moreover, one can verify that,

$$\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = \Theta^{-2} \qquad \langle \boldsymbol{\beta}, \dot{x} \rangle = \Theta^{-1}\dot{\Theta} \qquad \varpi = \Theta\dot{\Theta} \tag{2.48}$$

In terms of these objects the $\tilde{\boldsymbol{g}}$-*adapted equations for the conformal curves* are given by

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'} \tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{\beta}}^{\sharp}, \tag{2.49a}$$
$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'} \tilde{\boldsymbol{\beta}} = \tilde{\beta}^2 \tilde{\boldsymbol{x}}'^{\flat}, \tag{2.49b}$$

where $\tilde{\beta}^2 \equiv -\tilde{\boldsymbol{g}}^{\sharp}(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}})$ is constant along a given conformal geodesic.

2.4 Conformal evolution equations and hyperbolic reduction strategies

In view of the tensorial nature of the conformal Einstein field equations, to make assertions about the existence and properties of their solutions, it is necessary to derive from them a suitable evolution system to which the theory of hyperbolic partial differential equations can be applied. This procedure is known as a *hyperbolic reduction*. Part of the hyperbolic reduction procedure consists of a specification of the gauge inherent to the equations. A systematic way of proceeding to the specification of the gauge is through so-called *gauge source functions*. These functions are associated to derivatives of the field unknowns which are not determined by the field equations. This idea can be used to extract a first order symmetric hyperbolic system of equations for the field unknowns for the metric, frame and spinorial versions of the standard conformal Einstein field equations.

In the other hand, the extended conformal Einstein field equations are expressed in terms of Weyl connections and, thus, contain a bigger gauge freedom than the standard conformal equations. This opens the possibility of an alternative approach to gauge fixing; adapting the gauge to a congruence of conformal geodesics —see [37, 39]. As previously discussed, this is advantageous since, in vacuum, conformal geodesics allow to fix the conformal freedom by selecting a canonical representative of the conformal class $[\tilde{\boldsymbol{g}}]$. In this manner, one gains *a priori* knowledge of the conformal boundary of the spacetime.

Remark 6. The conformal factor given in Lemma 6 is *canonical* in the sense that if $\tilde{\boldsymbol{g}}$ is a vacuum solution to the Einstein field equations and $\boldsymbol{g} = \Theta^2\tilde{\boldsymbol{g}}$ then requiring $\boldsymbol{g}(\dot{\boldsymbol{x}}(\tau), \dot{\boldsymbol{x}}(\tau)) = 1$ fixes the form of the conformal factor Θ to be a quadratic function of τ.

2.4.1 Space spinor formalism

In what follows, let the Hermitian spinor $\tau^{AA'}$ denote the spinor counterpart of the vector $\sqrt{2}\boldsymbol{e_0}^a$. In addition, let $\{\epsilon_{\boldsymbol{A}}{}^A\}$ with $\epsilon_{\boldsymbol{0}}{}^A = o^A, \epsilon_{\boldsymbol{1}}{}^A = \iota^A$ denote a spinor dyad such that

$$\tau^{AA'} = \epsilon_{\boldsymbol{0}}{}^A\epsilon_{\boldsymbol{0}'}{}^{A'} + \epsilon_{\boldsymbol{1}}{}^A\epsilon_{\boldsymbol{1}'}{}^{A'}. \tag{2.50}$$

The normalisation $\tau^{AA'}\tau_{AA'} = 2$, has been chosen in accordance with the conventions of [17]. In what follows let $\tau^{\boldsymbol{AA'}}$ denote the components of $\tau^{AA'}$ respect to $\{\epsilon^{\boldsymbol{A}}{}_A\}$. The Hermitian spinor $\tau^{AA'}$ can be used to perform a *space spinor split* of the frame $\{\boldsymbol{e_{AA'}}\}$ and coframe $\{\boldsymbol{\omega^{AA'}}\}$. Namely, one can write

$$\boldsymbol{e_{AA'}} = \frac{1}{2}\tau^{\boldsymbol{AA'}}\boldsymbol{e} - \tau^{\boldsymbol{B}}{}_{\boldsymbol{A'}}\boldsymbol{e_{AB}}, \qquad \boldsymbol{\omega^{AA'}} = \frac{1}{2}\tau^{\boldsymbol{AA'}}\boldsymbol{\omega} + \tau_{\boldsymbol{C}}{}^{\boldsymbol{A'}}\boldsymbol{\omega^{CA}}, \tag{2.51}$$

where

$$\boldsymbol{e} \equiv \tau^{\boldsymbol{PP'}}\boldsymbol{e_{PP'}}, \qquad \boldsymbol{e_{AB}} \equiv \tau_{(\boldsymbol{A}}{}^{\boldsymbol{P'}}\boldsymbol{e_{B)P'}}, \qquad \boldsymbol{\omega} \equiv \tau_{\boldsymbol{PP'}}\boldsymbol{\omega^{PP'}}, \qquad \boldsymbol{\omega^{AB}} = -\tau^{(\boldsymbol{A}}{}_{\boldsymbol{P'}}\boldsymbol{\omega^{B)P'}}.$$

In this formalism one defines the *spatial Infeld-van der Waerden symbols* by $\sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}} \equiv \tau_{(\boldsymbol{A}}{}^{\boldsymbol{A'}}\sigma_{\boldsymbol{B})\boldsymbol{A'}}{}^{\boldsymbol{i}}$. A direct computation shows that the components of the Infeld-van der Waerden symbols can be read from the matrices

$$\begin{aligned}
&\sigma_{\boldsymbol{AB}}{}^{\boldsymbol{1}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{\boldsymbol{AB}}{}^{\boldsymbol{2}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad \sigma_{\boldsymbol{AB}}{}^{\boldsymbol{3}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
&\sigma^{\boldsymbol{AB}}{}_{\boldsymbol{1}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^{\boldsymbol{AB}}{}_{\boldsymbol{2}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \text{-i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad \sigma^{\boldsymbol{AB}}{}_{\boldsymbol{3}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{2.52}$$

It follows from the space spinor split of the frame encoded in equation (2.51) that the metric $\boldsymbol{g}$ admits the split

$$\boldsymbol{g} = \frac{1}{2}\boldsymbol{\omega}\otimes\boldsymbol{\omega} + h_{\boldsymbol{ABCD}}\boldsymbol{\omega^{AB}}\otimes\boldsymbol{\omega^{CD}}$$

where

$$h_{\boldsymbol{ABCD}} \equiv \boldsymbol{g}(\boldsymbol{e_{AB}}, \boldsymbol{e_{CD}}) = -\epsilon_{\boldsymbol{A}(\boldsymbol{C}}\epsilon_{\boldsymbol{D})\boldsymbol{B}}.$$

Similarly, any general connection $\breve{\boldsymbol{\nabla}}$ can be split as

$$\breve{\nabla}_{\boldsymbol{AA'}} = \frac{1}{2}\tau_{\boldsymbol{AA'}}\mathcal{P} - \tau_{\boldsymbol{A'}}{}^{\boldsymbol{Q}}\breve{\mathcal{D}}_{\boldsymbol{AQ}}, \tag{2.53}$$

where

$$\mathcal{P} \equiv \tau^{\boldsymbol{AA'}}\breve{\nabla}_{\boldsymbol{AA'}} \qquad \text{and} \qquad \breve{\mathcal{D}}_{\boldsymbol{AB}} \equiv \tau_{(\boldsymbol{B}}{}^{\boldsymbol{A'}}\breve{\nabla}_{\boldsymbol{A})\boldsymbol{A'}},$$

denote, respectively, the derivative along the direction given by $\tau^{\boldsymbol{AA'}}$ and $\breve{\mathcal{D}}_{\boldsymbol{AB}}$ is the *Sen connection* of $\breve{\nabla}$ relative to $\tau^{\boldsymbol{AA'}}$.

The Hermitian spinor $\tau^{AA'}$ induces a notion of Hermitian conjugation: given an arbitrary spinor with components $\mu_{\boldsymbol{AB}}$ its Hermitian conjugate has components

$$\mu^{\dagger}_{\boldsymbol{CD}} \equiv \tau_{\boldsymbol{C}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{B'}}\overline{\mu_{\boldsymbol{AB}}} = \tau_{\boldsymbol{C}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{B'}}\bar{\mu}_{\boldsymbol{A'B'}}, \tag{2.54}$$

where the bar denotes complex conjugation. In a similar manner, one can extend the definition to contravariant indices and higher valence spinors by requiring that $(\boldsymbol{\pi\mu})^{\dagger} = \boldsymbol{\pi}^{\dagger}\boldsymbol{\mu}^{\dagger}$. As a consequence of this definition, for a spinor $\mu_{\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_n}$ with a string of n indices, one has that

$$\mu^{\dagger\dagger}_{\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_n} = (-1)^n\mu_{\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_n}.$$

Additionally, the Hermitian conjugation operation allows to introduce the notion of *real and imaginary spinors*. If a spinor $\mu_{\boldsymbol{A}_1\boldsymbol{B}_1\ldots\boldsymbol{A}_n\boldsymbol{B}_n}$ with $2n$ indices satisfies

$$\mu^{\dagger}_{\boldsymbol{A}_1\boldsymbol{B}_1\ldots\boldsymbol{A}_n\boldsymbol{B}_n} = (-1)^n\mu_{\boldsymbol{A}_1\boldsymbol{B}_1\ldots\boldsymbol{A}_n\boldsymbol{B}_n},$$

it will be said to be real, while if it satisfies

$$\mu^{\dagger}_{\boldsymbol{A}_1\boldsymbol{B}_1\ldots\boldsymbol{A}_n\boldsymbol{B}_n} = (-1)^{n+1}\mu_{\boldsymbol{A}_1\boldsymbol{B}_1\ldots\boldsymbol{A}_n\boldsymbol{B}_n},$$

it will be said to be imaginary.

2.4.2 First order hyperbolic reduction for the standard conformal Einstein field equations

In [17] it was shown that introducing gauge source functions and exploiting the so-called, *space spinor formalism* one can extract from (2.10a)-(2.10g) a symmetric hyperbolic system of evolution equations. In this section the notion of gauge source functions is reviewed. For conciseness of the presentation, the hyperbolic reduction procedure is only sketched by means of a model equation —see [11, 13, 15–17] for a comprehensive discussion of the space spinor formalism and gauge source functions. To illustrate the general strategy of the hyperbolic reduction procedure consider the model equation

$$\nabla_{\boldsymbol{AA'}}\varphi_{\boldsymbol{BB'}} - \nabla_{\boldsymbol{BB'}}\varphi_{\boldsymbol{AA'}} = G_{\boldsymbol{AA'BB'}}. \tag{2.55}$$

Recall that, in general, one can decompose a spinor with the index structure of $G_{\boldsymbol{AA'BB'}}$, as follows

$$G_{\boldsymbol{AA'BB'}} = G_{(\boldsymbol{AB})(\boldsymbol{A'B'})} + \frac{1}{2}\epsilon_{\boldsymbol{A'B'}}G_{(\boldsymbol{A}|\boldsymbol{Q'}|\boldsymbol{B})}{}^{\boldsymbol{Q'}} + \frac{1}{2}\epsilon_{\boldsymbol{AB}}G_{\boldsymbol{Q}(\boldsymbol{A'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B'})} + \frac{1}{4}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}G_{\boldsymbol{QQ'}}{}^{\boldsymbol{QQ'}}.$$

Observe from equation (2.55) that $G_{\boldsymbol{AA'BB'}} = -G_{\boldsymbol{BB'AA'}}$. Exploiting this antisymmetry one can show that the first and last term in the last decomposition vanish. Now, define

$$G_{\boldsymbol{AB}\mathcal{K}} \equiv \frac{1}{2}G_{(\boldsymbol{A}|\boldsymbol{Q'}|\boldsymbol{B})}{}^{\boldsymbol{Q'}},$$

and observe that

$$\bar{G}_{\boldsymbol{A'B'}\mathcal{K}} = \frac{1}{2}\bar{G}_{(\boldsymbol{A'}|\boldsymbol{Q}|\boldsymbol{B'})}{}^{\boldsymbol{Q}} = \frac{1}{2}\bar{G}_{\boldsymbol{Q}(\boldsymbol{A'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B'})}.$$

Thus, the *irreducible decomposition* of $G_{\boldsymbol{AA'BB'}}$ is

$$G_{\boldsymbol{AA'BB'}} = \epsilon_{\boldsymbol{A'B'}}G_{\boldsymbol{AB}} + \epsilon_{\boldsymbol{AB}}\bar{G}_{\boldsymbol{AB}}.$$

In other words, the information of $G_{\boldsymbol{AA'BB'}}$ is encoded in the reduced spinor $G_{\boldsymbol{AB}}$ —this is analogous to the irreducible decomposition relating the Faraday tensor in terms of the Maxwell spinor as discussed in [12, 40]. Consequently, one can rewrite the model equation as

$$\nabla_{(\boldsymbol{A}|\boldsymbol{Q'}|}\varphi_{\boldsymbol{B})}{}^{\boldsymbol{Q'}} = G_{\boldsymbol{AB}}. \tag{2.56}$$

Notice that one has performed an irreducible decomposition of the spinors in equation (2.55), therefore equation (2.56) contains the same information as equation (2.55). Additionally, observe that

$$\nabla_{\boldsymbol{AQ'}}\varphi_{\boldsymbol{B}}{}^{\boldsymbol{Q'}} = \nabla_{(\boldsymbol{A}|\boldsymbol{Q'}|}\varphi_{\boldsymbol{B})}{}^{\boldsymbol{Q'}} + \frac{1}{2}\epsilon_{\boldsymbol{AB}}\nabla^{\boldsymbol{QQ'}}\varphi_{\boldsymbol{QQ'}}. \tag{2.57}$$

From the last expression one concludes that equation (2.56) does not contain information about the full derivative $\nabla_{\boldsymbol{AQ'}}\varphi_{\boldsymbol{B}}{}^{\boldsymbol{Q'}}$ but only about its symmetrised part. In other words, equation (2.56) leaves the divergence $\nabla^{\boldsymbol{QQ'}}\varphi_{\boldsymbol{QQ'}}$ completely unspecified. This observation leads to the notion of gauge source functions. Let $F(x)$ be an arbitrary smooth function of the coordinates. The divergence in equation (2.57), encoding the freedom left in the equation (2.56), will be generically called *gauge source function*. Using equation (2.56) and taking the above considerations into account one can construct the following equation for the unknown $\varphi_{\boldsymbol{QA}\mathcal{K}}$

$$\nabla_{\boldsymbol{AQ'}}\varphi_{\boldsymbol{B}}{}^{\boldsymbol{Q'}} = G_{\boldsymbol{AB}} + \frac{1}{2}\epsilon_{\boldsymbol{AB}}F(x). \tag{2.58}$$

The key observation is that one can extract a symmetric hyperbolic system of evolution equations from equation (2.58). This procedure can be generalised for an equation containing higher valence spinors with similar index structure as the simple model equation discussed above. The conformal Einstein field equations encoded in the zero-quantities (2.10a)-(2.10g) have a similar structure to that of the model equation and an analogous reduction procedure can be implemented. In the rest of this subsection the gauge source functions defined for the hyperbolic reduction of the conformal Einstein field equations are listed:

(i) The *coordinate gauge source function* is defined as

$$F^{\boldsymbol{a}}(x) \equiv \nabla^{\boldsymbol{QQ'}} e_{\boldsymbol{QQ'}}{}^{\boldsymbol{a}},$$

where $e^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}$ are the so-called *frame coefficients* defined via $\boldsymbol{e}_{\boldsymbol{AA'}} = e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\boldsymbol{c}_{\boldsymbol{a}}$ where $\{\boldsymbol{c}_{\boldsymbol{a}}\}$ is a smooth frame field on $\mathcal{M}$ and $\boldsymbol{e}_{\boldsymbol{AA'}}$ is a $\boldsymbol{g}$-orthonormal frame. The coordinate gauge source function can be succinctly written in terms of the frame coefficients and spin coefficients using that

$$\nabla_{\boldsymbol{AA'}} e_{\boldsymbol{BB'}}{}^{\boldsymbol{a}} = \boldsymbol{e}_{\boldsymbol{AA'}}(e_{\boldsymbol{BB'}}{}^{\boldsymbol{a}}) - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}} e_{\boldsymbol{QB'}}{}^{\boldsymbol{a}} - \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{B'}} e_{\boldsymbol{BQ'}}{}^{\boldsymbol{a}}. \tag{2.59}$$

(ii) The *frame gauge source function* is defined as

$$F_{\boldsymbol{AB}}(x) \equiv \nabla^{\boldsymbol{QQ'}} \Gamma_{\boldsymbol{QQ'AB}}, \tag{2.60}$$

where $\Gamma_{\boldsymbol{QQ'AB}}$ are the reduced connection coefficients. Similarly, the frame gauge source function can be succinctly written in terms of the frame coefficients and the spin coefficients via

$$\begin{aligned}\nabla_{\boldsymbol{EE'}} \Gamma_{\boldsymbol{FF'AB}} = \boldsymbol{e}_{\boldsymbol{EE'}}\big(\Gamma_{\boldsymbol{FF'AB}}\big) - \Gamma_{\boldsymbol{EE'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{F}} \Gamma_{\boldsymbol{QF'AB}} \\ - \bar{\Gamma}_{\boldsymbol{EE'}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{F}} \Gamma_{\boldsymbol{FQ'AB}} - \Gamma_{\boldsymbol{EE'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{A}} \Gamma_{\boldsymbol{FF'QB}} - \Gamma_{\boldsymbol{EE'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}} \Gamma_{\boldsymbol{FF'AQ}}.\end{aligned} \tag{2.61}$$

(iii) The *conformal gauge source function* is given by the Ricci scalar $R(x)$ of $\boldsymbol{g}$. This gauge source function fixes the freedom in choosing a representative from the conformal class $[\boldsymbol{g}]$. The relation between Ξ and R can be understood in terms of the conformal transformation law for the Ricci scalar since the latter implies a wave equation for the conformal factor Ξ where R acts as a source term —see [16] for further discussion of the role played by the conformal gauge source function in the formulation of the conformal Einstein field equations.

2.4.3 First order hyperbolic reduction for the extended conformal Einstein field equations

The space spinor formalism leads to a systematic split of the extended conformal Einstein field equations (2.32) into evolution and constraint equations. To this end, one performs a space spinor split for the fields $\boldsymbol{e_{AA'}}$, $f_{\boldsymbol{AA'}}$, $\hat{L}_{\boldsymbol{AA'}}$, $\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$. The frame coefficients $e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}$ satisfy formally identical splits to those in (2.51), where $\boldsymbol{e_{AA'}} = e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\boldsymbol{c_a}$ with $\boldsymbol{c_a} \in \{\boldsymbol{\partial}_\tau, \boldsymbol{c_i}\}$ represent a fixed frame field —the latter is not necessarily $\boldsymbol{g}$-orthonormal. Observe that, in terms of tensor frame components, the gauge condition (2.43) implies that $\boldsymbol{e_0}{}^{\boldsymbol{a}} = \delta_{\boldsymbol{0}}{}^{\boldsymbol{a}}$. The gauge conditions (2.44) and (2.43) are rewritten as

$$\tau^{\boldsymbol{AA'}}\boldsymbol{e_{AA'}} = \sqrt{2}\boldsymbol{\partial}_\tau, \qquad \tau^{\boldsymbol{AA'}}\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} = 0, \qquad \tau^{\boldsymbol{AA'}}\hat{L}_{\boldsymbol{AA'BB'}} = 0. \tag{2.62}$$

In addition, one defines

$$\hat{\Gamma}_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\hat{\Gamma}_{\boldsymbol{AA'CD}}, \quad \Gamma_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\Gamma_{\boldsymbol{AA'CD}}, \quad f_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}f_{\boldsymbol{AA'}}, \tag{2.63a}$$

$$L_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\hat{L}_{\boldsymbol{AA'CC'}}, \quad \Theta_{\boldsymbol{ABCD}} \equiv L_{\boldsymbol{AB(CD)}} \quad \Theta_{\boldsymbol{AB}} \equiv L_{\boldsymbol{ABQ}}{}^{\boldsymbol{Q}}. \tag{2.63b}$$

Recalling equation (2.2.3) one obtains

$$\hat{\Gamma}_{\boldsymbol{ABCD}} = \Gamma_{\boldsymbol{ABCD}} - \epsilon_{\boldsymbol{CA}}f_{\boldsymbol{DA'}}\tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}},$$

where $\Gamma_{\boldsymbol{ABCD}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\Gamma_{\boldsymbol{AA'CD}}$. This relation allows to write the equations in terms of the reduced connection coefficients of the Levi-Civita connection of $\boldsymbol{g}$ instead of the reduced connection coefficients of $\hat{\boldsymbol{\nabla}}$. Only the spinorial counterpart of the Schouten tensor of the connection $\hat{\boldsymbol{\nabla}}$ will not be written in terms of its Levi-Civita counterpart. Exploiting the notion of Hermitian conjugation given in equation (2.54) one defines

$$\chi_{\boldsymbol{ABCD}} \equiv -\frac{1}{\sqrt{2}}\left(\Gamma_{\boldsymbol{ABCD}} + \Gamma^{\dagger}_{\boldsymbol{ABCD}}\right), \qquad \xi_{\boldsymbol{ABCD}} \equiv \frac{1}{\sqrt{2}}\left(\Gamma_{\boldsymbol{ABCD}} - \Gamma^{\dagger}_{\boldsymbol{ABCD}}\right),$$

Observe that $\chi^{\dagger}_{\boldsymbol{ABCD}} = \chi_{\boldsymbol{ABCD}}$ while $\xi^{\dagger}_{\boldsymbol{ABCD}} = -\xi_{\boldsymbol{ABCD}}$. Consequently $\chi_{\boldsymbol{ABCD}}$ is real and $\xi_{\boldsymbol{ABCD}}$ is imaginary—see Section 2.4.1 for the notion of real and imaginary spinors. Notice that, consistent with these definitions, $\Gamma_{\boldsymbol{ABCD}}$ can be written as as

$$\Gamma_{\boldsymbol{ABCD}} = \frac{1}{\sqrt{2}}(\xi_{\boldsymbol{ABCD}} - \chi_{\boldsymbol{ABCD}}). \tag{2.64}$$

One proceeds with the rescaled Weyl spinor defining

$$\eta_{\boldsymbol{ABCD}} \equiv \frac{1}{2}\left(\phi_{\boldsymbol{ABCD}} + \phi^{\dagger}_{\boldsymbol{ABCD}}\right), \qquad \mu_{\boldsymbol{ABCD}} \equiv -\frac{1}{2}\mathrm{i}\left(\phi_{\boldsymbol{ABCD}} - \phi^{\dagger}_{\boldsymbol{ABCD}}\right).$$

Observe that $\eta^{\dagger}_{\boldsymbol{ABCD}} = \eta_{\boldsymbol{ABCD}}$ and $\mu^{\dagger}_{\boldsymbol{ABCD}} = \mu_{\boldsymbol{ABCD}}$ so that both $\eta_{\boldsymbol{ABCD}}$ and $\mu_{\boldsymbol{ABCD}}$ are real spinors. Consequently, one has

$$\phi_{\boldsymbol{ABCD}} = \eta_{\boldsymbol{ABCD}} + i\mu_{\boldsymbol{ABCD}} \tag{2.65}$$

The latter implies that $\eta_{\boldsymbol{ABCD}}$ and $\mu_{\boldsymbol{ABCD}}$ can be interpreted as the electric and magnetic parts of the rescaled Weyl spinor. Observe that the split (2.64) is not an electric-magnetic decomposition as that of equation (2.65). The gauge conditions (2.62) can be rewritten in terms of the spinors defined in (2.63a) as

$$f_{\boldsymbol{AB}} = f_{(\boldsymbol{AB})}, \qquad \Gamma_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{AB}} = -f_{\boldsymbol{AB}}, \qquad \hat{L}_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{AB}} = 0. \tag{2.66}$$

The last condition implies the decomposition

$$\hat{L}_{\boldsymbol{ABCD}} = \Theta_{\boldsymbol{ABCD}} + \frac{1}{2}\epsilon_{\boldsymbol{CD}}\Theta_{\boldsymbol{AB}},$$

for the components of the spinorial counterpart of the Schouten tensor of the Weyl connection where $\Theta_{\boldsymbol{ABCD}} \equiv \hat{L}_{(\boldsymbol{AB})(\boldsymbol{CD})}$ and $\Theta_{\boldsymbol{AB}} \equiv \hat{L}_{\boldsymbol{ABQ}}{}^{\boldsymbol{Q}}$.

The fields defined in the previous paragraphs allows to derive from the expressions

$$\tau^{\boldsymbol{AA'}}\hat{\Sigma}_{\boldsymbol{AA'}}{}^{\boldsymbol{PP'}}{}_{\boldsymbol{BB'}}\boldsymbol{e}_{\boldsymbol{PP'}}{}^{\boldsymbol{a}} = 0, \qquad \tau^{\boldsymbol{CC'}}\hat{\Xi}_{\boldsymbol{ABCC'DD'}} = 0, \tag{2.67a}$$

$$\tau^{\boldsymbol{AA'}}\hat{\Delta}_{\boldsymbol{AA'BB'CC'}} = 0, \qquad \tau_{(\boldsymbol{A}}{}^{\boldsymbol{A'}}\hat{\Lambda}_{|\boldsymbol{A'}|\boldsymbol{BCD})} = 0, \tag{2.67b}$$

as a set of evolution equations for the fields

$$\chi_{\boldsymbol{ABCD}}, \;\; \xi_{\boldsymbol{ABCD}}, \;\; \boldsymbol{e}_{\boldsymbol{AB}}{}^{\boldsymbol{0}}, \;\; \boldsymbol{e}_{\boldsymbol{AB}}{}^{\boldsymbol{i}}, \;\; f_{\boldsymbol{AB}}, \;\; \Theta_{\boldsymbol{ABCD}}, \;\; \Theta_{\boldsymbol{AB}}, \;\; \phi_{\boldsymbol{ABCD}}.$$

Explicitly, one has that

$$\partial_\tau e_{\boldsymbol{AB}}{}^{\boldsymbol{0}} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}}e_{\boldsymbol{PQ}}{}^{\boldsymbol{0}} - f_{\boldsymbol{AB}}, \tag{2.68a}$$

$$\partial_\tau e_{\boldsymbol{AB}}{}^{\boldsymbol{i}} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}}e_{\boldsymbol{PQ}}{}^{\boldsymbol{i}}, \tag{2.68b}$$

$$\partial_\tau \xi_{\boldsymbol{ABCD}} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}}\xi_{\boldsymbol{PQCD}} + \frac{1}{\sqrt{2}}(\epsilon_{\boldsymbol{AC}}\chi_{(\boldsymbol{BD})\boldsymbol{PQ}} + \epsilon_{\boldsymbol{BD}}\chi_{(\boldsymbol{AC})\boldsymbol{PQ}})f^{\boldsymbol{PQ}}, \tag{2.68c}$$

$$-\sqrt{2}\chi_{\boldsymbol{AB}(\boldsymbol{C}}{}^{\boldsymbol{E}}f_{\boldsymbol{D})\boldsymbol{E}} - \frac{1}{2}\left(\epsilon_{\boldsymbol{AC}}\Theta_{\boldsymbol{BD}} + \epsilon_{\boldsymbol{BD}}\Theta_{\boldsymbol{AC}}\right) - \mathrm{i}\Theta\mu_{\boldsymbol{ABCD}}, \tag{2.68d}$$

$$\partial_\tau f_{\boldsymbol{AB}} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}}f_{\boldsymbol{PQ}} + \frac{1}{\sqrt{2}}\Theta_{\boldsymbol{AB}}, \tag{2.68e}$$

$$\partial_\tau \chi_{(\boldsymbol{AB})\boldsymbol{CD}} = -\chi_{\boldsymbol{AB}}{}^{\boldsymbol{PQ}}\chi_{\boldsymbol{PQCD}} - \Theta_{\boldsymbol{ABCD}} + \Theta\eta_{\boldsymbol{ABCD}}, \tag{2.68f}$$

$$\partial_\tau \Theta_{\boldsymbol{ABCD}} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}}L_{\boldsymbol{PQ}(\boldsymbol{CD})} - \dot{\Theta}\eta_{\boldsymbol{ABCD}} + \mathrm{i}d^{\boldsymbol{P}}{}_{(\boldsymbol{A}}\mu_{\boldsymbol{B})\boldsymbol{CDP}}, \tag{2.68g}$$

$$\partial_\tau \Theta_{\boldsymbol{AB}} = -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{EF}}\Theta_{\boldsymbol{EF}} + \sqrt{2}d^{\boldsymbol{PQ}}\eta_{\boldsymbol{ABPQ}}, \tag{2.68h}$$

$$\partial_\tau \phi_{\boldsymbol{ABCD}} - \sqrt{2}\mathcal{D}_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\phi_{\boldsymbol{BCD})\boldsymbol{Q}} = 0. \tag{2.68i}$$

The following proposition relates the discussion of the conformal evolution equations and the full set of extended conformal field equations given by (2.21):

Lemma 8 (***propagation of the constraints and subsidiary system***)**.** *Let* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *be a spacetime where* $\tilde{\boldsymbol{g}}$ *is a solution to the vacuum Einstein field equations. Assume that an open set* $\mathcal{U} \subset \tilde{\mathcal{M}}$ *can be covered by a non-intersecting congruence of conformal geodesics and that the evolution equations extracted from equations* (2.67a)-(2.67b) *and the conformal Gauss gauge conditions* (2.66) *hold on* $\mathcal{U}$*. Then, the components of the zero quantities*

$$\hat{\Sigma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}}, \quad \hat{\Xi}_{\boldsymbol{ABCC'DD'}}, \quad \hat{\Delta}_{\boldsymbol{AA'BB'CC'}}, \quad \hat{\Lambda}_{\boldsymbol{AA'BC}} \quad \delta_{\boldsymbol{AA'}}, \quad \gamma_{\boldsymbol{AA'BB'}}, \quad \zeta_{\boldsymbol{AA'}},$$

which are not determined by the evolution equations or the gauge conditions, satisfy a first order symmetric hyperbolic system of equations (subsidiary system) whose lower order terms are algebraic and homogeneous in the zero-quantities on $\mathcal{U}$*.*

The proof of Lemma 8 can be found in [13, 22, 27] —see also [36] for a discussion of these equations in the presence of an electromagnetic field.

The most important consequence of Lemma 8 is that if the zero-quantities vanish at some initial hypersurface and the evolution equations (2.68a)-(2.68h) are satisfied, then the *full* extended conformal Einstein field equations encoded in (2.26a)-(2.26b) are satisfied in the development of the initial data. This is a consequence of the standard uniqueness result for *homogeneous* symmetric hyperbolic systems.

Remark 7. The evolution equations (2.68a)-(2.68i) are extracted from equations (2.67a)-(2.67b). In tensorial notation these correspond to the following components of the zero-quantities (2.22a)-(2.22d):

$$\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \quad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{a0b}} = 0, \quad \hat{\Delta}_{\boldsymbol{0bc}} = 0, \quad \hat{\Lambda}_{(\boldsymbol{a}|\boldsymbol{0}|\boldsymbol{b})} = 0, \quad \hat{\Lambda}^{*}_{(\boldsymbol{a}|\boldsymbol{0}|\boldsymbol{b})} = 0,$$

where $\hat{\Lambda}^{*}_{\boldsymbol{bcd}} = \frac{1}{2}\epsilon_{\boldsymbol{cd}}{}^{\boldsymbol{ef}}\hat{\Lambda}_{\boldsymbol{bef}}$ and the frame indices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ take values $0, 1, 2, 3$. The components of the zero-quantities (2.22a)-(2.22d) which are not determined by the evolution equations correspond to

$$\hat{\Sigma}_{\boldsymbol{i}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \quad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dib}} = 0, \quad \hat{\Delta}_{\boldsymbol{ibc}} = 0, \quad \hat{\Lambda}_{\boldsymbol{0ij}} = 0, \quad \hat{\Lambda}_{\boldsymbol{0i0}} = 0, \tag{2.69}$$

with $\boldsymbol{i}, \boldsymbol{j} = 1, 2, 3$. The lower order terms in the first symmetric hyperbolic system (subsidiary system) referred in Lemma 8 consist of algebraic expressions containing the zero-quantities in equation (2.69). The subsidiary system is not given explicitly in this thesis for conciseness. A detailed derivation of the subsidiary system and a comprehensive discussion can be found in [13, 22, 27] —see also [36].

Controlling the gauge

The derivation of the conformal evolution equations (2.68a)-(2.68h) is based on the assumption of the existence of a non-intersecting congruence of conformal geodesics. To verify this assumption one has to analyse the deviation vector of the congruence.

Let $\boldsymbol{z}$ denote the deviation vector of the congruence. One has then that

$$[\dot{\boldsymbol{x}}, \boldsymbol{z}] = 0. \tag{2.70}$$

Now, let $\boldsymbol{z}^{\boldsymbol{AA'}}$ denote the spinorial counterpart of the components $z^{\boldsymbol{a}}$ of $\boldsymbol{z}$ respect to a Weyl propagated frame $\{\boldsymbol{e_a}\}$. Following the spirit of the space spinor formalism one defines $z_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}} z_{\boldsymbol{AA'}}$. This spinor can be decomposed as

$$z_{\boldsymbol{AB}} = \frac{1}{2} z \epsilon_{\boldsymbol{AB}} + z_{(\boldsymbol{AB})}.$$

The evolution equations for the deviation vector can be readily deduced from the commutator (2.70). Expressing the latter in terms of the fields appearing in the extended conformal field equations one obtains

$$\partial_\tau z = f_{\boldsymbol{AB}} z^{(\boldsymbol{AB})}, \tag{2.71a}$$

$$\partial_\tau z_{(\boldsymbol{AB})} = \chi_{\boldsymbol{CD}(\boldsymbol{AB})} z^{(\boldsymbol{CD})}. \tag{2.71b}$$

The congruence of conformal geodesics is non-intersecting as long as $z_{(\boldsymbol{AB})} \neq 0$. Once one has solved equations (2.68a)-(2.68i) one can substitute $f_{\boldsymbol{AB}}$ and $\chi_{\boldsymbol{ABCD}}$ into equations (2.71a)-(2.71b) and analyse the evolution of the deviation vector —for further discussion see [41].

2.5 The conformal constraint equations

The *conformal constraint equations* encode the set of restrictions induced by the zero-quantities on the various fields on spacelike hypersurfaces of the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$. In what follows, one consider the setting where the 1-form $\boldsymbol{f}$ vanishes on one of these hypersurfaces, which is regarded as the initial hypersurface. Accordingly, the initial data for the *extended* conformal evolution equations (2.68a)-(2.68h) and those implied by the hyperbolic reduction of the *standard* conformal Einstein field equations using gauge source functions —see Section 2.4.2— are the same. Now, let $\tilde{\mathcal{S}}$ denote a 3-dimensional spacelike submanifold of $\tilde{\mathcal{M}}$. The metric $\tilde{\boldsymbol{g}}$ induces a 3-dimensional metric $\tilde{\boldsymbol{h}} = \tilde{\varphi}^* \tilde{\boldsymbol{g}}$ on $\tilde{\mathcal{S}}$, where $\tilde{\varphi} : \tilde{\mathcal{S}} \to \tilde{\mathcal{M}}$ is an embedding. Similarly, one can consider a 3-dimensional submanifold $\mathcal{S}$ of $\mathcal{M}$ with induced metric $\boldsymbol{h} = \varphi^* \boldsymbol{g}$, such that

$$\boldsymbol{h} = \Omega^2 \tilde{\boldsymbol{h}}, \tag{2.72}$$

where Ω denotes the restriction of the conformal factor to the initial hypersurface $\mathcal{S}$ —in Section 2.3.2 this restriction is denoted by $\Theta_\star$.

Let n_a and $\tilde{n}_a$ with $n_a = \Omega\tilde{n}_a$ be, respectively, the $\boldsymbol{g}$-unit and $\tilde{\boldsymbol{g}}$-unit normals, so that $n^a n_a = \tilde{n}^a\tilde{n}_a = 1$ —in accordance with the signature conventions introduced in Section 1.6 of Chapter 1 for a spacelike hypersurface. With these definitions, the second fundamental forms $\chi_{ab} \equiv h_a{}^c\nabla_c n_b$ and $\tilde{\chi}_{ab} \equiv \tilde{h}_a{}^c\tilde{\nabla}_c\tilde{n}_b$ are related by the formula

$$\chi_{ab} = \Omega(\tilde{\chi}_{ab} + \Sigma\tilde{h}_{ab}) \tag{2.73}$$

where $\Sigma \equiv n^a\nabla_a\Omega$.

The conformal constraint equations are conveniently expressed in terms of a frame $\{\boldsymbol{e_i}\}$ adapted to the hypersurface $\mathcal{S}$ —that is, the vectors $\boldsymbol{e_i}$ span $T\mathcal{S}$ and, thus, are orthogonal to its normal. All the fields appearing in the constraint equations are expressed in terms of this frame. The *conformal constraint equations* are then given by:

$$D_{\boldsymbol{i}}D_{\boldsymbol{j}}\Omega = -\Sigma\chi_{\boldsymbol{ij}} - \Omega L_{\boldsymbol{ij}} + s h_{\boldsymbol{ij}}, \tag{2.74a}$$
$$D_{\boldsymbol{i}}\Sigma = \chi_{\boldsymbol{i}}{}^{\boldsymbol{k}}D_{\boldsymbol{k}}\Omega - \Omega L_{\boldsymbol{i}}, \tag{2.74b}$$
$$D_{\boldsymbol{i}}s = -L_{\boldsymbol{i}}\Sigma - \Omega L_{\boldsymbol{i}}, \tag{2.74c}$$
$$\begin{aligned} D_{\boldsymbol{i}}L_{\boldsymbol{jk}} - D_{\boldsymbol{j}}L_{\boldsymbol{ik}} = {}& -\Sigma d_{\boldsymbol{ijk}} + d_{\boldsymbol{lkij}}D^{\boldsymbol{l}}\Omega - (\chi_{\boldsymbol{ik}}L_{\boldsymbol{j}} - \chi_{\boldsymbol{jk}}L_{\boldsymbol{i}}) \\ & -\big(\chi_{\boldsymbol{ik}}L_{\boldsymbol{j}} - \chi_{\boldsymbol{jk}}L_{\boldsymbol{i}}\big), \end{aligned} \tag{2.74d}$$
$$D_{\boldsymbol{i}}L_{\boldsymbol{j}} - D_{\boldsymbol{j}}L_{\boldsymbol{i}} = d_{\boldsymbol{lij}}D^{\boldsymbol{l}}\Omega + \chi_{\boldsymbol{i}}{}^{\boldsymbol{k}}L_{\boldsymbol{jk}} - \chi_{\boldsymbol{j}}{}^{\boldsymbol{k}}L_{\boldsymbol{ik}}, \tag{2.74e}$$
$$D^{\boldsymbol{k}}d_{\boldsymbol{kij}} = \chi^{\boldsymbol{k}}{}_{\boldsymbol{i}}d_{\boldsymbol{jk}} - \chi^{\boldsymbol{k}}{}_{\boldsymbol{j}}d_{\boldsymbol{ik}}, \tag{2.74f}$$
$$D^{\boldsymbol{i}}d_{\boldsymbol{ij}} = \chi^{\boldsymbol{ik}}d_{\boldsymbol{ijk}}, \tag{2.74g}$$
$$D_{\boldsymbol{j}}\chi_{\boldsymbol{ki}} - D_{\boldsymbol{k}}\chi_{\boldsymbol{ji}} = \Omega d_{\boldsymbol{ijk}} + h_{\boldsymbol{ij}}L_{\boldsymbol{k}} - h_{\boldsymbol{ik}}L_{\boldsymbol{j}}, \tag{2.74h}$$
$$l_{\boldsymbol{ij}} = \Omega d_{\boldsymbol{ij}} + L_{\boldsymbol{ij}} - \chi_{\boldsymbol{k}}{}^{\boldsymbol{k}}(\chi_{\boldsymbol{ij}} - \frac{1}{4}\chi h_{\boldsymbol{ij}}) + \chi_{\boldsymbol{ki}}\chi_{\boldsymbol{j}}{}^{\boldsymbol{k}} - \frac{1}{4}\chi_{\boldsymbol{kl}}\chi^{\boldsymbol{kl}}h_{\boldsymbol{ij}}, \tag{2.74i}$$
$$\lambda = 6\Omega s - 3\Sigma^2 - 3D_{\boldsymbol{k}}\Omega D^{\boldsymbol{k}}\Omega, \tag{2.74j}$$

where $\boldsymbol{D}$ is the Levi-Civita connection on $(\mathcal{S}, \boldsymbol{h})$, $l_{\boldsymbol{ij}}$ is the associated Schouten tensor of $\boldsymbol{D}$, $d_{\boldsymbol{ijk}} \equiv d_{\boldsymbol{i0jk}}$, $d_{\boldsymbol{ij}} \equiv d_{\boldsymbol{i0j0}}$, $L_{\boldsymbol{i}} \equiv L_{\boldsymbol{0i}}$ and s is the Friedrich scalar field on $\mathcal{S}$.

Definition. A *solution to the conformal constraint equations* on $\mathcal{S}$ is given by a collection $\boldsymbol{u}_\star \equiv (\Omega, \Sigma, s, \boldsymbol{e_i}, \gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}, \chi_{\boldsymbol{ij}}, L_{\boldsymbol{ij}}, L_{\boldsymbol{i}}, d_{\boldsymbol{ij}}, d_{\boldsymbol{ijk}})$ satisfying (2.74a)-(2.74j).

2.5.1 The Hamiltonian and momentum constraint equations

An alternative point of view for discussing the conformal constraint equations is to start with the usual Hamiltonian and momentum constraints in the physical representation $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$

$$\tilde{r} + \tilde{\chi}^2 - \tilde{\chi}_{ab}\tilde{\chi}^{ab} = 2\lambda, \tag{2.75a}$$
$$\tilde{D}^b\tilde{\chi}_{ab} - \tilde{D}_a\tilde{\chi} = 0, \tag{2.75b}$$

where $\tilde{r}$ is the Ricci scalar of $\tilde{\boldsymbol{h}}$, λ is the Cosmological constant. Considering the conformal rescaling $h = \Omega^2\tilde{h}_{ab}$, and an adapted frame $\boldsymbol{e_i}$ as in Section 2.5, a direct computation using equations (2.75a)- (2.75b) gives the so-called conformal Hamiltonian and momentum constraints:

$$2\Omega D_{\boldsymbol{i}}D^{\boldsymbol{i}}\Omega - 3D_{\boldsymbol{i}}\Omega D^{\boldsymbol{i}}\Omega + \tfrac{1}{2}\Omega^2 r - 3\Sigma^2 - \tfrac{1}{2}\Omega^2(\chi^2 - \chi_{\boldsymbol{ij}}\chi^{\boldsymbol{ij}}) + 2\Omega\Sigma\chi = \lambda, \tag{2.76a}$$
$$\Omega^3 D^{\boldsymbol{i}}(\Omega^{-2}\chi_{\boldsymbol{ik}}) - \Omega(D_{\boldsymbol{k}}\chi - 2\Omega^{-1}D_{\boldsymbol{k}}\Sigma) = 0, \tag{2.76b}$$

where $\tilde{r}$ is the Ricci scalar of $\boldsymbol{h}$. The relation between the conformal Hamiltonian and momentum constraint equations (2.76a)-(2.76b) and the conformal constraint equations (2.74a)-(2.74j) is the content of the following:

Lemma 9. *A solution* $(\mathcal{S}, \boldsymbol{u}_\star)$ *to the conformal constraint equations* (2.74a)-(2.74j) *implies a solution to the conformal Hamiltonian and momentum constraints* (2.76a)-(2.76b). *Conversely, a solution* $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{\chi}, \Omega, \Sigma)$ *of* (2.76a)-(2.76b) *gives rise to a solution to* (2.74a)-(2.74j) *on the points of* $\mathcal{S}$ *for which* $\Omega \neq 0$.

Remark 8. If one is to formulate a Cauchy problem for the conformal Einstein field equations, by prescribing initial data on a 3-dimensional manifold $\mathcal{S}$ in which $\Omega \neq 0$, Lemma 9 suggests to use equations (2.76a)-(2.76b) to determine initial data for the conformal evolution equations.

Lemma 9 and Remark 8 motivate the following definitions.

Definition (*basic initial data set*). A collection $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{\chi}, \Omega, \Sigma)$ where $\mathcal{S}$ denotes a 3-dimensional manifold, $\boldsymbol{h}$ a Riemannian 3-metric, $\boldsymbol{\chi}$ a symmetric rank-2 tensor, Ω and Σ scalar functions on $\mathcal{S}$ satisfying equations (2.76a)-(2.76b), will be called a *basic initial data set.*

Definition (*standard initial value problem for the conformal Einstein field equations*). The Cauchy problem for the evolution equations implied by the conformal Einstein field equations provided with a basic initial data set $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{\chi}, \Omega, \Sigma)$ will be called a *standard initial value problem.*

Remark 9. Observe that, in contrast with the conformal constraint equations (2.76a)-(2.76b), the conformal Hamiltonian and momentum constraint equations (2.76a)-(2.76b) are not formally regular at $\Omega = 0$ in the sense that they contain terms involving Ω^{-1} and Ω^{-2}. Consequently, for a Cauchy problem for the conformal Einstein field equations for which $\Omega(p) = 0$ for all $p \in \mathcal{S}$ —see definition of asymptotic initial value problem given below— equations (2.76a)-(2.76b) are not suitable to define initial data for this type of problems.

The last remark motivates the following definition:

Definition (*asymptotic initial value problem for the conformal Einstein field equations*). The Cauchy problem for the evolution equations implied by the conformal Einstein field equations provided with initial data consisting of $(\mathcal{S}, \boldsymbol{u}_\star)$ where $\mathcal{S}$ denotes a 3-dimensional manifold and $\boldsymbol{u}_\star$ is a solution to the conformal constraint equations on $\mathcal{S}$ for which $\Omega = 0$, will be called a *asymptotic initial value problem.*

2.5.2 The vacuum conformal constraint equations at the conformal boundary

In the last section it was pointed out the difficulty in obtaining initial data sets for an asymptotic initial value problem using (2.76a)-(2.76b). Nevertheless, in contrast with equations (2.76a)-(2.76b), the conformal constraint equations (2.74j)-(2.74j) are regular even when $\Omega = 0$. Moreover, the conformal constraint equations simplify considerably on spacelike hypersurfaces for which $\Omega = 0$. In this case equations (2.74a)-(2.74i) reduce to

$$s h_{ij} = \Sigma \chi_{ij}, \tag{2.77a}$$

$$D_i \Sigma = 0, \tag{2.77b}$$

$$D_i s = -L_i \Sigma, \tag{2.77c}$$

$$D_i L_{jk} - D_j L_{ik} = -\Sigma d_{ijk} - (\chi_{ik} L_j - \chi_{jk} L_i), \tag{2.77d}$$

$$D_i L_j - D_j L_i = \chi_i{}^k L_{jk} - \chi_j{}^k L_{ik}, \tag{2.77e}$$

$$D^k d_{kij} = \chi^k{}_i d_{jk} - \chi^k{}_j d_{ik}, \tag{2.77f}$$

$$\lambda = -3\Sigma^2, \tag{2.77g}$$

$$D^i d_{ij} = \chi^{ik} d_{ijk}, \tag{2.77h}$$

$$D_j \chi_{ki} - D_k \chi_{ji} = h_{ij} L_k - h_{ik} L_j, \tag{2.77i}$$

$$l_{ij} = L_{ij} - \chi(\chi_{ij} - \frac{1}{4}\chi h_{ij}) + \chi_{ki}\chi_j{}^k - \frac{1}{4}\chi_{kl}\chi^{kl} h_{ij}. \tag{2.77j}$$

In [9, 22] a procedure for obtaining solutions for these equations has been given. The main idea is to identify as free specifiable data the 3-metric $\boldsymbol{h}$ and a smooth function κ at $\mathcal{S}$ —encoding, essentially, the Friedrich scalar s at $\mathcal{S}$. Then, the remaining fields comprising a solution $\boldsymbol{u}_\star$ are derived from $\boldsymbol{h}$ and κ as follows:

$$\Sigma = \sqrt{\frac{|\lambda|}{3}}, \quad \Sigma_i = 0, \quad s = \Sigma\kappa, \quad \chi_{ij} = \kappa h_{ij}, \quad L_i = -D_i \kappa, \tag{2.78a}$$

$$L_{\boldsymbol{ij}} = l_{\boldsymbol{ij}} + \frac{1}{2}\kappa^2 h_{\boldsymbol{ij}}, \quad d_{\boldsymbol{ijk}} = -\Sigma^{-1} y_{\boldsymbol{ijk}}, \tag{2.78b}$$

where $y_{\boldsymbol{ijk}}$ denotes the components of the Cotton tensor of the metric $\boldsymbol{h}$. The only differential condition that has to be solved to obtain a full solution to the conformal constraint equations is

$$D^{\boldsymbol{i}} d_{\boldsymbol{ij}} = 0, \tag{2.78c}$$

where $d_{\boldsymbol{ij}}$ is a symmetric transverse-tracefree tensor encoding the initial data for the electric part of the rescaled Weyl tensor. This procedure is summarised in the following Lemma:

Lemma 10. *Given a Riemannian 3-metric $h_{\boldsymbol{ij}}$, a smooth scalar function κ and a $\boldsymbol{h}$-divergence-free symmetric tracefree tensor d_{ij} on $\mathcal{S}$, the tensor fields χ_{ij}, L_i, L_{ij} and d_{ijk} defined as in equations* (2.78a)-(2.78b) *constitute a solution $\boldsymbol{u}_\star$ to the conformal constraint equations with $\Omega = 0$.*

Remark 10. Observe in Lemma 10 that the choice of κ is irrespective of h_{ij} and d_{ij}.

Remark 11. Given a 3-metric $\boldsymbol{h}$ there is, in general, not a unique solution to $D^i d_{ij} = 0$. In other words, using Lemma (10), given $(\boldsymbol{h}, \kappa)$ one can construct several solutions to the conformal constraint equations $\boldsymbol{u}_\star$ with $\Omega = 0$ by considering different solutions to the equation $D^i d_{ij} = 0$. For instance, if $\boldsymbol{h}$ is conformally flat, the analysis given in [42] shows that all smooth solutions to $D^i d_{ij} = 0$, can be parametrised by four constants A, Q, P, J and an arbitrary function λ_2 of spin-weight two. Therefore, in the conformally flat case one can construct different solutions to $D^i d_{ij} = 0$ considering different values for A, Q, P, J or choosing different functions λ_2.

3 Second order hyperbolic reductions

3.1 Introduction

The first order hyperbolic reduction of the conformal Einstein field equations using gauge source functions, as briefly discussed in Chapter 2, was originally introduced in [15] —see also [17] for the hyperbolic reduction of the conformal Einstein-Yang-Mills equations. Additionally, in Chapter 2 the first order hyperbolic reduction of the extended conformal Einstein field equations employing conformal Gaussian systems was discussed. The latter hyperbolic reduction strategy was first introduced in [22] —see also[27]. Nevertheless, more recently, it has been shown that gauge source functions can be used to obtain, out of the metric conformal field equations, a system of quasilinear wave equations —see [18]. This particular construction requires the specification of a *coordinate gauge source function* and a *conformal gauge source function* and is close, in spirit, to the classical treatment of the Cauchy problem in General Relativity in [4] —see also [43]. Although, in principle, of general applicability, the construction of wave equations for the metric conformal field equations has been used, so far, only in the discussion of the asymptotic characteristic problem on a cone —see [19].

The discussion given in this chapter is based on the second order hyperbolic reduction procedure introduced in:

Gasperín E. and Valiente Kroon J.A., "Spinorial wave equations and stability of the Milne spacetime," *Classical and Quantum Gravity* **32** (Sept., 2015) 185021, `arXiv:1407.3317 [gr-qc]`.

In the latter reference it is shown how to deduce a system of quasilinear wave equations for the unknowns of the spinorial (standard) conformal Einstein field equations and its relation to the original set of field equations is analysed. The use of the spinorial formulation of the conformal Einstein field equations (or, in fact, the frame formulation of the conformal Einstein field equations) gives access to a wider set of gauge source functions consisting of *coordinate, frame and conformal gauge source functions*. Another advantage of the spinorial version of the conformal Einstein field equations is that they have a much simpler algebraic structure than the metric equations. To discuss this point let $\nabla_{AA'}$ denote the spinorial extension of the

Levi-Civita covariant derivative ∇_a of the unphysical metric $\boldsymbol{g}$. One of the features of the spinorial formalism simplifying the analysis is the use of the symmetric operator

$$\square_{AB} \equiv \nabla_{Q'(A}\nabla_{B)}{}^{Q'}$$

instead of the usual commutator of covariant derivatives $[\nabla_a, \nabla_b]$. As shown in this chapter, the use of spinors allows a more unified and systematic discussion of the construction of the wave equations and the so-called *subsidiary system* —needed to show that under suitable conditions a solution to the wave equations implies a solution to the conformal Einstein field equations. As already mentioned, in the spinorial formulation of the conformal Einstein field equations the metric is not part of the unknowns. This observation is important since, whenever the wave operator $\nabla_a\nabla^a$ is applied to any tensor of non-zero rank, there will appear derivatives of the connection which, in terms of the metric, represent second order derivatives. Thus, if the metric is part of the unknowns, the principal part of the operator $\nabla_a\nabla^a$ is altered by the presence of these derivatives. This is an extra complication that needs to be taken into account in the analysis of [18]. The use of a spinorial frame formalism allows to exploit the algebraic properties of the conformal field equations in a more systematic manner —as it will be seen in the sequel the construction of evolution and subsidiary equations becomes almost algorithmic. In addition, the use of a spinorial version of the equations allows the use of more general classes of gauges and may be more amenable to the inclusion of matter.

In view of the use of spinors, the wave equations considered in this chapter are expressed in terms of the spinorial extension of $\nabla_a\nabla^a$ —see [29]. Namely, one has

$$\square \equiv \nabla_{AA'}\nabla^{AA'}.$$

The operator $\square$ acting on spinors of non-zero rank will rise to terms involving frame derivatives of the spin connection coefficients. The operator $\square$ is the 2-spinor version of the square of the spin-Dirac derivative operator —see e.g. [44].

The construction of wave equations for the fields appearing in the conformal Einstein field equations gives access to a set of methods of the theory of partial differential equations alternative to that used for first order symmetric hyperbolic systems —see e.g. [7] for a discussion on this. For example, the discussion given in [18] is motivated by the analysis of the characteristic problem on a cone for which a detailed theory is available for quasilinear wave equations. An analogous construction of wave equations for the Dirac field on a curved spacetime using the 2-spinors formalism has been given in [45]. It is also worth mentioning that a similar construction of wave equations can be readily implemented for the Maxwell and Yang-Mills fields.

3.2 The spinorial wave equations

In this section a set of wave equations is derived from the spinorial version of the conformal Einstein field equations as given in Chapter 2. Since the approach for obtaining the equations is similar for most of the zero-quantities, a general discussion of the procedure is provided first. In the subsequent parts of this section the peculiarities of each equation are addressed. The results of this section are summarised in Proposition 1. Throughout this chapter only the standard formulation conformal Einstein field equations encoded in equations (2.11a)-(2.11b) of Chapter 2 are used. Therefore for conciseness, when referring to the conformal Einstein field equations it will be understood the standard (vacuum) conformal Einstein field equations. After the discussion of the model equation of Section 3.2.1, the impatient reader may jump to Section 3.2.7 for a summary of the results.

3.2.1 General procedure for obtaining the wave equations

Before deriving each of the wave equations it is illustrative to outline the general procedure with a model equation. To this end consider an equation of the form

$$\nabla^{\boldsymbol{E}}{}_{\boldsymbol{A}'}N_{\boldsymbol{E}\boldsymbol{A}\mathcal{K}} = 0, \tag{3.1}$$

where $N_{\boldsymbol{E}\boldsymbol{A}\mathcal{K}} \equiv \nabla_{(\boldsymbol{E}}{}^{\boldsymbol{B}'}M_{\boldsymbol{A})\boldsymbol{B}'\mathcal{K}}$ and $\mathcal{K}$ is an arbitrary string of spinor indices. The symmetries of the relevant quantities can be exploited using the following decomposition of a spinor of the same index structure

$$T_{\boldsymbol{E}\boldsymbol{A}\mathcal{K}} = T_{(\boldsymbol{E}\boldsymbol{A})\mathcal{K}} + \tfrac{1}{2}\epsilon_{\boldsymbol{E}\boldsymbol{A}}T_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}{}_{\mathcal{K}},$$

and recast $N_{\boldsymbol{E}\boldsymbol{A}\mathcal{K}}$ as

$$N_{\boldsymbol{E}\boldsymbol{A}\mathcal{K}} = \nabla_{\boldsymbol{E}}{}^{\boldsymbol{B}'}M_{\boldsymbol{A}\boldsymbol{B}'\mathcal{K}} + \frac{1}{2}\epsilon_{\boldsymbol{E}\boldsymbol{A}}\nabla^{\boldsymbol{Q}\boldsymbol{B}'}M_{\boldsymbol{Q}\boldsymbol{B}'\mathcal{K}}.$$

Remark 12. The model equation (3.1) determines the symmetrised derivative $\nabla_{(\boldsymbol{E}}{}^{\boldsymbol{B}'}M_{\boldsymbol{A})\boldsymbol{B}'\mathcal{K}}$, while the divergence $\nabla^{\boldsymbol{Q}\boldsymbol{B}'}M_{\boldsymbol{Q}\boldsymbol{B}'\mathcal{K}}$ can be freely specified.

In view of the last remark let $F_{\mathcal{K}}(x) \equiv \nabla^{\boldsymbol{Q}\boldsymbol{B}'}M_{\boldsymbol{Q}\boldsymbol{B}'\mathcal{K}}$ be a smooth but otherwise arbitrary spinor. This spinor, encoding the freely specifiable part of $N_{\boldsymbol{E}\boldsymbol{A}\mathcal{K}}$, is the *gauge source function* for the model equation. Taking this discussion into account, the model equation can be reexpressed as

$$\begin{aligned}\nabla^{\boldsymbol{E}}{}_{\boldsymbol{A}'}N_{\boldsymbol{E}\boldsymbol{A}\mathcal{K}} = \nabla^{\boldsymbol{E}}{}_{\boldsymbol{A}'}\nabla_{\boldsymbol{E}}{}^{\boldsymbol{B}'}M_{\boldsymbol{A}\boldsymbol{B}'\mathcal{K}} + \frac{1}{2}\nabla_{\boldsymbol{A}\boldsymbol{A}'}F_{\mathcal{K}}(x) = \nabla_{\boldsymbol{E}(\boldsymbol{A}'}\nabla_{\boldsymbol{B}')}{}^{\boldsymbol{E}}M_{\boldsymbol{A}}{}^{\boldsymbol{B}'}{}_{\mathcal{K}}\\ + \tfrac{1}{2}\epsilon_{\boldsymbol{A}'\boldsymbol{B}'}\nabla_{\boldsymbol{E}\boldsymbol{Q}'}\nabla^{\boldsymbol{E}\boldsymbol{Q}'}M_{\boldsymbol{A}}{}^{\boldsymbol{B}'}{}_{\mathcal{K}} + \frac{1}{2}\nabla_{\boldsymbol{A}\boldsymbol{A}'}F_{\mathcal{K}}(x) = 0\end{aligned} \tag{3.2}$$

where after the second equality sign, the decomposition of a 2-valence spinor in its symmetric and trace parts has been used. Finally, recalling the definition of the operators

$$\square \equiv \nabla_{\boldsymbol{AA'}}\nabla^{\boldsymbol{AA'}}, \qquad \square_{\boldsymbol{AB}} \equiv \nabla_{\boldsymbol{Q'}(\boldsymbol{A}}\nabla_{\boldsymbol{B})}{}^{\boldsymbol{Q'}},$$

equation (3.2) is rewritten as

$$\square M_{\boldsymbol{AA'}\mathcal{K}} - 2\square_{\boldsymbol{A'B'}}M_{\boldsymbol{A}}{}^{\boldsymbol{B'}}{}_{\mathcal{K}} - \nabla_{\boldsymbol{AA'}}F_{\mathcal{K}}(x) = 0. \tag{3.3}$$

The spinorial Ricci identities can be used to rewrite $\square_{\boldsymbol{A'}}{}^{\boldsymbol{B'}}M_{\boldsymbol{AB'}\mathcal{K}}$ in terms of the curvature spinors —namely, the Weyl spinor $\Psi_{\boldsymbol{ABCD}} = \Xi\phi_{\boldsymbol{ABCD}}$, the Ricci spinor $\Phi_{\boldsymbol{ABA'B'}}$, and the Ricci scalar and $M_{\boldsymbol{AB'}\mathcal{K}}$.

Remark 13. It is customary when using the spinorial Ricci identities to denote the Ricci scalar using the symbol Λ —see [12, 29]. More precisely, in accordance with the conventions used in this thesis one has that $\Lambda = -24R$.

In the rest of the section, it is discussed how to derive the particular wave equations implied by each of the zero-quantities following an analogous procedure as the one used for the model equation.

3.2.2 Wave equation for the frame (no-torsion condition)

The zero-quantity $\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{BB'}}$ encodes the no-torsion condition. The equation (2.10a) can be conveniently rewritten introducing an arbitrary frame $\{\boldsymbol{c_a}\}$, which allows to write $\boldsymbol{e_{AA'}} = e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\boldsymbol{c_a}$. Taking this into account one rewrites the zero-quantity for the no-torsion condition as

$$\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{BB'}}e_{\boldsymbol{QQ'}}{}^{\boldsymbol{c}} = \nabla_{\boldsymbol{BB'}}(e_{\boldsymbol{AA'}}{}^{\boldsymbol{c}}) - \nabla_{\boldsymbol{AA'}}(e_{\boldsymbol{BB'}}{}^{\boldsymbol{c}}) - C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}e_{\boldsymbol{BB'}}{}^{\boldsymbol{b}}, \tag{3.4}$$

where $C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ are the commutation coefficients of the frame, defined by the relation $[\boldsymbol{c_a}, \boldsymbol{c_b}] = C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{c_c}$. In the last expression $\nabla_{\boldsymbol{AA'}}e_{\boldsymbol{BB'}}{}^{\boldsymbol{c}}$ is to be interpreted as a short-hand for the longer expression given in equation (2.59). Using the irreducible decomposition of a spinor representing an antisymmetric tensor one obtains that

$$\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{BB'}}e_{\boldsymbol{QQ'}}{}^{\boldsymbol{c}} = \epsilon_{\boldsymbol{AB}}\bar{\Sigma}_{\boldsymbol{A'B'}}{}^{\boldsymbol{c}} + \epsilon_{\boldsymbol{A'B'}}\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}} \tag{3.5}$$

where

$$\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}} \equiv \frac{1}{2}\Sigma_{(\boldsymbol{A}|\boldsymbol{D'}|}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{B})}{}^{\boldsymbol{D'}}e_{\boldsymbol{QQ'}}{}^{\boldsymbol{c}}$$

is a reduced zero-quantity which can be written in terms of the frame coefficients using equation (3.4) as

$$\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}} = \nabla_{(\boldsymbol{A}}{}^{\boldsymbol{D'}}e_{\boldsymbol{B})\boldsymbol{D'}}{}^{\boldsymbol{c}} + \tfrac{1}{2}e_{(\boldsymbol{A}}{}^{\boldsymbol{D'a}}e_{\boldsymbol{B})\boldsymbol{D'}}{}^{\boldsymbol{b}}C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}.$$

Using the decomposition of a valence-2 spinor in the first term of the right-hand side renders

$$\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}} = \nabla_{\boldsymbol{A}}{}^{\boldsymbol{D}'} e_{\boldsymbol{BD}'}{}^{\boldsymbol{c}} + \tfrac{1}{2}\epsilon_{\boldsymbol{AB}} \nabla^{\boldsymbol{PD}'} e_{\boldsymbol{PD}'}{}^{\boldsymbol{c}} + \tfrac{1}{2} e_{(\boldsymbol{A}}{}^{\boldsymbol{D}'\boldsymbol{a}} e_{\boldsymbol{B})\boldsymbol{D}'}{}^{\boldsymbol{b}} C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}.$$

Introducing the *coordinate gauge source function* $F^{\boldsymbol{c}}(x) = \nabla^{\boldsymbol{PD}'} e_{\boldsymbol{PD}'}{}^{\boldsymbol{c}}$, a wave equation can then be deduced from the condition

$$\nabla^{\boldsymbol{A}}{}_{\boldsymbol{E}'} \Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}} = 0.$$

Observe that this equation is satisfied if $\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}} = 0$ —that is, if the corresponding conformal Einstein field equation is satisfied. Adapting the general procedure described in Section 3.2.1 as required, one obtains

$$\square e_{\boldsymbol{BE}'}{}^{\boldsymbol{c}} - 2\square_{\boldsymbol{E}'\boldsymbol{D}'} e_{\boldsymbol{B}}{}^{\boldsymbol{D}'\boldsymbol{c}} - \nabla_{\boldsymbol{BE}'} F^{\boldsymbol{c}}(\boldsymbol{x}) - \nabla^{\boldsymbol{A}}{}_{\boldsymbol{E}'} \left(e_{(\boldsymbol{A}}{}^{\boldsymbol{D}'\boldsymbol{a}} e_{\boldsymbol{B})\boldsymbol{D}'}{}^{\boldsymbol{b}} C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \right) = 0.$$

Finally, using the spinorial Ricci identities and rearranging the last term one finds the wave equation

$$\begin{aligned} \square e_{\boldsymbol{BE}'}{}^{\boldsymbol{c}} - 2e^{\boldsymbol{QD}'\boldsymbol{c}} \Phi_{\boldsymbol{QBE}'\boldsymbol{D}'} + 6\Lambda\, e_{\boldsymbol{BE}'}{}^{\boldsymbol{c}} - e_{(\boldsymbol{A}}{}^{\boldsymbol{D}'\boldsymbol{a}} e_{\boldsymbol{B})\boldsymbol{D}'}{}^{\boldsymbol{b}} \nabla^{\boldsymbol{A}}{}_{\boldsymbol{E}'} C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} & \\ - 2C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} e_{(\boldsymbol{A}}{}^{\boldsymbol{D}'\boldsymbol{a}} \nabla^{\boldsymbol{A}}{}_{|\boldsymbol{E}'|} e_{\boldsymbol{B})\boldsymbol{D}'}{}^{\boldsymbol{b}} - \nabla_{\boldsymbol{BE}'} F^{\boldsymbol{c}}(x) = 0. & \quad (3.6) \end{aligned}$$

3.2.3 Wave equation for the connection coefficients

The spinorial counterpart of the Riemann tensor can be decomposed as

$$R_{\boldsymbol{AA}'\boldsymbol{BB}'\boldsymbol{CC}'\boldsymbol{DD}'} = R_{\boldsymbol{ABCC}'\boldsymbol{DD}'} \epsilon_{\boldsymbol{B}'\boldsymbol{A}'} + \bar{R}_{\boldsymbol{A}'\boldsymbol{B}'\boldsymbol{CC}'\boldsymbol{DD}'} \epsilon_{\boldsymbol{BA}},$$

where the *reduced curvature spinor* $R_{\boldsymbol{ABCC}'\boldsymbol{DD}'}$ is expressed in terms of the spin connection coefficients as

$$\begin{aligned} R_{\boldsymbol{ABCC}'\boldsymbol{DD}'} + \Sigma_{\boldsymbol{CC}'}{}^{\boldsymbol{QQ}'}{}_{\boldsymbol{DD}'} \Gamma_{\boldsymbol{QQ}'\boldsymbol{AB}} = \nabla_{\boldsymbol{CC}'} \Gamma_{\boldsymbol{DD}'\boldsymbol{AB}} - \nabla_{\boldsymbol{DD}'} \Gamma_{\boldsymbol{CC}'\boldsymbol{AB}} & \\ + \Gamma_{\boldsymbol{CC}'}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}} \Gamma_{\boldsymbol{DD}'\boldsymbol{QA}} - \Gamma_{\boldsymbol{DD}'}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}} \Gamma_{\boldsymbol{CC}'\boldsymbol{QA}}. & \quad (3.7) \end{aligned}$$

In the last equation, $\nabla_{\boldsymbol{DD}'} \Gamma_{\boldsymbol{CC}'\boldsymbol{AB}}$ has been introduced for convenience as a shorthand for the longer expression given in equation (2.61). Observe that the zero quantity $\Xi_{\boldsymbol{ABCC}'\boldsymbol{DD}'}$ defined in equation (2.10b) has the symmetry $\Xi_{\boldsymbol{ABCC}'\boldsymbol{DD}'} = \Xi_{(\boldsymbol{AB})\boldsymbol{CC}'\boldsymbol{DD}'} = -\Xi_{(\boldsymbol{AB})\boldsymbol{DD}'\boldsymbol{CC}'}$. Exploiting this fact, the reduced spinors associated to the geometric and algebraic curvatures $R_{\boldsymbol{ABCC}'\boldsymbol{DD}'}$ and $\rho_{\boldsymbol{ABCC}'\boldsymbol{DD}'}$ can be split, respectively, as

$$R_{\boldsymbol{ABCC}'\boldsymbol{DD}'} \;\; = \;\; \epsilon_{\boldsymbol{C}'\boldsymbol{D}'} R_{\boldsymbol{ABCD}} + \epsilon_{\boldsymbol{CD}} R_{\boldsymbol{ABC}'\boldsymbol{D}'},$$

$$\rho_{ABCC'DD'} \;=\; \epsilon_{C'D'}\rho_{ABCD} + \epsilon_{CD}\rho_{ABC'D'},$$

where

$$R_{ABCD} = \tfrac{1}{2}R_{AB(C|E'|D)}{}^{E'}, \qquad R_{ABC'D'} = \tfrac{1}{2}R_{ABE(C'}{}^{E}{}_{D')},$$

are the *reduced geometric curvature spinors*. Analogous definitions are introduced for the algebraic curvature. Observe that in contrast with the split (3.5) used for the no-torsion condition, the reduced spinors R_{ABCD} and $R_{ABC'D'}$ are not complex conjugate of each other. Together, these two reduced geometric and algebraic curvature spinors give the reduced zero quantities

$$\Xi_{ABCD} = R_{ABCD} - \rho_{ABCD}, \qquad \Xi_{ABC'D'} = R_{ABC'D'} - \rho_{ABC'D'}.$$

Remark 14. Observe that although R_{ABCD} and $R_{ABC'D'}$ are independent, their derivatives are related through the *second Bianchi identity*, which implies that

$$\nabla^{C}{}_{D'}R_{ABCD} = \nabla^{C'}{}_{D}R_{ABC'D'}.$$

This observation is also true for the algebraic curvature as a consequence of the conformal field equations $\Delta_{CDBB'} = 0$ and $\Lambda_{B'BCD} = 0$ since they encode the second Bianchi identity written as differential conditions on the spinorial counterpart of the Schouten tensor and the Weyl spinor. To verify the last statement, recall that the equation for the Schouten tensor encoded in $\Delta_{CDBB'} = 0$ corresponds to the spinorial counterpart of the frame equation (2.7f). Using equation (2.7e), the latter can be rewritten as

$$\nabla_{a}C^{a}{}_{bcd} = \nabla_{c}L_{db} - \nabla_{d}L_{cb},$$

which corresponds to the second Bianchi identity written in terms of the Schouten and Weyl tensors. This can be easily verified, as the last equation is obtained from the substitution of the expression for the Riemann tensor in terms of the Weyl and Schouten tensors (i.e. the algebraic curvature) in the second Bianchi identity. This means that, as long as the conformal field equations $\Delta_{CDBB'} = 0$ and $\Lambda_{B'BCD} = 0$ are satisfied one can write

$$\nabla^{C}{}_{D'}\rho_{ABCD} = \nabla^{C'}{}_{D}\rho_{ABC'D'}.$$

Therefore, the reduced quantities Ξ_{ABCD} and $\Xi_{ACC'D'}$ are related via

$$\nabla^{C}{}_{D'}\Xi_{ABCD} = \nabla^{C'}{}_{D}\Xi_{ABC'D'}.$$

Now, one has to compute explicitly the reduced geometric and algebraic curvature. Recalling the definition of $\rho_{ABCC'DD'}$ in terms of the Weyl spinor and the spinorial counterpart of the Schouten tensor as given in equation (2.14b) it follows that

$$\rho_{\boldsymbol{ABCD}} = \Psi_{\boldsymbol{ABCD}} + L_{\boldsymbol{BE}'(\boldsymbol{D}}{}^{\boldsymbol{E}'}\epsilon_{\boldsymbol{C})\boldsymbol{A}}$$

or, equivalently

$$\rho_{\boldsymbol{ABCD}} = \Xi\phi_{\boldsymbol{ABCD}} + 2\Lambda(\epsilon_{\boldsymbol{DB}}\epsilon_{\boldsymbol{CA}} + \epsilon_{\boldsymbol{CB}}\epsilon_{\boldsymbol{DA}}).$$

Similarly,

$$\rho_{\boldsymbol{ABC}'\boldsymbol{D}'} = \Phi_{\boldsymbol{ABC}'\boldsymbol{D}'}. \tag{3.8}$$

A computation using the reduced version of the geometric curvature from expression (3.7) renders

$$R_{\boldsymbol{ABCD}} = -\tfrac{1}{2}\Sigma_{(\boldsymbol{C}|\boldsymbol{E}'|}{}^{\boldsymbol{QQ}'}{}_{\boldsymbol{D})}{}^{\boldsymbol{E}'}\Gamma_{\boldsymbol{QQ}'\boldsymbol{AB}} + \nabla_{(\boldsymbol{C}|\boldsymbol{E}'|}\Gamma_{\boldsymbol{D})}{}^{\boldsymbol{E}'}{}_{\boldsymbol{AB}} + \Gamma_{(\boldsymbol{C}|\boldsymbol{E}'}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}|}\Gamma_{\boldsymbol{D})}{}^{\boldsymbol{E}'}{}_{\boldsymbol{QA}}, \tag{3.9a}$$

$$R_{\boldsymbol{ABC}'\boldsymbol{D}'} = -\tfrac{1}{2}\Sigma_{\boldsymbol{E}(\boldsymbol{C}'}{}^{\boldsymbol{QQ}'\boldsymbol{E}}{}_{\boldsymbol{D}')}\Gamma_{\boldsymbol{QQ}'\boldsymbol{AB}} + \nabla_{\boldsymbol{E}(\boldsymbol{C}'}\Gamma_{\boldsymbol{D}')}{}^{\boldsymbol{E}}{}_{\boldsymbol{AB}} + \Gamma_{\boldsymbol{E}(\boldsymbol{C}'}{}^{\boldsymbol{Q}}{}_{|\boldsymbol{B}|}\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}')\boldsymbol{QA}}. \tag{3.9b}$$

If the no-torsion condition (3.4) is satisfied, then the first term in each of the last expressions vanishes. In this manner one obtains an expression for the reduced geometric curvature purely in terms of the reduced connection coefficients and, in turn, a wave equation from either $\nabla^{\boldsymbol{C}}{}_{\boldsymbol{D}'}\Xi_{\boldsymbol{ABCD}}$ or $\nabla^{\boldsymbol{C}'}{}_{\boldsymbol{D}}\Xi_{\boldsymbol{ABC}'\boldsymbol{D}'}$. In what follows, for concreteness only

$$\nabla^{\boldsymbol{C}'}{}_{\boldsymbol{D}}\Xi_{\boldsymbol{ABC}'\boldsymbol{D}'} = 0,$$

is considered. Adapting the procedure described in Section 3.2.1 and taking into account equations (3.8) and (3.9a) one obtains

$$\begin{aligned}\square\Gamma_{\boldsymbol{DD}'\boldsymbol{AB}} - 2\square_{\boldsymbol{DE}}\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}'\boldsymbol{AB}} - \nabla_{\boldsymbol{D}'\boldsymbol{D}}F_{\boldsymbol{AB}}(x)&\\ + 2\nabla^{\boldsymbol{C}'}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{E}(\boldsymbol{C}'}{}^{\boldsymbol{Q}}{}_{|\boldsymbol{B}|}\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}')\boldsymbol{QA}} &= 2\nabla^{\boldsymbol{C}'}{}_{\boldsymbol{D}}\Phi_{\boldsymbol{ABC}'\boldsymbol{D}'}.\end{aligned} \tag{3.10}$$

The gauge source function $F_{\boldsymbol{AB}}(x)$ that appears in the last expression is the *frame gauge source function* as defined in equation (2.60). Using the spinorial Ricci identities to replace $\square_{\boldsymbol{DE}}\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}'\boldsymbol{AB}}$ in equation (3.10) and exploiting the symmetry $\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}'\boldsymbol{AB}} = \Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}'(\boldsymbol{AB})}$ gives

$$\begin{aligned}\square_{\boldsymbol{DE}}\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}'\boldsymbol{AB}} = -3\Lambda\Gamma_{\boldsymbol{DD}'\boldsymbol{AB}} + \Gamma^{\boldsymbol{EH}'}{}_{\boldsymbol{AB}}\Phi_{\boldsymbol{D}'\boldsymbol{H}'\boldsymbol{DE}}&\\ + 2\Xi\phi_{\boldsymbol{DEH}(\boldsymbol{A}}\Gamma^{\boldsymbol{E}}{}_{|\boldsymbol{D}'|}{}^{\boldsymbol{H}}{}_{\boldsymbol{B})} - 2\Gamma_{(\boldsymbol{A}|\boldsymbol{D}'\boldsymbol{D}|\boldsymbol{B})} &- 2\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}'\boldsymbol{E}(\boldsymbol{B}}\epsilon_{|\boldsymbol{D}|\boldsymbol{A})}.\end{aligned} \tag{3.11}$$

Substituting the last expression into (3.10) one finds the wave equation

$$\begin{aligned}\square\Gamma_{\boldsymbol{DD}'\boldsymbol{AB}} - 2(\Gamma^{\boldsymbol{EH}'}{}_{\boldsymbol{AB}}\Phi_{\boldsymbol{D}'\boldsymbol{H}'\boldsymbol{DE}} - 3\Lambda\Gamma_{\boldsymbol{DD}'\boldsymbol{AB}} + 2\Xi\phi_{\boldsymbol{DEH}(\boldsymbol{A}}\Gamma^{\boldsymbol{E}}{}_{|\boldsymbol{D}'|}{}^{\boldsymbol{H}}{}_{\boldsymbol{B})}&\\ - 2\Gamma_{(\boldsymbol{A}|\boldsymbol{D}'\boldsymbol{D}|\boldsymbol{B})} - 2\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}'\boldsymbol{E}(\boldsymbol{B}}\epsilon_{|\boldsymbol{D}|\boldsymbol{A})}) + 2\nabla^{\boldsymbol{C}'}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{E}(\boldsymbol{C}'}{}^{\boldsymbol{Q}}{}_{|\boldsymbol{B}|}\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D}')\boldsymbol{QA}}&\\ - 2\nabla^{\boldsymbol{C}'}{}_{\boldsymbol{D}}\Phi_{\boldsymbol{BAC}'\boldsymbol{D}'} - \nabla_{\boldsymbol{D}'\boldsymbol{D}}F_{\boldsymbol{AB}}(x) &= 0.\end{aligned}$$

3.2.4 Wave equation for the Ricci spinor

The zero-quantity defined by equation (2.10e) is expressed in terms of the spinorial counterpart of the Schouten tensor. The spinor $L_{\boldsymbol{AA'BB'}}$ can be decomposed in terms of the Ricci spinor $\Phi_{\boldsymbol{AA'BB'}}$ and Λ as

$$L_{\boldsymbol{AA'BB'}} = \Phi_{\boldsymbol{AA'BB'}} - \Lambda \epsilon_{\boldsymbol{AB}} \epsilon_{\boldsymbol{A'B'}} \tag{3.12}$$

—see Appendix 3.4 for more details. In the context of the conformal Einstein field equations the field Λ can be regarded as a gauge source function. Thus, in what follows the equation $\Delta_{\boldsymbol{CABB'}} = 0$ is regarded as an expression encoding differential conditions on $\Phi_{\boldsymbol{AA'BB'}}$. In order to derive a wave equation for the Ricci spinor consider

$$\nabla^{\boldsymbol{C}}{}_{\boldsymbol{E'}} \Delta_{\boldsymbol{CDBB'}} = 0.$$

Proceeding, again, as described in Section 3.2.1 and using that $\nabla^{\boldsymbol{C}}{}_{\boldsymbol{E'}} \phi_{\boldsymbol{CDBQ}} = 0$ —that is, assuming that the equation encoded in the the zero-quantity $\Lambda_{\boldsymbol{C'DBQ}}$ is satisfied— gives

$$\square L_{\boldsymbol{DBE'B'}} - 2\square_{\boldsymbol{E'Q'}} L_{\boldsymbol{DB}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{B'}} - \nabla_{\boldsymbol{DE'}} \nabla^{\boldsymbol{EQ'}} L_{\boldsymbol{EQ'BB'}} - 2\phi_{\boldsymbol{CDBQ}} \nabla^{\boldsymbol{C}}{}_{\boldsymbol{E'}} \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}} \Xi = 0.$$

Using the decomposition (3.12) and symmetrising in $\boldsymbol{CD}$ one further obtains that

$$\square \Phi_{\boldsymbol{DBE'B'}} - 2\square_{\boldsymbol{E'Q'}} \Phi_{\boldsymbol{DB}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{B'}} - \nabla_{(\boldsymbol{D}|\boldsymbol{E'}} \nabla^{\boldsymbol{EQ'}} L_{\boldsymbol{EQ'}|\boldsymbol{B})\boldsymbol{B'}} - 2\phi_{\boldsymbol{CDBQ}} \nabla^{\boldsymbol{C}}{}_{\boldsymbol{E'}} \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}} \Xi = 0. \tag{3.13}$$

To find a satisfactory wave equation for the Ricci tensor it is necessary to rewrite the last three terms of equation (3.13). To compute the third term observe that the second contracted Bianchi identity as in equation (3.39) of Appendix 3.4 and the decomposition of the Schouten spinor given by equation (3.12) imply

$$\nabla^{\boldsymbol{EQ'}} L_{\boldsymbol{EQ'BB'}} = \nabla^{\boldsymbol{EQ'}} \Phi_{\boldsymbol{EQ'BB'}} - \epsilon_{\boldsymbol{EB}} \epsilon_{\boldsymbol{Q'B'}} \nabla^{\boldsymbol{EQ'}} \Lambda = -4\nabla_{\boldsymbol{BB'}} \Lambda.$$

Therefore, one finds that

$$\nabla_{(\boldsymbol{D}|\boldsymbol{E'}} \nabla^{\boldsymbol{EQ'}} L_{\boldsymbol{EQ'}|\boldsymbol{B})\boldsymbol{B'}} = -4\nabla_{\boldsymbol{E'}(\boldsymbol{D}} \nabla_{\boldsymbol{B})\boldsymbol{B'}} \Lambda. \tag{3.14}$$

This last expression is satisfactory since, as already mentioned, the Ricci scalar R (or equivalently Λ) can be regarded as a gauge source function —the so-called *conformal gauge source function* [15]. In order to replace the last term of equation (3.13) one exploits the field equation encoded in $Z_{\boldsymbol{AA'BB'}} = 0$ and the decomposition (3.12), to obtain

$$\phi_{\boldsymbol{CDBQ}} \nabla^{\boldsymbol{C}}{}_{\boldsymbol{E'}} \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{B'}} \Xi = -\Xi \phi_{\boldsymbol{CDBQ}} L^{\boldsymbol{C}}{}_{\boldsymbol{E'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B'}} = -\Xi \phi_{\boldsymbol{CDBQ}} \Phi^{\boldsymbol{C}}{}_{\boldsymbol{E'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B'}}. \tag{3.15}$$

Finally, computing $\square_{\boldsymbol{E'Q'}}\Phi_{\boldsymbol{DB}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{B'}}$ and substituting equations (3.14) and (3.15) gives

$$\square\Phi_{\boldsymbol{DBE'B'}} - 4\Phi^{\boldsymbol{P}}{}_{(\boldsymbol{B}}{}^{\boldsymbol{Q'}}{}_{|\boldsymbol{B'}|}\Phi_{\boldsymbol{D})\boldsymbol{PE'Q'}} + 6\Lambda\Phi_{\boldsymbol{DBE'B'}} - 2\Xi\bar{\phi}_{\boldsymbol{E'Q'B'H'}}\Phi_{\boldsymbol{DB}}{}^{\boldsymbol{Q'H'}}$$
$$+ 4\Lambda\Phi_{\boldsymbol{DB}}{}^{\boldsymbol{Q'}}{}_{(\boldsymbol{E'}}\epsilon_{\boldsymbol{Q'})\boldsymbol{B'}} + 2\phi_{\boldsymbol{CDBQ}}\Phi^{\boldsymbol{C}}{}_{\boldsymbol{E'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B'}} + 4\nabla_{\boldsymbol{E'}(\boldsymbol{D}}\nabla_{\boldsymbol{B})\boldsymbol{B'}}\Lambda = 0. \quad (3.16)$$

3.2.5 Wave equation for the rescaled Weyl spinor

Proceeding as in the previous subsections, consider the equation

$$\nabla_{\boldsymbol{D}}{}^{\boldsymbol{B'}}\Lambda_{\boldsymbol{B'BAC}} = 0. \quad (3.17)$$

Observe that in this case a gauge source function is not required since in the definition of $\Lambda_{\boldsymbol{B'BAC}}$ one already has a unsymmetrised derivative. Following the procedure described in Section 3.2.1 renders

$$\square\phi_{\boldsymbol{ABCD}} - 2\square_{\boldsymbol{DQ}}\phi_{\boldsymbol{ABC}}{}^{\boldsymbol{Q}} = 0.$$

Thus, to complete the discussion its necessary to compute $\square_{\boldsymbol{DQ}}\phi_{\boldsymbol{ABC}}{}^{\boldsymbol{Q}}$. Using the spinorial Ricci identities renders

$$\square_{\boldsymbol{DQ}}\phi_{\boldsymbol{ABC}}{}^{\boldsymbol{Q}} = \Xi\phi_{\boldsymbol{FQAD}}\phi_{\boldsymbol{BC}}{}^{\boldsymbol{FQ}} + \Xi\phi_{\boldsymbol{FQDB}}\phi_{\boldsymbol{AC}}{}^{\boldsymbol{FQ}} + \Xi\phi_{\boldsymbol{FQCD}}\phi_{\boldsymbol{AB}}{}^{\boldsymbol{FQ}} - 6\Lambda\phi_{\boldsymbol{ABCD}}$$

The symmetries of $\phi_{\boldsymbol{ABCD}}$ simplify the equation since

$$\square_{(\boldsymbol{D}|\boldsymbol{Q}}\phi_{\boldsymbol{A}|\boldsymbol{B})\boldsymbol{C}}{}^{\boldsymbol{Q}} = 3\Xi\phi^{\boldsymbol{FQ}}{}_{(\boldsymbol{AB}}\phi_{\boldsymbol{CD})\boldsymbol{FQ}} - 6\Lambda\phi_{\boldsymbol{ABCD}}.$$

Taking into account the last expression one obtains the following wave equation for the rescaled Weyl spinor

$$\square\phi_{\boldsymbol{ABCD}} - 6\Xi\phi^{\boldsymbol{FQ}}{}_{(\boldsymbol{AB}}\phi_{\boldsymbol{CD})\boldsymbol{FQ}} + 12\Lambda\phi_{\boldsymbol{ABCD}} = 0. \quad (3.18)$$

Observe that the wave equation for the rescaled Weyl spinor is remarkably simple.

3.2.6 Wave equation for the Friedrich scalar and the conformal factor

Since s is a scalar field, the general procedure described in Section 3.2.1 does not provide any computational advantage. The required wave equation is derived from considering

$$\nabla^{\boldsymbol{AA'}}Z_{\boldsymbol{AA'}} = 0.$$

Explicitly, the last equation can be written as

$$\square s + \nabla^{\boldsymbol{AA'}}\Phi_{\boldsymbol{ACA'C'}}\nabla^{\boldsymbol{CC}}\Xi + \Phi_{\boldsymbol{ACA'C'}}\nabla^{\boldsymbol{AA'}}\nabla^{\boldsymbol{CC'}}\Xi = 0.$$

Using the contracted second Bianchi identity (3.39) to replace the second term and the conformal field equation encoded in $Z_{\boldsymbol{AA'BB'}} = 0$ along with the decomposition (3.12) to replace the third term one obtains

$$\square s - \Xi\Phi_{\boldsymbol{ACA'C'}}\Phi^{\boldsymbol{ACA'C'}} - 3\nabla_{\boldsymbol{CC'}}\Lambda\nabla^{\boldsymbol{CC'}}\Xi = 0.$$

Finally, notice that a wave equation for the conformal factor follows directly from the contraction $Z_{\boldsymbol{AA'}}{}^{\boldsymbol{AA'}}$ and the decomposition (3.38):

$$\square\Xi = 4\left(s + \Lambda\Xi\right).$$

3.2.7 Summary of the analysis

The results of this section are summarised in the following proposition:

Proposition 1. *If the conformal Einstein field equations* (2.11a)-(2.11b) *in vacuum, are satisfied on $\mathcal{U} \subset \mathcal{M}$, and*

$$F^{\boldsymbol{a}}(x), \qquad F_{\boldsymbol{AB}}(x), \qquad \Lambda(x)$$

are smooth functions on $\mathcal{M}$ such that

$$\nabla^{\boldsymbol{QQ'}}e_{\boldsymbol{QQ'}}{}^{\boldsymbol{a}} = F^{\boldsymbol{a}}(x), \qquad \nabla^{\boldsymbol{QQ'}}\Gamma_{\boldsymbol{QQ'AB}} = F_{\boldsymbol{AB}}(x) \qquad \nabla^{\boldsymbol{QQ'}}\Phi_{\boldsymbol{PQP'Q'}} = -3\nabla_{\boldsymbol{PP'}}\Lambda(x).$$

then one has that

$$\square e_{\boldsymbol{BE'}}{}^{\boldsymbol{c}} - 2e^{\boldsymbol{QD'c}}\Phi_{\boldsymbol{QBE'D'}} + 6\Lambda\, e_{\boldsymbol{BE'}}{}^{\boldsymbol{c}} - e_{(\boldsymbol{A}}{}^{\boldsymbol{D'a}}e_{\boldsymbol{B})\boldsymbol{D'}}{}^{\boldsymbol{b}}\nabla^{\boldsymbol{A}}{}_{\boldsymbol{E'}}C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$$
$$- 2C_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}e_{(\boldsymbol{A}}{}^{\boldsymbol{D'a}}\nabla^{\boldsymbol{A}}{}_{|\boldsymbol{E'}|}e_{\boldsymbol{B})\boldsymbol{D'}}{}^{\boldsymbol{b}} - \nabla_{\boldsymbol{BE'}}F^{\boldsymbol{c}}(x) = 0, \qquad (3.19a)$$

$$\square\Gamma_{\boldsymbol{DD'AB}} - 2(\Gamma^{\boldsymbol{EH'}}{}_{\boldsymbol{AB}}\Phi_{\boldsymbol{D'H'DE}} - 3\Lambda\Gamma_{\boldsymbol{DD'AB}}$$
$$+ 2\Xi\phi_{\boldsymbol{DEH}(\boldsymbol{A}}\Gamma^{\boldsymbol{E}}{}_{|\boldsymbol{D'}|}{}^{\boldsymbol{H}}{}_{\boldsymbol{B})} - 2\Gamma_{(\boldsymbol{A}|\boldsymbol{D'D}|\boldsymbol{B})} - 2\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D'E}(\boldsymbol{B}}\epsilon_{|\boldsymbol{D}|\boldsymbol{A})})$$
$$+ 2\nabla^{\boldsymbol{C'}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{E}(\boldsymbol{C'}}{}^{\boldsymbol{Q}}{}_{|\boldsymbol{B}|}\Gamma^{\boldsymbol{E}}{}_{\boldsymbol{D'})\boldsymbol{QA}} - 2\nabla^{\boldsymbol{C'}}{}_{\boldsymbol{D}}\Phi_{\boldsymbol{BAC'D'}} - \nabla_{\boldsymbol{D'D}}F_{\boldsymbol{AB}}(x) = 0, \quad (3.19b)$$

$$\square\Phi_{\boldsymbol{DBE'B'}} - 4\Phi^{\boldsymbol{P}}{}_{(\boldsymbol{B}}{}^{\boldsymbol{Q'}}{}_{|\boldsymbol{B'}|}\Phi_{\boldsymbol{D})\boldsymbol{PE'Q'}} + 6\Lambda\Phi_{\boldsymbol{DBE'B'}} - 2\Xi\bar{\phi}_{\boldsymbol{E'Q'B'H'}}\Phi_{\boldsymbol{DB}}{}^{\boldsymbol{Q'H'}}$$
$$+ 4\Lambda\Phi_{\boldsymbol{DB}}{}^{\boldsymbol{Q'}}{}_{(\boldsymbol{E'}}\epsilon_{\boldsymbol{Q'})\boldsymbol{B'}} + 4\nabla_{\boldsymbol{E'}(\boldsymbol{D}}\nabla_{\boldsymbol{B})\boldsymbol{B'}}\Lambda + 2\phi_{\boldsymbol{CDBQ}}\Phi^{\boldsymbol{C}}{}_{\boldsymbol{E'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B'}} = 0, \quad (3.19c)$$

$$\square s - \Xi\Phi_{\boldsymbol{ACA'C'}}\Phi^{\boldsymbol{ACA'C'}} - 3\nabla_{\boldsymbol{CC'}}\Lambda\nabla^{\boldsymbol{CC'}}\Xi = 0, \qquad (3.19d)$$

$$\square\phi_{ABCD} - 6\Xi\phi^{FQ}{}_{(AB}\phi_{CD)FQ} + 12\Lambda\phi_{ABCD} = 0, \tag{3.19e}$$

$$\square\Xi - 4\left(s + \Lambda\Xi\right) = 0, \tag{3.19f}$$

hold on $\mathcal{U}$.

Remark 15. The unphysical metric is not part of the unknowns of the system of equations of the spinorial version of the conformal Einstein field equations. This observation is of relevance in the present context because when the operator $\square$ is applied to a spinor $N_{\mathcal{K}}$ of non-zero rank one obtains first derivatives of the connection —if the metric is part of the unknowns then these first derivatives of the connection representing second derivatives of $\boldsymbol{g}$ would enter into the principal part of the operator $\square$. Therefore, since in this setting the metric is not part of the unknowns, the principal part of the operator $\square$ is given by $\epsilon^{\boldsymbol{AB}}\epsilon^{\boldsymbol{A'B'}}\boldsymbol{e}_{\boldsymbol{AA'}}\boldsymbol{e}_{\boldsymbol{BB'}}$.

Remark 16. In the sequel let $\{\boldsymbol{e},\, \Gamma,\, \Phi,\, \boldsymbol{\phi}\}$ denote vector-valued unknowns encoding the independent components of $\{e_{\boldsymbol{AA'}}{}^{\boldsymbol{c}},\, \Gamma_{\boldsymbol{CC'AB}},\, \Phi_{\boldsymbol{AA'BB'}},\, \phi_{\boldsymbol{ABCD}}\}$ and let $\mathbf{u} \equiv (\boldsymbol{e},\, \Gamma,\, \Phi,\, \boldsymbol{\phi},\, s,\, \Xi)$. Additionally, let $\partial\mathbf{u}$ denote collectively the derivatives of $\mathbf{u}$. With this notation the wave equations of Proposition 1 can be recast as a quasilinear wave equation for $\mathbf{u}$ having, in local coordinates, the form

$$g^{\mu\nu}(\mathbf{u})\partial_\mu\partial_\nu\mathbf{u} + \boldsymbol{F}(x;\mathbf{u},\partial\mathbf{u}) = 0, \tag{3.20}$$

where $\boldsymbol{F}$ is a vector-valued function of its arguments and $g^{\mu\nu}$ denotes the components, in local coordinates, of contravariant version of a Lorentzian metric $\boldsymbol{g}$. In accordance with the current notation $g^{\mu\nu} \equiv \eta^{\boldsymbol{ab}}\boldsymbol{e}_{\boldsymbol{a}}{}^{\mu}\boldsymbol{e}_{\boldsymbol{b}}{}^{\nu}$ where, in local coordinates, one writes $\boldsymbol{e}_{\boldsymbol{a}} = \boldsymbol{e}_{\boldsymbol{a}}{}^{\mu}\boldsymbol{\partial}_\mu$.

3.3 Propagation of the constraints and the derivation of the subsidiary system

The starting point of the derivation of the wave equations discussed in the previous section was the conformal Einstein field equations. Therefore, any solution to the conformal Einstein field equations is a solution to the wave equations. It is now natural to ask: under which conditions a solution to the wave equations (3.19a)-(3.19f) will imply a solution to the conformal Einstein field equations? The general strategy to answer this question is to use the spinorial wave equations of Proposition 1 to construct a subsidiary system of homogeneous wave equations for the zero-quantities and impose vanishing initial conditions. Then, using a standard existence and uniqueness result for wave equations, the unique solution satisfying the data will be given by the vanishing of each zero-quantity. This means that under certain conditions (encoded in the initial data for the subsidiary system) a solution to the spinorial wave equations will imply a solution to the original conformal Einstein field equations. The procedure to construct the subsidiary equations for the zero quantities is similar to the construction of the wave equations of Proposition 1. There is, however, a key difference: the covariant derivative is, a priori, not assumed to be a Levi-Civita connection. Instead, one assumes that the connection is metric but not necessarily torsion-free. This derivative will be denoted by $\widehat{\nabla}$. Therefore, whenever a commutator of covariant derivatives appears, or in spinorial terms the operator $\widehat{\square}_{\boldsymbol{AB}} \equiv \widehat{\nabla}_{\boldsymbol{C}'(\boldsymbol{A}}\widehat{\nabla}_{\boldsymbol{B})}{}^{\boldsymbol{C}'}$, it is necessary to use the $\widehat{\nabla}$-spinorial Ricci identities involving a non-vanishing torsion spinor —this generalisation is given in Appendix 3.4 and is required in the discussion of the subsidiary equations where the torsion is, in itself, a variable for which a subsidiary equation needs to be constructed.

Remark 17. The introduction of a connection $\widehat{\nabla}$ which is not torsion-free is necessary for the discussion of the subsidiary system as the torsion $\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}}$ is part of the zero-quantities to be propagated.

As in the previous section, the procedure for obtaining the subsidiary system is similar for each zero-quantity. Therefore, a general outline of the procedure is given in the next section.

3.3.1 General procedure for obtaining the subsidiary system and the propagation of the constraints

In the general procedure described in Section 3.2.1, the spinor $N_{\boldsymbol{EA}\mathcal{K}}$ played the role of a zero-quantity, while the spinor $M_{\boldsymbol{AB}'\mathcal{K}}$ played the role of the variable for which the wave equation (3.3) was to be derived. In the construction of the subsidiary system one is not interested in finding an equation for $M_{\boldsymbol{AB}'\mathcal{K}}$ but in deriving an equation for $N_{\boldsymbol{EA}\mathcal{K}}$ under the hypothesis that the wave equation for $M_{\boldsymbol{AB}'\mathcal{K}}$ is satisfied. As already discussed, since the connection is not assumed to be torsion-free the equation for $N_{\boldsymbol{EA}\mathcal{K}}$ has to be written in terms of the metric connection $\widehat{\nabla}$.

Before deriving the subsidiary equation a couple of observations are in order. In Section 3.2.1 the quantity $N_{\boldsymbol{EA}\mathcal{K}} \equiv \nabla_{(\boldsymbol{E}}{}^{\boldsymbol{B}'}M_{\boldsymbol{A})\boldsymbol{B}'\mathcal{K}}$ was defined. Then, decomposing this quantity as usual one obtained

$$N_{\boldsymbol{EA}\mathcal{K}} = \nabla_{\boldsymbol{E}}{}^{\boldsymbol{B}'}M_{\boldsymbol{AB}'\mathcal{K}} + \tfrac{1}{2}\epsilon_{\boldsymbol{EA}}\nabla^{\boldsymbol{QB}'}M_{\boldsymbol{QB}'\mathcal{K}}.$$

At this point in the discussion of Section 3.2.1 a gauge source function $\nabla^{\boldsymbol{PQ}'}M_{\boldsymbol{PQ}'\mathcal{K}} = F_{\mathcal{K}}$ was introduced. Consequently, instead of directly deriving an equation for $N_{\boldsymbol{EA}\mathcal{K}}$

one has derived an equation using the modified quantity

$$\widehat{N}_{\boldsymbol{EA}\mathcal{K}} \equiv \nabla_{\boldsymbol{E}}{}^{\boldsymbol{B}'} M_{\boldsymbol{AB}'\mathcal{K}} + \tfrac{1}{2}\epsilon_{\boldsymbol{EA}} F_{\mathcal{K}}.$$

Accordingly, the wave equations discussed in Section 3.2 can be succinctly written as $\nabla^{\boldsymbol{E}}{}_{\boldsymbol{A}'}\widehat{N}_{\boldsymbol{EA}\mathcal{K}} = 0$. Later on, one has to show that, in fact, $\widehat{N}_{\boldsymbol{EA}\mathcal{K}} = N_{\boldsymbol{EA}\mathcal{K}}$ if the appropriate initial conditions are satisfied. In addition, observe that $\nabla^{\boldsymbol{A}}{}_{\boldsymbol{C}'}\widehat{N}_{\boldsymbol{EA}\mathcal{K}}$ can be written in terms of the connection $\widehat{\widehat{\nabla}}$ by means of a *transition spinor* $Q_{\boldsymbol{AA}'\boldsymbol{BC}}$ — see Appendix 3.5 for definitions. Using equation (3.50) of Appendix 3.5 one obtains

$$\begin{aligned}\widehat{\widehat{\nabla}}^{\boldsymbol{A}}{}_{\boldsymbol{C}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = \nabla^{\boldsymbol{A}}{}_{\boldsymbol{C}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} - Q^{\boldsymbol{A}}{}_{\boldsymbol{C}'\boldsymbol{A}}{}^{\boldsymbol{H}}\widehat{N}_{\boldsymbol{HB}\mathcal{K}} \\ - Q^{\boldsymbol{A}}{}_{\boldsymbol{C}'\boldsymbol{B}\mathcal{K}}{}^{\boldsymbol{H}}\widehat{N}_{\boldsymbol{AH}\mathcal{K}} - \cdots - Q^{\boldsymbol{A}}{}_{\boldsymbol{C}'\boldsymbol{K}}{}^{\boldsymbol{H}}\widehat{N}_{\boldsymbol{AB}\cdots\boldsymbol{H}}\end{aligned} \quad (3.21)$$

where $\boldsymbol{K}$ is the last index of the string $\mathcal{K}$. For a connection which is metric, the transition spinor can be written entirely in terms of the torsion as

$$Q_{\boldsymbol{AA}'\boldsymbol{BC}} \equiv -2\Sigma_{\boldsymbol{BAA}'\boldsymbol{C}} - 2\Sigma_{\boldsymbol{A}(\boldsymbol{C}|\boldsymbol{A}'|\boldsymbol{B})} - 2\bar{\Sigma}_{\boldsymbol{A}'(\boldsymbol{C}|\boldsymbol{Q}'}{}^{\boldsymbol{Q}'}\epsilon_{\boldsymbol{A}|\boldsymbol{B})}. \quad (3.22)$$

If the wave equation for $M_{\boldsymbol{AB}'\mathcal{K}}$ encoded in $\nabla^{\boldsymbol{A}}{}_{\boldsymbol{C}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$ is satisfied, the first term of equation (3.21) vanishes. Therefore, the wave equation discussed in Section 3.2.1 can be written in terms of the connection $\widehat{\widehat{\nabla}}$ as

$$\widehat{\widehat{\nabla}}^{\boldsymbol{A}}{}_{\boldsymbol{C}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = W_{\boldsymbol{BC}'\mathcal{K}}, \quad (3.23)$$

where

$$W_{\boldsymbol{BC}'\mathcal{K}} = -Q^{\boldsymbol{A}}{}_{\boldsymbol{C}'\boldsymbol{A}}{}^{\boldsymbol{H}}\widehat{N}_{\boldsymbol{HB}\mathcal{K}} - Q^{\boldsymbol{A}}{}_{\boldsymbol{C}'\boldsymbol{B}}{}^{\boldsymbol{H}}\widehat{N}_{\boldsymbol{AH}\mathcal{K}} - \ldots.. - Q^{\boldsymbol{A}}{}_{\boldsymbol{C}'\boldsymbol{K}}{}^{\boldsymbol{H}}\widehat{N}_{\boldsymbol{AB}\ldots\boldsymbol{H}}.$$

The subsidiary system

In this section it is shown that by setting the appropriate initial conditions, if the wave equation for $M_{\boldsymbol{AB}'\mathcal{K}}$ encoded in $\nabla^{\boldsymbol{A}}{}_{\boldsymbol{E}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$ holds then $\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$. The strategy will be to obtain an homogeneous wave equation for $\widehat{N}_{\boldsymbol{AB}\mathcal{K}}$ written in terms of the connection $\widehat{\widehat{\nabla}}$. First, observe that $\widehat{\widehat{\nabla}}^{\boldsymbol{Q}'}{}_{\boldsymbol{P}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}}$ can be decomposed as

$$\widehat{\widehat{\nabla}}^{\boldsymbol{Q}'}{}_{\boldsymbol{P}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = \widehat{\widehat{\nabla}}^{\boldsymbol{Q}'}{}_{(\boldsymbol{P}}\widehat{N}_{\boldsymbol{A})\boldsymbol{B}\mathcal{K}} + \tfrac{1}{2}\epsilon_{\boldsymbol{PA}}\widehat{\widehat{\nabla}}^{\boldsymbol{Q}'}{}_{\boldsymbol{E}}\widehat{N}^{\boldsymbol{E}}{}_{\boldsymbol{B}\mathcal{K}}. \quad (3.24)$$

Replacing the second term using equation (3.23) —i.e. using that the wave equation for $M_{\boldsymbol{AB}'\mathcal{K}}$ encoded in $\nabla^{\boldsymbol{A}}{}_{\boldsymbol{E}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$ holds— renders

$$\widehat{\widehat{\nabla}}^{\boldsymbol{Q}'}{}_{\boldsymbol{P}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = \widehat{\widehat{\nabla}}^{\boldsymbol{Q}'}{}_{(\boldsymbol{P}}\widehat{N}_{\boldsymbol{A})\boldsymbol{B}\mathcal{K}} + \tfrac{1}{2}\epsilon_{\boldsymbol{PA}}W^{\boldsymbol{Q}'}{}_{\boldsymbol{B}\mathcal{K}}.$$

Applying $\widehat{\nabla}^{\boldsymbol{P}}{}_{\boldsymbol{Q}'}$ to the previous equation and expanding the symmetrised term in the right-hand side one obtains

$$
\begin{aligned}
\widehat{\nabla}^{\boldsymbol{P}}{}_{\boldsymbol{Q}'}\widehat{\nabla}^{\boldsymbol{Q}'}{}_{\boldsymbol{P}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} &= \tfrac{1}{2}\widehat{\nabla}^{\boldsymbol{P}}{}_{\boldsymbol{Q}'}\left(\widehat{\nabla}^{\boldsymbol{Q}'}{}_{\boldsymbol{P}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} + \widehat{\nabla}^{\boldsymbol{Q}'}{}_{\boldsymbol{A}}\widehat{N}_{\boldsymbol{PB}\mathcal{K}}\right) + \tfrac{1}{2}\widehat{\nabla}_{\boldsymbol{AQ}'}W^{\boldsymbol{Q}'}{}_{\boldsymbol{B}\mathcal{K}},\\
&= -\tfrac{1}{2}\widehat{\square}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} - \tfrac{1}{2}\widehat{\nabla}_{\boldsymbol{PQ}'}\widehat{\nabla}^{\boldsymbol{Q}'}{}_{\boldsymbol{A}}\widehat{N}^{\boldsymbol{P}}{}_{\boldsymbol{B}\mathcal{K}} + \tfrac{1}{2}\widehat{\nabla}_{\boldsymbol{AQ}'}W^{\boldsymbol{Q}'}{}_{\boldsymbol{B}\mathcal{K}},\\
&= -\tfrac{1}{2}\widehat{\square}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} - \tfrac{1}{2}\left(\widehat{\square}_{\boldsymbol{PA}}\widehat{N}^{\boldsymbol{P}}{}_{\boldsymbol{B}\mathcal{K}} + \tfrac{1}{2}\epsilon_{\boldsymbol{PA}}\widehat{\square}\widehat{N}^{\boldsymbol{P}}{}_{\boldsymbol{B}\mathcal{K}}\right) + \tfrac{1}{2}\widehat{\nabla}_{\boldsymbol{AQ}'}W^{\boldsymbol{Q}'}{}_{\boldsymbol{B}\mathcal{K}}.
\end{aligned}
$$

From this expression, after some rearrangements one concludes that

$$
\widehat{\square}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 2\widehat{\square}_{\boldsymbol{PA}\mathcal{K}}\widehat{N}^{\boldsymbol{P}}{}_{\boldsymbol{B}\mathcal{K}} - 2\widehat{\nabla}_{\boldsymbol{AQ}'}W^{\boldsymbol{Q}'}{}_{\boldsymbol{B}\mathcal{K}}.
$$

It only remains to reexpress the right-hand side of the above equation using the $\widehat{\nabla}$-spinorial Ricci identities. This can be computed for each zero-quantity using the expressions given in Appendix 3.4. Observe that the result is always a homogeneous expression in the zero-quantities and its first derivatives. The last term also shares this property since the transition spinor can be completely written in terms of the torsion, as shown in equation (3.22), which is one of the zero-quantities. Finally, once the homogeneous wave equation is obtained one sets the initial conditions

$$
\widehat{N}_{\boldsymbol{AB}\mathcal{K}}|_{\mathcal{S}} = 0 \qquad \text{and} \qquad (\widehat{\nabla}_{\boldsymbol{EE}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}} = 0
$$

on a spacelike hypersurface $\mathcal{S}$, and using standard existence and uniqueness results for wave equations it follows that the unique solution satisfying this data is given by $\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$.

Remark 18. The crucial step in the last derivation was the assumption that the equation $\nabla^{\boldsymbol{A}}{}_{\boldsymbol{E}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$ is satisfied —i.e. the wave equation (3.3) for $M_{\boldsymbol{AB}\mathcal{K}}$.

Initial data for the subsidiary system

In this section the relations between the initial conditions

$$
\widehat{N}_{\boldsymbol{AB}\mathcal{K}}|_{\mathcal{S}} = 0, \qquad (\widehat{\nabla}_{\boldsymbol{EE}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}} = 0.
$$

are analysed. Additionally, in the subsequent discussion it is shown how to use these conditions to construct initial data for the wave equations of Proposition 1. More concretely, the main purpose of this section is to show that only $\widehat{N}_{\boldsymbol{AB}\mathcal{K}}|_{\mathcal{S}} = 0$ is essential, while $\widehat{\nabla}_{\boldsymbol{EE}'}\widehat{N}_{\boldsymbol{AB}}|_{\mathcal{S}} = 0$ holds by virtue of the condition $\nabla^{\boldsymbol{AA}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$. To do so, first observe that as the spatial derivatives of $\widehat{N}_{\boldsymbol{AB}\mathcal{K}}$ can be determined from $\widehat{N}_{\boldsymbol{AB}\mathcal{K}}|_{\mathcal{S}} = 0$. Then, it follows that $(\widehat{\nabla}_{\boldsymbol{EE}'}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}} = 0$ is equivalent to only specifying the derivative along the normal to the initial hypersurface $\mathcal{S}$.

Let $\tau^{\boldsymbol{AA}'}$ be an Hermitian spinor corresponding to a timelike vector such that $\tau^{\boldsymbol{AA}'}|_{\mathcal{S}}$ is the normal to $\mathcal{S}$. The spinor $\tau^{\boldsymbol{AA}'}$ can be used to perform a *space spinor* *split* of the derivative $\widehat{\nabla}_{\boldsymbol{AA'}}$ as discussed in Section 2.4.1 of Chapter 2. Using the split of $\widehat{\nabla}$ as in equation (2.53) and $\widehat{N}_{\boldsymbol{AB}\mathcal{K}}|_{\mathcal{S}} = 0$ it follows that

$$\widehat{\nabla}_{\boldsymbol{EE'}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}}|_{\mathcal{S}} = \tfrac{1}{2}(\tau_{\boldsymbol{EE'}}\mathcal{P}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}}.$$

Therefore, requiring $\widehat{\nabla}_{\boldsymbol{EE'}}\widehat{N}_{\boldsymbol{AB}}|_{\mathcal{S}} = 0$ is equivalent to $(\mathcal{P}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}} = 0$ as previously stated. Observe that the wave equation $\nabla^{\boldsymbol{AA'}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$ or, equivalently, $\widehat{\nabla}^{\boldsymbol{AA'}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = W^{\boldsymbol{A'}}{}_{\boldsymbol{B}\mathcal{K}}$ implies $(\widehat{\nabla}^{\boldsymbol{AA'}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}} = W^{\boldsymbol{A'}}{}_{\boldsymbol{B}\mathcal{K}}|_{\mathcal{S}}$ —recall that $W^{\boldsymbol{A'}}{}_{\boldsymbol{B}\mathcal{K}}|_{\mathcal{S}}$ is given entirely in terms of zero-quantities since the transition spinor can be written in terms of the torsion. Therefore, assuming that all the zero-quantities vanish on the initial hypersurface $\mathcal{S}$ it follows that $(\widehat{\nabla}^{\boldsymbol{AA'}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}} = 0$. Using, again, the space spinor decomposition of $\widehat{\nabla}_{\boldsymbol{AA'}}$ and considering $\widehat{N}_{\boldsymbol{AB}\mathcal{K}}|_{\mathcal{S}} = 0$ one obtains $(\tau^{\boldsymbol{AA'}}\mathcal{P}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}} = 0$ which also implies that $(\mathcal{P}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}} = 0$.

Summarising, the only the condition that is needed is that all the zero-quantities vanish on the initial hypersurface $\mathcal{S}$ since the condition $(\widehat{\nabla}_{\boldsymbol{EE'}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}})|_{\mathcal{S}} = 0$ is always satisfied by virtue of the wave equation $\nabla^{\boldsymbol{AA'}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$.

Propagation of the constraints

To close the argument one has to show that $\widehat{N}_{\boldsymbol{AB}} = N_{\boldsymbol{AB}}$. To do so, one writes

$$N_{\boldsymbol{AB}\mathcal{K}} - \widehat{N}_{\boldsymbol{AB}\mathcal{K}} = \tfrac{1}{2}\epsilon_{\boldsymbol{AB}}Q_{\mathcal{K}},$$

where $Q_{\mathcal{K}}$ encodes the difference between $\widehat{N}_{\boldsymbol{AB}\mathcal{K}}$ and $N_{\boldsymbol{AB}\mathcal{K}}$. Computing the trace of the last equation and taking into account the definition of $N_{\boldsymbol{AB}\mathcal{K}}$ one finds that $\widehat{N}^{\boldsymbol{A}}{}_{\boldsymbol{A}\mathcal{K}} = Q_{\mathcal{K}}$. Invoking the results derived in the last subsection it follows that if the wave equation $\nabla^{\boldsymbol{A}}{}_{\boldsymbol{E'}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$ is satisfied and all the zero-quantities vanish on the initial hypersurface $\mathcal{S}$ then $\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$. This observation also implies that if $\widehat{N}^{\boldsymbol{A}}{}_{\boldsymbol{A}\mathcal{K}}|_{\mathcal{S}} = 0$ then $\widehat{N}^{\boldsymbol{A}}{}_{\boldsymbol{A}\mathcal{K}} = 0$. The later result, expressed in terms of $Q_{\mathcal{K}}$ means that if $Q_{\mathcal{K}}|_{\mathcal{S}} = 0$ then $Q_{\mathcal{K}} = 0$. Therefore, requiring that all the zero-quantities vanish on $\mathcal{S}$ and that the wave equation $\nabla^{\boldsymbol{AA'}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$ holds everywhere, is enough to ensure that

$$\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = N_{\boldsymbol{AB}\mathcal{K}}$$

everywhere. Moreover, $\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 0$ implies that $N_{\boldsymbol{AB}\mathcal{K}} = 0$ and the gauge conditions hold. Namely, one has that

$$\nabla^{\boldsymbol{AB'}}M_{\boldsymbol{AB'}\mathcal{K}} = F_{\mathcal{K}}(x).$$

3.3.2 Subsidiary system and propagation of the constraints

The essential ideas of the Section 3.3.1 can be applied to every single zero-quantity. One only needs to take into account the particular index structure of each zero-quantity encoded in the string of spinor indices $\mathcal{K}$. The problem then reduces to the computation of

$$\widehat{\square}_{\boldsymbol{PA}}\widehat{N}^{\boldsymbol{P}}{}_{\boldsymbol{B}\mathcal{K}}, \qquad \widehat{\nabla}_{\boldsymbol{AQ'}}W^{\boldsymbol{Q'}}{}_{\boldsymbol{B}\mathcal{K}},$$

the result of which is to be substituted into

$$\widehat{\square}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 2\widehat{\square}_{\boldsymbol{PA}}\widehat{N}^{\boldsymbol{P}}{}_{\boldsymbol{B}\mathcal{K}} - 2\widehat{\nabla}_{\boldsymbol{AQ'}}W^{\boldsymbol{Q'}}{}_{\boldsymbol{B}\mathcal{K}}. \tag{3.25}$$

The latter can be succinctly computed using the equations (3.46a)-(3.46d) in Appendix 3.4. The explicit form can be easily obtained and renders long expressions for each zero-quantity. The key observation from these computations is that (3.25) leads to an homogeneous wave equation. The explicit form is given in Appendix 3.6. These results can be summarised in the following proposition:

Proposition 2. *Assume that the wave equations* (3.19a)-(3.19f) *of Proposition 1 encoded in*

$$\begin{aligned} \nabla^{\boldsymbol{A}}{}_{\boldsymbol{E'}}\widehat{\Sigma}_{\boldsymbol{AB}}{}^{\boldsymbol{c}} &= 0, & \nabla^{\boldsymbol{C'}}{}_{\boldsymbol{D}}\widehat{\Xi}_{\boldsymbol{ABC'D'}} &= 0,\\ \nabla^{\boldsymbol{C}}{}_{\boldsymbol{E'}}\widehat{\Delta}_{\boldsymbol{CDBB'}} &= 0, & \nabla_{\boldsymbol{E}}{}^{\boldsymbol{B'}}\Lambda_{\boldsymbol{B'BAC}} &= 0,\\ \nabla_{\boldsymbol{AA'}}Z^{\boldsymbol{AA'}} &= 0, & Z_{\boldsymbol{AA'}}{}^{\boldsymbol{AA'}} &= 0, \end{aligned}$$

are satisfied on $\mathcal{U} \subset \mathcal{M}$. Then the zero-quantities satisfy the homogeneous wave equations

$$\widehat{\square}\widehat{\Sigma}_{\boldsymbol{AB}}{}^{\boldsymbol{c}} - 2\widehat{\square}_{\boldsymbol{PA}}\widehat{\Sigma}^{\boldsymbol{P}}{}_{\boldsymbol{B}}{}^{\boldsymbol{c}} + 2\widehat{\nabla}_{\boldsymbol{AQ'}}W[\Sigma]^{\boldsymbol{Q'}}{}_{\boldsymbol{B}}{}^{\boldsymbol{c}} = 0, \tag{3.26}$$

$$\widehat{\square}\widehat{\Xi}_{\boldsymbol{ABC'D'}} - 2\widehat{\square}_{\boldsymbol{P'C'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{D'}} + 2\widehat{\nabla}_{\boldsymbol{C'Q}}W[\Xi]^{\boldsymbol{Q}}{}_{\boldsymbol{ABD'}} = 0, \tag{3.27}$$

$$\widehat{\square}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{DBB'}} - 2\widehat{\square}_{\boldsymbol{PC}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{DBB'}} + 2\widehat{\nabla}_{\boldsymbol{CQ'}}W[\Delta]^{\boldsymbol{Q'}}{}_{\boldsymbol{DBB'}} = 0, \tag{3.28}$$

$$\widehat{\square}\Lambda_{\boldsymbol{B'BAC}} - 2\widehat{\square}_{\boldsymbol{P'B'}}\Lambda^{\boldsymbol{P'}}{}_{\boldsymbol{BAC}} + 2\widehat{\nabla}_{\boldsymbol{B'Q}}W[\Lambda]^{\boldsymbol{Q}}{}_{\boldsymbol{BAC}} = 0, \tag{3.29}$$

$$\widehat{\nabla}_{\boldsymbol{AA'}}Z^{\boldsymbol{AA'}} - W[Z]^{\boldsymbol{AA'}}{}_{\boldsymbol{AA'}} = 0, \tag{3.30}$$

where

$$W[\Sigma]^{\boldsymbol{Q'}}{}_{\boldsymbol{B}}{}^{\boldsymbol{c}} \equiv \widehat{\nabla}^{\boldsymbol{Q'}}{}_{\boldsymbol{E}}\widehat{\Sigma}^{\boldsymbol{E}}{}_{\boldsymbol{B}}{}^{\boldsymbol{c}}, \quad W[\Xi]^{\boldsymbol{Q}}{}_{\boldsymbol{ABD'}} \equiv \widehat{\nabla}^{\boldsymbol{Q}}{}_{\boldsymbol{E'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{E'}}{}_{\boldsymbol{D'}}$$

$$W[\Delta]^{\boldsymbol{Q'}}{}_{\boldsymbol{DBB'}} \equiv \widehat{\nabla}^{\boldsymbol{Q'}}{}_{\boldsymbol{F}}\widehat{\Delta}^{\boldsymbol{F}}{}_{\boldsymbol{DBB'}}, \quad W[\Lambda]^{\boldsymbol{Q}}{}_{\boldsymbol{BAC}} \equiv \widehat{\nabla}_{\boldsymbol{E'}}{}^{\boldsymbol{Q}}\Lambda^{\boldsymbol{E'}}{}_{\boldsymbol{BAC}},$$

$$W[Z]^{\boldsymbol{AA'}}{}_{\boldsymbol{AA'}} \equiv \widehat{\nabla}^{\boldsymbol{AA'}}Z_{\boldsymbol{AA'}},$$

on $\mathcal{U}$.

In the following, the set of equations (3.26)-(3.30) given in Proposition 2 will be referred to as the *subsidiary system.* It should be noticed that the terms of the form $\widehat{\square}_{\boldsymbol{PA}}\widehat{N}^{\boldsymbol{P}}{}_{\boldsymbol{B}\mathcal{K}}$ and $W^{\boldsymbol{Q'}}{}_{\boldsymbol{B}\mathcal{K}}$ can be computed using the $\widehat{\nabla}$-Ricci identities and the transition spinor $Q_{\boldsymbol{AA'BC}}$ respectively. Using the subsidiary equations from the previous proposition one readily obtains the following *reduction lemma*:

Lemma 11. *If the initial data for the subsidiary system of Proposition 2 is given by*

$$\widehat{\Sigma}_{\boldsymbol{AB}}{}^{\boldsymbol{c}}|_{\mathcal{S}} = 0, \quad \widehat{\Xi}_{\boldsymbol{ABC'D'}}|_{\mathcal{S}} = 0, \quad \widehat{\Delta}_{\boldsymbol{ABCC'}}|_{\mathcal{S}} = 0, \quad \Lambda_{\boldsymbol{B'BAC}}|_{\mathcal{S}} = 0, \quad Z_{\boldsymbol{AA'}}|_{\mathcal{S}} = 0,$$

where $\mathcal{S}$ is a spacelike hypersurface and the wave equations of Proposition 2 are satisfied everywhere, then one has a solution to the vacuum conformal Einstein field equations —in other words

$$\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}} = 0, \quad \Xi_{\boldsymbol{ABC'D'}} = 0, \quad \Delta_{\boldsymbol{ABCC'}} = 0, \quad \Lambda_{\boldsymbol{B'BAC}} = 0, \quad Z_{\boldsymbol{AA'}} = 0,$$

in $D(\mathcal{S})$. Moreover, whenever $\Xi \neq 0$, the solution to the conformal Einstein field equations implies a solution to the vacuum Einstein field equations.

Proof. It can be verified, using the $\widehat{\nabla}$-Ricci identities given in the Appendix 3.4, that the equations of Proposition 2 are homogeneous wave equations for the zero-quantities. Notice, however, that the equation for $Z_{\boldsymbol{AA'}}$ is not a wave equation but of a first order homogeneous equation. Therefore, if the zero-quantities vanish on an initial spacelike hypersurface $\mathcal{S}$ then by the homogeneity of the equations one has that

$$\widehat{\Sigma}_{\boldsymbol{AB}}{}^{\boldsymbol{c}} = 0, \quad \widehat{\Xi}_{\boldsymbol{ABC'D'}} = 0, \quad \widehat{\Delta}_{\boldsymbol{ABCC'}} = 0, \quad \Lambda_{\boldsymbol{B'BAC}} = 0, \quad Z_{\boldsymbol{AA'}} = 0,$$

everywhere on $D(\mathcal{S})$. Moreover, since initially $\widehat{\Sigma}_{\boldsymbol{AB}}{}^{\boldsymbol{c}} = \Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}}$, $\widehat{\Xi}_{\boldsymbol{ABC'D'}} = \Xi_{\boldsymbol{ABC'D'}}$ and $\widehat{\Delta}_{\boldsymbol{ABCC'}} = \Delta_{\boldsymbol{ABCC'}}$, one has that $\Sigma_{\boldsymbol{AB}}{}^{\boldsymbol{c}} = 0$, $\Xi_{\boldsymbol{ABC'D'}} = 0$, $\Delta_{\boldsymbol{ABCC'}} = 0$ on $D(\mathcal{S})$. In addition, using that a solution to the conformal Einstein field equations implies a solution to the Einstein field equations whenever $\Xi \neq 0$ [15], it follows that a solution to the wave equations of Proposition 1 with initial data consistent with the initial conditions given in Lemma 11 will imply a solution to the vacuum Einstein field equations whenever $\Xi \neq 0$.

□

Remark 19. It is noticed that the initial data for the subsidiary equations give a way to specify the data for the wave equations of Proposition 1. This observation is readily implemented employing a space spinor formalism which mimics the hyperbolic reduction process to extract a first order hyperbolic system out of the conformal Einstein field equations —see e.g. [17]. In the following, to illustrate this procedure, initial data for the rescaled Weyl spinor encoded in $\Lambda_{\boldsymbol{A'BCD}}|_{\mathcal{S}} = 0$ is considered.

Initial data for the rescaled Weyl spinor

A convenient way to specify the initial data

$$\phi_{\boldsymbol{ABCD}}|_{\mathcal{S}}, \qquad \mathcal{P}\phi_{\boldsymbol{ABCD}}|_{\mathcal{S}}.$$

is to use the space spinor formalism to split the equations encoded in $\Lambda_{\boldsymbol{A'BCD}} = 0$. From this split, a system of evolution and constraint equations can be obtained. Recall that $\Lambda_{\boldsymbol{A'BCD}} \equiv \nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A'}}\phi_{\boldsymbol{ABCQ}}$. Making use of the the decomposition of $\nabla_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\nabla_{\boldsymbol{AA'}}$ in terms of the operators $\mathcal{P}$ and $\mathcal{D}_{\boldsymbol{AB}}$ one obtains

$$\Lambda_{\boldsymbol{ABCD}} = -\tfrac{1}{2}\mathcal{P}\phi_{\boldsymbol{ABCD}} + \mathcal{D}^{\boldsymbol{Q}}{}_{\boldsymbol{A}}\phi_{\boldsymbol{BCDQ}},$$

Evolution and constraint equations are obtained, respectively, from considering

$$E_{\boldsymbol{ABCD}} \equiv -2\Lambda_{\boldsymbol{ABCD}} = \mathcal{P}\phi_{\boldsymbol{ABCD}} - 2\mathcal{D}^{\boldsymbol{Q}}{}_{(\boldsymbol{A}}\phi_{\boldsymbol{BCD})\boldsymbol{Q}} = 0, \quad \text{(evolution equation)}$$
$$C_{\boldsymbol{CD}} \equiv \Lambda^{\boldsymbol{Q}}{}_{\boldsymbol{QCD}} = \mathcal{D}^{\boldsymbol{PQ}}\phi_{\boldsymbol{PQCD}} = 0. \quad \text{(constraint equation).}$$

Restricting the last equations to the initial hypersurface $\mathcal{S}$ it follows that the initial data $\phi_{\boldsymbol{PQCD}}|_{\mathcal{S}}$ must satisfy $C_{\boldsymbol{CD}}|_{\mathcal{S}} = 0$ and the initial data for $(\mathcal{P}\phi_{\boldsymbol{PQCD}})|_{\mathcal{S}}$ can be read form $E_{\boldsymbol{ABCD}}|_{\mathcal{S}} = 0$ i.e. $(\mathcal{P}\phi_{\boldsymbol{PQCD}})|_{\mathcal{S}} = 2\mathcal{D}^{\boldsymbol{Q}}{}_{(\boldsymbol{A}}\phi_{\boldsymbol{BCD})\boldsymbol{Q}}|_{\mathcal{S}}$.

The procedure for the other equations is analogous and can be succinctly obtained by revisiting the derivation of the first order hyperbolic equations derived from the conformal Einstein field equations using the space spinor formalism —see for instance [17].

Remark 20. The hyperbolic reduction given in [18] makes use of the metric version of the conformal Einstein field equations, consequently, since in this formulation the metric is part of the unknowns one has to append an equation for the metric. To do so, one follows the discussion given in Section 2.1.3 of Chapter 2. In other words one uses equation (2.2) and one considers R_{ab} in local coordinates (x^μ) as an expression involving second order derivatives of the metric components. However, the principal part of $R_{ab}[g]$, seen as an expression involving second order derivatives of the metric components, is not necessarily hyperbolic. To recast the system of second order equations (3.2)-(3.4) and (3.8)-(3.9) given in [18] it is necessary to impose the appropriate gauge condition. In [18] the gauge is fixed imposing a generalised wave-map gauge.

One of the advantages of deriving wave equations using the spinorial version of the conformal Einstein field equations is that the metric is not part of the unknowns so that the principal part of $\square$ is always hyperbolic. In this case, instead of writing an equation for the metric one has to write equations for the frame and connection coefficients denoted by $e_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}$ and $\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$ respectively. Another difference to be emphasised is that, in the spinorial approach put forward in this thesis, the algebraic structure of the spinorial version of the conformal Einstein field equations can be exploited to derive the wave equations in a systematic way. This can be noticed for instance in the derivation of the wave equation for the rescaled Weyl spinor $\phi_{\boldsymbol{ABCD}}$ in comparison with that for the rescaled Weyl tensor $d^{a}{}_{bcd}$.

3.4 Appendix: Spinorial relations

In this appendix several relations and identities that are used repeatedly throughout this chapter are recalled —see Sections 4.6, 4.7, 4.9 and 4.10 of [29]. In addition, using the remarks made in [46] a generalisation of the spinorial Ricci identities for a connection which is metric but not necessarily torsion-free is obtained.

3.4.1 The Levi-Civita case

In this section some well-known relations satisfied by the curvature spinors of a Levi-Civita connection are revisited. The discussion of this section follows [29], Sections 4.9 and 4.11. First recall the decomposition of a general curvature spinor

$$\grave{R}_{\boldsymbol{AA'BB'CC'DD'}} = \grave{R}_{\boldsymbol{ABCC'DD'}}\epsilon_{\boldsymbol{B'A'}} + \grave{\bar{R}}_{\boldsymbol{A'B'CC'DD'}}\epsilon_{\boldsymbol{BA}}.$$

In addition, the reduced spinor $\grave{R}_{\boldsymbol{ABCC'DD'}}$ can be decomposed as

$$\grave{R}_{\boldsymbol{ABCC'DD'}} = \grave{X}_{\boldsymbol{ABCD}}\epsilon_{\boldsymbol{C'D'}} + \grave{Y}_{\boldsymbol{ABC'D'}}\epsilon_{\boldsymbol{CD}},$$

where

$$\grave{X}_{\boldsymbol{ABCD}} \equiv \tfrac{1}{2}\grave{R}_{\boldsymbol{AB(C|E'|D)}}{}^{\boldsymbol{E'}} \qquad \grave{Y}_{\boldsymbol{ABCD}} \equiv \tfrac{1}{2}\grave{R}_{\boldsymbol{ABE(C'}}{}^{\boldsymbol{E}}{}_{\boldsymbol{D')}}.$$

In the above expressions the symbol $\grave{}$ over the kernel letter indicates that this relation is general —i.e. the connection is not necessarily neither metric nor torsion-free. The spinors $\grave{X}_{\boldsymbol{ABCD}}$ and $\grave{Y}_{\boldsymbol{ABCD}}$ are not necessarily symmetric in $\boldsymbol{AB}$.

It is well known that if that the connection is metric, then the spinors $\widehat{X}_{\boldsymbol{ABCD}}$ and $\widehat{Y}_{\boldsymbol{ABC'D'}}$ have the further symmetries:

$$\widehat{X}_{\boldsymbol{ABCD}} = \widehat{X}_{(\boldsymbol{AB})\boldsymbol{CD}}, \qquad \widehat{Y}_{\boldsymbol{ABC'D'}} = \widehat{Y}_{(\boldsymbol{AB})\boldsymbol{C'D'}}. \tag{3.31}$$

The symbol $\widehat{}$ is written over the kernel letter to denote that only the metricity of the connection is being assumed. If the connection is not only metric but, in addition, is torsion free (i.e. it is a Levi-Civita connection) then the *first Bianchi* *identity* $R_{\boldsymbol{a}[\boldsymbol{bcd}]} = 0$ can be written equivalently as

$$R^*{}_{\boldsymbol{ab}}{}^{\boldsymbol{cb}} = 0, \tag{3.32}$$

where $R^*{}_{\boldsymbol{abcd}} \equiv \frac{1}{2}\epsilon_{\boldsymbol{cd}}{}^{\boldsymbol{ef}} R_{\boldsymbol{abef}}$ and $\epsilon_{\boldsymbol{abcd}}$ is the totally antisymmetric Levi-Civita tensor. Notice that equation (3.32) can be written in spinorial terms as —see equations (4.6.7) and (4.6.14) of [29]

$$R_{\boldsymbol{AA'BB'}}{}^{\boldsymbol{CB'BC'}} = 0.$$

The last equation in turn implies that $X_{\boldsymbol{SP}}{}^{\boldsymbol{SP}} = \bar{X}_{\boldsymbol{S'P'}}{}^{\boldsymbol{S'P'}}$ and $Y_{\boldsymbol{ABA'B'}} = \bar{Y}_{\boldsymbol{A'B'AB}}$. Accordingly $X_{\boldsymbol{SP}}{}^{\boldsymbol{SP}}$ is a real scalar and $Y_{\boldsymbol{ABA'B'}}$ is a Hermitian spinor which, following the notation of [29], will be denoted by $\Phi_{\boldsymbol{ABA'D'}} = \bar{\Phi}_{\boldsymbol{A'B'AB}}$. Collecting all this information and decomposing in terms of irreducible components one obtains the usual decomposition of the curvature spinors

$$X_{\boldsymbol{ABCD}} = \Psi_{\boldsymbol{ABCD}} + \Lambda(\epsilon_{\boldsymbol{DB}}\epsilon_{\boldsymbol{CA}} + \epsilon_{\boldsymbol{CB}}\epsilon_{\boldsymbol{DA}}), \qquad Y_{\boldsymbol{ABC'D'}} = \Phi_{\boldsymbol{ABC'D'}},$$

where $\Psi_{\boldsymbol{ABCD}}$ is the Weyl spinor and $\Phi_{\boldsymbol{ABC'D'}}$ is the Ricci spinor. The latter is the spinorial counterpart of a world tensor (because of its Hermiticity which is consequence of the first Bianchi identity) and Λ is a real scalar (consequence of the first Bianchi identity again). Additionally, observe that $X_{\boldsymbol{A}(\boldsymbol{BC})}{}^{\boldsymbol{A}} = 0$. This is a consequence of the symmetry under the interchange of pairs $R_{\boldsymbol{abcd}} = R_{\boldsymbol{cdab}}$ of the Riemann tensor of a Levi-Civita connection. For a general connection the right hand side of last equation is not necessarily zero —the reason is that the interchange of pairs symmetry is a consequence of the antisymmetry in the first and second pairs of indices and the first Bianchi identity which for a general connection involves the torsion and its derivatives.

In the Levi-Civita case, the spinorial Ricci identities are the spinorial counterpart of

$$[\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}]v^{\boldsymbol{c}} = R^{\boldsymbol{c}}{}_{\boldsymbol{eab}}v^{\boldsymbol{e}}.$$

These identities are given in terms of the operator $\square_{\boldsymbol{AB}} = \nabla_{\boldsymbol{Q'}(\boldsymbol{A}}\nabla_{\boldsymbol{B})}{}^{\boldsymbol{Q'}}$. The spinorial Ricci identities are given in a rather compact form by

$$\square_{\boldsymbol{AB}}\xi^{\boldsymbol{C}} = X_{\boldsymbol{ABQ}}{}^{\boldsymbol{C}}\xi^{\boldsymbol{Q}}, \qquad \square_{\boldsymbol{A'B'}}\xi^{\boldsymbol{C}} = \Phi_{\boldsymbol{A'B'Q}}{}^{\boldsymbol{C}}\xi^{\boldsymbol{Q}}, \tag{3.33}$$

$$\square_{\boldsymbol{AB}}\xi_{\boldsymbol{C}} = -X_{\boldsymbol{ABC}}{}^{\boldsymbol{Q}}\xi_{\boldsymbol{Q}}, \qquad \square_{\boldsymbol{A'B'}}\xi_{\boldsymbol{C}} = -\Phi_{\boldsymbol{A'B'C}}{}^{\boldsymbol{Q}}\xi_{\boldsymbol{Q}}. \tag{3.34}$$

It is useful to combine these identities with the decomposition of $X_{\boldsymbol{ABCD}}$ to obtain a more detailed list of relations. The following expressions are repeatedly used in this chapter:

$$\square_{\boldsymbol{AB}}\xi_{\boldsymbol{C}} = \Psi_{\boldsymbol{ABCQ}}\xi^{\boldsymbol{Q}} - 2\Lambda\xi_{(\boldsymbol{A}}\epsilon_{\boldsymbol{B})\boldsymbol{C}}, \qquad \square_{(\boldsymbol{AB}}\xi_{\boldsymbol{C})} = \Psi_{\boldsymbol{ABCQ}}\xi^{\boldsymbol{Q}}, \tag{3.35}$$

$$\square_{AB}\xi^{B} = -3\Lambda\xi_{A}, \qquad \square_{A'B'}\xi_{C} = \xi^{Q}\Phi_{QCA'B'}. \tag{3.36}$$

Using these relations and the Jacobi identity (ϵ -identity) the second Bianchi identity can be expressed in terms of spinors as

$$\nabla^{A}{}_{B'}X_{ABCD} = \nabla^{A'}{}_{B}\Phi_{CDA'B'}.$$

For completeness, the relation between $\Phi_{ABA'B'}$ and the Ricci tensor and between R and Λ is given explicitly:

$$R_{ac} \mapsto R_{AA'CC'} = 2\Phi_{ACA'C'} - 6\Lambda\epsilon_{AC}\epsilon_{A'C'}, \qquad R = -24\Lambda. \tag{3.37}$$

From the above expressions it follows that $2\Phi_{ABA'B'}$ is the spinorial counterpart of the trace-free Ricci tensor $R_{\{ab\}} \equiv R_{ab} - \frac{1}{4}Rg_{ab}$. From this last observation, it follows that the spinorial counterpart of the Schouten tensor can be rewritten in terms of $\Phi_{AA'BB'}$ and Λ. Recalling the definition of the 4-dimensional Schouten tensor $L_{ab} = \frac{1}{2}R_{ab} - \frac{1}{12}Rg_{ab}$ and equation (3.37) one gets

$$L_{ABA'B'} = \Phi_{ACA'C'} - \Lambda\epsilon_{AC}\epsilon_{A'C'}. \tag{3.38}$$

The second contracted Bianchi identity can be recast in terms of these spinors as

$$\nabla^{CA'}\Phi_{CDA'B'} + 3\nabla_{DB'}\Lambda = 0. \tag{3.39}$$

3.4.2 Spinorial Ricci identities for a metric connection

It this section the case of a connection $\widehat{\nabla}$ which is metric but not torsion-free is considered. First, one needs to obtain a suitable generalisation of the operator $\square_{AB}$. In order to achieve this, observe that the relation $[\nabla_a, \nabla_b]u^d = R^{d}{}_{cab}u^c$ valid for a Levi-Civita connection extends to a connection with torsion as

$$[\widehat{\nabla}_a, \widehat{\nabla}_b]u^d = \widehat{R}^{d}{}_{cab}u^c + \Sigma_a{}^c{}_b\widehat{\nabla}_c u^d.$$

Another way to think the last equation is to define a modified commutator of covariant derivatives through

$$[\![\widehat{\nabla}_a, \widehat{\nabla}_b]\!]u^d \equiv \left([\widehat{\nabla}_a, \widehat{\nabla}_b] - \Sigma_a{}^c{}_b\widehat{\nabla}_c\right)u^d.$$

In this way one can recast the Ricci identities as

$$[\![\widehat{\nabla}_a, \widehat{\nabla}_b]\!]u^d = \widehat{R}^{d}{}_{cab}u^c.$$

This observation leads to an expression for the generalised operator

$$\widehat{\square}_{\boldsymbol{AB}} \equiv \widehat{\nabla}_{\boldsymbol{C}'(\boldsymbol{A}}\widehat{\nabla}_{\boldsymbol{B})}{}^{\boldsymbol{C}'}.$$

The relation between this operator and the commutator of covariant derivatives is

$$[\widehat{\nabla}_{\boldsymbol{AA}'}, \widehat{\nabla}_{\boldsymbol{BB}'}] = \epsilon_{\boldsymbol{A}'\boldsymbol{B}'}\widehat{\square}_{\boldsymbol{AB}} + \epsilon_{\boldsymbol{AB}}\widehat{\square}_{\boldsymbol{A}'\boldsymbol{B}'}.$$

One cannot directly write down the equivalent spinorial Ricci identities simply by replacing X and Y by $\widehat{X}$ and $\widehat{Y}$ because of appearance of the term $\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\widehat{\nabla}_{\boldsymbol{c}}u^{\boldsymbol{d}}$ in the commutator of the covariant derivatives. A way to get around this difficulty is to define a modified operator $\widehat{\boxminus}_{\boldsymbol{AB}}$ formed using the modified commutator of covariant derivatives instead of the usual commutator. In this way, one can directly translate the previous formulae simply by replacing X and Y by $\widehat{X}$ and $\widehat{Y}$. The relation between $\widehat{\boxminus}_{\boldsymbol{AB}}$ and $\widehat{\square}_{\boldsymbol{AB}}$ can clarified observing that

$$\begin{aligned}
\widehat{\boxminus}_{\boldsymbol{CD}} &= \tfrac{1}{2}\epsilon^{\boldsymbol{C}'\boldsymbol{D}'}[\![\widehat{\nabla}_{\boldsymbol{CC}'}, \widehat{\nabla}_{\boldsymbol{DD}'}]\!] \\
&= \tfrac{1}{2}\epsilon^{\boldsymbol{C}'\boldsymbol{D}'}\left([\widehat{\nabla}_{\boldsymbol{CC}'}, \widehat{\nabla}_{\boldsymbol{DD}'}] - \Sigma_{\boldsymbol{CC}'}{}^{\boldsymbol{EE}'}{}_{\boldsymbol{DD}'}\widehat{\nabla}_{\boldsymbol{EE}'}\right) \\
&= \tfrac{1}{2}\left(\widehat{\nabla}_{\boldsymbol{D}'\boldsymbol{C}}\widehat{\nabla}_{\boldsymbol{D}}{}^{\boldsymbol{D}'} + \widehat{\nabla}_{\boldsymbol{D}'\boldsymbol{D}}\widehat{\nabla}_{\boldsymbol{C}}{}^{\boldsymbol{D}'} - \Sigma_{\boldsymbol{CD}'}{}^{\boldsymbol{EE}'}{}_{\boldsymbol{D}}{}^{\boldsymbol{D}'}\widehat{\nabla}_{\boldsymbol{EE}'}\right).
\end{aligned} \tag{3.40}$$

Using the antisymmetry of the torsion spinor one has the decomposition

$$\Sigma_{\boldsymbol{AA}'}{}^{\boldsymbol{CC}'}{}_{\boldsymbol{BB}'} = \epsilon_{\boldsymbol{AB}}\Sigma_{\boldsymbol{A}}{}^{\boldsymbol{EE}'}{}_{\boldsymbol{B}} + \epsilon_{\boldsymbol{A}'\boldsymbol{B}'}\Sigma_{\boldsymbol{A}'}{}^{\boldsymbol{EE}'}{}_{\boldsymbol{B}'}, \tag{3.41}$$

where the reduced spinor is given by $\Sigma_{\boldsymbol{A}}{}^{\boldsymbol{EE}'}{}_{\boldsymbol{B}} = \frac{1}{2}\Sigma_{(\boldsymbol{A}|\boldsymbol{Q}'|}{}^{\boldsymbol{EE}'}{}_{\boldsymbol{B})}{}^{\boldsymbol{Q}'}$. Using this decomposition and symmetrising expression (3.40) in the indices $\boldsymbol{CD}$ one obtains

$$\widehat{\boxminus}_{\boldsymbol{CD}} = \widehat{\nabla}_{\boldsymbol{D}'(\boldsymbol{C}}\widehat{\nabla}_{\boldsymbol{D})}{}^{\boldsymbol{D}'} - \Sigma_{\boldsymbol{C}}{}^{\boldsymbol{EE}'}{}_{\boldsymbol{D}}\widehat{\nabla}_{\boldsymbol{EE}'} = \widehat{\square}_{\boldsymbol{CD}} - \Sigma_{\boldsymbol{C}}{}^{\boldsymbol{EE}'}{}_{\boldsymbol{D}}\widehat{\nabla}_{\boldsymbol{EE}'}.$$

Therefore

$$\widehat{\square}_{\boldsymbol{AB}} = \widehat{\boxminus}_{\boldsymbol{AB}} + \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{EE}'}{}_{\boldsymbol{B}}\widehat{\nabla}_{\boldsymbol{EE}'}. \tag{3.42}$$

In order to compute explicitly how $\widehat{\boxminus}_{\boldsymbol{AB}}$ acts on spinors it is sufficient to compute the generalised spinors $\widehat{X}_{\boldsymbol{ABCD}}$ and $\widehat{\Phi}_{\boldsymbol{ABC}'\boldsymbol{D}'}$.

As discussed in previous paragraphs, the fact that the connection is not torsion free is reflected in the symmetries of the curvature spinors. Notice that, the symmetries in equation (3.31) still hold due to the metricity of $\widehat{\nabla}$. Nevertheless, the interchange of pairs symmetry of the Riemann tensor, the reality condition on $\widehat{X}_{\boldsymbol{SP}}{}^{\boldsymbol{SP}}$ and the Hermiticity of $\widehat{\Phi}_{\boldsymbol{ABC}'\boldsymbol{D}'}$ do not longer hold as these properties rely on the the cyclic identity $R_{\boldsymbol{d}[\boldsymbol{abc}]} = 0$. In fact, the first Bianchi identity is, in general, given by

$$\grave{R}^{\boldsymbol{d}}{}_{[\boldsymbol{abc}]} + \grave{\nabla}_{[\boldsymbol{a}}\Sigma_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} + \Sigma_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\Sigma_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}} = 0.$$

It follows that $\widehat{X}_{\boldsymbol{A}(\boldsymbol{BC})}{}^{\boldsymbol{A}}$ does not necessarily vanish and, generically, it will depend on the torsion and its derivatives as can be seen from the last equation. Nonetheless, one labels, as usual, the remaining non-vanishing contractions of $\widehat{X}_{\boldsymbol{ABCD}}$

$$\widehat{X}_{\boldsymbol{AB}}{}^{\boldsymbol{AB}} = 6\widehat{\Lambda}, \qquad \widehat{X}_{(\boldsymbol{ABCD})} = \widehat{\Psi}_{\boldsymbol{ABCD}}, \qquad \widehat{X}_{\boldsymbol{A}(\boldsymbol{BC})}{}^{\boldsymbol{A}} = H_{\boldsymbol{BC}},$$

where $H_{\boldsymbol{BC}}$ is a spinor which, as discussed previously, depends on the torsion and its derivatives. The explicit form of $H_{\boldsymbol{BC}}$ will not be needed. Finally, recall the general decomposition in irreducible terms of a 4-valence spinor $\xi_{\boldsymbol{ABCD}}$:

$$\begin{aligned}\xi_{\boldsymbol{ABCD}} = \xi_{(\boldsymbol{ABCD})} + \tfrac{1}{2}\xi_{(\boldsymbol{AB})\boldsymbol{P}}{}^{\boldsymbol{P}} + \tfrac{1}{2}\xi_{\boldsymbol{P}}{}^{\boldsymbol{P}}{}_{(\boldsymbol{CD})}\epsilon_{\boldsymbol{AB}} + \tfrac{1}{4}\xi_{\boldsymbol{P}}{}^{\boldsymbol{P}}{}_{\boldsymbol{Q}}{}^{\boldsymbol{Q}}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{CD}} \\ +\tfrac{1}{2}\epsilon_{\boldsymbol{A}(\boldsymbol{C}}\xi_{\boldsymbol{D})\boldsymbol{B}} + \tfrac{1}{2}\epsilon_{\boldsymbol{B}(\boldsymbol{C}}\xi_{\boldsymbol{D})\boldsymbol{A}} - \tfrac{1}{3}\epsilon_{\boldsymbol{A}(\boldsymbol{C}}\epsilon_{\boldsymbol{D})\boldsymbol{B}}\xi.\end{aligned}$$

where

$$\xi_{\boldsymbol{AB}} \equiv \xi_{\boldsymbol{Q}(\boldsymbol{AB})}{}^{\boldsymbol{Q}}, \qquad \xi \equiv \xi_{\boldsymbol{PQ}}{}^{\boldsymbol{PQ}}.$$

Using the above formula one obtains the following expressions for the irreducible decomposition of the curvature spinor $\widehat{X}_{\boldsymbol{ABCD}}$:

$$\widehat{X}_{\boldsymbol{ABCD}} = \widehat{\Psi}_{\boldsymbol{ABCD}} + \widehat{\Lambda}\left(\epsilon_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}} + \epsilon_{\boldsymbol{AD}}\epsilon_{\boldsymbol{BC}}\right) + \tfrac{1}{2}\epsilon_{\boldsymbol{A}(\boldsymbol{C}}H_{\boldsymbol{D})\boldsymbol{B}} + \tfrac{1}{2}\epsilon_{\boldsymbol{B}(\boldsymbol{C}}H_{\boldsymbol{D})\boldsymbol{A}}. \quad (3.43)$$

In order to ease the comparisons with the Levi-Civita case let

$$\widehat{Y}_{\boldsymbol{ABC'D'}} = \widehat{\Phi}_{\boldsymbol{ABC'D'}}. \quad (3.44)$$

Observe that, in contrast with the case of a Levi-Civita connection, $\widehat{\Lambda}$ is not real and $\widehat{\Phi}_{\boldsymbol{ABC'D'}}$ is not Hermitian. In other words, one has that

$$\widehat{\Lambda} - \bar{\widehat{\Lambda}} \neq 0, \qquad \widehat{\Phi}_{\boldsymbol{ABC'D'}} - \bar{\widehat{\Phi}}_{\boldsymbol{A'B'CD}} \neq 0, \qquad H_{\boldsymbol{AB}} \neq 0. \quad (3.45)$$

In fact, the right hand side of the previous equations depends on the torsion and its derivatives —see [46]. However, its explicit expression is not required in the discussion of this chapter. Having found the curvature spinors, one can derive the spinorial Ricci identities. As discussed in the previous paragraph, the modified operator $\widehat{\boxminus}_{\boldsymbol{AB}}$ formed from the modified commutator of covariant derivatives satisfies a version of the spinorial Ricci identities which is obtained simply by replacing the curvature spinors $X_{\boldsymbol{ABCD}}$ and $\Phi_{\boldsymbol{ABC'D'}}$ by the spinors $\widehat{X}_{\boldsymbol{ABCD}}$ and $\widehat{\Phi}_{\boldsymbol{ABC'D'}}$. The Ricci identities with torsion are then given by

$$\begin{aligned}\widehat{\square}_{\boldsymbol{AB}}\xi^{\boldsymbol{C}} &= \widehat{X}_{\boldsymbol{ABQ}}{}^{\boldsymbol{C}}\xi^{\boldsymbol{Q}} + \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{PP'}}{}_{\boldsymbol{B}}\widehat{\nabla}_{\boldsymbol{PP'}}\xi^{\boldsymbol{C}}, \\ \widehat{\square}_{\boldsymbol{A'B'}}\xi^{\boldsymbol{C}} &= \bar{\widehat{\Phi}}_{\boldsymbol{A'B'Q}}{}^{\boldsymbol{C}}\xi^{\boldsymbol{Q}} + \bar{\Sigma}_{\boldsymbol{A'}}{}^{\boldsymbol{PP'}}{}_{\boldsymbol{B'}}\widehat{\nabla}_{\boldsymbol{PP'}}\xi^{\boldsymbol{C}}, \\ \widehat{\square}_{\boldsymbol{AB}}\xi_{\boldsymbol{C}} &= -\widehat{X}_{\boldsymbol{ABC}}{}^{\boldsymbol{Q}}\xi_{\boldsymbol{Q}} + \Sigma_{\boldsymbol{A}}{}^{\boldsymbol{PP'}}{}_{\boldsymbol{B}}\widehat{\nabla}_{\boldsymbol{PP'}}\xi_{\boldsymbol{C}},\end{aligned}$$

$$\widehat{\square}_{A'B'}\xi_C = -\bar{\widehat{\Phi}}_{A'B'C}{}^Q\xi_Q + \bar{\Sigma}_{A'}{}^{PP'}{}_{B'}\widehat{\nabla}_{PP'}\xi_C,$$

with $\widehat{X}_{ABCD}$ and $\widehat{\Phi}_{ABC'D'}$ given by equations (3.43) and (3.44). The primed version of the last expressions can be readily identified. More importantly, the detailed version (in terms of irreducible components) of the spinorial Ricci identities become

$$\widehat{\square}_{AB}\xi_C = \widehat{\Psi}_{ABCQ}\xi^Q - 2\widehat{\Lambda}\xi_{(A}\epsilon_{B)C} + U_{ABCQ}\xi^Q + \Sigma_A{}^{PP'}{}_B\widehat{\nabla}_{PP'}\xi_C, \tag{3.46a}$$

$$\widehat{\square}_{(AB}\xi_{C)} = \widehat{\Psi}_{ABCQ}\xi^Q + \Sigma_{(A}{}^{PP'}{}_B\widehat{\nabla}_{|PP'|}\xi_{C)}, \tag{3.46b}$$

$$\widehat{\square}_{AB}\xi^B = -3\widehat{\Lambda}\xi_A + H_{AB}\xi^B + \Sigma_A{}^{PP'}{}_B\widehat{\nabla}_{PP'}\xi^B, \tag{3.46c}$$

$$\widehat{\square}_{A'B'}\xi_C = \xi^Q\bar{\widehat{\Phi}}_{QCA'B'} + \bar{\Sigma}_{A'}{}^{PP'}{}_{B'}\widehat{\nabla}_{PP'}\xi_C. \tag{3.46d}$$

The above identities are supplemented by their complex conjugated version — keeping in mind the non-Hermiticity of $\widehat{\Phi}_{ABC'D'}$ and the non-reality of $\widehat{\Lambda}$ as stated in expression (3.45). In the last list of identities the quantity U_{ABCD} is defined as

$$U_{ABCD} \equiv \tfrac{1}{2}\epsilon_{A(C}H_{D)B} + \tfrac{1}{2}\epsilon_{B(C}H_{D)A}. \tag{3.47}$$

The Levi-Civita case can be readily recovered by setting $\Sigma_A{}^{PP'}{}_B = 0$ since in such case, the spinors H_{AB} and U_{ABCD} also vanish. Moreover, the pair interchange symmetry is recovered and the expressions in (3.45) become equalities.

3.5 Appendix: The transition tensor and the torsion tensor

In this appendix the transition spinor relating a Levi-Civita connection $\boldsymbol{\nabla}$ with a connection $\widehat{\boldsymbol{\nabla}}$ which is metric but not necessarily torsion-free is discussed. The general strategy behind this discussion can be found in [29]. Given two general connections $\grave{\boldsymbol{\nabla}}$ and $\acute{\boldsymbol{\nabla}}$ one has that

$$(\grave{\nabla}_a - \acute{\nabla}_a)\xi^b \equiv Q_a{}^b{}_c\xi^c$$

where $Q_a{}^b{}_c$ is the transition tensor. It is well known that for the case of a Levi-Civita connection $\boldsymbol{\nabla}$ and a metric connection $\widehat{\boldsymbol{\nabla}}$ one has

$$\Sigma_a{}^c{}_b = -2Q_{[a}{}^c{}_{b]} \qquad Q_{abc} = Q_{a[bc]}. \tag{3.48}$$

Therefore, the spinorial counterpart of the transition tensor can be decomposed as

$$Q_{AA'BB'CC'} = Q_{AA'BC}\epsilon_{B'C'} + \bar{Q}_{AA'B'C'}\epsilon_{BC} \tag{3.49}$$

where

$$Q_{\boldsymbol{AA'BC}} \equiv \tfrac{1}{2} Q_{\boldsymbol{AA'(B|Q'|C)}}{}^{\boldsymbol{Q'}}.$$

This expression allows to translate expressions containing the covariant derivative $\widehat{\boldsymbol{\nabla}}$ to expressions containing $\boldsymbol{\nabla}$ and the transition spinor $Q_{\boldsymbol{AA'BC}}$ as follows:

$$\widehat{\nabla}_{\boldsymbol{AA'}}\xi^{\boldsymbol{B}} = \nabla_{\boldsymbol{AA'}}\xi^{\boldsymbol{B}} + Q_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{Q}}\xi^{\boldsymbol{Q}}, \qquad \widehat{\nabla}_{\boldsymbol{AA'}}\xi_{\boldsymbol{C}} = \nabla_{\boldsymbol{AA'}}\xi_{\boldsymbol{C}} - Q_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{C}}\xi_{\boldsymbol{Q}}. \tag{3.50}$$

These expressions can be extended in a similar manner to spinors of any index structure. Now, from the equations in (3.48) it follows that

$$Q_{\boldsymbol{acb}} = -\Sigma_{\boldsymbol{a}[\boldsymbol{cb}]} - \tfrac{1}{2}\Sigma_{\boldsymbol{cab}}. \tag{3.51}$$

Using the above equation along with the decompositions (3.49) and (3.41) gives

$$Q_{\boldsymbol{AA'BC}} = -2\Sigma_{(\boldsymbol{B}|\boldsymbol{AA'}|\boldsymbol{C})} - 2\Sigma_{\boldsymbol{A}(\boldsymbol{C}|\boldsymbol{A'}|\boldsymbol{B})} - 2\bar{\Sigma}_{\boldsymbol{A'}(\boldsymbol{C}|\boldsymbol{Q'}}{}^{\boldsymbol{Q'}}\epsilon_{\boldsymbol{A}|\boldsymbol{B})}.$$

3.6 Appendix: Explicit expressions for the subsidiary equations

In Section 3.3.2 it was shown that the generic form of the equations in the subsidiary system is

$$\widehat{\widehat{\square}}\widehat{N}_{\boldsymbol{AB}\mathcal{K}} = 2\widehat{\widehat{\square}}_{\boldsymbol{PA}}\widehat{N}^{\boldsymbol{P}}{}_{\boldsymbol{B}\mathcal{K}} - 2\widehat{\widehat{\nabla}}_{\boldsymbol{AQ'}}W^{\boldsymbol{Q'}}{}_{\boldsymbol{B}\mathcal{K}}.$$

In this section results of Appendices 3.4 and 3.5 are used to compute explicitly the terms $\widehat{\widehat{\square}}_{\boldsymbol{PA}}\widehat{N}^{\boldsymbol{P}}{}_{\boldsymbol{B}\mathcal{K}}$ and $W^{\boldsymbol{Q'}}{}_{\boldsymbol{B}\mathcal{K}}$ for every zero-quantity. A direct computation using Appendices 3.4 and 3.5 render

$$\begin{aligned}
\widehat{\widehat{\square}}_{\boldsymbol{PA}}\widehat{\Sigma}^{\boldsymbol{P}}{}_{\boldsymbol{B}}{}^{\boldsymbol{c}} ={}& -3\widehat{\widehat{\Lambda}}\widehat{\Sigma}_{\boldsymbol{AB}}{}^{\boldsymbol{c}} + H_{\boldsymbol{PA}}\widehat{\Sigma}^{\boldsymbol{P}}{}_{\boldsymbol{B}}{}^{\boldsymbol{c}} + \widehat{\widehat{\Psi}}_{\boldsymbol{PABG}}\widehat{\Sigma}^{\boldsymbol{PG}\,\boldsymbol{c}} - 2\widehat{\widehat{\Lambda}}\widehat{\Sigma}^{\boldsymbol{P}}{}_{(\boldsymbol{P}}{}^{\boldsymbol{c}}\epsilon_{\boldsymbol{A})\boldsymbol{B}}\\
&+U_{\boldsymbol{PABQ}}\widehat{\Sigma}^{\boldsymbol{P}}{}_{(\boldsymbol{P}}{}^{\boldsymbol{c}}\epsilon_{\boldsymbol{A})\boldsymbol{B}} + 2\Sigma_{\boldsymbol{P}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{A}}\widehat{\widehat{\nabla}}_{\boldsymbol{QQ'}}\widehat{\Sigma}^{\boldsymbol{P}}{}_{\boldsymbol{B}}{}^{\boldsymbol{c}},\\
\widehat{\widehat{\square}}_{\boldsymbol{P'C'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{D'}} ={}& \widehat{\Xi}^{\boldsymbol{Q}}{}_{\boldsymbol{B}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{D'}}\bar{\widehat{\Phi}}_{\boldsymbol{QAP'C'}} + \bar{\Sigma}_{\boldsymbol{P'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{C'}}\widehat{\widehat{\nabla}}_{\boldsymbol{QQ'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{C'}}\\
&+\widehat{\Xi}_{\boldsymbol{A}}{}^{\boldsymbol{QP'}}{}_{\boldsymbol{D'}}\bar{\widehat{\Phi}}_{\boldsymbol{AQP'C'}} + \bar{\Sigma}_{\boldsymbol{P'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{C'}}\widehat{\widehat{\nabla}}_{\boldsymbol{QQ'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{C'}}\\
&-\bar{\widehat{\Lambda}}\widehat{\Xi}_{\boldsymbol{ABC'D'}} + \bar{H}_{\boldsymbol{P'C'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{D'}} + \bar{\Sigma}_{\boldsymbol{P'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{C'}}\widehat{\widehat{\nabla}}_{\boldsymbol{QQ'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{D'}}\\
&+\bar{\widehat{\Psi}}_{\boldsymbol{P'C'D'Q'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'Q'}} - \bar{\widehat{\Lambda}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'}}{}_{(\boldsymbol{P'}}\epsilon_{\boldsymbol{C'})\boldsymbol{D'}} + \bar{U}_{\boldsymbol{P'C'D'Q'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'Q'}}\\
&+\bar{\Sigma}_{\boldsymbol{P'}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{C'}}\widehat{\widehat{\nabla}}_{\boldsymbol{QQ'}}\widehat{\Xi}_{\boldsymbol{AB}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{D'}},\\
\widehat{\widehat{\square}}_{\boldsymbol{PC}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{DBB'}} ={}& -3\widehat{\widehat{\Lambda}}\widehat{\Delta}_{\boldsymbol{CDBB'}} + H_{\boldsymbol{PC}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{DBB'}} + \Sigma_{\boldsymbol{P}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{C}}\widetilde{\nabla}_{\boldsymbol{QQ'}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{DBB'}}\\
&+\widehat{\widehat{\Psi}}_{\boldsymbol{PCDQ}}\widehat{\Delta}^{\boldsymbol{PQ}}{}_{\boldsymbol{BB'}} - 2\widehat{\widehat{\Lambda}}\widehat{\Delta}^{\boldsymbol{P}}{}_{(\boldsymbol{P}|\boldsymbol{DBB'}|}\epsilon_{\boldsymbol{C})\boldsymbol{D}} + U_{\boldsymbol{PCDQ}}\widehat{\Delta}^{\boldsymbol{PQ}}{}_{\boldsymbol{BB'}}\\
&+\Sigma_{\boldsymbol{P}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{C}}\widehat{\widehat{\nabla}}_{\boldsymbol{QQ'}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{DBB'}} + \widehat{\widehat{\Psi}}_{\boldsymbol{PCBQ}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{D}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B'}} - 2\widehat{\widehat{\Lambda}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{D}(\boldsymbol{P}|\boldsymbol{B'}|}\epsilon_{\boldsymbol{C})\boldsymbol{B}}\\
&+U_{\boldsymbol{PCBQ}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{D}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{BB'}} + \Sigma_{\boldsymbol{P}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{C}}\widehat{\widehat{\nabla}}_{\boldsymbol{QQ'}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{DBB'}} + \widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{DB}}{}^{\boldsymbol{Q'}}\widehat{\widehat{\Phi}}_{\boldsymbol{Q'B'PC}}\\
&+\Sigma_{\boldsymbol{P}}{}^{\boldsymbol{QQ'}}{}_{\boldsymbol{C}}\widehat{\widehat{\nabla}}_{\boldsymbol{QQ'}}\widehat{\Delta}^{\boldsymbol{P}}{}_{\boldsymbol{DBB'}},
\end{aligned}$$

$$
\begin{aligned}
\widehat{\square}_{P'B'}\Lambda^{P'}{}_{BAC} = {} & \Lambda^{P'Q}{}_{AC}\bar{\widehat{\Phi}}_{QBP'B'} + \bar{\Sigma}_{P'}{}^{QQ'}{}_{B'}\widehat{\nabla}_{QQ'}\Lambda^{P'}{}_{BAC} - 3\widehat{\Lambda}\Lambda_{P'BAC} \\
& + \bar{H}_{P'B'}\Lambda^{P'}{}_{BAC} + \bar{\Sigma}_{P'}{}^{QQ'}{}_{B'}\widehat{\nabla}_{QQ'}\Lambda^{P'}{}_{BAC} + \Lambda^{P'}{}_{B}{}^{Q}{}_{C}\bar{\widehat{\Phi}}_{QAP'B'} \\
& + \bar{\Sigma}_{P'}{}^{QQ'}{}_{B'}\widehat{\nabla}_{QQ'}\Lambda^{P'}{}_{BAC} + \Lambda^{P'}{}_{BA}{}^{Q}\Phi_{QCP'B'} \\
& + \bar{\Sigma}_{P'}{}^{QQ'}{}_{B'}\widehat{\nabla}_{QQ'}\Lambda^{P'}{}_{BAC}.
\end{aligned}
$$

Moreover, one has that

$$
\begin{aligned}
W[\Sigma]^{Q'}{}_{B}{}^{c} \equiv {} & Q^{Q'}{}_{E}{}^{E}{}_{F}\widehat{\Sigma}^{F}{}_{B}{}^{c} - Q^{Q'}{}_{EB}{}^{F}\widehat{\Sigma}^{E}{}_{F}{}^{c}, \\
W[\Xi]^{Q}{}_{ABD'} \equiv {} & -Q^{Q}{}_{E'A}{}^{F}\widehat{\Xi}_{FB}{}^{E'}{}_{D} - Q^{Q}{}_{E'B}{}^{F}\widehat{\Xi}_{AF}{}^{E'}{}_{D} + \bar{Q}^{Q}{}_{E'}{}^{E'}{}_{F'}\widehat{\Xi}_{AB}{}^{F'}{}_{D} \\
& - Q^{Q}{}_{E'D}{}^{F}\widehat{\Xi}_{AB}{}^{E'}{}_{F}, \\
W[\Delta]^{Q'}{}_{DBB'} \equiv {} & Q^{Q'}{}_{E}\widehat{\Delta}^{F}{}_{DBB'} - Q^{Q'}{}_{ED}{}^{F}\widehat{\Delta}^{E}{}_{FBB'} - Q^{Q'}{}_{EB}{}^{F}\widehat{\Delta}^{E}{}_{DFB'} \\
& - \bar{Q}^{Q'}{}_{EB'}{}^{F'}\widehat{\Delta}^{E}{}_{DBF'}, \\
W[\Lambda]^{Q}{}_{BAC} \equiv {} & -Q_{E'}{}^{G}{}_{B}{}^{F}\Lambda^{E'}{}_{FAC} + \bar{Q}_{E'}{}^{GE'}{}_{F'}\Lambda^{F'}{}_{BAC} - Q_{E'}{}^{G}{}_{A}{}^{F}\Lambda^{E'}{}_{BFC} \\
& - Q_{E'}{}^{G}{}_{C}{}^{F}\Lambda^{E'}{}_{BAF}, \\
W[Z]^{AA'}{}_{AA'} \equiv {} & -Q_{AA'}{}^{A}{}_{E}Z^{EA'} - \bar{Q}_{AA'}{}^{A'}{}_{E}Z^{EE'},
\end{aligned}
$$

where the transition spinor is understood to be expressed in terms of the reduced torsion spinor which is, in itself, a zero-quantity —see equation (3.22).

4 Non-linear stability of the Milne spacetime

In this chapter an analysis of the non-linear stability of the Milne spacetime is given. This discussion is an application of the hyperbolic reduction procedure put forward in Chapter 3. The discussion given in this chapter is based on:

Gasperín E. and Valiente Kroon J.A., "Spinorial wave equations and stability of the Milne spacetime," *Classical and Quantum Gravity* **32** (Sept., 2015) 185021, `arXiv:1407.3317 [gr-qc]`.

The Milne Universe is a Friedman-Lemaître-Robertson-Walker (FLRW) solution to the Einstein field equations with vanishing Cosmological constant and negative spatial curvature —see e.g. [20]. The Milne Universe can be seen to be a part of the Minkowski spacetime written in comoving coordinates adapted to the world-line of a particle. Accordingly, analysing the non-linear stability of the Milne Universe is essentially equivalent to obtaining a proof of the semiglobal stability of the Minkowski spacetime —see [8]. The analysis of the semiglobal non-linear stability of the Minkowski spacetime given in [8] makes use of the standard conformal Einstein field equations and the first order hyperbolic reduction discussed in Section 2.4.2 of Chapter 2. Nevertheless, in principle, one could recover some of the classical results by H. Friedrich —say [8, 9]— using the system of wave equations discussed in Chapter 3 instead. With this motivation in mind, the Milne spacetime was chosen for analysis in this chapter to show how to use the system of wave equations of Chapter 3 on a specific application.

In the analysis of this chapter, the stability result follows from the general theory of quasilinear wave equations, in particular the property of Cauchy stability, as given in [21]. In broad terms, this stability result for the Milne Universe can be phrased as:

Main Result 1. *Initial data for the conformal wave equations close enough to the data for the Milne Universe give rise to a solution to the Einstein field equations which exist globally to the future and has an asymptotic structure similar to that of the Milne Universe.*

As in the case of first order hyperbolic reductions of the conformal Einstein field equations —see e.g. [17]— some of the methods discussed in this chapter are not

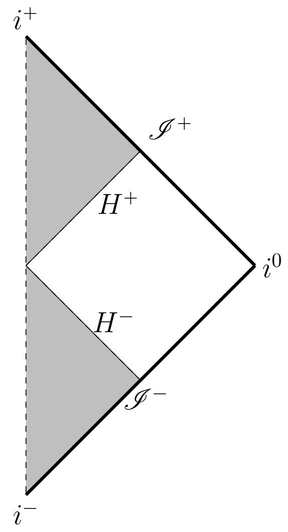

Figure 4.1: *Penrose Diagram for the Milne Universe. The diagram for the Milne Universe corresponds to a portion (shaded area) of the Penrose diagram of the Minkowski spacetime. The boundary $H^+ \cup H^-$ corresponds to the limit of the region where the coordinates (t, χ) are well defined. The region of spacetime obtained from evolving hyperboloidal initial data, as discussed in this chapter, does not correspond to all the shaded area in this diagram —compare with Figure 4.2.*

only applicable to the Milne spacetime but more generally to spacetime manifolds whose spatial sections are orientable compact manifolds. For this, one makes use the localisability property of solutions to hyperbolic equations —see e.g. [3] for further discussion on this type of constructions.

4.1 Basic properties of the Milne Universe

The Milne Universe is a Friedman-Lemaître-Robinson-Walker vacuum solution to the Einstein field equation with vanishing Cosmological constant, energy density and pressure. In fact, it represents a flat spacetime written in comoving coordinates of the worldlines starting at $t = 0$ —see [20]. This means that the Milne Universe can be seen as a portion of the Minkowski spacetime, which in turn can be conformally related to the *Einstein Cosmos*, $(\mathcal{M}_E \equiv \mathbb{R} \times \mathbb{S}^3, \mathring{\boldsymbol{g}})$ (sometimes also called the *Einstein cylinder*) —see Figure 4.1. The metric $\tilde{\mathring{\boldsymbol{g}}}$ of the Milne Universe is given in comoving coordinates $(t, \chi, \theta, \varphi)$ by

$$\tilde{\mathring{\boldsymbol{g}}} = \mathbf{d}t \otimes \mathbf{d}t - t^2 \left(\mathbf{d}\chi \otimes \mathbf{d}\chi + \sinh^2 \chi \left(\mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2 \theta \mathbf{d}\varphi \otimes \mathbf{d}\varphi \right) \right) \tag{4.1}$$

where

$$t \in (-\infty, \infty), \qquad \chi \in [0, \infty), \qquad \theta \in [0, \pi], \qquad \phi \in [0, 2\pi).$$

In fact, introducing the coordinates

$$\bar{r} \equiv t \sinh \chi, \qquad \bar{t} \equiv t \cosh \chi$$

the metric reads

$$\tilde{\mathring{\boldsymbol{g}}} = \mathbf{d}\bar{t} \otimes \mathbf{d}\bar{t} - \mathbf{d}\bar{r} \otimes \mathbf{d}\bar{r} - \bar{r}^2 \left(\mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2\theta \mathbf{d}\varphi \otimes \mathbf{d}\varphi\right).$$

Therefore, $\bar{t}^2 - \bar{r}^2 > 0$, and the Milne Universe corresponds to the interior of the light cone through the origin in the Minkowski spacetime as shown in the Penrose diagram of Figure 4.1. As already discussed, this metric is conformally related to the metric $\mathring{\boldsymbol{g}}$ of the Einstein Cosmos. More precisely, one has that

$$\mathring{\boldsymbol{g}} = \mathring{\Xi}^2 \tilde{\mathring{\boldsymbol{g}}}$$

where the metric of the Einstein cylinder, $\mathring{\boldsymbol{g}}$, is given by

$$\mathring{\boldsymbol{g}} \equiv \mathbf{d}T \otimes \mathbf{d}T - \hbar,$$

with $\hbar$ denoting the standard metric of $\mathbb{S}^3$

$$\hbar \equiv \mathbf{d}\psi \otimes \mathbf{d}\psi + \sin^2\psi \mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2\psi \sin^2\theta \mathbf{d}\varphi \otimes \mathbf{d}\varphi.$$

The conformal factor relating the metric of the Milne Universe to metric of the Einstein Universe is given by

$$\mathring{\Xi} = \cos T + \cos\psi,$$

and the coordinates (T, ψ) are related to $(\bar{t}, \bar{r})$ via

$$T = \arctan(\bar{t} + \bar{r}) + \arctan(\bar{t} - \bar{r}), \qquad \psi = \arctan(\bar{t} + \bar{r}) - \arctan(\bar{t} - \bar{r}).$$

Equivalently, in terms of the original coordinates t and χ one has

$$\chi = \arctan\left(\frac{\sin\psi}{\sin T}\right), \qquad t = \sqrt{\frac{\cos\psi - \cos T}{\cos\psi + \cos T}}.$$

Therefore, the Milne Universe is conformal to the domain

$$\tilde{\mathring{\mathcal{M}}} = \left\{p \in \mathcal{M}_E | \; 0 \leq \psi < \pi, \;\; \psi - \pi < T < \pi - \psi, \;\; |T| > \psi\right\}.$$

4.2 The Milne Universe as a solution to the wave equations of Proposition 1

Since the Milne Universe is a solution to the the Einstein field equations, it follows that the pair $(\mathring{\boldsymbol{g}}, \mathring{\Xi})$ implies a solution to the conformal Einstein field equations which, in turn, constitutes a solution to the wave equations of Proposition 1. Following the discussion of Sections 2.1.2 and 2.1.4, this solution consists of the frame fields

$$\{e_{\boldsymbol{a}}{}^{\boldsymbol{c}}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, L_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \Sigma_{\boldsymbol{a}}, \Xi, s\}$$

or, equivalently, the spinorial fields

$$\{e_{\boldsymbol{AA'}}{}^{\boldsymbol{c}}, \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}, \Phi_{\boldsymbol{AA'BB'}}, \phi_{\boldsymbol{ABCD}}, \Sigma_{\boldsymbol{AA'}}, \Sigma, \Xi, s\}$$

where one writes $\Sigma_{\boldsymbol{a}} \equiv \nabla_{\boldsymbol{a}}\Xi$ and $\nabla_{\boldsymbol{AA'}}\Xi \equiv \Sigma_{\boldsymbol{AA'}}$ as a shorthand for the derivative of the conformal factor.

For later use, notice that in the Einstein Cosmos $(\mathcal{M}_E, \mathring{\boldsymbol{g}})$ one has

$$\mathbf{Weyl}[\mathring{\boldsymbol{g}}] = 0, \qquad \mathbf{R}[\mathring{\boldsymbol{g}}] = -6, \qquad \mathbf{Schouten}[\mathring{\boldsymbol{g}}] = \tfrac{1}{2}\left(\mathbf{d}T \otimes \mathbf{d}T + \hbar\right).$$

The spinorial version of the above tensors can be more easily expressed in terms of a frame. To this end, now consider the class of geodesics on the Einstein Cosmos $(\mathcal{M}_E, \mathring{\boldsymbol{g}})$ given by

$$x(\tau) = (\tau, x_\star), \qquad \tau \in \mathbb{R},$$

where $x_\star \in \mathbb{S}^3$ is fixed. Using the congruence of geodesics generated by varying $x_\star$ over $\mathbb{S}^3$ one obtains a Gaussian system of coordinates (τ, x^α) on the Einstein cylinder $\mathbb{R}\times\mathbb{S}^3$ where (x^α) are some local coordinates on $\mathbb{S}^3$. In addition, in a slight abuse of notation *the standard time coordinate T on the Einstein cylinder is identified with the parameter τ of the geodesic.*

4.2.1 Frame expressions

A globally defined orthonormal frame on the Einstein Cosmos $(\mathcal{M}_E, \mathring{\boldsymbol{g}})$ can be constructed by first considering the linearly independent vector fields in $\mathbb{R}^4$

$$\begin{aligned}
\boldsymbol{c}_1 &\equiv w\frac{\partial}{\partial z} - z\frac{\partial}{\partial w} + x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\\
\boldsymbol{c}_2 &\equiv w\frac{\partial}{\partial y} - y\frac{\partial}{\partial w} + z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\\
\boldsymbol{c}_3 &\equiv w\frac{\partial}{\partial x} - x\frac{\partial}{\partial w} + y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},
\end{aligned}$$

where (w, x, y, z) are Cartesian coordinates in $\mathbb{R}^4$. The vectors $\{\boldsymbol{c_i}\}$ are tangent to $\mathbb{S}^3$ and form a global frame for $\mathbb{S}^3$ —see e.g. [8]. This spatial frame can be extended to a spacetime frame $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\}$ by setting $\mathring{\boldsymbol{e}}_{\boldsymbol{0}} \equiv \boldsymbol{\partial}_\tau$ and $\mathring{\boldsymbol{e}}_{\boldsymbol{i}} \equiv \boldsymbol{c_i}$. Using this notation one observes that the components of the basis respect to this frame are given by $\mathring{\boldsymbol{e}}_{\boldsymbol{a}} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}\boldsymbol{c_b} \equiv \mathring{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}}\boldsymbol{c_b}$. Respect to this orthogonal basis the components of the Schouten tensor read

$$\mathring{L}_{\boldsymbol{ab}} = \delta_{\boldsymbol{a}}{}^0\delta_{\boldsymbol{a}}{}^0 - \frac{1}{2}\eta_{\boldsymbol{ab}}.$$

so that the components of the traceless Ricci tensor are given by

$$\mathring{R}_{\{\boldsymbol{ab}\}} = 2\delta_{\boldsymbol{a}}{}^{0}\delta_{\boldsymbol{b}}{}^{0} - \frac{1}{2}\eta_{\boldsymbol{ab}}$$

where the curly bracket around the indices denote the symmetric trace-free part of the tensor. In addition,

$$\mathring{d}_{\boldsymbol{abcd}} = 0$$

since the Weyl tensor vanishes.

Let $\mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$ denote the connection coefficients of the Levi-Civita connection $\boldsymbol{D}$ of $\hbar$ with respect to the spatial frame $\{\boldsymbol{c_i}\}$. Observe that the structure coefficients defined by $[\boldsymbol{c_i}, \boldsymbol{c_j}] = C_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}\boldsymbol{c_k}$ are given by $C_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} = 2\epsilon_{\boldsymbol{ij}}{}^{\boldsymbol{k}}$, and, consequently

$$\mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} = -\epsilon_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$$

where $\epsilon_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$ is the 3-dimensional Levi-Civita totally antisymmetric tensor. Taking into account that $\mathring{\boldsymbol{e}}_{\boldsymbol{0}} = \boldsymbol{\partial}_\tau$ is a timelike Killing vector of $\mathring{\boldsymbol{g}}$, one can readily obtain the connection coefficients $\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$, of the Levi-Civita connection $\mathring{\boldsymbol{\nabla}}$ of the metric $\mathring{\boldsymbol{g}}$, with respect to the basis $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\}$. More precisely, one has that

$$\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = -\epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}.$$

For the conformal factor and its concomitants one readily obtains

$$\mathring{\Sigma} \equiv \mathring{\Sigma}_{\boldsymbol{0}} = -\sin\tau, \qquad \mathring{\Sigma}_{\boldsymbol{i}} = \boldsymbol{c_i}(\mathring{\Xi}), \qquad \mathring{s} = -\tfrac{1}{2}(\cos\tau + \cos\psi).$$

4.2.2 Spinorial expressions

In order to obtain the spinor frame form of the last expressions let $\tau^{AA'}$ denote the spinorial counterpart of the vector $\sqrt{2}\boldsymbol{\partial}_\tau$ so that $\tau_{AA'}\tau^{AA'} = 2$. With this choice, consider a spinor dyad $\{\epsilon_{\boldsymbol{A}}{}^{A}\} = \{o^A, \iota^A\}$ adapted to $\tau^{AA'}$ —i.e. a spinor dyad such that $\tau^{AA'}$ can be written as in equation (2.50). The spinor $\tau^{AA'}$ can be used to introduce a space spinor formalism similar to the one discussed in Section 2.4.1 of Chapter 2. One directly finds that in the present case

$$\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}} \equiv \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\mathring{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}} = \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}. \tag{4.2}$$

where $\sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}$ denotes the Infeld-van der Waerden symbols given in equations (2.9a)-(2.9b). Now, decomposing $\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}$ as in equation (2.51) and comparing with equation (4.2) one readily finds that the components of the space spinor split of the frame, $\mathring{e}^{\boldsymbol{a}}$ and $\mathring{e}_{\boldsymbol{AB}}{}^{\boldsymbol{a}}$, are given by

$$\mathring{e}^{\boldsymbol{0}} = \sqrt{2}, \qquad \mathring{e}_{\boldsymbol{AB}}{}^{\boldsymbol{0}} = 0,$$

$$\mathring{e}^{\boldsymbol{i}} = 0, \qquad \mathring{e}_{\boldsymbol{AB}}{}^{\boldsymbol{i}} = \sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}}.$$

where $\sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}}$ are the spatial Infeld-van der Waerden symbols given in equation (2.52). These expressions provide a direct way of recasting the frame expressions of Section 4.2.1 in spinorial terms. Denoting by $2\mathring{\Phi}_{\boldsymbol{AA'BB'}}$ the spinorial counterpart of $\mathring{R}_{\{\boldsymbol{ab}\}}$ one obtains

$$\mathring{\Phi}_{\boldsymbol{AA'BB'}} = \tfrac{1}{2}\sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\sigma_{\boldsymbol{BB'}}{}^{\boldsymbol{b}}\mathring{R}_{\{\boldsymbol{ab}\}} = \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{0}}\sigma_{\boldsymbol{BB'}}{}^{\boldsymbol{0}} - \tfrac{1}{4}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}.$$

From equation (2.50) one sees that $\tau_{\boldsymbol{AA'}} = \sqrt{2}\sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{0}}$. Accordingly,

$$\mathring{\Phi}_{\boldsymbol{AA'BB'}} = \tfrac{1}{2}\tau_{\boldsymbol{AA'}}\tau_{\boldsymbol{BB'}} - \tfrac{1}{4}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}.$$

To obtain the reduced spin connection coefficients one proceeds as follows: let $\mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}}$ denote the spinorial counterpart of $\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$. Since $\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = -\epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$, one can compute its spinorial counterpart by recalling the spinorial version of the volume form

$$\epsilon_{\boldsymbol{AA'BB'CC'DD'}} = \mathrm{i}(\epsilon_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}}\epsilon_{\boldsymbol{A'D'}}\epsilon_{\boldsymbol{B'C'}} - \epsilon_{\boldsymbol{AD}}\epsilon_{\boldsymbol{BC}}\epsilon_{\boldsymbol{A'C'}}\epsilon_{\boldsymbol{B'D'}}).$$

It follows then that

$$\mathring{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}} = -\tfrac{1}{\sqrt{2}}\tau^{\boldsymbol{AA'}}\epsilon_{\boldsymbol{AA'BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}} = -\tfrac{\mathrm{i}}{\sqrt{2}}(\tau^{\boldsymbol{C}}{}_{\boldsymbol{D'}}\epsilon_{\boldsymbol{BD}}\epsilon_{\boldsymbol{B'}}{}^{\boldsymbol{C'}} - \tau_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\epsilon_{\boldsymbol{B}}{}^{\boldsymbol{C}}\epsilon_{\boldsymbol{B'D'}}).$$

Combining the last expression with the definition of the reduced spin connection coefficients $\mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{B}} \equiv \tfrac{1}{2}\mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{CQ'}}{}_{\boldsymbol{BQ'}}$ one obtains

$$\mathring{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}} = -\tfrac{\mathrm{i}}{2\sqrt{2}}(\tau^{\boldsymbol{C}}{}_{\boldsymbol{Q'}}\epsilon_{\boldsymbol{BD}}\delta_{\boldsymbol{B'}}{}^{\boldsymbol{Q'}} - \tau_{\boldsymbol{D}}{}^{\boldsymbol{Q'}}\epsilon_{\boldsymbol{B'Q'}}\delta_{\boldsymbol{B}}{}^{\boldsymbol{C}}) = -\tfrac{\mathrm{i}}{2\sqrt{2}}(\epsilon_{\boldsymbol{BD}}\tau^{\boldsymbol{C}}{}_{\boldsymbol{B'}} + \tau_{\boldsymbol{DB'}}\delta_{\boldsymbol{B}}{}^{\boldsymbol{C}}).$$

Thus, one concludes that

$$\mathring{\Gamma}_{\boldsymbol{AA'BC}} = -\tfrac{\mathrm{i}}{\sqrt{2}}\epsilon_{\boldsymbol{A}(\boldsymbol{C}}\tau_{\boldsymbol{B})\boldsymbol{A'}}.$$

Finally, for the rescaled Weyl spinor one has

$$\mathring{\phi}_{\boldsymbol{ABCD}} = 0.$$

Gauge source functions for the Milne spacetime

The expressions for $\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ and $\mathring{e}_{\boldsymbol{b}}{}^{\boldsymbol{a}}$ derived in the previous sections allow to readily compute the gauge source functions associated to the conformal representation of the Milne Universe under consideration. Regarding $\mathring{e}_{\boldsymbol{b}}{}^{\boldsymbol{a}}$ as the component of a contravariant tensor one computes

$$\mathring{\nabla}^{\boldsymbol{b}}\mathring{e}_{\boldsymbol{b}}{}^{\boldsymbol{a}} = \eta^{\boldsymbol{cb}}\mathring{\nabla}_{\boldsymbol{c}}\mathring{e}_{\boldsymbol{b}}{}^{\boldsymbol{a}} = \eta^{\boldsymbol{cb}}(\mathring{\boldsymbol{e}}_{\boldsymbol{c}}(\mathring{e}_{\boldsymbol{b}}{}^{\boldsymbol{a}}) - \mathring{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\mathring{e}_{\boldsymbol{e}}{}^{\boldsymbol{a}})$$

where $\mathring{\boldsymbol{e}}_{\boldsymbol{c}} = \mathring{e}_{\boldsymbol{c}}{}^{\boldsymbol{e}}\boldsymbol{c}_{\boldsymbol{e}}$. Using that in this case $\mathring{e}_{\boldsymbol{b}}{}^{\boldsymbol{a}} = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}$, one obtains

$$\mathring{\nabla}^{\boldsymbol{b}}\mathring{e}_{\boldsymbol{b}}{}^{\boldsymbol{a}} = -\eta^{\boldsymbol{cb}}\mathring{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = \eta^{\boldsymbol{cb}}\epsilon_{\boldsymbol{0c}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0.$$

Therefore, the coordinate gauge source function vanishes. That is, one has that

$$\mathring{F}^{\boldsymbol{a}}(x) = \mathring{\nabla}^{\boldsymbol{AA'}}\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{a}} = 0.$$

The frame gauge source function can be obtained as follows

$$\begin{aligned}
\mathring{\nabla}^{\boldsymbol{a}}\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} &= \eta^{\boldsymbol{da}}\mathring{\nabla}_{\boldsymbol{d}}\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = \eta^{\boldsymbol{da}}\big(\mathring{\boldsymbol{e}}_{\boldsymbol{d}}(\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}) + \mathring{\Gamma}_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{e}}\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}} - \mathring{\Gamma}_{\boldsymbol{d}}{}^{\boldsymbol{e}}{}_{\boldsymbol{a}}\mathring{\Gamma}_{\boldsymbol{e}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} - \mathring{\Gamma}_{\boldsymbol{d}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}}\mathring{\Gamma}_{\boldsymbol{e}}{}^{\boldsymbol{b}}{}_{\boldsymbol{a}}\big)\\
&= -\eta^{\boldsymbol{da}}\mathring{\boldsymbol{e}}_{\boldsymbol{d}}(\epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}}) + \eta^{\boldsymbol{da}}\epsilon_{\boldsymbol{0d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{e}}\epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}} - \eta^{\boldsymbol{da}}\epsilon_{\boldsymbol{0d}}{}^{\boldsymbol{e}}{}_{\boldsymbol{a}}\epsilon_{\boldsymbol{0e}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} - \eta^{\boldsymbol{da}}\epsilon_{\boldsymbol{0d}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}}\epsilon_{\boldsymbol{0e}}{}^{\boldsymbol{b}}{}_{\boldsymbol{a}}\\
&= \epsilon_{\boldsymbol{0}}{}^{\boldsymbol{ab}}{}_{\boldsymbol{e}}\epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{c}} - \epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{e}}\epsilon_{\boldsymbol{0}}{}^{\boldsymbol{ae}}{}_{\boldsymbol{c}} = 0.
\end{aligned}$$

Therefore, using the irreducible decomposition of $\mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}}$ in terms of $\mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$ given in (2.12a), one concludes that

$$\mathring{F}_{\boldsymbol{AB}}(x) = \nabla^{\boldsymbol{QQ'}}\mathring{\Gamma}_{\boldsymbol{QQ'AB}} = 0.$$

Finally, the conformal gauge source function is determined by the value of the Ricci scalar, in this case $R = -6$. It follows then that

$$\Lambda = \tfrac{1}{4}.$$

Summary

The main results of this section are collected in the following proposition:

Proposition 3. *The fields* $(\mathring{\Xi}, \mathring{\Sigma}, \mathring{\Sigma}_{\boldsymbol{i}}, \mathring{s}, \mathring{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}}, \mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \mathring{L}_{\boldsymbol{ab}}, \mathring{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}})$ *given by*

$$\mathring{\Xi} = \cos\tau + \cos\psi, \qquad \mathring{\Sigma} = -\sin\tau, \qquad \mathring{\Sigma}_{\boldsymbol{i}} = \boldsymbol{c}_{\boldsymbol{i}}(\mathring{\Xi}), \qquad \mathring{e}_{\boldsymbol{a}}{}^{\boldsymbol{b}} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}, \qquad \mathring{d}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = 0,$$
$$\mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = -\epsilon_{\boldsymbol{0a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \qquad \mathring{L}_{\boldsymbol{ab}} = 2\delta_{\boldsymbol{a}}{}^{\boldsymbol{0}}\delta_{\boldsymbol{b}}{}^{\boldsymbol{0}} - \tfrac{1}{2}\eta_{\boldsymbol{ab}}, \qquad \mathring{s} = -\tfrac{1}{2}(\cos\tau + \cos\psi),$$

or, alternatively, in spinorial terms, the fields

$$(\mathring{\Xi}, \mathring{\Sigma}_{\boldsymbol{AA'}}, \mathring{s}, \mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}, \mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}, \mathring{\Phi}_{\boldsymbol{AA'BB'}}, \mathring{\phi}_{\boldsymbol{ABCD}})$$

with $\mathring{\Xi}$ *and* $\mathring{s}$ *as above and*

$$\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}} = \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}, \qquad \mathring{\Gamma}_{\boldsymbol{AA'BC}} = -\tfrac{\mathrm{i}}{\sqrt{2}}\epsilon_{\boldsymbol{A(B}}\tau_{\boldsymbol{C)A'}}, \qquad \mathring{\phi}_{\boldsymbol{ABCD}} = 0,$$

$$\mathring{\Phi}_{\boldsymbol{AA'BB'}} = \tfrac{1}{2}\tau_{\boldsymbol{AA'}}\tau_{\boldsymbol{BB'}} - \tfrac{1}{4}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{AC}}, \qquad \mathring{\Sigma}_{\boldsymbol{AA'}} = \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{a}}\boldsymbol{c_a}(\mathring{\Xi}),$$

defined on the Einstein cylinder $\mathbb{R}\times\mathbb{S}^3$ constitute a solution to the conformal Einstein field equations representing the Milne Universe. The gauge source functions associated to this representation are given by

$$\mathring{F}^{\boldsymbol{a}}(x) = 0, \qquad \mathring{F}_{\boldsymbol{AB}}(x) = 0, \qquad \mathring{\Lambda} = \tfrac{1}{4}.$$

The solution is smooth on $\tilde{\mathring{\mathcal{M}}}$ as defined Section 4.1. Moreover, one can smoothly extend this solution to the Einstein cylinder $\mathbb{R}\times\mathbb{S}^3$.

Initial data for the Milne spacetime

The expressions in Proposition 3 readily imply initial data for the wave equations (3.19a)-(3.19f) on the *hyperboloids*

$$\mathcal{H} \equiv \big\{p\in\mathbb{R}\times\mathbb{S}^3 \mid \tau(p) = \tau_\star,\ \cos\tau_\star + \cos\psi(p) \geq 0\big\}, \qquad \tau_\star \in [\tfrac{1}{2}\pi, \pi). \tag{4.3}$$

By construction, the development of this data is (a portion of) the Milne Universe.

In what follows, for simplicity, the discussion will be restricted to the initial data on the so-called *standard hyperboloid*

$$\mathcal{H}_\star \equiv \big\{p\in\mathbb{R}\times\mathbb{S}^3 \mid \tau(p) = \tfrac{1}{2}\pi,\ \cos\psi(p) \geq 0\big\}.$$

Nevertheless, this analysis can be readily extended to any of the hyperboloids $\mathcal{H}$ defined in expression (4.3). The intersection of the standard hyperboloid with the conformal boundary $\mathscr{I}^+$ is given by

$$\mathcal{Z}_\star \equiv \big\{p\in\mathcal{H}_\star \mid \cos\psi(p) = 0\big\}.$$

Restricting the expressions of Proposition 3 to $\mathcal{H}_\star$, one obtains the following initial data for the wave equations of Proposition 1:

$$\begin{gathered}
\mathring{\Xi}|_{\mathcal{H}_\star} = \cos\psi, \quad \mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}|_{\mathcal{H}_\star} = \sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}, \quad \mathring{\Gamma}_{\boldsymbol{AA'BC}}|_{\mathcal{H}_\star} = -\tfrac{\mathrm{i}}{\sqrt{2}}\epsilon_{\boldsymbol{A}(\boldsymbol{B}}\tau_{\boldsymbol{C})\boldsymbol{A'}}, \\
\mathring{\Phi}_{\boldsymbol{AA'BB'}}|_{\mathcal{H}_\star} = \tfrac{1}{2}\tau_{\boldsymbol{AA'}}\tau_{\boldsymbol{BB'}} - \tfrac{1}{4}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{AC}}, \quad \Sigma_{\boldsymbol{AA'}}|_{\mathcal{H}_\star} = \sigma_{\boldsymbol{AA}}{}^{\boldsymbol{a}}\boldsymbol{c_a}(\mathring{\Xi})|_{\mathcal{H}_\star}, \\
\mathring{s}|_{\mathcal{H}_\star} = -\tfrac{1}{2}\cos\psi, \quad \mathcal{P}\mathring{\Xi}|_{\mathcal{H}_\star} = \Sigma|_{\mathcal{H}_\star} = -\tfrac{1}{2}, \quad \mathcal{P}\mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}|_{\mathcal{H}_\star} = 0, \quad \mathring{\phi}_{\boldsymbol{ABCD}}|_{\mathcal{H}_\star} = 0, \\
\mathcal{P}\mathring{\Gamma}_{\boldsymbol{AA'BC}}|_{\mathcal{H}_\star} = 0, \quad \mathcal{P}\mathring{\Phi}_{\boldsymbol{AA'BB'}}|_{\mathcal{H}_\star} = 0, \\
\mathcal{P}\mathring{\phi}_{\boldsymbol{ABCD}}|_{\mathcal{H}_\star} = 0, \quad \mathcal{P}\mathring{s}|_{\mathcal{H}_\star} = \tfrac{1}{2}\sin\tau_\star.
\end{gathered}$$

Observe that the above data is, in fact, smooth on the whole of the hypersurface

$$\mathcal{S}_\star \equiv \big\{p\in\mathbb{R}\times\mathbb{S}^3 \mid \tau(p) = \tfrac{1}{2}\pi\big\} \supset \mathcal{H}_\star$$

of the Einstein cylinder. In what follows, the set $\mathcal{S}_\star$ will be the paradigmatical hypersurface on which initial data for the wave equations (3.19a)-(3.19f) will be prescribed.

4.2.3 Perturbation of initial data

To discuss the stability of the Milne Universe it is necessary to parametrise perturbations of initial data close to the data for the exact solution. To do so, consider a basic initial data set for the conformal field equations in vacuum and vanishing Cosmological constant, namely a collection $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \Omega, \Sigma)$ —here $\boldsymbol{K}$ is used instead of $\boldsymbol{\chi}$ to avoid confusion with the coordinate χ in equation (4.1)— satisfying (2.76a)-(2.76b) with $\lambda = 0$.

Hyperboloidal initial data

Motivated by the prototypical example of the Milne Universe, in what follows solutions to the conformal constraint equations (2.76a)-(2.76b) corresponding to hyperboloidal data will be considered. A solution $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \Omega, \Sigma)$ to equations (2.76a)-(2.76b) will be said to be *hyperboloidal data* if there exists a 3-dimensional manifold $\mathcal{H} \subset \mathcal{S}$ with boundary $\mathscr{Z}$, such that

$$\Omega > 0 \quad \text{on} \quad \text{int}\,\mathcal{H},$$
$$\Omega = 0, \qquad \mathbf{d}\Omega \neq 0, \qquad \Sigma < 0, \quad \text{on} \quad \mathscr{Z} \equiv \partial\mathcal{H} \approx \mathbb{S}^2.$$

The construction of hyperboloidal initial data has been analysed in [47, 48]. To discuss this point in more detail recall from Section 2.5 of Chapter 2 that a physical initial data set $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ is related to an unphysical data set $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \Omega, \Sigma)$ via

$$h_{ij} = \Omega^2 \tilde{h}_{ij}, \quad K_{ij} = \Omega(\tilde{K}_{ij} + \Sigma \tilde{h}_{ij}),$$

where $\Sigma = n^a \nabla_a \Omega$ and $n_a = \Omega \tilde{n}_a$ denote the $\boldsymbol{g}$-unit and $\tilde{\boldsymbol{g}}$-unit normals to the initial hypersruface. Now, consider initial data sets for which the physical second fundamental form is pure trace:

$$\tilde{K}_{ij} = \frac{1}{3}\tilde{K}\tilde{h}_{ij}.$$

Then, as a consequence of the momentum constraint (2.75b), $\tilde{K}$ is constant and the problem reduces to the analysis of the the Hamiltonian constraint. Observe that, in this case, equation (2.76a) reduces to

$$4\Omega D_i D^i \Omega - 6 D_i \Omega D^i \Omega + 2\Omega^2 r = \tilde{K}^2. \tag{4.4}$$

Let ϱ be a smooth function on $\mathcal{S}$ such that

$$\varrho|_{\partial\mathcal{S}} = 0, \qquad \mathbf{d}\varrho|_{\partial\mathcal{S}} \neq 0.$$

The function ϱ is regarded as a *boundary defining function.* Consider the Ansatz $\Omega = \rho\vartheta^{-2}$ with $\vartheta > 0$ on $\mathcal{S}$. With this Ansatz, equation (4.4) implies an elliptic equation for ϑ which is singular at $\partial\mathcal{S}$. This equation has been analysed in [47]. The conclusion of such analysis is the content of the following

Lemma 12. *Let $(\mathcal{S}, \boldsymbol{h})$ be a smooth Riemannian manifold with boundary $\partial\mathcal{S}$. Then, there exist a unique positive solution ϑ to the equation implied by (4.4) with $\Omega = \rho\vartheta^{-2}$. Moreover, the following are equivalent*

(i) The function ϑ and the tensors

$$L_{ij} = -\frac{1}{\Omega} D_{\{i}D_{j\}}\Omega + \frac{1}{12}(r + \frac{2}{3}K^2)h_{ij} \tag{4.5}$$

$$d_{ij} = \frac{1}{\Omega^2} D_{\{i}D_{j\}}\Omega + \frac{1}{\Omega} r_{\{ij\}} \tag{4.6}$$

determined on $\tilde{\mathcal{S}}$ by $\boldsymbol{h}$ and $\Omega = \rho\vartheta^{-2}$ extend smoothly to $\mathcal{S}$.

(ii) The Weyl tensor $C^a{}_{bcd}$ computed from data on $\mathcal{S}$ vanishes on $\partial\mathcal{S}$.

(iii) The conformal class $[\boldsymbol{h}]$ is such that the extrinsic curvature of $\partial\mathcal{S}$ with respect to its embedding in $(\mathcal{S}, \boldsymbol{h})$ is pure trace.

The expressions for L_{ij} and d_{ij} in Lemma 12 correspond to the spatial part of the (4-dimensional) Schouten tensor L_{ab} and electric part of the rescaled Weyl tensor $d^a{}_{bcd}$ as determined by the conformal constraint equations (2.74a)-(2.74j). The latter theorem has been extended to include more general forms of physical second fundamental forms $\tilde{K}_{ij}$ in [48, 49].

Remark 21. Despite the fact that the Milne spacetime corresponds to the region of the Einstein cylinder denoted by $\tilde{\mathring{\mathcal{M}}}$ —as given in Section 4.1— one can extend the fields describing the Milne solution in Proposition 3 to the whole Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$. In particular, the coordinate τ can be extended to $\tau \in \mathbb{R}$. Additionally, although the standard hyperboloid $\mathcal{H}_\star$ on the Milne spacetime —see expression 4.3— is completely contained in a hemisphere of $\mathbb{S}^3$ one can extend the data on $\mathcal{H}_\star$ to data on all $\mathbb{S}^3$ —see Remark 22. Observe that, in the case of the exact Milne spacetime, the 3-metric in the initial data corresponds to the standard metric on $\mathbb{S}^3$.

Motivated by observations in Remark 21, to study perturbations of exact Milne data, we will consider 3-manifolds $\mathcal{S}$ which are topologically (but not metrically) $\mathbb{S}^3$ —that is, $\mathcal{S} \approx \mathbb{S}^3$. Accordingly, one considers a diffeomorphism $\psi : \mathcal{S} \to \mathbb{S}^3$. This diffeomorphism and its inverse can be used to pull-back coordinates (x^α), the frame fields $\{\boldsymbol{c_i}\}$ and associated coframe fields $\{\boldsymbol{\alpha^i}\}$ on $\mathbb{S}^3$ to fields on $\mathcal{S}$. In a slight abuse of notation the coordinates, vector and covector fields on $\mathcal{S}$ are denoted again by (x^α), $\{\boldsymbol{c_i}\}$ and $\{\boldsymbol{\alpha^i}\}$. Observe that while $\{\boldsymbol{c_i}\}$ are orthonormal with respect to the standard metric of $\mathbb{S}^3$, they will not be orthonormal, in general, with respect to the metric $\boldsymbol{h}$ obtained from the solution to the conformal constraint equations (2.76a)-(2.76b).

The inherent freedom in the choice of the diffeomorphism ψ can be exploited to make ψ a harmonic map and the correspondence between coordinates of $\mathbb{S}^3$ and $\mathcal{S}$ the identity —see [17] for further details on this construction.

Parametrising the initial data

Assume one is given a hyperboloidal solution $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K}, \Omega, \Sigma)$ to the conformal constraint equations (2.76a)-(2.76b) defined on a $\mathcal{H} \subset \mathcal{S}$. Let $\{\boldsymbol{e_i}\}$ denote a $\boldsymbol{h}$-orthonormal frame over $\mathcal{S}$ and let $\{\boldsymbol{\omega^i}\}$ be the associated cobasis. Assume that there exist vector fields $\{\breve{\boldsymbol{e}}_{\boldsymbol{i}}\}$ such that an $\boldsymbol{h}$-orthonormal frame $\{\boldsymbol{e_i}\}$ is related to an $\hbar$-orthonormal frame $\boldsymbol{c_i}$ through $\boldsymbol{e_i} = \boldsymbol{c_i} + \breve{\boldsymbol{e}}_{\boldsymbol{i}}$. This last requirement is equivalent to introducing coordinates on $\mathcal{S}$ such that

$$\boldsymbol{h} = \mathring{\boldsymbol{h}} + \breve{\boldsymbol{h}} = \hbar + \breve{\boldsymbol{h}}. \tag{4.7}$$

Notice that the notation $\mathring{}$ is used to denote the value in the exact (background) solution while $\breve{}$ is used to denote the perturbation.

To measure the size of the perturbed initial data, one introduces Sobolev norms defined for any spinor quantity $N_{\mathcal{K}}$ with $_{\mathcal{K}}$ being an arbitrary string of frame spinor indices, as

$$||N_{\mathcal{K}}||_{\mathcal{H},m} \equiv \sum_{\mathcal{K}} ||N_{\mathcal{K}}||_{\mathcal{H},m}$$

where $\sum_{\mathcal{K}}$ is the sum over all the frame spinor indices encoded in $_{\mathcal{K}}$ and

$$||N_{\mathcal{K}}||_{\mathcal{H},m} = \left(\sum_{l=0}^{m} \sum_{\alpha_1,\ldots,\alpha_l}^{3} \int_{\mathcal{H}} (\partial_{\alpha_1} \ldots \partial_{\alpha_l} N_{\mathcal{K}})^2 \mathrm{d}\mu \right)^{1/2}.$$

where $\mathrm{d}\mu$ denotes the volume element associated to the standard metric on $\mathbb{S}^3$. Observe that since the indices in $_{\mathcal{K}}$ are frame indices, the quantities $N_{\mathcal{K}}$ are scalars. Notice, also that as $\mathcal{S} \approx \mathbb{S}^3$, then $\mathcal{H}$ can be regarded as a region of $\mathbb{S}^3$. Consistent with the split (4.7), one makes use of the above expressions to consider perturbations of the initial data for the Milne Universe on the standard hyperboloid $\mathcal{H}_\star$ of the form

$$\Xi|_{\mathcal{H}} = \mathring{\Xi}|_{\mathcal{H}} + \breve{\Xi}|_{\mathcal{H}}, \qquad e_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}|_{\mathcal{H}} = \mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}|_{\mathcal{H}} + \breve{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}|_{\mathcal{H}},$$
$$\Gamma_{\boldsymbol{AA'BC}}|_{\mathcal{H}} = \mathring{\Gamma}_{\boldsymbol{AA'BC}}|_{\mathcal{H}} + \breve{\Gamma}_{\boldsymbol{AA'BC}}|_{\mathcal{H}}, \qquad \Phi_{\boldsymbol{AA'BB'}}|_{\mathcal{H}} = \mathring{\Phi}_{\boldsymbol{AA'BB'}}|_{\mathcal{H}} + \breve{\Phi}_{\boldsymbol{AA'BB'}}|_{\mathcal{H}},$$
$$\phi_{\boldsymbol{ABCD}}|_{\mathcal{H}} = \breve{\phi}_{\boldsymbol{ABCD}}|_{\mathcal{H}}, \qquad s|_{\mathcal{H}} = \mathring{s}|_{\mathcal{H}} + \breve{s}|_{\mathcal{H}},$$

$$\Sigma_{\boldsymbol{AA'}}|_{\mathcal{H}} = \mathring{\Sigma}_{\boldsymbol{AA'}}|_{\mathcal{H}} + \breve{\Sigma}_{\boldsymbol{AA'}}|_{\mathcal{H}}.$$

together with

$$\Sigma|_{\mathcal{H}} = \mathring{\Sigma}|_{\mathcal{H}} + \breve{\Sigma}|_{\mathcal{H}}, \qquad \mathcal{P}e_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}|_{\mathcal{H}} = \mathcal{P}\breve{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}}|_{\mathcal{H}},$$
$$\mathcal{P}\Gamma_{\boldsymbol{AA'BC}}|_{\mathcal{H}} = \mathcal{P}\breve{\Gamma}_{\boldsymbol{AA'BC}}|_{\mathcal{H}}, \qquad \mathcal{P}\Phi_{\boldsymbol{AA'BB'}}|_{\mathcal{H}} = \mathcal{P}\breve{\Phi}_{\boldsymbol{AA'BB'}}|_{\mathcal{H}}$$
$$\mathcal{P}\phi_{\boldsymbol{ABCD}}|_{\mathcal{H}} = \mathcal{P}\breve{\phi}_{\boldsymbol{ABCD}}|_{\mathcal{H}}, \qquad \mathcal{P}s|_{\mathcal{H}} = \mathcal{P}\mathring{s}|_{\mathcal{H}} + \mathcal{P}\breve{s}|_{\mathcal{H}}.$$

Recall from the discussion of the space spinor formalism given in Section 2.4.1 of Chapter 2 that $\mathcal{P} \equiv \tau^{\boldsymbol{AA'}}\nabla_{\boldsymbol{AA'}} = \sqrt{2}\boldsymbol{\partial}_\tau$. Additionally observe that, by assumption, the above fields are solutions to the equations implied by the initial data for the subsidiary system given in Proposition 2. Thus, in particular, they ensure that the initial data for the subsidiary equations vanish. The above data will be collectively denoted by

$$\mathbf{w}_\star \equiv (\mathbf{u}_\star, \partial_\tau \mathbf{u}_\star).$$

The parametrisation into background and perturbed parts will be written as

$$\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star, \qquad \partial_\tau \mathbf{u}_\star = \partial_\tau \mathring{\mathbf{u}}_\star + \partial_\tau \breve{\mathbf{u}}_\star.$$

The perturbation part of the initial data $(\breve{\mathbf{u}}_\star, \partial_\tau \breve{\mathbf{u}}_\star)$ is only defined in the region $\mathcal{H}$ of $\mathcal{S}$. To apply the theory of quasilinear wave equations as described in Appendix 4.4, one needs data on the whole of $\mathcal{S} \approx \mathbb{S}^3$. The initial data can be extended invoking the *Extension Theorem* which states that there exists a linear operator

$$\mathcal{E} : H^m(\mathcal{H}, \mathbb{C}^N) \to H^m(\mathcal{S}, \mathbb{C}^N)$$

such that if $\mathbf{w}_\star \in H^m(\mathcal{H}, \mathbb{C}^N)$ then $\mathcal{E}\mathbf{w}_\star(x) = \mathbf{w}_\star(x)$ almost everywhere in $\mathcal{H}$ and

$$||\mathcal{E}\mathbf{w}_\star||_{m,\mathcal{S}} \leq K||\mathbf{w}_\star||_{m,\mathcal{H}}$$

where K is a universal constant for fixed m —see e.g. [50]. Hence, using equation (4.12), one can make $||\mathcal{E}\mathbf{w}_\star||_{m,\mathcal{S}}$ small as necessary by making $||\mathbf{w}_\star||_{m,\mathcal{H}}$ small —that is, the size of the extended data is controlled by the data on the initial hypersurface $\mathcal{H}$. Therefore, the extended data will be given by

$$\mathcal{E}\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \mathcal{E}\breve{\mathbf{u}}_\star \qquad \mathcal{E}\partial_\tau \mathbf{u}_\star = \partial_\tau \mathring{\mathbf{u}}_\star + \mathcal{E}\partial_\tau \breve{\mathbf{u}}_\star$$

which are well defined on $H^m(\mathcal{S}, \mathbb{C}^N)$. Therefore, if one assumes that

$$||\breve{\mathbf{u}}_\star||_{\mathcal{S},m} + ||\partial_\tau \breve{\mathbf{u}}_\star||_{\mathcal{S},m} < \varepsilon.$$

then
$$||\mathcal{E}\breve{\mathbf{u}}_\star||_{\mathcal{H},m} + ||\mathcal{E}\partial_\tau\breve{\mathbf{u}}_\star||_{\mathcal{H},m} \leq K\varepsilon.$$

Remark 22. The fact that the extension of the data obtained in the previous paragraph is not unique and it does not necessarily satisfy the constraints of Proposition 11 is not a problem in this analysis since

$$D^+(\mathcal{H}) \cap I^+(\mathcal{S}\backslash\mathcal{H}) = \emptyset$$

where D^+ denotes the domain of dependence and I^+ the chronological future of achronal sets —see [30] for a detailed discussion of these causal concepts. The proof of the last statement follows by contradiction. Let $q \in D^+(\mathcal{H}) \cap I^+(\mathcal{S}\backslash\mathcal{H})$. Then, on the one hand, one has that $q \in I^+(\mathcal{S}\backslash\mathcal{H})$, so that it follows that there exists a future timelike curve γ from $p \in \mathcal{S}\backslash\mathcal{H}$ to q. On the other hand $q \in D^+(\mathcal{H})$ which means that every past in extendible causal curve through q intersects $\mathcal{H}$, therefore $p \in \mathcal{H}$. This is a contradiction since $p \in \mathcal{S}\backslash\mathcal{H}$.

4.2.4 Construction of perturbed solutions

In this section a discussion of the construction of solutions to the wave equations (3.19a)-(3.19f) describing non-linear perturbations of the Milne Universe is provided.

Consistent with the discussion of the previous subsection, the unknowns in the wave equation (3.19a)-(3.19f) will be split into a background and a perturbation part. More precisely one writes

$$\Xi = \mathring{\Xi} + \breve{\Xi}, \qquad \Sigma_{\boldsymbol{AA'}} = \mathring{\Sigma}_{\boldsymbol{AA'}} + \breve{\Sigma}_{\boldsymbol{AA'}}, \qquad e_{\boldsymbol{AA'}}{}^{\boldsymbol{b}} = \mathring{e}_{\boldsymbol{AA'}}{}^{\boldsymbol{b}} + \breve{e}_{\boldsymbol{aA'}}{}^{\boldsymbol{b}}, \qquad s = \mathring{s} + \breve{s},$$
$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} = \mathring{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} + \breve{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}, \qquad \Phi_{\boldsymbol{AA'BB'}} = \mathring{\Phi}_{\boldsymbol{AA'BB'}} + \breve{\Phi}_{\boldsymbol{AA'BB'}},$$
$$\phi_{\boldsymbol{ABCD}} = \breve{\phi}_{\boldsymbol{ABCD}}.$$

Following the notation used in Remark 16, the independent components of the unknowns as a single vector-valued variable are collected in a vector-valued unknown $\mathbf{u}$. Consistent with this notation one has

$$\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}.$$

The components of the contravariant metric tensor $g^{\mu\nu}(x, \mathbf{u})$ in the vector-valued wave equation (3.20) can be written as the metric for the background solution $\mathring{\mathbf{u}}$ plus a term depending on the unknown $\mathbf{u}$

$$g^{\mu\nu}(x; \mathbf{u}) = \mathring{g}^{\mu\nu}(x; \mathring{\mathbf{u}}) + \breve{g}^{\mu\nu}(x; \mathbf{u}). \tag{4.8}$$

The latter can be expressed, alternatively, in spinorial terms as

$$g^{\mu\nu}(x;\mathbf{u}) = \epsilon^{\boldsymbol{AA'}}\epsilon^{\boldsymbol{BB'}}\boldsymbol{e}_{\boldsymbol{AA'}}{}^{\mu}\boldsymbol{e}_{\boldsymbol{BB'}}{}^{\nu} = \epsilon^{\boldsymbol{AA'}}\epsilon^{\boldsymbol{BB'}}(\mathring{\boldsymbol{e}}_{\boldsymbol{AA'}}{}^{\mu}\mathring{\boldsymbol{e}}_{\boldsymbol{BB'}}{}^{\nu} + \breve{e}_{\boldsymbol{AA'}}{}^{\mu}\breve{e}_{\boldsymbol{BB'}}{}^{\nu}). \quad (4.9)$$

Substituting the split (4.8) into equation (3.20) one obtains,

$$(g^{\mu\nu}(x;\mathring{\mathbf{u}}) + \breve{g}^{\mu\nu}(x;\mathbf{u}))\partial_\mu\partial_\nu(\mathring{\mathbf{u}} + \breve{\mathbf{u}}) + \boldsymbol{F}(x;\mathbf{u},\partial\mathbf{u}) = 0.$$

Noticing that $\mathring{\mathbf{u}}$ is, in fact, a solution to

$$\mathring{g}^{\mu\nu}(x;\mathring{\mathbf{u}})\partial_\mu\partial_\nu\mathring{\mathbf{u}} + \boldsymbol{F}(x;\mathring{\mathbf{u}},\partial\mathring{\mathbf{u}}) = 0,$$

it follows then that

$$\mathring{g}^{\mu\nu}(x;\mathring{\mathbf{u}})\partial_\mu\partial_\nu\breve{\mathbf{u}} + \breve{g}^{\mu\nu}(x;\mathbf{u})\partial_\mu\partial_\nu\mathring{\mathbf{u}} + \breve{g}^{\mu\nu}(x;\mathbf{u})\partial_\mu\partial_\nu\breve{\mathbf{u}} + \boldsymbol{F}(x;\mathbf{u},\partial\mathbf{u}) - \boldsymbol{F}(x;\mathring{\mathbf{u}},\partial\mathring{\mathbf{u}}) = 0.$$

Finally, since the background solution $\mathring{\mathbf{u}}$ is known then the last equation can be recast as

$$(\mathring{g}^{\mu\nu}(x) + \breve{g}^{\mu\nu}(x;\breve{\mathbf{u}}))\partial_\mu\partial_\nu\breve{\mathbf{u}} = \boldsymbol{F}(x;\breve{\mathbf{u}},\partial\breve{\mathbf{u}}). \quad (4.10)$$

The above equation is in a form where the local existence and Cauchy stability theory of quasilinear wave equations as given in, say, [21] can be applied. Notice that $\mathring{\boldsymbol{g}}(x)$ is Lorentzian since it corresponds to the metric of the background solution —i.e. the metric of the Einstein Cosmos. Now, consider initial data $(\mathbf{u}_\star, \partial_t\mathbf{u}_\star)$ close enough to initial data $(\mathring{\mathbf{u}}_\star, \partial_t\mathring{\mathbf{u}}_\star)$ for the Milne Universe —that is, take

$$(\mathbf{u}_\star, \partial_t\mathbf{u}_\star) \in B_\varepsilon(\mathring{\mathbf{u}}_\star, \partial_t\mathring{\mathbf{u}}_\star), \quad (4.11)$$

where the notion of closeness is encoded in

$$B_\varepsilon(\mathbf{u}_\star,\mathbf{v}_\star) \equiv \big\{(\mathbf{w}_1,\mathbf{w}_2) \in H^m(\mathcal{S},\mathbb{C}^N)\times H^m(\mathcal{S},\mathbb{C}^N) \mid \ \parallel \mathbf{w}_1 - \mathbf{u}_\star \parallel_{\mathcal{S},m} + \parallel \mathbf{w}_2 - \mathbf{v}_\star \parallel_{\mathcal{S},m} \leq \varepsilon\big\}.$$

Additionally, given $\delta > 0$ define

$$D_\delta \equiv \big\{(\mathbf{w}_1,\mathbf{w}_2) \in H^m(\mathcal{S},\mathbb{C}^N) \times H^m(\mathcal{S},\mathbb{C}^N) \mid \delta < |\det g_{\mu\nu}(\mathbf{w}_1)|\big\}.$$

Using that $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$, the requirement (4.11) is equivalent to say that the initial data for the perturbation is small in the sense that

$$||\breve{\mathbf{u}}_\star||_{\mathcal{S},m} + ||\partial_\tau\breve{\mathbf{u}}_\star||_{\mathcal{S},m} < \varepsilon. \quad (4.12)$$

With this remark in mind and recalling that $\mathring{\mathbf{u}}$ is explicitly known, observe that from equation (4.8) it follows that

$$g^{\mu\nu}(x;\mathbf{u}) = \mathring{g}^{\mu\nu}(x) + \breve{g}^{\mu\nu}(x;\breve{\mathbf{u}}).$$

Since the variable $\breve{\mathbf{u}}_\star$ is a vector-valued function collecting the independent components of the conformal fields, and in particular $(\breve{e}_{\boldsymbol{AA}'}{}^{\mu})_\star$, it follows that for sufficiently small initial $\breve{\mathbf{u}}_\star$ the perturbation $\breve{g}^{\mu\nu}$ will be small. Therefore, choosing ε small enough one can guarantee that the metric $\mathring{g}^{\mu\nu}(x) + \breve{g}^{\mu\nu}(x;\breve{\mathbf{u}}_\star)$ is initially Lorentzian.

To state the main result of this section, it will be convenient to shift the time coordinate τ by an amount of $\frac{1}{2}\pi$, namely $\widehat{\tau} \equiv \tau - \frac{1}{2}\pi$ so that the location of the standard hyperboloid of the Milne Universe is given by $\widehat{\tau} = 0$. At this point one is now in position to make use of a local existence and Cauchy stability result adapted from [21] —see Appendix 4.4 , to establish the following theorem:

Theorem 1 (***Existence and Cauchy stability***). *Let* $(\mathbf{u}_\star, \partial_\tau \mathbf{u}_\star) = (\mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star, \partial_\tau \mathring{\mathbf{u}}_\star + \partial_\tau \breve{\mathbf{u}}_\star)$ *be hyperboloidal initial data for the conformal wave equations on an 3-dimensional manifold* $\mathcal{H}$ *where* $(\mathring{\mathbf{u}}_\star, \partial_\tau \mathring{\mathbf{u}}_\star)$ *denotes initial data for the Milne Universe. Let* $(\mathcal{E}\mathbf{u}_\star, \mathcal{E}\partial_\tau \mathbf{u}_\star)$ *denote the extension of these data to* $\mathcal{S} \approx \mathbb{S}^3$*. Then, for* $m \geq 4$ *and* $\widehat{\tau}_\bullet \geq \frac{3}{2}\pi$ *there exist an* $\varepsilon > 0$ *such that:*

(i) For $||\breve{\mathbf{u}}_\star||_{\mathcal{H},m} + ||\partial_\tau \breve{\mathbf{u}}_\star||_{\mathcal{H},m} < \varepsilon$*, there exist a unique solution* $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ *to the wave equations of Proposition 1 with a minimal existence interval* $[0, \widehat{\tau}_\bullet]$ *and* $\mathbf{u} \in C^{m-2}([0, \widehat{\tau}_\bullet] \times \mathcal{S}, \mathbb{C}^N)$.

(ii) Given a sequence $(\mathbf{u}_\star^{(n)}, \mathbf{v}_\star^{(n)}) \in B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \cap D_\delta$ *such that*

$$\parallel \mathbf{u}_\star^{(n)} - \mathbf{u}_\star \parallel_{\mathcal{S},m} \to 0, \qquad \parallel \mathbf{v}_\star^{(n)} - \mathbf{v}_\star \parallel_{\mathcal{S},m} \to 0, \qquad \textit{as} \qquad n \to \infty,$$

then for the solutions $\mathbf{u}^{(n)}$ *with* $\mathbf{u}^{(n)} = \mathbf{u}_\star^{(n)}$ *and* $\partial_t \mathbf{u}^{(n)}(0,\cdot) = \mathbf{v}_\star^{(n)}$*, it holds that*

$$\parallel \mathbf{u}^{(n)}(\tau,\cdot) - \mathbf{u}(\tau,\cdot) \parallel_{\mathcal{S},m} \to 0 \qquad \textit{as} \qquad n \to \infty$$

uniformly in $\tau \in [0, \widehat{\tau}_\bullet)$ *as* $n \to \infty$.

(iii) The solution $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ *is unique in* $D^+(\mathcal{H})$ *and implies, wherever* $\Xi \neq 0$*, a* C^{m-2} *solution to the Einstein vacuum equations with vanishing Cosmological constant.*

Proof. Points *(i)* and *(ii)* are a direct application of Theorem 2 given in Appendix 4.4. The wave equation (4.10) is of the form covered by equation (4.17): the condition ensuring that $\mathring{\boldsymbol{g}}(x) + \breve{\boldsymbol{g}}(x;\mathbf{u})$ is Lorentzian is encoded in the requirement of the perturbation for the initial data being small as discussed in Section 4.2.4; moreover, the coefficients in equation (4.10) are smooth functions of their arguments. Notice that the background conformal representation of the Milne Universe, as given in Proposition 3, is a smooth solution to the wave equations of Chapter 3 on the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$. Thus, $\mathring{\boldsymbol{u}}$ is smooth on the time coordinate $\tau \in \mathbb{R}$. In particular, the solution $\mathring{\boldsymbol{u}}$ exist up to a time $\widehat{\tau}_\bullet \geq \frac{3}{2}\pi$. The theory contained in [21] —see also Appendix 4.4— ensures then, that the perturbations $\breve{\boldsymbol{u}}$ have the same existence time $\widehat{\tau}_\bullet$ as the reference solution $\mathring{\boldsymbol{u}}$.

The statement of point *(iii)* follows from the discussion of Section 3.3 for the propagation of the constraints and the subsidiary system as summarised in Proposition 1 and Lemma 11. In particular, in this section it was shown that a solution to the spinorial wave equations is a solution to the conformal Einstein field equations if initial data satisfies the appropriate conditions. As exemplified in Section 3.3.2 for the rescaled Weyl spinor, requiring the zero-quantities to vanish in the initial hypersurface renders conditions on the initial data. Finally, recall that a solution to the conformal Einstein field equations implies a solution to the Einstein field equations wherever $\Xi \neq 0$ —see [15].

□

Remark 23. The localisability property of solutions to wave equations allows to apply the methods leading to Theorem 1 to discuss the non-linear perturbations of background solutions whose spatial sections are orientable compact manifolds. For this one consider a finite cover of the base manifold. Solutions are then obtained on the Cauchy development of each of the elements of the cover. These solutions are then patched together to obtain a global in space solution. The geometric uniqueness of the setting ensures that solutions on the overlapping regions are compatible. The details of this well-known construction can be found in [3]. A discussion of the patching method for (first order) symmetric hyperbolic systems can be found in [17].

Remark 24. Point *(ii)* in the Theorem establishes the stability of the background (Milne) solution —i.e. the fact that the development of data close to Milne data will be, in a suitable sense, close to the Milne solution.

Remark 25. Observe that $\mathring{\Sigma} < 0$ for $\widehat{\tau} \in (0, \frac{1}{2}\pi)$ and $\mathring{\Sigma} > 0$ for $\widehat{\tau} \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$. Thus, recalling that $\Sigma = \mathring{\Sigma} + \breve{\Sigma}$ then $\Sigma < 0$ for $\widehat{\tau} \in (0, \frac{1}{2}\pi)$ and $\Sigma > 0$ for $\widehat{\tau} \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$ for ε small enough, therefore there is at least one point where $\Sigma = 0$. This fact will be used in the analysis of the conformal boundary given in the next section.

4.2.5 Structure of the conformal Boundary

In this section Theorem 1 is complemented by showing that the conformal boundary $\mathscr{I}$ coincides with the Cauchy horizon of $\mathcal{H}$. The argument of this section is based on analogous discussion in [8]. Since the Cauchy horizon $H(\mathcal{H}) = \partial(D^+(\mathcal{H}))$ is generated by null geodesics with endpoints on $\mathscr{Z}$ the null generators of $H(\mathcal{H})$ —i.e the null vectors tangent to $H(\mathcal{H})$— are given at $\mathscr{Z}$ by $\Sigma_{\boldsymbol{a}}|_{\mathscr{Z}}$. One defines two null vectors $(\boldsymbol{n}, \boldsymbol{l})$ on $\mathscr{Z}$ by setting

$$l_{\boldsymbol{a}\star} = \Sigma_{\boldsymbol{a}}|_{\mathscr{Z}}, \qquad \boldsymbol{n}_\star \perp \mathscr{Z}, \qquad \boldsymbol{g}(\boldsymbol{n}_\star, \boldsymbol{l}_\star) = 1 \qquad \text{on } \mathscr{Z}. \tag{4.13}$$

These pair of null vectors $\{\boldsymbol{l}_\star, \boldsymbol{n}_\star\}$, where $\boldsymbol{l}_\star$ is tangent to $H(\mathcal{H})$ on $\mathscr{Z}$ and $\boldsymbol{n}_\star$ is normal to $\mathscr{Z}$ is complemented with a pair of complex conjugate vectors $\boldsymbol{m}_\star$ and $\bar{\boldsymbol{m}}_\star$ tangent to $\mathscr{Z}$ such that $\boldsymbol{g}(\boldsymbol{m}_\star, \bar{\boldsymbol{m}}_\star) = 1$, so as to obtain the tetrad $\{\boldsymbol{l}_\star, \boldsymbol{n}_\star, \boldsymbol{m}_\star, \bar{\boldsymbol{m}}_\star\}$. In order to obtain a Newman-Penrose frame $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ off $\mathscr{Z}$ along the null generators of $H(\mathcal{H})$ one propagates them by parallel transport in the direction of $\boldsymbol{l}$ by requiring

$$l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}} l^{\boldsymbol{b}} = 0, \qquad l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}} n^{\boldsymbol{b}} = 0, \qquad l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}} m^{\boldsymbol{b}} = 0. \tag{4.14}$$

Now, suppose that one already has a solution to the conformal wave equations. Using the result of Lemma 11, one knows that the solution will also satisfy the conformal Einstein field equations. In this section the conformal Einstein field equations are employed to study the conformal boundary. From the tensorial (frame) version of the conformal Einstein field equations in vacuum as given in Section 2.1.2 of Chapter 2, one notices the subset of equations formed by equations (2.7c), (2.7d) and the definition of $\Sigma_{\boldsymbol{a}}$ as the gradient of the conformal factor:

$$\nabla_{\boldsymbol{a}}\Xi = \Sigma_{\boldsymbol{a}}, \tag{4.15a}$$
$$\nabla_{\boldsymbol{a}}\Sigma_{\boldsymbol{b}} = s g_{\boldsymbol{ab}} - \Xi L_{\boldsymbol{ab}}, \tag{4.15b}$$
$$\nabla_{\boldsymbol{a}} s = -L_{\boldsymbol{ab}}\Sigma^{\boldsymbol{b}}. \tag{4.15c}$$

Transvecting the first two equations, respectively, with $l^{\boldsymbol{a}}$, $l^{\boldsymbol{a}}l^{\boldsymbol{b}}$ and $l^{\boldsymbol{a}}m^{\boldsymbol{b}}$ renders

$$\begin{aligned}
&l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}\Xi = l^{\boldsymbol{a}}\Sigma_{\boldsymbol{a}},\\
&l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}(l^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}) = -\Xi L_{\boldsymbol{ab}} l^{\boldsymbol{a}} l^{\boldsymbol{b}},\\
&l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}(m^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}) = -\Xi L_{\boldsymbol{ab}} l^{\boldsymbol{a}} m^{\boldsymbol{b}},
\end{aligned}$$

where equation (4.14) and the fact that $\boldsymbol{l}$ is null and orthogonal to $\boldsymbol{m}$ have been used. The latter equations can be read as a system of homogeneous transport equations along the integral curves of $\boldsymbol{l}$ for a vector-valued variable containing as components Ξ , $\Sigma_{\boldsymbol{a}} l^{\boldsymbol{a}}$ and $\Sigma_{\boldsymbol{a}} m^{\boldsymbol{a}}$. Written in matricial form one has

$$\nabla_{\boldsymbol{l}} \begin{pmatrix} \Xi \\ \Sigma_{\boldsymbol{a}} l^{\boldsymbol{a}} \\ \Sigma_{\boldsymbol{a}} m^{\boldsymbol{a}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -L_{\boldsymbol{cd}} l^{\boldsymbol{c}} l^{\boldsymbol{d}} & 0 & 0 \\ -L_{\boldsymbol{cd}} l^{\boldsymbol{c}} m^{\boldsymbol{d}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \Xi \\ \Sigma_{\boldsymbol{a}} l^{\boldsymbol{a}} \\ \Sigma_{\boldsymbol{a}} m^{\boldsymbol{a}} \end{pmatrix}. \tag{4.16}$$

Observe that the column vector shown in the last equation is zero on $\mathscr{Z}$, since $\Xi|_{\mathscr{Z}}$ $=0$, $(l^{\boldsymbol{a}}\Sigma_{\boldsymbol{a}})|_{\mathscr{Z}} = (l^{\boldsymbol{a}}l_{\boldsymbol{a}})|_{\mathscr{Z}} = 0$ and $(\Sigma_{\boldsymbol{a}}m^{\boldsymbol{a}})|_{\mathscr{Z}} = (l_{\boldsymbol{a}}m^{\boldsymbol{a}})|_{\mathscr{Z}} = 0$ which follows from (4.13) and (4.14). Since equation (4.16) is homogeneous and it has vanishing initial data on $\mathscr{Z}$ one has that Ξ, $\Sigma_{\boldsymbol{a}}l^{\boldsymbol{a}}$ and $\Sigma_{\boldsymbol{a}}m^{\boldsymbol{a}}$ will be zero along $\boldsymbol{l}$ until one reaches a caustic point. Consequently, one concludes that the conformal Ξ factor vanishes in the portion of $H(\mathcal{H})$ which is free of caustics. Thus, this portion of $H(\mathcal{H})$ can be interpreted as the conformal boundary of the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. In addition, notice that from the vanishing of the column vector of equation (4.16) it follows that $\Sigma_{\boldsymbol{a}}l^{\boldsymbol{a}} = \Sigma_{\boldsymbol{a}}m^{\boldsymbol{a}} = 0$ on $H(\mathcal{H})$. Therefore, the only component of $\Sigma_{\boldsymbol{a}}$ that can be different from zero is $\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}$. Accordingly, $\Sigma^{\boldsymbol{a}}$ is parallel to $l^{\boldsymbol{a}}$ and $\Sigma^{\boldsymbol{a}} = (\Sigma_{\boldsymbol{c}}n^{\boldsymbol{c}})l^{\boldsymbol{a}}$. Moreover, since $\boldsymbol{g}(\boldsymbol{n}_{\star}, \boldsymbol{l}_{\star}) = 1$ it follows that $(\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}})|_{\mathscr{Z}} = 1$ —this can also be shown by noticing that $(n^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}})|_{\mathscr{Z}} = (n^{\boldsymbol{b}}l_{\boldsymbol{b}})|_{\mathscr{Z}} = 1$.

Now, in order to extract the information contained in $\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}$ one transvects (4.15b) with $l^{\boldsymbol{a}}n_{\boldsymbol{b}}$, to obtain

$$l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}(n^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}) = s g_{\boldsymbol{ab}}l^{\boldsymbol{a}}n^{\boldsymbol{b}} - \Xi L_{\boldsymbol{ab}}l^{\boldsymbol{a}}n^{\boldsymbol{b}}.$$

Using that $\boldsymbol{g}(\boldsymbol{l}, \boldsymbol{n}) = 1$ and that Ξ vanishes on $H(\mathcal{H})$ one concludes that

$$\nabla_{\boldsymbol{l}}(\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}) = s \qquad \text{on } H(\mathcal{H}).$$

One can obtain a further equation transvecting (4.15c) with $l^{\boldsymbol{a}}$

$$l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}s = -L_{\boldsymbol{ab}}l^{\boldsymbol{a}}\Sigma^{\boldsymbol{b}} = -L_{\boldsymbol{ab}}l^{\boldsymbol{a}}\Sigma_{\boldsymbol{f}}n^{\boldsymbol{f}}l^{\boldsymbol{b}} \qquad \text{on } H(\mathcal{H}).$$

It follows then that one has the system

$$\nabla_{\boldsymbol{l}}\begin{pmatrix}\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}\\ s\end{pmatrix} = \begin{pmatrix}0 & 1\\ -L_{\boldsymbol{cd}}l^{\boldsymbol{c}}l^{\boldsymbol{d}} & 0\end{pmatrix}\begin{pmatrix}\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}\\ s\end{pmatrix}.$$

Since $(\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}})|_{\mathscr{Z}} = 1$ (i.e. non-vanishing), the solution for the column vector formed by s and $\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}$ cannot be zero. Accordingly, s and $\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}$ cannot vanish simultaneously. Finally, transvecting equation (4.15b) with $m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}$ one gets

$$m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}\nabla_{\boldsymbol{a}}\Sigma_{\boldsymbol{b}} = -\Xi m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}L_{\boldsymbol{ab}} + s g_{\boldsymbol{ab}}m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}.$$

Using that $\boldsymbol{g}(\boldsymbol{m}, \bar{\boldsymbol{m}}) = 1$ and restricting to $H(\mathcal{H})$ where $\Xi = 0$ renders

$$\bar{m}^{\boldsymbol{b}}m^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}\Sigma_{\boldsymbol{b}} = s \qquad \text{on } H(\mathcal{H}).$$

Using $\boldsymbol{g}(\bar{\boldsymbol{m}}, \boldsymbol{l}) = 0$ it follows that the left hand side of the last equation is equivalent to

$$\begin{aligned} m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}\nabla_{\boldsymbol{a}}\Sigma_{\boldsymbol{b}} &= m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}\nabla_{\boldsymbol{a}}(\Sigma_{\boldsymbol{c}}n^{\boldsymbol{c}}l_{\boldsymbol{b}}) \\ &= m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}\Sigma_{\boldsymbol{c}}n^{\boldsymbol{c}}\nabla_{\boldsymbol{a}}l_{\boldsymbol{b}} + m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}l_{\boldsymbol{b}}\nabla_{\boldsymbol{a}}\Sigma_{\boldsymbol{c}}n^{\boldsymbol{c}} \\ &= \Sigma_{\boldsymbol{c}}n^{\boldsymbol{c}}m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}\nabla_{\boldsymbol{a}}l_{\boldsymbol{b}}. \end{aligned}$$

Finally, recalling the definition of the expansion $\rho \equiv -m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}\nabla_{\boldsymbol{a}}l_{\boldsymbol{b}}$ (in the Newman-Penrose notation [12]) one finally obtains

$$\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}\rho = -s \qquad \text{on } H(\mathcal{H}).$$

Since the only possible non-zero component of the gradient of Ξ is $\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}$ and it cannot vanish simultaneously with s, one has that $\mathbf{d}\Xi = 0$ implies $\rho \to \infty$ on $H(\mathcal{H})$. To be able to identify the point i^+ where $\mathbf{d}\Xi = 0$ with timelike infinity one needs to calculate the Hessian of the conformal factor. Observe that this information is contained in the conformal field equation (2.7c). Considering this equation at $H(\mathcal{H})$, where it has already been shown that the conformal factor vanishes, one gets

$$\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi = s g_{\boldsymbol{ab}}.$$

Now, as it has been shown that s and $\Sigma_{\boldsymbol{a}}n^{\boldsymbol{a}}$ (or, equivalently, $\mathbf{d}\Xi$) do not vanish simultaneously one concludes that $s \neq 0$ and that $\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi$ is non-degenerate. Thus, the point i^+ on $H(\mathcal{H})$ where both Ξ and $\mathbf{d}\Xi$ vanish can be regarded as representing future timelike infinity for the physical spacetime $(\tilde{\mathcal{M}}, \tilde{g})$.

Remark 26. Observe that the construction discussed in the previous paragraphs crucially assumes that $\Xi_\star$ is zero on the boundary $\mathcal{Z}$ of the initial hypersurface $\mathcal{H}$. This construction cannot be repeated if one were to take another hypersurface $\mathcal{H}'$ with boundary $\mathcal{Z}'$ where the conformal factor does not vanish. This is the case of an initial hypersurface that intersects the cosmological horizon, where for the reference solution the conformal factor does not vanish —see Figure 4.2.

The results of the analysis of this section are summarised in the following:

Proposition 4. ***(Structure of the conformal boundary)*** *Let* $\mathbf{u}$ *denote a solution to the conformal wave equations equations constructed as described in Theorem 1. Then, there exists a point* $i^+ \in H(\mathcal{H})$ *where* $\Xi|_{i^+} = 0$ *and* $\mathbf{d}\Xi|_{i^+} = 0$ *but the Hessian* $\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi|_{i^+}$ *is non-degenerate. In addition,* $\mathbf{d}\Xi \neq 0$ *on* $\mathscr{I}^+ = H(\mathcal{H}) \setminus \{i^+\}$. *Moreover* $D^+(\mathcal{H}) = J^-(i^+)$.

Proof. From the conclusions of Theorem 1 and the discussion of Section 4.2.5 it follows that if one has a solution to the conformal wave equations which, in turn implies a solution to the conformal field equations, then there exists a point i^+ in

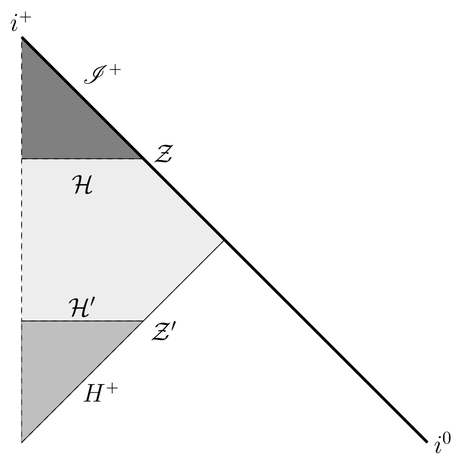

Figure 4.2: *Portion of the Penrose diagram of the Milne Universe showing the initial hypersurface $\mathcal{H}$ where the the hyperboloidal data is prescribed. At $\mathcal{Z}$ the conformal factor vanishes and the argument of Section 4.2.5 can be applied. The dark grey area represents the development of the data on $\mathcal{H}$. Compare with the case of the hypersurface $\mathcal{H}'$ which intersects the horizon at $\mathcal{Z}'$ where the argument cannot be applied. Analogous hypersurfaces can be depicted for the lower diamond of the complete diagram of Figure 4.1.*

$H(\mathcal{H})$ where both the conformal factor and its gradient vanish but $\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi$ is non-degenerate. This means that i^+ can be regarded as future timelike infinity for the physical spacetime. In addition, null infinity $\mathscr{I}^+$ will be located at $H(\mathcal{H})\setminus\{i^+\}$ where the conformal factor vanishes but its gradient does not. □

Remark 27. Inspection of the argument leading to Proposition 4 requires two continuous derivatives of the conformal fields involved. This is precisely the minimal regularity provided by Theorem 1.

4.3 Conclusions

The discussion given in this chapter shows how the wave equations derived in Chapter 3 can be employed to study the semiglobal non-linear stability of the Milne Universe. This analysis, in particular, exemplifies how the extraction of a system of quasilinear wave equations out of the conformal Einstein field equations allows to readily make use of the general theory of partial differential equations to obtain non-trivial statements about the global existence of solutions to the Einstein field equations. The analysis of Chapter 3 has been restricted to the vacuum case. However, a similar procedure can be carried out, in the non-vacuum case, for some suitable matter models with trace-free energy-momentum tensor —see e.g. [17].

In addition, this analysis has been restricted to the case of the so-called standard conformal Einstein field equations. Nevertheless, as discussed in Chapter 2, there exists a more general version of the conformal Einstein field equations, the so-called, extended conformal Einstein field equations in which the various equations are expressed in terms of a Weyl connection.

The hyperbolic reduction procedures for the extended conformal Einstein field equations available in the literature do not make use of gauge source functions. Instead, one makes use of conformal Gaussian systems based on the congruence of privileged curves known as conformal geodesics to extract a first order symmetric hyperbolic system. It is an interesting open question to see whether it is possible to use conformal Gaussian systems to deduce wave equations for the conformal fields in the extended conformal Einstein field equations. Nevertheless, the latter will not be pursued in this thesis.

4.4 Appendix: Basic existence and stability theory for quasilinear wave equations

In this appendix an adapted version of a theorem for quasilinear wave equations given in [21] is given. The particular formulation has been chosen so as to simplify comparison with an analogous result for first order symmetric hyperbolic as given in [8] —see Theorem 3.1 in that reference.

In what follows, one will consider open, connected subsets $\mathcal{U} \subset \mathcal{M}_T \equiv [0,T) \times \mathcal{S}$ for some $T > 0$ and $\mathcal{S} \approx \mathbb{S}^3$ an oriented, compact 3-dimensional manifold. On $\mathcal{U}$ one can introduce local coordinates $x = (x^\mu) = (t, x^\alpha)$. Given a fixed $N \in \mathbb{N}$, in what follows, let $\mathbf{u} : \mathcal{M}_T \to \mathbb{C}^N$ denote a $\mathbb{C}^N$-valued function. The derivatives of $\mathbf{u}$ will be denoted, collectively, by $\partial\mathbf{u}$. The discussion will be restricted to quasilinear wave equations of the form

$$g^{\mu\nu}(x;\mathbf{u})\partial_\mu\partial_\nu\mathbf{u} = \mathbf{F}(x;\mathbf{u},\partial\mathbf{u}), \tag{4.17}$$

where $g^{\mu\nu}(x;\mathbf{u})$ denotes the contravariant version of a Lorentzian metric $g_{\mu\nu}(x;\mathbf{u})$ which depends smoothly on the unknown $\mathbf{u}$ and the coordinates x and $\mathbf{F}$ is a smooth $\mathbb{C}^N$-valued function of its arguments. Separating the fields into real and imaginary parts one can regard $\mathbf{u}$ as a $\mathbb{R}^{2N}$-valued function.

In order to formulate a Cauchy problem for equation (4.17) it is necessary to supplement it with initial data corresponding to the value of $\mathbf{u}$ and $\partial_t\mathbf{u}$ on the initial hypersurface $\mathcal{S}$. For simplicity, choose coordinates such that $\mathcal{S}$ is described by the condition $t = 0$. Given two functions $\mathbf{u}_\star$, $\mathbf{v}_\star \in H^m(\mathcal{S},\mathbb{C}^N)$, $m \geq 2$, one defines the ball of radius ε centred around $(\mathbf{u}_\star, \mathbf{v}_\star)$ as the set

$$B_\varepsilon(\mathbf{u}_\star,\mathbf{v}_\star) \equiv \big\{(\mathbf{w}_1,\mathbf{w}_2) \in H^m(\mathcal{S},\mathbb{C}^N)\times H^m(\mathcal{S},\mathbb{C}^N) \mid \ \| \mathbf{w}_1-\mathbf{u}_\star \|_{\mathcal{S},m} + \| \mathbf{w}_2-\mathbf{v}_\star \|_{\mathcal{S},m}\leq \varepsilon\big\}.$$

Also, given $\delta > 0$ define

$$D_\delta \equiv \big\{ (\mathbf{w}_1, \mathbf{w}_2) \in H^m(\mathcal{S}, \mathbb{C}^N) \times H^m(\mathcal{S}, \mathbb{C}^N) \mid \delta < |\det g_{\mu\nu}(\mathbf{w}_1)| \big\}.$$

The basic existence and Cauchy stability theory for equations of the form (4.17) has been given in [21]. The following theorem is based on Theorem III of the later reference —the presentation follows Theorem 3.1 of [8]:

Theorem 2. *Given an orientable, compact, 3-dimensional manifold $\mathcal{S} \approx \mathbb{S}^3$, consider the the Cauchy problem*

$$\begin{aligned} &g^{\mu\nu}(x; \mathbf{u}) \partial_\mu \partial_\nu \mathbf{u} = \mathbf{F}(x; \mathbf{u}, \partial \mathbf{u}), \\ &\mathbf{u}(0, x) = \mathbf{u}_\star(x) \in H^{m+1}(\mathcal{S}, \mathbb{C}^N), \\ &\partial_t \mathbf{u}(0, x) = \mathbf{v}_\star(x) \in H^m(\mathcal{S}, \mathbb{C}^N), \qquad m \geq 4, \end{aligned}$$

and assume that $g_{\mu\nu}(x; \mathbf{u}_\star)$ is a Lorentzian metric such that $(\mathbf{u}_\star, \mathbf{v}_\star) \in D_\delta$ for some $\delta > 0$. Then:

(i) There exists $T > 0$ and a unique solution to the Cauchy problem defined on $[0, T) \times \mathcal{S}$ such that

$$\mathbf{u} \in C^{m-2}([0, T) \times \mathcal{S}, \mathbb{C}^N).$$

Moreover, $(\mathbf{u}(t, \cdot), \partial_t \mathbf{u}(t, \cdot)) \in D_\delta$ for $t \in [0, T)$.

(ii) There is a $\varepsilon > 0$ such that a common existence time T can be chosen for all initial data conditions on $B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \cap D_\delta$.

(iii) If the solution $\mathbf{u}$ with initial data $\mathbf{u}_\star$ exists on $[0, T)$ for some $T > 0$, then the solutions to all initial conditions in $B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \cap D_\delta$ exist on $[0, T]$ if $\varepsilon > 0$ is sufficiently small.

(iv) If ε and T are chosen as in (i) and one has a sequence $(\mathbf{u}_\star^{(n)}, \mathbf{v}_\star^{(n)}) \in B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \cap D_\delta$ such that

$$\| \mathbf{u}_\star^{(n)} - \mathbf{u}_\star \|_{\mathcal{S},m} \to 0, \qquad \| \mathbf{v}_\star^{(n)} - \mathbf{v}_\star \|_{\mathcal{S},m} \to 0, \qquad \text{as} \qquad n \to \infty,$$

then for the solutions $\mathbf{u}^{(n)}(t, \cdot)$ with $\mathbf{u}^{(n)}(0, \cdot) = \mathbf{u}_\star^{(n)}$ and $\partial_t \mathbf{u}^{(n)}(0, \cdot) = \mathbf{v}_\star^{(n)}$, it holds that

$$\| \mathbf{u}^{(n)}(t, \cdot) - \mathbf{u}(t, \cdot) \|_{\mathcal{S},m} \to 0 \qquad \text{as} \qquad n \to \infty$$

uniformly in $t \in [0, t)$ as $n \to \infty$.

Remark 28. The particular formulation of the theorem has been chosen to ease the comparison with Theorem 3.1 of [8] where an analogous existence and Cauchy stability result for first order symmetric hyperbolic systems is given.

Remark 29. The hypotheses of Theorem III in [21], on which Theorem 2 is based, require the positivity away from zero of the coefficients of the second derivatives with respect to time in the wave equations. It also requires that the second order spatial partial derivatives give rise to a strongly elliptic system. These requirements are satisfied for equation (4.17) if the matrix $g_{\mu\nu}$ is a non-degenerate Lorentzian metric. This requirement is encoded in the set D_δ. The regularity requirements of the coefficients in the quasilinear wave equations with respect to their arguments required by Theorem III in [21] are satisfied by the smoothness assumptions on the functional form of equation (4.17).

Remark 30. Theorem III in [21] establishes the well-posedness of the Cauchy problem for quasilinear wave equations of the form (4.17). It contains two main statements. The first establishes the local existence and uniqueness of solutions to initial value problem —this is essentially the content of point *(i)* in Theorem 2. The second statement asserts the continuous dependence of solutions and existence times with respect to the initial data. In particular, if a known solution to the initial value problem has a minimal existence time T, then any initial data sufficiently close to the reference solution will have the same existence time —this is essentially the statement in points *(ii)* and *(iii)* of Theorem 2. The non-existence of nearby data with solutions having the same existence time would be in contradiction with the uniform continuity of the map relating initial data and solutions. The dependence of the solutions on the initial data is continuous in the topology of $H^{m+1} \times H^m$ uniformly in the common existence interval —this is the statement in point *(iv)* of Theorem 2. Notice that this is expressed, for concreteness, in terms of the convergence of a sequence.

Remark 31. In view of the applications of the Theorem, the regularity of the solutions has been expressed in terms of standard derivatives rather than Sobolev spaces —see also, the remarks after Theorem 3.1 in [8].

Remark 32. The regularity in both the hypothesis and conclusions of the Theorem are not optimal. The reader interested is referred to [51–54].

Remark 33. Alternatively, rewriting equation (4.17) as a first order symmetric hyperbolic system, one can obtain Theorem 2 from Theorem 3.1 of [8]. Similar ideas have been used in [55, 56].

Remark 34. Using the method of patching solutions, Theorem III can be extended to any compact orientable 3-manifold $\mathcal{S}$ —see e.g. the discussion in [17].

5 Perturbations of the asymptotic region of the Schwarzschild-de Sitter spacetime.

5.1 Introduction

The stability of black hole spacetimes is, arguably, one of the outstanding problems in mathematical General Relativity. The challenge in analysing the stability of black hole spacetimes lies in both the mathematical problems as well as in the physical concepts to be grasped. By contrast, the non-linear stability of Minkowski spacetime —see e.g. [8, 10]— and de Sitter spacetimes —see [8, 9]— are well understood.

The results in [8, 9] show that conformal Einstein field equations are a powerful tool for the analysis of the stability and global properties of vacuum asymptotically simple spacetimes —see [8, 9, 11, 57]. In particular, the analysis given in [8, 9], makes use of the standard conformal Einstein field equations. More recently, in [41], it was shown that the extended conformal field equations can be used to obtain an alternative proof of the semiglobal non-linear stability of the Minkowski spacetime and of the global non-linear stability of the de-Sitter spacetime —see [41]. In view of these results, a natural question is whether conformal methods can be used in the global analysis of spacetimes containing black holes. The discussion in this chapter is based on

Gasperín E. and Valiente Kroon J.A., "Perturbations of the asymptotic region of the Schwarzschild–de Sitter spacetime," *Annales Henri Poincaré* (2017) 1–73. `http://dx.doi.org/10.1007/s00023-016-0544-z`.

where a first step in this direction is given by analysing certain aspects of the conformal structure of the Schwarzschild-de Sitter spacetime using conformal methods. The current approaches for analysing stability properties of black hole spacetimes do not make use of conformal formulations of the Einstein field equations, consequently, the use of conformal methods for the stability analysis of solutions of the Einstein field equations represents a new and unexploited venue. Despite the fact the result obtained in this chapter does not fully address the outstanding stability of the Schwarzschild-de Sitter, the constructed class of solutions is non-trivial. In addition, as previously mentioned, it does represent a first step in implementing conformal methods for the the analysis of black hole spacetimes.

5.1.1 The Schwarzschild-de Sitter spacetime

The Schwarzschild-de Sitter spacetime is a spherically symmetric solution to the vacuum Einstein field equations with Cosmological constant. It depends on two parameters: the Cosmological constant λ and the mass parameter m. The assumption of spherical symmetry almost completely singles out the Schwarzschild-de Sitter spacetime among the vacuum solutions to the Einstein field equations with de Sitter-like Cosmological constant. The other admissible solution is the so-called Nariai spacetime. This observation can be regarded as a generalisation of Birkhoff's theorem —see [58] for a modern discussion on this classical result. For small values of the area radius r, the solution behaves like the Schwarzschild spacetime and for large values its behaviour resembles that of the de Sitter spacetime. In the Schwarzschild-de Sitter spacetime the relation between the mass and Cosmological constant determines the location of the *Cosmological* and *black hole horizons*.

The presence of a Cosmological constant makes the Schwarzschild-de Sitter solution a convenient candidate for a global analysis by means of the extended conformal Einstein field equations —see Section 2.2 in Chapter 2— as the solution is an example of a spacetime which admits a smooth conformal extension towards the future (respectively, the past) —see Figures 5.3, 5.4 and 5.5 in this chapter. This type of spacetimes are called future (respectively, past) asymptotically de Sitter —see Section 5.2.1 for definitions and [59, 60] for a more extensive discussion. As the Cosmological constant takes a de Sitter-like value, the conformal boundary of the spacetime is spacelike and, moreover, there exists a conformal representation in which the induced 3-metric on the conformal boundary $\mathscr{I}$ is homogeneous. Thus, it is possible to integrate the extended conformal field equations along single conformal geodesics.

In this chapter *the Schwarzschild-de Sitter spacetime as a solution to the extended conformal Einstein field equations is analysed*. The insights thus obtained are used to discuss non-linear perturbations of the spacetime. A natural starting point for this discussion is the analysis of conformal geodesic equations on the spacetime. The results of this analysis can, in turn, be used to rewrite the spacetime in the conformal gauge associated to these curves. However, despite the fact that the conformal geodesic equations for spherically symmetric spacetimes can be written in quadratures [37], in general, the integrals involved cannot be solved analytically. In view of this difficulty, in this chapter the conformal properties of the exact Schwarzschild-de Sitter spacetime are analysed by means of an asymptotic initial value problem for the conformal field equations. Accordingly, initial data implied by the Schwarzschild-de Sitter spacetime on the conformal boundary is obtained and used to analyse the

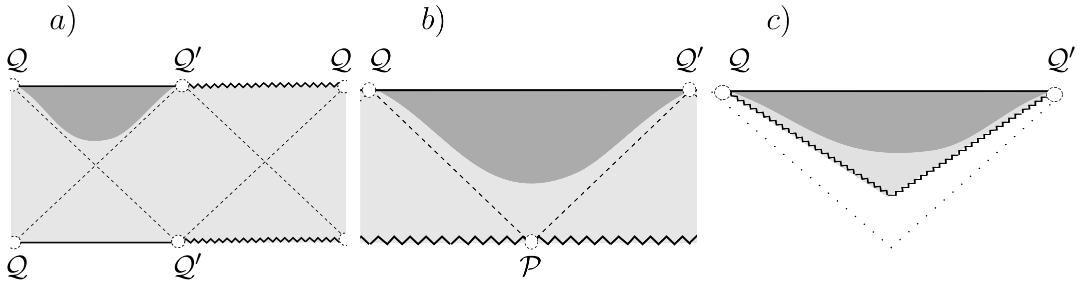

Figure 5.1: *Schematic depiction of the Main Result 2. The dark grey area in panels a) b) and c) illustrates the region covered by the development of asymptotic initial data close to that of the Schwarzschild-de Sitter spacetime —in the global representation— for the subextremal, extremal and hyperextremal cases respectively. The light grey area represents the exact Schwarzschild-de Sitter spacetime. For the exact Schwarzschild-de Sitter spacetime the initial metric is $\hbar$, the standard metric on $\mathbb{S}^3$, and the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ are excluded (denoted by empty circles in the diagram). See also Figures 5.3, 5.4 and 5.5.*

Figure 5.2: *Schematic depiction of the Main Result 2. Development of asymptotic initial data close to that of the Schwarzschild-de Sitter spacetime in the representation in which Theorem 3 is obtained. For the exact Schwarzschild-de Sitter spacetime the initial metric is $\boldsymbol{h}$, the standard metric on $\mathbb{R}\times\mathbb{S}^2$, and the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ are at infinity with respect to $\boldsymbol{h}$ —since $\hbar$ and $\boldsymbol{h}$ are conformally flat one has $\boldsymbol{h} = \omega^2\hbar$. The initial data for the subextremal, extremal and hyperextremal cases is formally identical. For small enough perturbations the development have the same asymptotic structure as the reference spacetime*

behaviour of the conformal evolution equations. An important property of these evolution equations is that their essential dynamics is governed by a *core system* of equations. Consequently, an important aspect of this discussion consists of the analysis of the formation of singularities in the core system. This analysis is irrespective of the relation between $\lambda \neq 0$ and m. This allows to formulate a result which is valid for the subextremal, extremal and hyperextremal Schwarzschild-de Sitter spacetime characterised by the conditions $0 < 9m^2|\lambda| < 1$, $9m^2|\lambda| = 1$ and $9m^2|\lambda| > 1$ respectively.

5.1.2 The main result

The analysis of the conformal properties of the Schwarzschild-de Sitter spacetime allows to formulate a result concerning the existence of solutions to the asymptotic initial value problem for the Einstein field equations with de Sitter-like Cosmological constant which can be regarded as perturbations of the *asymptotic region* of the

Schwarzschild-de Sitter spacetime —see Figures 5.1 and 5.2. The existence result proven in this chapter can be stated as:

Main Result 2 (***asymptotically de Sitter spacetimes close to the asymptotic region of the SdS spacetime***). *Given asymptotic initial data which is suitably close to data for the Schwarzschild-de Sitter spacetime there exists a solution to the Einstein field equations which exists towards the future (past) and has an asymptotic structure similar to that of the Schwarzschild-de Sitter spacetime —that is, the solution is future (past) asymptotically de Sitter.*

Remark 35. A detailed formulation of the Main Result of this chapter can be found in Section 5.4.4 —see Theorem 3.

The analysis of the conformal evolution equations governing the dynamics of the background solution given in this chapter provides explicit minimal existence intervals for the solutions. These intervals are certainly not optimal. An interesting question related to this class of solutions to the Einstein field equations is to obtain their maximal development. To address this problem one requires different methods of the theory of partial differential equations and it will be discussed elsewhere. A schematic depiction of the Main Result is given in Figure 5.1.

Part of the analysis of the background solution requires deriving asymptotic initial data for the Schwarzschild-de Sitter spacetime. The construction of this initial data allows to study in detail the singular behaviour of the conformal structure of the family of background spacetimes at the *asymptotic points* $\mathcal{Q}$ and $\mathcal{Q}'$, where the horizons of the spacetime meet the conformal boundary. As a consequence of the singular behaviour of the asymptotic initial data, the discussion of the asymptotic initial value problem has to exclude these points. In view of this, it turns out that a more convenient conformal representation to analyse the conformal evolution equations for both the exact Schwarzschild-de Sitter spacetime and its perturbations is one in which the the conformal boundary is metrically $\mathbb{R}\times\mathbb{S}^2$ rather than $\mathbb{S}^3\setminus\{\mathcal{Q},\mathcal{Q}'\}$ so that the problematic asymptotic points are *sent* to infinity —see Figure 5.2.

5.1.3 Related results

The properties of the Schwarzschild-de Sitter spacetime have been systematically probed by means of an analysis of the solutions of the scalar wave equation using *vector field methods* —see [61]. This type of analysis requires special care when discussing the behaviour of the solution close to the horizons. In the asymptotic initial value problem considered in this chapter, the domain of influence of the initial data is contained in the region corresponding to the asymptotic region of the Schwarzschild-de Sitter spacetime.

The properties of the Nariai spacetime —the other solution appearing in the generalisation of Birkhoff's theorem to spacetimes with a de Sitter-like Cosmological constant— have been analysed by means of both analytic and numerical methods in [62, 63]. In particular, in the former reference it is shown that the Nariai solution does not admit a smooth conformal extension —see also [57]. Thus, it cannot be obtained from an asymptotic initial value problem.

Finally, it is pointed out that the singularity of the Schwarzschild-de Sitter spacetime is not a conformal gauge singularity since $\tilde{C}_{abcd}\tilde{C}^{abcd} \to \infty$ as $r \to 0$. Accordingly, theory of the extendibility of conformal gauge singularities as developed in [64] cannot be applied in the case analysed in this chapter. For any of the possible conformal gauges available, one either has a singularity of the Weyl tensor arising at a finite value of the parameter of a conformal geodesic or one has an inextendible conformal geodesic along which the Weyl tensor is always smooth.

5.2 The asymptotic initial value problem in General Relativity

In this section the notion of asymptotically de Sitter spacetimes is revisited —see [2, 59, 60]. In particular, it is discussed how to use the conformal field equations expressed in terms of a conformal Gaussian system —recall the conformal evolution and constraint equations discussed in Sections 2.4.3 and 2.5 of Chapter 2— to set up an *asymptotic initial value problem* for a spacetime with a spacelike conformal boundary. This section concludes with a discussion of the structural properties of the conformal evolution equations in the framework of the theory of symmetric hyperbolic systems contained in [23].

5.2.1 Asymptotically de Sitter spacetimes

The following definition of future asymptotically de Sitter spacetimes will be frequently used in this chapter.

Definition. *A spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *satisfying the vacuum Einstein field equations*

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}, \tag{5.1}$$

is future asymptotically de Sitter if there exist a spacetime with boundary $(\mathcal{M}, \boldsymbol{g})$, *a smooth conformal factor* Ξ *and a diffeomorphism* $\varphi : \tilde{\mathcal{M}} \to \mathcal{U} \subseteq \mathcal{M}$, *such that:*

$$\begin{aligned} &\Xi > 0 \quad \textit{in} \quad \mathcal{U}, \\ &\Xi = 0 \quad \textit{and} \quad \mathbf{d}\Xi \neq 0 \quad \textit{on} \quad \mathscr{I}^+ \equiv \partial\mathcal{U}, \\ &\mathscr{I}^+ \quad \textit{is spacelike —i.e.} \quad \boldsymbol{g}(\mathbf{d}\Xi, \mathbf{d}\Xi) > 0 \quad \textit{on} \quad \mathscr{I}^+, \end{aligned}$$

$$\mathscr{I}^+ \quad \textit{lies to the future of } \tilde{\mathcal{M}} \quad \textit{—i.e.} \quad \mathscr{I}^+ \subset I^+(\tilde{\mathcal{M}}).$$

Observe that this definition does not restrict the topology of $\mathscr{I}^+$. In particular, it does not have to be compact —see [60]. The notion of past asymptotically de Sitter is defined in analogous way. Additionally, $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is asymptotically de Sitter if it is *future and past asymptotically de Sitter.* Notice that a spacetime which is asymptotically de Sitter is not necessarily *asymptotically simple* —see [2] for a precise definition of asymptotically simple spacetimes. In the following, in a slight abuse of notation, the mapping $\varphi : \tilde{\mathcal{M}} \to \mathcal{U} \subseteq \mathcal{M}$ will be omitted in the notation so that one writes

$$\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}. \tag{5.2}$$

Furthermore, the term *asymptotic region* will be used to refer to the set $J^-(\mathscr{I}^+)$ of a future asymptotically de Sitter spacetime or $J^+(\mathscr{I}^-)$ of a past asymptotically de Sitter spacetime.

5.2.2 The formulation of an asymptotic initial value problem

In this section it is shown how the conformal Gaussian gauge can be used to formulate an asymptotic initial value problem for the extended conformal Einstein field equations. Thus, in the sequel an initial hypersurface on which the conformal factor vanishes, so that it corresponds to the conformal boundary of a hypothetical spacetime, is considered. Accordingly, this initial hypersurface will be denoted by $\mathscr{I}$.

The conformal boundary

Following Lemma 6 one can set, without loss of generality, $\tau_\star = 0$ on $\mathscr{I}$. Moreover, it will be assumed that $f_{\boldsymbol{a}}$ vanishes initially. Accordingly, one has the initial condition $\boldsymbol{\beta}_\star = \Theta_\star^{-1}\mathbf{d}\Theta_\star$. Recalling that $\boldsymbol{d} = \Theta\boldsymbol{\beta}$, and $\tilde{\boldsymbol{g}}^\sharp = \Theta^2 \boldsymbol{g}^\sharp$, and using the constraints in equation (2.41) of Lemma 6 it readily follows, for the asymptotic initial value problem (in which $\Theta_\star = 0$), that

$$\dot{\Theta}_\star = \sqrt{\frac{|\lambda|}{3}}.$$

Moreover, using again that $\boldsymbol{d} = \Theta\boldsymbol{\beta}$ and requiring $\dot{\boldsymbol{x}}_\star$ to be orthogonal to $\mathscr{I}$ (so that $\dot{\boldsymbol{x}}_\star = \boldsymbol{e}_0$), one obtains $d_{0\star} = \dot{\Theta}_\star$. Consequently

$$d_{0\star} = \sqrt{\frac{|\lambda|}{3}}.$$

The coefficient $\ddot{\Theta}_\star$ is fixed by the requirement $s = \Sigma\kappa$ on $\mathscr{I}$ —see [65]. From the definition of s and $\Sigma_{\boldsymbol{a}} \equiv \nabla_{\boldsymbol{a}}\Theta$ it follows that

$$
\begin{aligned}
s_\star &= \left(\frac{1}{4}\nabla_{\boldsymbol{a}}\nabla^{\boldsymbol{a}}\Theta + \frac{1}{24}R\Theta\right)_\star = \frac{1}{4}(\boldsymbol{e}_{\boldsymbol{a}}\Sigma^{\boldsymbol{a}})_\star + \frac{1}{4}(\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}}\Sigma^{\boldsymbol{b}})_\star \\
&= \frac{1}{4}\eta^{\boldsymbol{ab}}(\boldsymbol{e}_{\boldsymbol{a}}\boldsymbol{e}_{\boldsymbol{b}}\Theta)_\star + \frac{1}{4}\dot{\Theta}_\star(\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{a}}{}_{\boldsymbol{0}})_\star.
\end{aligned}
\tag{5.3}
$$

Taking into account that Θ and $\Sigma_{\boldsymbol{i}}$ vanish at $\mathscr{I}$ one has that $\eta^{\boldsymbol{ab}}(\boldsymbol{e}_{\boldsymbol{a}}\boldsymbol{e}_{\boldsymbol{b}}\Theta)_\star = \ddot{\Theta}_\star$. Using the solution to the constraints given in equations (2.78a)-(2.78b) of Chapter 2 and exploiting the properties of the adapted orthonormal frame one obtains $(\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{a}}{}_{\boldsymbol{0}})_\star = (\Gamma_{\boldsymbol{i}}{}^{\boldsymbol{i}}{}_{\boldsymbol{0}})_\star = (\chi_{\boldsymbol{i}}{}^{\boldsymbol{i}})_\star = \kappa\delta_{\boldsymbol{i}}{}^{\boldsymbol{i}} = 3\kappa$. Substituting into equation (5.3) and using that $s_\star = \dot{\Theta}_\star\kappa$ one gets

$$
\ddot{\Theta}_\star = \dot{\Theta}_\star\kappa.
$$

Summarising, for an asymptotic initial value problem the conformal factor implied by the conformal Gaussian gauge is given by

$$
\Theta(\tau) = \sqrt{\frac{|\lambda|}{3}}\,\tau\left(1 + \frac{1}{2}\kappa\tau\right).
\tag{5.4}
$$

The conformal factor given by equation (5.4) is, in a certain sense, universal —see Remark 37. It does not encode any information about the particular details of the spacetime to be evolved from $\mathscr{I}$. As such, it can be used to analyse any spacetime with de Sitter-like Cosmological constant as long as the spacetime has at least one component of the conformal boundary. If $\kappa \neq 0$ the conformal boundary has two components located at

$$
\tau = 0 \quad \text{and} \quad \tau = -\frac{2}{\kappa}.
$$

The first zero corresponds to the initial hypersurface $\mathscr{I}$. The physical spacetime corresponds to the region where $\Theta \neq 0$. Therefore, the roots of Θ render two different regions of $(\mathcal{M}, \boldsymbol{g})$ corresponding to two different conformal representation of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. One of these representations corresponds to the region covered by the conformal geodesics with $\tau \in [-2/|\kappa|, 0]$ or $\tau \in [0, 2/|\kappa|]$ and other corresponds to the region covered by the conformal geodesics with $\tau \in [0, \infty)$ or $\tau \in (-\infty, 0]$ depending on the sign of κ.

Remark 36. The discussion of the previous paragraphs is formal: the component of the conformal boundary given by $\tau = -2/\kappa$ may not be realised in a specific spacetime. This is, in particular, the case of the extremal and hyperextremal Schwarzschild-de Sitter spacetimes in which the singularity precludes reaching the second conformal infinity —see Figure 5.4.

Remark 37. The expression (5.4) is universal in the sense that it is the conformal factor singled out by a congruence of conformal geodesics starting orthogonally from $\mathscr{I}$—see Section 2.3.2 in Chapter 2. Observe that the location of the conformal boundary is known apriori in the sense that the values of τ for which the conformal factor of Lemma 6 vanishes can be written in terms of $\Theta_\star$, $\dot{\Theta}_\star$ and $\ddot{\Theta}_\star$. Furthermore, notice that the values of $\Theta_\star$, $\dot{\Theta}_\star$ and $\ddot{\Theta}_\star$ are fixed by the value of λ and κ in the asymptotic initial value problem.

Exploiting the conformal gauge freedom

The conformal freedom of the setting allows us to further simplify the solution to the conformal constraint equations at $\mathscr{I}$. Given a solution to the conformal Einstein field equations associated to a metric $\boldsymbol{g}$, it follows from the conformal covariance of the equations and fields that the conformally related metric $\boldsymbol{g}' \equiv \vartheta^2 \boldsymbol{g}$ for some ϑ is also a solution. On an initial hypersurface $\mathcal{S}$ the latter implies implies $\boldsymbol{h}' = \vartheta_\star^2 \boldsymbol{h}$. From the definition of the Friedrich scalar s —see Section 2.1.2 in Chapter 2— and the conformal transformation rule for the Ricci scalar one has that

$$s'_\star = \vartheta_\star^{-1} s_\star + \vartheta_\star^{-2} (\nabla_{\boldsymbol{c}} \vartheta)_\star (\nabla^{\boldsymbol{c}} \Theta)_\star .$$

Thus, the condition $s' = 0$ can be solved locally for $\vartheta_\star$. Accordingly, one chooses $\vartheta_\star$ so that $\kappa = 0$. In this gauge $\chi'_{\boldsymbol{ij}}$ and $L'_{\boldsymbol{i}}$ vanish and $L'_{\boldsymbol{ij}} = l'_{\boldsymbol{ij}}$ at $\mathscr{I}$. In addition, the conformal factor reduces to

$$\Theta(\tau) = \sqrt{\frac{|\lambda|}{3}}\tau .$$

In this representation Θ has only one zero and the second component of the conformal boundary (if any) is located at an infinite distance with respect to the parameter τ.

5.2.3 The general structure of the conformal evolution equations

One of the advantages of the hyperbolic reduction of the extended conformal Einstein field equations by means of conformal Gaussian systems is that it provides *a priori* knowledge of the location of the conformal boundary of the solutions to the conformal field equations —see Remark 37. Following the discussion in Section 2.3.2 of Chapter 2, the conformal geodesics fix the gauge through equations (2.44) and (2.43). The last condition corresponds to the requirement on the spacetime to possess a congruence of conformal geodesics and a Weyl propagated frame —i.e. equations (2.34) and (2.35) are satisfied. As already mentioned, the system of evolution equations (2.68a)-(2.68h) constitutes a symmetric hyperbolic system. This is the key property for analysing the existence and stability of perturbations of suitable spacetimes using the extended conformal Einstein field equations.

To discuss the structure of the conformal evolution system in more detail, let $\boldsymbol{e}$

denote the components of the frame $\boldsymbol{e_{AB}}$, $\boldsymbol{\Gamma}$ the independent components of $\chi_{\boldsymbol{ABCD}}$ and $\xi_{\boldsymbol{ABCD}}$, and $\boldsymbol{\phi}$ the independent components of the rescaled Weyl spinor $\phi_{\boldsymbol{ABCD}}$. Then the evolution equations (2.68a)-(2.68h) can be written as

$$\partial_\tau \boldsymbol{\upsilon} = \mathbf{K}\boldsymbol{\upsilon} + \mathbf{Q}(\boldsymbol{\Gamma})\boldsymbol{\upsilon} + \mathbf{L}(x)\boldsymbol{\phi}, \tag{5.5a}$$

$$(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e}))\partial_\tau \boldsymbol{\phi} + \mathbf{A}^{\boldsymbol{i}}\partial_{\boldsymbol{i}}\boldsymbol{\phi} = \mathbf{B}(\boldsymbol{\Gamma}), \tag{5.5b}$$

where $\boldsymbol{\upsilon}$ represents the independent components of the spinors in the conformal evolution equations except for the rescaled Weyl spinor whose components are represented by $\boldsymbol{\phi}$. In addition, $\mathbf{I}$ is the 5×5 identity matrix, $\mathbf{K}$ is a constant matrix, $\mathbf{Q}$, $\mathbf{A}^0$, $\mathbf{A}^{\boldsymbol{i}}$, and $\mathbf{B}$ are smooth matrix valued functions of its arguments and $\mathbf{L}(x)$ is a matrix valued function depending on the coordinates. To have an even more compact notation let $\mathbf{u} \equiv (\boldsymbol{\upsilon}, \boldsymbol{\phi})$. Consistent with this notation, let $\mathring{\mathbf{u}}$ denote a solution to the evolution equations (5.5a)-(5.5b) arising from data $\mathring{\mathbf{u}}_\star$ prescribed on an hypersurface $\mathcal{S}$. The solution $\mathring{\mathbf{u}}$ will be regarded as the *reference solution.* Consider a general perturbation succinctly written as $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$. Equivalently, one considers

$$\boldsymbol{e} = \mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}, \qquad \boldsymbol{\Gamma} = \mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}}, \qquad \boldsymbol{\phi} = \mathring{\boldsymbol{\phi}} + \breve{\boldsymbol{\phi}}. \tag{5.6}$$

Recalling that $\mathring{\mathbf{u}}$ is a solution to the conformal evolution equations (5.5a)-(5.5b) and making use of the split (5.6) one obtains that

$$\partial_\tau \breve{\boldsymbol{\upsilon}} = \mathbf{K}\breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\upsilon}} + \mathbf{L}(x)\breve{\boldsymbol{\phi}}, \tag{5.7a}$$

$$\begin{aligned}&(\mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}))\partial_\tau \breve{\boldsymbol{\phi}} + (\mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}))\partial_\tau \mathring{\boldsymbol{\phi}} + \mathbf{A}^{\boldsymbol{i}}(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}})\partial_{\boldsymbol{i}}\breve{\boldsymbol{\phi}} + \\ &\mathbf{A}^{\boldsymbol{i}}(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}})\partial_{\boldsymbol{i}}\mathring{\boldsymbol{\phi}} = \mathbf{B}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\phi}} + \mathbf{B}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\phi}}.\end{aligned} \tag{5.7b}$$

Equations (5.7a) and (5.7b) are read as equations for the components of the perturbed fields $\breve{\boldsymbol{\upsilon}}$ and $\breve{\boldsymbol{\phi}}$. These equations are in a form where the theory of first order symmetric hyperbolic systems in [23] can be applied to obtain a existence and stability result for small perturbations of the initial data $\mathring{\mathbf{u}}_\star$. This requires however, the introduction of the appropriate norms measuring size of the perturbed initial data $\breve{\mathbf{u}}_\star$. This general discussion will not be developed further, instead, the discussion will be particularised in Section 5.4.3 introducing the appropriate norms required to analyse the Schwarzschild-de Sitter spacetime as an asymptotic initial value problem.

5.3 The Schwarzschild-de Sitter spacetime and its conformal structure

In this section the general properties of the Schwarzschild-de Sitter spacetime that will be relevant for the main analysis are briefly reviewed.

5.3.1 The Schwarzschild-de Sitter spacetime

The *Schwarzschild-de Sitter spacetime* is the spherically symmetric solution to the Einstein field equations (5.1) with, in the signature conventions used in this thesis, a *negative* Cosmological constant given in *static coordinates* (t, r, θ, φ) by

$$\tilde{\boldsymbol{g}}_{SdS} = F(r)\mathbf{d}t \otimes \mathbf{d}t - F(r)^{-1}\mathbf{d}r \otimes \mathbf{d}r - r^2\boldsymbol{\sigma}, \tag{5.8}$$

where the function $F(r)$ is given by

$$F(r) \equiv 1 - \frac{2m}{r} + \frac{1}{3}\lambda r^2, \tag{5.9}$$

and $\boldsymbol{\sigma}$ is the standard metric on the 2-sphere $\mathbb{S}^2$

$$\boldsymbol{\sigma} \equiv \mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2\theta \mathbf{d}\varphi \otimes \mathbf{d}\varphi,$$

with $t \in (-\infty, \infty),\ r \in (0, \infty),\ \theta \in [0, \pi],\ \varphi \in [0, 2\pi)$. This solution reduces to the *de Sitter spacetime* when $m = 0$ and to the *Schwarzschild solution* when $\lambda = 0$.

Remark 38. In the following, only the case $m > 0$ will be considered. Furthermore, it will be always assumed a de Sitter-like value for the Cosmological constant λ.

The location of the roots of the polynomial $r - 2m + \frac{1}{3}\lambda r^3$ are determined by the relation between m and λ; whenever $0 < 9m^2|\lambda| < 1$ this polynomial has two distinct positive roots r_b, r_c and a negative root r_- located at

$$\begin{aligned}
r_b &\equiv \frac{2}{\sqrt{|\lambda|}}\cos\left(\frac{\alpha}{3} + \frac{4\pi}{3}\right), \\
r_c &\equiv \frac{2}{\sqrt{|\lambda|}}\cos\left(\frac{\alpha}{3}\right), \\
r_- &\equiv \frac{2}{\sqrt{|\lambda|}}\cos\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right),
\end{aligned}$$

where $\cos\alpha = -3m\sqrt{|\lambda|}$. The positive roots $0 < r_b \leq r_c$ correspond, respectively, to a black hole-like horizon and a Cosmological-like horizon. One can classify this 2-parameter family of solutions to the Einstein field equations depending on the relation between the parameters m and λ . The *subextremal Schwarzschild-de Sitter*

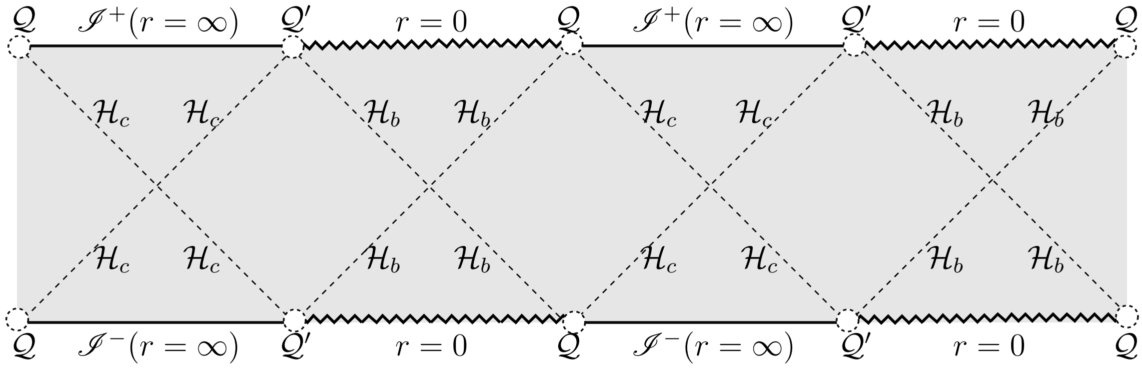

Figure 5.3: *Penrose diagram for the subextremal Schwarzschild-de Sitter spacetime. The excluded points $\mathcal{Q}$, $\mathcal{Q}'$ represent asymptotic regions where the Cosmological horizon appear to meet $\mathscr{I}$. As discussed in Section 5.3.1 this region of the spacetime does not belong to $\mathscr{I}$.*

spacetime arises when the relation between m and λ satisfies

$$0 < 9m^2|\lambda| < 1. \tag{5.10}$$

If condition (5.10) holds, one can verify that $F(r) > 0$ for $r_b < r < r_c$ while $F(r) < 0$ in the regions $0 \leq r < r_b$ and $r > r_c$. Consequently, the solution is static for $r_b < r < r_c$ —see [66]. The *extremal Schwarzschild-de Sitter* spacetime is obtained by setting

$$|\lambda| = 1/9m^2. \tag{5.11}$$

If the extremal condition (5.11) holds, then the black hole and Cosmological horizons degenerate into a *single Killing horizon* at $r = 3m$. Moreover, one has that $F(r) < 0$ for $0 \leq r < \infty$ so that the hypersurfaces of constant coordinate r are spacelike while those of constant t are timelike and there are no static regions. In the extremal case the function $F(r)$ can be factorised as

$$F(r) = -\frac{(r-3m)^2(r+6m)}{27m^2r}. \tag{5.12}$$

In the *hyperextremal Schwarzschild-de Sitter* spacetime one considers

$$9m^2|\lambda| > 1. \tag{5.13}$$

In this case one has again $F(r) < 0$ for $0 \leq r < \infty$ so that similar remarks as those for the extremal case hold. The crucial difference with the extremal case is that in the hyperextremal case there are no horizons. Finally, at $r = 0$ it can be verified that the spacetime has a *curvature singularity* irrespective of the relation between m and λ —in particular, the scalar $\tilde{C}_{abcd}\tilde{C}^{abcd}$, with $\tilde{C}^a{}_{bcd}$ the Weyl tensor of the metric $\tilde{\boldsymbol{g}}_{SdS}$, blows up.

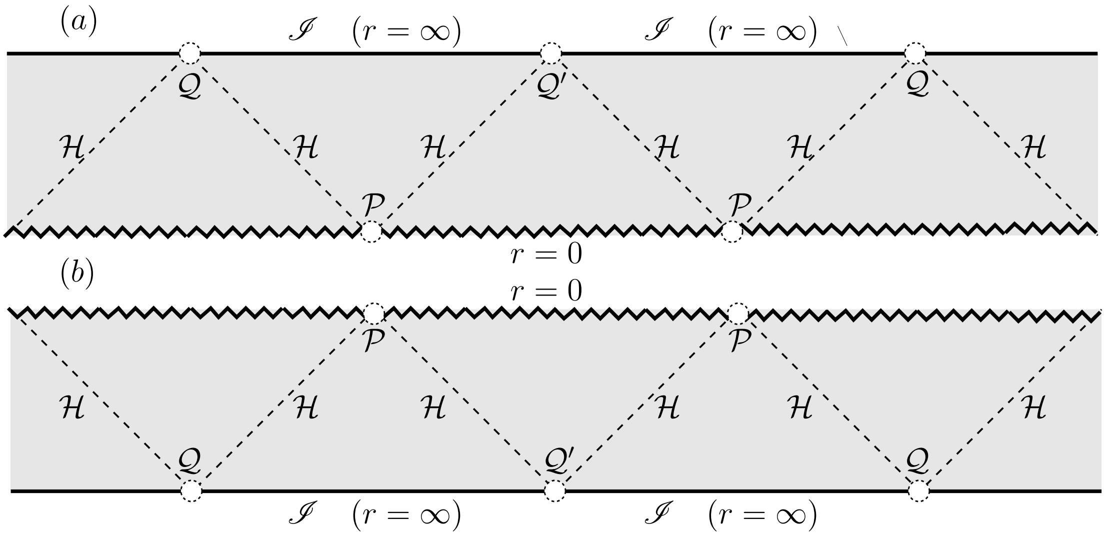

Figure 5.4: *Penrose diagrams for the extremal Schwarzschild-de Sitter spacetime. Case (a) corresponds to a white hole which evolves towards a de Sitter final state while case (b) is a model of a black hole with a future singularity. The continuous black line denotes the conformal boundary; the serrated line denotes the location of the singularity; the dashed line shows the location of the Killing horizons $\mathcal{H}$ at $r = 3m$. The excluded points $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{P}$ represent asymptotic regions of the spacetime that do not belong to $\mathscr{I}$ or the singularity $r = 0$.*

5.3.2 The $\mathbb{S}^3 \backslash \{\mathcal{Q}, \mathcal{Q}'\}$-representation

The basic conformal structure of the subextremal and extremal Schwarzschild-de Sitter spacetimes has already been discussed in [66, 67] and [68] respectively. Coordinate and Penrose diagrams have been also provided in [69] for the subextremal, extremal and hyperextremal cases. This section provides a concise discussion, adapted to the conventions used in this thesis, of the conformal structure of the Schwarzschild-de Sitter spacetime in the subextremal, extremal and hyperextremal cases. In this section the discussion starts showing that irrespective of the relation of m and λ the induced metric at the conformal boundary for the Schwarzschild de Sitter spacetime can be identified with the standard metric on $\mathbb{S}^3$. As discussed in more detail in Section 5.3.3, this construction depends on the particular conformal representation being considered. In the subextremal case one cannot obtain simultaneously an analytic extension regular near both r_b and r_c—see [67]. Since one is interested only in the asymptotic region, in this section only the region $r > r_c$ is considered. For the extremal and hyperextremal cases such considerations are not necessary.

In the following the null coordinates

$$u \equiv \sqrt{|\lambda|}(t - \mathfrak{r}), \qquad v \equiv \sqrt{|\lambda|}(t + \mathfrak{r}),$$

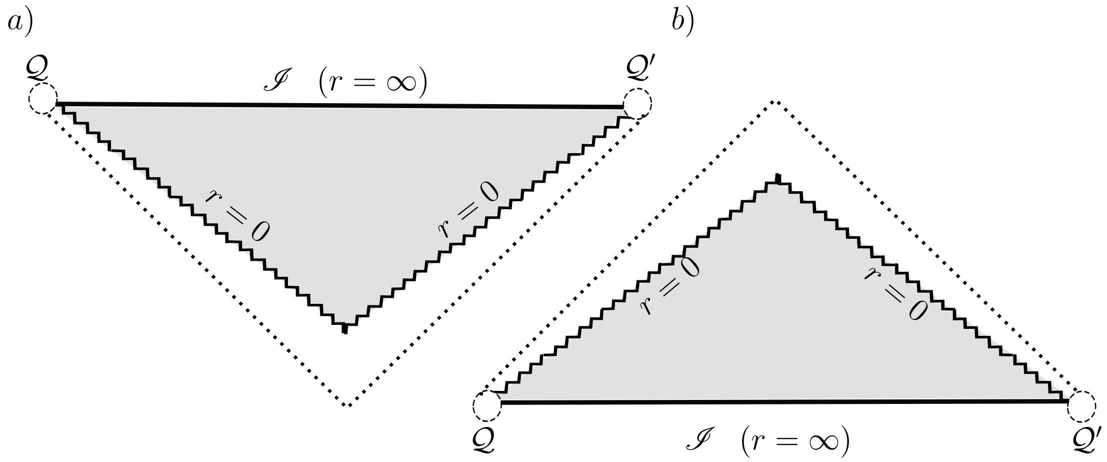

Figure 5.5: *Penrose diagram for the hyperextremal Schwarzschild-de Sitter spacetime. The singularity is of spacelike nature. Dotted lines at 45 ° and 135 ° have been included for visualisation. Case (a) corresponds to a white hole which evolves to a final de-Sitter state. Case (b) corresponds to a black hole with a future spacelike singularity.*

are introduced, where $\mathfrak{r}$ is a *tortoise* coordinate given by

$$\mathfrak{r} \equiv \int \frac{1}{F(r)} dr. \tag{5.14}$$

This integral can be computed explicitly —see [66, 67]. The particular form of $\mathfrak{r}$ depends on the relation between λ and m. As discussed in [66, 68] the integration constant can always be chosen so that $\mathfrak{r} \to 0$ as $r \to \infty$. Defining $\tan U \equiv u, \tan V \equiv v$, with $U, V \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ one gets the line element

$$\tilde{\boldsymbol{g}}_{SdS} = \frac{1}{2}\frac{F(r)}{|\lambda|}\sec^2 U \sec^2 V \left(\mathbf{d}U \otimes \mathbf{d}V + \mathbf{d}V \otimes \mathbf{d}U\right) - r^2 \boldsymbol{\sigma}. \tag{5.15}$$

As discussed in [66, 67], one can construct *Kruskal type* coordinates covering the black hole horizon by choosing appropriately the integration constant in equation (5.14). Analogously, choosing a different integration constant, one can construct Kruskal type coordinates covering the cosmological horizon. Nevertheless in the subextremal case, as emphasised in [67], it is not possible to construct Kruskal type coordinates covering simultaneously both horizons. To construct the Penrose diagram for this spacetime, one considers as building blocks the Penrose diagrams for the regions $0 \leq r \leq r_b$, $r_b \leq r \leq r_c$ and $r_c \leq r < \infty$ which are then glued together using the corresponding Kruskal type coordinates to cross each horizon —see [67, 69] for a detailed discussion on the construction the Penrose diagram and Kruskal type coordinates in the Schwarzschild-de Sitter spacetime. Consistent with the above discussion and given that one is only interested in the asymptotic region, the analysis is restricted, in the subextremal case, to $r > r_c$. In the extremal case one has, however, that $r_b = r_c = 3m$ and one can verify that

$$\lim_{r\to 3m}\frac{\cos U}{r-3m} = \lim_{r\to 3m}\frac{\cos V}{r-3m} = C,$$

where $C \neq 0$ is a constant depending on m and the integration constant chosen in the definition of $\mathfrak{r}$. Consequently, in the extremal case, the metric (5.15) is well defined for the whole range of the coordinate r: $0 < r < \infty$ —see [68]. Introducing the coordinates $(\bar{U}, \bar{V})$ defined via

$$\tan U \equiv \ln\tan\left(\frac{\pi}{4}+\frac{\bar{U}}{2}\right), \qquad \tan V \equiv \ln\tan\left(\frac{\pi}{4}+\frac{\bar{V}}{2}\right)$$

one obtains

$$\tilde{\boldsymbol{g}}_{SdS} = \frac{1}{2}\frac{F(r)}{|\lambda|}\sec\bar{U}\sec\bar{V}\left(\mathbf{d}\bar{U}\otimes\mathbf{d}\bar{V}+\mathbf{d}\bar{V}\otimes\mathbf{d}\bar{U}\right) - r^2\boldsymbol{\sigma}.$$

Recalling that in the subextremal case $F(r) \leq 0$ for $r \geq r_c$ while for the extremal and hyperextremal cases $F(r) \leq 0$ for $0 < r < \infty$, one identifies the conformal factor

$$\Xi^2 = \frac{|\lambda|}{|F(r)|}\cos\bar{U}\cos\bar{V}.$$

Therefore, one obtains the conformal metric $\boldsymbol{g}_{SdS} = \Xi^2\tilde{\boldsymbol{g}}_{SdS}$ with

$$\boldsymbol{g}_{SdS} = -\frac{1}{2}\left(\mathbf{d}\bar{U}\otimes\mathbf{d}\bar{V}+\mathbf{d}\bar{V}\otimes\mathbf{d}\bar{U}\right) - \frac{|\lambda|r^2}{|F(r)|}\cos\bar{U}\cos\bar{V}\boldsymbol{\sigma}. \tag{5.16}$$

Introducing the coordinates

$$T \equiv \bar{U}+\bar{V}, \qquad \Psi \equiv \bar{V}-\bar{U},$$

one gets

$$\boldsymbol{g}_{SdS} = \frac{1}{4}\left(\mathbf{d}\Psi\otimes\mathbf{d}\Psi - \mathbf{d}T\otimes\mathbf{d}T\right) - \frac{|\lambda|r^2}{|F(r)|}\cos\frac{1}{2}(T+\Psi)\cos\frac{1}{2}(T-\Psi)\,\boldsymbol{\sigma}.$$

The analysis in [67] shows that the conformal factor Ξ tends to zero as $r\to\infty$. Hence, to identify the induced metric at $\mathscr{I}$ it is sufficient to analyse such limit. Noticing that

$$\mathfrak{r} = \frac{1}{2\sqrt{|\lambda|}}(v-u) = \frac{1}{2\sqrt{|\lambda|}}\ln\left(\frac{\tan(\pi/4+\bar{V})}{\tan(\pi/4+\bar{U})}\right)$$

and recalling that

$$\lim_{r\to\infty}\mathfrak{r} = 0,$$

one concludes that $r\to\infty$ implies $\Psi = 0$ as long as $\bar{U}\neq\pm\frac{1}{2}\pi$ and $\bar{V}\neq\pm\frac{1}{2}\pi$. Using equation (5.9) one can verify that

$$\lim_{r\to\infty}\frac{|\lambda|r^2}{|F(r)|}=1.$$

Consequently, the induced metric on $\mathscr{I}$ is given by

$$\boldsymbol{h}=-\frac{1}{4}\mathbf{d}T\otimes\mathbf{d}T-\cos^2\frac{T}{2}\boldsymbol{\sigma}$$

which can be written in a more recognisable form introducing $\xi\equiv\frac{1}{2}(T+\pi)$ so that

$$\hbar=-\mathbf{d}\xi\otimes\mathbf{d}\xi-\sin^2\xi\boldsymbol{\sigma}. \tag{5.17}$$

The metric $\hbar$ is the standard metric on $\mathbb{S}^3$. Observe that the excluded points in the discussion of this section $(\bar{U},\bar{V})=(\pm\frac{1}{2}\pi,\pm\frac{1}{2}\pi)$ correspond to $\xi=0$ and $\xi=\pi$ —the North and South poles of $\mathbb{S}^3$. The Penrose diagrams of the subextremal, extremal and hyperextremal Schwarzschild-de Sitter spacetime are given in Figures 5.3, 5.4 and 5.5. The conformal boundary $\mathscr{I}$ of the (subextremal, extremal and hyperextremal) Schwarzschild-de Sitter spacetime, defined by the condition $\Xi=0$, is *spacelike* consistent with the fact that the Cosmological constant of the spacetime is de Sitter-like —see e.g. [12, 70]. Moreover, the singularity at $r=0$ is of a *spacelike nature* —see [68, 69]. As pointed out in [20, 67], the Schwarzschild-de Sitter spacetime can be interpreted as the model of a *white hole* singularity towards a final de Sitter state. Alternatively, making use of a reflection

$$u\mapsto -u, \qquad v\mapsto -v,$$

one obtains a model of a *black hole* with a future singularity —see Figures 5.3, 5.4 and 5.5.

In what follows, the white hole point of view for the extremal and hyperextremal cases will be adopted so that $\mathscr{I}$ corresponds to future conformal infinity and one will consider a *backward asymptotic initial value problem*. Consistent with this point of view, for the subextremal case we consider asymptotic initial data on $\mathscr{I}^+$ and study the development of such data towards the curvature singularity located at $r=0$ —see Figure 5.1.

5.3.3 The $\mathbb{R}\times\mathbb{S}^2$-representation

In Section 5.3.2 it was shown that there exist a conformal representation in which the induced metric on the conformal boundary corresponds to the standard metric on $\mathbb{S}^3$. A quick inspection shows that the metric (5.17) is *conformally flat*. In this section this observation is put in a wider perspective and it is shown how this follows as a consequence of the the spherical symmetry of the spacetime. In addition, a conformal representation in which the induced metric at the conformal boundary corresponds to the standard metric on $\mathbb{R}\times\mathbb{S}^2$ is discussed. This conformal representation will be of particular importance in the subsequent analysis.

The conformal boundary of spherically symmetric and asymptotically de Sitter spacetimes

Following an argument similar to the one given in [71] one has the following construction for a spherically symmetric spacetime with spacelike conformal boundary: if a spacetime $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ is spherically symmetric then the metric $\tilde{\boldsymbol{g}}$ can be written in a warped product form

$$\tilde{\boldsymbol{g}} = \tilde{\boldsymbol{\gamma}} - \tilde{\rho}^2\boldsymbol{\sigma}, \tag{5.18}$$

where $\tilde{\boldsymbol{\gamma}}$ is the 2-metric on the quotient manifold $\tilde{\mathcal{Q}}\equiv\tilde{\mathcal{M}}/SO(3)$, $\boldsymbol{\sigma}$ is the standard metric of $\mathbb{S}^2$ and $\tilde{\rho}:\tilde{\mathcal{Q}}\rightarrow\mathbb{R}$. If $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ are conformally related, $\boldsymbol{g}=\Theta^2\tilde{\boldsymbol{g}}$, then the spherical symmetry condition for $\boldsymbol{g}$ is translated into the requirement that $\boldsymbol{g}$ can be written in the form

$$\boldsymbol{g} = \boldsymbol{\gamma} - \rho^2\boldsymbol{\sigma},$$

where $\boldsymbol{\gamma}\equiv\Theta\tilde{\boldsymbol{\gamma}}$ and $\rho\equiv\Theta\tilde{\rho}$, where Θ does not depend on the coordinates on $\mathbb{S}^2$. Near $\mathscr{I}$ one introduces local coordinates (Θ,ψ) on the quotient manifold $\mathcal{Q}\equiv\mathcal{M}/SO(3)$ so that $\Theta=0$ denotes the locus of $\mathscr{I}$. Since the conformal boundary is spacelike one has that $\boldsymbol{g}(\mathbf{d}\Theta,\mathbf{d}\Theta)>0$. Therefore, the metric induced on $\mathscr{I}$ by $\boldsymbol{g}$ is of the form

$$\boldsymbol{h} = -A(\psi)\mathbf{d}\psi\otimes\mathbf{d}\psi - \rho^2(\psi)\boldsymbol{\sigma},$$

where $A(\psi)$ is a positive function. Redefining the coordinate ψ one can rewrite $\boldsymbol{h}$ as

$$\boldsymbol{h} = -\rho^2(\psi)(\mathbf{d}\psi\otimes\mathbf{d}\psi+\boldsymbol{\sigma}).$$

It can be readily verified —say, by calculating the Cotton tensor of $\boldsymbol{h}$— that the metric $\boldsymbol{h}$ is conformally flat. In Section 5.3.4 it will be shown that, in view of the conformal freedom of the setting, a convenient choice is to consider a conformal representation in which the the 3-metric on $\mathscr{I}$ is given by

$$\boldsymbol{h} = -\mathbf{d}\psi\otimes\mathbf{d}\psi - \boldsymbol{\sigma}. \tag{5.19}$$

This metric is the standard metric of the cylinder $\mathbb{R}\times\mathbb{S}^2$ with $\psi\in(-\infty,\infty)$. It can be verified that this conformal representation is related to the one discussed in Section 5.3.2 via $\boldsymbol{h}=\omega^2\hbar$, where the conformal factor ω and the relation between the coordinates are given by

$$\psi(\xi) = \psi_\star - \ln|\csc\xi+\cot\xi|, \qquad \omega(\xi)=\csc(\xi). \tag{5.20}$$

Equivalently, one has that

$$\xi(\psi) = \arccos\left(\frac{e^{2(\psi_\star-\psi)}-1}{e^{2(\psi_\star-\psi)}+1}\right), \qquad \omega(\psi) = \frac{e^{\psi}}{2e^{\psi_\star}}(e^{2\psi_\star}+e^{2\psi}),$$

where $\psi_\star$ is a constant of integration. Observe that in this representation $\xi = 0$ and $\xi = \pi$ correspond to $\psi = -\infty$ and $\psi = \infty$, respectively.

The extrinsic curvature of the conformal boundary in the $\mathbb{R}\times\mathbb{S}^2$ representation

A particularly simple conformal representation for the Schwarzschild-de Sitter spacetime can be obtained using the discussion of Section 5.3.3. Accordingly, take the metric of the Schwarzschild-de Sitter spacetime as written in equation (5.8) with $F(r)$ as given by the relation (5.9) and consider the conformal factor $\widehat{\Xi} \equiv 1/r$. Introducing the coordinates $\varrho \equiv 1/r$ and $\zeta \equiv \sqrt{|\lambda|/3}t$, the conformal metric

$$\widehat{\boldsymbol{g}} \equiv \widehat{\Xi}^2 \tilde{\boldsymbol{g}}_{SdS}$$

is given by

$$\widehat{\boldsymbol{g}} = \frac{3}{|\lambda|}\Big(\varrho^2 - 2m\varrho^3 - \frac{1}{3}|\lambda|\Big)\mathbf{d}\zeta\otimes\mathbf{d}\zeta - \Big(\varrho^2 - 2m\varrho^3 - \frac{1}{3}|\lambda|\Big)^{-1}\mathbf{d}\varrho\otimes\mathbf{d}\varrho - \boldsymbol{\sigma}.$$

The induced metric on the hypersurface described by the condition $\widehat{\Xi} = 0$ is given by

$$\widehat{\boldsymbol{h}} = -\mathbf{d}\zeta\otimes\mathbf{d}\zeta - \boldsymbol{\sigma}.$$

It can be verified that $\widehat{\boldsymbol{g}}$ *satisfies a conformal gauge for which the conformal boundary has vanishing extrinsic curvature.* To see this, consider a $\widehat{\boldsymbol{g}}$-orthonormal coframe $\{\boldsymbol{\omega}^{\boldsymbol{a}}\}$ with

$$\boldsymbol{\omega}^0 = \sqrt{\frac{3}{|\lambda|}}\left(\varrho^2 - 2m\varrho^3 - \frac{1}{3}|\lambda|\right)^{1/2}\mathbf{d}\zeta, \qquad \boldsymbol{\omega}^3 = \left(\varrho^2 - 2m\varrho^3 - \frac{1}{3}|\lambda|\right)^{-1/2}\mathbf{d}\varrho,$$

and $\{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2\}$ a $\boldsymbol{\sigma}$-orthonormal coframe. Denote by $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ the corresponding dual frame. Using this frame one can directly compute the Friedrich scalar $\widehat{s} \equiv \frac{1}{4}\widehat{\nabla}^{\boldsymbol{c}}\widehat{\nabla}_{\boldsymbol{c}}\widehat{\Xi} + \frac{1}{24}\widehat{R}\,\widehat{\Xi}$. The computation of the Ricci scalar yields

$$\widehat{R} = -12m\varrho. \tag{5.21}$$

A direct calculation using

$$\widehat{\nabla}_\mu\widehat{\nabla}^\mu\Xi = \frac{1}{\sqrt{-\det\widehat{\boldsymbol{g}}}}\partial_\mu(\sqrt{-\det\widehat{\boldsymbol{g}}}\,\widehat{g}^{\mu\nu}\partial_\nu\Xi)$$

shows that $\widehat{\nabla}_a\widehat{\nabla}^a\Xi = 6m\varrho^2 - 2\varrho$. Consequently, the scalar $\widehat{s}$ vanishes at the hypersurface defined by $\widehat{\Xi} = \varrho = 0$. Contrasting this result with the solution to the conformal constraints given in equations (2.78a)-(2.78b) of Chapter 2 one concludes that in this representation the hypersurface described by $\widehat{\Xi} = 0$ has vanishing extrinsic curvature as claimed.

Remark 39. Notice that, in this representation the curvature singularity, located at $r = 0$, corresponds to $\varrho = \infty$. Consequently, the singularity is at an infinite distance from the conformal boundary.

Observe that, the components of the Weyl tensor with respect to the orthonormal frame $\{\boldsymbol{e_a}\}$ as described above are given by

$$C_{1212} = -2m\varrho,\; C_{1313} = m\varrho,\; C_{1010} = -m\varrho,\; C_{2323} = m\varrho,\; C_{2020} = -m\varrho,\; C_{3030} = 2m\varrho.$$

This information will be required in the discussion of the initial data for the rescaled Weyl tensor —see Section 5.3.4. Using now that $d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = \Xi^{-1}C^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ with $\widehat{\Xi} = \varrho$ and exploiting the fact that the computations have been carried out in an orthonormal frame so that $C^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = \eta^{\boldsymbol{af}}C_{\boldsymbol{fbcd}}$, one gets

$$d_{1212} = -2m, \quad d_{1313} = m, \quad d_{1010} = -m, \quad d_{2323} = m, \quad d_{2020} = -m, \quad d_{3030} = 2m.$$

Finally, considering $d_{ij} \equiv d_{i0j0}$ one obtains

$$d_{11} = -m, \quad d_{22} = -m, \quad d_{33} = 2m. \tag{5.22}$$

5.3.4 Identifying asymptotic regular data

As discussed in Section 5.3.1, there is a conformal representation in which the induced metric on the conformal boundary of the Schwarzschild-de Sitter is the standard metric $\hbar$ on $\mathbb{S}^3$. Nevertheless, the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$, as depicted in the Penrose diagram of Figure 5.3, are associated to the behaviour of those timelike geodesics which never cross the horizon —see Appendix 5.6. Despite that, from the point of view of the intrinsic geometry of $\mathscr{I}$ these asymptotic regions —corresponding to the North and South poles of $\mathbb{S}^3$— are regular, from a spacetime point of view they are not. This issue will be further discussed in this section. In particular, it will be shown that the initial data for the electric part of the rescaled Weyl tensor is singular at $\mathcal{Q}$ and $\mathcal{Q}'$. Fortunately, as exposed in Sections 5.2.2 one can exploit the inherent conformal freedom of the setting to select any representative of the conformal class $[\hbar]$ to construct a solution to the conformal constraint equations —see Section 2.5 of Chapter 2. Taking into account the previous remarks it will be convenient to choose the conformal representation discussed in Section

5.3.3, $\boldsymbol{h} = \omega^2 \hbar$ with ω and $\boldsymbol{h}$ given in equations (5.19) and (5.20), in which the points $\mathcal{Q}$ and $\mathcal{Q}'$ are at infinity respect to the metric $\boldsymbol{h}$.

A frame for the induced metric at $\mathscr{I}$

Consistent with the discussion of the last section, on $\mathscr{I}$ one considers an adapted frame $\{\boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ such that the metric (5.19) can be written in the form

$$\boldsymbol{h} = -(\boldsymbol{l} \otimes \boldsymbol{l} + \boldsymbol{\sigma})$$

where

$$\boldsymbol{l} = \mathbf{d}\psi, \qquad \boldsymbol{\sigma} = \frac{1}{2}(\boldsymbol{m} \otimes \bar{\boldsymbol{m}} + \bar{\boldsymbol{m}} \otimes \boldsymbol{m}).$$

In terms of abstract index notation one has

$$h_{ij} = -l_i l_j - 2m_{(i}\bar{m}_{j)}. \tag{5.23}$$

The frame $\{\boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ satisfies the pairings

$$l_j l^j = -1, \qquad m_j \bar{m}^j = -1, \qquad l_j m^j = l_j \bar{m}^j = m_j m^j = \bar{m}_j \bar{m}^j = 0. \tag{5.24}$$

Initial data for the rescaled Weyl tensor

The procedure for the construction of a solution to the conformal constraints at the conformal boundary requires, in particular, a solution to the divergence equation (2.78c) for the electric part of the rescaled Weyl tensor —see Section 2.5.2 in Chapter 2. The requirement of spherical symmetry of the spacetime can be succinctly incorporated using the results in [72]. If the unphysical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ possesses a Killing vector $\boldsymbol{X}$ then the initial data encoded in the symmetric tracefree tensor d_{ij} must satisfy the condition

$$\pounds_{\boldsymbol{X}} d_{ij} = 0, \tag{5.25}$$

where $\pounds_{\boldsymbol{X}}$ denotes the Lie derivative in the direction of $\boldsymbol{X}$ on the initial hypersurface. If d_{ij} is to be compatible with the symmetries of $\mathbb{R} \times \mathbb{S}^2$ then it is of form

$$d_{ij} = \frac{1}{2}\varsigma(3l_i l_j + h_{ij}). \tag{5.26}$$

where $\varsigma = d_{ij} l^i l^j$. The most important property of the geometry of the conformal boundary of the Schwarzschild-de Sitter spacetime is the fact that it is conformally flat. The latter is particularly convenient as the general form of symmetric, tracefree and divergence-free tensors (i.e. *TT-tensors*) in a conformally flat setting are well-known —see e.g. [42, 73]. This opens the possibility of determining the required d_{ij} not only in the $\mathbb{R} \times \mathbb{S}^2$-representation but to explore solutions to equation (2.78c) in other conformal representations which will be relevant for the subsequent discussion.

TT-tensors on $\mathbb{R}^3$. For convenience of the reader, in this short paragraph, the conventions and discussion of TT-tensors on Euclidean space given in [42, 73] are adapted to the present setting. The general solutions to the equation

$$\grave{D}^{\boldsymbol{i}}\grave{d}_{\boldsymbol{ij}} = 0, \tag{5.27}$$

where $\grave{\boldsymbol{h}} \equiv -\boldsymbol{\delta}$ is the flat metric has been given in [42]. One can introduce Cartesian coordinates $(x^{\boldsymbol{\alpha}})$ with the origin of $\mathbb{R}^3$ located at a fiduciary position $\mathcal{O}$. Additionally, polar coordinates defined via $\rho^2 = \delta_{\boldsymbol{\alpha\beta}}x^{\boldsymbol{\alpha}}x^{\boldsymbol{\beta}}$ are introduced. The flat metric in these coordinates reads

$$\grave{\boldsymbol{h}} = -\mathbf{d}\rho \otimes \mathbf{d}\rho - \rho^2\boldsymbol{\sigma}. \tag{5.28}$$

Using this notation and taking into account the requirement of spherical symmetry encoded in equation (5.25) the flat space counterpart of the required solution is

$$\grave{\boldsymbol{d}} = \frac{A_\star}{\rho^3}\left(3\mathbf{d}\rho \otimes \mathbf{d}\rho + \grave{\boldsymbol{h}}\right),$$

where $A_\star$ is a constant. In order to obtain an analogous solution in conformally related 3-manifolds one can exploit the conformal properties of equation (5.27) using the following:

Lemma 13. *Let $\bar{d}_{ij}$ be a tracefree symmetric solution to $\bar{D}^i\bar{d}_{ij} = 0$ where $\bar{D}$ is the Levi-Civita connection of $\bar{\boldsymbol{h}}$. Let $\boldsymbol{h} = \omega^2\bar{\boldsymbol{h}}$, then $d_{ij} = \omega^{-1}\bar{d}_{ij}$ is a symmetric tracefree solution to $D^id_{ij} = 0$ where D is the Levi-Civita connection of $\boldsymbol{h}$.*

This lemma can be found in [42]. Here the statements have been adapted to agree with the conventions of this thesis.

TT-tensors on $\mathbb{S}^3$ and $\mathbb{R}\times\mathbb{S}^2$. One can exploit Lemma 13 to derive spherically symmetric solutions of the divergence equation (5.27) in conformally flat 3-manifolds. In particular, the metrics $\hbar$ and $\grave{\boldsymbol{h}}$ as given in equations (5.17) and (5.28) are related via

$$\hbar = \omega^2\grave{\boldsymbol{h}},$$

where

$$\rho(\xi) = \cot(\xi/2), \qquad \omega(\xi) = 2\sin^2(\xi/2), \tag{5.29}$$

The coordinate transformation $\rho(\xi)$ corresponds to the stereographic projection in which the origin $\mathcal{O}$ of $\mathbb{R}^3$ is mapped to the South pole on $\mathbb{S}^3$. Alternatively, one can also derive

$$\rho(\xi) = \tan(\xi/2), \qquad \omega(\xi) = 2\cos^2(\xi/2), \tag{5.30}$$

corresponding to the stereographic projection in which the origin of $\mathbb{R}^3$ is mapped to the North pole of $\mathbb{S}^3$. Using Lemma 13 with equations (5.29) or (5.30) one obtains

$$\grave{\boldsymbol{d}} = \frac{A_\star}{2\sqrt{1-\omega^2(\xi)}}\left(3\mathbf{d}\xi\otimes\mathbf{d}\xi+\hbar\right). \tag{5.31}$$

Observe that $\grave{d}_{ij}$ is singular when $\omega(\xi)=1$ which corresponds to $\xi=0$ and $\xi=\pi$ according to equations (5.29) and (5.30), respectively. Therefore, in this conformal representation the electric part of the rescaled Weyl tensor is singular at the North and South poles of $\mathbb{S}^3$. Proceeding in a analogous way as in the previous paragraphs one can observe that the metrics $\boldsymbol{h}$ and $\grave{\boldsymbol{h}}$ given in equations (5.19) and (5.28) are related via

$$\boldsymbol{h} = \omega^2\grave{\boldsymbol{h}}$$

where

$$\rho(\psi)=e^{\psi}, \qquad \omega(\psi)=e^{-\psi}.$$

A straightforward computation using Lemma 13 renders

$$\boldsymbol{d} = A_\star\left(3\mathbf{d}\psi\otimes\mathbf{d}\psi+\boldsymbol{h}\right). \tag{5.32}$$

Moreover, one can verify that $\pounds_{\boldsymbol{X}} d_{ij}=0$. Finally, comparing expression (5.32) with equation (5.22) one can recognise that $A_\star = m$. Observe that this identification is irrespective of the extrinsic curvature of $\mathscr{I}$ since the latter is fixed by κ which does not play a role in the determination of a solution to $D^i d_{ij}=0$ —see Lemma 10 and Remark 10.

5.3.5 Asymptotic initial data for the Schwarzschild-de Sitter spacetime

In the last section it was shown that the $\mathbb{R}\times\mathbb{S}^2$-conformal representation leads to regular asymptotic data for the rescaled Weyl tensor. In this section the discussion of the asymptotic initial data for the Schwarzschild-de Sitter spacetime is completed for this conformal representation. To do so, one makes use of the procedure to solve the conformal constraints at the conformal boundary, as discussed in Section 2.5.2 of Chapter 2, and the specific properties of the Schwarzschild-de Sitter spacetime.

Initial data for the Schouten tensor

Computing the Schouten tensor $\boldsymbol{Sch}[\boldsymbol{h}]$ of $\boldsymbol{h}$ one obtains

$$\boldsymbol{Sch}[\boldsymbol{h}] = -\frac{1}{2}\mathbf{d}\psi\otimes\mathbf{d}\psi+\frac{1}{2}\boldsymbol{\sigma}.$$

Equivalently, in abstract index notation one writes

$$l_{ij} = -l_i l_j - \frac{1}{2} h_{ij}.$$

Thus, recalling the solution to the conformal constraints given in equation (2.78b) of Chapter 2, one gets

$$L_{ij} = -l_i l_j - \frac{1}{2}(1-\kappa^2) h_{ij}.$$

Initial data for the connection coefficients

In order to compute the connection coefficients associated with the coframe $\{\boldsymbol{\omega}_{\boldsymbol{i}}\}$ recall that $\boldsymbol{\omega}^3 = \mathbf{d}\psi$ and $\{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2\}$ are $\boldsymbol{\sigma}$-orthonormal. Equivalently, one has that $\{\boldsymbol{e}_{\boldsymbol{i}}\} = \{\boldsymbol{\partial}_\psi, \boldsymbol{e}_{\boldsymbol{1}}, \boldsymbol{e}_{\boldsymbol{2}}\}$ with

$$\boldsymbol{e}_{\boldsymbol{1}} = \frac{1}{\sqrt{2}}(\boldsymbol{m} + \bar{\boldsymbol{m}}), \qquad \boldsymbol{e}_{\boldsymbol{2}} = \frac{\mathrm{i}}{\sqrt{2}}(\boldsymbol{m} - \bar{\boldsymbol{m}}),$$

where $\boldsymbol{\sigma} = \boldsymbol{m} \otimes \bar{\boldsymbol{m}} + \bar{\boldsymbol{m}} \otimes \boldsymbol{m}$, so that

$$\boldsymbol{h} = -\boldsymbol{\omega}^1 \otimes \boldsymbol{\omega}^1 - \boldsymbol{\omega}^2 \otimes \boldsymbol{\omega}^2 - \boldsymbol{\omega}^3 \otimes \boldsymbol{\omega}^3.$$

The connection coefficients can be obtained using the first structure equation (5.88a) given in Appendix 5.8.1. Proceeding in this manner, by a straightforward computation, one can show that the only non-zero connection coefficient is $\gamma_2{}^2{}_1$. In terms of the Ricci-rotation coefficients, the latter corresponds to $2\sqrt{2}\,\mathrm{Re}(\alpha_\star)$ where $\alpha_\star = -\frac{1}{2}\bar{m}^a \bar{\delta} m_a$ in the standard NP notation —see [12]. Therefore, the only no-trivial initial data for the connection coefficients is

$$\gamma_2{}^2{}_1 = \sqrt{2}(\alpha_\star + \bar{\alpha}_\star).$$

Remark 40. The frame over the cylinder $\mathbb{R} \times \mathbb{S}^2$ introduced in this section is not a global one. Nevertheless, it is possible to construct an atlas covering $\mathbb{R} \times \mathbb{S}^2$ such that one each of the charts one has a well defined frame of the required form.

Spinorial initial data

In this section the spinorial counterpart of the asymptotic initial data computed in the previous sections is discussed.

Spin connection coefficients

The spinorial counterpart of the asymptotic initial data constructed in the previous sections is readily obtained by suitable contraction with the spatial Infeld-van der Waerden symbols —see Appendix 5.8.3. Following the discussion of Section 5.3.5, let $\boldsymbol{\omega}^3 = \mathbf{d}\psi$ and let $\{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2\}$ denote a $\boldsymbol{\sigma}$-orthonormal coframe. Using equations (5.95b) of Appendix 5.8.3 one has that the spinorial coframe is given by

$$\boldsymbol{\omega}^{\boldsymbol{AB}} = \sigma_{\boldsymbol{i}}{}^{\boldsymbol{AB}}\boldsymbol{\omega}^{\boldsymbol{i}} = (y^{\boldsymbol{AB}} + z^{\boldsymbol{AB}})\boldsymbol{\omega}^1 + \mathrm{i}(y^{\boldsymbol{AB}} - z^{\boldsymbol{AB}})\boldsymbol{\omega}^2 - x^{\boldsymbol{AB}}\boldsymbol{\omega}^3. \tag{5.33}$$

Alternatively, one has that the *spinorial frame* is given by

$$\boldsymbol{e_{AB}} = x_{\boldsymbol{AB}} e_x{}^3 \boldsymbol{\partial}_\psi + \sqrt{2} y_{\boldsymbol{AB}} e_y{}^+ \bar{\boldsymbol{m}}^\flat + \sqrt{2} z_{\boldsymbol{AB}} e_z{}^- \boldsymbol{m}^\flat$$

where $e_x{}^3$, $e_y{}^+$, $e_z{}^-$ denote the only non-vanishing frame coefficients. Equation (5.33) allows to compute the reduced connection coefficients $\gamma_{\boldsymbol{A}}{}^{\boldsymbol{B}}{}_{\boldsymbol{CD}}$ using the first Cartan structure equation (5.94a) in Appendix 5.8.3. Alternatively, one can use the results of Section 5.3.5 and the spatial Infeld-van der Waerden symbols to compute

$$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}} \equiv \gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}} \sigma^{\boldsymbol{CD}}{}_{\boldsymbol{j}} \sigma_{\boldsymbol{EF}}{}^{\boldsymbol{k}},$$

where

$$\gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} = \delta_{\boldsymbol{i}}{}^2 \delta_1{}^{\boldsymbol{j}} \delta_{\boldsymbol{k}}{}^2 \gamma_2{}^1{}_2 + \delta_{\boldsymbol{i}}{}^2 \delta_2{}^{\boldsymbol{j}} \delta_{\boldsymbol{k}}{}^1 \gamma_2{}^2{}_1,$$

with

$$\gamma_2{}^1{}_2 = -\sqrt{2}(\alpha_\star + \bar{\alpha}_\star), \qquad \gamma_2{}^2{}_1 = \sqrt{2}(\alpha_\star + \bar{\alpha}_\star).$$

Using the identities (5.95a)-(5.95b) in Appendix 5.8.3 one obtains

$$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}} = 2\sqrt{2}(\alpha_\star + \bar{\alpha}_\star)(y_{\boldsymbol{AB}} - z_{\boldsymbol{AB}})(y_{\boldsymbol{EF}} z^{\boldsymbol{CD}} - y^{\boldsymbol{CD}} z_{\boldsymbol{EF}}).$$

Thus, the reduced connection coefficients are given by

$$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{D}}{}_{\boldsymbol{F}} \equiv \frac{1}{2}\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{CD}} = (\alpha_\star + \bar{\alpha}_\star) x^{\boldsymbol{D}}{}_{\boldsymbol{F}} (y_{\boldsymbol{AB}} - z_{\boldsymbol{AB}}). \tag{5.34}$$

By computing the spinor version of the *connection form* $\boldsymbol{\gamma}^{\boldsymbol{D}}{}_{\boldsymbol{F}} \equiv \gamma_{\boldsymbol{AB}}{}^{\boldsymbol{D}}{}_{\boldsymbol{F}} \boldsymbol{\omega}^{\boldsymbol{AB}}$ using equations (5.34) and (5.33) one can readily verify that the first structure equation is satisfied. Additionally, using the reality conditions,

$$x_{\boldsymbol{AB}}{}^\dagger = -x_{\boldsymbol{AB}}, \qquad y_{\boldsymbol{AB}}{}^\dagger = z_{\boldsymbol{AB}}, \qquad z_{\boldsymbol{AB}}{}^\dagger = y_{\boldsymbol{AB}} \tag{5.35}$$

one can verify that $\gamma_{\boldsymbol{ABCD}}$ is an imaginary spinor —as is to be expected from the space spinor formalism. The field $\gamma_{\boldsymbol{ABCD}}$ represents the initial data for the field $\xi_{\boldsymbol{ABCD}}$ —the imaginary part of the reduced connection coefficient $\Gamma_{\boldsymbol{ABCD}}$. The real part of $\Gamma_{\boldsymbol{ABCD}}$ corresponds to the Weingarten spinor $\chi_{\boldsymbol{ABCD}}$ which, in accordance with equation (2.78a) of Chapter 2, is given initially by

$$\chi_{\boldsymbol{ABCD}} = \kappa h_{\boldsymbol{ABCD}}.$$

Rewriting the reduced connection coefficients (5.34) in terms of the basic valence-4 spinors as defined in Appendix 5.8.2 one obtains for $\xi_{\boldsymbol{ABCD}} = \gamma_{\boldsymbol{ABCD}}$ the explicit expression

$$\xi_{\boldsymbol{ABCD}} = -(\alpha_\star + \bar{\alpha}_\star)(\epsilon^1{}_{\boldsymbol{ABCD}} + \epsilon^3{}_{\boldsymbol{ABCD}}) \\ + \frac{1}{2\sqrt{2}}(\alpha_\star + \bar{\alpha}_\star)\epsilon_{\boldsymbol{AC}}(y_{\boldsymbol{BD}} + z_{\boldsymbol{BD}}) + \frac{1}{2\sqrt{2}}(\alpha_\star + \bar{\alpha}_\star)\epsilon_{\boldsymbol{BD}}(y_{\boldsymbol{AC}} + z_{\boldsymbol{AC}}).$$

Spinorial counterpart of the Schouten tensor

The spinorial counterpart of the Schouten tensor l_{ij} can be directly read from the expressions in Section 5.3.5. Observe that the elementary spinor $x^{\boldsymbol{AB}}$ corresponds to the components of l_i with respect to the coframe (5.33) since

$$\boldsymbol{\omega}^{\boldsymbol{AB}} x_{\boldsymbol{AB}} = -x^{\boldsymbol{AB}} x_{\boldsymbol{AB}} \boldsymbol{\omega}^3 = \boldsymbol{\omega}^3 = \mathbf{d}\psi = \boldsymbol{l}.$$

Replacing $h_{\boldsymbol{ij}}$ by its space spinor counterpart $h_{\boldsymbol{ABCD}}$ one obtains

$$l_{\boldsymbol{ij}} \mapsto l_{\boldsymbol{ABCD}} = -x_{\boldsymbol{AB}} x_{\boldsymbol{CD}} - \frac{1}{2} h_{\boldsymbol{ABCD}}.$$

Equivalently, recalling that the space spinor counterpart of the tracefree part of a tensor $l_{\{\boldsymbol{ij}\}} \equiv l_{\boldsymbol{ij}} - \frac{1}{3} l h_{\boldsymbol{ij}}$ corresponds to the totally symmetric spinor $l_{(\boldsymbol{ABCD})}$ it follows then from

$$l_{\boldsymbol{ij}} = l_{\{\boldsymbol{ij}\}} + \frac{1}{3} l h_{\boldsymbol{ij}},$$

that

$$l_{\boldsymbol{ABCD}} = l_{(\boldsymbol{ABCD})} + \frac{1}{3} l h_{\boldsymbol{ABCD}}.$$

Using that $l \equiv h^{\boldsymbol{ij}} l_{\boldsymbol{ij}} = \frac{1}{4} r$, where r is the Ricci scalar of $\boldsymbol{h}$, and that for the metric (5.19) one has $r = -2$ it follows that $l = -\frac{1}{2}$. Exploiting that $l_{(\boldsymbol{ABCD})} = -x_{(\boldsymbol{AB}} x_{\boldsymbol{CD})} = -2\epsilon^2_{\boldsymbol{ABCD}}$ one obtains

$$l_{\boldsymbol{ABCD}} = -2\epsilon^2{}_{\boldsymbol{ABCD}} - \frac{1}{6} h_{\boldsymbol{ABCD}}. \tag{5.36}$$

Finally, recalling the expressions for the components of the spacetime Schouten tensor, as given in equation (2.78b) of Chapter 2, one concludes

$$L_{\boldsymbol{ABCD}} = -2\epsilon^2{}_{\boldsymbol{ABCD}} - \frac{1}{6}(1 - 3\kappa^2) h_{\boldsymbol{ABCD}}.$$

Initial data for the rescaled Weyl spinor

Following the approach employed in last section, the spinorial counterpart of expression (5.32) is given by

$$d_{\boldsymbol{ABCD}} = A_\star (3 l_{\boldsymbol{AB}} l_{\boldsymbol{CD}} + h_{\boldsymbol{ABCD}}).$$

However, the trace-freeness condition simplifies the last expression since $d^{\boldsymbol{i}}{}_{\boldsymbol{i}} = 0$ implies that $d_{\boldsymbol{ij}} = d_{\{\boldsymbol{ij}\}}$. Therefore $d_{\boldsymbol{ABCD}} = d_{(\boldsymbol{ABCD})} = 3A_\star l_{(\boldsymbol{AB}} l_{\boldsymbol{CD})}$. As the elementary spinor $x_{\boldsymbol{AB}}$ can be associated to the components of $\boldsymbol{l}$ respect to the coframe (5.33) one gets that

$$d_{\boldsymbol{ABCD}} = 3A_\star x_{(\boldsymbol{AB}} x_{\boldsymbol{CD})}.$$

This last expression can be equivalently written in terms of the basic valence-4 space spinors, defined in Appendix 5.8.2, as

$$\phi_{\boldsymbol{ABCD}} = 6m\epsilon^2{}_{\boldsymbol{ABCD}}.$$

where, in the absence of a magnetic part, one has identified $\phi_{\boldsymbol{ABCD}}$ initially with $d_{\boldsymbol{ABCD}}$. Observe that $A_\star = m$ has been set, consistent with the discussion of Section 5.3.4.

5.4 The solution to the asymptotic initial value problem for the Schwarzschild-de Sitter spacetime and perturbations

As already discussed in the introductory section, recasting explicitly the Schwarzschild-de Sitter spacetime as a solution to the system of conformal evolution equations (2.68a)-(2.68i) of Chapter 2 requires solving, in an explicit manner, the conformal geodesic equations. This, as discussed in Appendix 5.6.2, is not possible in general. Instead, an alternative approach is to study directly the conformal evolution equations (2.68a)-(2.68i) making explicit the spherical symmetry of the solution and the asymptotic initial data corresponding to the Schwarzschild-de Sitter spacetime. This approach does not only extract the required information about the reference solution —in the conformal Gaussian gauge— but, in addition, is a model for the general structure of the conformal evolution equations. The relevant analysis is discussed in Sections 5.4.1 and 5.4.2. As a complementary analysis, the formation of singularities in the evolution equations is also studied. In order to have a more compact discussion leading to the Main Result of this chapter, the analysis of the formation of singularities is presented in Appendix 5.7. Finally, in Section 5.4.3, the theory of symmetric hyperbolic systems contained in [23] is used to obtain a existence and stability result for the development of small perturbations to the asymptotic initial data of the Schwarzschild-de Sitter spacetime.

5.4.1 The spherically symmetric evolution equations

Hitherto, the discussion of the extended conformal Einstein field equations and the conformal constraint equations has been completely general. Since one is interested in analysing the Schwarzschild-de Sitter spacetime as a solution to the conformal field equations it is necessary to incorporate specific properties of this spacetime. The most important assumption for this analysis is that of the *spherical symmetry* of the spacetime. As mentioned before, under this assumption, a generalisation of Birkhoff's theorem for vacuum spacetimes with de Sitter-like Cosmological constant shows that the spacetime must be locally isometric to either the Nariai or the Schwarzschild-de Sitter solutions —see [58]. As the Nariai solution is known to not admit a smooth conformal boundary [57, 62], then the formulation of an asymptotic initial value problem readily selects the Schwarzschild-de Sitter spacetime.

To incorporate the assumption of spherical symmetry into the conformal field equations encoded in the spinorial zero-quantities (2.25a)-(2.25d) of Chapter 2 one has to reexpress the requirement of spherical symmetry in terms of the space spinor formalism. In order to ease the presentation one can simply introduce a consistent Ansatz for spherical symmetry —a similar approach has been taken in [71]. More precisely, one sets

$$\phi_{\boldsymbol{ABCD}} = \phi_2\, \epsilon^2{}_{\boldsymbol{ABCD}}, \tag{5.37a}$$

$$\Theta_{\boldsymbol{AB}} = \sqrt{2}\Theta_x{}^T\, x_{\boldsymbol{AB}}, \tag{5.37b}$$

$$\Theta_{\boldsymbol{ABCD}} = \Theta_2{}^S\, \epsilon^2{}_{\boldsymbol{ABCD}} + \frac{1}{3}\Theta_h{}^S\, h_{\boldsymbol{ABCD}}, \tag{5.37c}$$

$$\begin{aligned}\xi_{\boldsymbol{ABCD}} = {} & \xi_1\, \epsilon^1{}_{\boldsymbol{ABCD}} + \xi_2\, \epsilon^2{}_{\boldsymbol{ABCD}} + \xi_3\, \epsilon^3{}_{\boldsymbol{ABCD}} + \frac{1}{3}\xi_h\, h_{\boldsymbol{ABCD}} \\ & + \frac{\xi_x}{\sqrt{2}}(x_{\boldsymbol{BD}}\epsilon_{\boldsymbol{AC}} + x_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}}) + \frac{\xi_y}{\sqrt{2}}(y_{\boldsymbol{BD}}\epsilon_{\boldsymbol{AC}} + y_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}}) \\ & + \frac{\xi_z}{\sqrt{2}}(z_{\boldsymbol{BD}}\epsilon_{\boldsymbol{AC}} + z_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}}),\end{aligned} \tag{5.37d}$$

$$\chi_{\boldsymbol{ABCD}} = \chi_2\, \epsilon^2_{\boldsymbol{ABCD}} + \frac{1}{3}\chi_h\, h_{\boldsymbol{ABCD}}, \tag{5.37e}$$

$$e^0{}_{\boldsymbol{AB}} = e^0_x\, x_{\boldsymbol{AB}}, \quad e^3_{\boldsymbol{AB}} = e^3_x\, x_{\boldsymbol{AB}}, \quad e^+_{\boldsymbol{AB}} = e^+_y\, y_{\boldsymbol{AB}}, \quad e^-_{\boldsymbol{AB}} = e^-_z\, z_{\boldsymbol{AB}}, \tag{5.37f}$$

$$f_{\boldsymbol{AB}} = f_x\, x_{\boldsymbol{AB}}, \tag{5.37g}$$

$$d_{\boldsymbol{AB}} = d_x\, x_{\boldsymbol{AB}}. \tag{5.37h}$$

The *elementary spinors* $x_{\boldsymbol{AB}}$, $y_{\boldsymbol{AB}}$, $z_{\boldsymbol{AB}}$, $\epsilon^2_{\boldsymbol{ABCD}}$ and $h_{\boldsymbol{ABCD}}$ used in the above Ansatz are defined in Appendix 5.8.2. For further details on the construction of a general spherically symmetric Ansatz see [27, 74]. Alternatively, one can follow a procedure similar to that of Section 5.3.5 —by writing a consistent spherically symmetric Ansatz for the orthonormal frame one can identify the non-vanishing components of the required tensors. The transition to the spinorial version of such Ansatz can be obtained by contracting appropriately with the Infeld-van der

Waerden symbols taking into account equations (5.95a)-(5.95b), (5.92a)-(5.92d) and (5.93b)-(5.93g) of Appendices 5.8.2 and 5.8.3.

The Ansatz for spherical symmetry encoded in equations (5.37a)-(5.37h) combined with the evolution equations (2.68a)-(2.68i) leads, after suitable contraction with the elementary spinors of Appendix 5.8.2, to a set of evolution equations for the fields

$$\phi_2,\ {\Theta_x}^T,\ {\Theta_2}^S,\ {\Theta_h}^S, \xi_1,\ \xi_3,\ \xi_x,\ \xi_y,\ \xi_z,\ \ e^0_x,\ e^3_x,\ e^+_z, e^-_y,\ f_x.$$

This lengthy computation has been carried out using the suite `xAct` for tensor and spinorial manipulations in `Mathematica` —see [75]. At the end of the day one obtains the following evolution equations:

$$\partial_\tau e^0_x = \tfrac{1}{3}\chi_2 e^0_x - \tfrac{1}{3}\chi_h e^0_x - f_x, \tag{5.38a}$$
$$\partial_\tau e^3_x = \tfrac{1}{3}\chi_2 e^3_x - \tfrac{1}{3}\chi_h e^3_x, \tag{5.38b}$$
$$\partial_\tau e^+_y = -\tfrac{1}{6}\chi_2 e^+_y - \tfrac{1}{3}\chi_h e^+_y, \tag{5.38c}$$
$$\partial_\tau e^-_z = -\tfrac{1}{6}\chi_2 e^-_z - \tfrac{1}{3}\chi_h e^-_z, \tag{5.38d}$$
$$\partial_\tau f_x = \tfrac{1}{3}\chi_2 f_x - \tfrac{1}{3}\chi_h f_x + \Theta^T_x, \tag{5.38e}$$
$$\partial_\tau \chi_2 = \tfrac{1}{6}\chi_2^2 - \tfrac{2}{3}\chi_2\chi_h - \Theta^S_2 - \Theta\phi_2, \tag{5.38f}$$
$$\partial_\tau \chi_h = -\tfrac{1}{6}\chi_2^2 - \tfrac{1}{3}\chi_h^2 - \Theta^S_h, \tag{5.38g}$$
$$\partial_\tau \xi_3 = \tfrac{1}{12}\chi_2\xi_3 - \tfrac{1}{3}\chi_h\xi_3 - \tfrac{1}{2}\chi_2\xi_y, \tag{5.38h}$$
$$\partial_\tau \xi_1 = \tfrac{1}{12}\chi_2\xi_1 - \tfrac{1}{3}\chi_h\xi_1 - \tfrac{1}{2}\chi_2\xi_z, \tag{5.38i}$$
$$\partial_\tau \xi_x = -\tfrac{1}{2}\chi_2 f_x - \Theta^T_x - \tfrac{1}{6}\chi_2\xi_x - \tfrac{1}{3}\chi_h\xi_x, \tag{5.38j}$$
$$\partial_\tau \xi_y = -\tfrac{1}{8}\chi_2\xi_3 + \tfrac{1}{12}\chi_2\xi_y - \tfrac{1}{3}\chi_h\xi_y, \tag{5.38k}$$
$$\partial_\tau \xi_z = -\tfrac{1}{8}\chi_2\xi_1 + \tfrac{1}{12}\chi_2\xi_z - \tfrac{1}{3}\chi_h\xi_z, \tag{5.38l}$$
$$\partial_\tau \Theta^T_x = \tfrac{1}{3}\chi_2\Theta^T_x - \tfrac{1}{3}\chi_h\Theta^T_x + \tfrac{1}{3}d_x\phi_2, \tag{5.38m}$$
$$\partial_\tau \Theta^S_2 = \tfrac{1}{6}\chi_2\Theta^S_2 - \tfrac{1}{3}\chi_h\Theta^S_2 - \tfrac{1}{3}\chi_2\Theta^S_h + \dot{\Theta}\phi_2, \tag{5.38n}$$
$$\partial_\tau \Theta^S_h = -\tfrac{1}{6}\chi_2\Theta^S_2 - \tfrac{1}{3}\chi_h\Theta^S_h, \tag{5.38o}$$
$$\partial_\tau \phi_2 = -\tfrac{1}{2}\chi_2\phi_2 - \chi_h\phi_2. \tag{5.38p}$$

The results of the analysis of Section 5.3.5 provide the asymptotic initial data for the above spherically symmetric evolution equations. The resulting expressions are collected in the following lemma:

Lemma 14. *There exists a conformal gauge in which asymptotic initial data for the Schwarzschild-de Sitter spacetime can be expressed, in terms of the fields defined by the Ansatz* (5.37a)-(5.37h), *as*

$$\phi_2 = 6m, \qquad {\Theta_x}^T = 0, \qquad {\Theta_2}^S = -2, \qquad {\Theta_h}^S = -\frac{1}{2}(1-3\kappa^2),$$

$$\xi_1 = -(\alpha_\star + \bar{\alpha}_\star), \qquad \xi_3 = -(\alpha_\star + \bar{\alpha}_\star), \qquad \xi_x = \frac{1}{2\sqrt{2}}(\alpha_\star + \bar{\alpha}_\star), \qquad \xi_y = \frac{1}{2\sqrt{2}}(\alpha_\star + \bar{\alpha}_\star),$$

$$\xi_z = \frac{1}{2\sqrt{2}}(\alpha_\star + \bar{\alpha}_\star), \qquad \chi_2 = 0, \qquad \chi_h = 3\kappa, \qquad \chi_x = 0,$$

$$e_x^0 = 0, \qquad e_x^3 = 1, \qquad e_z^+ = 1, \qquad e_y^- = 1,$$

$$f_x = 0.$$

5.4.2 The Schwarzschild-de Sitter spacetime in the conformal Gaussian gauge

In this section the spherically symmetric evolution equations derived in the previous section is analysed in some detail. In particular, it is shown that there is a subsystem of equations that decouples from the rest —which will be called the *core system*— and controls the essential dynamics of the system (5.38a)-(5.38p).

As the Schwarzschild-de Sitter spacetime possesses a curvature singularity at $r = 0$, one expects, in general, the conformal evolution equations to develop singularities. To explain this in more detail observe that if Schwarzschild-de Sitter spacetime metric were written in conformal Gaussian coordinates the curvature singularity located at $r = 0$ would be reached at a certain value of the unphysical proper time $\tau = \tau_{\not{\,\,}}$. Since the conformal evolution equations (5.38a)-(5.38p) with the initial data of Lemma 14 describe the Schwarzschild-de Sitter spacetime in the conformal Gaussian gauge, then, the existence of a singularity at $\tau_{\not{\,\,}}$ should be already encoded in equations (5.38a)-(5.38p) with the initial data of Lemma 14. Moreover, since the two essential parameters appearing in the initial data given in Lemma 14 are m and κ —the function $\alpha_\star$ only encodes the connection on $\mathbb{S}^2$— one expects, in general, that the congruence of conformal geodesics reaches the curvature singularity at $\tau = \tau_{\not{\,\,}}(m,\kappa)$. Nevertheless, numerical evaluations suggest that for $\kappa = 0$ the core system does not develop any singularity —observe that this is consistent with Remark 39. Furthermore, an estimation for the time of existence $\tau_{\circledcirc}$ of the solution to the conformal evolution equations (5.38a)-(5.38p) with initial data in the case $\kappa = 0$ is given. A discussion of the mechanism for the formation of singularities in the core system ($\kappa \neq 0$) and the role of the parameter κ is given in Appendix 5.7.

The core system

Inspection of the system (5.38a)-(5.38p) reveals that there is a subsystem of equations that decouples from the rest. In the sequel these equations will be referred as the *core system*. Defining the fields

$$\chi \equiv \frac{1}{3}\left(\frac{1}{2}\chi_2 + \chi_h\right), \qquad L \equiv -\frac{1}{3}\left(\frac{1}{2}{\Theta_2}^S + {\Theta_h}^S\right), \qquad \phi \equiv \frac{1}{3}\phi_2, \tag{5.39}$$

the system (5.38p)-(5.38a) can be shown to imply the equations

$$\dot{\phi} = -3\chi\phi, \tag{5.40a}$$
$$\dot{\chi} = -\chi^2 + L - \frac{1}{2}\Theta\phi, \tag{5.40b}$$
$$\dot{L} = -\chi L - \frac{1}{2}\dot{\Theta}\phi, \tag{5.40c}$$

where the overdot denotes differentiation with respect to τ and

$$\Theta(\tau) = \sqrt{\frac{|\lambda|}{3}}\tau\left(1 + \frac{1}{2}\kappa\tau\right), \qquad \dot{\Theta} = \sqrt{\frac{|\lambda|}{3}}(1 + \kappa\tau).$$

The initial data for this system is given by

$$\phi(0) = 2m, \qquad \chi(0) = \kappa, \qquad L(0) = \frac{1}{2}(1 - \kappa^2). \tag{5.41}$$

As it will be seen in the remainder of this chapter, equations (5.40a)-(5.40c) with initial data (5.41) govern the dynamics of the complete system (5.38a)-(5.38p). The evolution of the remaining fields can be understood once the core system has been investigated.

Analysis of the Core System

This section will be concerned with an analysis of the initial value problem for the core system (5.40a)-(5.40c) with initial data given by (5.41). As it will be seen in the following, the essential feature driving the dynamics of the core system (5.40a)-(5.40c) is the fact that the function χ satisfies a Riccati equation coupled to two further fields. One also has the following:

Observation 1. The core equation (5.40a) can be integrated to yield

$$\phi(\tau) = 2m \exp\left(-3\int_0^\tau \chi(\mathrm{s})\mathrm{ds}\right). \tag{5.42}$$

Hence, $\phi(\tau) > 0$ if $m \neq 0$.

In the remaining of this section, the behaviour of the core system is analysed in the case where the extrinsic curvature of $\mathscr{I}$ vanishes.

As discussed in Section 5.2.2 in the case $\kappa = 0$ the conformal factor reduces to $\Theta(\tau) = \sqrt{|\lambda|/3}\tau$ —thus, one has only one root corresponding to the initial hypersurface $\mathscr{I}$. To simplify the notation recall that $\dot{\Theta}_\star = \sqrt{|\lambda|/3}$ so that $\Theta(\tau) = \dot{\Theta}_\star\tau$. Accordingly, the core system (5.40a)- (5.40c) can be rewritten as

$$\dot{\phi} = -3\chi\phi, \tag{5.43a}$$

$$\dot{\chi} = -\chi^2 + L - \frac{1}{2}\dot{\Theta}_\star \tau\phi, \tag{5.43b}$$
$$\dot{L} = -\chi L - \frac{1}{2}\dot{\Theta}_\star \phi. \tag{5.43c}$$

Moreover, the initial data reduces to

$$\chi(0) = 0, \qquad L(0) = \frac{1}{2}, \qquad \phi(0) = 2m.$$

Observation 2. A direct inspection shows that equations (5.43a)-(5.43c) imply that

$$\chi(\tau) = \tau L(\tau).$$

This relation can be easily verified by direct substitution into equations (5.43b) and (5.43c). Observe that $L(\tau) = \chi(\tau)/\tau$ is well defined at $\mathscr{I}$ where $\tau = 0$ and $\chi(0) = 0$ since the initial conditions ensure that

$$\lim_{\tau\to 0} \frac{\chi(\tau)}{\tau} = \frac{1}{2}.$$

Taking into account the above observation the core system reduces to

$$\dot{L} = -\tau L^2 - \frac{1}{2}\dot{\Theta}_\star \phi, \tag{5.44a}$$
$$\dot{\phi} = -3\tau L\phi, \tag{5.44b}$$

with initial data

$$L(0) = \frac{1}{2}, \qquad \phi(0) = 2m. \tag{5.45}$$

Observation 3. One can integrate (5.44b) to

$$\phi(\tau) = 2m \exp\Big(-\int_0^\tau \mathrm{s}L(\mathrm{s})\mathrm{ds}\Big) \tag{5.46}$$

and conclude that $\phi(\tau) > 0$ for $\tau > 0$.

To prove the boundedness of the solutions to the core system one starts by proving some basic estimates:

Lemma 15. *If* $\kappa = 0$*, then the solution of* (5.40a)-(5.40c) *with initial data* (5.41) *satisfies the bound*

$$L(\tau) \geq \phi(\tau)\left(\frac{1}{4m} - \frac{1}{2}\dot{\Theta}_\star\tau\right) \qquad \textit{for} \qquad \tau \geq 0.$$

Proof. Using equations (5.44a) and (5.44b) the following expression is derived

$$\phi\dot{L} - L\dot{\phi} = 2\tau L^2\phi - \frac{1}{2}\dot{\Theta}_\star\phi^2 \geq -\frac{1}{2}\dot{\Theta}_\star\phi^2 \qquad \text{for} \qquad \tau \geq 0. \tag{5.47}$$

Since $\phi(\tau) > 0$ one can consider the derivative of L/ϕ. Notice that

$$\phi^2 \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{L}{\phi}\right) = \phi\dot{L} - L\dot{\phi}.$$

This observation and inequality (5.47) gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{L}{\phi}\right) \geq -\frac{1}{2}\dot{\Theta}_\star \qquad \text{for} \qquad \tau \geq 0.$$

Integrating the last differential inequality from $\tau = 0$ to $\tau > 0$ taking into account the initial conditions leads to

$$L(\tau) \geq \phi(\tau)\left(\frac{1}{4m} - \frac{1}{2}\dot{\Theta}_\star\tau\right) \qquad \text{for} \qquad \tau \geq 0.$$

□

Observe that the last estimate ensures that $L(\tau)$ is non-negative for $\tau \in [0, 8m/\dot{\Theta}_\star]$. It turns out that finding an upper bound for $L(\tau)$ is relatively simple:

Lemma 16. *If* $\kappa = 0$ *then, for the solution of* (5.40a)-(5.40c) *with initial data* (5.41), *one has that*

$$L(\tau) \leq \frac{2}{\tau^2 + 4} \qquad \textit{for} \qquad \tau \geq 0.$$

Proof. Assume $\tau \geq 0$. Using that $\phi(\tau) > 0$ and equation (5.44a) one obtains the differential inequality

$$\dot{L}(\tau) \leq -\tau L^2(\tau).$$

Using that $L(\tau) > 0$ for $\tau \geq 0$ one gets

$$\frac{\dot{L}(\tau)}{L^2(\tau)} \leq -\tau.$$

The last expression can be integrated giving an upper bound for $L(\tau)$:

$$L(\tau) \leq \frac{2}{\tau^2 + 4}.$$

□

A simple bound on a finite interval can be found for the field $\phi(\tau)$ as follows:

Lemma 17. *If* $\kappa = 0$ *then, for the solution of* (5.40a)-(5.40c) *with initial data* (5.41) *and for* $0 \leq \tau \leq 1/(2\sqrt[3]{\dot{\Theta}_\star m})$, *the field* $\phi(\tau)$ *satisfies*

$$\phi(\tau) \leq \frac{2m}{1 - \dot{\Theta}_\star m\tau^3}.$$

Proof. Assume $\tau \geq 0$. From the estimate of Lemma 15 one has that

$$L \geq -\frac{1}{2}\dot{\Theta}_\star \tau \phi.$$

Therefore

$$-3\tau L\phi \leq \frac{3}{2}\dot{\Theta}_\star \tau^2 \phi^2.$$

Using equation (5.44b) one obtains the differential inequality

$$\dot{\phi} \leq \frac{3}{2}\dot{\Theta}_\star \tau^2 \phi^2.$$

Since $\phi(\tau) > 0$ the last expression can be integrated to yield,

$$\phi(\tau) \leq \frac{2m}{1 - \dot{\Theta}_\star m \tau^3}.$$

Therefore, for $0 < \tau < 1/\sqrt[3]{\dot{\Theta}_\star m}$, the field, $\phi(\tau)$ is bounded by above. Consequently, one can take $0 \leq \tau \leq 1/(2\sqrt[3]{\dot{\Theta}_\star m})$. □

The results of Lemmas 15, 16 and 17 can be summarised in the following:

Lemma 18. *The solution to the core system* (5.40a)-(5.40c) *with initial data* (5.41)*, in the case* $\kappa = 0$*, is bounded for* $0 \leq \tau \leq \tau_\bullet$*, where*

$$\tau_\bullet \equiv \min\left\{\frac{8m}{\dot{\Theta}_\star}, \frac{1}{2\sqrt[3]{\dot{\Theta}_\star m}}\right\}. \tag{5.48}$$

Remark 41. A plot of the numerical evaluation of the solutions to the core system (5.40a)-(5.40c) with initial data (5.41) in the case $\kappa = 0$ is shown in Figure 5.6.

Behaviour of the remaining fields in the conformal evolution equations

In this section the analysis of the conformal evolution equations is completed. In particular, it is shown that the dynamics of the whole evolution equations is driven by the core system. To this end, the following fields are introduced

$$\bar{\chi} \equiv \frac{1}{3}(\chi_2 - \chi_h), \qquad \bar{L} \equiv \frac{1}{3}(\Theta^S_2 - \Theta^S_h).$$

The evolution equations for these variables are

$$\dot{\bar{\chi}} = \bar{\chi}^2 - \bar{L} - \Theta\phi, \tag{5.49a}$$

$$\dot{\bar{L}} = \bar{\chi}\bar{L} + \dot{\Theta}\phi, \tag{5.49b}$$

with initial data

$$\bar{\chi}(0) = -\kappa, \qquad \bar{L}(0) = -\frac{1}{2}(1+\kappa^2).$$

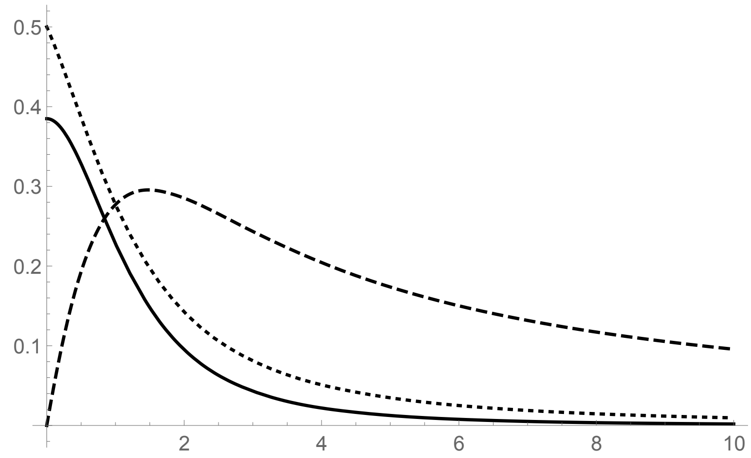

Figure 5.6: *Numerical solution of the core system in the $\kappa = 0$ case with $|\lambda| = 3$ and $m = 1/3\sqrt{3}$. The solid line corresponds to ϕ, the dashed line to χ and the dotted line to L. Observe that in contrast to the $\kappa > 1$ and $\kappa < -1$ cases, numeric evaluations suggest that in the case $\kappa = 0$ the fields of the core system are bounded for all times —see Figures 5.11 and 5.12 of Appendix 5.7.*

Notice that despite these equations resembling those of the core system, the field ϕ is not determined by the equations (5.49a)-(5.49b) —thus, this subsystem will be called the *supplementary system*. Once the core system has been solved, ϕ can be regarded as a source term for the system (5.49a)-(5.49b). If $\bar{\chi}$ and $\bar{L}$ are known then one can write the remaining unknowns in quadratures —the analysis of the supplementary system (5.49a)-(5.49b) will be given later in this section. Defining

$$\xi^+_{y3} \equiv \xi_y + \frac{1}{2}\xi_3, \qquad \xi^-_{y3} \equiv \xi_y - \frac{1}{2}\xi_3,$$
$$\xi^+_{z1} \equiv \xi_z + \frac{1}{2}\xi_1, \qquad \xi^-_{z1} \equiv \xi_z - \frac{1}{2}\xi_1,$$

one finds that the equations for these fields can be formally solved to give

$$\xi^+_{y3}(\tau) = \xi^+_{y3}(0)\exp\left(-\int_0^\tau \chi(s)\mathrm{ds}\right), \qquad \xi^-_{y3} = \xi^-_{y3}(0)\exp\left(-\int_0^\tau \bar{\chi}(s)\mathrm{ds}\right),$$
$$\xi^+_{z1}(\tau) = \xi^+_{z1}(0)\exp\left(-\int_0^\tau \chi(s)\mathrm{ds}\right), \qquad \xi^-_{z1} = \xi^-_{z1}(0)\exp\left(-\int_0^\tau \bar{\chi}(s)\mathrm{ds}\right).$$

The role of the the subsystem formed by Θ^T_x, f_x and e^3_x is analysed in the following result.

Lemma 19. *Given asymptotic initial data for the Schwarzschild-de Sitter spacetime, if $\partial_\psi\kappa = 0$ on $\mathscr{I}$ then*

$$f_x(\tau) = e_x{}^0(\tau) = \Theta_x{}^T(\tau) = 0.$$

Proof. This result follows directly from equations (5.38a), (5.38e), (5.38m) and the initial data given in Lemma 14. To see this, first recall that

$$d_x \equiv x^{\boldsymbol{AB}} e_{\boldsymbol{AB}}{}^{\boldsymbol{i}} \boldsymbol{e_i}(\Theta) = e_x{}^0\partial_0\Theta + e_x{}^3\partial_3\Theta.$$

Assuming then that $\boldsymbol{e_3}(\kappa) = 0$ one has that $\boldsymbol{e_3}(\Theta) = 0$ and therefore

$$d_x = \sqrt{2} e_x{}^0 \dot{\Theta}.$$

Observing that equations (5.38a), (5.38e) and (5.38m) form an homogeneous system of equations for the fields $e_x{}^0, f_x, \Theta_x{}^T$ with vanishing initial data then, using a standard existence and uniqueness argument for ordinary differential equations, it follows that the unique solution to this subsystem is the trivial solution, namely

$$f_x(\tau) = e_x{}^0(\tau) = \Theta_x{}^T(\tau) = 0.$$

□

Using the result of Lemma 19 one can formally integrate equation (5.38j) to yield

$$\xi_x(\tau) = \xi_x(0) \exp\left(-\int_0^\tau \chi(s)\mathrm{ds} \right).$$

The frame coefficients can also be found by quadratures

$$e_x^3(\tau) = e_x^3(0) \exp\left(\int_0^\tau \bar{\chi}(s)\mathrm{ds} \right), \qquad e_y^+(\tau) = e_y^+(0) \exp\left(-\int_0^\tau \chi(s)\mathrm{ds} \right),$$
$$e_z^+(\tau) = e_z^+(0) \exp\left(-\int_0^\tau \chi(s)\mathrm{ds} \right).$$

Since one can write

$$\begin{aligned}
\chi_2 &= 2(\chi + \bar{\chi}), & \chi_h &= 2\chi - \bar{\chi}, & \Theta_2^S &= 2(\bar{L} - L), & \Theta_h^S &= -\bar{L} - 2L, \\
\xi_y &= \frac{1}{2}(\xi_{y3}^+ + \xi_{y3}^-), & \xi_z &= \frac{1}{2}(\xi_{z1}^+ + \xi_{z1}^-), & \xi_1 &= 2(\xi_{z1}^+ - \xi_{z1}^-), & \xi_3 &= 2(\xi_{y3}^+ - \xi_{y3}^-).
\end{aligned}$$

then, it only remains to study the behaviour of $\bar{\chi}$ and $\bar{L}$ to completely characterise the evolution equations (5.38a)-(5.38p).

Remark 42. In the analysis of the core system of Appendix 5.7 the mechanism for the formation of singularities at finite time in the case $\kappa \neq 0$ is identified. Since ϕ acts as a source term for the supplementary system (5.49a)-(5.49b) one expects the solution to this system to be singular at finite time if the solutions to the core system develop a singularity. Clearly, the behaviour of the core system is independent from the behaviour of the supplementary system. Observe that, nevertheless, $\bar{L}$ and $\bar{\chi}$ could blow up earlier than ϕ. The analysis of the supplementary system and an estimation of a existence time is given later on in this section.

Deviation equation for the congruence

As discussed in Section 2.4.3 of Chapter 2, the evolution equations (2.68a)-(2.68h) are derived under the assumption of the existence of a non-intersecting congruence of conformal geodesics. In this section the solutions to the deviation equations are analysed.

As a consequence of Lemma 19 one has $f_{\boldsymbol{AB}} = 0$. Following the spirit of the space spinor formalism, the deviation spinor $z_{\boldsymbol{AB}}$ can be written in terms of elementary valence 2 spinors as

$$z_{(\boldsymbol{AB})} = z_x x_{\boldsymbol{AB}} + z_y y_{\boldsymbol{AB}} + z_z z_{\boldsymbol{AB}}.$$

Substituting expression (5.37e) into equation (2.71b) of Chapter 2 and using the identities given in equation (5.93g) of Appendix 5.8.2 one obtains

$$\partial_\tau z_x = 0, \qquad \partial_\tau z_z = 0, \qquad \partial_\tau z_y = -\frac{1}{12}\chi_2 z_y - \frac{1}{6}\chi_h z_y.$$

One can formally integrate these equations to obtain

$$z_x(\tau) = z_{x\star}, \qquad z_z(\tau) = z_{z\star}, \qquad z_y(\tau) = z_{y\star} \exp\left(-\frac{1}{2}\int_0^\tau \chi(s)\mathrm{d}s\right).$$

In the last equation, $z_{x\star}$, $z_{y\star}$ and $z_{z\star}$ denote the initial value of $z_x(\tau)$, $z_y(\tau)$ and $z_z(\tau)$ respectively. It follows that the deviation vector is non-zero and regular as long as the initial data $z_{x\star}$, $z_{y\star}$ and $z_{z\star}$ are non-vanishing and $\chi(\tau)$ is regular. Accordingly, *the congruence of conformal geodesics will be non-intersecting.*

Analysis of the supplementary system

As in the case of the core system, the supplementary system is simpler in the gauge in which $\kappa = 0$. In such case, direct inspection shows that equations (5.49a)-(5.49b) imply

$$\bar{\chi} = -\tau\bar{L}.$$

This can be verified by direct substitution into equations (5.49a) and (5.49b). Notice that $\bar{L}(\tau)$ is well defined at $\mathscr{I}$ where $\tau = 0$ and $\bar{\chi}(0) = 0$ since the initial conditions ensure that

$$\lim_{\tau\to 0}\frac{\bar{\chi}(\tau)}{\tau} = \frac{1}{2}.$$

Taking into account this observation, the system (5.49a)-(5.49b) reduces to the equation

$$\dot{\bar{L}} = -\tau\bar{L}^2 + \dot{\Theta}_\star\phi, \tag{5.50}$$

with initial data

$$\bar{L}(0) = -\frac{1}{2}. \tag{5.51}$$

Using that ϕ is only determined by the core system, together with the analysis of the core system given in the beginning of Section 5.4.2 one obtains the following result:

Lemma 20. *The solution to equation* (5.50) *with initial data* (5.51) *is bounded for* $0 \leq \tau \leq \tau_{\circledcirc}$ *with*

$$\tau_{\circledcirc} \equiv min\{\tau_{\circ}, \tau_{\bullet}\}, \qquad where \qquad \tau_{\circ} \equiv \sqrt{\dot{\Theta}_\star^{-1/2}\left(\frac{\pi}{2} + 2\arctan\left(\frac{1}{2}\dot{\Theta}_\star^{-1/2}\right)\right)} \tag{5.52}$$

Proof. To prove that $\bar{L}(\tau)$ is bounded from above one proceeds by contradiction. Assume that $\bar{L} \to \infty$ for some finite $\tau_{\not{\iota}} \in [0, \tau_\bullet]$, then $\dot{\bar{L}} \to \infty$ at $\tau_{\not{\iota}}$. Now, equation (5.50) can be rewritten as

$$\dot{\bar{L}} + \tau \bar{L}^2 = \dot{\Theta}_\star \phi.$$

Therefore, since $\tau \geq 0$, the last expression implies that $\phi \to \infty$ at $\tau_{\not{\iota}}$. However, in the analysis of the core system in Section 5.4.2 it was shown that ϕ is finite for $\tau \in [0, \tau_\bullet]$. This is a contradiction, and one cannot have $\bar{L} \to \infty$ at $\tau_{\not{\iota}} \in [0, \tau_\bullet]$. Consequently $L(\tau)$ is bounded from above for $0 \leq \tau \leq \tau_\bullet$. To show that $\bar{L}(\tau)$ is bounded from below, for $0 \leq \tau \leq \tau_\circ$ with $\tau_\circ$ as given by relation (5.52), observe that $\phi(\tau) > -\tau$ for $\tau \geq 0$ since $\phi(\tau) > 0$. Using this observation, equation (5.50) implies the differential inequality

$$\dot{\bar{L}} \geq -\tau(\bar{L}^2 + \dot{\Theta}_\star).$$

Since $\dot{\Theta}_\star > 0$ one has that $(\bar{L}^2 + \dot{\Theta}_\star) > 0$. Thus, one can rewrite the last inequality as

$$\frac{\dot{\bar{L}}}{(\bar{L}^2 + \dot{\Theta}_\star)} \geq -\tau,$$

which can be integrated using the initial data (5.51) to give

$$L(\tau) \geq -\sqrt{\Theta_\star}\tan\left(\frac{1}{2}\sqrt{\dot{\Theta}_\star}\tau^2 + \arctan\left(\frac{1}{2\sqrt{\dot{\Theta}_\star}}\right)\right).$$

Since the function tan is bounded if its argument lies in $[0, \pi/4]$ one concludes that $L(\tau)$ is bounded from below for $0 \leq \tau \leq \tau_\circ$. Finally, taking the minimum of $\tau_\bullet$ and $\tau_\circ$ one obtains the result. □

Remark 43. Numerical evaluations of the solutions to the supplementary system show that it should be possible to improve Lemmas 18 and 20 and conclude that the solutions do not blow up in finite time. These results, however, will not be required to formulate the Main Result of this thesis.

5.4.3 Perturbations of the Schwarzschild-de Sitter spacetime

In the sequel, one will consider perturbations of the Schwarzschild-de Sitter spacetime which can be covered by a congruence of conformal geodesics so that Lemma 6 of Chapter 2 can be applied. In particular, this means that the functional form of the conformal factor is the same for for both the background and the perturbed spacetime.

The discussion of Section 5.3.4 brings to the foreground the difficulties in setting up an asymptotic initial value problem for the Schwarzschild-de Sitter spacetime in a representation in which the initial hypersurface contains the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$: on the one hand, the initial data for the rescaled Weyl tensor is singular at both $\mathcal{Q}$ and $\mathcal{Q}'$; and, on the other hand, the curves in a congruence of timelike conformal geodesics become asymptotically null as they approach $\mathcal{Q}$ and $\mathcal{Q}'$ —see Appendix 5.6.

Consistent with the above remarks, the analysis of the conformal evolution equations (2.68a)-(2.68h) has been obtained in a conformal representation in which the metric on $\mathscr{I}$ is the standard one on $\mathbb{R}\times\mathbb{S}^2$. In this particular conformal representation the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ are at infinity respect to the 3-metric of $\mathscr{I}$ and the initial data for the Schwarzschild-de Sitter spacetime is homogeneous. In this section non-linear perturbations of the Schwarzschild-de Sitter spacetime are analysed by means of suitably posed initial value problems. More precisely, one is interested in analysing the development of perturbed initial data close to that of the Schwarzschild-de Sitter spacetime in the above described conformal representation. Then, using the conformal evolution equations (2.68a)-(2.68h) and the theory of first order symmetry hyperbolic systems contained in [23] one can obtain a existence and stability result for a reference solution corresponding to the asymptotic region of the Schwarzschild-de Sitter spacetime —see Figure 5.1.

Perturbations of asymptotic data for the Schwarzschild-de Sitter spacetime

In what follows, let $\mathcal{S}$ denote a 3-dimensional manifold with $\mathcal{S}\approx\mathbb{R}\times\mathbb{S}^2$. By assumption, there exists a diffeomorphism $\psi:\mathcal{S}\rightarrow\mathbb{R}\times\mathbb{S}^2$ which can used to pull-back a coordinate system $x=(x^\alpha)$ on $\mathbb{R}\times\mathbb{S}^2$ to obtain a coordinate system on $\mathcal{S}$ —i.e. $\widehat{x}=\psi^*x=x\circ\psi$. Exploiting the fact that ψ is a diffeomorphism one can define not only the pull-back $\psi^*:T^*(\mathbb{R}\times\mathbb{S}^2)\rightarrow T^*\mathcal{S}$ but also the push-forward of its inverse $(\psi^{-1})_*:T(\mathbb{R}\times\mathbb{S}^2)\rightarrow T\mathcal{S}$. Using this mapping, one can push-forward vector fields $\boldsymbol{c_i}$ on $T(\mathbb{R}\times\mathbb{S}^2)$ and pull-back their covector fields $\boldsymbol{\alpha^i}$ on $T^*(\mathbb{R}\times\mathbb{S}^2)$ via

$$\widehat{\boldsymbol{c}}_{\boldsymbol{i}}=(\psi^{-1})_*\boldsymbol{c_i},\qquad \widehat{\boldsymbol{\alpha}}^{\boldsymbol{i}}=\psi^*\boldsymbol{\alpha^i}.$$

In a slight abuse of notation, the fields $\widehat{\boldsymbol{c}}_{\boldsymbol{i}}$ and $\widehat{\boldsymbol{\alpha}}^{\boldsymbol{i}}$ will be simply denoted by $\boldsymbol{c_i}$ and $\boldsymbol{\alpha^i}$.

In the following, all the fields discussed previously for the exact Schwarzschild-de Sitter spacetime will be referred as the *background solution*. This fields will be distinguished with a $\mathring{\ }$ over the Kernel letter —e.g. $\mathring{\boldsymbol{h}}$ will denote the standard metric on $\mathbb{R}\times\mathbb{S}^2$ given in equation (5.19). Similarly, the perturbation to the corresponding field will be identified with a $\breve{}$ over the Kernel letter. Notice that although the frame $\{\boldsymbol{c_i}\}$ is $\mathring{\boldsymbol{h}}$-orthonormal, it is not necessarily orthogonal respect to the intrinsic 3-metric $\boldsymbol{h}$ on $\mathcal{S}$.

Let $\{\boldsymbol{e_i}\}$ denote a $\boldsymbol{h}$-orthonormal frame over $T\mathcal{S}$ and let $\{\boldsymbol{\omega^i}\}$ be the associate cobasis. Assume that there exist vector fields $\{\breve{\boldsymbol{e}}_{\boldsymbol{i}}\}$ such that an $\boldsymbol{h}$-orthonormal frame $\{\boldsymbol{e_i}\}$ is related to an $\mathring{\boldsymbol{h}}$-orthonormal frame $\{\boldsymbol{c_i}\}$ through the relation

$$\boldsymbol{e_i}=\boldsymbol{c_i}+\breve{\boldsymbol{e}}_{\boldsymbol{i}}.$$

This last requirement is equivalent to introducing coordinates on $\mathcal{S}$ such that

$$\boldsymbol{h}=\mathring{\boldsymbol{h}}+\breve{\boldsymbol{h}}. \tag{5.53}$$

Now, consider a solution

$$(h_{\boldsymbol{ij}},\ \chi_{\boldsymbol{ij}},\ L_{\boldsymbol{i}},\ L_{\boldsymbol{ij}},\ d_{\boldsymbol{ijk}},\ d_{\boldsymbol{ij}})$$

to the asymptotic conformal constraint equations (2.77a)-(2.77i) of Chapter 2 which is, in some sense to be determined, close to initial data for the Schwarzschild-de Sitter spacetime so that one can write

$$\begin{aligned} h_{\boldsymbol{ij}}|_{\mathcal{S}}&=\mathring{h}_{\boldsymbol{ij}}|_{\mathcal{S}}+\breve{h}_{\boldsymbol{ij}}|_{\mathcal{S}}, \qquad \chi_{\boldsymbol{ij}}|_{\mathcal{S}}=\mathring{\chi}_{\boldsymbol{ij}}|_{\mathcal{S}}+\breve{\chi}_{\boldsymbol{ij}}|_{\mathcal{S}}, \qquad L_{\boldsymbol{i}}|_{\mathcal{S}}=\mathring{L}_{\boldsymbol{i}}|_{\mathcal{S}}+\breve{L}_{\boldsymbol{i}}|_{\mathcal{S}}\\ L_{\boldsymbol{ij}}|_{\mathcal{S}}&=\mathring{L}_{\boldsymbol{ij}}|_{\mathcal{S}}+\breve{L}_{\boldsymbol{ij}}|_{\mathcal{S}}, \qquad d_{\boldsymbol{ijk}}|_{\mathcal{S}}=\mathring{d}_{\boldsymbol{ijk}}|_{\mathcal{S}}+\breve{d}_{\boldsymbol{ijk}}|_{\mathcal{S}}, \qquad d_{\boldsymbol{ij}}|_{\mathcal{S}}=\mathring{d}_{\boldsymbol{ij}}|_{\mathcal{S}}+\breve{d}_{\boldsymbol{ij}}|_{\mathcal{S}}. \end{aligned}$$

A spinorial version of these data can be obtained using the spatial Infeld-van der Waerden symbols. Accordingly, one writes

$$\eta_{\boldsymbol{ABCD}}|_{\mathcal{S}}=\mathring{\eta}_{\boldsymbol{ABCD}}|_{\mathcal{S}}+\breve{\eta}_{\boldsymbol{ABCD}}|_{\mathcal{S}}, \qquad \mu_{\boldsymbol{ABCD}}|_{\mathcal{S}}=\breve{\mu}_{\boldsymbol{ABCD}}|_{\mathcal{S}}, \tag{5.54a}$$

$$L_{\boldsymbol{ABCD}}|_{\mathcal{S}}=\mathring{L}_{\boldsymbol{ABCD}}|_{\mathcal{S}}+\breve{L}_{\boldsymbol{ABCD}}|_{\mathcal{S}}, \ \xi_{\boldsymbol{ABCD}}|_{\mathcal{S}}=\mathring{\xi}_{\boldsymbol{ABCD}}|_{\mathcal{S}}+\breve{\xi}_{\boldsymbol{ABCD}}|_{\mathcal{S}}, \tag{5.54b}$$

$$L_{\boldsymbol{AB}}|_{\mathcal{S}}=\breve{L}_{\boldsymbol{AB}}|_{\mathcal{S}}, \qquad \chi_{\boldsymbol{ABCD}}|_{\mathcal{S}}=\mathring{\chi}_{\boldsymbol{ABCD}}|_{\mathcal{S}}+\breve{\chi}_{\boldsymbol{ABCD}}|_{\mathcal{S}}, \tag{5.54c}$$

$$\boldsymbol{e_{AB}}|_{\mathcal{S}}=\mathring{\boldsymbol{e}}_{\boldsymbol{AB}}|_{\mathcal{S}}+\breve{\boldsymbol{e}}_{\boldsymbol{AB}}|_{\mathcal{S}}, \qquad f_{\boldsymbol{AB}}|_{\mathcal{S}}=\breve{f}_{\boldsymbol{AB}}|_{\mathcal{S}}. \tag{5.54d}$$

Observe that all the objects appearing in expressions (5.54a)-(5.54d) are scalars. Using the notation introduced in Section 5.2.3, the initial data (5.54a)-(5.54d) can be compactly denoted by

$$\boldsymbol{u}|_{\mathcal{S}}=\mathring{\boldsymbol{u}}|_{\mathcal{S}}+\breve{\boldsymbol{u}}|_{\mathcal{S}}.$$

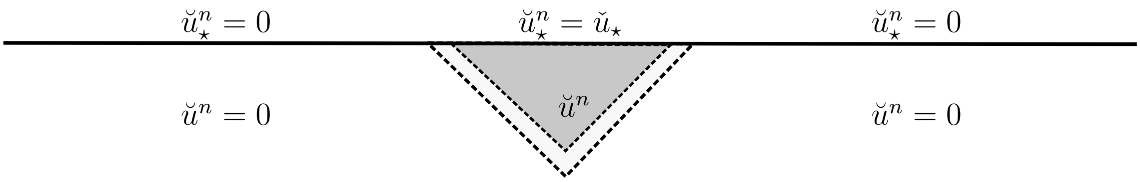

Figure 5.7: *Schematic depiction of the extended data. The perturbed initial data* $\breve{\boldsymbol{u}}^0_\star$ *coincides with* $\breve{\boldsymbol{u}}_\star$ *on* $\mathcal{S}_0$. *In the transition region,* $\mathcal{Z}_0 \setminus \mathcal{S}_0$, $\breve{\boldsymbol{u}}^0_\star$ *is extended smoothly until it vanishes on* $\mathcal{S} \setminus \mathcal{Z}_0$. *The domain of dependence of the original data on* $\mathcal{S}_0$ *is represented by the shaded area while the domain of dependence of the extended data on* $\mathcal{Z}_0$ *corresponds to the outer dashed triangle.*

The basic cylinder

Consider the following countable covers of $\mathbb{R}\times\mathbb{S}^2$ with sets of the form $I_n \equiv [-\frac{1}{2}\tau_\circledcirc + \frac{1}{2}\tau_\circledcirc n, \frac{1}{2}\tau_\circledcirc + \frac{1}{2}\tau_\circledcirc n] \times \mathbb{S}^2$ and $Y_n = [-\tau_\circledcirc + \frac{1}{2}\tau_\circledcirc n, \tau_\circledcirc + \frac{1}{2}\tau_\circledcirc n] \times \mathbb{S}^2$ with $n \in \mathbb{Z}$:

$$\mathbb{R} \times \mathbb{S}^2 = \bigcup_{n\in\mathbb{Z}} I_n = \bigcup_{n\in\mathbb{Z}} Y_n.$$

Additionally, notice that $I_n \subset Y_n$. In the sequel, the sets $I_0 = [-\frac{1}{2}\tau_\circledcirc, \frac{1}{2}\tau_\circledcirc] \times \mathbb{S}^2$ and $Y_0 \equiv [-\tau_\circledcirc, \tau_\circledcirc] \times \mathbb{S}^2$ will be called the *basic cylinder and the extended basic cylinder*, respectively. The diffeomorphism $\psi : \mathcal{S} \to \mathbb{R} \times \mathbb{S}^2$ of last subsection allow us to define the collection of sets $\mathcal{Z}_n \equiv \psi^{-1}(Y_n)$ and $\mathcal{S}_n \equiv \psi^{-1}(I_n)$ which can be used to obtain countable covers of $\mathcal{S}$:

$$\mathcal{S} = \bigcup_{n\in\mathbb{Z}} \mathcal{S}_n = \bigcup_{n\in\mathbb{Z}} \mathcal{Z}_n.$$

To have a more compact notation let $\boldsymbol{u}_\star, \mathring{\boldsymbol{u}}_\star$ and $\breve{\boldsymbol{u}}_\star$ represent $\boldsymbol{u}|_\mathcal{S}, \mathring{\boldsymbol{u}}|_\mathcal{S}$ and $\breve{\boldsymbol{u}}|_\mathcal{S}$ so that given perturbed initial data on $\mathcal{S}$ one can write

$$\boldsymbol{u}_\star = \mathring{\boldsymbol{u}}_\star + \breve{\boldsymbol{u}}_\star.$$

Now, define

$$\boldsymbol{u}^n_\star = \mathring{\boldsymbol{u}}_\star + \breve{\boldsymbol{u}}^n_\star,$$

with

$$\breve{\boldsymbol{u}}^n_\star = \begin{cases} \breve{\boldsymbol{u}}_\star & x \in \mathcal{S}_n \ , \\ h(x)\breve{\boldsymbol{u}}_\star & x \in \mathcal{Z}_n \setminus \mathcal{S}_n, \\ 0 & x \in \mathcal{S} \setminus \mathcal{Z}_n, \end{cases} \tag{5.55}$$

where $h(x)$ is smooth function such that $h(x) = 1$ for $x \in \partial\mathcal{S}_n$ and $h(x) = 0$ for $x \in \partial\mathcal{Z}_n$. Notice that on $\mathcal{S}_n$ the initial data $\boldsymbol{u}^n_\star$ satisfies the conformal constraint equations while it does not in $\mathcal{Z}_n \setminus \mathcal{S}_n$. However, due to the finite speed of propagation property the corresponding solution arising from this data on the domain of dependence of $\mathcal{S}_n$ is not influenced by the extended data on $\mathcal{Z}_n \setminus \mathcal{S}_n$.

Observe that $(\mathbb{R}\times\mathbb{S}^2, \mathring{\boldsymbol{h}})$ possesses a translation symmetry that can be exploited to regard initial data on each $\mathcal{Z}_n$ as initial data on $\mathcal{Z}_0$. To see this more clearly, notice that Y_n can be obtained by translation of the extended basic cylinder Y_0. Consequently, for each Y_n there exist a bijection $\Psi^n : Y_0 \to Y_n$. Moreover, the diffeomorphism $\psi : \mathcal{S} \to \mathbb{R}\times\mathbb{S}^2$, in turn, induces a bijection $\varphi^n : \mathcal{Z}_0 \to \mathcal{Z}_n$. Thus, one can exploit φ^n to pullback initial data on $\mathcal{Z}_n$ to $\mathcal{Z}_0$. In other words, $\varphi^n_*(\breve{\boldsymbol{u}}^n_\star)$ can be taken as a collection of perturbed initial data on $\mathcal{Z}_0$.

Controlling the size of the perturbation

In this subsection the necessary notions and definitions to measure the size of the perturbation of the initial data are introduced. The discussion will be given for initial data on $\mathcal{Z}_0 \approx Y_0$ for conciseness. Nevertheless, as discussed above, any perturbation on $\mathcal{Z}_n$ can be pulled-back to $\mathcal{Z}_0$. Let $\mathcal{A}\equiv\{(\phi_+,\mathcal{U}_+),(\phi_-,\mathcal{U}_-)\}$ with $\phi_+ : \mathcal{U}_+ \to \mathbb{R}^3$ and $\phi_- : \mathcal{U}_- \to \mathbb{R}^3$ be an atlas for Y_0. Let $\mathcal{V}_+ \subset \mathcal{U}_+$, $\mathcal{V}_- \subset \mathcal{U}_-$ be closed sets such that $Y_0 \subset \mathcal{V}_+ \cup \mathcal{V}_-$. In addition, define the functions

$$\eta_+(x) = \begin{cases} 1 & x\in\phi_1(\mathcal{V}_+) \\ 0 & x\in\mathbb{R}^3/\phi_1(\mathcal{V}_+) \end{cases}, \qquad \eta_-(x) = \begin{cases} 1 & x\in\phi_2(\mathcal{V}_-) \\ 0 & x\in\mathbb{R}^3/\phi_2(\mathcal{V}_-) \end{cases}. \tag{5.56}$$

Observe that any point $p\in\mathcal{Z}_0$ is described in local coordinates by $x_p = (\phi\circ\psi)(p)$ with $x_p\in\phi(\mathcal{U})$ where $\psi : \mathcal{S}\to\mathbb{R}\times\mathbb{S}^2$ is the diffeomorphism of the last subsection and $(\phi,\mathcal{U})\in\mathcal{A}$. Consequently, any smooth function $Q : \mathcal{Z}_0 \to \mathbb{C}^N$ can be regarded in local coordinates as $Q(x) : \phi(\mathcal{U}) \to \mathbb{C}^N$. Let $Q_i(x)$ denote the restriction of $Q(x)$ to one the open sets $\phi_i(\mathcal{U}_i)$ for $i = +,-$. Then, we define the norm of Q as

$$\parallel Q \parallel_{\mathcal{Z}_0,m} \equiv \parallel \eta_+(x)Q_+(x) \parallel_{\mathbb{R}^3,m} + \parallel \eta_-(x)Q_-(x) \parallel_{\mathbb{R}^3,m}$$

where

$$\parallel Q \parallel_{\mathbb{R}^3,m} = \left(\sum_{l=0}^{m} \sum_{\alpha_1,\dots,\alpha_l}^{3} \int_{\mathbb{R}^3} (\partial_{\alpha_1}\dots\partial_{\alpha_l}Q)^2 \mathrm{d}\mu \right)^{1/2}.$$

where $\mathrm{d}\mu$ is the volume form associated to the Euclidean metric $\boldsymbol{\delta}$ on $\mathbb{R}^3$. Now, one can use these notions to define Sobolev norms for any quantity $Q_{\mathcal{K}}$ with ${}_{\mathcal{K}}$ being an arbitrary string of frame spinor indices as

$$\parallel Q_{\mathcal{K}} \parallel_{\mathcal{S}_0,m} \equiv \sum_{\kappa} \parallel Q_\kappa \parallel_{\mathcal{Z}_0,m}.$$

In the last expression m is a positive integer and the sum is carried over all the independent components of $Q_{\mathcal{K}}$ which have been denoted by Q_κ.

Formulation of the evolution problems

Consistent with the split (5.54a)-(5.54d) for the initial data, one looks for solutions to the conformal evolution equations (5.5a)-(5.5b) of the form

$$\eta_{\boldsymbol{ABCD}} = \mathring{\eta}_{\boldsymbol{ABCD}} + \breve{\eta}_{\boldsymbol{ABCD}}, \qquad \mu_{\boldsymbol{ABCD}} = \breve{\mu}_{\boldsymbol{ABCD}}, \tag{5.57a}$$

$$L_{\boldsymbol{ABCD}} = \mathring{L}_{\boldsymbol{ABCD}} + \breve{L}_{\boldsymbol{ABCD}}, \qquad \xi_{\boldsymbol{ABCD}} = \mathring{\xi}_{\boldsymbol{ABCD}} + \breve{\xi}_{\boldsymbol{ABCD}}, \tag{5.57b}$$

$$L_{\boldsymbol{AB}} = \breve{L}_{\boldsymbol{AB}}, \qquad \chi_{\boldsymbol{ABCD}} = \mathring{\chi}_{\boldsymbol{ABCD}} + \breve{\chi}_{\boldsymbol{ABCD}}, \tag{5.57c}$$

$$\boldsymbol{e}_{\boldsymbol{AB}} = \boldsymbol{c}_{\boldsymbol{AB}} + \breve{\boldsymbol{e}}_{\boldsymbol{AB}}, \qquad f_{\boldsymbol{AB}} = \breve{f}_{\boldsymbol{AB}}. \tag{5.57d}$$

Following with the notation introduced before, the perturbed initial data $\breve{\boldsymbol{u}}$ in $\mathcal{Z}_0$ —as given in equation (5.55)— will be represented as $\breve{\boldsymbol{u}}^0_\star$ and the development of such data will be denoted by $\breve{\boldsymbol{u}}^0$. Now, let

$$\| \breve{\mathbf{u}}^0_\star \|_{\mathcal{Z}_0,m} \equiv \| \breve{\chi}_{\boldsymbol{ABCD}} \|_{\mathcal{Z}_0,m} + \| \breve{\xi}_{\boldsymbol{ABCD}} \|_{\mathcal{Z}_0,m} + \| \breve{L}_{\boldsymbol{ABCD}} \|_{\mathcal{Z}_0,m} + \| \breve{L}_{\boldsymbol{AB}} \|_{\mathcal{Z}_0,m} + \| \breve{\boldsymbol{e}}_{\boldsymbol{AB}} \|_{\mathcal{Z}_0,m} + \| \breve{f}_{\boldsymbol{AB}} \|_{\mathcal{Z}_0,m} + \| \breve{\phi}_{\boldsymbol{ABCD}} \|_{\mathcal{Z}_0,m} .$$

Perturbations on the extended basic cylinder

The main analysis of the background solution in Section 5.4.2 was performed in a conformal representation in which the asymptotic initial data is homogeneous and the extrinsic curvature of $\mathscr{I}$ vanishes —i.e. $\kappa = 0$. The general evolution equations (5.5a)-(5.5b) consist of transport equations for $\boldsymbol{\upsilon}$ coupled with a system of partial differential equations for $\boldsymbol{\phi}$. However, as shown in Section 5.4.2, the assumption of *spherical symmetry* implies that the only independent component of the spinorial field $\phi_{\boldsymbol{ABCD}}$ is ϕ_2. Consequently, the system (5.5a)-(5.5b) reduces, for the background fields $\mathring{\mathbf{u}} = (\mathring{\boldsymbol{\upsilon}}, \mathring{\boldsymbol{\phi}})$, to a system of ordinary differential equations. The *Piccard-Lindelöf theorem* can be applied to discuss local existence of the latter system. However, one does not have, *a priori*, control on the smallness of the existence time. To obtain statements concerning *the existence time of the perturbed solution*, one recalls that the discussion of the evolution equations of Section 5.4.2 shows that the components of solution $\mathring{\mathbf{u}}$ are regular for $\tau \in [0, \tau_\circledcirc]$ with $\tau_\circledcirc$ as given in equation (5.52), so that the guaranteed existence time is not arbitrarily small.

The analysis of the core system in Section 5.4.2 was restricted to the case $\kappa = 0$, in which the conformal boundary has vanishing extrinsic curvature. In this case, an explicit existence time $\tau_\circledcirc$ for the solution to the conformal evolution equations was obtained. In contrast, the analysis given in Appendix 5.7 shows that in general, for $\kappa \neq 0$, the core system develops a singularity at finite $\tau_{\not{\downarrow}}$. Since the results given in Section 5.4.2 for the conformal deviation equations hold not only for $\kappa = 0$, but for any κ as long as $\partial_\psi \kappa = 0$, one has that the congruence of conformal geodesics is non-intersecting in the $\kappa \neq 0$ case as well. This shows that, the singularities in the core system in the case $\kappa \neq 0$ are not gauge singularities.

In this section it is shown how one can exploit these observations, together with the theory for symmetric hyperbolic systems, to prove the existence of solutions to the general conformal evolution equations with the same existence time $\tau_{\circledcirc}$ for small perturbations of asymptotic initial data close to that of the Schwarzschild-de Sitter reference solution. By construction, the development of this perturbed data will be contained in the domain of influence which corresponds, in this case, to the asymptotic region of the spacetime —see Figure 5.9.

Taking into account the above remarks and using the theory of symmetric hyperbolic systems contained in [23] on $\mathcal{Z}_0$ —see Figure 5.8— one can formulate the following lemma

Lemma 21 (***existence and Cauchy stability for perturbations of asymptotic initial data for the Schwarzschild-de Sitter spacetime on the extended basic cylinder***). *Let* $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ *denote asymptotic initial data for the extended conformal Einstein field equations on a 3-dimensional manifold* $\mathcal{S} \approx \mathbb{R} \times \mathbb{S}^2$ *where* $\mathring{\mathbf{u}}_\star$ *denotes the asymptotic initial data for the Schwarzschild-de Sitter spacetime (subextremal, extremal and hyperextremal cases) with* $\kappa = 0$ *in which the asymptotic points* $\mathcal{Q}$ *and* $\mathcal{Q}'$ *are at infinity —*κ *encodes the trace of the extrinsic curvature of* $\mathcal{S}$*. Let* $\psi : \mathcal{S} \to \mathbb{R} \times \mathbb{S}^2$ *be a diffeomorphism and define the sets* $I_0 \equiv [-\frac{1}{2}\tau_{\circledcirc}, \frac{1}{2}\tau_{\circledcirc}] \times \mathbb{S}^2$, $Y_0 \equiv [-\tau_{\circledcirc}, \tau_{\circledcirc}] \times \mathbb{S}^2$*. Additionally, let* $\mathcal{S}_0 \equiv \psi^{-1}(I_0)$ *and* $\mathcal{Z}_0 \equiv \psi^{-1}(Y_0)$ *and* $\boldsymbol{u}^0_\star = \mathring{\boldsymbol{u}}_\star + \breve{\boldsymbol{u}}^0_\star$ *with* $\breve{\boldsymbol{u}}^0_\star$ *as in equation* (5.55)*. Then, for* $m \geq 4$ *and* $\tau_{\circledcirc}$ *as given in equation* (5.52)*, there exists* $\epsilon > 0$ *such that:*

(i) for $||\breve{\mathbf{u}}^0_\star||_{\mathcal{Z}_0,m} < \epsilon$ *, there exist a unique solution* $\breve{\mathbf{u}}^0$ *to the conformal evolution equations* (5.7a)-(5.7b) *with a minimal existence interval* $[0, \tau_{\circledcirc}]$ *and*

$$\breve{\mathbf{u}}^0 \in C^{m-2}([0, \tau_{\circledcirc}] \times \mathcal{Z}_0, \mathbb{C}^N),$$

and the associated congruence of conformal geodesics contains no conjugate points in $[0, \tau_{\circledcirc}]$*;*

(ii) given a sequence of perturbed data $\{^{(n)}\breve{\mathbf{u}}^0_\star\}$ *such that*

$$||^{(n)}\ \breve{\mathbf{u}}^0_\star\ ||_{\mathcal{Z}_0,m} \to 0 \qquad as \qquad n \to \infty,$$

then the corresponding solutions $\{^{(n)}\breve{\mathbf{u}}^0\}$ *have a minimum existence interval* $[0, \tau_{\circledcirc}]$ *and it holds that*

$$||^{(n)}\ \breve{\mathbf{u}}^0\ ||_{\mathcal{Z}_0,m} \to 0 \qquad as \qquad n \to \infty$$

uniformly in $\tau \in [0, \tau_{\circledcirc}]$ *as* $n \to \infty$.

Proof. Points *(i)* and *(ii)* are a direct application of the theory contained in [23] where it is used that the background solution $\mathring{\mathbf{u}}$ is regular on $\tau \in [0, \tau_{\circledcirc}]$. The initial data for the Schwarzschild-de Sitter spacetime encoded in $\mathbf{u}_\star$ is in a representation in which the points $\mathcal{Q}$ and $\mathcal{Q}'$ are at infinity. Observe that the asymptotic initial data, as derived in Section 5.3.5, for the subextremal, extremal and hyperextremal cases are formally the same —in particular, notice that the initial data for the electric part of the rescaled Weyl tensor contains information about the mass m while the conformal factor Θ carries information about λ. The arguments in the analysis of Section 5.4.2 are irrespective of the relation between λ and m. The key observation in the proof is that one can apply the general theory of symmetric hyperbolic systems of [23] for each open set and chart of an atlas for Y_0. Then, these local solutions can be patched together to obtain the required global solution over $[0, \tau_{\circledcirc}] \times \mathcal{Z}_0$ —it is sufficient to cover Y_0 with two patches as discussed in previous subsections. Details of a similar construction in the context of characteristic problems can be found in [17]. □

5.4.4 Main result

As briefly mentioned in Section 5.4.3 the translation invariance of $(\mathbb{R} \times \mathbb{S}^2, \mathring{\boldsymbol{h}})$ allow us to define a bijection $\varphi^n : \mathcal{Z}_0 \to \mathcal{Z}_n$ and regard $\varphi^n_*(\breve{\boldsymbol{u}}^n_\star)$ as initial data on $\mathcal{Z}_0$. In addition, due to the coordinate invariance, the conformal evolution equations when acted by the pullback φ^n_* can be shown to coincide with the evolution equations over $\mathcal{Z}_0$ as there is no explicit spatial coordinates in the equations and the partial derivatives are invariant when considering the change of coordinates induced by φ_n. Therefore, the restriction of the initial value problem on $\mathcal{Z}_n$ induces an initial value problem on $\mathcal{Z}_0$ with initial data $\varphi^n_*(\breve{\boldsymbol{u}}^n_\star)$.

Remark 44. Notice that, in general, $\varphi^n_*(\breve{\boldsymbol{u}}^n_\star)$ do not coincide with each other as they represent general perturbations of the initial data.

Observe that one can follow the discussion of Section 5.4.3 to define the norm $\| \breve{\boldsymbol{u}}_\star \|_{\mathcal{S},m}$, by replacing $\mathcal{Z}_0$ and Y_0 by $\mathcal{S}$ and $\mathbb{R} \times \mathbb{S}^2$, respectively. In addition, notice that

$$\| \breve{\boldsymbol{u}}_\star \|_{\mathcal{S},m} < \varepsilon,$$

for $\varepsilon > 0$, implies that there exist some $0 < \varepsilon' < \varepsilon$ such that

$$\| \varphi^n_* \breve{\boldsymbol{u}}^n_\star \|_{\mathcal{Z}_0,m} < \varepsilon'.$$

With the last observation one can then apply the theory of symmetric hyperbolic systems contained in [23] to obtain an analogous result to that of Lemma 21 for each $\varphi^n_* \breve{\boldsymbol{u}}^n_\star$. Let $\boldsymbol{w}^n$ denote the corresponding developments of the perturbed initial data $\varphi^n_* \breve{\boldsymbol{u}}^n_\star$ on $\mathcal{Z}_0$. The theory of symmetric hyperbolic systems of [23] ensures that these

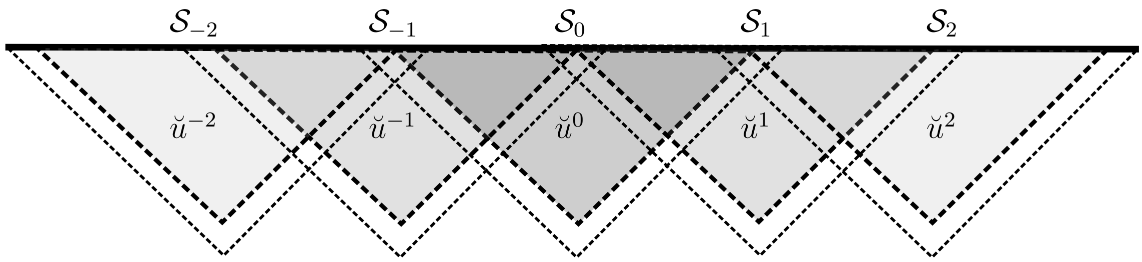

Figure 5.8: *Schematic depiction of the sets $\mathcal{S}_n \subset \mathcal{Z}_n$ and their corresponding domains of dependence. The translation invariance of $(\mathbb{R} \times \mathbb{S}^2, \mathring{\boldsymbol{h}})$ can be exploited to regard each Y_n as a copy of the Y_0. Theory of symmetric hyperbolic systems of [23] can be applied to $\mathcal{Z}_0$ exploiting the minimal existence time $\tau_{\circledcirc}$ to obtain a local solution $\breve{\boldsymbol{u}}^0$. Moreover, each of the developments $\breve{\boldsymbol{u}}^n$ share the same existence time of $\breve{\boldsymbol{u}}^0$. To obtain a global solution depending on the original initial data given on each $\mathcal{S}_n$ one has to consider a smaller time of existence $0 < \tau_\natural < \tau_{\circledcirc}$. Removing the overlapping regions appropriately, these local solutions can be patched together to obtain a global solution $\breve{\boldsymbol{u}}$.*

developments share a minimal existence time $\tau_{\circledcirc}$ —see proof of Lemma 21. In order to recast the solution $\boldsymbol{w}^n$ as a solution on $[0, \tau_{\circledcirc}] \times \mathcal{Z}_n$ define $\varphi^n_\tau : [0, \tau_{\circledcirc}] \times \mathcal{Z}_0 \to [0, \tau_{\circledcirc}] \times \mathcal{Z}_n$ by requiring that the action of φ^n to remain constant along conformal geodesics. Then, one can use the pullback of the inverse of φ^n_τ to define a solution on $[0, \tau_{\circledcirc}] \times \mathcal{Z}_n$ as $\breve{\boldsymbol{u}}^n \equiv ((\varphi^n_\tau)^{-1})_*(\boldsymbol{w}^n)$.

Remark 45. Recall that $\mathcal{S} = \bigcup_{n\in\mathbb{Z}} \mathcal{S}_n$ so one can construct a solution depending only on data given in $\mathcal{S}_n$ by removing the part of the solution determined by the extended data in $\mathcal{Z}_n \setminus \mathcal{S}_n$. Notice that, this may require shrinking the time of existence to some $\tau_\natural$ with $0 < \tau_\natural < \tau_{\circledcirc}$.

Remark 46. Observe that the initial data $\varphi^n_* \breve{\boldsymbol{u}}^{n+1}_\star$ and $\varphi^n_* \breve{\boldsymbol{u}}^n_\star$ coincide on

$$Q_n \equiv \mathcal{S}_n \bigcap \mathcal{S}_{n+1},$$

therefore, as a consequence of the uniqueness property for symmetric hyperbolic system one has that the developments $\breve{\boldsymbol{u}}^n$ and $\breve{\boldsymbol{u}}^{n+1}$ coincide in the corresponding domain of dependence of Q_n.

Remark 46 shows that if one is to construct a a global solution by adding each individual contribution $\breve{\boldsymbol{u}}^n$ one has to excise not only the part of the solution arising from the extended data given on $\mathcal{Z}_n \setminus \mathcal{S}_n$ but also the solution $\breve{\boldsymbol{u}}^{n+1}$ in the domain of dependence of Q_n. To do so, let $J_n \equiv (-\frac{1}{4}\tau_{\circledcirc} + \frac{1}{2}\tau_{\circledcirc} n, \frac{1}{4}\tau_{\circledcirc} + \frac{1}{2}\tau_{\circledcirc} n) \times \mathbb{S}^2$ and consider the sets $\mathcal{J}_n \equiv \psi^{-1}(J_n)$. Then the global solution over $[0, \tau_\natural] \times \mathcal{S}$ is given by

$$\breve{\boldsymbol{u}} = \sum_{n\in\mathbb{Z}} p^n(x) \tag{5.58}$$

where

$$p^n(x) = \begin{cases} \breve{\boldsymbol{u}}^n & x \in [0, \tau_\natural] \times \mathcal{J}_n \\ 0 & x \notin [0, \tau_\natural] \times \mathcal{J}_n. \end{cases} \tag{5.59}$$

The above discussion leads to the following theorem:

Theorem 3 (***existence and Cauchy stability for perturbations of asymptotic initial data for the Schwarzschild-de Sitter spacetime***)**.** *Let* $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ *denote asymptotic initial data for the extended conformal Einstein field equations on a 3-dimensional manifold* $\mathcal{S} \approx \mathbb{R} \times \mathbb{S}^2$ *where* $\mathring{\mathbf{u}}_\star$ *denotes the asymptotic initial data for the Schwarzschild-de Sitter spacetime (subextremal, extremal and hyperextremal cases) with* $\kappa = 0$ *in which the asymptotic points* $\mathcal{Q}$ *and* $\mathcal{Q}'$ *are at infinity —*κ *encodes the trace of the extrinsic curvature of* $\mathcal{S}$*. Then, for* $m \geq 4$ *and for some* $0 < \tau_\natural < \tau_\circledcirc$ *with* $\tau_\circledcirc$ *as given in equation* (5.52)*, there exists* $\varepsilon > 0$ *such that:*

(i) for $||\breve{\mathbf{u}}_\star||_{\mathcal{S},m} < \varepsilon$ *, there exist a unique solution* $\breve{\mathbf{u}}$ *to the conformal evolution equations* (5.7a)-(5.7b) *with a minimal existence interval* $[0, \tau_\natural]$ *and*

$$\breve{\mathbf{u}} \in C^{m-2}([0, \tau_\natural] \times \mathcal{S}, \mathbb{C}^N),$$

and the associated congruence of conformal geodesics contains no conjugate points in $[0, \tau_\natural]$*;*

(ii) given a sequence of perturbed data $\{{}^{(n)}\breve{\mathbf{u}}_\star\}$ *such that*

$$||^{(n)}\, \breve{\mathbf{u}}_\star \,||_{\mathcal{S},m} \to 0 \qquad as \qquad n \to \infty,$$

then the corresponding solutions $\{{}^{(n)}\breve{\mathbf{u}}\}$ *have a minimum existence interval* $[0, \tau_\natural]$ *and it holds that*

$$||^{(n)}\, \breve{\mathbf{u}} \,||_{\mathcal{S},m} \to 0 \qquad as \qquad n \to \infty$$

uniformly in $\tau \in [0, \tau_\natural]$ *as* $n \to \infty$*;*

(iii) the solution $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ *is unique in* $[0, \tau_\natural] \times \mathcal{S}$ *and implies a* C^{m-2} *solution* $(\tilde{\mathcal{M}}_{\tau_\natural}, \tilde{\boldsymbol{g}})$ *to the Einstein vacuum equations with the same de Sitter-like Cosmological constant as the background solution where*

$$\tilde{\mathcal{M}}_{\tau_\natural} \equiv (0, \tau_\natural) \times \mathcal{S}.$$

Moreover, the hypersurface $\mathscr{I} \equiv \{0\} \times \mathcal{S}$ *represents the conformal boundary of the spacetime.*

Proof. To prove points *(i)* and *(ii)* observe that the smallness assumption on the initial data $\| \breve{\boldsymbol{u}}_\star \|_{\mathcal{S},m} < \varepsilon$ ensures in particular that there exist $0 < \varepsilon' < \varepsilon$ such that

$\| \varphi^n_* \breve{\boldsymbol{u}}^n_\star \|_{\mathcal{Z}_0,m} < \varepsilon'$. Then, one can apply the theory contained in [23] to obtain an analogous result to that of Lemma 21 for a collection of perturbed initial data $\varphi^n_* \breve{\boldsymbol{u}}^n_\star$ on $\mathcal{Z}_0$. The theory of [23] ensures that the corresponding developments $\boldsymbol{w}^n$ share the same minimal existence time $\tau_\circledcirc$ —see the proof and previous discussion leading to Lemma 21. The previously defined map $\varphi^n_\tau : [0, \tau_\circledcirc] \times \mathcal{Z}_0 \to [0, \tau_\circledcirc] \times \mathcal{Z}_n$ can be used to recast the corresponding developments $\boldsymbol{w}^n$ in $[0, \tau_\circledcirc] \times \mathcal{Z}_0$ as solutions $\breve{\boldsymbol{u}}^n$ on $[0, \tau_\circledcirc] \times \mathcal{Z}_n$. Observe that one can remove the part of the solution arising from the extended data on $\mathcal{Z}_n \setminus \mathcal{S}_n$ considering the solution on $[0, \tau_\natural] \times \mathcal{S}_n$ for some $\tau_\natural$ with $0 < \tau_\natural < \tau_\circledcirc$. Then a global solution depending only on the original data $\breve{\boldsymbol{u}}_\star$ can be constructed. In order to obtain the required global solution over $[0, \tau_\natural] \times \mathcal{S}$ one defines $\breve{\boldsymbol{u}}$ adding each individual contribution $\breve{\boldsymbol{u}}^n$ as in equation (5.58). The function $p^n(x)$ has a double purpose, in the one hand, it removes the part of the solution arising from the extended data on $\mathcal{Z}_n \setminus \mathcal{S}_n$ and, in the other hand, it ensures that the solution is not added more than once in the overlapping regions. The resulting solution will belong to H^m_{loc} for fixed τ and as a consequence of Sobolev embedding theorems

$$\breve{\mathbf{u}} \in C^{m-2}([0, \tau_\natural] \times \mathcal{S}, \mathbb{C}^N).$$

Given a sequence $\{^{(n)}\breve{\boldsymbol{u}}_\star\}$ one can identify sequences $\{^{(n)}(\varphi^{n'}_* \breve{\boldsymbol{u}}^{n'}_\star)\}$ with $n, n' \in \mathbb{Z}$. Then, one can apply the theory of [23] as it was done in Lemma 21. The global solutions $\{^{(n)}\breve{\boldsymbol{u}}\}$ are constructed as in equation (5.58). To prove point *(iii)* first observe that from Lemma 8 of Chapter 2 the solution to the conformal evolution system (5.7a)-(5.7b) implies a solution $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ to the extended conformal Einstein field equations on $[0, \tau_\natural] \times \mathcal{S}$ if $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ solves the conformal constraint equations on the initial hypersurface. This solution implies, using Lemma 3 of Chapter 2, a solution to the Einstein field equations whenever the conformal factor is not vanishing. General results of the theory of asymptotics implies then that the initial hypersurface $\mathcal{S}$ can be interpreted as the conformal boundary of the physical spacetime $(\tilde{\mathcal{M}}_{\tau_\natural}, \tilde{\boldsymbol{g}})$ —see [12, 13]. □

Remark 47. An explicit class of perturbed asymptotic initial data sets can be constructed, keeping the initial metric fixed to be standard one on $\mathbb{R} \times \mathbb{S}^2$, using the analysis of [42] as follows: introduce Cartesian coordinates $(x^{\boldsymbol{\alpha}})$ in $\mathbb{R}^3$ with origin located at a fiduciary position $\mathcal{Q}$ and define a polar coordinate via $\rho^2 \equiv \delta_{\boldsymbol{\alpha\beta}} x^{\boldsymbol{\alpha}} x^{\boldsymbol{\beta}}$. Let $l^i = x^i/\rho$ and denote by m^i and $\bar{m}^i$ a pair of complex null vector such that the flat metric on $\mathbb{R}^3$ reads

$$\acute{h}_{ij} = -l_i l_j - 2m_i \bar{m}_j.$$

The general solution of the equation

$$\acute{D}^i \acute{d}_{ij} = 0,$$

where $\acute{D}^i$ is the Levi-Civita connection of the flat metric on $\mathbb{R}^3$, can be parametrised

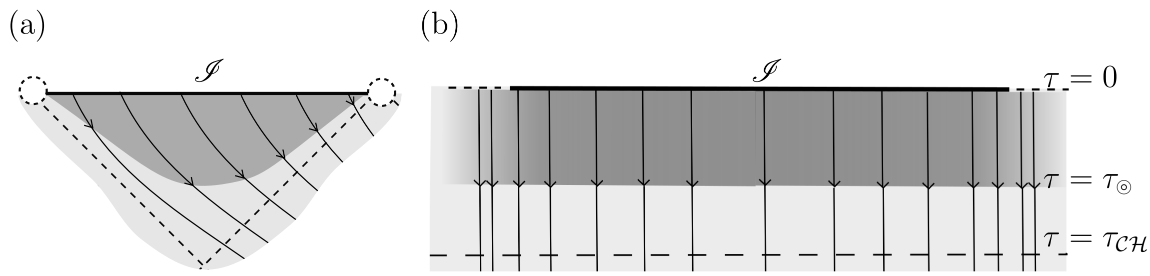

Figure 5.9: *Schematic depiction of the development of perturbed initial data for the Schwarzschild-de Sitter spacetime and the congruence of conformal geodesics. In* (a) *the evolution of asymptotic initial data is depicted in the conformal representation in which the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ are at a finite distance respect to the metric on $\mathscr{I}$. Figure* (b) *shows a schematic depiction of the evolution of asymptotic initial data in the conformal representation in which Theorem 3 has been formulated. In contrast to the conformal representation leading to Figure* (a) *, the initial data is homogeneous and formally identical for the subextremal, extremal or hyperextremal cases. In both diagrams, the dashed line corresponds to the location of a hypothetical Cauchy horizon of the development.*

as

$$\rho^3 \acute{d}_{ij} = \xi(3l_il_j + \acute{h}_{ij}) + \sqrt{2}\eta_1 l_{(i}\bar{m}_{j)} + \sqrt{2}\bar{\eta}_1 l_{(i}m_{j)} + \bar{\mu}_2 m_i m_j + \mu_2 \bar{m}_i\bar{m}_j \qquad (5.60)$$

with

$$\xi = \bar{\eth}^2\lambda_2^R + A + \rho Q + \frac{1}{\rho}P,$$
$$\eta_1 = -2\rho\partial_\rho\bar{\eth}\lambda_2^R + \bar{\eth}\lambda_2^I + \rho\eth Q - \frac{1}{\rho}\eth P + \mathrm{i}\eth J,$$
$$\mu_2 = 2\rho\partial_\rho(\rho\partial_\rho\lambda_2^R) - 2\lambda_2^R + \eth\bar{\eth}\lambda_2^R - \rho\partial_\rho\lambda_2^I.$$

where A, P, Q, J are arbitrary constants, $\lambda_2^R = \mathrm{Re}(\lambda_2(x^{\boldsymbol{\alpha}}))$ and $\lambda_2^I = \mathrm{iIm}(\lambda_2(x^{\boldsymbol{\alpha}}))$ with $\lambda_2(x^{\boldsymbol{\alpha}})$ representing a smooth function of spin-weight 2 —see [42] for a detailed derivation and [12] for definitions of the $\eth$ and $\bar{\eth}$ operators. Let $\acute{d}_{ij}^{(\lambda)}$ denote the part of $\acute{d}_{ij}$ associated with λ_2 —namely setting $A = Q = P = 0$ in equation (5.60). Observe that $\acute{d}_{ij}^{(\lambda)}$ can have, in general, any behaviour near $\mathcal{Q}$ —see [42]. However, setting $\lambda_2 = \mathcal{O}(\rho^n)$ with $n \geq 3$ the term $\acute{d}_{ij}^{(\lambda)}$ is regular near $\mathcal{Q}$. Using the frame version of the conformal transformation rule of Lemma 13 and either equation (5.29) or (5.30) one can verify that the corresponding term in the $\mathbb{S}^3$-representation is $\grave{d}_{\boldsymbol{ij}}^{(\lambda)} = \mathcal{O}(\rho^{n+3})$. Similarly, using the conformal transformation formulae, given in Section 5.3.3, relating the $\mathbb{S}^3$ and $\mathbb{R}\times\mathbb{S}^2$-representations of the initial data, one obtains $d_{\boldsymbol{ij}}^{(\lambda)} = \mathcal{O}(\rho^{n+6})$. Observe that regular behaviour of perturbed initial data in the $\mathbb{R}\times\mathbb{S}^2$-representation does not necessarily correspond to regular behaviour in the $\mathbb{S}^3$-representation nor in the $\mathbb{R}^3$-representation.

5.5 Conclusions

This chapter contains an analysis of the Schwarzschild-de Sitter family of spacetimes as a solution to the extended conformal Einstein field equations expressed in terms of a conformal Gaussian system. Given that, in principle, it is not possible to explicitly express the spacetimes in this gauge, an alternative strategy has been adopted; formulating an asymptotic initial value problem for a spherically symmetric spacetime with a de Sitter-like Cosmological constant. The generalisation of Birkhoff's theorem to vacuum spacetimes with Cosmological constant then ensures that the resulting solutions are necessarily a member of the Schwarzschild-de Sitter spacetime.

As part of the formulation of an asymptotic initial value problem for the Schwarzschild-de Sitter spacetime it was necessary to specify suitable initial data for the conformal evolution equations. The rather simple form that the conformal constraint equations acquire in the framework considered in this chapter allows to study in detail the conformal properties of the Schwarzschild-de Sitter spacetime at the conformal boundary and, in particular, at the asymptotic points where the conformal boundary *meets* the horizons. The key observation from this analysis is that the conformal structure is singular at these points and cannot be regularised in an obvious manner. Accordingly, any satisfactory formulation of the asymptotic initial value problem will exclude these points.

An interesting property of the conformal evolution equations under the assumption of spherical symmetry is that the system reduces to a set of transport equations along the conformal geodesics covering the spacetime. The essential dynamics, and in particular the formation of singularities in the solutions to this system, is governed by a *core system* of three equations —one of them a Riccati equation. As discussed in Appendix 5.7, this core system provides a mechanism for the formation of singularities in the exact solution. The analysis of the core system allows not only to study the properties on the Schwarzschild-de Sitter spacetime expressed in terms of a conformal Gaussian gauge system, but also to understand the effects that the *gauge data* has on the properties of the conformal representation arising as a solution to the conformal evolution equations. Despite the fact that the core system discussed in this chapter is related to the spherical symmetry assumption, it is of interest to explore the whether or not the exist an analogous structure in the general equations —without spherical symmetry.

The conformal representation of the Schwarzschild-de Sitter spacetime obtained in this chapter has been used to show that it is possible to construct, say, *future asymptotically de Sitter* solutions to the Einstein vacuum Einstein with a minimum existence time —as measured by the proper time of the conformal geodesics used to construct the gauge system— which can be understood as perturbations of a member of the Schwarzschild-de Sitter family of spacetimes. As already mentioned in the main text, it is an interesting problem to determine the maximal Cauchy development to these spacetimes. In order to obtain the maximal Cauchy development of suitably small perturbations of asymptotic data for the Schwarzschild-de Sitter one would require the use of more refined methods of the theory of hyperbolic partial differential equations as one is, basically, confronted with a *global existence problem* for the conformal evolution equations. In this respect, it could be conjectured that the *time symmetric conformal representation* in which $\kappa = 0$ together with the *global stability* methods of [76] should allow to make inroads into this issue. Closely related to the construction of the maximal development of perturbations of asymptotic initial data of the Schwarzschild-de Sitter spacetime is the question whether there is a Cauchy horizon associated to the boundary of this development. If this is the case, one would like to investigate the properties of this horizon. Intuitively, the answer to these issues should depend on the relation between the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ and the conformal structure of the spacetime. In particular, one would like to know whether the singularities of the rescaled Weyl tensor at these points generically propagate along the boundary of the perturbed solution —notice, that they do not for the background solution. If one were able to use the $\mathbb{R} \times \mathbb{S}^2$-representation of the conformal boundary of perturbations of asymptotic initial data for the Schwarzschild-de Sitter to construct a maximal development and to gain sufficient control on the asymptotic behaviour of the various conformal fields, one could then rescale this solution to obtain a representation with a conformal boundary of the form $\mathbb{S}^3 \setminus \{\mathcal{Q}, \mathcal{Q}'\}$. As discussed in the main text, in this representation some fields are singular at $\mathcal{Q}$ and $\mathcal{Q}'$. This observation suggests that this construction could shed some light regarding the propagation (or lack thereof) of singularities near the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$.

5.6 Appendix: The asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ and conformal geodesics in the Schwarzschild-de Sitter spacetime

5.6.1 Analysis of the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$

In Section 5.3.2 it was shown that there exist a conformal representation of the Schwarzschild-de Sitter spacetime in which the metric at the conformal boundary is $\hbar$ —i.e. the standard metric on $\mathbb{S}^3$. In addition, it was observed that the North and South poles of $\mathbb{S}^3$ correspond to special points in the conformal structure that have been labelled as $\mathcal{Q}$ and $\mathcal{Q}'$. These asymptotic regions are represented in the Penrose diagram for the subextremal, extremal and hyperextremal Schwarzschild-de Sitter spacetime as the points where the conformal boundary and the Cosmological horizon, Killing horizon and singularity, respectively, seem to meet —see Figures 5.3

and 5.4 and 5.5. As discussed in Section 5.3.2 these points correspond to $(\bar{U}, \bar{V}) = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ for which the tortoise coordinate $\mathfrak{r}$ is not well defined. In Section 5.3.4 it was shown that in the conformal representation in which the initial metric is $\hbar$ the data for the electric part of the rescaled Weyl tensor $d\!\!\!/_{ij}$, as given in equation (5.31), is singular precisely at $\mathcal{Q}$ and $\mathcal{Q}'$. Observe that written in spinorial terms the initial data for the rescaled Weyl spinor in this conformal representation is given by

$$\phi\!\!\!/_{ABCD} = \frac{6m}{\sqrt{1-\omega^2(\xi)}}\epsilon^2{}_{ABCD}$$

which is singular at both $\mathcal{Q}$ and $\mathcal{Q}'$. This situation resembles that of the geometry near spacelike infinity i^0 of the Minkowski spacetime and the construction of the *cylinder at infinity* given in [27] which allows to regularise the data for the rescaled Weyl spinor. However, some experimentation reveals that this type of regularisation procedure (in contrast with the analysis of Schwarzschild spacetime given in [27]) cannot be implemented in the analysis of the Schwarzschild-de Sitter spacetime without spoiling the regular behaviour of the conformal factor. Since the hyperbolic reduction procedure for the extended conformal Einstein field equations is based on the existence of a congruence of conformal geodesics in spacetime, the singular behaviour of the initial data for the rescaled Weyl spinor suggest that the congruence of conformal geodesics does not cover the region of the spacetime corresponding to $\mathcal{Q}$ and $\mathcal{Q}'$. To clarify this point, in the remaining of this section the behaviour of conformal geodesics as they approach the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ is analysed.

5.6.2 Geodesics in Schwarzschild-de Sitter spacetime

The method for the hyperbolic reduction for the extended conformal Einstein field equations available in the literature requires adapting the gauge to a congruence of conformal geodesics. The behaviour of metric geodesics in the Schwarzschild-de Sitter spacetime has been already studied [77, 78] and an analysis of conformal geodesics in Schwarzschild-de Sitter and anti-de-Sitter spacetimes is carried out in [79]. In static coordinates (t, r, θ, φ) the equation for radial timelike geodesics, $(\theta = \theta_\star, \varphi = \varphi_\star)$ with $\theta_\star$ and $\varphi_\star$ constant, are

$$\frac{\mathrm{d}r}{\mathrm{d}\tilde{\tau}} = \sqrt{\gamma^2 - F(r)}, \qquad \frac{\mathrm{d}t}{\mathrm{d}\tilde{\tau}} = \frac{\gamma}{F(r)}. \tag{5.62}$$

The first equation can be formally integrated as

$$\tilde{\tau} - \tilde{\tau}_\star = \int_{r_\star}^{r} \frac{1}{\sqrt{\gamma^2 - F(s)}}\mathrm{d}s \tag{5.63}$$

where $\tilde{\tau}$ is the $\tilde{\boldsymbol{g}}_{SdS}$-proper time and γ is a constant of motion which can be identified with the energy of a particle moving along the geodesic. The equation for t can be solved once equation (5.63) has been integrated. As pointed out in [66, 68], by choosing $\gamma = 1$ one can explicitly solve this integral. However in general, for arbitrary γ, the integral is complicated and cannot be written in terms of elementary functions. A side observation is that if $r \neq r_b$ and $r \neq r_c$ then the curves of constant t correspond to geodesics with $\gamma = 0$. Finally, its worth noticing that geodesics with constant r are characterised by the condition

$$\gamma^2 - F(r) = 0. \tag{5.64}$$

This last type of curves, which will be called *critical curves*, are analysed in Section 5.6.4. In general, the properties of conformal geodesics differ from their metric counterparts. However, in the case of an Einstein spacetime with spacelike conformal boundary any conformal geodesic leaving $\mathscr{I}$ orthogonally is, up to reparametrisation, a metric geodesic —see [39] and Lemma 7 of Chapter 2.

5.6.3 A special class of conformal geodesics in the Schwarzschild-de Sitter spacetime

As briefly mentioned in Section 5.6.2 and pointed out in [66, 68], in general, the integral (5.63) cannot be written in terms of elementary functions except for the special case when $\gamma = 1$ where it yields

$$r(\tilde{\tau}) = \mathcal{C}e^{\tilde{\tau}}\left(1 - \left(\frac{3m}{2|\lambda|}\right)\mathcal{C}^{-3}e^{-3\tilde{\tau}}\right)^{2/3}, \tag{5.65}$$

where $\mathcal{C}$ is an integration constant. The last expression is valid irrespective of the relation between m and λ. One can also use this expression to integrate the second equation in (5.62) to obtain the geodesic parametrised as $(r(\tilde{\tau}), t(\tilde{\tau}))$. The integration of t will not be required for the purposes of the analysis of this section. A complete analysis of conformal geodesics in the Schwarzschild-de Sitter and anti-de Sitter spacetimes will be given in [79]. By virtue of Lemma 7 one can recast the geodesic with $\gamma = 1$ as a conformal geodesic by reparametrising it in terms of the unphysical proper time as determined by equation (2.42) given in Lemma 7 and equation (5.4) of Section 5.2.2 . A straightforward computation yields

$$\tilde{\tau}(\tau) = \sqrt{\frac{3}{|\lambda|}}\ln\left|\frac{\tau}{2+\kappa\tau}\right|. \tag{5.66}$$

Equivalently, assuming either $\kappa > 0$ and $\tau \geq 0$ or $\kappa < 0$ and $0 \leq \tau \leq -2/\kappa$ one obtains in both cases

$$\tau(\tilde{\tau}) = \frac{2\exp\left(\sqrt{\frac{|\lambda|}{3}}\tilde{\tau}\right)}{1 - \kappa\exp\left(\sqrt{\frac{|\lambda|}{3}}\tilde{\tau}\right)}. \tag{5.67}$$

From the last expression one can verify that

$$\lim_{\tilde{\tau}\to-\infty}\tau(\tilde{\tau})=0, \qquad \lim_{\tau\to\infty}\tau(\tilde{\tau})=-2/\kappa,$$

as expected. Rewriting equation (5.65) in terms of the unphysical proper time one obtains

$$r(\tau)=\frac{1}{(m|\lambda|)^{2/3}}\frac{(m|\lambda|\mathcal{C}^3\tau^3-6(2+\kappa\tau)^3)^{2/3}}{\mathcal{C}\tau(\tau+2\kappa\tau)}. \tag{5.68}$$

From the last expressions one can verify that one has $r\to\infty$ as $\tau\to0$ and $\tau\to-2/\kappa$. The location of the singularity $r=0$ is determined by

$$\tau_{\not{\iota}}=\frac{2}{(m|\lambda|)^{1/3}\mathcal{C}-\kappa}.$$

Recalling that $\mathcal{C}$ is an integration constant which depends on the initial data for the congruence, since the only freedom left in the conformal factor is encoded in κ, one realises that $\mathcal{C}=\mathcal{C}(\kappa)$. So one cannot draw any precise conclusion about the location of the singularity unless one further identifies explicitly $\mathcal{C}(\kappa)$. In particular, considering constant κ and setting $\mathcal{C}$ to be proportional to κ, say $\mathcal{C}=\frac{(2\varkappa+1)}{(m|\lambda|)^{1/3}}\kappa$ for some proportionality constant $\varkappa$, one obtains

$$\tau_{\not{\iota}}=\frac{1}{\varkappa\kappa},$$

which is in agreement the with the qualitative behaviour of the core system as shown in Figures 5.6, 5.11, and 5.12. Notice, however, that the arguments of the core system given in Section 5.4.2 and Appendix 5.7 do not rely on integrating (5.63) explicitly.

5.6.4 Critical curves on the Schwarzschild-de Sitter spacetime

In order to clarify the role of the asymptotic points, in this section it is shown that there are not timelike conformal geodesics reaching $\mathcal{Q}$ and $\mathcal{Q}'$ orthogonally. More precisely, it is shown that a timelike conformal geodesic becomes asymptotically null as it approaches either $\mathcal{Q}$ or $\mathcal{Q}'$. This is in stark tension with the conditions for constructing a conformal Gaussian system of coordinates in the neighbourhood of $\mathcal{Q}$ and $\mathcal{Q}'$.

As shown in the Penrose diagram of Figure 5.10 in the subextremal case the curves of constant $t=t_\star$ accumulate in the *bifurcation spheres* $\mathcal{B}$ and $\mathcal{B}'$ while the curves of constant r accumulate in the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$. By contrast, in the extremal case the curves with constant $t=t_\star$ approach the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ —see [69] for an extensive discussion on the Penrose diagram for Schwarzschild-de Sitter spacetime. It follows from the geodesic equation (5.62) that the curves of

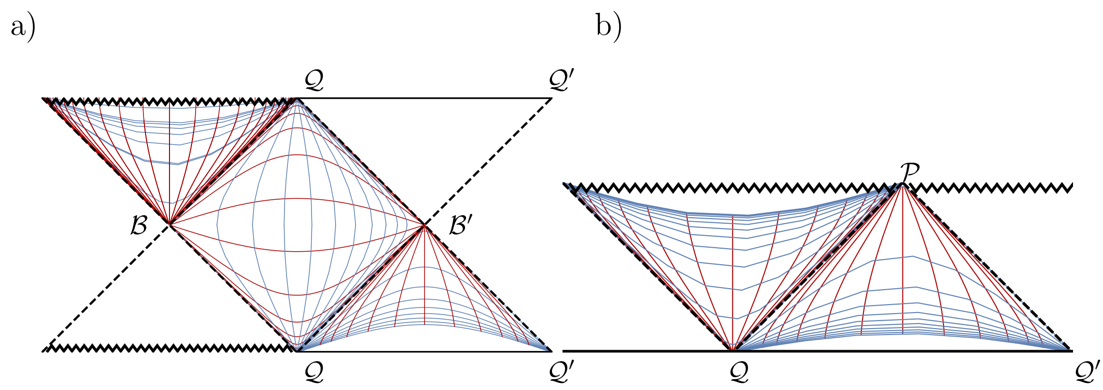

Figure 5.10: *Curves of constant r and t in the Schwarzschild-de Sitter spacetime. a) Curves with constant t and r (red and blue respectively) are plotted on the Penrose diagram of the Subextremal Schwarzschild-de Sitter spacetime. Curves of constant t accumulate at the bifurcation spheres $\mathcal{B}$, $\mathcal{B}'$ while the curves of constant r accumulate at the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$. b) Curves with constant t and r (red and blue respectively) are plotted on the Penrose diagram of the extremal Schwarzschild-de Sitter spacetime. In contrast with the subextremal case, curves with constant t in starting from some $r_\star < 3m$ accumulate at the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ while those starting from $r_\star > 3m$ accumulate at $\mathcal{P}$. The hyperextremal case is qualitatively similar to the extremal one and has been omitted.*

constant r correspond to geodesics whenever the condition (5.64) is satisfied, this equation explicitly reads

$$|\lambda| r^3 + 3(\gamma^2 - 1)r + 6m = 0. \tag{5.69}$$

Observe that for $\gamma = 1$ the last condition reduces to $|\lambda| r^3 + 6m = 0$ which cannot be solved for positive r.

In this section an analysis of the behaviour of the critical curves on the Schwarzschild-de Sitter spacetime is performed. Notice that in the hyperextremal case the are no timelike geodesics with constant r since for $|\lambda| > 1/9m^2$ one has strictly $F(r) < 0$ so that the condition (5.64) can never be satisfied.

Critical curves in the extremal Schwarzschild-de Sitter spacetime

To start the analysis consider the simpler case in which $|\lambda| = 1/9m^2$ so that $F(r)$ is given as in equation (5.12) and the condition (5.64) reduces to considering $r = 3m$ and $\gamma = 0$. Observe that the curves with $\gamma = 0$ and $r \neq 3m$ correspond to curves with constant $t = t_\star$ which, as discussed in previous paragraphs, approach asymptotically the points $\mathcal{Q}$ and $\mathcal{Q}'$. Notice that for $\gamma = 0$ the expression (5.63) can easily be integrated to yield

$$\tilde{\tau} - \tilde{\tau}_\star = 3m \ln\left(H(r)/H(r_\star)\right) \tag{5.70}$$

where
$$H(r) \equiv \frac{\sqrt{3r}+\sqrt{r+6m}}{(\sqrt{3r}-\sqrt{r+6m})(\sqrt{r}+\sqrt{r+6m})^{2\sqrt{3}}}.$$
Observe that equation (5.70), as pointed out in [68], implies that the geodesics with $\gamma = 0$ never cross the horizon since $\tilde{\tau} \to \infty$ as $r \to 3m$. For simplicity, let $M_\star \equiv H(r_\star) + \exp(\tilde{\tau}_\star/3m)$ with $r_\star \neq 3m$ so that $\tilde{\tau} = 3m \ln|H(r)/M_\star|$. Reparametrising using equation (5.67) and that $|\lambda| = 1/9m^2$ renders

$$\tau(r) = \frac{2W(r)}{M_\star^p - \kappa W(r)}$$

with $W(r) = H(r)^{1/\sqrt{3}}$. Using L'Hôpital rule one can verify that $\tau \to -2/\kappa$ as $r \to 3m$. To analyse the behaviour of these curves as they approach the points $\mathcal{Q}$ and $\mathcal{Q}'$ one considers r such that $r = 3m + \epsilon$. Then, one has for small $\epsilon > 0$ that

$$W(r) = \left(\frac{m}{r-3m}\right)^{1/\sqrt{3}} \left(\frac{C_1}{m} + \frac{C_2}{m^2}(r-3m) + \mathcal{O}((r-3m)^2)\right)$$

where C_1 and C_2 are numerical factors whose explicit form is not relevant for the subsequent discussion. Hence, to leading order $W(r) = C/\epsilon^p$ where C is a non-zero constant depending on m only and $p = 1/\sqrt{3}$. Consequently, to leading order

$$\frac{\mathrm{d}\tau}{\mathrm{d}\epsilon} = -\frac{pC\kappa\epsilon^{p-2}}{(M_\star^p\epsilon^p - \kappa C)} - \frac{pC\epsilon}{M_\star^p\epsilon^p - \kappa C}.$$

Therefore, since $p < 2$ one has that $\mathrm{d}\tau/\mathrm{d}\epsilon$ diverges as $\epsilon \to 0$ so that the curves with $\gamma = 0$ become tangent to the horizon as they approach $\mathcal{Q}$ or $\mathcal{Q}'$ —that is, they become null as they approach $\mathcal{Q}$ or $\mathcal{Q}'$. This is analogous to the behaviour of the critical curve in the Schwarzschild spacetime pointed out in [35], and the subextremal Reissner-Nordström spacetime in [80] —in contrast, in the extremal Reissner-Nordström spacetime one has $\mathrm{d}\tau/\mathrm{d}\epsilon = 0$ as $\epsilon \to 0$ as discussed in [80] .

Critical curves in the subextremal Schwarzschild-de Sitter spacetime

For the subextremal case one could parametrise the roots of the depressed cubic (5.69) using Vieta's formulae and choose some $\gamma \neq 1$ for which there is at least one positive root. However, notice that fixing a value for γ is equivalent to prescribe initial data for the congruence:

$$t(\tilde{\tau}) = t_\star, \qquad r(\tilde{\tau}) = r_\star, \qquad \left.\frac{\mathrm{d}r}{\mathrm{d}\tilde{\tau}}\right|_{r_\star} = \sqrt{\gamma^2 - F(r_\star)}, \qquad \left.\frac{\mathrm{d}t}{\mathrm{d}\tilde{\tau}}\right|_{r_\star} = \frac{\gamma}{F(r_\star)}. \qquad (5.71)$$

Restricting the analysis to the static region $r_b < r_\star < r_c$ for which $F(r_\star) > 0$ and setting
$$\left.\frac{\mathrm{d}r}{\mathrm{d}\tilde{\tau}}\right|_{r_\star} = 0,$$

one gets

$$\gamma = \sqrt{F(r_\star)},$$

and condition (5.64) is equivalent to

$$F(r_\star) - F(r) = \frac{|\lambda|(r-r_\star)}{3r} Q(r),$$

where $Q(r)$ is the polynomial

$$Q(r) \equiv r^2 + r_\star r - \frac{6m}{|\lambda| r_\star}.$$

Notice that $Q(r)$ can be factorised as

$$Q(r) = (r - \alpha_-(r_\star))(r - \alpha_+(r_\star)),$$

where

$$\alpha_\pm(r_\star) \equiv \frac{r_\star}{2}\left(-1 \pm \sqrt{1 + \frac{24m}{|\lambda| r_\star^3}}\right).$$

In addition, observe that

$$\begin{aligned} Q(r_\star) &> 0 \quad \text{for} \quad r_\circledast < r_\star < r_c, \\ Q(r_\star) &< 0, \quad \text{for} \quad r_b < r_\star < r_\circledast, \\ Q(r_\star) &= 0, \quad \text{for} \quad r_\star = r_\circledast, \end{aligned}$$

where $r_\circledast \equiv \left(\frac{3m}{|\lambda|}\right)^{1/3}$. In the extremal case one has $r_b = r_c = r_\circledast = 3m$. The curve $r = r_\circledast$, as in the extremal case, will be called the *critical curve*. With the above notation the integral (5.63) can be then rewritten as

$$\tilde{\tau} - \tilde{\tau}_\star = \int_{r_\star}^{r} \sqrt{\frac{s}{(s-r_\star)(s-\alpha_-(r_\star))(s-\alpha_+(r_\star))}} \mathrm{d}s. \tag{5.72}$$

To study the behaviour close to the critical curve consider $r_\star = (1+\epsilon) r_\circledast$ For small $\epsilon > 0$ and considering $s > r_\star$ one can expand the right hand side of equation (5.72) in Taylor series as

$$\tilde{\tau} - \tilde{\tau}_\star = \int_{r_\star}^{r} \sqrt{\frac{s}{s + 2r_\circledast}} \left(\frac{1}{s - r_\circledast} - \frac{3 r_\circledast^2 s \epsilon^2}{2(s - r_\circledast)^3}\right) \mathrm{d}s + \mathcal{O}(\epsilon^3). \tag{5.73}$$

Integrating one obtains

$$\tilde{\tau} - \tilde{\tau}_\star = -\frac{2}{\sqrt{3}} \operatorname{arctanh}\left(\sqrt{3}\sqrt{\frac{1+\epsilon}{3+\epsilon}}\right) + 2\ln\left(\sqrt{r_\circledast(1+\epsilon)} + \sqrt{r_\circledast(3+\epsilon)}\right)$$

$$-\frac{2}{\sqrt{3}}\operatorname{arctanh}\left(\frac{3r}{r+2r_{\circledast}}\right)+2\ln\left(\sqrt{r}+\sqrt{r+2r_{\circledast}}\right)\ -\frac{3}{4}r_{\circledast}\sqrt{1+2r_{\circledast}}(1+2\epsilon)$$
$$-\frac{3}{4}{r_{\circledast}}^{2}\sqrt{1+2r_{\circledast}}\frac{(2r-r_{\circledast})\epsilon^2}{(r_{\circledast}-r)^2}+\mathcal{O}(\epsilon^3).$$

As $\epsilon \to 0$ the last expression diverges —as is to be expected. The divergent term can be expanded for small $\epsilon > 0$ as

$$\operatorname{arctanh}\left(\sqrt{3}\sqrt{\frac{1+\epsilon}{3+\epsilon}}\right)=\frac{1}{2}\ln\left(\left|-\frac{6}{\epsilon}+4+\frac{\epsilon}{6}+\mathcal{O}(\epsilon^2)\right|\right)$$

and the second term can be expanded as

$$\ln\left(\sqrt{r_{\circledast}(1+\epsilon)}+\sqrt{r_{\circledast}(3+\epsilon)}\right)=\ln\left((1+\sqrt{3})\sqrt{r_{\circledast}}\right)+\frac{\epsilon}{2\sqrt{3}}-\frac{\epsilon^2}{6\sqrt{3}}+\mathcal{O}(\epsilon^3).$$

Hence, to leading order one has

$$\tilde{\tau}(r)=\frac{1}{\sqrt{3}}\ln\epsilon+f(r)+\mathcal{O}(\epsilon)$$

where

$$f(r)=\tilde{\tau}_{\star}+2\ln\left((1+\sqrt{3})\sqrt{r_{\circledast}}\right)-\frac{2}{\sqrt{3}}\operatorname{arctanh}\left(\frac{3r}{r+2r_{\circledast}}\right)$$
$$+2\ln\left(\sqrt{r}+\sqrt{r+2r_{\circledast}}\right)-\frac{3}{4}r_{\circledast}\sqrt{1+2r_{\circledast}}.$$

Reparametrising respect to the unphysical proper time using (5.67) one gets

$$\tau(r)=\frac{2\exp\left(\sqrt{|\lambda|/3}f(r)+\mathcal{O}(\epsilon)\right)\epsilon^{p}}{1-\kappa\exp\left(\sqrt{|\lambda|/3}f(r)+\mathcal{O}(\epsilon)\right)\epsilon^{p}}$$

with $p=1/\sqrt{3}$. Thus one gets

$$\frac{\mathrm{d}\tau}{\mathrm{d}\epsilon}=\frac{2p\exp\left(\sqrt{|\lambda|/3}f(r)+\mathcal{O}(\epsilon)\right)\epsilon^{p-1}}{\left(1-\exp\left(\sqrt{|\lambda|/3}f(r)+\mathcal{O}(\epsilon)\right)\kappa\epsilon^{p}\right)^2}.$$

Observe that since $p<1$ then one has that $\mathrm{d}\tau/\mathrm{d}\epsilon$ diverges as $\epsilon \to 0$.

5.7 Appendix: The conformal evolution equations in the case $\kappa \neq 0$ and reparametrisations

In Section 5.4.2 the case $\kappa = 0$ was analysed —this corresponds to a conformal boundary with vanishing extrinsic curvature. Nevertheless, as discussed in Section 5.2.2, κ is a conformal gauge quantity arising from the conformal transformation properties of the conformal field equations. Consequently, it is of interest to analyse the behaviour of the core system in the case $\kappa \neq 0$. For simplicity, in the remainder of this section, κ will be assumed to be a constant on the initial hypersurface corresponding to $\tau = 0$. In first instance, the analysis will be restricted to $|\kappa| > 1$ and then it will be discussed how to exploit the conformal covariance of the equations to extend these results for $\kappa \in [-1, 0) \cup (0, 1]$.

5.7.1 Analysis of the core system with $\kappa > 1$

To start the discussion of this case observe that, for $\kappa > 1$, one has that $\Theta(\tau) \geq 0$ and $\dot{\Theta}(\tau) > 0$ for $\tau \geq 0$. Using this simple observation and the core equations (5.40a)-(5.40c) one obtains the following:

Lemma 22. *For a solution to the core system* (5.40a)-(5.40c) *with initial data given by* (5.41) *and* $\kappa > 1$ *one has that* $L(\tau) < 0$ *for* $\tau \geq 0$.

Proof. One proceeds by contradiction. Assume that there exists $0 < \tau_L < \infty$ such that $L(\tau_L) = 0$. Without loss of generality one can assume that τ_L corresponds to the first zero of $L(\tau)$. Since for $\kappa > 1$ one has $L(0) < 0$ then by continuity it follows that $\dot{L}(\tau_L) \geq 0$ —$\dot{L}(\tau_L)$ cannot be negative since this would imply that $L(\tau)$ crossed the τ-axis at some time $\tau < \tau_L$ but this is not possible since τ_L is the first zero of $L(\tau)$. It follows then from equation (5.40c) that

$$0 \leq \dot{L}(\tau_L) = -\chi(\tau_L)L(\tau_L) - \frac{1}{2}\dot{\Theta}(\tau_L)\phi(\tau_L).$$

Since $L(\tau_L) = 0$ and $\dot{\Theta}(\tau_L) > 0$, the last inequality implies that $\phi(\tau_L) \leq 0$ but this is a contradiction since it is already known from Observation 1 —which is valid for any value of κ— that $\phi(\tau) > 0$ for any τ. □

Observation 4. Using that $\dot{\Theta}(\tau) \geq 0$ for $\kappa > 1$ and $\tau \geq 0$ and that $\phi(\tau) > 0$ one derives from equation (5.40c) the differential inequality

$$\dot{L}(\tau) \leq -\chi(\tau)L(\tau).$$

Observing Lemma 22 one has that $L(\tau) < 0$. Thus, one can formally integrate the last differential inequality and obtain

$$L(\tau) \leq L(0)\exp\left(-\int_0^\tau \chi(\mathrm{s})\mathrm{ds}\right). \tag{5.74}$$

With these observations one can show that the function $\chi(\tau)$ which is initially positive must necessarily have a zero.

Lemma 23. *For a solution to the core system* (5.40a)-(5.40c) *with initial data given by* (5.41) *and* $\kappa > 1$ *there exist* $0 < \tau_\chi < \infty$ *such that* $\chi(\tau_\chi) = 0$.

Proof. One proceeds again by contradiction. Assume that $\chi(\tau)$ never vanishes. Since $\chi(0) = \kappa > 0$ then $\chi(\tau) > 0$ for $\tau \geq 0$. From Lemma 22 it follows that $L(\tau) < 0$. In addition, one knows that $\Theta(\tau)\phi(\tau) \geq 0$. With these observations equation (5.40b) gives

$$\dot{\chi}(\tau) < -\chi^2(\tau) \quad \text{for} \quad \tau > 0.$$

Since one is assuming that $\chi(\tau)$ never vanishes then

$$\frac{\dot{\chi}(\tau)}{\chi^2(\tau)} < -1.$$

Integrating from 0 to $\tau > 0$ and using the initial data (5.41) one gets

$$\chi(\tau) < \frac{1}{\tau + 1/\kappa} \quad \text{for} \quad \tau > 0. \tag{5.75}$$

In a similar way, one can consider equation (5.40b) and obtain the differential inequality

$$\dot{\chi}(\tau) < -\frac{1}{2}\Theta(\tau)\phi(\tau) \quad \text{for} \quad \tau \geq 0.$$

Using now equation (5.42) one gets

$$\dot{\chi} < -m\Theta(\tau)\exp\left(-3\int_0^\tau \chi(\mathrm{s})\mathrm{ds}\right) \quad \text{for} \quad \tau \geq 0.$$

Integrating the from 0 to $\tau > 0$ renders

$$\chi(\tau) < \kappa - m\int_0^\tau \Theta(\mathrm{s})\exp\left(-3\int_0^s \chi(\mathrm{s}')\mathrm{ds}'\right)\mathrm{ds} \quad \text{for} \quad \tau \geq 0. \tag{5.76}$$

On the other hand, integrating expression (5.75) one has

$$\int_0^\tau \chi(\mathrm{s})\mathrm{ds} < \ln(\kappa\tau + 1).$$

Consequently,

$$-m\Theta(\tau)\exp\left(-3\int_0^\tau \chi(\mathrm{s}')\mathrm{ds}'\right) < -m\sqrt{\frac{|\lambda|}{3}}\frac{\tau(1+\frac{1}{2}\kappa\tau)}{(1+\kappa\tau)^3}.$$

Integrating one obtains

$$-m\int_0^\tau \Theta(\mathrm{s})\exp\left(-3\int_0^s \chi(\mathrm{s}')\mathrm{ds}'\right)\mathrm{ds} < -\frac{m}{2\kappa^2}\sqrt{\frac{|\lambda|}{3}}\left(\frac{1}{(\kappa\tau+1)^2} + \ln(\kappa\tau+1) - 1\right).$$

Substituting the above result into the inequality (5.76) one obtains

$$\chi(\tau) < \kappa - \frac{m}{2\kappa^2}\sqrt{\frac{|\lambda|}{3}}\left(\frac{1}{(\kappa\tau+1)^2} + \ln(\kappa\tau+1) - 1\right).$$

The right hand side of the last expression becomes negative for some sufficiently large τ. This is a contradiction as it was assumed that $\chi(\tau)$ never vanishes and $\chi(0) > 0$. □

Observation 5. Combining Lemma 22 and Observation 1, one concludes that $L(\tau) < 0$ and $\Theta(\tau)\phi(\tau) > 0$ for $\tau > 0$. Using these properties in equation (5.40b) renders

$$\dot{\chi}(\tau) < 0 \quad \text{for} \quad \tau \geq 0.$$

Thus, $\chi(\tau)$ is always decreasing. From Lemma 23 one knows that there exists a finite $\tau_\chi > 0$ such that $\chi(\tau_\chi) = 0$. Then, by continuity, for any $\tau > \tau_\chi$ one has that $\chi(\tau) < 0$.

With this last observation one is in the position of proving the main result of this section:

Proposition 5. *There exists* $0 < \tau_\natural < \infty$ *such that the solution of* (5.40a)-(5.40c) *with initial data given by* (5.41) *and* $\kappa > 1$ *satisfies*

$$\chi \to -\infty, \quad L \to -\infty, \quad \phi \to \infty \quad \textit{as} \quad \tau \to \tau_\natural .$$

Proof. From Lemma 23 one knows that there exists a finite τ_χ for which $\chi(\tau)$ vanishes. By Observation 5, one has that $\chi(\tau_\Diamond) < 0$ for any $\tau_\Diamond > \tau_\chi$. Let $\chi_\Diamond \equiv \chi(\tau_\Diamond) < 0$. One can assume that $\chi_\Diamond$ is finite, otherwise there is nothing to prove. Now, using Lemma 22 and that $\Theta(\tau)\phi(\tau) > 0$ one obtains

$$\dot{\chi}(\tau) < -\chi^2(\tau) \quad \text{for} \quad \tau \geq 0.$$

Using that $\chi(\tau) < 0$ for any $\tau > \tau_\Diamond$ one obtains

$$\frac{\dot{\chi}(\tau)}{\chi^2(\tau)} < -1.$$

Integrating the last expression form $\tau = \tau_\Diamond$ to $\tau > \tau_\Diamond$ renders

$$\chi(\tau) < \frac{1}{\tau - \tau_\Diamond + 1/\chi_\Diamond} \quad \text{for} \quad \tau > \tau_\Diamond. \tag{5.77}$$

From inequality (5.77) one can conclude that $\chi(\tau) \to -\infty$ for some finite time $\tau_\natural < \tau_\Diamond - 1/\chi_\Diamond$. Additionally, observe that $\tau_\Diamond - 1/\chi_\Diamond > \tau_\Diamond > 0$ since $\chi_\Diamond < 0$.

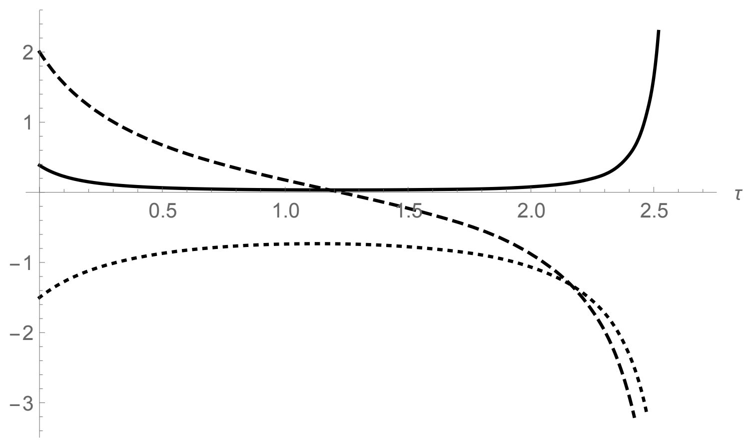

Figure 5.11: *Numerical solutions of the core system* (5.40a) -(5.40c) *with initial data given by* (5.41) *in the case* $\kappa = 2$ *and* $|\lambda| = 3$*,* $m = 1/3\sqrt{3}$*. The solid line describes the evolution of* ϕ*, the dashed line that of* χ *and the dotted line that of* L*. One can observe the formation of a singularity at* $\tau \approx 2.6392$.

Now, given that $\chi \to -\infty$ as $\tau \to \tau_{\lightning}$ it follows from equation (5.42) that $\phi \to \infty$ as $\tau \to \tau_{\lightning}$. Similarly, from inequality (5.74) and that $L(0) < 0$ it follows that $L \to -\infty$ as $\tau \to \tau_{\lightning}$. □

Remark 48. A plot of the numerical evaluation of the solutions to the core system (5.40a)-(5.40c) with initial data (5.41) in the case $\kappa > 1$ can be seen in Figure 5.11.

5.7.2 Analysis of the core system with $\kappa < -1$

In this section a similar approach to that followed in Section 5.7.1 is used to show that the fields in the core system diverge for some finite time if $\kappa < -1$. An interesting feature of this case is that, assuming one knows that there exists a singularity in the development, there exists an *a priori* upper bound for the time of its appearance —namely, the location of second component of the conformal boundary at $\tau = 2/|\kappa|$. As a byproduct of the analysis of this section an improvement of this basic bound is obtained.

An important remark concerning the case $\kappa < -1$ is that if $\tau \in [0, 1/|\kappa|]$ then both $\Theta(\tau)$ and $\dot{\Theta}(\tau)$ are non-negative. Based on this observation the first result in this section is:

Lemma 24. *If* $\kappa < -1$ *then the solution to the core system* (5.40a)-(5.40c) *with initial data* (5.41) *satisfies* $L(\tau) < 0$ *for* $\tau \in [0, 1/|\kappa|]$.

Proof. One proceeds by contradiction. Assume that there exists $0 < \tau_L \leq 1/|\kappa|$ such that $L(\tau_L) = 0$. Without loss of generality one can assume that τ_L is the first zero of $L(\tau)$. Since $L(0) < 0$ for $\kappa < -1$ then by continuity $\dot{L}(\tau_L) \geq 0$. Therefore, proceeding as in Lemma 22 one gets from equation (5.40c)

$$0 \leq \dot{L}(\tau_L) = -\chi(\tau_L)L(\tau_L) - \frac{1}{2}\dot{\Theta}(\tau_L)\phi(\tau_L) \quad \text{for} \quad \tau \in [0, 1/|\kappa|].$$

Since $L(\tau_L) = 0$ and $\dot{\Theta}(\tau_L) > 0$ the last inequality implies that $\phi(\tau_L) \leq 0$. This is a contradiction since $\phi(\tau) > 0$ —cfr. Observation 1. □

Lemma 25. *If* $\kappa < -1$ *then the solution to the core system* (5.40a)-(5.40c) *with initial data* (5.41) *satisfies* $\chi(\tau) < 0$ *for* $\tau \in [0, 1/|\kappa|]$.

Proof. Again, one proceeds by contradiction. Assume that there exists $0 < \tau_\chi \leq 1/|\kappa|$ such that $\chi(\tau_\chi) = 0$. Without lost of generality one can assume that τ_χ is the first zero of $\chi(\tau)$. Then, by continuity, one has that $\dot{\chi}(\tau_\chi) \geq 0$. Using equation (5.40b) one has

$$0 \leq \dot{\chi}(\tau_\chi) = -\chi(\tau_\chi)^2 + L(\tau_\chi) - \frac{1}{2}\Theta(\tau_\chi)\phi(\tau_\chi) \quad \text{for} \quad \tau \in [0, 1/|\kappa|].$$

Therefore, since $\chi(\tau_\chi) = 0$ one has

$$L(\tau_\chi) \geq \frac{1}{2}\Theta(\tau_\chi)\phi(\tau_\chi) > 0.$$

This is a contradiction since by Lemma 24 one has that $L(\tau) < 0$ for $\tau \in [0, 1/|\kappa|]$. □

Observation 6. Proceeding as in Observation 4 one readily has that for $\kappa < -1$

$$L(\tau) \leq L(0) \exp\Big(-\int_0^\tau \chi(\mathrm{s})\mathrm{ds}\Big) \qquad \text{for} \qquad \tau \in (0, 1/|\kappa|].$$

This last observation is used, in turn, to prove the main result of this section:

Proposition 6. *If* $\kappa < -1$*, then for the solution of* (5.40a)-(5.40c) *with initial data* (5.41) *there exists* $0 < \tau_{\not{\iota}} < 1/|\kappa|$ *such that*

$$\chi(\tau) \to -\infty, \quad L(\tau) \to -\infty, \quad \textit{and} \quad \phi(\tau) \to \infty \quad \textit{as} \quad \tau \to \tau_{\not{\iota}}.$$

Proof. Consider equation (5.40b) on the interval $\tau \in [0, 1/|\kappa|]$. Using Lemma 24 one knows that $L(\tau) < 0$. This observation and the fact that $\phi(\tau) > 0$ leads to the differential inequality

$$\dot{\chi}(\tau) < -\chi^2(\tau) \quad \text{for} \quad \tau \in [0, 1/|\kappa|].$$

Since by Lemma 25, one knows that $\chi(\tau) \neq 0$ for $\tau \in [0, 1/|\kappa|]$ one can rewrite the last expression as

$$\frac{\dot{\chi}(\tau)}{\chi^2(\tau)} < -1 \quad \text{for} \quad \tau \in [0, 1/|\kappa|].$$

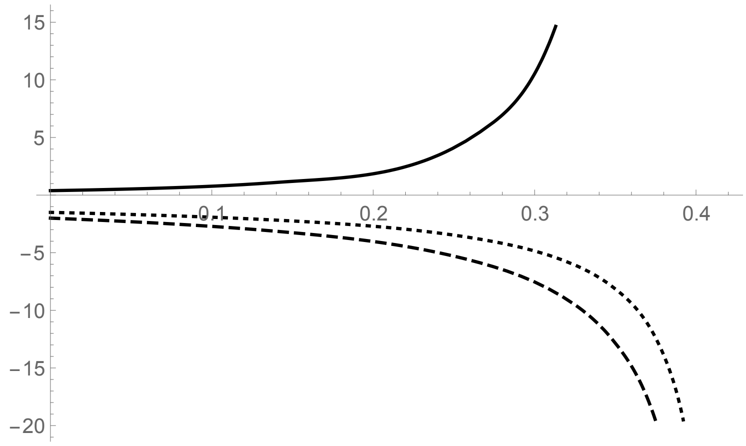

Figure 5.12: *Numerical solution of the core system* (5.40a)-(5.40c) *with* $|\lambda| = 3$, $m = 1/3\sqrt{3}$ *in the case* $\kappa = -2$. *The solid line corresponds to* ϕ, *the dashed line to* χ *and the dotted line to* L. *One can observe a singularity at* $\tau \approx 0.4203$.

Integrating from $\tau = 0$ to $1/|\kappa|$ and using the initial data (5.41) one gets

$$\chi(\tau) < \frac{1}{\tau} - \frac{1}{|\kappa|}. \tag{5.78}$$

From inequality (5.78) one concludes that $\chi(\tau) \to -\infty$ for some $0 < \tau_{\not{\iota}} \leq 1/|\kappa|$. Finally, using Observation 6 and Observation 1 one concludes that $L(\tau) \to -\infty$ and $\phi(\tau) \to \infty$ as $\tau \to \tau_{\not{\iota}}$ for some $0 < \tau_{\not{\iota}} \leq 1/|\kappa|$. □

Notice that this upper bound for the location of the singularity is not trivial and improves the basic bound $\tau \leq 2/|\kappa|$ given by the location of the second component of the conformal boundary.

Remark 49. *A plot of the numerical evaluation of the solutions to the core system* (5.40a)-(5.40c) *with initial data* (5.41) *in the case* $\kappa < -1$ *can be seen in Figure 5.12.*

5.7.3 Exploiting the conformal gauge

In Lemma 19 it was shown that if $\partial_\psi \kappa = 0$ then the evolution equations imply, in particular, $f_x = 0$. Due to the spherical symmetry Ansatz, the component f_x is the only potentially non-zero component of $\boldsymbol{f}$. Thus, one concludes that $\boldsymbol{f} = 0$. In Section this section this feature will be exploited to extract further information about κ and s. These results are then used to discuss the conformal gauge freedom of the extended conformal field equations and the role played by reparametrisations of conformal geodesics.

The relation between the Weyl and Levi-Civita connections

As discussed in Section 2.2.1 of Chapter 2, the Weyl connection $\hat{\boldsymbol{\nabla}}$ expressing the extended conformal field equations is related to the Levi-Civita connection $\boldsymbol{\nabla}$ of the unphysical metric $\boldsymbol{g}$ via the 1-form $\boldsymbol{f}$. If $\boldsymbol{f}$ vanishes then $\hat{\boldsymbol{\nabla}} = \boldsymbol{\nabla}$. Exploiting this simple observation one obtain the following results:

Lemma 26. *If $\boldsymbol{f} = 0$ then the conformal gauge conditions* (2.34) *and* (2.35) *imply that $s = \ddot{\Theta}$. Moreover, s is constant along the conformal geodesics.*

Proof. As discussed in Section 2.2.1 of Chapter 2, if $\boldsymbol{f} = 0$ then $\hat{L}_{ab} = L_{ab}$ and $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$. Using the conformal gauge condition (2.34) it follows that $L_{\boldsymbol{0a}} = 0$ and $\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0$. Now, the standard vacuum conformal field equations (2.7c) and (2.7d) render

$$\nabla_{\boldsymbol{0}}\nabla_{\boldsymbol{0}}\Theta + \Theta L_{\boldsymbol{00}} - s\eta_{\boldsymbol{00}} = 0, \tag{5.79a}$$

$$\nabla_{\boldsymbol{0}}s = -L_{\boldsymbol{0b}}\nabla^{\boldsymbol{b}}\Theta. \tag{5.79b}$$

Using $L_{\boldsymbol{0a}} = 0$ and $\Gamma_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0$ in equation (5.79a) one concludes $\ddot{\Theta} = s$. Similarly, from equation (5.79b) one gets $\dot{s} = 0$. Therefore s is constant along the conformal geodesics. □

Remark 50. *In the asymptotic initial value problem the initial value of s is given by $s_\star = \sqrt{|\lambda|/3}\kappa$ —see equation* (2.78a)*. Thus, if $\boldsymbol{f} = 0$ then $s = \sqrt{|\lambda|/3}\kappa$ along the conformal geodesics.*

Finally, one has the following:

Lemma 27. *In the asymptotic initial value problem, if $\boldsymbol{f} = 0$, then the conformal gauge conditions* (2.34) *and* (2.35) *together with the conformal Einstein field equations imply that $\boldsymbol{e_i}(\kappa) = 0$ —that is, κ is a constant.*

Proof. Using $\boldsymbol{f} = 0$ and the gauge conditions (2.34) one gets from the conformal field equation (2.7g) that

$$6\Theta s - 3\dot{\Theta}^2 + 3\delta^{\boldsymbol{ij}}\boldsymbol{e_i}\Theta\boldsymbol{e_j}\Theta = \lambda. \tag{5.80}$$

Using Lemma 26 one has $s = \ddot{\Theta}$. Therefore, substituting $\Theta(\tau) = \dot{\Theta}_\star\tau(1+\kappa\tau/2)$ into equation (5.80) and recalling $\dot{\Theta}_\star = \sqrt{|\lambda|/3}$ one obtains

$$\tau^4\delta^{\boldsymbol{ij}}\boldsymbol{e_i}(\kappa)\boldsymbol{e_j}(\kappa) = 0.$$

Observe that the last equation is trivially satisfied on $\mathscr{I}$ as $\tau = 0$. Off the initial hypersurface, where $\tau \neq 0$, the last equation implies

$$\delta^{\boldsymbol{ij}}\boldsymbol{e_i}(\kappa)\boldsymbol{e_j}(\kappa) = 0.$$

Therefore, one concludes that $\boldsymbol{e_i}(\kappa) = 0$. □

Changing the conformal gauge

The analysis of the core system given in Sections 5.7.1, 5.7.2 and Section 5.4.2 covers the cases for which $|\kappa| > 1$ and $\kappa = 0$. As a consequence of the conformal covariance of the extended conformal Einstein field equations one has the freedom of performing conformal rescalings and of reparametrising the conformal geodesics — thus, effectively changing the representative of the conformal class $[\tilde{\boldsymbol{g}}]$ one is working with. This conformal freedom can be exploited to extend the analysis given in Sections 5.7.1 and 5.7.2 to the case where $\kappa \in [-1, 0) \cup (0, 1]$.

Following the discussion in the previous paragraph, any two spacetimes $(\mathcal{M}, \boldsymbol{g})$ and $(\bar{\mathcal{M}}, \bar{\boldsymbol{g}})$ with $\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$ and $\bar{\boldsymbol{g}} = \bar{\Theta}^2 \tilde{\boldsymbol{g}}$ representing two solutions to the extended conformal Einstein field equations for different choices of parameter κ are conformally related. From Lemmas 5 and 6 of Chapter 2 one has that

$$\Theta(\tau) = \sqrt{\frac{|\lambda|}{3}}\tau\Big(1 + \frac{1}{2}\kappa\tau\Big), \qquad \bar{\Theta}(\bar{\tau}) = \sqrt{\frac{|\lambda|}{3}}\bar{\tau}\Big(1 + \frac{1}{2}\bar{\kappa}\bar{\tau}\Big), \tag{5.81}$$

with

$$\bar{\tau} = \frac{a\tau}{c\tau + d}. \tag{5.82}$$

The free parameter b in the fractional transformation of Lemma 5 has been set to $b = 0$ in order to ensure that Θ and $\bar{\Theta}$ vanish at $\tau = 0$ and $\bar{\tau} = 0$, respectively. Thus, the conformal boundary $\mathscr{I}$ is equivalently represented by the hypersurfaces with $\tau = 0$ or $\bar{\tau} = 0$. As $\boldsymbol{g}$ and $\bar{\boldsymbol{g}}$ are conformally related one can write

$$\bar{\boldsymbol{g}} = \omega^2 \boldsymbol{g} \quad \text{with} \quad \omega \equiv \bar{\Theta}\Theta^{-1}.$$

Using equations (5.81) and (5.82) one obtains, after a calculation, that

$$\omega(\tau) = \frac{a\left(1 + \dfrac{a\bar{\kappa}\tau}{2(c\tau + d)}\right)}{\left((c\tau + d)\left(1 + \dfrac{1}{2}\kappa\tau\right)\right)}. \tag{5.83}$$

The conformal transformation law for the field s can be seen to be given by

$$\bar{s} = \omega^{-1}s + \omega^{-2}\nabla_{\boldsymbol{c}}\omega\nabla^{\boldsymbol{c}}\Theta + \frac{1}{2}\omega^{-3}\Theta\nabla_{\boldsymbol{c}}\omega\nabla^{\boldsymbol{c}}\omega.$$

As discussed in Section 5.7.3, in the analysis of the Schwarzschild-de Sitter spacetime one can assume that $\partial_\psi \kappa = 0$ and $\boldsymbol{f} = 0$. Now, Lemmas 26 and 27 imply that $s = \sqrt{|\lambda|/3}\kappa$ and $\bar{s} = \sqrt{|\lambda|/3}\bar{\kappa}$ are constant. Exploiting this observation, the transformation law for s can be read as an equation for ω —namely

$$\Theta\dot{\omega}^2 + 2\omega\dot{\Theta}\dot{\omega} + \omega^2 s - \omega^3\bar{s} = 0. \tag{5.84}$$

Substituting expression (5.83) into equation (5.84) one gets the condition

$$2c + a\bar{\kappa} - d\kappa = 0. \tag{5.85}$$

One can read equation (5.85) as the transformation law for $\bar{\kappa}$ so that

$$\bar{\kappa} = \frac{d\kappa - 2c}{a}.$$

In order to have a meaningful transformation law between $\bar{\tau}$ and τ, neither a nor d can vanish. Substituting equation (5.85) into the reparametrisation formula (5.82) and expression (5.83) one can observe that a/d actually corresponds to $\omega(0) \equiv \omega_\star$. Therefore, one has that

$$\bar{\tau}(\tau) = \frac{2\omega_\star \tau}{(\omega_\star\bar{\kappa} - \kappa)\tau - 2}, \qquad \omega(\tau) = \frac{4\omega_\star}{\big((\omega_\star\bar{\kappa} - \kappa)\tau - 2\big)^2}. \tag{5.86}$$

From the last expression one can identify $\dot{\omega}_\star \equiv \dot{\omega}(0) = \omega_\star(\omega_\star\bar{\kappa} - \kappa)$. In addition, notice that $\bar{\tau} \to \infty$ and $\omega \to \infty$ as $\tau \to 2/(\omega_\star\bar{\kappa} - \kappa)$. Therefore, the hypersurface defined by $\tau = 2/(\omega_\star\bar{\kappa} - \kappa)$ is at an infinite distance from the conformal boundary as measured with respect to the $\bar{\boldsymbol{g}}$-proper time.

Remark 51. An alternative approach to deduce equations (5.85) and (5.86) is to write $\bar{\Theta}(\bar{\tau}(\tau)) = \omega(\tau)\Theta(\tau)$ and use equations (5.81) and (5.82) to identify κ and ω.

5.8 Appendix: Cartan's structure equations and space spinor formalism

In this appendix a brief discussion of Cartan's structure equations and the space spinor formalism is given.

5.8.1 Cartan's structure equations in frame formalism

Consider a $\boldsymbol{h}$-orthonormal frame $\{\boldsymbol{e_i}\}$ with corresponding coframe $\{\boldsymbol{\omega^i}\}$. By construction, one has $\langle \boldsymbol{\omega^i}, \boldsymbol{e_j}\rangle = \delta_{\boldsymbol{i}}{}^{\boldsymbol{j}}$. The connection coefficients of the Levi-Civita connection $\boldsymbol{D}$ of $\boldsymbol{h}$ respect to this frame are defined as

$$\langle \boldsymbol{\omega^j}, D_{\boldsymbol{i}}\boldsymbol{e_k}\rangle \equiv \gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}.$$

As a consequence of the metricity of $\boldsymbol{D}$ it follows that $\gamma_{\boldsymbol{ijk}} = -\gamma_{\boldsymbol{ikj}}$. The connection form is accordingly defined as

$$\boldsymbol{\gamma^j}{}_{\boldsymbol{k}} \equiv \gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \wedge \boldsymbol{\omega^i}. \tag{5.87}$$

With these definitions, the first and second Cartan's structure equations are, respectively, given by

$$\mathbf{d}\boldsymbol{\omega}^{\boldsymbol{i}} = -\boldsymbol{\gamma}^{\boldsymbol{i}}{}_{\boldsymbol{j}} \wedge \boldsymbol{\omega}^{\boldsymbol{j}}, \tag{5.88a}$$

$$\mathbf{d}\gamma^{\boldsymbol{i}}{}_{\boldsymbol{j}} = -\boldsymbol{\gamma}^{\boldsymbol{i}}{}_{\boldsymbol{k}} \wedge \boldsymbol{\gamma}^{\boldsymbol{k}}{}_{\boldsymbol{j}} + \boldsymbol{\Omega}^{\boldsymbol{i}}{}_{\boldsymbol{j}}, \tag{5.88b}$$

where $\boldsymbol{\Omega}^{\boldsymbol{i}}{}_{\boldsymbol{j}}$ is the curvature 2-form defined as

$$\boldsymbol{\Omega}^{\boldsymbol{i}}{}_{\boldsymbol{j}} \equiv R^{\boldsymbol{i}}{}_{\boldsymbol{jkl}}\boldsymbol{\omega}^{\boldsymbol{k}} \wedge \boldsymbol{\omega}^{\boldsymbol{l}}. \tag{5.89}$$

5.8.2 Basic spinors

In the space spinor formalism, given a spin basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ where $\boldsymbol{A=0,1}$, any of the spinorial fields appearing in the extended conformal Einstein field equations can be decomposed in terms of basic irreducible spinors. The basic valence-2 symmetric spinors are:

$$x_{\boldsymbol{AB}} \equiv \sqrt{2}\epsilon_{(\boldsymbol{A}}{}^{\boldsymbol{0}}\epsilon_{\boldsymbol{B})}{}^{\boldsymbol{1}}, \qquad y_{\boldsymbol{AB}} \equiv -\frac{1}{\sqrt{2}}\epsilon_{(\boldsymbol{A}}{}^{\boldsymbol{1}}\epsilon_{\boldsymbol{B})}{}^{\boldsymbol{1}}, \qquad z_{\boldsymbol{AB}} \equiv \frac{1}{\sqrt{2}}\epsilon_{(\boldsymbol{A}}{}^{\boldsymbol{0}}\epsilon_{\boldsymbol{B})}{}^{\boldsymbol{0}}. \tag{5.90}$$

The basic valence 4 spinors are given by

$$\epsilon_{\boldsymbol{AC}}x_{\boldsymbol{BD}} + \epsilon_{\boldsymbol{BD}}x_{\boldsymbol{AC}}, \qquad \epsilon_{\boldsymbol{AC}}y_{\boldsymbol{BD}} + \epsilon_{\boldsymbol{BD}}y_{\boldsymbol{AC}}, \qquad \epsilon_{\boldsymbol{AC}}z_{\boldsymbol{BD}} + \epsilon_{\boldsymbol{BD}}z_{\boldsymbol{AC}}, \tag{5.91a}$$

$$h_{\boldsymbol{ABCD}} \equiv -\epsilon_{\boldsymbol{A}(\boldsymbol{C}}\epsilon_{\boldsymbol{D})\boldsymbol{B}}, \qquad \epsilon^{\boldsymbol{i}}{}_{\boldsymbol{ABCD}} = \epsilon_{(\boldsymbol{A}}{}^{(\boldsymbol{E}}\epsilon_{\boldsymbol{B}}{}^{\boldsymbol{F}}\epsilon_{\boldsymbol{C}}{}^{\boldsymbol{G}}\epsilon_{\boldsymbol{D})}{}^{\boldsymbol{H})_{\boldsymbol{i}}}. \tag{5.91b}$$

In the last expression $^{(\boldsymbol{ABCD})_{\boldsymbol{i}}}$ indicates that an $\boldsymbol{i}$ number of indices are set equal to $\mathbf{1}$ after symmetrisation. Any valence 4 spinor ζ_{ABCD} with the symmetries $\zeta_{(AB)(CD)}$ can be expanded in terms of these basic spinors. One has the identities

$$x_{\boldsymbol{AB}}x_{\boldsymbol{CD}} = 2\epsilon^{2}{}_{\boldsymbol{ABCD}}, \quad y_{\boldsymbol{AB}}y_{\boldsymbol{CD}} = \frac{1}{2}\epsilon^{4}{}_{\boldsymbol{ABCD}}, \quad z_{\boldsymbol{AB}}z_{\boldsymbol{CD}} = \frac{1}{2}\epsilon^{0}{}_{\boldsymbol{ABCD}}, \tag{5.92a}$$

$$x_{\boldsymbol{AB}}y_{\boldsymbol{CD}} = -\epsilon^{3}{}_{\boldsymbol{ABCD}} + \frac{1}{2\sqrt{2}}(\epsilon_{\boldsymbol{AB}}y_{\boldsymbol{BD}} + \epsilon_{\boldsymbol{BD}}y_{\boldsymbol{AC}}), \tag{5.92b}$$

$$x_{\boldsymbol{AB}}z_{\boldsymbol{CD}} = \epsilon^{1}{}_{\boldsymbol{ABCD}} - \frac{1}{2\sqrt{2}}(\epsilon_{\boldsymbol{AB}}z_{\boldsymbol{BD}} + \epsilon_{\boldsymbol{BD}}z_{\boldsymbol{AC}}), \tag{5.92c}$$

$$y_{\boldsymbol{AB}}z_{\boldsymbol{CD}} = -\frac{1}{2}\epsilon^{4}{}_{\boldsymbol{ABCD}} - \frac{1}{4\sqrt{2}}(\epsilon_{\boldsymbol{AB}}x_{\boldsymbol{BD}} + \epsilon_{\boldsymbol{BD}}x_{\boldsymbol{AC}}) - \frac{1}{6}h_{\boldsymbol{ABCD}}. \tag{5.92d}$$

Another set of identities used in the main text is given by

$$x_{\boldsymbol{AB}}x^{\boldsymbol{AB}} = 1, \quad x_{\boldsymbol{AB}}y^{\boldsymbol{AB}} = 0, \tag{5.93a}$$

$$x_{\boldsymbol{AB}}z^{\boldsymbol{AB}} = 0, \quad z_{\boldsymbol{AB}}z^{\boldsymbol{AB}} = 0, \quad y_{\boldsymbol{AB}}z^{\boldsymbol{AB}} = -\frac{1}{2}, \tag{5.93b}$$

$$x_{\boldsymbol{A}}{}^{\boldsymbol{Q}}x_{\boldsymbol{BQ}} = \frac{1}{2}\epsilon_{\boldsymbol{AB}}, \quad y_{\boldsymbol{A}}{}^{\boldsymbol{Q}}x_{\boldsymbol{BQ}} = \frac{1}{\sqrt{2}}y_{\boldsymbol{AB}}, \tag{5.93c}$$

$$z_{\boldsymbol{A}}{}^{\boldsymbol{Q}} x_{\boldsymbol{BQ}} = -\frac{1}{\sqrt{2}} z_{\boldsymbol{AB}}, \quad y_{\boldsymbol{A}}{}^{\boldsymbol{Q}} y_{\boldsymbol{BQ}} = 0, \tag{5.93d}$$

$$y_{\boldsymbol{A}}{}^{\boldsymbol{Q}} z_{\boldsymbol{BQ}} = -\frac{1}{2\sqrt{2}} x_{\boldsymbol{AB}} + \frac{1}{4}\epsilon_{\boldsymbol{AB}}, \quad z_{\boldsymbol{A}}{}^{\boldsymbol{Q}} z_{\boldsymbol{BQ}} = 0, \tag{5.93e}$$

$$\epsilon^2_{\boldsymbol{ABCD}} x^{\boldsymbol{CD}} = -\frac{1}{3} x_{\boldsymbol{AB}}, \quad \epsilon^2_{\boldsymbol{ABCD}} y^{\boldsymbol{CD}} = \frac{1}{6} y_{\boldsymbol{AB}}, \tag{5.93f}$$

$$\epsilon^2_{\boldsymbol{ABCD}} z^{\boldsymbol{CD}} = \frac{1}{6} z_{\boldsymbol{AB}}. \tag{5.93g}$$

These identities and a more exhaustive list has been given in [81].

5.8.3 Cartan's structure equations in spinor form

The space spinor counterpart of coframe and connection coefficients can be obtained succinctly by contraction with the spatial Infeld-van der Waerden symbols —see equation (2.52) of Chapter 2, as $\boldsymbol{\omega}^{\boldsymbol{AB}} \equiv \boldsymbol{\omega}^{\boldsymbol{i}} \sigma_{\boldsymbol{i}}{}^{\boldsymbol{AB}}$ and $\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}} = \gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}} \sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}} \sigma_{\boldsymbol{j}}{}^{\boldsymbol{CD}} \sigma^{\boldsymbol{k}}{}_{\boldsymbol{EF}}$. With these definitions the spinorial version of the Cartan structure equations is given by

$$\mathbf{d}\boldsymbol{\omega}^{\boldsymbol{AB}} = -\boldsymbol{\gamma}^{\boldsymbol{A}}{}_{\boldsymbol{B}} \wedge \boldsymbol{\omega}^{\boldsymbol{BE}} - \gamma^{\boldsymbol{B}}{}_{\boldsymbol{E}} \wedge \boldsymbol{\omega}^{\boldsymbol{AE}}, \tag{5.94a}$$

$$\mathbf{d}\boldsymbol{\gamma}^{\boldsymbol{A}}{}_{\boldsymbol{B}} = -\boldsymbol{\gamma}^{\boldsymbol{A}}{}_{\boldsymbol{E}} \wedge \boldsymbol{\gamma}^{\boldsymbol{E}}{}_{\boldsymbol{B}} + \Omega^{\boldsymbol{A}}{}_{\boldsymbol{B}}, \tag{5.94b}$$

where

$$\boldsymbol{\gamma}^{\boldsymbol{A}}{}_{\boldsymbol{B}} \equiv \frac{1}{2}\gamma_{\boldsymbol{CD}}{}^{\boldsymbol{AQ}}{}_{\boldsymbol{BQ}} \boldsymbol{\omega}^{\boldsymbol{CD}},$$

and $\Omega^{\boldsymbol{A}}{}_{\boldsymbol{B}}$ is the spinor version of the curvature 2-form, with

$$\Omega^{\boldsymbol{A}}{}_{\boldsymbol{B}} \equiv \frac{1}{2} r^{\boldsymbol{A}}{}_{\boldsymbol{BCDEF}} \omega^{\boldsymbol{CD}} \wedge \boldsymbol{\omega}^{\boldsymbol{EF}}.$$

In the last expression the spinor $r_{\boldsymbol{ABCDEF}}$ can be decomposed as

$$r_{\boldsymbol{ABCDEF}} = \left(\frac{1}{2} s_{\boldsymbol{ABCD}} - \frac{1}{12} r h_{\boldsymbol{ABCE}}\right)\epsilon_{\boldsymbol{DF}} + \left(\frac{1}{2} s_{\boldsymbol{ABDF}} - \frac{1}{12} r h_{\boldsymbol{ABDF}}\right)\epsilon_{\boldsymbol{CE}}$$

where $s_{\boldsymbol{ABCD}}$ and r correspond to the space spinor version of the trace-free part of the Ricci tensor and Ricci scalar of $\boldsymbol{h}$, respectively.

To relate the previous discussion with the basic spinors $x_{\boldsymbol{AB}}$, $y_{\boldsymbol{AB}}$ and $z_{\boldsymbol{AB}}$, observe that using equation (5.90) and the expression for the spatial Infeld-van der Waerden symbols as given in equation (2.52) one obtains that

$$\sigma_{\boldsymbol{AB}}{}^{\boldsymbol{1}} = -z_{\boldsymbol{AB}} - y_{\boldsymbol{AB}}, \quad \sigma_{\boldsymbol{AB}}{}^{\boldsymbol{2}} = \mathrm{i}(z_{\boldsymbol{AB}} - y_{\boldsymbol{AB}}), \quad \sigma_{\boldsymbol{AB}}{}^{\boldsymbol{3}} = x_{\boldsymbol{AB}}, \tag{5.95a}$$

$$\sigma^{\boldsymbol{AB}}{}_{\boldsymbol{1}} = z^{\boldsymbol{AB}} + y^{\boldsymbol{AB}}, \quad \sigma^{\boldsymbol{AB}}{}_{\boldsymbol{2}} = \mathrm{i}(-z_{\boldsymbol{AB}} + y_{\boldsymbol{AB}}), \quad \sigma^{\boldsymbol{AB}}{}_{\boldsymbol{3}} = -x^{\boldsymbol{AB}}. \tag{5.95b}$$

6 Asymptotic initial data for the Kerr-de Sitter spacetime

6.1 Introduction

In Chapter 5 the asymptotic initial value problem for the Schwarzschild-de Sitter spacetime was studied. One of the main features that was exploited in this analysis was the fact that, in the appropriate conformal representation the induced metric at the conformal boundary is conformally flat. Despite the fact that in the representation in which the initial 3-metric $\boldsymbol{h}$ is the standard metric on $\mathbb{S}^3$ the initial data for the rescaled Weyl spinor is singular at the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ it was shown that there exist a conformal representation in which the initial data for the rescaled Weyl spinor is regular. In view of these remarks, it is natural to explore if one can perform a similar analysis for the Kerr-de Sitter spacetime. In this short chapter this question in explored. Therefore, this discussion represents a partial generalisation of the analysis given in Chapter 5. In this chapter, in particular, asymptotic initial data for the Kerr-de Sitter spacetime in a conformal representation for which this initial data is regular is given. Then, using the theory on symmetric hyperbolic systems of [23] one obtains a local existence result of small perturbations of asymptotic initial data close to the Kerr-de Sitter spacetime. Nevertheless, in contrast with the analysis given in Chapter 5 an estimation of the time of existence is not given.

6.2 The Kerr-de Sitter spacetime and its conformal structure

In this section, the general properties of the Kerr-de Sitter spacetime that will be relevant for the main analysis of this chapter are discussed.

6.2.1 The Kerr-de Sitter spacetime

The *Kerr-de Sitter spacetime* is an axisymmetric solution to the vacuum Einstein field equations —as given in equation (5.1)— with a de-Sitter like Cosmological constant. The metric of the Kerr-de Sitter spacetime in *Boyer-Lindquist-type coor-*

dinates (t, r, θ, φ) is described by the line element

$$\tilde{\boldsymbol{g}}_{KdS} = \frac{\Delta_r}{\Delta_\lambda^2 \varrho^2}\big(\mathbf{d}t - a\sin^2\theta\mathbf{d}\varphi\big) \otimes \big(\mathbf{d}t - a\sin^2\theta\mathbf{d}\varphi\big) - \frac{\varrho^2}{\Delta_r}\mathbf{d}r \otimes \mathbf{d}r - \frac{\varrho^2}{\Delta_\theta}\mathbf{d}\theta \otimes \mathbf{d}\theta - \frac{\Delta_\theta \sin^2\theta}{\Delta_\lambda^2 \varrho^2}\Big(a\mathbf{d}t - \big(r^2 + a^2\big)\mathbf{d}\varphi\Big) \otimes \Big(a\mathbf{d}t - \big(r^2 + a^2\big)\mathbf{d}\varphi\Big), \quad (6.1)$$

where,

$$\varrho^2 \equiv r^2 + a^2\cos^2\theta, \quad \Delta_r \equiv (r^2 + a^2)\Big(1 - \frac{1}{3}|\lambda| r^2\Big) - 2mr,$$
$$\Delta_\theta \equiv 1 + \frac{1}{3}|\lambda| a^2\cos^2\theta, \quad \Delta_\lambda \equiv 1 + \frac{1}{3}|\lambda| a^2,$$

with $t \in (-\infty, \infty)$, $r \in (0, \infty)$, $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$ —see [20, 82]. As in the conventions used in this thesis $\lambda < 0$ for a de-Sitter value of the Cosmological constant the above expressions are written in terms of $|\lambda|$ to avoid confusion. This solution reduces to the Schwarzschild-de Sitter spacetime when $a = 0$ and to the de Sitter spacetime when $m = 0$ —see [82]. The location of the black hole and cosmological horizons is determined by the condition

$$\Delta_r = (r - r_+)(r - r_-)(r - r_c)(r - r_f) = 0, \quad (6.2)$$

where $r_\pm$ corresponds to the Kerr black hole horizons while r_c correspond to a cosmological horizon. Additionally $r_f < 0$ represents an additional cosmological horizon. Notice, nevertheless, that the curvature singularity is located at $r = 0$ —see [82]. The principal null directions $\tilde{\boldsymbol{\ell}}$ and $\tilde{\boldsymbol{n}}$ of the Kerr-de Sitter spacetimes have been determined in [82]. In accordance with the signature conventions used in this thesis, these vectors are given in Boyer-Lindquist coordinates by

$$\tilde{\boldsymbol{\ell}} = \frac{1}{\sqrt{2}}\frac{\Delta_\lambda(a^2 + r^2)}{\sqrt{|\Delta_r|}\,\varrho}\boldsymbol{\partial}_t + \frac{1}{\sqrt{2}}\frac{\sqrt{|\Delta_r|}}{\varrho}\boldsymbol{\partial}_r + \frac{1}{\sqrt{2}}\frac{a\Delta_\lambda}{\sqrt{|\Delta_r|}\,\varrho}\boldsymbol{\partial}_\varphi, \quad (6.3a)$$

$$\tilde{\boldsymbol{n}} = \frac{1}{\sqrt{2}}\frac{\Delta_\lambda(a^2 + r^2)}{\sqrt{|\Delta_r|}\,\varrho}\boldsymbol{\partial}_t - \frac{1}{\sqrt{2}}\frac{\sqrt{|\Delta_r|}}{\varrho}\boldsymbol{\partial}_r + \frac{1}{\sqrt{2}}\frac{a\Delta_\lambda}{\sqrt{|\Delta_r|}\varrho}\boldsymbol{\partial}_\varphi. \quad (6.3b)$$

A direct computation using the metric (6.1) shows that one can complement $\{\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{n}}\}$, as given above, with the following pair of complex null vectors

$$\tilde{\boldsymbol{m}} = \frac{1}{\sqrt{2}}\frac{\Delta_\lambda a\sin\theta}{\sqrt{\Delta_\theta}\,\varrho}\boldsymbol{\partial}_t + \frac{\mathrm{i}}{\sqrt{2}}\frac{\sqrt{\Delta_\theta}}{\varrho}\boldsymbol{\partial}_\theta + \frac{1}{\sqrt{2}}\frac{\Delta_\lambda}{\sqrt{\Delta_\theta}\varrho\sin\theta}\boldsymbol{\partial}_\varphi, \quad (6.3c)$$

$$\bar{\tilde{\boldsymbol{m}}} = \frac{1}{\sqrt{2}}\frac{\Delta_\lambda a\sin\theta}{\sqrt{\Delta_\theta}\,\varrho}\boldsymbol{\partial}_t - \frac{\mathrm{i}}{\sqrt{2}}\frac{\sqrt{\Delta_\theta}}{\varrho}\boldsymbol{\partial}_\theta + \frac{1}{\sqrt{2}}\frac{\Delta_\lambda}{\sqrt{\Delta_\theta}\varrho\sin\theta}\boldsymbol{\partial}_\varphi. \quad (6.3d)$$

It can be verified that $\{\tilde{\boldsymbol{\ell}}, \tilde{\boldsymbol{n}}, \tilde{\boldsymbol{m}}, \bar{\tilde{\boldsymbol{m}}}\}$ satisfies the pairings

$$\tilde{m}^a \bar{\tilde{m}}_a = \bar{\tilde{m}}^a \tilde{m}_a = -1, \tag{6.4}$$

$$\tilde{l}^a \tilde{n}_a = \tilde{n}_a \tilde{l}^a = \begin{cases} 1 & \Delta_r > 0, \\ -1 & \Delta_r < 0, \end{cases} \tag{6.5}$$

while all the other contractions vanish. Consequently, $\{\tilde{\boldsymbol{\ell}}, \tilde{\boldsymbol{n}}, \tilde{\boldsymbol{m}}, \bar{\tilde{\boldsymbol{m}}}\}$ constitutes a null tetrad adapted to the principal null directions of the Kerr-de Sitter spacetime. With this information at hand one can compute the Weyl curvature components using the NP formalism. A straight forward computation using that

$$\Psi_2 = \frac{1}{2} C_{abcd}\big(\tilde{n}^a \tilde{l}^b \tilde{n}^c \tilde{l}^d - \tilde{n}^a \tilde{l}^b \tilde{m}^c \bar{\tilde{m}}^d\big), \tag{6.6}$$

renders

$$\Psi_2 = \frac{m}{(r - \mathrm{i} a \cos\theta)^3}, \tag{6.7}$$

while $\Psi_1 = \Psi_3 = \Psi_4 = 0$. This is consistent with the fact that the Kerr-de Sitter spacetime is of Petrov type D. Moreover, Ψ_2 as given in (6.7) coincides with the corresponding expression for the Kerr spacetime. Observe that the information about the Cosmological constant λ is contained in the null tetrad as determined by expressions (6.3a)-(6.3d).

6.3 The rescaled Weyl spinor for Petrov type D spacetimes

Given a conformal rescaling

$$\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}} \tag{6.8}$$

and a spin dyad $\{\tilde{\epsilon}_{\boldsymbol{A}}{}^A\}$ associated to the *physical null tetrad* $\{\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{n}}, \tilde{\boldsymbol{m}}, \bar{\tilde{\boldsymbol{m}}}\}$ one can define an *unphysical spin dyad* $\{\epsilon_{\boldsymbol{A}}{}^A\}$ via

$$\epsilon_{\boldsymbol{A}}{}^A = \Xi^{-1/2} \tilde{\epsilon}_{\boldsymbol{A}}{}^A. \tag{6.9}$$

Observe that this choice is not unique and equation (6.9) gives equal conformal weight to the elements of the spin basis. The corresponding *unphysical null tetrad* $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ is related to the physical null tetrad via

$$l_a = \Xi \tilde{l}_a, \quad n_a = \Xi \tilde{n}_a, \quad m_a = \Xi \tilde{m}_a, \quad \bar{m}_a = \Xi \bar{\tilde{m}}_a. \tag{6.10}$$

As in the case of the spin dyad, this choice is not unique but gives equal conformal weight to both $\boldsymbol{l}$ and $\boldsymbol{n}$ —consistent with equation (6.9). If the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$

is of Petrov type D and the associated spin basis $\{\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}\}$ correspond to the principal null directions then the Weyl spinor can be expressed as

$$\Psi_{ABCD} = \Psi_2 \tilde{o}_{(A}\tilde{o}_{B}\tilde{\iota}_{C}\tilde{\iota}_{D)}. \tag{6.11}$$

The Weyl spinor corresponds to the spinor counterpart of the anti-self dual Weyl tensor $\mathcal{C}_{abcd}$

$$\mathcal{C}_{abcd} \equiv C_{abcd} + \mathrm{i}C^{*}_{abcd} \tag{6.12}$$

where, C_{abcd} is the Weyl tensor and $C^{*}_{abcd} = \frac{1}{2}\epsilon_{ab}{}^{ef}C_{efcd}$ is the left-dual Weyl tensor. Consequently, the tensor equivalent of equation (6.11) is given by

$$\mathcal{C}_{abcd} = \Psi_2\Big(\tilde{V}_{ab}\tilde{U}_{ab} + \tilde{U}_{ab}\tilde{V}_{cd} + \tilde{W}_{ab}\tilde{W}_{cd}\Big), \tag{6.13}$$

where $\mathcal{C}_{abcd}$ is the anti-self dual Weyl tensor while $\tilde{V}_{ab}$, $\tilde{U}_{ab}$ and $\tilde{W}_{ab}$ are a basis of self-dual bivectors related to the null tetrad —see [83] for a discussion on the decomposition of the Weyl tensor in terms of self-dual 2-forms:

$$\tilde{U}_{ab} = -\tilde{l}_a\bar{\tilde{m}}_b + \tilde{l}_b\bar{\tilde{m}}_a, \tag{6.14a}$$
$$\tilde{V}_{ab} = \tilde{n}_a\tilde{m}_b - \tilde{n}_b\tilde{m}_a, \tag{6.14b}$$
$$\tilde{W}_{ab} = \tilde{m}_a\bar{\tilde{m}}_b - \tilde{m}_b\bar{\tilde{m}}_a - \tilde{n}_a\tilde{l}_b + \tilde{n}_b\tilde{l}_a. \tag{6.14c}$$

As consequence of equation (6.12) one can obtain an expression for the Weyl tensor C_{abcd} replacing Ψ_2 with $\mathrm{Re}(\Psi_2)$ in equation (6.13). Since the rescaled Weyl spinor associated to the conformal representation (6.8) is defined via

$$\phi_{ABCD} = \Xi^{-1}\Psi_{ABCD},$$

using equation (6.9) one concludes that the components of the rescaled Weyl spinor respect to the unphysical spin basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ are related to the components of the Weyl spinor in the physical spin basis $\{\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}\}$ through

$$\phi_{\boldsymbol{ABCD}} = \Xi^{-3}\Psi_{\boldsymbol{ABCD}}. \tag{6.15}$$

Consequently, the only non-zero component of the rescaled Weyl spinor is given by

$$\phi_2 = \Xi^{-3}\Psi_2. \tag{6.16}$$

Next, define an Hermitian $\tau_{AA'}$ spinor via

$$\tau^{AA'} \equiv \epsilon_{\boldsymbol{0}}{}^{A}\epsilon_{\boldsymbol{0}'}{}^{A'} + \epsilon_{\boldsymbol{1}}{}^{A}\epsilon_{\boldsymbol{1}'}{}^{A'}. \tag{6.17}$$

The spinor $\tau^{AA'}$ with normalisation $\tau^{AA'}\tau_{AA'} = 2$ corresponds to the spinor counterpart of a timelike vector $\sqrt{2}\tau^a$ and can be used to perform a space spinor split as discussed in Sections 2.4.1 and 2.4.2 of Chapter 2 to decompose the rescaled Weyl spinor in electric and magnetic parts. A straightforward computation shows then that the electric and magnetic parts of the rescaled Weyl spinor respect to $\tau^{AA'}$ as determined by equation (6.17) read

$$\eta_{ABCD} = 3(\phi_2 + \bar{\phi}_2)o_{(A}o_B\iota_C\iota_{D)}, \qquad \mu_{ABCD} = -3\mathrm{i}(\phi_2 - \bar{\phi}_2)o_{(A}o_B\iota_C\iota_{D)}. \tag{6.18}$$

In tensorial notation using equations (6.13) and (6.10) one arrives at

$$d_{abcd} = \Xi^{-3}\mathrm{Re}(\Psi_2)\Big(V_{ab}U_{ab} + U_{ab}V_{cd} + W_{ab}W_{cd}\Big),$$

where

$$V_{ab} = \Xi^2\tilde{V}_{ab}, \quad U_{ab} = \Xi^2\tilde{U}_{ab}, \quad W_{ab} = \Xi^2\tilde{W}_{ab}.$$

For the subsequent discussion it will be useful at this point to define timelike and spacelike covectors associated to the principal null tetrad via

$$\tilde{s}_a \equiv \frac{1}{\sqrt{2}}(\tilde{l}_a + \tilde{n}_a), \qquad \tilde{\tau}_a \equiv \frac{1}{\sqrt{2}}(\tilde{l}_a - \tilde{n}_a).$$

Observe that

$$\tilde{g}^{ab}\tilde{s}_a\tilde{s}_b = -\tilde{g}^{ab}\tilde{\tau}_a\tilde{\tau}_b = \begin{cases} 1 & \tilde{l}^a\tilde{n}_a = 1, \\ -1 & \tilde{l}^a\tilde{n}_a = -1. \end{cases}$$

while all the other contractions vanish. Similarly, one defines the corresponding unphysical counterparts through

$$s_a = \Xi\tilde{s}_a, \quad \tau_a = \Xi\tilde{\tau}_a, \tag{6.19}$$

satisfying the pairings

$$g^{ab}s_as_b = -g^{ab}\tau_a\tau_b = \begin{cases} 1 & l^an_a = 1, \\ -1 & l^an_a = -1, \end{cases} \tag{6.20}$$

while all the other contractions vanish. For conciseness, consider the case $l^an_a = -1$ in which τ_a is timelike. Respect to this vector one can define the projector tensor as

$$h_{ab} \equiv g_{ab} - \tau_a\tau_b. \tag{6.21}$$

The electric and magnetic parts E_{ab} and B_{ab} of the rescaled Weyl spinor d_{abcd} respect to τ_a are defined via

$$E_{ab} \equiv \tau^q\tau^d h_a{}^p h_b{}^c d_{pqcd}, \qquad B^*_{ab} \equiv \tau^b\tau^d h_p{}^a h_q{}^c d^*_{abcd} \tag{6.22}$$

A direct computation using equations (6.10), (6.14a)-(6.14c), (6.20) and (6.21) shows that for a Petrov type D spacetime one can rewrite the electric part of the rescaled Weyl tensor as

$$E_{ab} = \frac{1}{2}\mathrm{Re}(\phi_2)(3s_a s_b + h_{ab}). \tag{6.23}$$

A computation along the above lines leads to an analogous expression for the magnetic part of the rescaled Weyl tensor. Nevertheless, this calculation is not pursued further as such expressions will not be required in the subsequent analysis.

6.4 The $\mathbb{R}\times\mathbb{S}^2$-conformal representation

As discussed in Chapter 5, to derive the asymptotic initial data for the conformal Einstein field equations it is necessary to identify a representative of the conformal class $[\boldsymbol{h}]$ and a trace-free tensor d_{ij} satisfying $D^i d_{ij} = 0$ where $\boldsymbol{D}$ denotes the covariant derivative of $\boldsymbol{h}$. The tensor d_{ij} encodes the initial data electric part of the rescaled Weyl tensor d_{abcd}. In the discussion of the asymptotic initial value problem for the Schwarzschild-de Sitter spacetime given in Chapter 5, it was shown that the conformal representation for which the induced metric at the conformal boundary is the standard metric for $\mathbb{R}\times\mathbb{S}^2$ leads to a regular representation of the asymptotic initial data for the Schwarzschild-de Sitter spacetime. As discussed in Chapter 5 this is in stark contrast with the representation of asymptotic initial data for the Schwarzschild-de Sitter spacetime for which the induced metric at the conformal boundary $\boldsymbol{h}$ corresponds to the standard metric on $\mathbb{S}^3$ —in this representation the initial data for the electric part of the rescaled Weyl tensor diverges at the North and South poles of $\mathbb{S}^3$. In this section a similar approach is followed to obtain a regular representation of the asymptotic initial data for the Kerr-de Sitter spacetime.

6.4.1 Asymptotic initial data-tensorial description

In analogy to the case of the Schwarzschild-de Sitter spacetime one considers the conformal rescaling

$$\widehat{\boldsymbol{g}} = \widehat{\Xi}^2 \tilde{\boldsymbol{g}}_{KdS}$$

where $\widehat{\Xi} = 1/r$ and uses the conformal factor as a coordinate. In other words, one introduces $\xi \equiv 1/r$ as a new coordinate. Following the discussion of Section 6.3, in this representation the only non-zero component of the rescaled Weyl spinor —see equation (6.16), reads

$$\phi_2 = \frac{m}{(1 - ia\xi\cos\theta)^3}.$$

Observe that, in this representation the asymptotic initial data for the non-zero component of the rescaled Weyl spinor does not contain information about the angular momentum a nor the Cosmological constant λ as

$$\phi_2|_{\mathscr{I}} = m.$$

Using equation (6.18) one concludes that the only non-zero component of the electric part of the rescaled Weyl spinor is then given by

$$\eta_2|_{\mathscr{I}} = 6m, \tag{6.24}$$

while the the magnetic part of the rescaled Weyl spinor vanishes at $\mathscr{I}$. Despite the fact that the use of spinors leads directly to the above conclusions it is instructive to recover this discussion from the tensorial expression (6.13). Additionally, analysing the behaviour of the null frame at the conformal boundary is better carried out in tensor frame notation. To do so, observe that the physical timelike and spacelike covectors $\tilde{s}_a$ and $\tilde{\tau}_a$ are given in Boyer-Lindquist coordinates by

$$\tilde{\tau}_a = -\frac{\mathrm{sgn}(\Delta_r)\varrho}{\sqrt{|\Delta_r|}}\mathrm{d}r, \qquad \tilde{s}_a = \frac{\mathrm{sgn}(\Delta_r)\sqrt{|\Delta_r|}}{\Delta_\lambda\varrho}\big(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi\big).$$

In the coordinate system (t,ξ,θ,φ) their unphysical counterparts $s_a = \widehat{\Xi}\tilde{s}_a$ and $\tau_a = \widehat{\Xi}\tilde{\tau}_a$ read

$$\tau_a = \frac{\mathrm{sgn}(\Delta_r)\sqrt{a^2\xi^2\cos^2\theta+1}}{\sqrt{|(a^2\xi^2+1)(\xi^2-\lambda/3)-2m\xi^3|}}\mathrm{d}\xi,$$
$$s_a = \frac{\mathrm{sgn}(\Delta_r)\sqrt{|(a^2\xi^2+1)(\xi^2-\lambda/3)-2m\xi^3|}}{\Delta_\lambda\sqrt{a^2\xi^2\cos^2\theta+1}}\big(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi\big).$$

Since for $r>r_c$ one has that $\Delta_r<0$ it follows that τ_a is timelike and s_a is spacelike in the asymptotic region. Furthermore, at the conformal boundary $\mathscr{I}$ one has

$$\tau_a|_{\mathscr{I}} = -\sqrt{\frac{3}{|\lambda|}}\mathrm{d}\xi, \qquad s_a|_{\mathscr{I}} = -\frac{1}{\Delta_\lambda}\sqrt{\frac{|\lambda|}{3}}\big(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi\big). \tag{6.25}$$

Notice that τ_a is parallel to $\mathbf{d}\widehat{\Xi}$. In other words, τ_a is orthogonal to $\mathscr{I}$. The induced metric at the conformal boundary can be found writing

$$\widehat{\boldsymbol{g}} = \widehat{\Xi}\tilde{\boldsymbol{g}}_{KdS},$$

in the coordinates (t,ξ,θ,φ) and considering the limit $\xi\to 0$. A direct computation yields

$$\boldsymbol{h} = -\frac{|\lambda|}{3\Delta_\lambda^2}(\mathbf{d}t - a\sin^2\theta\mathbf{d}\varphi)\otimes(\mathbf{d}t - a\sin^2\theta\mathbf{d}\varphi) - \frac{1}{\Delta_\theta}\mathbf{d}\theta\otimes\mathbf{d}\theta - \frac{\Delta_\theta\sin^2\theta}{\Delta_\lambda^2}\mathbf{d}\varphi\otimes\mathbf{d}\varphi. \tag{6.26}$$

The last computation has also been reported in [84] and [24]. A direct computation shows that $\boldsymbol{h}$ is conformally flat as the Cotton tensor of $\boldsymbol{h}$ vanishes. From the metric (6.26) one readily identifies an orthonormal basis for $\boldsymbol{h}$;

$$\boldsymbol{h} = -\boldsymbol{\omega}^3\otimes\boldsymbol{\omega}^3 - \boldsymbol{\omega}^1\otimes\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2\otimes\boldsymbol{\omega}^2,$$

where

$$\boldsymbol{\omega^3} = \frac{1}{\Delta_\lambda}\sqrt{\frac{|\lambda|}{3}}\big(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi\big), \quad \boldsymbol{\omega^1} = \frac{1}{\sqrt{\Delta_\theta}}\mathrm{d}\theta, \quad \boldsymbol{\omega^2} = \frac{\sqrt{\Delta_\theta}\sin\theta}{\Delta_\lambda}\mathrm{d}\varphi.$$

Observe that the covector $\boldsymbol{\omega^3}_a$ corresponds to $s_a|_{\mathscr{I}}$ as given in equation (6.25). With this information, observe that the initial data for the electric part of the rescaled Weyl tensor $E_{ab}|_{\mathscr{I}}$ can be directly read off from equations (6.23), (6.26) and (6.25). On the other hand, as discussed in Chapter 5 the initial data for the magnetic rescaled Weyl tensor can be read off from the Cotton tensor of the induced metric at the conformal boundary $\boldsymbol{h}$. As $\boldsymbol{h}$ is conformally flat it follows that the initial data for *the magnetic part of the rescaled Weyl tensor* vanishes. This is in agreement with the observation, as discussed above, that for a Petrov type D spacetime the initial data for magnetic part of the rescaled Weyl spinor corresponds to the imaginary part of $\phi_2|_{\mathscr{I}}$ —see equation (6.18)— which vanishes in this case. To complete the asymptotic initial data observe that the frame dual to $\{\boldsymbol{\omega^a}\}$ determined by $\langle\boldsymbol{\omega^a},\boldsymbol{e_b}\rangle = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}$ is given by

$$\boldsymbol{e_3} = \Delta_\lambda\sqrt{\frac{3}{|\lambda|}}\boldsymbol{\partial}_t, \qquad \boldsymbol{e_1} = \sqrt{\Delta_\theta}\boldsymbol{\partial}_\theta, \qquad \boldsymbol{e_2} = \frac{\Delta_\lambda}{\sqrt{\Delta_\theta}}(\csc\theta\boldsymbol{\partial}_\varphi + a\sin\theta\boldsymbol{\partial}_t).$$

The components of the 3-dimensional Schouten tensor $l_{ij}\equiv r_{ij} - \frac{1}{4}rh_{ij}$ respect to this frame read

$$l_{\boldsymbol{33}} = \frac{1}{2}(\Delta_\lambda - \Delta_\theta - 1), \qquad l_{\boldsymbol{11}} = \frac{1}{2}(3\Delta_\theta - \Delta_\lambda - 1),$$
$$l_{\boldsymbol{22}} = \frac{1}{2}(3\Delta_\theta - \Delta_\lambda - 1), \qquad l_{\boldsymbol{32}} = -a\sqrt{\frac{\lambda}{3}}\sqrt{\Delta_\theta}\sin\theta,$$

while all the other components vanish. The only non-zero connection coefficients respect to the frame this frame are

$$\gamma_{212} = \frac{\cot\theta}{\sqrt{\Delta_\theta}}(2\Delta_\theta - \Delta_\lambda), \qquad \gamma_{123} = -\gamma_{213} = -\gamma_{312} = a\sqrt{\frac{|\lambda|}{3}}\cos\theta,$$

Finally, for completeness, one can compute the Friedrich scalar in this conformal representation to obtain

$$\widehat{s} = \frac{1}{4}\widehat{\nabla}_{\boldsymbol{c}}\widehat{\nabla}^{\boldsymbol{c}}\widehat{\Xi} + \frac{1}{24}\tilde{R}\widehat{\Xi}. \tag{6.27}$$

Recalling the conformal transformation law for the Ricci scalar

$$\widehat{R}-\frac{1}{\widehat{\Xi}^2}\tilde{R}=-\frac{6}{\widehat{\Xi}}\widehat{\nabla}_c\widehat{\nabla}^c\widehat{\Xi}+\frac{12}{\widehat{\Xi}^2}\widehat{\nabla}_c\widehat{\Xi}\widehat{\nabla}^c\widehat{\Xi}, \tag{6.28}$$

one observes that that the Friedrich scalar can be rewritten as

$$\widehat{s}=\frac{1}{24\widehat{\Xi}}\tilde{R}+\frac{1}{2\widehat{\Xi}}\widehat{\nabla}_c\widehat{\Xi}\widehat{\nabla}^c\widehat{\Xi}. \tag{6.29}$$

Alternatively one can write the last expression as

$$\widehat{s}=\frac{1}{24\widehat{\Xi}}\tilde{R}+\frac{1}{2\widehat{\Xi}^3}\tilde{g}^{ac}\tilde{\nabla}_a\widehat{\Xi}\tilde{\nabla}_c\widehat{\Xi}. \tag{6.30}$$

A direct computation using the above expression renders

$$\widehat{s}=\frac{\xi(3-a^2\lambda+a^2\lambda\cos\theta^2-6m\xi+3a^2\xi^2)}{6(1+a^2\cos\theta^2\xi^2)}. \tag{6.31}$$

Thus one concludes that, in this conformal representation, the Friedrich scalar $\widehat{s}$ vanishes at the conformal boundary. Observe that as a consequence of the conformal constraint equations, given in Section 2.5.2 of Chapter 2, one has then $\chi_{ij}=0$ at $\mathscr{I}$.

6.4.2 Asymptotic initial data; spinorial description

The initial data for the electric and magnetic part of the rescaled Weyl is

$$\eta_{\boldsymbol{ABCD}}=6mo_{(\boldsymbol{A}}o_{\boldsymbol{B}}\iota_{\boldsymbol{C}}\iota_{\boldsymbol{D})}, \qquad \mu_{\boldsymbol{ABCD}}=0.$$

As discussed before, the only non-zero contribution to the initial data for the rescaled Weyl spinor comes from its electric part. In terms of the valence-4 basic spinors of the space spinor formalism —see Appendix 5.8.2 of Chapter 5, the initial data for the rescaled Weyl spinor then reads

$$\phi_{\boldsymbol{ABCD}}=6m\epsilon^2{}_{\boldsymbol{ABCD}}.$$

The spinorial counterpart of the other objects described in the previous section can be found by suitable contraction with the Infeld-van der Waerden symbols using that

$$T_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}}=\gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}\sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}}\sigma_{\boldsymbol{j}}{}^{\boldsymbol{CD}}\sigma^{\boldsymbol{k}}{}_{\boldsymbol{EF}}, \tag{6.32}$$

for a generic spatial tensor $T_i{}^j{}_k$ with components $T_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$. Following the conventions used in Chapter 5 for the frame, one introduces a pair of complex null vectors $\boldsymbol{e}_+$

and $\boldsymbol{e}_-$ determined by

$$\boldsymbol{e}_+ \equiv \frac{1}{\sqrt{2}}(\boldsymbol{e_1} - \mathrm{i}\boldsymbol{e_2}), \quad \boldsymbol{e}_- \equiv \frac{1}{\sqrt{2}}(\boldsymbol{e_1} + \mathrm{i}\boldsymbol{e_2}).$$

A direct computation using equation (6.32) and equation (5.95a)-(5.95b) given in Appendix of Chapter 5 to express the spatial Infeld-van der Waerden symbols in terms of of the basic valence-2 symmetric spinors renders

$$\boldsymbol{e_{AB}} = x_{\boldsymbol{AB}}\boldsymbol{e_3} - \sqrt{2}y_{\boldsymbol{AB}}\boldsymbol{e}_- - \sqrt{2}z_{\boldsymbol{AB}}\boldsymbol{e}_+.$$

A similar computation then shows that the spinorial counterpart of the connection coefficients $\gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$ can be encoded in

$$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{HF}} = 2\mathrm{i}\Big(\gamma_{123}x_{\boldsymbol{HF}}(y_{\boldsymbol{CD}}z_{\boldsymbol{AB}} - y_{\boldsymbol{AB}}z_{\boldsymbol{CD}}) + \gamma_{123}x_{\boldsymbol{CD}}(-y_{\boldsymbol{HF}}z_{\boldsymbol{AB}} + y_{\boldsymbol{AB}}z_{\boldsymbol{HF}}) - (\gamma_{123}x_{\boldsymbol{AB}} + \mathrm{i}\gamma_{212}(y_{\boldsymbol{AB}} - z_{\boldsymbol{AB}}))(y_{\boldsymbol{H}}z_{\boldsymbol{CD}} - y_{\boldsymbol{CD}}z_{\boldsymbol{HF}})\Big).$$

The reduced connection defined as $\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{C}}{}_{\boldsymbol{F}} \equiv \frac{1}{2}\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{CF}}$ is given by

$$\gamma_{\boldsymbol{ABDF}} = \frac{1}{\sqrt{2}}\gamma_{212}x_{\boldsymbol{DF}}(y_{\boldsymbol{AB}} - x_{\boldsymbol{DF}}z_{\boldsymbol{AB}}) + \gamma_{123}\frac{\mathrm{i}}{\sqrt{2}}\Big(-x_{\boldsymbol{AB}}x_{\boldsymbol{DF}} + 2\Big(y_{\boldsymbol{DF}}z_{\boldsymbol{AB}} + y_{\boldsymbol{AB}}z_{\boldsymbol{DF}}\Big)\Big).$$

This can be rewritten in its irreducible parts by introducing the basic valence-4 spinors —see Appendix 5.8.2 of Chapter 5. A computation renders

$$\gamma_{\boldsymbol{ABDF}} = -\gamma_{212}\Big(\frac{1}{\sqrt{2}}(\epsilon^1_{\boldsymbol{ABCF}} + \epsilon^3_{\boldsymbol{ABDF}}) + \frac{1}{4}(y_{\boldsymbol{BF}}\epsilon_{\boldsymbol{AD}} + y_{\boldsymbol{AD}}\epsilon_{\boldsymbol{BF}}) + \frac{1}{4}(z_{\boldsymbol{BF}}\epsilon_{\boldsymbol{AD}} + z_{\boldsymbol{AD}}\epsilon_{\boldsymbol{BF}})\Big) - \mathrm{i}\sqrt{2}\gamma_{123}\Big(2\epsilon^2_{\boldsymbol{ABDF}} + \frac{1}{3}h_{\boldsymbol{ABDF}}\Big).$$

Additionally, using the reality conditions (5.35) given in Chapter 5, it can be verified that $\gamma_{\boldsymbol{ABCD}}$ is an imaginary spinor —as is to be expected from the space spinor formalism. The last spinor corresponds to the initial data for the spinor field $\xi_{\boldsymbol{ABCD}}$ representing the imaginary part of the reduced connection $\Gamma_{\boldsymbol{ABCD}}$. The real part of $\Gamma_{\boldsymbol{ABCD}}$ is encoded in the Weingarten spinor $\chi_{\boldsymbol{ABCD}}$ whose initial data is given by

$$\chi_{\boldsymbol{ABCD}} = \kappa h_{\boldsymbol{ABCD}}.$$

The spinorial counterpart of the 3-dimensional Schouten tensor is

$$l_{\boldsymbol{ABCD}} = l_{33}x_{\boldsymbol{AB}}x_{\boldsymbol{CD}} + 2l_{11}(y_{\boldsymbol{CD}}z_{\boldsymbol{AB}} + y_{\boldsymbol{AB}}z_{\boldsymbol{CD}}) - \mathrm{i}l_{23}x_{\boldsymbol{CD}}(y_{\boldsymbol{AB}} - z_{\boldsymbol{AB}}),$$

which written in its irreducible parts is

$$l_{\boldsymbol{ABCD}} = 2l_{33}\epsilon^2{}_{\boldsymbol{ABCD}} - 2l_{11}\Big(\epsilon^2_{\boldsymbol{ABCD}} + \frac{1}{3}h_{\boldsymbol{ABCD}}\Big)$$
$$+\mathrm{i}l_{23}\Big(\epsilon^1_{\boldsymbol{ABCD}} + \epsilon^3_{\boldsymbol{ABCD}} + \frac{1}{2\sqrt{2}}(y_{\boldsymbol{BD}}\epsilon_{\boldsymbol{AC}} + y_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}}) + \frac{1}{2\sqrt{2}}(z_{\boldsymbol{BC}}\epsilon_{\boldsymbol{AC}} + z_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}})\Big).$$

Thus, using that $L_{\boldsymbol{ij}} = l_{\boldsymbol{ij}} + \frac{1}{2}\kappa^2 h_{\boldsymbol{ij}}$, one obtains

$$L_{\boldsymbol{ABCD}} = 2l_{33}\epsilon^2{}_{\boldsymbol{ABCD}} - l_{11}\Big(-2\epsilon^2_{\boldsymbol{ABCD}} + \frac{4-3\kappa^2}{6}h_{\boldsymbol{ABCD}}\Big)$$
$$+\mathrm{i}l_{23}\Big(\epsilon^1_{\boldsymbol{ABCD}} + \epsilon^3_{\boldsymbol{ABCD}} + \frac{1}{2\sqrt{2}}(y_{\boldsymbol{BD}}\epsilon_{\boldsymbol{AC}} + y_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}}) + \frac{1}{2\sqrt{2}}(z_{\boldsymbol{BC}}\epsilon_{\boldsymbol{AC}} + z_{\boldsymbol{AC}}\epsilon_{\boldsymbol{BD}})\Big).$$

6.4.3 Changing the conformal representation

The most important feature of the asymptotic initial data discussed in Section 6.4.1 is the fact that the $\boldsymbol{h}$ is conformally flat. Tracefree tensors satisfying the equation

$$D^{\boldsymbol{i}}d_{\boldsymbol{ij}} = 0 \tag{6.33}$$

have been analysed in the conformally flat setting in [42]. The conformal invariance of the last equation —see Lemma 13 of Chapter 5, reduces the problem to that of analysing the above equation in a flat background, in other words for $\boldsymbol{h} = -\boldsymbol{\delta}$, and then suitably rescaling the solution. In [42], all smooth solutions to equation (6.33) in flat space have been parametrised in terms of five quantities: A, J, P, Q and λ_2 where the first four quantities are constants while λ_2 denotes an arbitrary smooth function of spin-weight two. The latter quantity can be expressed alternatively as $\lambda_2 = \eth^2\lambda_0$ where λ_0 is a smooth scalar function of spin-weight zero and $\eth$ and $\bar{\eth}$ denote the $\eth$ and $\bar{\eth}$ operators of the Newman-Penrose formalism —see [12] for a general discussion on the Newman-Penrose formalism. In Section 5.3.4 of Chapter 5 the solution to equation (6.33) on $\mathbb{R}^3$ subject to the condition (5.25) was discussed. In this paragraph, the latter discussion is extended —dropping the spherical symmetry condition encoded in equation (5.25)— adapting the analysis given in [42] to the present setting and notation. Consider the equation

$$\grave{D}^{\boldsymbol{i}}\grave{d}_{\boldsymbol{ij}} = 0, \tag{6.34}$$

where $\grave{\boldsymbol{h}} \equiv -\boldsymbol{\delta}$ is the flat metric. Following [42], one can introduce Cartesian coordinates $(x^{\boldsymbol{\alpha}})$ with the origin of $\mathbb{R}^3$ located at a fiduciary position $\mathcal{O}$. Additionally, one can introduce polar coordinates defined via $\rho = \delta_{\boldsymbol{\alpha\beta}}x^{\boldsymbol{\alpha}}x^{\boldsymbol{\beta}}$. The flat metric in these coordinates reads

$$\grave{\boldsymbol{h}} = -\mathbf{d}\rho\otimes\mathbf{d}\rho - \rho^2\boldsymbol{\sigma}. \tag{6.35}$$

where $\boldsymbol{\sigma}$ is the standard metric on $\mathbb{S}^2$. Considering an arbitrary pair of complex null vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ such that

$$\boldsymbol{\sigma}^\sharp = (\boldsymbol{m}\otimes\bar{\boldsymbol{m}} + \bar{\boldsymbol{m}}\otimes\boldsymbol{m})$$

and denoting the radial direction $\boldsymbol{\partial}_\rho$ by $\boldsymbol{n}$ one reexpresses the metric as $\grave{\boldsymbol{h}}^\sharp = -\boldsymbol{n}\otimes\boldsymbol{n} - \boldsymbol{\sigma}$. Introducing a spin-dyad adapted to the above described frame one can express the metric in terms of the basic valence-2 spinors of the space spinor formalism as

$$\grave{h}_{\boldsymbol{ABCD}} = -\grave{x}_{\boldsymbol{AB}}\grave{x}_{\boldsymbol{BC}} - 2\grave{y}_{\boldsymbol{AB}}\grave{z}_{\boldsymbol{CD}} - 2\grave{y}_{\boldsymbol{CD}}\grave{z}_{\boldsymbol{AB}}.$$

With this notation the general solution to equation (6.34) can be expressed in the space spinor formalism as

$$\rho^3\grave{d}_{\boldsymbol{ABCD}} = 6\xi\grave{\epsilon}^2_{\boldsymbol{ABCD}} + 2\eta_1\grave{\epsilon}^1_{\boldsymbol{ABCD}} - 2\eta_1\grave{\epsilon}^3_{\boldsymbol{ABCD}} + 2\bar{\mu}_2\grave{\epsilon}^4_{\boldsymbol{ABCD}} + 2\mu_2\grave{\epsilon}^0_{\boldsymbol{ABCD}}$$

where

$$\begin{aligned}
\xi &= \bar{\eth}^2\lambda_2^R + A + \rho Q + \frac{1}{\rho}P,\\
\eta_1 &= -2\rho\partial_\rho\bar{\eth}\lambda_2^R + \bar{\eth}\lambda_2^I + \rho\eth Q - \frac{1}{\rho}\eth P + \mathrm{i}\eth J,\\
\mu_2 &= 2\rho\partial_\rho(\rho\partial_\rho\lambda_2^R) - 2\lambda_2^R + \eth\bar{\eth}\lambda_2^R - \rho\partial_\rho\lambda_2^I.
\end{aligned}$$

with $\lambda_2^R = \mathrm{Re}(\lambda_2)$ and $\lambda_2^I = \mathrm{iIm}(\lambda_2)$ —see [42] for a detailed derivation. To extend the above result to the conformally flat setting introduce $\boldsymbol{e_i}$ and $\bar{\boldsymbol{e}}_{\boldsymbol{i}}$ denoting a $\boldsymbol{h}$ and $\bar{\boldsymbol{h}}$ orthonormal frames, with

$$\boldsymbol{h} = \omega^2\bar{\boldsymbol{h}},$$

so that

$$\bar{\boldsymbol{e}}_{\boldsymbol{i}} = \omega\boldsymbol{e}_{\boldsymbol{i}}. \tag{6.37}$$

Using equation (6.37) one observes that the frame version of the transformation law for TT-tensors given in Lemma 13 of Chapter 5 reads

$$d_{\boldsymbol{ij}} = \omega^{-3}\bar{d}_{\boldsymbol{ij}}. \tag{6.38}$$

From the discussion of Sections 6.4.1 and 6.4.2 one knows that there exist a representation in which the asymptotic initial data for the rescaled Weyl spinor for the Kerr-de Sitter spacetime can be expressed as

$$\phi_{\boldsymbol{ABCD}} = 6m\epsilon^2_{\boldsymbol{ABCD}},$$

and the induced metric at the conformal boundary is determined by equation (6.26).

As the Cotton tensor of $\boldsymbol{h}$ vanishes, there exists a conformal factor ω such that

$$\grave{\boldsymbol{h}} = \omega^2 \boldsymbol{h}$$

with $\boldsymbol{h}$ and $\grave{\boldsymbol{h}}$ given by equations (6.26) and (6.35) respectively. Consequently, one concludes that in the representation in which the induced metric at the conformal boundary is flat, $\grave{\boldsymbol{h}} = -\boldsymbol{\delta}$, the initial data for the rescaled Weyl spinor reads

$$\grave{\phi}_{\boldsymbol{ABCD}} = \omega^{-3}\phi_{\boldsymbol{ABCD}}$$

in other words, the initial data for the Kerr-de Sitter spacetime in the flat representation is determined by four constants A, P, Q, J and a smooth function of spin-weight 2, $\lambda_2 = \lambda_2^R + \lambda_2^I$, satisfying

$$\bar{\eth}^2\lambda_2^R + A + \rho Q + \frac{1}{\rho}P = \frac{m}{\rho^3\omega^3}, \tag{6.39a}$$

$$-2\rho\partial_\rho\bar{\eth}\lambda_2^R + \bar{\eth}\lambda_2^I + \rho\bar{\eth}Q - \frac{1}{\rho}\bar{\eth}P + \mathrm{i}\bar{\eth}J = 0, \tag{6.39b}$$

$$2\rho\partial_\rho(\rho\partial_\rho\lambda_2^R) - 2\lambda_2^R + \eth\bar{\eth}\lambda_2^R - \rho\partial_\rho\lambda_2^I = 0. \tag{6.39c}$$

In Section 5.3.4 of Chapter 5 it was shown that for the Schwarzschild-de Sitter spacetime $\omega = \rho^{-1}$. This in turn, using equations (6.39a)-(6.39c), imply $P = Q = J = \lambda_2 = 0$ and $A = m$ which then characterise the asymptotic initial data for the Schwarzschild-de Sitter spacetime. For the Kerr-de Sitter spacetime the conformal factor ω cannot be determined explicitly and the discussion for the Schwarzschild-de Sitter spacetime given in Chapter 5 suggests that the initial data for the rescaled Weyl spinor will be singular in the flat representation. Consequently, the discussion of the initial data will be restricted to the one associated with conformal representation described in Sections 6.4.1 and 6.4.2.

Notice that the above discussion is consistent with the characterisations of *asymptotically Kerr-de Sitter like* spacetimes given in [85]. In the latter reference, Kerr-de Sitter-like spacetimes are characterised in terms of a conformal Killing vector v_i at $\mathscr{I}$ arising from a Killing vector X_i on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ where $\tilde{\boldsymbol{g}}$ is a solution to the Einstein field equations with positive Cosmological constant. In [85] it is shown that the so-called *rescaled Mars-Simon tensor* vanishes if the following conditions hold

$$C_{ij} = \sqrt{\frac{\lambda}{3}}C_{mag}|v|^{-5}(v_iv_j + |v|^2\frac{1}{3}h_{ij}), \tag{6.40a}$$

$$d_{ij} = C_{el}|v|^{-5}(v_iv_j + |v|^2\frac{1}{3}h_{ij}), \tag{6.40b}$$

where $|v|^2 \equiv v_iv^i$, C_{ij} is the Cotton-York tensor of the induced metric at the conformal boundary, d_{ij} denotes the electric part of the rescaled Weyl tensor and C_{el} and C_{mag} are two undetermined constants —see [85] for a detailed discussion. Equations (6.40a) and (6.40b) have been adapted to the signature conventions used in this thesis. Notice additionally that, the second expression can be written using the space spinor formalism as

$$d_{\boldsymbol{ABCD}} = C_{el}|v|^{-5}v_{(\boldsymbol{AB}}v_{\boldsymbol{CD})}.$$

This is consistent with the discussion of Section 6.4.1 if the spin-dyad $\epsilon_{\boldsymbol{A}}{}^{A}$ is aligned with the conformal Killing vector v^i so that, up to a normalisation factor, $v_{\boldsymbol{AB}}$ corresponds to $x_{\boldsymbol{AB}}$.

6.5 Perturbations of the Kerr-de Sitter spacetime

In the following, a similar discussion to that given in Chapter 5 for perturbations of the Schwarzschild-de Sitter spacetime will be given. In particular, one will consider perturbations of the Kerr-de Sitter spacetimes which can be covered by a congruence of conformal geodesics. The last assumption implies, in particular, that the functional form of the conformal factor will be the same as that of the background solution. Observe that the asymptotic initial data obtained in Sections 6.4.1 and 6.4.2 correspond to a conformal representation in which the initial hypersurface $\tau = 0$ representing $\mathscr{I}$ is topologically $\mathbb{R} \times \mathbb{S}^2$ with a metric $\boldsymbol{h}$ given by (6.26). Contrast with the case of the Schwarzschild-de Sitter spacetime, analysed in Chapter 5, in which the initial hypersurface is not only topologically but also metrically $\mathbb{R}\times\mathbb{S}^2$. In this section non-linear perturbations of the Kerr-de Sitter spacetimes are analysed by means of a suitably posed initial value problem. In other words, the development of perturbed initial data close to that of the Kerr-de Sitter spacetime, in the above described conformal representation, is discussed. In view of the symmetric hyperbolicity of the conformal evolution equations (5.5a)-(5.5b) of Chapter 5 one can exploit the theory of first order symmetry hyperbolic systems contained in [23] to obtain an existence result for the asymptotic region of the Kerr-de Sitter spacetime. Proceeding in an analogous way as in Section 5.4.3 of Chapter 5 one obtains the following result

Theorem 4 (***existence of perturbations of asymptotic initial data for the Kerr-de Sitter spacetime***). *Let* $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ *denote asymptotic initial data for the extended conformal Einstein field equations on a 3-dimensional manifold* $\mathcal{S} \approx \mathbb{R} \times \mathbb{S}^2$ *where* $\mathring{\mathbf{u}}_\star$ *denotes the asymptotic initial data for the Kerr-de Sitter spacetime in the time-symmetric conformal representation* $\kappa = 0$. *Then, for* $m \geq 4$ *there exist a small* $\tau_\circ > 0$ *and* $\varepsilon > 0$ *such that:*

(i) for $||\breve{\mathbf{u}}_\star||_{\mathcal{S},m} < \varepsilon$ *, there exist a unique solution* $\breve{\mathbf{u}}$ *to the conformal evolution*

equations (5.5a)-(5.5b) *with a minimal existence interval* $[0,\tau_\circ]$ *and*

$$\breve{\mathbf{u}} \in C^{m-2}([0,\tau_\circ]\times\mathcal{S},\mathbb{C}^N),$$

and the associated congruence of conformal geodesics contains no conjugate points in $[0,\tau_\circ]$*;*

(ii) the solution $\mathbf{u}=\mathring{\mathbf{u}}+\breve{\mathbf{u}}$ *is unique in* $[0,\tau_\circ]\times\mathcal{S}$ *and implies a* C^{m-2} *solution* $(\tilde{\mathcal{M}}_{\tau_\circ},\tilde{\boldsymbol{g}})$ *to the Einstein vacuum equations with the same de Sitter-like Cosmological constant as the background solution where*

$$\tilde{\mathcal{M}}_{\tau_\circ}\equiv(0,\tau_\circ)\times\mathcal{S}.$$

Moreover, the hypersurface $\mathscr{I}\equiv\{0\}\times\mathcal{S}$ *represents the conformal boundary of the spacetime.*

Proof. The proof of this theorem is analogous to that given for the Schwarzschild-de Sitter spacetime in Theorem 3.

6.6 Conclusions

In this chapter an analysis of perturbations of the Kerr-de Sitter spacetime arising from suitably posed asymptotic initial value problems is given. To do so, initial data for the conformal Einstein field equations representing asymptotic initial data for the Kerr-de Sitter spacetime was found. Then, by introducing the appropriate norms —see Section 5.4.3 of Chapter 5— small perturbations of asymptotic initial data close to that of the Kerr-de Sitter spacetime were considered. Exploiting the structure of the conformal evolution equations and the theory of symmetric hyperbolic systems contained in [23] an existence result for perturbations of the Kerr-de Sitter spacetime valid in their corresponding asymptotic region was obtained. The asymptotic initial data for the Kerr-de Sitter spacetime discussed in this chapter was obtained in the *time symmetric conformal representation* $\kappa=0$ —alternatively characterised by the vanishing of the Friedrich scalar at $\mathscr{I}$. Nonetheless, the conformal properties of the conformal constraint equations open the possibility to consider other conformal representations. In particular, as the induced metric at the conformal boundary of the Kerr-de Sitter spacetime $\boldsymbol{h}$ is conformally flat, one could in principle consider a representation in which $\grave{\boldsymbol{h}}$ is the flat metric. In the latter representation, all smooth solutions of the Gauss constraint (TT-tensors) determining the initial data for the electric part of the rescaled Weyl tensor have been parametrised in terms of four constants A, J, Q, P and a smooth complex function of spin weight two λ_2 in [42]. As discussed in more detail in Chapter 5 and briefly in this chapter, for the Schwarzschild-de Sitter spacetime, the constant A can be identified with the mass parameter m of the exact solution. Intuition would suggest that the initial data for the electric part of the rescaled Weyl tensor spinor for the Kerr-de Sitter spacetime could be obtained by simple superposition of the solution containing with $A \neq 0$ and $J \neq 0$ and all the other parameters vanishing. Nevertheless, as shown in this chapter this is not the case as the asymptotic initial data for electric part of the rescaled Weyl spinor for the Kerr-de Sitter spacetime in this conformal representation is characterised by equations (6.39a)-(6.39c). Moreover, it would be interesting to investigate the representation for which the initial data for the induced metric at the conformal boundary is the standard metric of $\mathbb{S}^3$ since, as discussed in Chapter 5, in this conformal representation the asymptotic initial data for the Schwarzschild-de Sitter spacetime is singular in the region of the spacetime where the horizons meet the conformal boundary. The singular behaviour of d_{ij} is not observed here as in the $\mathbb{R} \times \mathbb{S}^2$-representation the points where the conformal boundary meets the horizon (asymptotic points) are send to infinity. By working on this representation the smallness requirement on the initial data, using the theory contained in [23], imposes certain decay of the perturbations at these points. Despite the results of this chapter constitute a generalisation of the analysis given in Chapter 5 for the Schwarzschild-de Sitter spacetime, in the latter chapter an analysis of the time of existence for the solutions was given, the latter requires a deeper analysis of the conformal evolution equations describing the exact Kerr-de Sitter spacetime as a solution to the extended conformal Einstein field equations expressed in terms of a conformal Gaussian system. An analysis addressing the above raised questions for the Kerr-de Sitter spacetime will be addressed elsewhere. Nevertheless, this chapter shows that it is possible to construct *future asymptotically de Sitter spacetimes* whose asymptotic initial data lies on an open ball close to the Kerr-de Sitter spacetime. In particular, the latter implies that the initial data for the induced metric at the conformal boundary of the perturbed spacetimes is not necessarily conformally flat, and consequently, according to the theory of asymptotics given in [24] these spacetimes represent non-trivial examples for the theory of asymptotics for de Sitter-like spacetimes allowing for gravitational radiation at $\mathscr{I}$.

7 Zero rest-mass fields and the Newman-Penrose constants on flat space

7.1 Introduction

The concept of asymptotic simplicity is central for the understanding of isolated systems in general relativity. In this regard, Penrose's proposal [33] is an attempt to characterise the fall-off behaviour of the gravitational field in a geometric manner —see also [34]. As discussed in Chapter 1, in Penrose's proposal to study the asymptotic region of the *physical spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfying the Einstein field equations one considers an *unphysical spacetime* $(\mathcal{M}, \boldsymbol{g})$, where $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ are related via

$$\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}},$$

The set of points where $\Xi = 0$ but $\mathbf{d}\Xi \neq 0$ is called the conformal boundary. If $\tilde{\boldsymbol{g}}$ satisfies the vacuum Einstein field equations (with vanishing Cosmological constant) near $\mathscr{I}$, then the conformal boundary defines a smooth null hypersurface of $\mathcal{M}$ and one calls $\mathscr{I}$ *null infinity* —see [12, 34]. One can identify two disjoint pieces of $\mathscr{I}$: $\mathscr{I}^-$ and $\mathscr{I}^+$ correspond to the past and future end points of null geodesics. If every null geodesic acquires two distinct endpoints at $\mathscr{I}$, the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is said to be *asymptotically simple* —see [12, 13, 34] for precise definitions. The Minkowski spacetime, $(\mathbb{R}^4, \tilde{\boldsymbol{\eta}})$ is the prototypical example of an asymptotically simple spacetime. In the standard conformal representation of the Minkowski spacetime, the unphysical spacetime can be identified with the Einstein cylinder $(\mathcal{M}_E, \boldsymbol{g}_E)$ where $\mathcal{M}_{\mathcal{E}} \approx \mathbb{R} \times \mathbb{S}^3$ and

$$\boldsymbol{g}_E = \mathbf{d}T \otimes \mathbf{d}T - \mathbf{d}\psi \otimes \mathbf{d}\psi - \sin^2\psi \boldsymbol{\sigma}, \qquad \Xi = \cos(T) + \cos(\psi),$$

where $-\pi < T < \pi$, $0 < \psi < \pi$ and $\boldsymbol{\sigma}$ is the standard metric on $\mathbb{S}^2$. In this conformal representation $\mathscr{I}^{\pm}$ correspond to the sets of points on the Einstein cylinder, $\mathcal{M}_{\mathcal{E}} \equiv \mathbb{R} \times \mathbb{S}^3$, for which $0 < \psi < \pi$ and $T = \pm(\pi - \psi)$. One can directly verify that $\Xi|_{\mathscr{I}^{\pm}} = 0$ while $\mathbf{d}\Xi|_{\mathscr{I}^{\pm}} \neq 0$ —see [12]. Consequently, a special region in the conformal structure of the Minkowski spacetime is *spatial infinity* i^0 for which both $\Xi|_{i^0}$ and

$\mathbf{d}\Xi|_{i^0}$ vanish. In this conformal representation, spatial infinity corresponds to a point in the Einstein cylinder with coordinates $\psi = \pi$ and $T = 0$.

A natural problem to be considered is the existence of spacetimes whose conformal structure resembles that of the Minkowski spacetime. The conformal Einstein field equations introduced originally in [11] provide a convenient framework for discussing global existence of *asymptotically simple* solutions to the Einstein field equations. An important application of these equations is the proof of the semi-global non-linear stability of the Minkowski spacetime given in [11]. In the latter work, the evolution of perturbed initial data close to exact Minkowski data is analysed. Nevertheless, the initial data is not prescribed on a Cauchy hypersurface $\tilde{\mathcal{S}}$ but on a hyperboloid $\tilde{\mathcal{H}}$ whose conformal extension in $\mathcal{M}$ intersects $\mathscr{I}$ —see Chapter 4 for a similar discussion of the non-linear stability the Milne spacetime. Therefore, an open problem in the framework of the conformal Einstein field equations is the analysis of the evolution of initial data prescribed on a Cauchy hypersurface $\mathcal{S}$ intersecting i^0 —see [10] for the proof of the global non-linear stability of the Minkowski spacetime employing different methods. One of the main difficulties in establishing a global result for the stability of the Minkowski spacetime using conformal methods lies on the fact that the initial data for the conformal Einstein field equations is not smooth at i^0. This is not unexpected since, as observed by Penrose —see [33, 86]— the conformal structure of spacetimes with non-vanishing mass becomes singular at spatial infinity —in the sense that the rescaled Weyl tensor becomes singular at i^0. A milestone in the resolution of this problem is the construction, originally introduced in [27], of a new representation of spatial infinity known as the *cylinder at spatial infinity*. In this representation, spatial infinity is not represented as a point but as set whose topology is that of a cylinder. This representation is well adapted to exploit the properties of curves with special conformal properties: *conformal geodesics*. In addition, it allows to formulate a regular finite initial value problem for the conformal Einstein field equations. Other approaches for analysing the gravitational field near spatial infinity using different representations of spatial infinity have been also proposed in literature —see [87–90].

The framework of the cylinder at spatial infinity and its connection with the conformal Einstein field equations have been exploited in an analysis of the *gravitational Newman-Penrose (NP) constants* in [81]. The NP constants, originally introduced in [91], are defined in terms of integrals over cuts $\mathcal{C} \approx \mathbb{S}^2$ of $\mathscr{I}$. The integrands in the expressions defining the NP constants are, however, written in a particular gauge adapted to $\mathscr{I}$ (the so-called *NP-gauge*) while the natural gauge used in the framework of the cylinder at spatial infinity (the so-called *F-gauge* in [81]), is adapted to a congruence of conformal geodesics and hinged at a Cauchy hypersurface $\mathcal{S}$. This fact, which at first instance looks as an obstacle to analyse the NP constants, turns out to be advantageous since, once the relation between the NP-gauge and the F-gauge is clarified, one can relate the initial data prescribed on $\mathcal{S}$ with the gravitational NP constants at $\mathscr{I}$.

In a recent work [28], the authors exploit the notion of these conserved quantities at $\mathscr{I}$ to make inroads into the problem of the information paradox —see [92–94]. In the [28], the concept of *soft hair* is motivated by means of an analysis of the conservation laws and symmetries of abelian gauge theories in Minkowski space. These conservation laws correspond essentially to the electromagnetic version of the gravitational NP constants. With this motivation, in this chapter zero rest-mass fields propagating on flat space and their corresponding NP constants are studied. The discussion of this chapter is based on

Gasperín E. and Valiente Kroon J.A., "Zero rest-mass fields and the Newman-Penrose constants on flat space," *ArXiv e-prints* (Aug., 2016) , `arXiv:1608.05716 [gr-qc]`.

In this chapter, two physically relevant fields are analysed: the spin-1 and spin-2 zero rest-mass fields. The spin-1 field provides a description of the electromagnetic field while the spin-2 field on the Minkowski spacetime describes linearised gravity. It is shown how the framework of the cylinder at spatial infinity can be exploited to relate the corresponding NP constants with the initial data on a Cauchy hypersurface intersecting i^0 —see Propositions 11 and 12 for the spin-1 case and Proposition 13 and 14 for the spin-2 case. Additionally, it is shown that, for the class of initial data considered, the NP constants at $\mathscr{I}^+$ and $\mathscr{I}^-$ coincide —see Theorems 5 and 3. Moreover, it is discussed how this identification arises from a delicate interplay between the evolution and constraint equations associated to these fields. In particular, the analysis given in this chapter highlights the connection between the smoothness of the fields at null infinity and the finiteness of the conserved quantities.

7.2 The cylinder at spatial infinity and the F-gauge

In this section a conformal representation of the Minkowski spacetime that is adapted to a congruence of conformal geodesics is discussed. This conformal representation, introduced originally in [27], is well suited for analysing the behaviour of fields near spatial infinity. In broad terms, in this representation spatial infinity i^0, which corresponds to a point in the standard compactification of the Minkowski spacetime, is blown up to a set I with the topology of $\mathbb{R} \times \mathbb{S}^2$. In the subsequent discussion this representation will be referred as the *cylinder at spatial infinity*. The discussion of the cylinder at spatial infinity as presented in [27] is given in the language of fibre bundles. In particular, the construction of the so-called *extended bundle space* is required —see [27, 95]. Nevertheless, a discussion which does not make use of this construction is presented in the following.

7.2.1 The cylinder at spatial infinity

Consider the Minkowski metric $\tilde{\boldsymbol{\eta}}$ in Cartesian coordinates $\tilde{x}^\alpha = (\tilde{t}, \tilde{x}^i)$,

$$\tilde{\boldsymbol{\eta}} = \eta_{\mu\nu}\mathbf{d}\tilde{x}^\mu \otimes \mathbf{d}\tilde{x}^\nu,$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Introducing polar coordinates defined by $\tilde{\rho}^2 = \delta_{ij}\tilde{x}^i\tilde{x}^j$ where $\delta_{ij} = \mathrm{diag}(1,1,1)$, and an arbitrary choice of coordinates on $\mathbb{S}^2$, the metric $\tilde{\boldsymbol{\eta}}$ can be written as

$$\tilde{\boldsymbol{\eta}} = \mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} - \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} - \tilde{\rho}^2\boldsymbol{\sigma},$$

with $\tilde{t} \in (-\infty,\infty)$, $\tilde{\rho} \in [0,\infty)$ and $\boldsymbol{\sigma}$ denotes the standard metric on $\mathbb{S}^2$. A common procedure to obtain a conformal representation of the Minkowski spacetime close to i^0 is to introduce *inversion coordinates* $x^\alpha = (t, x^i)$ defined by —see [12],

$$x^\mu = -\tilde{x}^\mu/\tilde{X}^2, \qquad \tilde{X}^2 \equiv \tilde{\eta}_{\mu\nu}\tilde{x}^\mu\tilde{x}^\nu.$$

The inverse transformation is given by

$$\tilde{x}^\mu = -x^\mu/X^2, \qquad X^2 = \eta_{\mu\nu}x^\mu x^\nu.$$

Using these coordinates one readily identifies the following conformal representation of the Minkowski spacetime

$$\boldsymbol{g}_I = \Xi^2\,\tilde{\boldsymbol{\eta}}, \tag{7.1}$$

where $\boldsymbol{g}_I = \eta_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$ and $\Xi = X^2$. Notice, additionally that, $X^2 = 1/\tilde{X}^2$. Introducing an *unphysical polar coordinate* defined as $\rho^2 = \delta_{ij}x^ix^j$, one observes that the rescaled metric $\boldsymbol{g}_I$ and conformal factor Ξ read

$$\boldsymbol{g}_I = \mathbf{d}t \otimes \mathbf{d}t - \mathbf{d}\rho \otimes \mathbf{d}\rho - \rho^2\boldsymbol{\sigma}, \qquad \Xi = t^2 - \rho^2, \tag{7.2}$$

with $t \in (-\infty,\infty)$ and $\rho \in (0,\infty)$. In this conformal representation, spatial infinity i^0 corresponds to a point located at the origin. For future reference, observe that $\tilde{t}$ and $\tilde{\rho}$ are related to t and ρ via

$$\tilde{t} = -\frac{t}{t^2-\rho^2}, \qquad \tilde{\rho} = \frac{\rho}{t^2-\rho^2}. \tag{7.3}$$

Then, one introduces a time coordinate τ defined via $t = \rho\tau$. In the coordinate system determined by τ and ρ the metric $\boldsymbol{g}_I$ is written as

$$\boldsymbol{g}_I = \rho^2 \mathbf{d}\tau \otimes \mathbf{d}\tau - (1-\tau^2)\mathbf{d}\rho \otimes \mathbf{d}\rho + \rho\tau \mathbf{d}\rho \otimes \mathbf{d}\tau + \rho\tau \mathbf{d}\tau \otimes \mathbf{d}\rho - \rho^2 \boldsymbol{\sigma}.$$

The required conformal representation is obtained by considering the rescaled metric

$$\boldsymbol{g}_C \equiv \frac{1}{\rho^2}\boldsymbol{g}_I. \tag{7.4}$$

Introducing $\varrho^\star \equiv -\ln\rho$ the metric $\boldsymbol{g}_C$ explicitly reads

$$\boldsymbol{g}_C = \mathbf{d}\tau \otimes \mathbf{d}\tau - (1-\tau^2)\mathbf{d}\varrho^\star \otimes \mathbf{d}\varrho^\star - \tau \mathbf{d}\tau \otimes \mathbf{d}\varrho^\star - \tau \mathbf{d}\varrho^\star \otimes \mathbf{d}\tau - \boldsymbol{\sigma}.$$

Observe that spatial infinity i^0, which is at infinity respect to the metric $\boldsymbol{g}_C$, corresponds to a set which has the topology of $\mathbb{R}\times\mathbb{S}^2$ —see [27, 95]. In what follows the coordinates (τ,ρ) will be preferred and will be referred as the *F-coordinates*. Following the conformal rescalings previously introduced one considers the conformal extension $(\mathcal{M}, \boldsymbol{g}_C)$ where

$$\boldsymbol{g}_C = \Theta^2\tilde{\boldsymbol{\eta}}, \qquad \Theta = \rho(1-\tau^2),$$

and

$$\mathcal{M} \equiv \{p \in \mathbb{R}^4 \mid -1 \leq \tau \leq 1, \;\; \rho(p) \geq 0\}.$$

In this representation future and past null infinity are located at

$$\mathscr{I}^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1\}, \qquad \mathscr{I}^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1\},$$

and the physical Minkowski spacetime can be identified with the region

$$\tilde{\mathcal{M}} \equiv \{p \in \mathcal{M} \mid -1 < \tau(p) < 1, \;\; \rho(p) > 0\}.$$

In addition, the following sets will be distinguished:

$$I \equiv \{p \in \mathcal{M} \mid \; |\tau(p)| < 1, \; \rho(p) = 0\}, \qquad I^0 \equiv \{p \in \mathcal{M} \mid \tau(p) = 0, \; \rho(p) = 0\},$$

$$I^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1, \; \rho(p) = 0\}, \qquad I^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1, \; \rho(p) = 0\}.$$

Notice that spatial infinity i^0, which originally was a point in the $\boldsymbol{g}_I$−representation, can be identified with the set I in the $\boldsymbol{g}_C$−representation. In addition, one can intuitively think of the *critical sets* I^+ and I^- as the region where spatial infinity "touches" $\mathscr{I}^+$ and $\mathscr{I}^-$ respectively. Similarly, I^0 represents the intersection of i^0 and the initial hypersurface $\mathcal{S} \equiv \{\tau = 0\}$ —see Figure 7.1. See also [27, 81] and [95] for further discussion of the framework of the cylinder at spatial infinity implemented

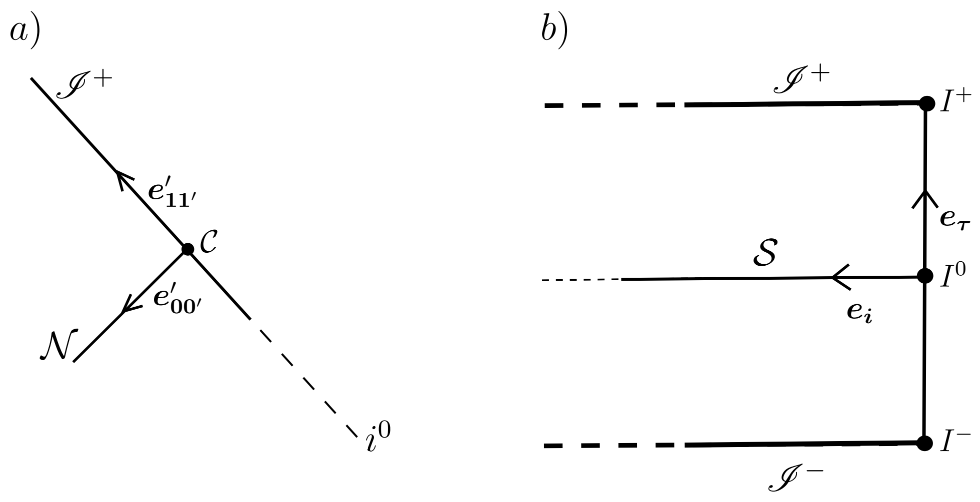

Figure 7.1: *Figure a) illustrates the geometric setting in which the NP-frame $\boldsymbol{e}'_{\boldsymbol{AA'}}$ is constructed: $\boldsymbol{e}'_{\boldsymbol{11'}}$ is paralelly propagated along $\mathscr{I}^+$ while $\boldsymbol{e}'_{\boldsymbol{00'}}$ is tangent to the generators of a null hypersurface $\mathcal{N}$ transverse to $\mathscr{I}^+$. On $\mathcal{C} = \mathcal{N} \cap \mathscr{I}^+$ a complex null frame $\boldsymbol{e}_{\boldsymbol{01'}}$ and $\boldsymbol{e}_{\boldsymbol{10'}}$ is chosen and paralelly propagated along $\mathcal{N}$. Figure b) shows a schematic depiction of the cylinder at spatial infinity and the F-frame $\{\boldsymbol{e_\tau}, \boldsymbol{e_i}\}$. In this representation spatial infinity i^0 is blown-up to a set I with the topology of $\mathbb{R} \times \mathbb{S}^2$. The location of $I^0 = \mathcal{S} \cap I$ and the critical sets $I^\pm$ where the cylinder at spatial infinity I meets future and past null infinity $\mathscr{I}^\pm$ are also shown in this figure.*

for stationary spacetimes.

7.2.2 The F-gauge

In this section a brief discussion of the so-called F-gauge is provided —see [81, 95] for a discussion of the F-gauge in the language of fibre bundles. Following the philosophy of the previous section the discussion presented here will not make use of the extended bundle space —see [81, 95] for definitions. One of the motivations for the introduction of this gauge is that it exploits the properties of conformal geodesics. More precisely, in this framework, one introduces an orthonormal frame (from which one can construct an associated null frame) whose timelike leg corresponds to the tangent of a conformal geodesic starting from a fiduciary spacelike hypersurface $\mathcal{S} = \{\tau = 0\}$ —see Section 2.3.2 in Chapter 2.

To start the discussion, consider the conformal extension $(\mathcal{M}, \boldsymbol{g}_C)$ of the Minkowski spacetime and the F-coordinate system introduced in Section 7.2.1. Observe that the induced metric on the surface $\mathcal{Q} \equiv \{\tau = \tau_\star, \rho = \rho_\star,\}$, with $\tau_\star, \rho_\star$ fixed, is the standard metric on $\mathbb{S}^2$. Consequently, one can introduce a complex null frame $\{\boldsymbol{\partial}_+, \boldsymbol{\partial}_-\}$ on $\mathcal{Q}$ as described in Appendix 7.9. To propagate this frame off $\mathcal{Q}$ one requires that

$$[\boldsymbol{\partial}_\tau, \boldsymbol{\partial}_\pm] = 0, \qquad [\boldsymbol{\partial}_\rho, \boldsymbol{\partial}_\pm] = 0.$$

Taking into account the above construction one writes, in spinorial notation, the spacetime frame

$$\boldsymbol{e}_{\mathbf{00'}} = \frac{\sqrt{2}}{2}\big((1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho\big), \qquad \boldsymbol{e}_{\mathbf{11'}} = \frac{\sqrt{2}}{2}\big((1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho\big), \tag{7.5a}$$

$$\boldsymbol{e}_{\mathbf{01'}} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_+, \qquad \boldsymbol{e}_{\mathbf{10'}} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_-. \tag{7.5b}$$

The corresponding dual coframe is given by

$$\boldsymbol{\omega}^{\mathbf{00'}} = \frac{\sqrt{2}}{2}\Big(\mathbf{d}\tau - \frac{1}{\rho}(1-\tau)\mathbf{d}\rho\Big), \qquad \boldsymbol{\omega}^{\mathbf{11'}} = \frac{\sqrt{2}}{2}\Big(\mathbf{d}\tau + \frac{1}{\rho}(1+\tau)\mathbf{d}\rho\Big),$$
$$\boldsymbol{\omega}^{\mathbf{01'}} = \sqrt{2}\boldsymbol{\omega}^+, \qquad \boldsymbol{\omega}^{\mathbf{10'}} = \sqrt{2}\boldsymbol{\omega}^-.$$

One can directly verify that

$$\boldsymbol{g}_C = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}\boldsymbol{\omega}^{\boldsymbol{AA'}}\boldsymbol{\omega}^{\boldsymbol{BB'}}.$$

The above construction and frame will be referred in the following discussion as the *F-gauge*. A direct computation using the Cartan structure equations shows that the only non-zero reduced connection coefficients are given by

$$\Gamma_{\mathbf{00'}}{}^{\mathbf{1}}{}_{\mathbf{1}} = \Gamma_{\mathbf{11'}}{}^{\mathbf{1}}{}_{\mathbf{1}} = \frac{\sqrt{2}}{4}, \qquad \Gamma_{\mathbf{00'}}{}^{\mathbf{0}}{}_{\mathbf{0}} = \Gamma_{\mathbf{11'}}{}^{\mathbf{0}}{}_{\mathbf{0}} = -\frac{\sqrt{2}}{4},$$
$$\Gamma_{\mathbf{10'}}{}^{\mathbf{1}}{}_{\mathbf{1}} = -\Gamma_{\mathbf{10'}}{}^{\mathbf{0}}{}_{\mathbf{0}} = \frac{\sqrt{2}}{4}\varpi, \qquad \Gamma_{\mathbf{01'}}{}^{\mathbf{0}}{}_{\mathbf{0}} = -\Gamma_{\mathbf{01'}}{}^{\mathbf{1}}{}_{\mathbf{1}} = \frac{\sqrt{2}}{4}\overline{\varpi}.$$

7.3 The electromagnetic field in the F-gauge

In this section the Maxwell equations on $(\mathcal{M}, \boldsymbol{g}_C)$ are discussed. After rewriting the equations in terms of the $\eth$ and $\bar{\eth}$ operators, a general solution is obtained by expanding the fields in spin-weighted spherical harmonics. The resulting equations for the coefficients of the expansion, satisfy ordinary differential equations which can be explicitly solved in terms of special functions. The analysis given here is similar to the one for the Maxwell field on a Schwarzschild background in [96] and the gravitational field in [27]. Notice that, in contrast with the analysis presented in this section, in the latter references the equations and relevant structures are lifted to the extended bundle space. Additionally, the initial data considered in this analysis is generic and in particular is not assumed to be time symmetric.

7.3.1 The spinorial Maxwell equations

The Maxwell equations in the 2-spinor formalism take the form of the spin-1 equation

$$\nabla_{A'}{}^{A}\phi_{AB} = 0. \tag{7.6}$$

Let $\epsilon_{\boldsymbol{A}}{}^{A}$ with $\epsilon_{\boldsymbol{0}}{}^{A} = o^{A}$ and $\epsilon_{\boldsymbol{1}}{}^{A} = \iota^{A}$ denote a spin dyad adapted to the F-gauge so that $e_{\boldsymbol{AA'}}{}^{AA'} = \epsilon_{\boldsymbol{A}}{}^{A}\epsilon_{\boldsymbol{A'}}{}^{A}$, corresponds to the null frame introduced in Section 7.2.2. A direct computation shows that equation (7.6) implies a set of equations for the components of ϕ_{AB} respect to $\epsilon_{\boldsymbol{A}}{}^{A}$: $\phi_0 \equiv \phi_{AB}o^Ao^B$, $\phi_1 \equiv \phi_{AB}o^A\iota^B$ and $\phi_2 \equiv \phi_{AB}\iota^A\iota^B$, which can be split into a system of evolution equations

$$(1+\tau)\partial_\tau\phi_0 - \rho\partial_\rho\phi_0 - \partial_+\phi_1 = -\phi_0, \tag{7.7a}$$

$$\partial_\tau\phi_1 - \frac{1}{2}\big(\partial_+\phi_2 + \partial_-\phi_0\big) = \frac{1}{2}\big(\overline{\varpi}\phi_2 + \varpi\phi_0\big), \tag{7.7b}$$

$$(1-\tau)\partial_\tau\phi_2 + \rho\partial_\rho\phi_2 - \partial_-\phi_1 = \phi_2, \tag{7.7c}$$

and a constraint equation

$$\tau\partial_\tau\phi_1 - \rho\partial_\rho\phi_1 + \frac{1}{2}\big(\partial_-\phi_0 - \partial_+\phi_2\big) = \frac{1}{2}\big(\overline{\varpi}\phi_2 - \varpi\phi_0\big). \tag{7.7d}$$

One can systematically solve the above equations decomposing the fields ϕ_0, ϕ_1, ϕ_2 in spin-weighted spherical harmonics. To do so, one has to rewrite these equations in terms of the $\eth$ and $\bar{\eth}$ operators of Newman and Penrose. Using equation (7.106) of Appendix 7.10 and the fact that ϕ_0, ϕ_1 and ϕ_2 have spin weights 1, 0 and -1, respectively, one finds that equations (7.7a)-(7.7d) can be rewritten as the following evolution equations

$$(1+\tau)\partial_\tau\phi_0 - \rho\partial_\rho\phi_0 + \eth\phi_1 = -\phi_0, \tag{7.8a}$$

$$\partial_\tau\phi_1 + \frac{1}{2}\big(\eth\phi_2 + \bar{\eth}\phi_0\big) = 0, \tag{7.8b}$$

$$(1-\tau)\partial_\tau\phi_2 + \rho\partial_\rho\phi_2 + \bar{\eth}\phi_1 = \phi_2, \tag{7.8c}$$

and the constraint equation

$$\tau\partial_\tau\phi_1 - \rho\partial_\rho\phi_1 + \frac{1}{2}\big(\eth\phi_2 - \bar{\eth}\phi_0\big) = 0. \tag{7.8d}$$

7.3.2 The transport equations for the electromagnetic field on the cylinder at spatial infinity

In order to analyse the behaviour of solutions of the Maxwell equations in a neighbourhood of the cylinder at spatial infinity one assumes that ϕ_0, ϕ_1 and ϕ_2 are smooth functions of τ and ρ. Moreover, taking into account equation (7.108) of Appendix 7.10 one makes the Ansatz:

Assumption 1. The components of the Maxwell field admit a Taylor-like expansion around $\rho = 0$ of the form

$$\phi_n = \sum_{p=|1-n|}^{\infty} \sum_{\ell=|1-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p;\ell,m}(\tau) Y_{1-n;\ell-1,m} \rho^p, \tag{7.9}$$

where $a_{n,p;\ell m} : \mathbb{R} \to \mathbb{C}$ and with $n = 0, 1, 2$.

Remark 52. Recalling that $Y_{s';\ell',m'} = 0$ for $l' < |s'|$, one notices that the lowest order in the expansion for ϕ_0 is $\mathcal{O}(\rho^2)$. This observation will play a role in Section 7.6 when the electromagnetic NP constants are computed in terms of the initial data. Expression (7.9) is not the most general Ansatz which is compatible with the Maxwell constraints. However, more general expansions, like

$$\phi_n = \sum_{p=|1-n|}^{\infty} \sum_{\ell=|1-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p;\ell,m}(\tau) Y_{1-n;\ell,m} \rho^p,$$

which follow from general multipolar expansions in electrostatics and magnetostatics and allow for higher harmonics at each order in p can be seen to have, in general, divergent Newman-Penrose constants.

To simplify the notation of the subsequent analysis let

$$\phi_n^{(p)} \equiv \left. \frac{\partial^p \phi_n}{\partial \rho^p} \right|_{\rho=0}, \tag{7.10}$$

with $n = 0, 1, 2$. Formally differentiating equations (7.8a)-(7.8d) with respect to ρ and evaluating at the cylinder I one obtains

$$(1+\tau)\dot{\phi}_0^p - (p-1)\phi_0^p + \eth\phi_1^p = 0, \tag{7.11a}$$

$$\dot{\phi}_1^{(p)} + \frac{1}{2}\big(\eth\phi_2^{(p)} + \bar{\eth}\phi_0^{(p)}\big) = 0, \tag{7.11b}$$

$$(1-\tau)\dot{\phi}_2^{(p)} + (p-1)\phi_2^{(p)} + \bar{\eth}\phi_1^{(p)} = 0, \tag{7.11c}$$

$$\tau\dot{\phi}_1^{(p)} - p\phi_1^{(p)} + \frac{1}{2}\big(\eth\phi_2^{(p)} - \bar{\eth}\phi_0^{(p)}\big) = 0, \tag{7.11d}$$

where the dot denotes a derivative respect to τ. Using equations (7.109a)-(7.109b) of Appendix 7.10 and the expansions encoded in equation (7.9) one obtains the following equations for $a_{n,p;\ell m}$:

$$(1+\tau)\dot{a}_{0,p;\ell m} + \sqrt{\ell(\ell+1)}a_{1,p;\ell m} - (p-1)a_{0,p;\ell m} = 0, \tag{7.12}$$

$$\dot{a}_{1,p;\ell m} + \frac{1}{2}\sqrt{\ell(\ell+1)}(a_{2,p;\ell m} - a_{0,p;\ell m}) = 0, \tag{7.13}$$

$$(1-\tau)\dot{a}_{2,p;\ell m} - \sqrt{\ell(\ell+1)}a_{1,p;\ell m} + (p-1)a_{2,p;\ell,m} = 0, \tag{7.14}$$

$$\tau\dot{a}_{1,p;\ell m} - \frac{1}{2}\sqrt{\ell(\ell+1)}(a_{2,p;\ell m} + a_{0,p;\ell m}) - pa_{1,p;\ell m} = 0, \tag{7.15}$$

for $p \geq 1$, $1 \leq \ell \leq p$, $-\ell \leq m \leq \ell$. Notice that equations (7.12)-(7.15) correspond, essentially, to the homogeneous part of the equations reported in [96]. Furthermore, $a_{1,p;\ell,m}$ can be solved from (7.13) and (7.15) in terms of $a_{0,p;\ell m}$ and $a_{2,p;\ell,m}$ to obtain

$$a_{1,p;\ell m} = \frac{\sqrt{\ell(\ell+1)}}{2p}\big((1-\tau)a_{2,p;\ell.m} + (1+\tau)a_{0,p;\ell,m}\big). \tag{7.16}$$

Substituting $a_{1,p;\ell,m}$ as given in (7.16) into equations (7.12) and (7.14) one obtains

$$(1+\tau)\dot{a}_{0,p;\ell,m} + \Big(\frac{1}{2p}\ell(\ell+1)(1+\tau) - (p-1)\Big)a_{0,p;\ell,m} + \frac{1}{2p}\ell(\ell+1)(1-\tau)a_{2,p;\ell m} = 0, \tag{7.17a}$$

$$(1-\tau)\dot{a}_{2,p;\ell m} - \frac{1}{2p}\ell(\ell+1)(1+\tau)a_{0,p;\ell m} - \Big(\frac{1}{2p}\ell(\ell+1)(1-\tau) - (p-1)\Big)a_{2,p;\ell m} = 0. \tag{7.17b}$$

At this point one can follow the procedure discussed in [96] to obtain a fundamental matrix for the system (7.17a)-(7.17b): a direct computation shows that one can decouple the last system of first order equations and obtain the second order equations

$$(1-\tau^2)\ddot{a}_{0,p;\ell,m} + 2(1-(1-p)\tau)\dot{a}_{0,p;\ell,m} + (p+\ell)(\ell-p+1)a_{0,p;\ell,m} = 0, \tag{7.18a}$$

$$(1-\tau^2)\ddot{a}_{2,p;\ell,m} - 2(1+(1-p)\tau)\dot{a}_{2,p;\ell,m} + (p+\ell)(\ell-p+1)a_{2,p;\ell,m} = 0. \tag{7.18b}$$

Dropping temporarily the subindices p, ℓ, m observe that, if $a_2(\tau)$ solves (7.18b) then $a_2(-\tau)$ solves equation (7.18a). Equations (7.18a)-(7.18b) are particular examples of *Jacobi ordinary differential equations.* Following the discussion of [96] one obtains the following:

Proposition 7. *For $p \geq 2$, $\ell < p$, $-\ell \leq m \leq \ell$ the solutions to the Jacobi equations* (7.18a)-(7.18b) *are polynomial in τ. For $p \geq 2$, $\ell = p$, $-p \leq m \leq p$ one has*

$$a_{0,p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{p+1}\left(\frac{1+\tau}{2}\right)^{p-1}\left(C_{p,m} + C^{\circledast}_{p,m}\int_0^\tau \frac{ds}{(1+s)^p(1-s)^{p+2}}\right), \tag{7.19a}$$

$$a_{2,p;p,m}(\tau) = \left(\frac{1+\tau}{2}\right)^{p+1}\left(\frac{1-\tau}{2}\right)^{p-1}\left(D_{p,m} + D^{\circledast}_{p,m}\int_0^\tau \frac{ds}{(1-s)^p(1+s)^{p+2}}\right). \tag{7.19b}$$

where $C_{p,m}$, $C^{\circledast}_{p,m}$ and $D_{p,m}$, $D^{\circledast}_{p,m}$ are integration constants.

Remark 53. Observe that, for non-vanishing $C^{\circledast}_{p,m}$ and $D^{\circledast}_{p,m}$, the solutions $a_{0,p;p,m}(\tau)$ and $a_{2,p;p,m}(\tau)$ with $p \geq 2$, $-p \leq m \leq p$, contain terms which diverge logarithmically near $\tau = \pm 1$.

Remark 54. The expressions of Proposition 7 are solutions to the Jacobi equations.

To obtain a solution to the original system it is necessary to evaluate these expressions in the coupled system (7.17a)-(7.17b). In turn, this gives rise to restrictions on the integration constants.

Remark 55. The convergence of the expansions encoded in (7.9) follows from the results of [97].

7.3.3 Initial data for the Maxwell equations

Evaluating the constraint equation (7.8d) at $\tau = 0$ gives the following equation

$$\rho\partial_\rho\phi_1 - \frac{1}{2}\big(\eth\phi_2 - \bar{\eth}\phi_0\big) = 0. \tag{7.20}$$

Consistent with the expressions encoded in equation (7.9) one considers on the initial hypersurface $\mathcal{S}$ fields $\phi_n|_\mathcal{S}$, with $n = 0, 1, 2$, which can be expanded as

$$\phi_n|_\mathcal{S} = \sum_{p=|1-n|}^{\infty}\sum_{\ell=|1-n|}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}a_{n,p;\ell,m}(0)Y_{1-n;\ell-1m}\rho^p. \tag{7.21}$$

Observe that once $a_{0,p;\ell,m}(0)$ and $a_{2,p;\ell,m}(0)$ are given, $a_{1,p;\ell,m}(0)$ is already determined by virtue of equation (7.16) as

$$a_{1,p;\ell m}(0) = \frac{\sqrt{\ell(\ell+1)}}{2p}\big(a_{2,p;\ell.m}(0) + a_{0,p;\ell,m}(0)\big).$$

In addition, observe that equations (7.17a)-(7.17b) are first order while equations (7.18a)-(7.18b) are second order. Consequently, the initial data $\dot{a}_{0,p,\ell,m}(0)$ and $\dot{a}_{2,p,\ell,m}(0)$ are determined, by virtue of equations (7.17a)-(7.17b) restricted to $\mathcal{S}$, by the initial data $a_{0,p,\ell,m}(0)$ and $a_{2,p,\ell,m}(0)$.

The following remark plays an important role for the subsequent discussion of the electromagnetic NP constants:

Remark 56. For general p, ℓ and m, the free data is encoded in $a_{0,p,\ell,m}(0)$ and $a_{2,p;\ell,m}(0)$. Nevertheless, for $p = \ell$, a direct substitution of the solution (7.19a)-(7.19b) into equations (7.17a)-(7.17b) shows that $C^{\circledast}_{p,m} = D^{\circledast}_{p,m} = 0$ with $p \geq 2$, $-p \leq m \leq p$. Consequently, the potentially divergent terms in expressions (7.19a)-(7.19b) do not contribute to the electromagnetic field. Additionally, one has that

$$a_{0,p;p,m}(0) = a_{2,p;p,m}(0) = C_{p,m} = D_{p,m}, \tag{7.22}$$

with $p \geq 2$, $-p \leq m \leq p$. Observe that the initial data considered is generic and the restriction (7.22) is a consequence of the interplay of the evolution and constraint equations. In other words, this condition does not arise from restricting the class of initial data.

7.4 The massless spin-2 field equations in the F-gauge

In Section 7.3 the Maxwell equations (in the F-gauge) were discussed, these correspond in spinorial formalism to the spin-1 equations. In this section, a similar analysis is performed for a spin-2 field propagating on the Minkowski spacetime. As discussed in [98] the spin-2 equations on the Minkowski spacetime can be used to describe the linearised gravitational field. In [98] these equations were written in terms the lifts of the relevant structures to the extended bundle space. In this section, following the spirit of this chapter, the equations will be discussed without making use of these structures. In a similar way as in the electromagnetic case studied in Section 7.3, after rewriting the equations in terms of the $\eth$ and $\bar{\eth}$ operators, a general solution is obtained by expanding the fields in spin-weighted spherical harmonics. The resulting equations for the coefficients of the expansion satisfy ordinary differential equations which can be explicitly solved in terms of special functions.

7.4.1 The spin-2 equation

As discussed in [98], the linearised gravitational field over the Minkowski spacetime can be described with the so-called massless spin-2 field equation

$$\nabla_{A'}{}^{A}\phi_{ABCD} = 0. \tag{7.23}$$

Following an approach analogous to the one described in Section 7.3.1 for the electromagnetic field, it can be shown that equation (7.23) implies the following evolution equations for the components of the spinor ϕ_{ABCD}

$$(1+\tau)\partial_\tau\phi_0 - \rho\partial_\rho\phi_0 - \partial_+\phi_1 + \bar{\varpi}\phi_1 = -2\phi_0, \tag{7.24a}$$
$$\partial_\tau\phi_1 - \frac{1}{2}\partial_+\phi_2 - \frac{1}{2}\partial_-\phi_0 - \varpi\phi_0 = -\phi_1, \tag{7.24b}$$
$$\partial_\tau\phi_2 - \frac{1}{2}\partial_-\phi_1 - \frac{1}{2}\partial_+\phi_3 - \frac{1}{2}\varpi\phi_1 - \frac{1}{2}\bar{\varpi}\phi_3 = 0, \tag{7.24c}$$
$$\partial_\tau\phi_3 - \frac{1}{2}\partial_+\phi_4 - \frac{1}{2}\partial_-\phi_2 - \bar{\varpi}\phi_4 = \phi_3, \tag{7.24d}$$
$$(1-\tau)\partial_\tau\phi_4 + \rho\partial_\rho\phi_4 - \partial_-\phi_3 + \varpi\phi_3 = 2\phi_4, \tag{7.24e}$$

and the constraint equations

$$\tau\partial_\tau\phi_1 - \rho\partial_\rho\phi_1 - \frac{1}{2}\partial_+\phi_2 + \frac{1}{2}\partial_-\phi_0 + \varpi\phi_0 = 0, \tag{7.25a}$$
$$\tau\partial_\tau\phi_2 - \rho\partial_\rho\phi_2 - \frac{1}{2}\partial_+\phi_3 + \frac{1}{2}\partial_-\phi_1 - \frac{1}{2}\bar{\varpi}\phi_3 + \frac{1}{2}\varpi\phi_1 = 0, \tag{7.25b}$$
$$\tau\partial_\tau\phi_3 - \rho\partial_\rho\phi_3 - \frac{1}{2}\partial_+\phi_4 + \frac{1}{2}\partial_-\phi_2 - \bar{\varpi}\phi_4 = 0, \tag{7.25c}$$

where the five components $\phi_0, \phi_1, \phi_2, \phi_3$ and ϕ_4, given by

$$\phi_0 \equiv \phi_{ABCD}o^A o^B o^C o^D, \qquad \phi_1 \equiv \phi_{ABCD}o^A o^B o^C \iota^D,$$
$$\phi_2 \equiv \phi_{ABCD}o^A o^B \iota^C \iota^D, \qquad \phi_3 \equiv \phi_{ABCD}o^A \iota^B \iota^C \iota^D,$$
$$\phi_4 \equiv \phi_{ABCD}\iota^A \iota^B \iota^C \iota^D,$$

have spin weight of $2, 1, 0, -1, -2$ respectively. Taking into account this observation and equations (7.106) and (7.107) given in Appendix 7.10 one can rewrite (7.24a)-(7.25c) in terms of the $\eth$ and $\bar{\eth}$ as done for the electromagnetic case. A direct computation renders the following evolution equations

$$(1+\tau)\partial_\tau\phi_0 - \rho\partial_\rho\phi_0 + \eth\phi_1 = -2\phi_0, \tag{7.26a}$$
$$\partial_\tau\phi_1 + \frac{1}{2}\bar{\eth}\phi_0 + \frac{1}{2}\eth\phi_2 = -\phi_1, \tag{7.26b}$$
$$\partial_\tau\phi_2 + \frac{1}{2}\bar{\eth}\phi_1 + \frac{1}{2}\eth\phi_3 = 0, \tag{7.26c}$$
$$\partial_\tau\phi_3 + \frac{1}{2}\bar{\eth}\phi_2 + \frac{1}{2}\eth\phi_4 = \phi_3, \tag{7.26d}$$
$$(1-\tau)\partial_\tau\phi_4 + \rho\partial_\rho\phi_4 + \bar{\eth}\phi_3 = 2\phi_4, \tag{7.26e}$$

and the constraint equations

$$\tau\partial_\tau\phi_1 - \rho\partial_\rho\phi_1 + \frac{1}{2}\eth\phi_2 - \frac{1}{2}\bar{\eth}\phi_0 = 0, \tag{7.27a}$$
$$\tau\partial_\tau\phi_2 - \rho\partial_\rho\phi_2 + \frac{1}{2}\eth\phi_3 - \frac{1}{2}\bar{\eth}\phi_1 = 0, \tag{7.27b}$$
$$\tau\partial_\tau\phi_3 - \rho\partial_\rho\phi_3 + \frac{1}{2}\eth\phi_4 - \frac{1}{2}\bar{\eth}\phi_2 = 0. \tag{7.27c}$$

With the equations already written in this way, one can follow the discussion of [98] for parametrising the solutions to equations (7.26a)-(7.27c).

7.4.2 The transport equations for the massless spin-2 field on the cylinder at spatial infinity

One proceeds in analogous way as in the electromagnetic case and assumes that the fields ϕ_n with $n = 0, 1, 2, 3, 4$, are smooth functions of τ and ρ. Taking into account equation (7.108) of Appendix 7.10, it is assumed one can express the components of the linearised gravitational field in a Taylor-like expansion around $\rho = 0$. More precisely, we make the Ansatz:

Assumption 2. *In what follows it is assumed that the components of the spin-2* *field have the expansions*

$$\phi_n = \sum_{p=|2-n|}^{\infty} \sum_{\ell=|2-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p;\ell,m}(\tau) Y_{2-n;\ell-1,m} \rho^p \tag{7.28}$$

where $a_{n,p;\ell,m} : \mathbb{R} \to \mathbb{C}$ *and* $n = 0, \ldots, 4$.

Remark 57. Recalling that $Y_{s';\ell',m'} = 0$ for $l' < |s'|$ then one notices that the lowest order in the expansion for ϕ_0 is $\mathcal{O}(\rho^3)$. As in the case of the spin-1 field, one can consider more general expressions which are compatible with the spin-2 constraints which admit higher harmonics at every order. Some experimentation reveals, however, that these more general expansions lead to divergent NP constants —cf. Remark 52.

For the remaining part of this section, the p-th derivative respect to ρ of the fields ϕ_n with $n = 0, 1, 2, 3, 4$ evaluated at the cylinder I, is denoted using the same notation as in equation (7.10). Then, by formally differentiating equations (7.26a)-(7.27c) respect to ρ and evaluating at the cylinder I, one obtains the equations

$$(1+\tau)\partial_\tau \phi_0^{(p)} + \eth \phi_1^{(p)} (p-2)\phi_0^{(p)} = 0, \tag{7.29a}$$

$$\partial_\tau \phi_1^{(p)} + \frac{1}{2}\bar{\eth}\phi_0^{(p)} + \frac{1}{2}\eth\phi_2^{(p)} + \phi_1^{(p)} = 0, \tag{7.29b}$$

$$\partial_\tau \phi_2 + \frac{1}{2}\bar{\eth}\phi_1^{(p)} + \frac{1}{2}\eth\phi_3^{(p)} = 0, \tag{7.29c}$$

$$\partial_\tau \phi_3 + \frac{1}{2}\bar{\eth}\phi_2^{(p)} + \frac{1}{2}\eth\phi_4^{(p)} - \phi_3^{(p)} = 0, \tag{7.29d}$$

$$(1-\tau)\partial_\tau \phi_4^{(p)} + \bar{\eth}\phi_3^{(p)} + (p-2)\phi_4^{(p)} = 0, \tag{7.29e}$$

and

$$\tau\partial_\tau \phi_1 + \frac{1}{2}\eth\phi_2^{(p)} - \frac{1}{2}\bar{\eth}\phi_0^{(p)} - p\phi_1^{(p)} = 0, \tag{7.30a}$$

$$\tau\partial_\tau \phi_2 + \frac{1}{2}\eth\phi_3^{(p)} - \frac{1}{2}\bar{\eth}\phi_1^{(p)} - p\phi_2^{(p)} = 0, \tag{7.30b}$$

$$\tau\partial_\tau \phi_3 + \frac{1}{2}\eth\phi_4^{(p)} - \frac{1}{2}\bar{\eth}\phi_2^{(p)} - p\phi_3^{(p)} = 0. \tag{7.30c}$$

The last set of equations along with the expansion (7.28), in turn, imply the following equations for $a_{n,p;\ell,m}$ with $p \geq 2$ and $2 \leq \ell \leq p$:

$$(1+\tau)\dot{a}_0 + \lambda_1 a_1 - (p-2)a_0 = 0, \tag{7.31a}$$

$$\dot{a}_1 - \frac{1}{2}\lambda_1 a_0 + \frac{1}{2}\lambda_0 a_2 + a_1 = 0, \tag{7.31b}$$

$$\dot{a}_2 - \frac{1}{2}\lambda_0 a_1 + \frac{1}{2}\lambda_0 a_3 = 0, \tag{7.31c}$$

$$\dot{a}_3 - \frac{1}{2}\lambda_0 a_2 + \frac{1}{2}\lambda_1 a_4 - a_3 = 0, \tag{7.31d}$$

$$(1-\tau)\dot{a}_4 - \lambda_1 a_3 + (p-2)a_4 = 0, \tag{7.31e}$$

and

$$\tau\dot{a}_1 + \frac{1}{2}\lambda_0 a_2 + \frac{1}{2}\lambda_1 a_0 - p a_1 = 0, \tag{7.32a}$$
$$\tau\dot{a}_2 + \frac{1}{2}\lambda_0 a_3 + \frac{1}{2}\lambda_0 a_1 - p a_2 = 0, \tag{7.32b}$$
$$\tau\dot{a}_3 + \frac{1}{2}\lambda_1 a_4 + \frac{1}{2}\lambda_0 a_2 - p a_3 = 0, \tag{7.32c}$$

where $\lambda_1 \equiv \sqrt{(\ell-1)(\ell+2)}$ and $\lambda_0 \equiv \sqrt{\ell(\ell+1)}$ and the labels $p;\ell,m$ have been suppressed for conciseness. From equations (7.31b)-(7.31d) and (7.32a)-(7.32c) one obtains an algebraic system which can be written as

$$\begin{bmatrix} p+\tau & -\frac{1}{2}(1-\tau)\lambda_0 & 0 \\ -\frac{1}{2}(1+\tau)\lambda_0 & p & -\frac{1}{2}(1-\tau)\lambda_0 \\ 0 & -\frac{1}{2}(1+\tau)\lambda_0 & p-\tau \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{2}\lambda_1 \begin{bmatrix} (1+\tau)a_0 \\ 0 \\ (1-\tau)a_4 \end{bmatrix}. \tag{7.33}$$

Solving the above system and substituting a_0, a_1 and a_3 written in terms of a_0 and a_4 into equations (7.31a) and (7.31e) one obtains

$$(1+\tau)\dot{a}_0 + (-(p-2) + f(\tau,p,\ell))a_0 + g(\tau,p,\ell)a_4 = 0, \tag{7.34a}$$
$$(1-\tau)\dot{a}_4 + (-(p-2) + f(-\tau,p,\ell))a_4 + g(-\tau,p,\ell)a_0 = 0, \tag{7.34b}$$

where

$$f(\tau,p,\ell) \equiv \frac{(1+\tau)(\ell-1)(\ell+2)[4p^2 - 4p\tau + \ell(\ell+1)(\tau^2-1)]}{4p(2p^2 - \ell(\ell+1) + (\ell-1)(\ell+2)\tau^2)},$$
$$g(\tau,p,\ell) \equiv \frac{(1-\tau)^3\ell(\ell+1)(\ell-1)(\ell+2)}{4p(2p^2 - \ell(\ell+1) + (\ell-1)(\ell+2)\tau^2)}.$$

Together, the last equations entail the decoupled equations

$$(1-\tau^2)\ddot{a}_0 + (4 + 2(p-1)\tau)\dot{a}_0 + (p+\ell)(p-\ell+1)a_0 = 0, \tag{7.35a}$$
$$(1-\tau^2)\ddot{a}_4 + (-4 + 2(p-1)\tau)\dot{a}_4 + (p+\ell)(p-\ell+1)a_4 = 0. \tag{7.35b}$$

It can be verified that if $a_0(\tau)$ solves (7.35a) then $a_0(-\tau)$ solves equation (7.35b). As in the electromagnetic case, these equations are Jacobi ordinary differential equations. For the solutions to these equations one has the following:

Proposition 8. *For $p \geq 3$, $p > \ell$, $-\ell \leq m \leq \ell$ the solutions to equations* (7.35a)-(7.35b) *are polynomial. For $p \geq 3$, $p = \ell$, $-p \leq m \leq p$ one has*

$$a_{0,p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{p+2}\left(\frac{1+\tau}{2}\right)^{p-2}\left(C_{p,m} + C^{\circledast}_{p,m}\int_0^{\tau}\frac{ds}{(1+s)^{p-1}(1-s)^{p+3}}\right), \tag{7.36a}$$

$$a_{4,p;p,m}(\tau) = \left(\frac{1+\tau}{2}\right)^{p+2}\left(\frac{1-\tau}{2}\right)^{p-2}\left(D_{p,m} + D^{\circledast}_{p,m}\int_0^{\tau}\frac{ds}{(1-s)^{p-1}(1+s)^{p+3}}\right). \tag{7.36b}$$

where $C_{p,m}$, $C^{\circledast}_{p,m}$ and $D_{p,m}$, $D^{\circledast}_{p,m}$ are integration constants.

Remark 58. Notice that for non-vanishing $C^{\circledast}_{p,m}$ and $D^{\circledast}_{p,m}$ the above solution diverges logarithmically near $\tau = \pm 1$. The expressions of Proposition 8 are solutions to the Jacobi equations. To obtain a solution to the original system it is necessary to evaluate these expressions in the coupled system (7.34a)-(7.34b). In turn, this shows that the integration constants are not independent of each other.

Remark 59. The convergence of the expansions (7.28) follows from the results given in [98].

7.4.3 Initial data for the spin-2 equations

Consistent with equations (7.28) one considers on the initial hypersurface $\mathcal{S}$ fields $\phi_n|_{\mathcal{S}}$, with $n = 0,1,2,3,4$ which can be expanded as

$$\phi_n|_{\mathcal{S}} = \sum_{p=|2-n|}^{\infty}\sum_{\ell=|2-n|}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}a_{n,p;\ell,m}(0)Y_{2-n;\ell-1m}\rho^p. \tag{7.37}$$

Observe that, by virtue of equation (7.33), the initial data $a_{1,p;\ell,m}(0)$, $a_{2,p;\ell,m}(0)$ and $a_{3,p;\ell,m}(0)$ are determined by $a_{0,p,\ell,m}(0)$ and $a_{4,p,\ell,m}(0)$. In addition, notice that, equations (7.34a)-(7.34b) are first order while equations (7.35a)-(7.35b) are second order. Therefore, the initial data $\dot{a}_{0,p,\ell,m}(0)$ and $\dot{a}_{4,p,\ell,m}(0)$ are determined, as a consequence of equations (7.34a)-(7.34b) restricted to $\mathcal{S}$, by the initial data $a_{0,p,\ell,m}(0)$ and $a_{4,p,\ell,m}(0)$. The following remarks plays an important role for the subsequent discussion of the spin-2 NP constants:

Remark 60. For general p, ℓ and m, the free initial data is encoded in $a_{0,p,\ell,m}(0)$ and $a_{4,p;\ell,m}(0)$. However, for $p = \ell$, a direct substitution of the solution (7.36a)-(7.36b) into equations (7.34a)-(7.34b) shows that $C^{\circledast}_{p,m} = D^{\circledast}_{p,m}$ and $C_{p,m} = D_{p,m}$. In other words, for $p \geq 3$, $-p \leq m \leq p$,

$$a_{0,p,p,m}(0) = a_{4,p,p,m}(0). \tag{7.38}$$

Remark 61. In contrast with the electromagnetic case, in principle, initial data with $C^{\circledast}_{p,m} = D^{\circledast}_{p,m} \neq 0$, is admissible and consequently, for generic initial data the appearance of logarithmic singularities is expected. Nevertheless, for the computation of the NP constants $C^{\circledast}_{p,m} = D^{\circledast}_{p,m} = 0$ will be assumed —otherwise the expressions defining the NP constants diverge —see Section 7.7.

Remark 62. The solutions to the constraint equations correspond, in tensor frame notation to solutions to the equation

$$D^i \phi_{ij} = 0,$$

where D_i denotes the covariant Levi-Civita derivative of the metric h_{ij} intrinsic to the initial hypersurface $\mathcal{S}$ and ϕ_{ij} corresponds to the tensorial counterpart of the field ϕ_{ABCD}. In the conformally flat setting, the solutions to these equations are known —see [42]. Moreover, in the latter reference, a general parametrisation to the solutions to this equation was given. Consequently, one could, in principle, rewrite the initial data considered in this section using this parametrisation.

7.5 The NP-gauge

In this section, an adapted frame satisfying the NP-gauge conditions and Bondi coordinates are constructed for the conformal extension $(\mathcal{M}, \boldsymbol{g}_I)$ introduced in Section 7.2.1. For convenience of the reader, a general discussion of the NP-gauge conditions and the construction of Bondi coordinates is provided in the first part of this section.

7.5.1 The NP-gauge conditions and Bondi coordinates

This section provides a general discussion of the NP-gauge conditions and the construction of Bondi coordinates. A more comprehensive discussion of these gauge conditions and their consequences can be found in [12, 13, 81].

Let $(\mathcal{M}, \boldsymbol{g}, \Xi)$ denote a conformal extension of an asymptotically simple spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ where $\tilde{\boldsymbol{g}}$ satisfies the vacuum Einstein field equations with vanishing Cosmological constant. It is a general result in the theory of asymptotics that for vacuum spacetimes with vanishing Cosmological constant the conformal boundary $\mathscr{I}$, with locus given by $\Xi = 0$, consists of two disjoint null hypersurfaces $\mathscr{I}^+$ and $\mathscr{I}^-$ each one having the topology of $\mathbb{R} \times \mathbb{S}^2$—see [12, 13]. In this section the discussion will be particularised to $\mathscr{I}^+$. Nevertheless, the time dual results and constructions can be formulated for $\mathscr{I}^-$ in an analogous manner. To simplify the notation, the symbol $\simeq$ will be used to denote equality at $\mathscr{I}$, e.g. if w is a scalar field on $\mathcal{M}$ that vanishes at $\mathscr{I}$ one writes $w \simeq 0$. Let $\{\check{\boldsymbol{e}}_{\boldsymbol{AA'}}\}$ denote a frame satisfying $\boldsymbol{g}(\check{\boldsymbol{e}}_{\boldsymbol{AA'}}, \check{\boldsymbol{e}}_{\boldsymbol{BB'}}) = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}$ in a neighbourhood $\mathcal{U} \subset \mathcal{M}$ of $\mathscr{I}^+$. Additionally, let $\check{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$ denote the reduced connection coefficients of the Levi-Civita connection of

$\boldsymbol{g}$ defined with respect to $\check{\boldsymbol{e}}_{\boldsymbol{AA'}}$. The frame $\check{\boldsymbol{e}}_{\boldsymbol{AA'}}$ is an adapted frame at $\mathscr{I}^+$ if the following conditions hold:

(i) The vector $\check{\boldsymbol{e}}_{\boldsymbol{11'}}$ is tangent to and parallely propagated along $\mathscr{I}^+$, i.e.,

$$\check{\nabla}_{\boldsymbol{11'}}\check{\boldsymbol{e}}_{\boldsymbol{11'}} \simeq 0.$$

(ii) On $\mathcal{U}$ there exists a smooth function u inducing an affine parameter on the null generators of $\mathscr{I}^+$, namely $\check{\boldsymbol{e}}_{\boldsymbol{11'}}(u) \simeq 1$. The vector $\check{\boldsymbol{e}}_{\boldsymbol{00'}}$ is then defined as $\check{\boldsymbol{e}}_{\boldsymbol{00'}} = \boldsymbol{g}(\mathbf{d}u, \cdot)$ so that it is tangent to the null generators of the hypersurfaces transverse to $\mathscr{I}$ defined by

$$\mathcal{N}_{u_\circ} \equiv \{p \in \mathcal{U} \mid u(p) = u_\circ\},$$

with constant $u_\circ$.

(iii) The frame $\{\check{\boldsymbol{e}}_{\boldsymbol{AA'}}\}$ is tangent to the cuts $\mathcal{C}_{u_\circ} \equiv \mathcal{N}_{u_\circ} \cap \mathscr{I}^+ \approx \mathbb{S}^2$ and parallely propagated along $\mathcal{N}_{u_\circ}$, namely

$$\check{\nabla}_{\boldsymbol{00'}}\check{\boldsymbol{e}}_{\boldsymbol{AA'}} = 0 \qquad \text{on} \qquad \mathcal{N}_{u_\circ}.$$

Conditions (i)-(iii) can be encoded in the following requirements on the reduced connection coefficients $\check{\Gamma}_{\boldsymbol{AA'CD}}$:

Proposition 9 (***adapted frame at*** $\mathscr{I}^+$)**.** *Let* $(\mathcal{M}, \boldsymbol{g}, \Xi)$ *be a conformal extension of an asymptotically simple spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *with vanishing Cosmological constant. On a neighbourhood* $\mathcal{U} \subset \mathcal{M}$ *of* $\mathscr{I}^+$ *it is always possible to find a* $\boldsymbol{g}$*-null frame* $\{\check{\boldsymbol{e}}_{\boldsymbol{AA'}}\}$ *for which*

$$\check{\Gamma}_{\boldsymbol{10'11}} \simeq 0, \qquad \check{\Gamma}_{\boldsymbol{11'11}} \simeq 0, \tag{7.39a}$$

$$\check{\Gamma}_{\boldsymbol{1000}} = \bar{\bar{\check{\Gamma}}}_{\boldsymbol{1'0'0'0'}}, \qquad \check{\Gamma}_{\boldsymbol{11'00}} = \bar{\bar{\check{\Gamma}}}_{\boldsymbol{1'00'1'}} + \check{\Gamma}_{\boldsymbol{01'01}}, \qquad \check{\Gamma}_{\boldsymbol{00'AB}} = 0 \quad \textit{on} \quad \mathcal{U}. \tag{7.39b}$$

The conformal freedom of the setting, i.e. the fact that instead of $(\mathcal{M}, \boldsymbol{g}, \Xi)$ one can consider $(\mathcal{M}', \boldsymbol{g}', \Xi')$ with

$$\boldsymbol{g}' \mapsto \theta^2 \boldsymbol{g}, \qquad \Xi \mapsto \Xi' = \theta\Xi,$$

can be exploited to obtain an *improved* frame $\boldsymbol{e}'_{\boldsymbol{AA'}}$ leading to further simplifications to the conditions given in Proposition 9. If in addition, one introduces an arbitrary function $\varkappa$ constant along the generators of $\mathscr{I}^+$ and sets

$$\boldsymbol{e}'_{\boldsymbol{11'}} \simeq \theta^{-2}\varkappa\check{\boldsymbol{e}}_{\boldsymbol{11'}}, \qquad \text{on} \qquad \mathscr{I}^+, \tag{7.40}$$

one is lead to define an affine parameter $u'(u)$ such that $\boldsymbol{e}'_{\boldsymbol{11'}}$ is parallely propagated and $\boldsymbol{e}'_{\boldsymbol{11'}}(u') = 1$. This in turn implies $\mathrm{d}u'/\mathrm{d}u = \kappa^{-1}\theta^2$ which, integrating along the null generators of $\mathscr{I}^+$, renders

$$u'(u) = \frac{1}{\varkappa}\int_{u_\star}^{u} \theta^2(s)\mathrm{d}s + u'_\star, \tag{7.41}$$

where the integration constants $u_\star$ and $u'_\star$ identify a fiduciary cut $\mathcal{C}_\star \equiv \mathcal{C}_{u_\star}$. Observing that equation (7.40) also holds on $\mathcal{C}_\star$, one prescribes the remaining part of the frame on $\mathcal{C}_\star$ as

$$\boldsymbol{e}'_{\boldsymbol{00'}} = \varkappa^{-1}\check{\boldsymbol{e}}_{\boldsymbol{00'}}, \qquad \boldsymbol{e}'_{\boldsymbol{01'}} = \theta^{-1}\check{\boldsymbol{e}}_{\boldsymbol{01}}, \qquad \boldsymbol{e}'_{\boldsymbol{10'}} = \theta^{-1}\check{\boldsymbol{e}}_{\boldsymbol{10'}} \qquad \text{on} \qquad \mathcal{C}_\star. \tag{7.42}$$

Observe that, using equations (7.40) and (7.42), it can be verified that $\boldsymbol{g}(\boldsymbol{e}'_{\boldsymbol{AA'}}, \boldsymbol{e}'_{\boldsymbol{BB'}}) = \epsilon_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{BB'}}$ on $\mathcal{C}_\star$. Using these expressions, one can exploit the freedom in choosing $\varkappa$ and θ along with the conformal transformation laws for the relevant fields (connection coefficients and curvature spinors) and a general rotation of $\boldsymbol{e}'_{\boldsymbol{01'}}$ and $\boldsymbol{e}'_{\boldsymbol{10'}}$ of the form

$$\boldsymbol{e}'_{\boldsymbol{01'}} \mapsto e^{\mathrm{i}c}\boldsymbol{e}'_{\boldsymbol{01'}}, \qquad \boldsymbol{e}'_{\boldsymbol{10'}} \mapsto e^{-\mathrm{i}c}\boldsymbol{e}_{\boldsymbol{10'}}, \tag{7.43}$$

where c is a scalar function such that $c = 0$ at $\mathcal{C}_\star$, to obtain a improved frame $\boldsymbol{e}'_{\boldsymbol{AA'}}$ that satisfies the following conditions:

Proposition 10 (***NP-gauge conditions at*** $\mathscr{I}^+$)**.** *Let* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *be an asymptotically simple spacetime. Locally, it is always possible to find a conformal extension* $(\mathcal{M}', \boldsymbol{g}', \Xi')$ *for which there exist a* $\boldsymbol{g}'$*-null frame* $\{\boldsymbol{e}'_{\boldsymbol{AA'}}\}$ *such that the reduced spin connection coefficients of the Levi-Civita connection of* $\boldsymbol{g}'$ *with respect to* $\boldsymbol{e}'_{\boldsymbol{AA'}}$ *satisfy the gauge conditions:*

$$\Gamma'_{\boldsymbol{00'BC}} \simeq 0, \qquad \Gamma'_{\boldsymbol{11'BC}} \simeq 0, \tag{7.44a}$$

$$\Gamma'_{\boldsymbol{01'11}} \simeq 0, \qquad \Gamma'_{\boldsymbol{10'00}} \simeq 0, \qquad \Gamma'_{\boldsymbol{10'11}} \simeq 0, \tag{7.44b}$$

$$\bar{\Gamma}'_{\boldsymbol{1'00'1'}} + \Gamma_{\boldsymbol{01'01}} \simeq 0. \tag{7.44c}$$

Moreover, for the curvature one has

$$R' \simeq 0, \qquad \Phi'_{\boldsymbol{12}} \simeq 0, \qquad \Phi'_{\boldsymbol{22}} \simeq 0. \tag{7.44d}$$

where R' *and* $\Phi'_{\boldsymbol{AA'BB'}}$ *are, respectively, the Ricci scalar and the components (with respect to* $\boldsymbol{e}'_{\boldsymbol{AA'}}$*) of the trace-free Ricci spinor of the Levi-Civita connection of* $\boldsymbol{g}'$*. Additionally,* $\boldsymbol{e}'_{\boldsymbol{00'}}(\Xi')$ *is constant on* $\mathscr{I}^+$*.*

A frame $\boldsymbol{e}'_{\boldsymbol{AA'}}$ satisfying the conditions of Proposition 10 will be said to be a NP-frame. The proof of this proposition can be found in [81] and [13]. The proof, in addition gives a procedure to determine θ and $\varkappa$ by prescribing data on $\mathcal{C}_\star$ which is extended along $\mathscr{I}$ solving ordinary differential equations. Observing equations (7.40), (7.42) and (7.43) one concludes that in general, frames $\check{\boldsymbol{e}}_{\boldsymbol{AA'}}$ and $\boldsymbol{e}'_{\boldsymbol{AA'}}$ of Propositions 9 and 10 respectively, are related via a conformal transformation $\boldsymbol{g}' =$

$\theta^2 \boldsymbol{g}$ and a Lorentz transformation encoded in $(\varkappa, c)$ so that

$$\boldsymbol{e}'_{\boldsymbol{11'}} \simeq \theta^{-2}\varkappa \check{\boldsymbol{e}}_{\boldsymbol{11'}}, \quad \boldsymbol{e}'_{\boldsymbol{00'}} = \varkappa^{-1}\check{\boldsymbol{e}}_{\boldsymbol{00'}}, \quad \boldsymbol{e}'_{\boldsymbol{01'}} = e^{\mathrm{i}c}\theta^{-1}\check{\boldsymbol{e}}_{\boldsymbol{01}}, \quad \boldsymbol{e}'_{\boldsymbol{10'}} = e^{-\mathrm{i}c}\theta^{-1}\check{\boldsymbol{e}}_{\boldsymbol{10'}} \quad \text{on} \quad \mathcal{U}.$$

The function $\varkappa$ corresponds a boost while c encodes a spin.

In the discussion of the NP-gauge, is customary to complete the construction introducing *Bondi coordinates* as follows: choose an arbitrary coordinate system $\vartheta^{\mathfrak{a}}$ with $\mathfrak{a} = 2, 3$ on the cut $\mathcal{C}_\star \approx \mathbb{S}^2$. Extend this coordinate system to $\mathscr{I}^+$ so that they remain constant along its null generators. Recalling that u', as defined in equation (7.41), corresponds to an affine parameter along the generators of $\mathscr{I}^+$ fixed by the condition $\boldsymbol{e}'_{\boldsymbol{11'}}(u') = 1$, is then natural to use as an affine parameter on the hypersurfaces $\mathcal{N}_u$, transverse to $\mathscr{I}^+$, a parameter r' fixed by the conditions $\boldsymbol{e}'_{\boldsymbol{00'}}(r') = 1$ and $r' \simeq 0$. Using r' and u' defined as previously described, $(r', u', \vartheta^{\mathfrak{a}})$ defines a Bondi coordinate system.

7.5.2 The NP frame and Bondi coordinates for the conformal extension $(\mathcal{M}, \boldsymbol{g}_I)$

To implement the procedure described in Section 7.5.1 for the conformal extension $(\mathcal{M}, \boldsymbol{g}_I)$ it is convenient to introduce null coordinates $u = t - \rho$ and $v = t + \rho$. Observe that the unphysical null coordinates u and v are related to the physical null coordinates $\tilde{u} = \tilde{t} - \tilde{\rho}$ and $\tilde{v} = \tilde{t} + \tilde{\rho}$ via $u = -1/\tilde{u}$ and $v = -1/\tilde{v}$. In these coordinates, the metric $\boldsymbol{g}_I$ and conformal factor Ξ read

$$\boldsymbol{g}_I = \frac{1}{2}\left(\mathbf{d}u \otimes \mathbf{d}v + \mathbf{d}v \otimes \mathbf{d}u\right) - \frac{1}{4}(v-u)^2 \boldsymbol{\sigma}, \qquad \Xi = uv.$$

In this representation, future null infinity $\mathscr{I}^+$ is located at $v = 0$ while past null infinity $\mathscr{I}^-$ is located at $u = 0$. Additionally, a $\boldsymbol{g}_I$-null frame is given by

$$\check{\boldsymbol{e}}_{\boldsymbol{00'}} = \sqrt{2}\boldsymbol{\partial}_v, \qquad \check{\boldsymbol{e}}_{\boldsymbol{11'}} = \sqrt{2}\boldsymbol{\partial}_u, \qquad \check{\boldsymbol{e}}_{\boldsymbol{01'}} = \frac{\sqrt{2}}{v-u}\boldsymbol{\partial}_+, \qquad \check{\boldsymbol{e}}_{\boldsymbol{10'}} = \frac{\sqrt{2}}{v-u}\boldsymbol{\partial}_-.$$

A direct computation using the Cartan structure equations, one can verify that the only non-zero spin coefficients are

$$\check{\Gamma}_{\boldsymbol{10'10}} = -\frac{\sqrt{2}}{2}\frac{\varpi}{v-u}, \qquad \check{\Gamma}_{\boldsymbol{01'10}} = \frac{\sqrt{2}}{2}\frac{\overline{\varpi}}{v-u}, \qquad \check{\Gamma}_{\boldsymbol{01'11}} = \check{\Gamma}_{\boldsymbol{10'00}} = -\frac{\sqrt{2}}{v-u}.$$

A direct inspection reveals that the frame $\{\check{\boldsymbol{e}}_{\boldsymbol{AA'}}\}$ does satisfy all the conditions of Proposition 10. In order to construct a frame satisfying the conditions defining the NP gauge one has to introduce a conformal rescaling a Lorentz transformation as follows: consider a conformal rescaling

$$\boldsymbol{g}' = \theta^2 \boldsymbol{g}_I, \tag{7.45}$$

and the following $\boldsymbol{g}'$-null frame,

$$\boldsymbol{e}'_{\boldsymbol{00}} = \varkappa^{-1}\check{\boldsymbol{e}}_{\boldsymbol{00'}}, \quad \boldsymbol{e}'_{\boldsymbol{11'}} = \varkappa\theta^{-2}\check{\boldsymbol{e}}_{\boldsymbol{11'}}, \quad \boldsymbol{e}'_{\boldsymbol{01'}} = \theta^{-1}e^{\mathrm{ic}}\check{\boldsymbol{e}}_{\boldsymbol{01'}}, \quad \boldsymbol{e}'_{\boldsymbol{10'}} = \theta^{-1}e^{\text{-ic}}\check{\boldsymbol{e}}_{\boldsymbol{10'}}. \quad (7.46)$$

Some experimentation reveals that setting

$$\theta = \frac{2}{v-u}, \qquad \varkappa = \frac{4u^2}{(v-u)^2}, \qquad c = 0,$$

one obtains the non-zero spin coefficients

$$\Gamma'_{\boldsymbol{11'10}} = \frac{\sqrt{2}uv}{u-v}, \qquad \Gamma'_{\boldsymbol{10'10}} = -\frac{\sqrt{2}}{4}\varpi, \qquad \Gamma'_{\boldsymbol{01'10}} = \frac{\sqrt{2}}{4}\overline{\varpi}. \qquad (7.47)$$

In addition, observe that $\Gamma'_{\boldsymbol{11'10}} \simeq 0$. A further computation, using the NP-equations as given in [12] and equation (7.102) of Appendix 7.9, shows that

$$R' = 0, \qquad \Phi'_{\boldsymbol{00}} = \Phi'_{\boldsymbol{01}} = \Phi'_{\boldsymbol{02}} = \Phi'_{\boldsymbol{22}} = 0, \quad \Phi'_{\boldsymbol{11}} = \frac{1}{2}.$$

An inspection of the conditions of Proposition 10 shows that $\boldsymbol{e}'_{\boldsymbol{AA'}}$ constitutes a frame in the NP-gauge. To round up the discussion one can introduce Bondi coordinates (r', u') fixed by the requirements

$$\boldsymbol{e}'_{\boldsymbol{00'}}(r') = 1, \qquad \boldsymbol{e}_{\boldsymbol{11'}}(u') = 1, \qquad r' \simeq 0.$$

A direct computation shows that

$$r' = \frac{4}{\sqrt{2}}\Big(\frac{uv}{u-v}\Big), \qquad u' = -\frac{1}{\sqrt{2}}\frac{1}{u}.$$

In these coordinates the frame $\boldsymbol{e}'_{\boldsymbol{AA'}}$ reads

$$\boldsymbol{e}'_{\boldsymbol{00'}} = \boldsymbol{\partial}_{r'}, \qquad \boldsymbol{e}'_{\boldsymbol{11'}} = -\frac{1}{2}r'^2\boldsymbol{\partial}_{r'} + \boldsymbol{\partial}_{u'}, \qquad \boldsymbol{e}'_{\boldsymbol{01'}} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_{+}, \qquad \boldsymbol{e}'_{\boldsymbol{01'}} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_{-}. \quad (7.48)$$

Observe that the Bondi coordinates (r', u') are related to the physical coordinates $\tilde{\rho}$ and $\tilde{u}$, as introduced in Section 7.2.1, through

$$r' = -\frac{2}{\sqrt{2}}\frac{1}{\tilde{\rho}}, \qquad u' = \frac{1}{\sqrt{2}}\tilde{u}.$$

For future reference, notice that in the physical coordinates $(\tilde{\rho}, \tilde{u})$ the NP-frame $\boldsymbol{e}'_{\boldsymbol{AA'}}$ is given by

$$\boldsymbol{e}'_{\boldsymbol{00'}} = \sqrt{2}\tilde{\rho}^2\boldsymbol{\partial}_{\tilde{\rho}}, \quad \boldsymbol{e}'_{\boldsymbol{11'}} = \sqrt{2}\boldsymbol{\partial}_{\tilde{u}} - \sqrt{2}\boldsymbol{\partial}_{\tilde{\rho}}, \quad \boldsymbol{e}'_{\boldsymbol{01'}} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_{+}, \quad \boldsymbol{e}'_{\boldsymbol{01'}} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_{-}. \quad (7.49)$$

7.5.3 Relating the NP-gauge to the F-gauge

In general, a frame in the F-gauge and the NP-gauge will not coincide since, while the former is based on a Cauchy hypersurface, the latter is adapted to $\mathscr{I}$ —see Figure 7.1. However, as $\boldsymbol{g}_C$ and $\boldsymbol{g}'$ are conformally related, $\boldsymbol{g}' = \kappa^2 \boldsymbol{g}_C$, then the frames $\boldsymbol{e}_{\boldsymbol{AA}'}$ and $\boldsymbol{e}'_{\boldsymbol{AA}'}$ are related through a conformal rescaling and a Lorentz transformation

$$\boldsymbol{e}'_{\boldsymbol{AA}'} = \kappa^{-1} \Lambda^{\boldsymbol{B}}{}_{\boldsymbol{A}} \bar{\Lambda}^{\boldsymbol{B}'}{}_{\boldsymbol{A}'} \boldsymbol{e}_{\boldsymbol{BB}'}. \tag{7.50}$$

To determine explicitly κ and $\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{B}}$ observe that the frame $\check{\boldsymbol{e}}_{\boldsymbol{AA}'}$, introduced in Section 7.5.2, written in the F-coordinates, reads

$$\check{\boldsymbol{e}}_{\boldsymbol{00}'} = \frac{\sqrt{2}}{2\rho}\Big((1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho\Big), \; \check{\boldsymbol{e}}_{\boldsymbol{11}'} = \frac{\sqrt{2}}{2\rho}\Big((1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho\Big), \; \check{\boldsymbol{e}}_{\boldsymbol{01}'} = \frac{\sqrt{2}}{2\rho}\boldsymbol{\partial}_+, \; \check{\boldsymbol{e}}_{\boldsymbol{10}'} = \frac{\sqrt{2}}{2\rho}\boldsymbol{\partial}_-.$$

In addition, one has

$$\theta = \frac{1}{\rho}, \qquad \varkappa = (1-\tau)^2. \tag{7.51}$$

Then, from a direct comparison of equation (7.4) and (7.45) one concludes that

$$\boldsymbol{g}' = \boldsymbol{g}_C. \tag{7.52}$$

Moreover, using equations (7.46) and (7.51) the NP frame $\{\boldsymbol{e}'_{\boldsymbol{AA}'}\}$ in the F-coordinates reads

$$\boldsymbol{e}'_{\boldsymbol{00}'} = \frac{\sqrt{2}}{2}\frac{1}{\rho(1-\tau)^2}\Big((1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho\Big), \qquad \boldsymbol{e}'_{\boldsymbol{11}'} = \frac{\sqrt{2}}{2}\rho(1-\tau)^2\Big((1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho\Big),$$

$$\boldsymbol{e}'_{\boldsymbol{01}'} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_+, \qquad \boldsymbol{e}'_{\boldsymbol{10}'} = \frac{\sqrt{2}}{2}\boldsymbol{\partial}_-.$$

Comparing the last expressions for $\boldsymbol{e}'_{\boldsymbol{AA}'}$ and $\boldsymbol{e}_{\boldsymbol{AA}'}$ as given in equations (7.5a)-(7.5b) one concludes that

$$\Lambda^{\boldsymbol{0}}{}_{\boldsymbol{0}} = \frac{1}{\rho^{1/2}(1-\tau)}, \qquad \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{1}} = \rho^{1/2}(1-\tau), \qquad \kappa = 1. \tag{7.53}$$

7.6 The electromagnetic NP constants

Consider the Minkowski spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{\eta}})$ described through the physical coordinates $(\tilde{u}, \tilde{\rho})$ as defined in Sections 7.2.1 and 7.5.2. In these coordinates one has

$$\tilde{\boldsymbol{\eta}} = \mathbf{d}\tilde{u} \otimes \mathbf{d}\tilde{u} + \mathbf{d}\tilde{u} \otimes \mathbf{d}\tilde{\rho} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{u} - \tilde{\rho}^2 \boldsymbol{\sigma}.$$

From equations (7.1) and (7.45) one has that $\boldsymbol{g}' = \theta^2\Xi^2\tilde{\boldsymbol{\eta}}$ and using equations (7.2), (7.3) and (7.51) one concludes that

$$\boldsymbol{g}' = \frac{1}{\tilde{\rho}^2}\tilde{\boldsymbol{\eta}}. \tag{7.54}$$

Let $\epsilon'_{\boldsymbol{A}}{}^{A}$, with $\epsilon'_{\boldsymbol{0}}{}^{A} = o'^{A}$ and $\epsilon'_{\boldsymbol{1}}{}^{A} = \iota'^{A}$, denote a spin dyad so that $e'_{\boldsymbol{AA'}}{}^{AA'} = \epsilon'_{\boldsymbol{A}}{}^{A}\epsilon'_{\boldsymbol{A'}}{}^{A'}$ constitutes the NP-frame given in equation (7.49). Let $\{\tilde{o}^A, \tilde{\iota}^A\}$ denote a spin dyad denoted by $\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}$ and defined via

$$o^A = \tilde{\rho}\tilde{o}^A, \qquad \iota^A = \tilde{\iota}^A. \tag{7.55}$$

Notice that, by virtue of equation (7.54), the spin dyad $\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}$ is normalised respect to $\tilde{\boldsymbol{\eta}}$. To introduce the electromagnetic NP constants as defined in [91] consider the physical Maxwell spinor $\tilde{\phi}_{AB}$ satisfying

$$\tilde{\nabla}_{A'}{}^{A}\tilde{\phi}_{AB} = 0,$$

where $\tilde{\nabla}_{AA'}$ denotes the Levi-Civita connection respect to $\tilde{\boldsymbol{\eta}}$. The components the physical Maxwell spinor respect to the spin dyad $\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}$ will be denoted, as usual, by $\tilde{\phi}_0 \equiv \tilde{\phi}_{AB}\tilde{o}^A\tilde{o}^B$, $\tilde{\phi}_1 \equiv \tilde{\phi}_{AB}\tilde{o}^A\tilde{\iota}^B$, $\tilde{\phi}_2 \equiv \tilde{\phi}_{AB}\tilde{\iota}^A\tilde{\iota}^B$.

Assumption 3. Following [91], the $\tilde{\phi}_0$ component is assumed to have an expansion

$$\tilde{\phi}_0 = \sum_{n=0}^{N}\frac{\tilde{\phi}_0^n}{\tilde{\rho}^{3+n}} + o\Big(\frac{1}{\tilde{\rho}^{3+N}}\Big), \tag{7.56}$$

where the coefficients $\tilde{\phi}_0^n$ do not depend on $\tilde{\rho}$.

The electromagnetic NP constants are defined through the following integrals over cuts $\mathcal{C}$ of null infinity:

$$F_m^{n,k} \equiv \int_{\mathcal{C}} \bar{Y}_{1;n+1,m}\tilde{\phi}_0^{n+1}\mathrm{d}S,$$

where n, $m \in \mathbb{Z}$ with $n \geq 0$, $|m| \leq n+1$ and $\mathrm{d}S$ denotes the area element respect to $\boldsymbol{\sigma}$. In flat space, F_m^n are absolutely conserved in the sense that their value is independent of the cut $\mathcal{C}$ on which they are evaluated —see [91]. From these, only those given by $n = 0$ and $m = -1, 0, 1$ are conserved in the general non-linear Einstein Maxwell theory —see [91].

7.6.1 Translation to the F-gauge

In view of equation (7.54), one has that, as a consequence of the standard conformal transformation law for the spin-1 equation —see [12], the spinor ϕ'_{AB}, satisfying

$$\nabla'_{A'}{}^{A}\phi'_{AB} = 0,$$

where $\nabla'_{AA'}$ is the Levi-Civita connection of $\boldsymbol{g}'$, is related to $\tilde{\phi}_{AB}$ via

$$\phi'_{AB} = \tilde{\rho}\tilde{\phi}_{AB}. \tag{7.57}$$

Therefore, using equations (7.56), (7.55) and (7.57), one obtains

$$\phi'_0 = \sum_{n=0}^{N} \frac{\tilde{\phi}_0^n}{\tilde{\rho}^n} + o\Big(\frac{1}{\tilde{\rho}^N}\Big),$$

where $\phi'_0 \equiv \phi'_{AB}o'^A o'^B$. From equation (7.49), one has that $\boldsymbol{e}'_{\mathbf{00}'} = \sqrt{2}\tilde{\rho}^2\boldsymbol{\partial}_{\tilde{\rho}}$ and consequently

$$\frac{1}{\sqrt{2}}\boldsymbol{e}'_{\mathbf{00}'}(\phi'_0) = -\tilde{\phi}_0^1 + o(\tilde{\rho}^{-1}).$$

The repeated application of $\boldsymbol{e}'_{\mathbf{00}'}$ to the above relation shows that in general

$$\frac{1}{2^{q/2}}\boldsymbol{e}'^{(q)}_{\mathbf{00}'}(\phi'_0) = (-1)^q q!\,\tilde{\phi}_0^q + \sum_{i=q+1}^{N}(-1)^q\frac{(i+1)!}{(i-q+1)!}\,\frac{\tilde{\phi}_0^i}{\tilde{\rho}^{i-q}} + o\Big(\frac{1}{\tilde{\rho}^{N-q}}\Big),$$

where $\boldsymbol{e}^{(q)}_{\mathbf{00}'}(\phi'_0)$ denotes q consecutive applications of $\boldsymbol{e}'_{\mathbf{00}'}$ to ϕ_0. Thus, the quantities F^n_m can be written as

$$F^n_m = \frac{(-1)^{n+1}}{(n+1)!\,2^{(n+1)/2}}\int_{\mathcal{C}} \bar{Y}_{1;n+1,m}\,\boldsymbol{e}'^{(n+1)}_{\mathbf{00}'}(\phi'_0)\,\mathrm{d}S. \tag{7.58}$$

Observe that the constants F^n_m in the previous equation are expressed in terms of $\boldsymbol{g}'$-associated quantities. In order to obtain a general expression for the electromagnetic NP quantities in the F-gauge one has to rewrite expression (7.58) in terms of $\boldsymbol{g}_{\mathcal{C}}$-related quantities. As discussed before, the frames $\boldsymbol{e}_{\boldsymbol{AA}'}$ and $\boldsymbol{e}'_{\boldsymbol{AA}'}$ are related through a conformal rescaling and a Lorentz transformation as given in equation (7.50). For the sake of generality, the first part of the discussion will be carried out for general κ and $\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{B}}$.

Remark 63. In [91] it is shown that the Newman-Penrose constants at $\mathscr{I}^+$ of a purely outgoing field propagating on Minkowksi spacetime vanish. A more recent discussion of this phenomenon was given in [99]. As discussed in [99], if $\tilde{\phi}_0^{out}$ is a purely retarded field then it can be expressed as

$$\tilde{\phi}_0^{out} = K\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} Y_{2;\ell,m}\tilde{\rho}^{(\ell-2)}(\partial_{\tilde{\rho}} - 2\partial_{\tilde{u}})^{\ell-2}\left(\frac{b^{\ell m}(\tilde{u})}{\tilde{\rho}^{\ell+3}}\right) \tag{7.59}$$

where $b^{\ell m}(\tilde{u})$ are smooth functions which depend only on $\tilde{u}$ and K is an unimportant numerical constant. A short argument given in Proposition 3 of [99] shows that the NP constants associated to the latter field vanish —see also [91]. In order to compare (7.59) with the formal expansion for $\tilde{\phi}_0$ implied by Assumption 2 observe that, recalling that $\tilde{u} = \tilde{t} - \tilde{\rho}$, equations in (7.3) render

$$\rho = \frac{\tilde{\rho}}{\tilde{u}(\tilde{u}+2\tilde{\rho})}, \qquad \tau = -\frac{\tilde{\rho}+\tilde{u}}{\tilde{\rho}}. \tag{7.60}$$

In addition, notice that the discussion of Section 7.5 implies that

$$\tilde{\phi}_0 = \tilde{\rho}^{-3}(\Lambda_{\mathbf{0}}{}^{\mathbf{0}})^4\phi_0. \tag{7.61}$$

Therefore, using equations (7.60), (7.61) and (7.53) one sees that the formal expansions (7.28) imply

$$\tilde{\phi}_0 = \frac{\tilde{u}^2\tilde{\rho}^2}{(\tilde{u}+2\tilde{\rho})^2}\sum_{p=2}^{\infty}\sum_{\ell=2}^{p}\sum_{m=-\ell}^{\ell}\frac{Y_{2;\ell-1,m}}{p!}\,\frac{\tilde{\rho}^{p-3}}{\tilde{u}^p(\tilde{u}+2\tilde{\rho})^p}\,a_{0,p;\ell,m}\left(\frac{\tilde{\rho}+\tilde{u}}{\tilde{\rho}}\right), \tag{7.62}$$

where that $a_{0,p;\ell,m}$ is the solution to the differential equation (7.18a). These solutions can be written explicitly in terms of Jacobi polynomials —see [96]. Expanding the first few terms in expressions (7.62) and (7.59) one can check that by Assumption 2 does not imply purely retarded fields.

Remark 64. The time dual of Remark 63 follows *mutatis mutandis*: the NP constants on $\mathscr{I}^-$ of a purely the advanced field $\tilde{\phi}_4^{in}$ vanish. The formal expansions of Assumption 2 do not imply purely advanced fields.

Explicit computation of the first three constants

Let $\epsilon_{\boldsymbol{A}}{}^A$, with $\epsilon_{\mathbf{0}}{}^A = o^A$ and $\epsilon_{\mathbf{1}}{}^A = \iota^A$, denote a spin dyad normalised respect to $\boldsymbol{g}_C$ as defined in Section 7.3. As a consequence of equation (7.50), the spin dyads $\epsilon_{\boldsymbol{A}}{}^A$ and $\epsilon'_{\boldsymbol{A}}{}^A$, giving rise to $\boldsymbol{e}_{\boldsymbol{AA'}}$ and $\boldsymbol{e}'_{\boldsymbol{AA'}}$, are related via

$$\epsilon'_{\boldsymbol{A}}{}^A = \kappa^{-1/2}\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{A}}\epsilon_{\boldsymbol{B}}{}^A. \tag{7.63}$$

Additionally, the spinor field ϕ_{AB}, satisfying

$$\nabla_{A'}{}^A\phi_{AB} = 0,$$

where $\nabla_{AA'}$ is the Levi-Civita connection respect to $\boldsymbol{g}_C$, is related to ϕ'_{AB} via

$$\phi'_{AB} = \kappa^{-1}\phi_{AB}.$$

Therefore, one has that

$$\phi'_0 = \kappa^{-2}\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}\phi_{\boldsymbol{CD}},$$

where $\phi_{\boldsymbol{CD}} \equiv \epsilon_{\boldsymbol{C}}{}^C\epsilon_{\boldsymbol{D}}{}^D\phi_{CD}$. Using the Leibniz rule one obtains

$$
\boldsymbol{e}'_{\mathbf{00}}(\phi'_0) = \kappa^{-2}\Big(\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}\boldsymbol{e}'_{\mathbf{00'}}(\phi_{\boldsymbol{CD}}) + 2\phi_{\boldsymbol{CD}}\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\boldsymbol{e}'_{\mathbf{00'}}(\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}) - 2\kappa^{-1}\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}\phi_{\boldsymbol{CD}}\boldsymbol{e}'_{\mathbf{00'}}(\kappa)\Big). \tag{7.64}
$$

Notice that, in the above expression, all the quantities except for the frame derivative $\boldsymbol{e}'_{\mathbf{00}}$ are $\boldsymbol{g}_{\mathcal{C}}$-related quantities, namely, given in the F-gauge and the F-coordinates. Using equation (7.50) one can expand expression (7.64). This leads to the following expression for the conserved quantities:

$$
F^0_m = -\frac{1}{\sqrt{2}}\int_{\mathcal{C}} \bar{Y}_{1;1,m}\kappa^{-3}\Big(\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{B}}{}_{\mathbf{0}}\bar{\Lambda}^{\boldsymbol{B'}}{}_{\mathbf{0'}}\boldsymbol{e}_{\boldsymbol{BB'}}(\phi_{\boldsymbol{CD}}) + 2\kappa\phi_{\boldsymbol{CD}}\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\boldsymbol{e}'_{\mathbf{00'}}(\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}) - 2\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}\phi_{\boldsymbol{CD}}\boldsymbol{e}'_{\mathbf{00}}(\kappa)\Big)\mathrm{d}S. \tag{7.65}
$$

for $m = -1, 0, 1$. These correspond to the three electromagnetic NP quantities that remain conserved in the non-linear Einstein Maxwell theory. The last expression represents the electromagnetic counterpart of the gravitational NP quantities in the F-gauge as reported in [81] in equation (III.5). The last expression is general can be used, in principle, to find the electromagnetic NP constants in the F-gauge in the non-linear case. Nevertheless, particularising the discussion to the case analysed in this chapter simplifies the expressions considerably. To verify this, observe that, using the results of Section 7.5.3, equation (7.64) reduces to

$$
\boldsymbol{e}'_{\mathbf{00'}}(\phi'_0) = (\Lambda_{\mathbf{0}}{}^{\mathbf{0}})^4(\boldsymbol{e}_{\mathbf{00'}}(\phi_0)) + 2\phi_0(\Lambda_{\mathbf{0}}{}^{\mathbf{0}})\boldsymbol{e}'_{\mathbf{00'}}(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}). \tag{7.66}
$$

Using equations (7.5a) and (7.53) one observes that

$$
\boldsymbol{e}'_{\mathbf{00'}}(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}) = \frac{\sqrt{2}}{4}\Big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\Big)^3, \tag{7.67}
$$

and more generally

$$
\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}) = \left(\frac{\sqrt{2}}{4}\right)^n (2n-1)!!(\Lambda_{\mathbf{0}}{}^{\mathbf{0}})^{2n+1}. \tag{7.68}
$$

Using equation (7.67) one gets

$$
\boldsymbol{e}'_{\mathbf{00'}}(\phi'_0) = (\Lambda_{\mathbf{0}}{}^{\mathbf{0}})^4\Big(\boldsymbol{e}_{\mathbf{00'}}(\phi_0) + \frac{\sqrt{2}}{2}\phi_0\Big). \tag{7.69}
$$

In order to write explicitly the first term of the last expression one uses equation (7.5a) and obtains

$$
\boldsymbol{e}_{\mathbf{00'}}(\phi_0) = \frac{1}{\sqrt{2}}\Big((1-\tau)\boldsymbol{\partial}_\tau\phi_0 + \rho\boldsymbol{\partial}_\rho\phi_0\Big). \tag{7.70}
$$

Substituting equations (7.53) and (7.70) into equation (7.69) renders

$$\boldsymbol{e}'_{\mathbf{00'}}(\phi'_0) = \frac{1}{\sqrt{2}}\rho^{-2}(1-\tau)^{-4}\Big((1-\tau)\boldsymbol{\partial}_\tau\phi_0 + \rho\boldsymbol{\partial}_\rho\phi_0 + \phi_0\Big).$$

Using the last expression, the quantities F^0_m as determined in equation (7.58) are rewritten as

$$F^0_m = \lim_{\substack{\rho\to 0\\ \tau\to 1}}\Bigg(-\frac{1}{2}\int_{\mathbb{S}^2}\bar{Y}_{1;1,m}\rho^{-2}(1-\tau)^{-4}\Big((1-\tau)\boldsymbol{\partial}_\tau\phi_0 + \rho\boldsymbol{\partial}_\rho\phi_0 + \phi_0\Big)\mathrm{d}S\Bigg). \quad (7.71)$$

Substituting the expansion (7.9) for ϕ_0 into equation (7.71) and using the orthogonality relation

$$\int_{\mathbb{S}^2} Y_{s;\ell',m'}\bar{Y}_{s;\ell,m} = \delta_{\ell,\ell'}\delta_{m,m'}, \quad (7.72)$$

one obtains

$$F^0_m = \lim_{\tau\to 1}\Bigg(-\frac{1}{2\times 2!}(1-\tau)^{-4}\Big((1-\tau)\dot{a}_{0,2;2,m} + 3a_{0,2;2,m}\Big)\Bigg). \quad (7.73)$$

Using the solution for $a_{0,p;\ell,m}$ as given in equation (7.19a) and the discussion of the initial data of Section 7.3.3 showing that $C^{\circledast}_{p,m} = 0$, one gets

$$F^0_m = -\frac{1}{2\times 2!\times 16}C_{2,m}. \quad (7.74)$$

where $C_{2,m}$ is the integration constant of Proposition 7.

Remark 65. As discussed in Section 7.3.3 the condition $C^{\circledast}_{p,m} = 0$ with $p \geq 2$, $-p \leq m \leq p$, does not represent a restriction on the class of initial data but arises as a necessary condition ensuring that the solutions (7.19a)-(7.19b) to the Jacobi equation correspond to a solution to the original equations (7.17a)-(7.17b). In the spin-2 case the analogous condition, in contrast, does represent a restriction on the class of initial data.

Proceeding in an analogous way, one can compute the next set of constants in the hierarchy, i.e., F^1_m. A direct computation using equations (7.69) and (7.67) renders

$$\boldsymbol{e}'^{(2)}_{\mathbf{00}}(\phi'_0) = \big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\big)^6\Big(\boldsymbol{e}^{(2)}_{\mathbf{00'}}(\phi_0) + \frac{3\sqrt{2}}{2}\boldsymbol{e}_{\mathbf{00'}}(\phi_0) + \phi_0\Big). \quad (7.75)$$

Using expression (7.5a) one has

$$\boldsymbol{e}^{(2)}_{\mathbf{00'}}(\phi_0) = \frac{1}{2}\Big((1-\tau)^2\boldsymbol{\partial}^2_\tau\phi_0 + 2\rho(1-\tau)\boldsymbol{\partial}_\tau\boldsymbol{\partial}_\rho\phi_0 + \rho^2\boldsymbol{\partial}^2_\rho\phi_0 - (1-\tau)\boldsymbol{\partial}_\tau\phi_0 + \rho\boldsymbol{\partial}_\rho\phi_0\Big). \quad (7.76)$$

Substituting equations, (7.53), (7.70) and (7.76) into (7.75) renders

$$
\begin{aligned}
\boldsymbol{e}'^{(2)}_{\mathbf{00}'}(\phi'_0) = \frac{1}{2}\rho^{-3}(1-\tau)^{-6}\Big((1-\tau)^2\boldsymbol{\partial}^2_\tau\phi_0 + 2\rho(1-\tau)\boldsymbol{\partial}_\tau\boldsymbol{\partial}_\rho\phi_0 + \rho^2\boldsymbol{\partial}^2_\rho\phi_0 \\
+ 2(1-\tau)\boldsymbol{\partial}_\tau\phi_0 + 4\rho\boldsymbol{\partial}_\rho\phi_0 + 2\phi_0\Big).
\end{aligned}
$$

Using the last expression the integral of equation (7.58) reads

$$
\begin{aligned}
F^1_m = \lim_{\substack{\rho\to 0\\ \tau\to 1}}\Bigg(\frac{1}{8}\int_{\mathbb{S}^2}\bar{Y}_{1;2,m}\rho^{-3}(1-\tau)^{-6}\Big((1-\tau)^2\boldsymbol{\partial}^2_\tau\phi_0 + 2\rho(1-\tau)\boldsymbol{\partial}_\tau\boldsymbol{\partial}_\rho\phi_0 \\
+ \rho^2\boldsymbol{\partial}^2_\rho\phi_0 + 2(1-\tau)\boldsymbol{\partial}_\tau\phi_0 + 4\rho\boldsymbol{\partial}_\rho\phi_0 + 2\phi_0\Big)\mathrm{d}S\Bigg). \quad (7.77)
\end{aligned}
$$

Exploiting the orthogonality conditions (7.72) one gets

$$
F^1_m = \lim_{\tau\to 1}\left(\frac{1}{8\times 3!}(1-\tau)^{-6}\Big((1-\tau)^2\ddot{a}_{0,3;3,m} + 6(1-\tau)\dot{a}_{0,3;3,m} + 20a_{0,3;3,m}\Big)\right). \quad (7.78)
$$

Consequently, using equation (7.19a) with $C^{\circledast}_{p,m} = 0$ —see Remark 65, one obtains

$$
F^1_m = \frac{1}{8\times 3!\times 32}C_{3,m}.
$$

where $C_{3,m}$ is the integration constant of Proposition 7. It is instructive to find explicitly one order more in this hierarchy —namely F^2_m. A computation using equations (7.75) and (7.67) renders

$$
\boldsymbol{e}'^{(3)}_{\mathbf{00}'}(\phi'_0) = \big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\big)^8\left(\boldsymbol{e}^{(3)}_{\mathbf{00}'}(\phi_0) + 3\sqrt{2}\boldsymbol{e}^{(2)}_{\mathbf{00}'}(\phi_0) + \frac{11}{2}\boldsymbol{e}_{\mathbf{00}'}(\phi_0) + \frac{3\sqrt{2}}{2}\phi_0\right). \quad (7.79)
$$

Applying $\boldsymbol{e}_{\mathbf{00}'}$ to equation (7.76) one obtains

$$
\begin{aligned}
\boldsymbol{e}^{(3)}_{\mathbf{00}'}(\phi_0) = \frac{1}{2\sqrt{2}}\Big((1-\tau)^3\boldsymbol{\partial}^3_\tau\phi_0 + \rho^3\boldsymbol{\partial}^3_\rho\phi_0 + 3\rho(1-\tau)^2\boldsymbol{\partial}_\rho\boldsymbol{\partial}^2_\tau\phi_0 \\
+ 3\rho^2(1-\tau)\boldsymbol{\partial}^2_\rho\boldsymbol{\partial}_\tau\phi_0 - 3(1-\tau)^2\boldsymbol{\partial}^2_\tau\phi_0 + 3\rho^2\boldsymbol{\partial}^2_\rho\phi_0 \\
+ (1-\tau)\boldsymbol{\partial}_\tau\phi_0 + \rho\boldsymbol{\partial}_\rho\phi_0\Big). \quad (7.80)
\end{aligned}
$$

Using the last expression along with equations (7.53), (7.70), (7.76) one gets

$$
\begin{aligned}
\boldsymbol{e}'^{(3)}_{\mathbf{00}'}(\phi'_0) = \frac{1}{2\sqrt{2}}\rho^{-4}(1-\tau)^{-8}\Big((1-\tau)^3\boldsymbol{\partial}^3_\tau(\phi_0) + \rho^3\boldsymbol{\partial}^3_\rho(\phi_0) + 3\rho^2(1-\tau)\boldsymbol{\partial}^2_\rho\boldsymbol{\partial}_\tau(\phi_0) \\
+ 3\rho(1-\tau)^2\boldsymbol{\partial}_\rho\boldsymbol{\partial}^2_\tau(\phi_0) + 9\rho^2\boldsymbol{\partial}^2_\rho(\phi_0) + 3(1-\tau)^2\boldsymbol{\partial}^2_\tau(\phi_0) + 12\rho(1-\tau)\boldsymbol{\partial}_\rho\boldsymbol{\partial}_\tau(\phi_0) \\
+ 18\rho\boldsymbol{\partial}_\rho(\phi_0) + 6(1-\tau)\boldsymbol{\partial}_\tau(\phi_0) + 6\phi_0\Big).
\end{aligned}
$$

Consequently, the quantities F^2_m as given in equation (7.58) read

$$F^2_m = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Bigg(\frac{1}{48}\int_{\mathbb{S}^2} \bar{Y}_{1;3,m}\rho^{-4}(1-\tau)^{-8}\Big((1-\tau)^3\boldsymbol{\partial}^3_\tau(\phi_0) + \rho^3\boldsymbol{\partial}^3_\rho(\phi_0)$$

$$+3\rho^2(1-\tau)\boldsymbol{\partial}^2_\rho\boldsymbol{\partial}_\tau(\phi_0)+3\rho(1-\tau)^2\boldsymbol{\partial}_\rho\boldsymbol{\partial}^2_\tau(\phi_0)+9\rho^2\boldsymbol{\partial}^2_\rho(\phi_0)+3(1-\tau)^2\boldsymbol{\partial}^2_\tau(\phi_0)$$

$$+ 12\rho(1-\tau)\boldsymbol{\partial}_\rho\boldsymbol{\partial}_\tau(\phi_0) + 18\rho\boldsymbol{\partial}_\rho(\phi_0) + 6(1-\tau)\boldsymbol{\partial}_\tau(\phi_0) + 6\phi_0\Big)\mathrm{d}S\Bigg). \quad (7.81)$$

Exploiting the orthogonality condition (7.72) the last expression simplifies to

$$F^2_m = \lim_{\tau\to 1}\Bigg(\frac{1}{48\times 4!}(1-\tau)^{-8}\Big((1-\tau)^3\dddot{a}_{0,4;4,m} + 15(1-\tau)^2\ddot{a}_{0,4;4,m}$$

$$+ 90(1-\tau)\dot{a}_{0,4;4,m} + 210a_{0,4;4,m}\Big)\Bigg). \quad (7.82)$$

Finally, using equation (7.19a) with $C^{\circledast}_{p,m} = 0$ —see Remark 65, one obtains

$$F^2_m = \frac{3}{48\times 4!\times 128}C_{4,m}.$$

where $C_{4,m}$ is the integration constant of Proposition 7.

The general case

The previous discussion suggests that, in principle, it should be possible to obtain a general formula for F^n_m. Revisiting the calculation of F^0_m, F^1_m and F^2_m one can obtain the following results concerning the overall structure of the electromagnetic NP constants in flat space:

Lemma 28. *For any integer $n \geq 1$*

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = \Big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\Big)^{2(n+1)}\sum_{i=1}^{n} A_i\boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0)$$

for some coefficients A_i independent of ρ and τ.

Proof. To prove this result one proceeds by induction. Equations (7.69), (7.75) and (7.79) already show that the result is valid for $n = 1$, $n = 2$ and $n = 3$. This constitutes the basis of induction. Now, assume that

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = \Big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\Big)^{2(n+1)}\sum_{i=1}^{n} A_i\boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0),$$

then, applying $\boldsymbol{e}'_{\mathbf{00'}}$ to the last expression one has

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0) = \Big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\Big)^{2(n+1)}\sum_{i=1}^{n} A_i\boldsymbol{e}'_{\mathbf{00'}}(\boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0))+2(n+1)\Big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\Big)^{2n+1}\Big(\boldsymbol{e}'_{\mathbf{00'}}\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\Big)\sum_{i=1}^{n} A_i\boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0).$$

Using equations (7.50), (7.53) and (7.67) one obtains

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0) = \Big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\Big)^{2(n+2)} \sum_{i=1}^{n} A_i \boldsymbol{e}^{(i+1)}_{\mathbf{00'}}(\phi_0) + \frac{\sqrt{2}}{2}(n+1)\Big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\Big)^{2(n+2)} \sum_{i=1}^{n} A_i \boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0).$$

One can rearrange the last expression into

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0) = \Big(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\Big)^{2(n+2)} \sum_{i=1}^{n+1} \bar{A}_i \boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0),$$

where $\bar{A}_1 = A_1$ and $\bar{A}_i = \frac{\sqrt{2}}{2}(n+1)A_i + A_{i-1}$ for $i \geq 2$. □

Lemma 29. *For any integer $n \geq 1$*

$$\boldsymbol{e}^{(n)}_{\mathbf{00'}}(\phi_0) = \sum_{\substack{i+j=k \\ k=1}}^{k=n} B_{ij}\rho^i(1-\tau)^j \boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0,$$

for some coefficients B_{ij} independent of ρ and τ.

Proof. As in the proof of Lemma 28, one argues inductively. Equations (7.70), (7.76) and (7.80) serve as the basis of induction. Assume that

$$\boldsymbol{e}^{(n)}_{\mathbf{00'}}(\phi_0) = \sum_{\substack{i+j=k \\ k=1}}^{k=n} B_{ij}\rho^i(1-\tau)^j \boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0.$$

then, applying $\boldsymbol{e}_{\mathbf{00'}}$ to the last expression renders

$$\boldsymbol{e}^{(n+1)}_{\mathbf{00'}}(\phi_0) = \sum_{\substack{i+j=k \\ k=1}}^{k=n} B_{ij}\Big(\rho^i(1-\tau)^j \boldsymbol{e}_{\mathbf{00'}}(\boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0) + (\boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0)\boldsymbol{e}_{\mathbf{00'}}(\rho^i(1-\tau)^j)\Big).$$

Using that

$$\boldsymbol{e}_{\mathbf{00'}}(\rho^i(1-\tau)^j) = \frac{1}{\sqrt{2}}(i+j)\rho^i(1-\tau)^j$$

and

$$\boldsymbol{e}_{\mathbf{00'}}(\boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0) = \frac{1}{\sqrt{2}}\Big((1-\tau)\boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j+1)}_\tau \phi_0 + \rho \boldsymbol{\partial}^{(i+1)}_\rho \boldsymbol{\partial}^{j}_\tau \phi_0\Big),$$

one obtains

$$\begin{aligned}\boldsymbol{e}^{(n+1)}_{\mathbf{00'}}(\phi_0) = \sum_{\substack{i+j=k \\ k=1}}^{k=n} \frac{1}{\sqrt{2}} B_{ij}\Big(&\rho^i(1-\tau)^{j+1}\boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j+1)}_\tau \phi_0 + \rho^{i+1}(1-\tau)^j \boldsymbol{\partial}^{(i+1)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0 \\ &+ (i+j)\rho^i(1-\tau)^j \boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0\Big).\end{aligned}$$

The last expression can be rearranged as

$$\boldsymbol{e}^{(n+1)}_{\mathbf{00'}}(\phi_0) = \sum_{\substack{i+j=k\\k=1}}^{k=n+1} \bar{B}_{ij}\rho^i(1-\tau)^j \boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0,$$

for some coefficients $\bar{B}_{ij}$ which depend only on B_{ij}, i and j. □

Remark 66. In the following the label $+$ is added to the constants F^{n+}_m to remind that the quantities correspond to the NP constants at $\mathscr{I}^+$.

Proposition 11. *If the electromagnetic constants F^{n+}_m at $\mathscr{I}^+$ are finite, then F^{n+}_m depends only on the initial datum $a_{0;n+2,n+2,m}(0)$ —that is, one has*

$$F^{n+}_m = Q^+(m,n)C_{n+2,m},$$

where $Q^+(m,n)$ is a numerical coefficient and $C_{n+2,m}$ is the integration constant of Proposition 7.

Proof. Using equation (7.53) and the results from Lemmas 28 and 29 one has that

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = \rho^{-(n+1)}(1-\tau)^{-2(n+1)} \sum_{q=1}^{n} \sum_{\substack{i+j=k\\k=1}}^{k=q} E_{ij}\rho^i(1-\tau)^j \boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0,$$

for some coefficients E_{ij} independent of ρ and τ. Using the expansion for ϕ_0 given in equation (7.9) one has

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = \rho^{-(n+1)}(1-\tau)^{-2(n+1)} \sum_{q=1}^{n} \sum_{\substack{i+j=k\\k=1}}^{k=q} \sum_{p=1}^{\infty}\sum_{\ell'=1}^{p} \sum_{m'=-\ell'}^{\ell'} \left\{ \frac{1}{p!} E_{ij}\rho^i(1-\tau)^j Y_{1;\ell'-1,m'} \right.$$
$$\left. \boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau (\rho^p a_{0,p;\ell',m'}(\tau)) \right\}.$$

Noticing that

$$\boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau (\rho^p a_{0,p;\ell',m'}(\tau)) = \boldsymbol{\partial}^{(i)}_\rho(\rho^p) \boldsymbol{\partial}^{(j)}_\tau (a_{0,p;\ell',m'}(\tau)), \tag{7.83}$$

and using that

$$\boldsymbol{\partial}^{(i)}_\rho \rho^p = \frac{(p+1)!}{(p-i+1)!}\rho^{p-i}, \tag{7.84}$$

one finds

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = (1-\tau)^{-2(n+1)} \sum_{q=1}^{n} \sum_{\substack{i+j=k\\k=1}}^{k=q} \sum_{p=1}^{\infty}\sum_{\ell'=1}^{p} \sum_{m'=-\ell'}^{\ell'} \left\{ E_{ijp}\, \rho^{p-(n+1)}(1-\tau)^j \right.$$

$$Y_{1;\ell'-1,m}\boldsymbol{\partial}_\tau^{(j)}(a_{0,p;\ell',m'}(\tau))\Bigg\},$$

where $E_{ijp} = E_{ij}(p+1)/(p-i+1)!$. Notice that, the terms with $p < n+1$ diverge when $\rho \to 0$ while the terms with $p > n+1$ vanish when $\rho \to 0$. Integrating the last expression with $\bar{Y}_{1;n,m}$ and using the the orthogonality condition (7.72) one obtains

$$\int_{\mathbb{S}^2} \bar{Y}_{1;n,m}\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0)\mathrm{d}S = (1-\tau)^{-2(n+1)}\sum_{q=1}^{n}\sum_{\substack{i+j=k\\k=1}}^{k=q}\sum_{p=1}^{\infty}\sum_{\ell'=1}^{p}\sum_{m'=-\ell'}^{\ell'}\Bigg\{E_{ijp}\,\rho^{p-(n+1)}(1-\tau)^j$$
$$\delta_{\ell'-1,n}\delta_{m',m}\boldsymbol{\partial}_\tau^{(j)}(a_{0,p;\ell',m'}(\tau))\Bigg\}.$$

Noticing that only the terms with $\ell' = n+1$ and $m = m'$ contribute to the sum and recalling that $\ell' \leq p$ one realises that all the potentially diverging terms with $p < n+1$ vanish. Taking this into account this observation one concludes that

$$\lim_{\rho\to 0}\int_{\mathbb{S}^2} \bar{Y}_{1;n,m}\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0)\mathrm{d}S = (1-\tau)^{-2(n+1)}\sum_{q=1}^{n}\sum_{\substack{i+j=k\\k=1}}^{k=q} E_{ijn+1}(1-\tau)^j\boldsymbol{\partial}_\tau^{(j)}(a_{0,n+1;n+1,m}(\tau)).$$

Taking into account the expression for the electromagnetic NP quantities F^n_m in the F-gauge as given in equation (7.58), consistently with this definition, one replaces n with $n+1$ to obtain

$$F^{n+}_m = \lim_{\tau\to 1}\left[(1-\tau)^{-2(n+2)}\sum_{q=1}^{n+1}\sum_{\substack{i+j=k\\k=1}}^{k=q} E_{ijn+2}(1-\tau)^j\boldsymbol{\partial}_\tau^{(j)}(a_{0,n+2;n+2,m}(\tau))\right]. \quad (7.85)$$

Therefore, if F^n_m is finite then it can only depend on the initial datum $a_{0;n+2,n+2,m}(0)$. Moreover, since $C^{\circledast}_{n+2,m} = 0$. One concludes that

$$F^{n+}_m = Q^+(m,n)C_{n+2,m},$$

where $Q^+(m,n)$ is a numerical coefficient and $C_{n+2,m}$ is the integration constant of Proposition 7. □

Remark 67. Notice that to show that F^{n+}_m is always finite then one would need to analyse the limit given in equation (7.85). This, however, requires a detailed analysis of the coefficients E_{ijp} which in addition would determine explicitly the numerical coefficient $Q^+(m,n)$. The latter requires a lengthy computation which will not be pursued here.

7.6.2 The constants at $\mathscr{I}^-$

The analysis carried out in Sections 7.5 and 7.6 for the electromagnetic constants defined at $\mathscr{I}^+$, can be performed in a completely analogous way for $\mathscr{I}^-$. To do so, consider a formal replacement $\tau \rightarrow -\tau$ and consistently $\boldsymbol{\partial}_\tau \rightarrow -\boldsymbol{\partial}_\tau$. Upon this formal replacement the roles of $\boldsymbol{\ell} = \boldsymbol{e_{00'}}$ and $\boldsymbol{n} = \boldsymbol{e_{11'}}$ as defined in (7.5a) and ϕ_0 and ϕ_2 are essentially interchanged. Then, following the discussion of Sections 7.5.3 and 7.6, one obtains *mutatis mutandis* the time dual of Proposition 11:

Proposition 12. *If the electromagnetic constants F_m^{n-} at $\mathscr{I}^-$ are finite, then F_m^{n-} depends only on the initial datum $a_{2;n+2,n+2,m}(0)$. Moreover,*

$$F_m^{n-} = Q^-(m,n) D_{n+2,m},$$

where $Q^-(m,n)$ is a numerical coefficient and $D_{n+2,m}$ is the integration constant of Proposition 7.

Finally, recalling the results of Propositions 11 and 12 and the discussion of the initial data given in Section 7.3.3 one obtains the following:

Theorem 5. *If the electromagnetic NP constants F_m^{n+} and F_m^{n-} at $\mathscr{I}^+$ and $\mathscr{I}^-$, are finite, then, up to a numerical factor $Q^+(m,n)/Q^-(m,n)$, coincide.*

Remark 68. Observe that the conclusion of Theorem 5, which at first instance would seem to hold only for time-symmetric data, holds for generic initial data and is a consequence of the interplay between the evolution and constraint equations as discussed in Section 7.3.3.

Remark 69. The computations at order $n = 0, 1, 2$ given in Section 7.6.1 suggest that in fact $Q^+(m,n) = Q^-(m,n)$. Nevertheless, explicitly determining these factors require a lengthy computation which will not be pursued here.

7.7 The NP constants for the massless spin-2 field

In this section an analogous analysis to that given in Section 7.6 is performed for the case of the spin-2 massless field. The same notation as the one introduced in Section 7.6 will be used. In particular, the spin dyads $\tilde{\epsilon}_{\boldsymbol{A}}{}^A$, $\epsilon'_{\boldsymbol{A}}{}^A$ and $\epsilon_{\boldsymbol{A}}{}^A$ associated to $\tilde{\boldsymbol{\eta}}$, $\boldsymbol{g}'$ and $\boldsymbol{g}_C$ will be employed. To introduce the gravitational NP constants originally introduced in [91], let $\tilde{\phi}_0$, $\tilde{\phi}_1$, $\tilde{\phi}_2$, $\tilde{\phi}_3$ and $\tilde{\phi}_4$ denote the components of the spin-2 massless field $\tilde{\phi}_{ABCD}$ respect to $\tilde{\epsilon}_{\boldsymbol{A}}{}^A$. The spin-2 equation reads

$$\tilde{\nabla}_{A'}{}^A \tilde{\phi}_{ABCD} = 0. \tag{7.86}$$

Assumption 4. Following [91], the component ϕ_0 is assumed to have the expansion

$$\tilde{\phi}_0 = \sum_{n=0}^{N} \frac{\tilde{\phi}_0^n}{\tilde{\rho}^{5+n}} + o\Big(\frac{1}{\tilde{\rho}^{5+N}}\Big), \tag{7.87}$$

where the coefficients $\tilde{\phi}_0^n$ do not depend on $\tilde{\rho}$.

As already mentioned, the field $\tilde{\phi}_{ABCD}$ provides a description of the linearised gravitational field over the Minkowski spacetime. In the full non-linear theory, the linear field $\tilde{\phi}_{ABCD}$ is replaced by the Weyl spinor Ψ_{ABCD} and the analogue of equation (7.86) encodes the second Bianchi identity in vacuum —see [91]. The spin-2 NP quantities are defined through the following integrals over cuts $\mathcal{C}$ of null infinity:

$$G_m^n \equiv \int_{\mathcal{C}} \bar{Y}_{2;n+2,m} \tilde{\phi}_0^{n+1} \mathrm{d}S,$$

where $n, m \in \mathbb{Z}$ with $n \geq 0$, $|m| \leq n+2$ and $\mathrm{d}S$ denotes the area element respect to $\boldsymbol{\sigma}$. The NP constants G_m^n are absolutely conserved in the sense that their value is independent on the cut $\mathcal{C}$ on which they are evaluated.

Remark 70. In particular, the constants G_m^0 are also conserved in the full non-linear case of the gravitational field where $\tilde{\phi}_0$ is replaced by the component Ψ_0 of the Weyl spinor Ψ_{ABCD} —see [91]. These are the only constants of the hierarchy which are generically inherited in the non-linear case.

7.7.1 Translation to the F-gauge

An expression for the gravitational NP constants in the F-gauge has been given in Section III of [81]. In order to provide a self-contained discussion and for the ease of comparison with the analysis made in Section 7.6 the analogue of Formula (III.5) of [81] will be derived in accordance with the notation and conventions used in this chapter. In view of equation (7.54), one has that, as a consequence of the standard conformal transformation law for the spin-2 equation —see [12], the spinor ϕ'_{ABCD}, satisfying

$$\nabla'_{A'}{}^{A} \phi'_{ABCD} = 0,$$

where $\nabla'_{AA'}$ is the Levi-Civita connection of $\boldsymbol{g}'$, is related to ϕ_{ABCD} via

$$\phi'_{ABCD} = \tilde{\rho} \tilde{\phi}_{ABCD}. \tag{7.88}$$

Therefore, using equations (7.87),(7.55) and (7.88), one obtains

$$\phi'_0 = \sum_{n=0}^{N} \frac{\tilde{\phi}_0^n}{\tilde{\rho}^n} + o\Big(\frac{1}{\tilde{\rho}^N}\Big),$$

where $\phi'_0 \equiv \phi'_{ABCD} o'^A o'^B o'^C o'^D$. Using the last expansion and recalling that $\boldsymbol{e}'_{\mathbf{00'}} = \sqrt{2}\tilde{\rho}^2 \boldsymbol{\partial}_{\tilde{\rho}}$ one obtains, after consecutive applications of $\boldsymbol{e}'_{\mathbf{00'}}$, the expression

$$G^n_m = -\frac{(-1)^{n+1}}{(n+1)!2^{(n+1)/2}} \int_{\mathcal{C}} \bar{Y}_{2;n+2,m} \boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0)\mathrm{d}S. \tag{7.89}$$

To derive an expression for the spin-2 NP constants in the F-gauge one recalls the relation between the $\boldsymbol{g}'$ and $\boldsymbol{g}_C$ representations and their associated spin dyads encoded in equation (7.63). Once again, as a consequence of the conformal transformation laws for the spin-2 equation one has that the spinor field ϕ_{ABCD} related to ϕ'_{ABCD} through

$$\phi'_{ABCD} = \kappa^{-1}\phi_{ABCD},$$

satisfies

$$\nabla_{A'}{}^{A}\phi_{ABCD} = 0,$$

where $\nabla_{AA'}$ represents the Levi-Civita connection respect to $\boldsymbol{g}_C$. Additionally, one has that

$$\phi'_0 = \kappa^{-3}\Lambda^{\boldsymbol{A}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{B}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}\phi_{\boldsymbol{ABCD}},$$

where $\phi_{\boldsymbol{ABCD}} \equiv \epsilon_{\boldsymbol{A}}{}^{A}\epsilon_{\boldsymbol{B}}{}^{B}\epsilon_{\boldsymbol{C}}{}^{C}\epsilon_{\boldsymbol{D}}{}^{D}\phi_{ABCD}$.

Explicit computation of the first constant

Using equation (7.89) and the Leibniz rule one obtains the analogue of Equation (III.5) of [81] written in accordance with the notation and conventions used in this thesis

$$\begin{aligned} G^0_m = -\frac{1}{\sqrt{2}} \int_{\mathcal{C}} \bar{Y}_{2;2,m}\kappa^{-4}\Big(&\Lambda^{\boldsymbol{A}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{B}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}}\big(\Lambda^{\boldsymbol{E}}{}_{\mathbf{0}}\bar{\Lambda}^{\boldsymbol{E}'}{}_{\mathbf{0}'}\boldsymbol{e}_{\boldsymbol{EE}'}(\phi_{\boldsymbol{ABCD}}) \\ &- 3\phi_{\boldsymbol{ABCD}}\boldsymbol{e}'_{\mathbf{00'}}(\kappa)\big) + 4\kappa\Lambda^{\boldsymbol{A}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{B}}{}_{\mathbf{0}}\Lambda^{\boldsymbol{C}}{}_{\mathbf{0}}\phi_{\boldsymbol{ABCD}}\boldsymbol{e}'_{\mathbf{00'}}(\Lambda^{\boldsymbol{D}}{}_{\mathbf{0}})\Big)\mathrm{d}S. \end{aligned} \tag{7.90}$$

Particularising the discussion to the case of the Minkowski spacetime, simplifies the expressions considerably. To see this, observe that, using the results of Section 7.5.3 and equation (7.67) one has that

$$\boldsymbol{e}'_{\mathbf{00'}}(\phi'_0) = (\Lambda_{\mathbf{0}}{}^{\mathbf{0}})^6(\boldsymbol{e}_{\mathbf{00'}}(\phi_0) + \sqrt{2}\phi_0). \tag{7.91}$$

A direct computation using equation (7.5a) and (7.69) renders

$$\boldsymbol{e}'_{\mathbf{00'}}(\phi'_0) = \frac{\sqrt{2}}{\rho^3(1-\tau)^6}\Big(\frac{1}{2}(1-\tau)\boldsymbol{\partial}_\tau\phi_0 + \frac{1}{2}\rho\boldsymbol{\partial}_\rho\phi_0 + \phi_0\Big).$$

Using the last expression, the quantities G^0_m as determined in equation (7.89) are rewritten as

$$G_m^0 = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \left(-\frac{1}{2}\int_{\mathbb{S}^2} \bar{Y}_{2;2,m}\rho^{-3}(1-\tau)^{-6}\Big((1-\tau)\boldsymbol{\partial}_\tau\phi_0 + \rho\boldsymbol{\partial}_\rho\phi_0 + 2\phi_0\Big)\mathrm{d}S\right). \quad (7.92)$$

Substituting the expansion for ϕ_0 as succinctly encoded in (7.28)

$$\phi_0 = \sum_{p=2}^{\infty}\sum_{\ell=2}^{p}\sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{0,p;\ell,m}(\tau) Y_{2;\ell-1m}\rho^p, \quad (7.93)$$

into equation (7.71) renders

$$G_m^0 = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \left(-\frac{1}{2}(1-\tau)^{-6}\sum_{p=2}^{\infty}\sum_{\ell=2}^{p}\sum_{m=-\ell}^{\ell}\frac{1}{p!}\rho^{n-3}\int_{\mathbb{S}^2}\bar{Y}_{2;2,m}\Big((1-\tau)\dot{a}_{0,p;\ell,m} + (p+2)a_{0,p;\ell,m}\Big)\mathrm{d}S\right).$$

Using the orthogonality relation (7.72) one obtains

$$G_m^0 = \lim_{\tau\to 1}\left(-\frac{1}{2\times 3!}(1-\tau)^{-6}\Big((1-\tau)\dot{a}_{0,3;3,m} + 5a_{0,3;3,m}\Big)\right).$$

Remark 71. The above expression is general and makes no assumption on the form of the initial data. An explicit calculation shows, however, that the limit will diverge unless one discards the logarithmic part of the solution in (7.36a). This observation brings to the forefront the close relation between the regularity at the conformal boundary (and in particular at i^0) and the NP constants.

The previous remark motivates the following assumption:

Assumption 5. *The initial data* (7.37) *is assumed to satisfy the regularity condition*

$$C^{\circledast}_{p,m} = D^{\circledast}_{p,m} = 0 \qquad \textit{for} \qquad p \geq 3, \ -p \leq m \leq p.$$

so that no logarithmic singularities arise in the solutions to the Jacobi equation (7.34a)-(7.34b).

Substituting the solution for $a_{0,p;\ell,m}$ as given in equation (7.36a) for $p = \ell = 3$, taking into account the discussion of the initial data of Section 7.4.3 and setting $C^{\circledast}_{3,m} = 0$, consistent with Assumption 5, one obtains

$$G_m^0 = -\frac{1}{2\times 3!\times 64}C_{3,m}. \quad (7.94)$$

where $C_{3,m}$ is the integration constant of Proposition 8.

The general case

One can obtain in similar way to compute higher constants in the hierarchy G^n_m. In order to obtain a general expression for the overall structure of G^n_m one proceeds inductively —in a similar way to the discussion of the electromagnetic NP constants F^n_m.

Lemma 30. *For any integer $n \geq 1$*

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = \left(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\right)^{2(n+2)} \sum_{i=1}^{n} A_i \boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0)$$

for some coefficients A_i independent of ρ and τ.

Proof. As before, one argues by induction. Equation (7.91) for the case $n = 0$ constitutes the basis of induction. Assuming that

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = \left(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\right)^{2(n+2)} \sum_{i=1}^{n} A_i \boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0),$$

and applying $\boldsymbol{e}'_{\mathbf{00'}}$ one obtains

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0) = \left(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\right)^{2(n+2)} \sum_{i=1}^{n} A_i \boldsymbol{e}'_{\mathbf{00'}}(\boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0)) + 2(n+2)\left(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\right)^{2n+3}\left(\boldsymbol{e}'_{\mathbf{00'}}\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\right) \sum_{i=1}^{n} A_i \boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0).$$

Making use of equations (7.50), (7.53) and (7.67) one gets

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0) = \left(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\right)^{2(n+3)} \sum_{i=1}^{n} A_i \boldsymbol{e}^{(i+1)}_{\mathbf{00'}}(\phi_0) + \frac{\sqrt{2}}{2}(n+1)\left(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\right)^{2(n+3)} \sum_{i=1}^{n} A_i \boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0).$$

One can rearrange the last expression into

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0) = \left(\Lambda_{\mathbf{0}}{}^{\mathbf{0}}\right)^{2(n+3)} \sum_{i=1}^{n+1} \bar{A}_i \boldsymbol{e}^{(i)}_{\mathbf{00'}}(\phi_0),$$

where $\bar{A}_1 = A_1$ and $\bar{A}_i = \frac{\sqrt{2}}{2}(n+2)A_i + A_{i-1}$ for $i \geq 2$. □

Remark 72. Observe that the conclusion of Lemma 29 in Section 7.6 is valid for any scalar field ϕ on $\mathcal{M}$. Consequently, it can be applied without further change for the spin-2 case.

Proposition 13. *If the NP constants $G^{n}_{m}{}^{+}$ associated to a spin-2 field on the Minkowski spacetime at $\mathscr{I}^+$ are finite, then $G^{n}_{m}{}^{+}$ depends only on the initial datum $a_{0;n+3,n+3,m}(0)$ —that is, one has*

$$G^{n}_{m}{}^{+} = Q^+(m,n)C_{n+3,m},$$

where $Q^+(m,n)$ is a numerical coefficient and $C_{n+3,m}$ is the integration constant of Proposition 8.

Proof. Taking into account Remark 72 and equation (7.53) one obtains using Lemmas 29 and 30 that

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = \rho^{-(n+2)}(1-\tau)^{-2(n+2)} \sum_{q=1}^{n} \sum_{\substack{i+j=k \\ k=1}}^{k=q} E_{ij}\rho^i(1-\tau)^j \boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau \phi_0,$$

for some coefficients E_{ij} independent of ρ and τ. Using the expansion for ϕ_0 given in equation (7.93) one has

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = \rho^{-(n+2)}(1-\tau)^{-2(n+2)} \sum_{q=1}^{n} \sum_{\substack{i+j=k \\ k=1}}^{k=q} \sum_{p=2}^{\infty}\sum_{\ell'=2}^{p}\sum_{m'=-\ell'}^{\ell'} \bigg\{ \frac{1}{p!} E_{ij}\rho^i(1-\tau)^j Y_{2;\ell'-1,m'} \boldsymbol{\partial}^{(i)}_\rho \boldsymbol{\partial}^{(j)}_\tau (\rho^p a_{0,p;\ell',m'}(\tau)) \bigg\}.$$

Using equations (7.83) and (7.84) one finds

$$\boldsymbol{e}'^{(n)}_{\mathbf{00'}}(\phi'_0) = (1-\tau)^{-2(n+2)} \sum_{q=1}^{n} \sum_{\substack{i+j=k \\ k=1}}^{k=q} \sum_{p=2}^{\infty}\sum_{\ell'=2}^{p}\sum_{m'=-\ell'}^{\ell'} \bigg\{ E_{ijp}\ \rho^{p-(n+2)}(1-\tau)^j Y_{2;\ell'-1,m} \boldsymbol{\partial}^{(j)}_\tau (a_{0,p;\ell',m'}(\tau)) \bigg\},$$

where $E_{ijp} = E_{ij}(p+1)/(p-i+1)!$. In view of the expression for the spin-2 NP constants given in equation (7.89), one replaces n with $n+1$ to obtain

$$\boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0) = (1-\tau)^{-2(n+3)} \sum_{q=1}^{n+1} \sum_{\substack{i+j=k \\ k=1}}^{k=q} \sum_{p=2}^{\infty}\sum_{\ell'=2}^{p}\sum_{m'=-\ell'}^{\ell'} \bigg\{ E_{ijp}\ \rho^{p-(n+3)}(1-\tau)^j Y_{2;\ell'-1,m} \boldsymbol{\partial}^{(j)}_\tau (a_{0,p;\ell',m'}(\tau)) \bigg\}.$$

Integrating the last expression with $\bar{Y}_{2;n+2,m}$ and using the the orthogonality condition (7.72) one obtains

$$\int_{\mathbb{S}^2} \bar{Y}_{2;n+2,m} \boldsymbol{e}'^{(n+1)}_{\mathbf{00'}}(\phi'_0)\mathrm{d}S = (1-\tau)^{-2(n+3)} \sum_{q=1}^{n+1} \sum_{\substack{i+j=k \\ k=1}}^{k=q} \sum_{p=2}^{\infty}\sum_{\ell'=2}^{p}\sum_{m'=-\ell'}^{\ell'} \bigg\{ E_{ijp}\delta_{\ell'-1,n+2}\delta_{m',m} \rho^{p-(n+3)}(1-\tau)^j \boldsymbol{\partial}^{(j)}_\tau (a_{0,p;\ell',m'}(\tau)) \bigg\}.$$

Observe that only the terms with $\ell' = n+3$ and $m = m'$ contribute to the sum and $\ell' \leq p$, consequently all the potentially diverging terms with $p < n+3$ vanish. Taking this into account one concludes that

$$\lim_{\rho\to 0}\int_{\mathbb{S}^2} \bar{Y}_{2;n+3,m}\boldsymbol{e}'^{(n+1)}_{\boldsymbol{00'}}(\phi'_0)\mathrm{d}S = (1-\tau)^{-2(n+3)}\sum_{q=1}^{n+1}\sum_{\substack{i+j=k\\k=1}}^{k=q} E_{ijn+3}(1-\tau)^j\boldsymbol{\partial}_\tau^{(j)}(a_{0,n+3;n+3,m}(\tau)).$$

Therefore, one concludes that

$$G^{n\;+}_{m} = \lim_{\tau\to 1}\left[(1-\tau)^{-2(n+3)}\sum_{q=1}^{n+1}\sum_{\substack{i+j=k\\k=1}}^{k=q} E_{ijn+3}(1-\tau)^j\boldsymbol{\partial}_\tau^{(j)}(a_{0,n+3;n+3,m}(\tau))\right] \tag{7.95}$$

At this point, a necessary condition for the above limit to be finite is to set $C^{\circledast}_{n+3,m} = 0$ to avoid the appearance of logarithmic singularities. In accordance with Assumption 5 one concludes that, if G^{n}_{m} is finite then it can only depend on the initial datum $a_{0;n+3,n+3,m}(0)$. Moreover,

$$G^{n\;+}_{m} = Q^+(m,n)C_{n+3,m},$$

where $Q^+(m,n)$ is a numerical coefficient and $C_{n+3,m}$ is the integration constant of Proposition 8. In the last line the label + has been added to remind that the quantities correspond to the NP constants at $\mathscr{I}^+$. □

Remark 73. Notice that Assumption 5 is a necessary condition if the full hierarchy of constants $G^{n\;+}_{m}$ is required. Nevertheless, if one is only interested on a finite subset of these constants, say $G^{n\;+}_{m}$ at fixed order n', then the restriction imposed by Assumption 5 to the initial data can be relaxed to $C^{\circledast}_{p,m} = D^{\circledast}_{p,m} = 0$ for $p = n'+3$, $-p \leq m \leq p$.

7.7.2 The constants on $\mathscr{I}^-$

The time dual result can be obtained succinctly considering a formal replacement $\tau \to -\tau$ and consistently $\boldsymbol{\partial}_\tau \to -\boldsymbol{\partial}_\tau$. As previously discussed, upon this formal replacement the roles of $\boldsymbol{\ell} = \boldsymbol{e}_{\boldsymbol{00'}}$ and $\boldsymbol{n} = \boldsymbol{e}_{\boldsymbol{11'}}$ as defined in (7.5a) and ϕ_0 and ϕ_4 are essentially interchanged. Finally, one obtains *mutatis mutandis* that the time dual of Proposition 13

Proposition 14. *If the NP constants $G^{n\;-}_{m}$ associated to a spin-2 field on the Minkowski spacetime at $\mathscr{I}^-$ are finite, then $G^{n\;-}_{m}$ depends only on the initial datum $a_{4;n+3,n+3,m}(0)$ —that is, one has*

$$G^{n\;-}_{m} = Q^-(m,n)D_{n+3,m},$$

where $Q^-(m,n)$ is a numerical coefficient and $D_{n+3,m}$ is the integration constant of

Proposition 8.

Remark 74. A necessary condition for $G^{n\,\pm}_{m}$ to be finite is that the regularity condition of Assumption 5 is satisfied —see Remark 73. Nevertheless, to show that this condition is sufficient requires a detailed analysis of the coefficients E_{ijn+3} in equation (7.95), which in addition would determine explicitly the numerical coefficient $Q^{\pm}(m,n)$. The latter requires a lengthy computation which will not be pursued here.

Recalling Propositions 13 and 14 and the discussion of the initial data given in Section 7.4.3 one obtains the following:

Main Result 3. *If the spin-2 NP constants $G^{n\,+}_{m}$ and $G^{n\,-}_{m}$ at $\mathscr{I}^{+}$ and $\mathscr{I}^{-}$ in Minkowski spacetime, are finite, then, up to a numerical factor $Q^{+}(m,n)/Q^{-}(m,n)$, coincide.*

Remark 75. This conclusion, which at first instance would seem to hold only for time-symmetric data, is a consequence of the field equations and, as discussed in Section 7.4.3, holds for generic initial data satisfying the regularity condition given in Assumption 5 —see also Remark 73.

Remark 76. A similar symmetric behaviour has been observed in the gravitational case in [100]. In that reference the Newman-Penrose constants at future and past null infinity of the spacetime arising from Bowen-York initial data have been computed.

7.7.3 The time symmetric case

It is of interest to analyse the case when the initial data is time-symmetric. An analysis of a spin-2 field on Minkowski spacetime with time-symmetric initial data in the framework of the cylinder at spatial infinity was given in [98]. In this reference it is shown that for time-symmetric initial data one has, for $p \geq 3$, $-p \leq m \leq p$, that

$$a_{0,p,p,m}(0) = -a_{4,p,p,m}(0). \tag{7.96}$$

Nevertheless, as shown in Section 7.4.3 if the regularity condition $C^{\circledast}_{p,m} = D^{\circledast}_{p,m} = 0$ holds, then necessarily $a_{0,p,p,m}(0) = a_{4,p,p,m}(0)$. Combining this observation with the condition (7.96) valid for time-symmetric data, one concludes, for $p \geq 3$, $-p \leq m \leq m$, that

$$a_{0,p,p,m}(0) = a_{4,p,p,m}(0) = 0. \tag{7.97}$$

Therefore $C_{p,m} = D_{p,m} = 0$ and using Propositions 13 and 14 one concludes that all the constants in the hierarchy $G^{n\,\pm}_{m}$ vanish.

Proposition 15. *Given time-symmetric initial data for the spin-2 field on the Minkowski spacetime, if the regularity conditions* (7.97) *hold for $p \geq 3$, then the* *gravitational NP constants at $\mathscr{I}^+$ and $\mathscr{I}^-$, denoted by $G^{n\,+}_m$ and $G^{n\,-}_m$, are finite and vanish —that is, one has that*

$$G^{n\,+}_m = G^{n\,-}_m = 0.$$

Remark 77. In the time-symmetric case the regularity condition of Assumption 5 imposes conditions on initial data which can be written covariantly in terms of the value of the linearised Bach tensor and its derivatives at spatial infinity. More precisely, in the time-symmetric case, the regularity condition $C^{\circledast}_{p,m} = D^{\circledast}_{p,m} = 0$ at fix order p is equivalent to

$$D_{(\boldsymbol{A}_q\boldsymbol{B}_q}\cdots D_{\boldsymbol{A}_1\boldsymbol{B}_1}B_{\boldsymbol{ABCD})}(i^0) = 0, \quad q = 0\ldots p,$$

where $B_{\boldsymbol{ABCD}}$ is the linearised Bach spinor given in terms of $\phi_{\boldsymbol{ABCD}}$ via

$$B_{\boldsymbol{ABCD}} = 2D_{\boldsymbol{E}(\boldsymbol{A}}\Omega\phi_{\boldsymbol{BCD})}{}^{\boldsymbol{E}} + \Omega D_{\boldsymbol{E}(\boldsymbol{A}}\phi_{\boldsymbol{BCD})}{}^{\boldsymbol{E}},$$

where $\Omega = \rho^2$ and D is the Levi-Civita connection respect to the induced metric by $\boldsymbol{g}_C$ at $\mathcal{S}$ —see Section 7.1. This result was given in [96] and follows from direct linearisation of Theorem 4.1 in [27]. In the non-linear case the above regularity condition is a necessary but not sufficient condition for the regularity of $\mathscr{I}$ as shown in [101].

Remark 78. Whether or not there exists an analogous covariant representation of the regularity condition of Assumption 5 for the non time-symmetric case is still a research question which is not investigated in this thesis.

7.8 Conclusions

In this chapter the correspondence between initial data given on a Cauchy hypersurface $\mathcal{S}$ intersecting i^0 on Minkowski spacetime for the spin-1 (electromagnetic) and spin-2 fields and their associated NP constants was analysed. This analysis has been done for the full hierarchy of NP constants F^n_m and G^n_m in the Minkowski spacetime.

For the electromagnetic case, it was shown that, once the initial data for the Maxwell spinor is written as an expansion of the form (7.21), the electromagnetic NP constants F^{n+}_m at $\mathscr{I}^+$ can be identified with the initial datum $a_{0,p;\ell,m}(0)$ with $p = \ell = n+2$. Since $1 \leq \ell \leq p$, one concludes that F^n_m are in correspondence with the highest harmonic but are irrespective of the initial data for the lower modes $\ell < p$. In an analogous way, one can identify the electromagnetic NP constants F^{n-}_m at $\mathscr{I}^-$ with the initial datum $a_{2,p;\ell,m}(0)$. Notice that the only restriction imposed on the initial data is to have the appropriate decay at infinity so that the electromagnetic NP constants can be defined. Apart from this requirement, the initial data encoded in (7.21) is completely general. As a by-product of the analysis of Section 7.6 and the discussion of the field equations given in Section 7.3.3 one concludes that the electromagnetic NP constants at $\mathscr{I}^+$ and $\mathscr{I}^-$ coincide —up to an irrelevant numerical factor. In this discussion, the field equations $\mathcal{S}$ played a dual role: on the one hand they allow to conclude that for $p=\ell$ one has that $C^{\circledast}_{p,m}=D^{\circledast}_{p,m}=0$ so that the potentially singular part of the solutions (7.19a) and (7.19b) does not contribute to the electromagnetic field. On the other hand, the field equations further imply that $a_{0,p;\ell,m}(0)=a_{2,p;\ell,m}(0)=C_{p,m}=D_{p,m}$. This last observation is the one which ultimately relates the electromagnetic NP constants at $\mathscr{I}^+$ and $\mathscr{I}^-$. Observe that this result is irrespective of the initial data being time symmetric or not.

An analogous analysis was performed for a spin-2 field on a Minkowski background. In contrast with the electromagnetic case, for the spin-2 field, the divergent terms at $\tau=\pm1$ in expressions (7.36a)-(7.36b) are solutions which contribute to the field. In other words, $C^{\circledast}_{p,m}=D^{\circledast}_{p,m}\neq 0$ represents, in principle, admissible initial data. Consequently, for generic initial data, logarithmic singularities at null infinity arise. In such cases the spin-2 field does not have the appropriate decay and the associated NP constants are divergent. Thus, if the initial data for the field is written as an expansion of the form (7.37) and satisfy the regularity condition $C^{\circledast}_{p,m}=D^{\circledast}_{p,m}=0$, then the spin-2 NP constants $G^{n\,+}_{m}$ and $G^{n\,-}_{m}$ at $\mathscr{I}^+$ and $\mathscr{I}^-$ can be identified with the initial data $a_{0,n+3;n+3,m}(0)$ and $a_{4,n+3,n+3,m}(0)$, respectively. Moreover, as discussed in Section 7.4.3, if the regularity condition is satisfied, the field equations imply that $a_{0,p,p,m}(0)=a_{4,p,p,m}(0)$. Consequently, up to a numerical constant, $G^{n\,+}_{m}$ and $G^{n\,-}_{m}$ coincide. Notice that this result is irrespective of the initial data being time symmetric or not. Furthermore, a direct consequence of this analysis is that, for time-symmetric data satisfying the regularity condition, the spin-2 NP constants vanish.

7.9 Appendix: The connection on $\mathbb{S}^2$

In this section expressions for the connection coefficients (of the Levi-Civita connection) respect to a complex null frame which do not make reference to any particular coordinate system on $\mathbb{S}^2$ are obtained. To do so, the Cartan structure equations as given in Appendix 5.8 of Chapter 5 will be employed. Let $\{\boldsymbol{\partial}_+,\boldsymbol{\partial}_-\}$ be a complex null frame on $\mathbb{S}^2$ with corresponding dual covectors $\{\boldsymbol{\omega}^+,\boldsymbol{\omega}^-\}$. Namely, one considers

$$\boldsymbol{\sigma}=2(\boldsymbol{\omega}^+\otimes\boldsymbol{\omega}^-+\boldsymbol{\omega}^-\otimes\boldsymbol{\omega}^+),\qquad \boldsymbol{\sigma}^\sharp=\frac{1}{2}(\boldsymbol{\partial}_+\otimes\boldsymbol{\partial}_-+\boldsymbol{\partial}_-\otimes\boldsymbol{\partial}_+),$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^\sharp$ denote the covariant and contravariant version of the standard metric on $\mathbb{S}^2$. Furthermore, one assumes that

$$\boldsymbol{\partial}_+ = \overline{\boldsymbol{\partial}_-}, \tag{7.98}$$

and consequently $\boldsymbol{\omega}^+ = \overline{\boldsymbol{\omega}^-}$. To start the discussion observe that $[\boldsymbol{\partial}_+, \boldsymbol{\partial}_-]$ and its complex conjugate can be expressed as a linear combination of the basis vectors $\boldsymbol{\partial}_+$ and $\boldsymbol{\partial}_-$. A direct inspection, taking into account the condition encoded in equation (7.98), reveals that

$$[\boldsymbol{\partial}_+, \boldsymbol{\partial}_-] = \varpi \boldsymbol{\partial}_+ - \overline{\varpi} \boldsymbol{\partial}_-, \tag{7.99}$$

where ϖ is a scalar field over $\mathbb{S}^2$. Using the no-torsion condition of the Levi-Civita connection $\not{\nabla}$ on $\mathbb{S}^2$ one obtains from equation (7.99) that

$$\not{\nabla}_+ \boldsymbol{\partial}_- - \not{\nabla}_- \boldsymbol{\partial}_+ = \varpi \boldsymbol{\partial}_+ - \overline{\varpi} \boldsymbol{\partial}_-, \tag{7.100}$$

where $\not{\nabla}_+$ and $\not{\nabla}_-$ denote a covariant derivative in the direction of $\boldsymbol{\partial}_+$ and $\boldsymbol{\partial}_-$ respectively. Using equation (7.100) and the metricity conditions $\not{\nabla}_+ \boldsymbol{\sigma} = 0, \not{\nabla}_- \boldsymbol{\sigma} = 0$, one finds that the only non-zero connection coefficients are all encoded in the scalar field ϖ:

$$\Gamma_-{}^-{}_- = \overline{\Gamma_+{}^+{}_+} = -\Gamma_-{}^+{}_+ = -\overline{\Gamma_+{}^-{}_-} = \varpi.$$

The connection can be compactly encoded in the curvature 1-form $\boldsymbol{\gamma}^{\boldsymbol{a}}{}_{\boldsymbol{b}}$ as defined in equations (5.89) and (5.87) in Appendix 5.8 of Chapter 5. A direct computation renders

$$\boldsymbol{\gamma}^+{}_+ = \overline{\boldsymbol{\gamma}^-{}_-} = \overline{\varpi}\boldsymbol{\omega}^+ - \varpi\boldsymbol{\omega}^-, \qquad \boldsymbol{\gamma}^+{}_- = \boldsymbol{\gamma}^-{}_+ = 0.$$

Using the first Cartan structure equation as given in equation (5.88a) in Appendix 5.8 of Chapter 5, one obtains

$$\mathbf{d}\boldsymbol{\omega}^+ = -\varpi\boldsymbol{\omega}^+ \wedge \boldsymbol{\omega}^-, \qquad \mathbf{d}\boldsymbol{\omega}^- = \overline{\varpi}\boldsymbol{\omega}^+ \wedge \boldsymbol{\omega}^-. \tag{7.101}$$

For completeness, using the above expressions and the second Cartan structure equation as given in (5.88b) in Appendix 5.8 of Chapter 5, one can directly compute the curvature form $\boldsymbol{\Omega}^{\boldsymbol{a}}{}_{\boldsymbol{b}}$:

$$\boldsymbol{\Omega}^+{}_+ = \overline{\boldsymbol{\Omega}^-{}_-} = -2(|\varpi|^2 + \frac{1}{2}(\boldsymbol{\partial}_+\varpi + \boldsymbol{\partial}_-\overline{\varpi}))\boldsymbol{\omega}^+ \wedge \boldsymbol{\omega}^-.$$

Notice that in order to find further information about ϖ one can exploit the fact that the Riemann curvature for maximally symmetric spaces $(\mathcal{N}, \boldsymbol{h})$ is given by

$$R_{\boldsymbol{abcd}} = \frac{1}{2}R(h_{\boldsymbol{ab}}h_{\boldsymbol{bd}} - h_{\boldsymbol{ad}}h_{\boldsymbol{bc}}),$$

where R is the Ricci scalar of the Levi-Civita connection of the metric $\boldsymbol{h}$ on $\mathcal{N}$. Since the Ricci scalar for $\mathbb{S}^2$ is $R = -2$, using equations (5.89) and (5.87) as given in Appendix 5.8 of Chapter 5, one finds that

$$\boldsymbol{\Omega}^+{}_+ = \overline{\boldsymbol{\Omega}^-{}_-} = 2\boldsymbol{\omega}^+ \wedge \boldsymbol{\omega}^-.$$

Consequently, one concludes that the scalar field ϖ satisfies

$$|\varpi|^2 + \frac{1}{2}(\boldsymbol{\partial}_+\varpi + \boldsymbol{\partial}_-\overline{\varpi}) = -1. \tag{7.102}$$

7.10 Appendix: The ð and $\bar{ð}$ operators

In this appendix, the operators $\boldsymbol{\partial}_+$ and $\boldsymbol{\partial}_-$ are written in terms of the ð and $\bar{ð}$ operators of Newman and Penrose. To fix the notation and conventions, let $ð_P$ and $\bar{ð}_P$ denote the ð and $\bar{ð}$ operators as defined in [12]. In the language of the NP-formalism [12, 29, 102], given a null frame represented by $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ their corresponding covariant directional derivatives are denoted by $\{D, \Delta, \delta, \bar{\delta}\}$. The operators $ð_P$ and $\bar{ð}_P$ acting on a quantity η with spin weight s can be written in terms of the δ and $\bar{\delta}$ derivatives as —see [12],

$$ð_P\eta = \delta\eta + s(\bar{\alpha} - \beta)\eta, \qquad \bar{ð}_P\eta = \bar{\delta}\eta - s(\alpha - \bar{\beta})\eta, \tag{7.103}$$

where α and β denote the spin coefficients as defined in the NP formalism. The action of the directional derivatives δ and $\bar{\delta}$ on the vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$, projected into the tangent space $T(\mathcal{Q}) \subset T(\mathcal{M})$ spanned by $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$, is encoded in

$$\delta m^a = -(\bar{\alpha} - \beta)m^a, \qquad \delta\bar{m}^a = (\bar{\alpha} - \beta)\bar{m}^a \qquad \text{on} \qquad \mathcal{Q}. \tag{7.104}$$

The directional derivatives $\nabla\!\!\!/_+$ and $\nabla\!\!\!/_-$ as defined on Appendix 7.9 are related to δ and $\bar{\delta}$ via

$$\delta = \frac{1}{\sqrt{2}}\nabla\!\!\!/_+, \qquad \bar{\delta} = \frac{1}{\sqrt{2}}\nabla\!\!\!/_-.$$

It follows from the discussion of Appendix 7.9 and equation (7.104) that

$$\bar{\alpha} - \beta = -\frac{1}{\sqrt{2}}\overline{\varpi}, \qquad \text{on} \qquad \mathcal{Q} \tag{7.105}$$

Using equations (7.104) and (7.105) one obtains

$$\nabla\!\!\!/_+\eta = \sqrt{2}ð_P\eta + s\overline{\varpi}\eta, \qquad \nabla\!\!\!/_-\eta = \sqrt{2}\bar{ð}_P\eta - s\varpi\eta, \tag{7.106}$$

To align the discussion with the conventions of [81, 96, 98] is convenient to define ð and $\bar{ð}$ by rescaling $ð_P$ and $\bar{ð}_P$ as

$$ð \equiv -\frac{1}{\sqrt{2}}ð_P, \qquad \bar{ð} \equiv -\frac{1}{\sqrt{2}}\bar{ð}_P. \tag{7.107}$$

The corresponding eigenfunctions $Y_{s;\ell m}$ of the operator $ð\bar{ð}$, defining the spin-weighted spherical harmonics, will be assumed to be rescaled in accordance with equation (7.107). Exploiting that $\{Y_{s;\ell m}\}$, with $0 \leq |s| \leq \ell$ and $-\ell \leq m \leq \ell$, form a complete basis for functions of spin-weight s over $\mathbb{S}^2$, given a scalar field $\xi : \mathcal{Q} \to \mathbb{R}$, with spin-weight s, one can expand ξ as

$$\xi = \sum_{\ell=s}^{\infty} \sum_{m=-\ell}^{\ell} C_{s\ell m}\, Y_{s;\ell m}. \tag{7.108}$$

In addition, one has that

$$ð(Y_{s;\ell m}) = \sqrt{(\ell - s)(\ell + s + 1)}\; Y_{s+1;\ell,m}, \tag{7.109a}$$
$$\bar{ð}(Y_{s;\ell,m}) = -\sqrt{(\ell + s)(\ell - s + 1)}\; Y_{s-1;\ell m}. \tag{7.109b}$$

Notice that equation (7.108) as well as equations (7.109a)-(7.109b) do not depend on the specific choice of coordinates on $\mathcal{Q}$.

8 Conclusions, perspectives and future work

In this thesis a variety of applications of the conformal Einstein field equations has been given. These equations were motivated and presented in Chapters 1 and 2 respectively. The main strength of the use of the conformal Einstein field equations as a tool for the analysis of global properties of solutions to the Einstein field equations resides in their behaviour under conformal transformations. This property, in turn, opens the possibility to study the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ through the analysis of its conformal extension $(\mathcal{M}, \boldsymbol{g})$. From the point of view of the initial value problem it allows to reduce, in certain cases, global problems into local problems, e.g., the proof of the global non-linear stability of the de-Sitter spacetime given in [9] and the semiglobal non-linear stability of the Minkowski spacetime given in [8]. Moreover, it allows to formulate asymptotic initial value problems, i.e., an initial value problem in which the initial data is given at the conformal boundary. From a more physical point of view, to be able to analyse the behaviour of the gravitational field at the conformal boundary is of great important as several quantities of physical interest such as the Bondi mass, the Newman-Penrose constants and the notion of gravitational radiation are defined in terms of structures at $\mathscr{I}$.

As discussed in Chapter 2, there are two versions of the conformal Einstein field equations: the *standard conformal Einstein field equations* and the *extended conformal Einstein field equations*. In the former case the gauge is fixed by introducing *gauge source functions* while in the latter by exploiting the notion of *conformal Gaussian systems of coordinates*. In both cases, one obtains a first order system of symmetric hyperbolic evolution equations. Nevertheless, in the classical discussion of the Cauchy problem in general relativity in [4], the hyperbolic reduction of the Einstein field equations using *harmonic coordinates* —corresponding to a particular choice of the coordinate gauge source function— renders a system of wave equations for the metric components. Consequently, in this regard, it is natural to obtain a second order hyperbolic reduction for the conformal Einstein field equations. In [18], a system of wave equations equivalent to the metric formulation of the conformal Einstein field equations has been given. In the latter reference the gauge is fixed exploiting a generalised wave-map gauge and is closer in spirit to the classical treatment of the Cauchy problem in General Relativity.

Chapter 3 contains the first application of the conformal Einstein field equations discussed in this thesis. In this chapter, a second order hyperbolic reduction of the spinorial formulation of the conformal Einstein field equations was given. The analysis given in this chapter shows how the spinorial formulation leads to a systematic construction of the wave equations for the relevant fields. The use of the spinorial formulation is advantageous as it gives access to a wider set of gauge source functions (coordinate, frame and conformal gauge source functions). Moreover, in the spinorial formulation, the equations posses a simpler algebraic structure than in the metric formulation. In particular, the equation for the rescaled Weyl spinor, which can be regarded as the most fundamental field in the equations, satisfies a particularly simple wave equation. Chapter 4 constitutes an application of the wave equations introduced in Chapter 3. In this chapter small perturbations of hyperboloidal initial data of the Milne spacetime were discussed. This analysis is similar to the discussion of the Minkowski spacetime given in [8]. Notice that the analysis given in Chapter 3 is restricted to the vacuum case. Nevertheless, the standard conformal Einstein field equations can be formulated for some suitable matter models for which the energy-momentum tensor is tracefree. Consequently, a natural generalisation to the analysis given in Chapter 3 is to obtain a system of wave equations for the standard conformal Einstein field equations coupled with tracefree matter, e.g., the electromagnetic field. Additionally, in view of the discussion of the extended Einstein field equations in Chapter 2 an open problem is whether it is possible to use conformal Gaussian systems of coordinates to obtain a system of wave equations for the fields in the extended conformal Einstein field equations.

Presumably, one of the most important questions for conformal methods in General Relativity is whether the conformal Einstein field equations can be employed not only for the analysis of asymptotically simple spacetimes —say as in [8, 9, 11, 57]— but if they can be used to make inroads into the stability analysis of black hole spacetimes. From the physical point of view, observations have established that the Cosmological constant is positive in our universe. Consequently, spacetimes describing isolated systems embedded in a de-Sitter universe constitutes a class of physically relevant spacetimes to be analysed. In this regard, in Chapter 5 constitutes an application of conformal methods for the stability analysis of non-linear perturbations of the Schwarzschild-de Sitter spacetime. In this chapter, initial data for an asymptotic initial value problem —initial data given on the (spacelike) conformal boundary— for the Schwarzschild-de Sitter spacetime was obtained. In particular, it was shown how the initial data allows to understand the singular behaviour of the conformal structure at the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ where the horizons of the Schwarzschild-de Sitter spacetime meet the conformal boundary. Using the insights gained from the analysis of the Schwarzschild-de Sitter spacetime in a conformal Gaussian gauge, non-linear perturbations close to the Schwarzschild-de Sitter spacetime in the asymptotic region were considered. It was shown that small enough perturbations of asymptotic initial data for the Schwarzschild de-Sitter spacetime give rise to a solution to the Einstein field equations which exists to the future and has an asymptotic structure similar to that of the Schwarzschild-de Sitter spacetime. The analysis given in Chapter 5 shows some of the main features and difficulties in using the conformal Einstein field equations in the analysis of perturbations of black hole spacetimes particularly for those with de-Sitter like asymptotics. Despite the fact this result does not fully address the outstanding stability of the Schwarzschild-de Sitter, the constructed class of solutions is non-trivial. Moreover, these perturbed spacetimes constitute a large class of asymptotically Schwarzschild-de Sitter spacetimes —see [24] for a definition of asymptotically Schwarzschild-de Sitter spacetime— which are dynamical and represent non-trivial examples of spacetimes allowing for gravitational radiation at $\mathscr{I}$. In particular, for the non-linear perturbations of the Schwarzschild de-Sitter spacetime analysed in Chapter 5 the induced metrics at the conformal boundary h_{ab} are, in principle, in contrast with the exact Schwarzschild-de Sitter asymptotic datum $\mathring{h}_{ab}$, not conformally flat, as h_{ab} can lie anywhere in an open ball centred at $\mathring{h}_{ab}$ in terms of suitably defined norms. Consequently they serve as non-trivial examples for the theory of asymptotics for de-Sitter like spacetimes introduced in [24, 103, 104].

Nonetheless, the result obtained in Chapter 5 does not exhaust the full domain of dependence associated to this asymptotic initial value problem. Thus, a natural generalisation of these results is to obtain the maximal Cauchy development of the small perturbed data discussed in Chapter 5. A possible approach would be to exploit the *time-symmetric conformal representation* in which $\kappa = 0$ and use the global stability methods contained in [76]. Related to this problem is the question of whether there exist a Cauchy horizon associated to this development and if the singularities in the rescaled Weyl tensor at the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ propagate through this boundary. To perform an analysis of the gravitational field close to the asymptotic points a possible venue would be to exploit the mean curvature foliation of the Schwarzschild-de Sitter spacetime discussed in [105, 106]. As discussed in [105] there exist a particular hypersurface of this foliation for which the extrinsic curvature is pure trace and the induced 3-metric metric is formally the same as the 3-metric in the the time-symmetric slice in the Schwarzschild spacetime. This hypersurface has an asymptotic end which, in this case, corresponds to one the asymptotic points in the Schwarzschild-de Sitter spacetime —see the Figure 2 in [105] and Figures 2 and 3 in [106]. Analysing the evolution of initial data for the conformal Einstein field equation on this slice and exploiting the techniques used to analyse spatial infinity —the framework of the *cylinder at spatial infinity* as discussed in [27]— would bee useful in understanding the behaviour of the gravitational field close to the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$.

In Chapter 6 small perturbations of the Kerr-de Sitter spacetime were discussed. This constitutes a partial generalisation of the analysis of Chapter 5. Following the spirit of Chapter 5 an asymptotic initial value problem was formulated and small perturbations of exact asymptotic initial data for the Kerr-de Sitter spacetime were considered. Then, using the theory of symmetric hyperbolic systems contained in [23] an existence result for small perturbations was obtained. Nevertheless, the discussion given in Chapter 6 only constitutes a partial generalisation of the results given in Chapter 5 as the conformal evolution equations governing the exact Kerr-de Sitter spacetime in the conformal Gaussian gauge were not analysed. In particular, notice that, in contrast with the analysis give in Chapter 5, an estimation of the time of existence of the solutions was not obtained. Instead, the emphasis of the analysis of Chapter 6 was on discussing the existence of conformal representation for which the associated asymptotic initial data is regular. In this regard, the most important observation is that although in this case the initial 3-metric $\boldsymbol{h}$ does not correspond to the standard metric on $\mathbb{R}\times\mathbb{S}^2$, as in the case of the Schwarzschild-de Sitter spacetime, it is conformally flat. Additionally, it was shown that, similar to the case of the Schwarzschild-de Sitter spacetime, this conformal representation is associated to the time-symmetric representation for which $\kappa = 0$ —equivalently characterised by the vanishing of the Friedrich scalar at $\mathscr{I}$. It is of interest to investigate whether the initial data for the rescaled Weyl tensor becomes singular at the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ in other conformal representation; in particular in the $\mathbb{S}^3/\{\mathcal{Q},\mathcal{Q}'\}$ conformal representation. To clarify this point, one would need to analyse the behaviour of the conformal factor relating both representations. This analysis will be done elsewhere.

An additional observation related to the analysis of the Kerr-de Sitter spacetime using conformal methods is that, despite that at first sight the conformal Einstein field equations expressed in components respect to an arbitrary frame seem to be very complicated, symmetry assumptions —spherical symmetry in the case of the Schwarzschild-de Sitter spacetime— can reduce the the number of equations to be analysed. In the case of Petrov type D spacetimes, the symmetry of the spacetime is closely related to the existence of Killing spinors. In this regard, it is of interest to analyse the relation of symmetries at conformal infinity in de-Sitter like spacetimes. In other words conformal Killing vectors and Killing spinors at the conformal boundary in de-Sitter like spacetimes —e.g., the Schwarzschild de-Sitter and Kerr-de Sitter spacetimes. In [85] an analysis has been given exploiting the *unphysical Killing initial data equations* introduced in [72, 107] and a characterisation of the Kerr-de Sitter spacetime via the *Mars-Simon tensor* —see [108]. Nevertheless, as Killing spinors can be considered as a more fundamental quantity than Killing vectors, it is natural to try to find a similar characterisation as the one given in [85] but exploiting the notion of Killing spinors instead. Furthermore, as the *Mars-Simon* *tensor* and the *rescaled Mars-Simon tensor* introduced in [85] share the same symmetries of the Weyl tensor, a spinorial approach seems appropriate. The interplay between the existence of Killing spinors at the conformal boundary of de-Sitter like spacetimes can potentially be related with the properties of the asymptotic initial data for these spacetimes. These notions can be, ultimately, useful for making inroads into the resolution of the uniqueness problem of black holes with a de-Sitter like Cosmological constant.

One of the conclusions from the analysis of the global structure of spacetimes using the conformal methods can be condensed in the observation that, generically, initial data for the conformal Einstein field equations will not be smooth at least in one point in the conformal extension of the spacetime. In the case of the proof of the semiglobal non-linear stability of the Minkowski spacetime of [8] this point corresponds to spatial infinity i^0; for the Schwarzschild-de Sitter spacetime the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$. In the case of the problem of spatial infinity i^0, a milestone in the resolution of this problem is the construction, originally introduced in [27], of a new representation of spatial infinity known as the *cylinder at spatial infinity*. In this representation, spatial infinity is not represented as a point but as set whose topology is that of a cylinder. In addition, it allows to formulate a regular finite initial value problem for the conformal Einstein field equations. In this regard, Chapter 7 represents an application of the framework of the cylinder at spatial infinity into the analysis of the Newman-Penrose constants. The analysis of these constants has gained some interest recently due to the discussion given in [28] where the concept of soft-hair on black holes is put forward. In the latter reference, the discussion is motivated by analysing the Maxwell equations on a flat background and relating conserved charges which are constructed as surface integrals of the electromagnetic field at future and past null infinity. These quantities correspond, essentially, to the electromagnetic Newman-Penrose constants on flat space. With this motivation, in Chapter 7 the framework of the cylinder at spatial infinity was used to clarify the correspondence between data on a spacelike hypersurface for the spin-1 and spin-2 fields —the former represents the Maxwell spinor and the latter can be interpreted as the linearised gravitational field— propagating on a Minkowski background and the value of their corresponding Newman-Penrose constants at future and past null infinity. In particular, it was shown that the electromagnetic NP constants at future and past null infinity case, are related to each other as they arise from the same terms in the initial data. Moreover, it was shown that this observation is true for data which is not necessarily time-symmetric. A similar result was obtained for the NP constants associated to the spin-2 field. However, in the latter case, for generic initial data, logarithmic singularities at null infinity arise. In such case, the integrals defining the NP constants are divergent unless one imposes an regularisation condition restricting the initial data.

Most of the applications of the conformal Einstein field equations for the stability analysis of solutions to the Einstein field equations make use of the theory of symmetric hyperbolic systems contained in [23]. Nevertheless, to make inroads in to the resolution of the outstanding problem on the full global non-linear stability of black hole spacetimes, say the Kerr-de Sitter spacetime, one first needs to implement more refined theory of partial differential equations and have more control on the behaviour of the perturbations by obtaining suitable estimates. The natural approach would be to import some techniques used in the vector field methods approach used in [10, 61, 109, 110]. Combining these two different approaches; the conformal Einstein field equations and the standard vector field method, is a promising venue as one could exploit the strengths of both approaches. In the current standard applications of vector field methods for analysing perturbations of de-Sitter like spacetimes containing black holes —see [61]— determining the location of the conformal boundary is a delicate issue. In contrast, using the extended conformal Einstein field equations and conformal Gaussian systems of coordinates can be advantageous as in the formulation of the asymptotic initial value problem studied in Chapter 5 one has apriori knowledge of the location of the conformal boundary. Since the implementation of vector field methods for analysing black hole spacetimes using the conformal Einstein field equations would be a long term program, in order to make inroads into this problem one can start with a more modest problem in which one can probe and test these techniques now in the conformal setting. In this regard, one can start considering perturbations of spacetimes which can be conformally embedded into the Einstein cylinder, e.g., the de-Sitter spacetime. One could then use vector field methods to analyse the extended conformal Einstein field equations to obtain more detailed information about the behaviour of the perturbations than that obtained by direct application of theory symmetric hyperbolic systems contained in [23].

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