Ordering Ambiguous Acts [1]

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January 2017

[1] We thank D. Ahn, M. Amarante, E. Dekel, H. Ergin, F. Gul, I. Gilboa, C. Gollier, S. Grant, J-Y. Jaffray, E. Karni, P. Klibanoff, F. Maccheroni, M. Marinacci, S. Morris, W. Pesendorfer, T. Post, M. Ryan, U. Segal, C. Shannon, M. Siniscalchi, and seminar participants at Berkeley, Bocconi, British Columbia, Caltech, DTEA, Gerzensee, Munich, Oxford, Paris (Roy), Rice, RUD and Southampton for helpful discussions.

Abstract

We investigate what it means for one act to be more ambiguous than another. The question is evidently analogous to asking what makes one prospect riskier than another, but beliefs are neither objective nor representable by a unique probability. Our starting point is an abstract class of preferences constructed to be (strictly) partially ordered by a more ambiguity averse relation. First, we define two notions of more ambiguous with respect to such a class. A more ambiguous (I) act makes an ambiguity averse decision maker (DM) worse off but does not affect the welfare of an ambiguity neutral DM. A more ambiguous (II) act adversely affects a more ambiguity averse DM more, as measured by the compensation they require to switch acts. Unlike more ambiguous (I), more ambiguous (II) does not require indifference of ambiguity neutral elements to the acts being compared. Second, we implement the abstract definitions to characterize more ambiguous (I) and (II) for two explicit preference families: α-maxmin expected utility and smooth ambiguity. Thirdly, we give applications to the comparative statics of more ambiguous in a standard portfolio problem and a consumption-saving problem.

JEL Classification Numbers: C44, D800, D810, G11

Keywords: Ambiguity, Uncertainty, Knightian Uncertainty, Ambiguity Aversion, Uncertainty aversion, Ellsberg paradox, Comparative statics, Single-crossing, More ambiguous, Portfolio choice.

1 Introduction

Consider a decision maker (DM) choosing among *acts*, choices with contingent consequences. Following intuitive arguments of Knight (1921) and Ellsberg (1961), pioneering formalizations by Schmeidler (1989) and Gilboa and Schmeidler (1989), and a body of subsequent work, modern decision theory distinguishes two categories of subjectively uncertain belief: *unambiguous* and *ambiguous*. An ambiguous belief cannot be expressed using a single probability distribution. Intuitively, an event is deemed (subjectively) ambiguous if the DM's belief about the event, as revealed by his preferences, cannot be expressed as a unique probability.[1] The usual interpretation is that the DM is uncertain about the 'true' probability of the ambiguous event (and takes this uncertainty into account when making his choice). A DM considers an act to be unambiguous if, for each set of consequences, its inverse image is unambiguous. Otherwise, the act is ambiguous. In this paper we investigate what makes one act *more ambiguous* than another.

One focus of the recent literature applying ideas of ambiguity to economic contexts, finance and macroeconomics in particular, is on how equilibrium trade in financial assets is affected when agents seek assets that are 'robust' to the perceived ambiguity. A comparative static question of interest in such models is, naturally, that of more ambiguous.[2] We need concepts of more ambiguous just as concepts of orders of riskiness were needed to facilitate comparative statics of 'more risky' in economic analysis. A first challenge arises in formulating general defining principles of more ambiguous which, in keeping with revealed preference traditions, are preference based but not tied down to particular parametric preference forms. A second, is to characterize the definitions for preference families commonly encountered in applications.

We explore two related but different ideas for revealing (via choice behavior) whether an act is more affected by ambiguity than another act. These ideas lead to two different orders on the space of acts: *more ambiguous (I)* and *more ambiguous (II)*. Essentially, an act f is more ambiguous (I) than act g if an ambiguity averse DM prefers g to f but an ambiguity neutral DM is indifferent between the acts. Roughly put, an act f is more ambiguous (II) than act g if the more ambiguity averse agent requires more compensation to give up g for f. More ambiguous (I) is akin to the Rothschild-Stiglitz notion of more risky. What lies at the heart of more ambiguous (II) is a strengthening of a single crossing condition: the more ambiguity averse the 'type' of agent the greater the compensation she requires to switch to the more ambiguous act. An advantage of the first definition is it allows us to identify acts which may be differentiated purely and solely in terms how much they are affected by ambiguity. An advantage of the second definition is it allows us to compare acts on the basis of being differently affected by ambiguity even when they are different in other dimensions, such as two assets with different average returns. In both instances the order of more ambiguous arises on the back of a relation on preferences, *more ambiguity averse than*. In the first definition, we compare the choice made by an ambiguity neutral preference with that by an ambiguity averse preference; the second definition compares the choice made by one preference to that by another which is more ambiguity averse. Fixing a preference class partially ordered by a more ambiguity averse relation, the defining

[1] There is an extensive literature discussing the definition of ambiguous events, e.g., Epstein (1999), Ghirardato and Marinacci (2002), Nehring (2001) and Klibanoff, Marinacci and Mukerji (2005).

[2] See, e.g., Hansen (2007), Caballero and Krishnamurthy (2008), Epstein and Schneider (2008), Hansen and Sargent (2010), Uhlig (2010), Ju and Miao (2012), Collard, Mukerji, Tallon, and Sheppard (2011), Gollier (2011).

properties determine whether that *class* deems an act to be more ambiguous than another act.

We also study the case of *events*. Since bets on events are acts with binary outcomes, the two notions of more ambiguous acts are shown to define analogous notions of *more ambiguous events*. Conceptually, these notions take forward the literature on definitions of ambiguous events and are likely to be of interest in applications too; such as, when investigating the effect of more ambiguous on contingent contracts.

The abstract definitions are implemented to characterize more ambiguous (I) and (II) for two families of preferences prominent in applications: *α-maxmin expected utility* preferences (α-MEU) and *smooth ambiguity* preferences. To fix ideas, in this introductory section we focus on the results applying to the former family; as will be seen the results for the other family are closely related. The α-MEU family generalizes the well known maxmin expected utility preferences due to Gilboa and Schmeidler (1989). For these preferences the DM's belief is represented by a compact, convex subset $\Pi \subseteq \Delta$, of the set of all probability measures on the state space. Acts are evaluated by a weighted average of the maximum and minimum expected utility ranging over the set Π of probabilities. Fixing the belief, the greater the weight on the minimum expected utility, the greater the ambiguity aversion.

Each characterization result is predicated on a specified (strictly partially ordered) class of preferences. Typically, elements of a class come from a single family, without further restriction on ambiguity and risk attitudes but with a fixed belief. Fixing belief is natural and necessary given that we are in a framework with subjective belief on states and choice objects are acts, rather than lotteries involving given distributions on outcomes. In this framework one needs such a restriction even when working with expected utility (say, to investigate risk orders on *acts*) since under different beliefs the same act will induce correspondingly different lotteries over outcomes; an act that is riskier under one belief can be the opposite under another belief.[3]

The resulting characterizations are very natural. First, note that for the α-MEU class with subjective belief fixed at Π, preferences between a bet on an event E or an event E' are determined by the closed intervals $\Pi(E) \equiv \{\pi(E) \mid \pi \in \Pi\}$, $\Pi(E') \equiv \{\pi(E') \mid \pi \in \Pi\}$. We show that E is a more ambiguous (I) event than E' if and only if $\Pi(E) \supset \Pi(E')$, with the intervals sharing a common center. The characterization of E is a more ambiguous (II) event than E' is that $\frac{\max \Pi(E)}{\min \Pi(E)} \geq \frac{\max \Pi(E')}{\min \Pi(E')}$, and similarly for the complementary events (since our definition of more ambiguous event is symmetric between events and their complements). The characterization of ($\mathscr{P}$)-more ambiguous (I) events makes precise one sense in which the probabilities of a more ambiguous event vary more as π ranges on Π. For a ($\mathscr{P}$)-more ambiguous (II) event there is a different sense of "vary more" : more elastic, a sense familiar to us from elementary economic theory. We show the characterizing condition for more ambiguous (II) may be formally interpreted as saying that the interval $\Pi(E)$ is more elastic than $\Pi(E')$. Our results, therefore, make concrete two senses in which a DM's belief about a more ambiguous[4] event is less precise.

We next proceed to characterizing more ambiguous acts. The findings echo those for

[3]Our ideas, in particular those related to more ambiguous (II), transfer quite straightforwardly to domains involving, purely, "objective" ambiguity as considered, e.g., in Olszewski (2007) and in Ahn (2008). There preferences are formulated directly over given sets of lotteries and make no reference to a state space, acts nor to any subjective belief. See Section 6 for a discussion and Appendix A.5 for formal details.

[4]It follows from these characterizations that for α-MEU preferences, a more ambiguous (I) event is also a more ambiguous (II) event. However, for smooth ambiguity preferences a more ambiguous (I) event is *not* generally a more ambiguous (II) event. See footnote 10 and Examples 3.1 and 3.2.

events. For instance, for an α-MEU class with a given subjective belief Π and risk attitude (but admitting a range of ambiguity attitudes), a more ambiguous (I) act will have a centered expansion of the interval of corresponding expected utilities. A more ambiguous (II) act f has a translation, $f+p$ such that, the interval of expected utilities of the less ambiguous act g is contained in the interval of expected utilities for $f+p$. Thinking back to the case of *more risky*, a virtue of that theory is that it supplies a constructive procedure to go from a less risky lottery to a more risky lottery (e.g., through a sequence of mean-preserving spreads). In that spirit, we provide sufficient conditions that show how more ambiguous acts may be constructed. For more ambiguous (I), it is sufficient that the lottery induced by less ambiguous act for each $\pi \in \Pi$ is constructed as a garbling of the set of lotteries induced by the more ambiguous act corresponding to the set Π, while ensuring that centers of the sets of lotteries induced by the two acts coincide. For more ambiguous (II), such a sufficient condition is that as π ranges on Π, the set of lotteries induced by the more ambiguous act contains the set of lotteries induced by the less ambiguous act.

We obtain characterizations under an additional restriction on the events and acts being compared: *belief comonotonicity*. A pair of events, E, E', is belief comonotone on Π if for all $\pi_1, \pi_2 \in \Pi$, $(\pi_1(E)-\pi_2(E))(\pi_1(E')-\pi_2(E')) \geq 0$. Hence, belief comonotonicity for a pair of events imposes a linear order $\leqq$ on the set of probability measures Π. Evidently, this condition gives a sense in which two events are (stochastically) similar, e.g., a bet on the S&P increasing by 5% at close on a particular day and an analogous bet on the FTSE, but not a bet on a stock market index and a bet on the outcome of a boxing match. That ambiguity affects two belief comonotone events in a qualitatively similar way is evident from behavior. We show that a pair of belief comonotone events are *characterized*, in preference terms, by a lack of mutual hedging possibilities for (α-MEU) DMs. Belief comonotonicity facilitates particularly clear and succinct characterizations. For example, the characterizing condition for E being more ambiguous (II) than E', is identical for smooth ambgiuity and α-MEU preferences: $\pi(E)$ is more elastic than $\pi(E')$ (as functions of $\pi \in \Pi$).

There is a natural extension of the idea of belief comonotonicity to acts which enables the analysis of the ambiguity of acts to be disaggregated into the analysis of the ambiguity of certain families of (worse outcome) events defined by the acts. By this device more powerful characterizations than are available in the general case become available. Notably, the conditions obtained make no reference to the DMs' risk attitudes. Take the case of *more ambiguous (I)* acts. We are able to define, for each act f, a *joint distribution* over outcomes and beliefs which serves as a sort of 'sufficient statistic' for the act in that it contains all the information relevant to choices between acts for the preference class. Given these joint distributions, it is shown that more ambiguous (I) is characterized by the existing statistical notion of *concordance*. That is, conditioning on an "event" of the kind $\{\pi' \in \Delta \mid \pi' \leqq \pi\}$ makes the conditional distribution of outcomes worse by first-order stochastic dominance for the more ambiguous act than the less ambiguous act. Therefore, a variation in π affects the probability distribution on outcomes more strongly in the case of the more ambiguous act. We show, by an example, how the result can be applied to understand why a particular preference pattern may be chosen by ambiguity averse DMs in Machina's "reflection" example (Machina (2009)).

Finally, we discuss and illustrate the applicability of the theory developed here in some standard economic problems under the belief comonotonicity assumption. We analyze the standard portfolio choice problem with one safe and one uncertain asset and consider the comparative static effect on the optimal portfolio weight when the uncertain asset is replaced another which more ambiguous (I). We identify conditions that yield the "expected" comparative static for the α-MEU case and for the smooth ambiguity case. Thirdly, we analyze an optimal savings problem in which future income is ambiguous, for a class of α-MEU and a class of smooth ambiguity preferences with risk attitude the CARA class. We explore the impact on savings as future income becomes more ambiguous (II) and show that while for the α-MEU class there is no impact, for the smooth ambiguity class the optimal savings increase.

Related literature The literature on more ambiguous is rather spare. Segal (1987) analyzes preferences over binary acts, e.g., $(x, E; 0, \neg E)$, where you win x if the event E occurs, 0 otherwise. The ambiguity concerning the probability of E is represented by a probability distribution F^* on $[0,1]$ governing the probability that E occurs. Then, to rank "degrees of ambiguity" , Segal adopts the criterion that G^* is more ambiguous than F^* if F^* crosses G^* only at their common mean from below. Segal writes, referring to an ambiguity averse DM, "one is tempted to assume that if G^* is more ambiguous than F^*, then the value of $(x, E; 0, \neg E)$ under F^* is greater than its value under G^*" , but shows that this is not generally true. Segal's counterexample naturally inspired us to think of preferences as the appropriate starting point for primitive notions of more ambiguous. The analysis in Grant and Quiggin (2005) is also related, but less so. It proceeds in a direction opposite to the one taken in this paper: starting with a primitive notion of a more uncertain act it goes on to characterize corresponding dual notions of *more uncertainty averse* for various preference models where *uncertainty* is an encompassing notion that does not distinguish between ambiguity and risk. Gul and Pesendorfer (2014) define a notion closely related to that of a more ambiguous event and characterize it for the *Expected Uncertain Utility* theory of decision making under ambiguity proposed and axiomatized in their paper. In our language, their idea is captured by saying a class of preferences sharing the same belief considers an event E to be (Gul-Pesendorfer) more ambiguous than another event E' if there is a pair of preferences $\hat{\succeq}, \succeq$ in the class with $\hat{\succeq}$ more ambiguity averse than $\succeq$ such that $\succeq$ strictly prefers a bet on E to a (same stake) bet on E' while $\hat{\succeq}$ prefers the bet on E'. For the class of α-MEU preferences with a given belief Π, $\alpha \in [0,1]$, the definition is characterized by the subinterval condition $\Pi(E') \subset \Pi(E)$. It is revealing to compare this with the characterizations of more ambiguous (I) and (II) events (for α-MEU). Evidently, since this is an expansion, nested but without requiring a common center, it implies more ambiguous (II) but is implied by more ambiguous (I).[5]

The rest of the paper is organized as follows. In Section 2, following a statement of the formal setting, we present the definitions of more ambiguous acts and events in terms abstract preference classes and describe, more fully, the two explicit, parametric preference families applied in characterizations. Section 3 implements the definitions to characterize more ambiguous events, while Section 4 does the same for more ambiguous acts. Section 5 explores the belief comonotonicity condition and characterizations that obtain with that condition. Section 6 discusses applications and Section 7 concludes.

[5]See Remark 5.4 for a further discussion of the relation with more ambiguous (II).

2 Decision theoretic considerations

2.1 Preliminaries

Let $\boldsymbol{X}$ be a compact subinterval of $\mathbb{R}$ and $\boldsymbol{L}$ the set of distributions over $\boldsymbol{X}$ with finite supports:

$$\boldsymbol{L} = \left\{ l : \boldsymbol{X} \longrightarrow [0,1] \mid l(x) \neq 0 \text{ for finitely many } x\text{'s in } \boldsymbol{X} \text{ and } \sum_{x \in \boldsymbol{X}} l(x) = 1 \right\}.$$

Let $\boldsymbol{S}$ be a separable metric space and let $\boldsymbol{\Sigma}$ be the Borel σ-algebra of subsets of $\boldsymbol{S}$. Denote by $\mathscr{F}_0$ the set of all $\boldsymbol{\Sigma}$-measurable functions from $\boldsymbol{S}$ to $\boldsymbol{L}$. Let $\mathscr{F}$ be a convex subset of $\boldsymbol{L}^{\boldsymbol{S}}$ which includes all constant functions in $\mathscr{F}_0$. In the usual decision theoretic nomenclature, elements of $\boldsymbol{X}$ are (deterministic) *outcomes*, elements of $\boldsymbol{L}$ are *lotteries*, elements of $\boldsymbol{S}$ are *states* and elements of $\boldsymbol{\Sigma}$ are *events*. Elements of $\mathscr{F}$ are *acts* whose state contingent *consequences* are elements of $\boldsymbol{L}$. Hence, given $f \in \mathscr{F}$ and $s \in \boldsymbol{S}$, $f(s)$ is a (finitely supported) probability distribution on $\boldsymbol{X}$ while $f(s)(x)$ denotes the probability of $x \in \boldsymbol{X}$ under $f(s)$. As usual, we may think of an element of $\boldsymbol{L}$ as a *constant act*, i.e., an act with the same consequence in every state. Given an $x \in \boldsymbol{X}$, $\delta_x \in \boldsymbol{L}$ denotes a degenerate lottery such that $\delta_x(x) = 1$. Given $y \in \mathbb{R}$, let $\left(f + \delta_y\right)$ denote a uniform translation of the contingent distributions on outcomes, that is an act such that, $\left(f + \delta_y\right)(s)\left(x + y\right) = f(s)(x)$, $s \in \boldsymbol{S}$, $x \in \boldsymbol{X}$. When there is no possibility of confusion, we will sometimes denote the lottery degenerate at $y \in \boldsymbol{X}$ simply by y, in particular we sometimes write $f + y$ to denote $f + \delta_y$. When translating acts, it is necessary to avoid hitting the bounds of $\boldsymbol{X}$. Let $\boldsymbol{L_J} \subset \boldsymbol{L}$ be the set of all finitely supported lotteries for which outcomes lie in a subinterval $\boldsymbol{J}$ of $\boldsymbol{X}$ with $|\boldsymbol{J}| < |\boldsymbol{X}|/3$ and center coinciding with the center of $\boldsymbol{X}$. Let $\mathscr{F}_{\boldsymbol{J}} \subset \mathscr{F}$ denote acts with consequences whose outcomes lie in $\boldsymbol{L_J}$. For most of the paper we will not need to appeal to the full armory of Anscombe-Aumann acts, and acts with degenerate lottery consequences suffice: let $\hat{\mathscr{F}} \subset \mathscr{F}$ be the set of all acts mapping states to degenerate lotteries over outcomes in $\boldsymbol{X}$ and let $\hat{\mathscr{F}}_{\boldsymbol{J}} \subset \hat{\mathscr{F}}$ be the set of acts with (degenerate) consequences in $\boldsymbol{L_J}$. If $x, y \in \boldsymbol{X}$ and $E \in \Sigma$, xEy denotes the binary act with a (degenerate) consequence x if the realized state $s \in E$ and y otherwise. We say xEy is a bet on E if $x > y$.

Let $\pi : \boldsymbol{\Sigma} \longrightarrow [0,1]$ be a countably additive probability. The set of all such probabilities, π, is denoted by Δ. Let $C(\boldsymbol{S})$ be the set of all continuous and bounded real-valued functions on $\boldsymbol{S}$. Using $C(\boldsymbol{S})$ we equip Δ with the vague topology, that is, the coarsest topology on Δ that makes the following functionals continuous: $\pi \mapsto \int \psi \, d\pi$, for each $\psi \in C(\boldsymbol{S})$ and $\pi \in \Delta$. Let $\mathscr{B}_\Delta$ denote the Borel σ-algebra on Δ generated by the vague topology. Let $\mathscr{B}_{\mathbb{R}}$ and $\mathscr{B}_{\boldsymbol{X}}$ denote the Borel σ-algebras of $\mathbb{R}$ and of $\boldsymbol{X}$, respectively. Given $\pi \in \Delta$, any act $f \in \mathscr{F}$ induces a corresponding lottery, a probability distribution over outcomes. Formally, for the act f, defines the stochastic kernel[6] $(\pi, B) \mapsto P_\pi^f(B)$ from $(\Delta, \mathscr{B}_\Delta)$ to $(\boldsymbol{X}, \mathscr{B}_{\boldsymbol{X}})$ such that

$$P_\pi^f(B) = \int_{\boldsymbol{S}} \sum_{x \in B} f(s)(x) \, d\pi(s), B \in \mathscr{B}_{\boldsymbol{X}}. \tag{1}$$

To save on notation, we sometimes write $P_\pi^f(x)$ to denote the distribution function induced by the act f given a probability π. Specifically, we write $P_\pi^f(x)$ to denote $P_\pi^f((-\infty, x] \cap X)$.

[6] P_π^f defines a stochastic kernel since it is a probability measure on $(\boldsymbol{X}, \mathscr{B}_{\boldsymbol{X}})$ for each $\pi \in \Delta$ and for each $B \in \mathscr{B}_{\boldsymbol{X}}$, $\pi \mapsto P_\pi^f(B)$ is a measurable function.

Note that $x \mapsto P^f_\pi(x)$ is, therefore, well-defined on $\mathbb{R} \supset \boldsymbol{X}$. It is useful to note, given a compact, convex $\Pi \subset \Delta$, $f \in \mathscr{F}$, the kernel P^f_π is *mixture linear* in $\pi \in \Pi$, i.e.,

$$P^f_{\lambda\pi'+(1-\lambda)\pi''} = \lambda P^f_{\pi'} + (1-\lambda)P^f_{\pi''}, \pi', \pi'' \in \Pi \subset \Delta, \lambda \in [0,1]. \tag{2}$$

A binary relation $\succeq$ over $\mathscr{F}$ denotes a DM's preference ordering over acts, the objects of choice. Throughout, we will assume a DM's preferences satisfy properties of weak order and monotonicity, defined below.

Axiom 1. *[Weak order]* The preference $\succeq$ is complete and transitive.

Axiom 2. *[Monotonicity] (i)* If $x, y \in \boldsymbol{X}$ and $x \geq y$ then $\delta_x \succeq \delta_y$.
(ii) For every $l, l' \in \boldsymbol{L}$, if $l \succ l'$ and $0 \leq \beta < \alpha \leq 1$, then $\alpha l + (1-\alpha)l' \succ \beta l + (1-\beta)l'$.
(iii) For every $f, g \in \mathscr{F}$, $f(s) \succeq g(s)$ for all $s \in \boldsymbol{S}$ implies $f \succeq g$.

Note, (i) and (ii) of Axiom 2 ensures that preferences over lotteries respect first order stochastic dominance, while (iii) ensures that preferences are state independent.

2.2 Defining more ambiguous

We define ordinal measures of how much the (subjective) evaluation of an act is affected, relative to another act, by (subjectively perceived) ambiguity. The measures are calibrated with reference to individual preferences by comparing how acts are evaluated by two preferences, one of which is *more ambiguity averse* than the other. Hence, our starting point is a notion of comparative ambiguity aversion. We adopt an established notion. Definition 2.1 is, essentially, a restatement of Epstein (1999) and Ghirardato and Marinacci (2002) definitions of comparative uncertainty/ambiguity aversion which were, in turn, a natural adaptation of Yaari (1969)'s classic formulation of comparative (subjective) risk aversion. Just as the definition of comparative risk aversion requires an a priori definition of a risk-free act, here the analogous role for "ambiguity-free" acts is played by lotteries.

Definition 2.1. Let $\mathscr{P}$ be a class of preferences over $\mathscr{F}$. Let $\succeq_A, \succeq_B \in \mathscr{P}$. We say $\succeq_B$ is ($\mathscr{P}$)-**more ambiguity averse** than $\succeq_A$ if, for all $l \in \boldsymbol{L}$ and $f \in \mathscr{F}$, $f \succeq_B l \Rightarrow f \succeq_A l$ and $f \preceq_A l \Rightarrow f \preceq_B l$.

Remark 2.1. The above definition implies that if two preferences can be ordered in terms of ambiguity aversion then they must rank lotteries in the same way.

As Epstein (1999) notes, to define absolute (rather than comparative) risk aversion, it is necessary to adopt a "normalization" for risk neutrality. The standard normalization is expected value. Analogously, to obtain a notion of absolute ambiguity aversion it is necessary to adopt a normalization for *ambiguity neutrality*. There are two normalizations prominent in the literature. Ghirardato and Marinacci (2002) say a preference is ambiguity neutral if it is a subjective expected utility (SEU) preference. That is, for any $f, g \in \mathscr{F}$, there exists a utility function, $u : \boldsymbol{X} \longrightarrow \mathbb{R}$, and a subjective belief associated with the preference, $\pi \in \Delta$, such that,

$$f \succeq g \Leftrightarrow \int_{\boldsymbol{S}} \left[\sum_{x \in \boldsymbol{X}} u(x) f(s)(x) \right] d\pi(s) \geq \int_{\boldsymbol{S}} \left[\sum_{x \in \boldsymbol{X}} u(x) g(s)(x) \right] d\pi(s).$$

In Epstein (1999), a preference $\succeq$ is ambiguity neutral if it is *probabilistically sophisticated*, that is, a preference that ranks acts or lotteries solely on the basis of their implied probability distributions over outcomes (Machina and Schmeidler (1992)). More precisely, letting $\mathbb{P}$ be the set of all Borel probability measures on $(\boldsymbol{X}, \mathscr{B}_{\boldsymbol{X}})$, $\succeq$ is probabilistically sophisticated if there exists a function $W : \mathbb{P} \longrightarrow \mathbb{R}$, and an associated belief $\pi \in \Delta$, such that, $f \succeq g \Leftrightarrow W\left(P_\pi^f\right) \geq W\left(P_\pi^g\right)$, $f, g \in \mathscr{F}$. We may use either of the above two normalizations of ambiguity neutrality to obtain a *corresponding* notion of (absolutely) ambiguity averse: an *ambiguity averse* preference is one that is more ambiguity averse than an ambiguity neutral preference.

Although Definition 2.1 says $\mathscr{P}$ is partially ordered by a more ambiguity averse relation, this does not necessarily imply that there exists any distinct pair of preferences in $\mathscr{P}$ which are ordered by the relation.

Definition 2.2. Let $\mathscr{P}$ be a class of preferences over $\mathscr{F}$. We say $\mathscr{P}$ is **strictly partially ordered** by ($\mathscr{P}$)-more ambiguity averse if for each $\succeq \in \mathscr{P}$ there exists $\succeq' \in \mathscr{P}$, $\succeq \neq \succeq'$, such that $\succeq$ is ($\mathscr{P}$)-more ambiguity averse than $\succeq'$ or $\succeq'$ is ($\mathscr{P}$)-more ambiguity averse than $\succeq$.

The first notion of more ambiguous we offer is akin to Rothschild and Stiglitz (1970)'s notion of more risky: given $l_1, l_2 \in \boldsymbol{L}$, l_1 is *riskier* than l_2 if l_1, l_2 have the same mean and every risk averter prefers l_2 to l_1. Given an ambiguity neutral DM and another who is more ambiguity averse, we require that the ambiguity neutral DM be indifferent between the two acts being compared while the ambiguity averse DM disprefers the more ambiguous act. The definition is meant to work with either of the two notions of ambiguity neutrality discussed.

Definition 2.3. Let $\mathscr{P}$ be a class of preferences over $\mathscr{F}$ strictly partially ordered by ($\mathscr{P}$)-more ambiguity averse and such that each $\succeq \in \mathscr{P}$ is related to an ambiguity neutral element of $\mathscr{P}$. Given $f, g \in \mathscr{F}$, we say f is a ($\mathscr{P}$)-**more ambiguous (I) act** than g, denoted f ($\mathscr{P}$)-m.a.(I) g, if the following conditions are satisfied:

(i)if $\succeq \in \mathscr{P}$ is ambiguity neutral then $g \sim f$;

(ii)for all $\succeq_A, \succeq_B \in \mathscr{P}$ such that $\succeq_A$ is an ambiguity neutral preference and $\succeq_B$ is ($\mathscr{P}$)-more (respectively, less) ambiguity averse than $\succeq_A$ we have $g \succeq_B (\preceq_B) f$.

The first condition ensures that the acts being compared do not differ in *any* aspect that may be considered relevant by the sub-class that does not care about ambiguity. In *conjunction with* condition (i), condition (ii) allows us to infer that f is more ambiguous; by itself condition (ii) is not good enough, e.g., it may be satisfied if g has greater average returns. The notion of an act being more ambiguous than another is calibrated with respect to a reference class $\mathscr{P}$, restricted to be a strictly partially ordered class of preferences. A restriction on the class of preferences admitted is necessary in order that the definition not be vacuous. It may be helpful to compare with the familiar study of *risk* (e.g. Rothschild and Stiglitz (1970)). Risk is generally studied for lotteries, rather than for acts defined on a state space with subjective beliefs which may differ for different preferences. However, evidently, the analysis of risk may be extended from lotteries to acts if the class of preferences under discussion share the same subjective beliefs (so that an act corresponds to a unique lottery). In Definition 2.3 while $\mathscr{P}$ may well include several ambiguity neutral preferences, incorporating different subjective beliefs and/or risk attitudes, by condition (i) however, each ambiguity neutral preference must deem the acts being compared equivalent thereby restricting the subjective belief associated with the ambiguity neutral preferences included.

Furthermore, every preference included in the reference class may be ordered as more or less ambiguity averse than *some* ambiguity neutral preference in $\mathscr{P}$.

The requirement in Definition 2.3 that ambiguity neutral agents be indifferent between the acts being compared is very natural but it has two drawbacks. First, we may wish to compare acts with respect to how they are affected by ambiguity, even though they may differ on other dimensions.[7] Second, there are reference classes $\mathscr{P}$ of interest which do not contain ambiguity neutral elements. For example, the set of all α-MEU preferences sharing the same set of priors in the representation functional in general will not include an ambiguity neutral sub-class (see Section 2.3). These two considerations motivate our second definition of more ambiguous.

Definition 2.4. Let $\mathscr{P}$ be a class of preferences over $\mathscr{F}$ strictly partially ordered by ($\mathscr{P}$)-more ambiguity averse. Given acts $f, g \in \mathscr{F}_{\boldsymbol{J}}$, we say f is a ($\mathscr{P}$)-**more ambiguous (II) act** than g, denoted f ($\mathscr{P}$)-m.a.(II) g, if for all $p \in \mathbb{R}$ with $\left|p\right| \leq |\boldsymbol{J}|$, $g \succeq_A \left(f + \delta_p\right) \Rightarrow g \succeq_B \left(f + \delta_p\right)$, and $g \succ_A \left(f + \delta_p\right) \Rightarrow g \succ_B \left(f + \delta_p\right)$, whenever $\succeq_B$ is ($\mathscr{P}$)-more ambiguity averse than $\succeq_A$.

First, consider the case where $g \sim_A \left(f + \delta_p\right) \Rightarrow g \succeq_B \left(f + \delta_p\right)$. In this case, the amount p may be interpreted as a "compensating premium" ; it measures, behaviorally, A's welfare loss in giving up g for f. Hence, in this case, the defining property for f to be m.a.(II) than g is that the compensating premium good enough for A is not good enough for B, who is more ambiguity averse than A: an act is identified as more ambiguous (II) if the compensation required to switch to it increases with ambiguity aversion.[8] In general, since we have not assumed continuity, there might not exist p such that indifference, $g \sim_A \left(f + \delta_p\right)$, obtains. Suppose $g \succ_A \left(f + \delta_p\right)$, and so p is an amount that does not sweeten f enough to persuade A to give up g. Then, the definition requires that p won't be enough to persuade B either, whose preference is more ambiguity averse than A's. More abstractly, an equivalent way of thinking of the m.a.(II) definition is that it requires that arbitrary translations of acts being compared satisfy a single crossing property for ambiguity:

Definition 2.5. Let $\mathscr{P}$ be a class of preferences over $\mathscr{F}$. Let $f, g \in \mathscr{F}$. The ordered pair of acts $\left(f, g\right)$, satisfies the **single crossing property for ambiguity** with respect to $\mathscr{P}$, if for all $\succeq_B$ ($\mathscr{P}$)-more ambiguity averse than $\succeq_A$, $g \succeq_A (\succ_A) f \Rightarrow g \succeq_B (\succ_B) f$.

This property identifies an act as being more affected by ambiguity simply by checking the effect of increasing ambiguity aversion on the direction of preference. It defines a fundamental comparative static in that it applies even when the acts being compared differ in other aspects; in contrast to m.a.(I), which applies only when the acts are identical apart for the way they are affected by ambiguity.[9] However, single crossing is not generally

[7]Analogous issues limit the applicability of the Rothschild-Stiglitz notion of more risky. For example, according to this notion, a lottery is not more risky than a degenerate lottery unless support of the latter is its mean. Such issues led to the development other notions of increasing risk, e.g., Jewitt (1989).

[8]Notice such "inter-preferential" comparison is very much a part of m.a.(I) too, and therefore of *more risky* (Rothschild-Stiglitz). In m.a.(I) we compare the preference of a ambiguity neutral DM with that of a DM who is not ambiguity neutral.

[9]The analog of Definition 2.5 for risk (with subjective beliefs) allows that the acts differ in aspects other than riskiness (such as different means) but as risk aversion increases, f tends to become less attractive than g due to f having a greater riskiness component. If $\mathscr{P}$ is taken to be SEU preferences with nondecreasing vNM utility and identical belief, π, the condition is equivalent to the distribution functions $P_\pi^f(x)$, $P_\pi^g(x)$, satisfying a single crossing property, see e.g. Gollier (2001), chapter 7. We make use of this fact below (Lemma A.2).

transitive. Hence, m.a.(II) is constructed to be 'single crossing plus', where the 'plus' is the requirement that single crossing continues to be satisfied following arbitrary translations of f. The role of this extra requirement is to ensure transitivity of the (m.a.(II)) relation. Note, given Monotonicity, f is m.a.(II) g if, whenever $\succeq_B$ is more ambiguity averse than $\succeq_A$ and δ_p and δ_q are such that $g \sim_A \left(f+\delta_p\right)$, $g \sim_B \left(f+\delta_q\right)$ then $q \geq p$. Thus, a m.a.(II) act adversely affects a more ambiguity averse DM more, as measured by the compensation they require to switch acts. By focusing on the *marginal* effect (on the welfare loss in giving up g for f) *exclusively* due to an increase in ambiguity aversion, the defining property is effective even when g and f differ in ways other than how they are affected by ambiguity.[10]

2.2.1 More ambiguous events

As noted in the Introduction, it is of interest to define (comparative) ambiguity of *events*. Preferences for betting on one event rather than another should reveal (a subjective view) as to how much the event is affected by ambiguity compared to the other event.

Definition 2.6. Let $\mathscr{P}$ be a class of preferences over $\mathscr{F}$ strictly partially ordered by ($\mathscr{P}$)-more ambiguity averse. Given events E, $E' \in \mathbf{\Sigma}$, we say E is a ($\mathscr{P}$)-**more ambiguous** (**I**) **event** than E' if xEy is a ($\mathscr{P}$)-more ambiguous (I) act than $xE'y$, for all $x, y \in \boldsymbol{X}$. Similarly, we say E is a ($\mathscr{P}$)-**more ambiguous** (**II**) **event** than E' if xEy is a ($\mathscr{P}$)-more ambiguous (I) act than $xE'y$, for all $x, y \in \boldsymbol{X}$.

Thus, any bet on a more ambiguous event has to be a more ambiguous act, *and* we require the same to hold of bets on corresponding complementary events. The m.a.(I) notion would require an ambiguity neutral preference to be indifferent between betting on either event while an ambiguity averse preference dis-prefers the more ambiguous event. On the other hand, m.a.(II) does not require an ambiguity neutral preference to be so indifferent and would conclude E was more ambiguous than E' if the compensation demanded for swapping a bet on E' for a bet on E increased with ambiguity aversion.

2.3 Parametric families of preferences considered in characterizations

We will apply the definitions to characterize more ambiguous for two parametric families of preferences, the α-maxmin expected utility (α-MEU) family and the smooth ambiguity family. Next, we provide a brief description of these families.

The α-MEU model (Hurwicz (1951), Ghirardato, Maccheroni, and Marinacci (2004), henceforth, GMM)[11] represents preferences over acts in $\mathscr{F}$ according to,

$$V_{\Pi,\alpha,u}(f) = \alpha \min_{\pi\in\Pi} \int_{\boldsymbol{S}} \left[\sum_{x\in\boldsymbol{X}} u(x) f(s)(x)\right] d\pi(s) + (1-\alpha) \max_{\pi\in\Pi} \int_{\boldsymbol{S}} \left[\sum_{x\in\boldsymbol{X}} u(x) f(s)(x)\right] d\pi(s), \quad (3)$$

[10]Note, the two definitions of more ambiguous are distinct in that neither relation is strictly weaker than the other. The first definition, requires an ambiguity neutral benchmark, unlike the second. The second definition satisfies a single crossing property. Just as the Rothschild-Stiglitz notion does not generally satisfy single crossing, neither does the relation generated by Definition 2.3.

[11]The functional form was first suggested by Hurwicz. GMM axiomatizes a functional form of which the α-MEU form is a special case. However, Eichberger, Grant, Kelsey, and Koshevoy (2011) show that the GMM axiomatization does not provide a complete foundation to the special α-MEU case, in particular when the state space, $\boldsymbol{S}$ is finite. Klibanoff, Mukerji, and Seo (2011) suggest a preference based foundation for interpreting the set Π as belief.

where $u:\boldsymbol{X}\longrightarrow\mathbb{R}$ is a nondecreasing (Bernoulli) utility function representing risk attitude, $\Pi\subset\Delta$ is weak*-compact, convex and represents the belief associated with the preference, and $\alpha\in[0,1]$ is a weight interpreted as an index of ambiguity attitude. Let $\mathscr{P}^{M}_{u,\Pi}$ denote a class α-MEU preferences with given u and Π but with α ranging over the interval $[0,1]$.[12] When u ranges over a set U, $\mathscr{P}^{M}_{U,\Pi}\equiv\cup_{u\in U}\mathscr{P}^{M}_{u,\Pi}$. In the characterizations of more ambiguous to follow, we typically set $U=U_1$, the set of nondecreasing utilities. Let $\succeq_A,\succeq_B\in\mathscr{P}^{M}_{U_1,\Pi}$. By Proposition 12 in GMM, $\succeq_A$ is ($\mathscr{P}^{M}_{U_1,\Pi}$)-more ambiguity averse than $\succeq_B\Leftrightarrow\alpha_A\geq\alpha_B$, and u_A and u_B are equal up to an affine transformation, where α_A,u_A, and α_B,u_B are associated with $\succeq_A$ and $\succeq_B$, respectively.

The *smooth ambiguity* model (Klibanoff, Marinacci, and Mukerji (2005), henceforth, KMM)[13] represents preferences over acts according to,

$$V_{\mu,\phi,u}(f)=\int_{\Delta}\phi\left(\int_{S}\left[\sum_{x\in\boldsymbol{X}}u(x)f(s)(x)\right]d\pi(s)\right)d\mu(\pi),\tag{4}$$

where, $u:\boldsymbol{X}\longrightarrow\mathbb{R}$ is a nondecreasing utility function representing risk attitude, $\mu:\mathscr{B}_{\Delta}\longrightarrow[0,1]$, a probability measure on Δ represents the belief associated with the preference and $\phi:u(\boldsymbol{X})\longrightarrow\mathbb{R}$ is a strictly increasing function representing ambiguity attitude. Let $\mathscr{P}^{S}_{u,\mu,\Phi}$, denote the class of smooth ambiguity preferences with a given μ and u and where ϕ ranges over some set $\Phi(u)$ of functions ϕ. When u ranges over a set U, $\mathscr{P}^{S}_{U,\mu,\Phi}\equiv\bigcup_{u\in U}\mathscr{P}^{S}_{u,\mu,\Phi}$. In the characterizations of more ambiguous to follow, we typically set $\Phi(u)=\Phi_1(u)$, the set of strictly increasing ϕ and write $\mathscr{P}^{S}_{U_1,\mu,\Phi_1(u)}\equiv\mathscr{P}^{S}_{U_1,\mu}$. Let $\succeq_A$, $\succeq_B\in\mathscr{P}^{S}_{U_1,\mu}$. Then, by Theorem 2 in KMM, $\succeq_A$ is ($\mathscr{P}^{S}_{U_1,\mu}$)-more ambiguity averse than $\succeq_B\Leftrightarrow\phi_A=h\circ\phi_B$, where $h:\phi_B(u(\boldsymbol{X}))\rightarrow\mathbb{R}$ is concave, and u_A and u_B are equal up to an affine transformation, where u_A,ϕ_A and u_B,ϕ_B are associated with $\succeq_A$ and $\succeq_B$, respectively.

A belief μ, associated with a smooth ambiguity preference, put together with an act, induces a joint probability measure on outcomes and probability distributions over states. For each act $f\in\mathscr{F}$, and $B\in\mathscr{B}_{\boldsymbol{X}}$, $\pi\longmapsto P^{f}_{\pi}(B)$ is a $\mathscr{B}_{\Delta}$ measurable function. The measure μ therefore uniquely[14] defines, for each act $f\in\mathscr{F}$, a probability measure $P^{f,\mu}$ on $(X\times\Delta,\mathscr{B}_{\boldsymbol{X}}\times\mathscr{B}_{\Delta})$ such that for every $C\in\mathscr{B}_{\Delta}, B\in\mathscr{B}_{\boldsymbol{X}}$,

$$P^{f,\mu}(B\times C)=\int_{C}P^{f}_{\pi}(B)\,d\mu(\pi).\tag{5}$$

Recall, the definition of m.a.(I) invokes the existence of an ambiguity neutral element in the relevant preference class. The class $\mathscr{P}^{S}_{U_1,\mu}$ includes an SEU preference: the case where ϕ is affine. However, for a *given* compact, convex $\Pi\subset\Delta$, $\mathscr{P}^{M}_{U_1,\Pi}$, does not in general contain an SEU preference. Rogers and Ryan (2012) shows the α-MEU preference $(\Pi,0.5,u)$ is an ambiguity neutral (SEU) preference if and only if Π is *centrally symmetric*.[15]

[12]Henceforth, whenever we refer to a compact $\Pi\subset\Delta$, representing belief associated with an α-MEU preference, we mean it is weak*-compact.

[13]For other preference models with similar representations see Ergin and Gul (2009), Nau (2006), Neilson (2010) and Seo (2009).

[14]See, e.g., Meyer (1966), T14, p.15.

[15]An α-MEU preference represented with a given centrally symmetric Π, may be alternatively parameterized as an MEU preference or a maxmax EU preference with a *different* representation set of priors (a set that varies with the α in the original representation). In the alternative parametric representation attitudes to ambiguity and beliefs are not separated, which makes it less useful for conducting comparative static exercises

Definition 2.7. A set $\Pi \subseteq \Delta$ is **centrally symmetric** if there exists $\pi^\star \in \Pi$(called the center of Π) such that, for any $\pi \in \Delta$, $\pi \in \Pi \Leftrightarrow \pi^\star - (\pi - \pi^\star) \in \Pi$.

As noted in KMM, SEU preferences are the only probabilistically sophisticated preferences within the smooth ambiguity class (so long as preferences over lotteries are expected utility). Marinacci (2002), shows SEU preferences are the only probabilistically sophisticated preferences within the class of α-MEU preferences defined over acts whose domain includes at least one unambiguous event which is assigned a strictly positive probability by the subjective belief(s) associated with the preferences in the class. Hence, essentially, it is without loss of generality for us to assume SEU as the benchmark model for ambiguity neutrality for α-MEU preferences.

3 Characterizing more ambiguous events

3.1 More ambiguous (I)

α-MEU preferences At the outset, it is important to note that, since application of Definition 2.6 requires the existence of an ambiguity neutral element in the preference class, ambiguity neutrality is required for the *full* set of acts $\mathcal{F}$ and not just on bets on the events being compared. Hence, we characterize the definition for a class of preferences corresponding to a belief described by a compact, convex, centrally symmetric $\Pi \subset \Delta$.[16] Given an event $E \in \Sigma$, since Π is compact convex, the set of points $\pi(E) \in [0,1]$ as π ranges over Π is a closed interval which we denote as $\Pi(E) = \{\pi(E) \mid \pi \in \Pi\} = [\min \Pi(E), \max \Pi(E)] \subset [0,1]$. This interval has center $\frac{1}{2}\min \Pi(E) + \frac{1}{2}\max \Pi(E)$. It is easy to check, denoting the center of Π as $\pi^\star$, that $\frac{1}{2}\min \Pi(E) + \frac{1}{2}\max \Pi(E) = \pi^\star(E)$.

Proposition 3.1. *Let $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$, where Π is a compact, convex centrally symmetric subset of Δ with center $\pi^\star$. Consider two events, $E, E' \in \Sigma$. The following are equivalent:*
(i) E is a ($\mathscr{P}$)-more ambiguous (I) event than E';
(ii) $\Pi(E') \subset \Pi(E)$ and $\pi^\star(E') = \pi^\star(E)$.

One naturally expects $\Pi(E)$ to expand in some way as the event E is substituted for a more ambiguous one. The above proposition shows that $\Pi(E)$ expands while retaining the same center. The retention of the same center ensures that ambiguity neutral elements in the preference class consider a bet on E to be indifferent to a (same stakes) bet on E'.

Smooth ambiguity preferences To assist with intuition, we state the analog for smooth ambiguity preferences for the case where μ has finite support.

Proposition 3.2. *Let $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$, where $\operatorname{supp}\mu = \{\pi_i \in \Delta \mid i = 1, ..., m\}$. Consider two events, $E, E' \in \Sigma$. The following are equivalent:*

(involving more ambiguity averse/ more ambiguous). At the same time, the reparameterization shows the central symmetry assumption puts a strong restriction on the class of preferences admitted. Note central symmetry invoked for characterizing m.a.(I) only; the restriction is not required for m.a.(II). Finally, Siniscalchi (2009) shows if we add his Complementary Independence Axiom to the MEU model the set of representing priors are centrally symmetric, thus indicating a behavioral basis for assuming Central Symmetry.

[16]Note, the characterizations of more ambiguous derived depend on the ordinal properties of the referenced preference class, rather than any particular representation chosen for it. See Appendix B for details.

(i) E is a $(\mathscr{P})$-more ambiguous (I) event than E';
(ii) There exists a row stochastic matrix $[k_{ij}]_{i,j=1,\dots,m}$ such that

$$\pi_j(E') = \sum_{j=1}^{m} \pi_i(E) k_{ij} \tag{6}$$

$$\mu_i = \sum_{j=1}^{m} k_{ij}\mu_j. \tag{7}$$

Equation (6) implies each $\pi(E')$, $\pi \in \Pi$, is contained in the convex hull of $\{\pi(E) \mid \pi \in \Pi\}$; given Π, the corresponding event probabilities of E' lie in a more circumscribed set than those for E. Hence, as counterpart to $\Pi(E') \subset \Pi(E)$ in condition (ii) of Proposition 3.1 we have:

$$\text{co}\{\pi_1(E'),\dots,\pi_m(E')\} \subset \text{co}\{\pi_1(E),\dots,\pi_m(E)\}. \tag{8}$$

Notice, conditions (6 and 8) make no reference to a second order probability distribution beyond the determination of its support. Condition (7) however, implies that the μ-average of the event probabilities is the same whether one considers E or E':

$$\sum_{i=1}^{m} \mu_i \pi_i(E') = \sum_{i=1}^{m} \sum_{j=1}^{m} k_{ij}\pi_i(E)\mu_j = \sum_{j=1}^{m} \mu_j \pi_j(E). \tag{9}$$

Hence, analogous to the requirement $\pi^\star(E') = \pi^\star(E)$ in Proposition 3.1 here it is required that the convex hulls of E, E', share the same barycenter, reflecting the condition that ambiguity neutral elements of $\mathscr{P}$ consider the bets on E and E' indifferently. Evidently, the characterizing condition may be understood as saying that the (second order) distribution on probabilities of the more ambiguous event is a 'mean preserving spread'[17] of the distribution on the probabilities of the less ambiguous event.

3.2 More Ambiguous (II)

The characterization of $(\mathscr{P})$-more ambiguous (I) events makes precise one sense in which the event probabilities of more ambiguous events vary more with π, the distribution on the state space. For $(\mathscr{P})$-more ambiguous (II) events there will be a different sense of "vary more" which is also very natural: more elastic. To set the scene, recall the elementary theory of supply, in which one nondecreasing supply function, $\widetilde{S}(p)$, is said to be more elastic than another, $S(p)$, if for the same increase in price, the proportionate increase in $\widetilde{S}$ is greater than the proportionate increase in S: equivalently, the ratio $\frac{\widetilde{S}}{S}$ is increasing in the price.[18] The latter condition evidently implies that there exists a nondecreasing function η such that $S = \eta \circ \widetilde{S}$ and (since $\widetilde{S}$ and S are both nondecreasing) that the map $\widetilde{S} \mapsto \frac{\widetilde{S}}{\eta \circ \widetilde{S}}$ is

[17] Rothschild and Stiglitz (1970), give a number of conditions equivalent to *Increasing Risk* including one defined by a sequence mean preserving spreads. Müller and Stoyan (2002) give a useful summary of the mathematical literature. Our characterization takes a somewhat different form which is more insightful in the present context. For instance, it is useful in demonstrating the close connection to the characterization in Proposition 3.1 through equation (6), which does not refer to the second order prior (beyond its support).

[18] The equivalence may be seen as follows. Let the ratio of two supply functions, $\frac{\widetilde{S}(p)}{S(p)}$, be increasing in p; equivalently, $\ln\left(\frac{\widetilde{S}(p)}{S(p)}\right)$ is increasing in $\ln p$. But,

$$\frac{d}{d\ln p}\left[\ln\left(\frac{\widetilde{S}(p)}{S(p)}\right)\right] > 0 \Leftrightarrow \frac{d\ln \widetilde{S}(p)}{d\ln p} > \frac{d\ln S(p)}{d\ln p}.$$

nondecreasing. Since η maps a more elastic function to a less elastic one, we might say its defining property is that it is elasticity reducing. In our context, for example, if there is such a function $\eta : \Pi(E) \to [0,1]$ for which $\pi(E') = \eta(\pi(E))$, then E' is evidently less sensitive to variations in $\pi \in \Pi$ than is E. Next, we formally define a map to have the elasticity reducing property in a way that it makes it readily applicable to our context where points in the domain and range of the map are event probabilities.[19] In particular, we will require that this property holds for the complements of the events, as well as for the events themselves. This is natural since our definition of more ambiguous is symmetric between events and their complements.

Definition 3.1. *Let $A \subset [0,1]$. We will say a nondecreasing function $\eta : A \to [0,1]$ is **elasticity reducing** if for all $a_1, a_2 \in A$ such that $0 < a_1 < a_2 < 1$, $\frac{a_1}{\eta(a_1)} \leq \frac{a_2}{\eta(a_2)}$ and $\frac{1-a_1}{1-\eta(a_1)} \geq \frac{1-a_2}{1-\eta(a_2)}$.*

α-MEU preferences Since the m.a.(II) notion does not impose any requirement on an ambiguity neutral preference (and, in fact, does not even require the inclusion of any such preference in the reference class of preferences), we are not restricted to preferences with centrally symmetric Π. In this case, it is intuitive to expect the characterization generalizes the condition of Proposition 3.1 by not requiring the expansion to be centered.

Proposition 3.3. *Let $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$, where Π is a compact, convex subset of Δ. Consider two events, $E, E' \in \boldsymbol{\Sigma}$. The following are equivalent:*

(i) E is a ($\mathscr{P}$)-more ambiguous (II) event than E';

(ii) There exists an elasticity reducing function $\eta : \Pi(E) \to [0,1]$ such that $\min\Pi(E') = \eta(\min\Pi(E))$ and $\max\Pi(E') = \eta(\max\Pi(E))$.

In other words, $\frac{\max\Pi(E)}{\min\Pi(E)} \geq \frac{\max\Pi(E')}{\min\Pi(E')}$, and similarly for the complements (which follows from the definition of elasticity reducing). Thus, restricting attention to the extremal event probabilities, which are all we need to look at given these preferences, the probability of the more ambiguous event (and its complement) varies proportionately more as π ranges on Π. Notice, the the subinterval condition $\Pi(E') \subset \Pi(E)$ implies the condition (ii) in the above proposition; condition (ii) makes evident a more general sense in which the DM may have a less precise belief. For α-MEU preferences, a more ambiguous (I) event is also a more ambiguous (II) event but the converse is false.

Example 3.1. *Consider two events E and E' such that $\Pi(E) = [0.3, 0.7]$ and $\Pi(E') = [0.6, 0.8]$. Clearly, for the preference class $\mathscr{P}^M_{U_1,\Pi}$, where Π is compact, convex, E is an m.a.(II) event than E' since $\frac{\max\Pi(E)}{\min\Pi(E)} \geq \frac{\max\Pi(E')}{\min\Pi(E')}$. But E is not an m.a.(I) event than E' since the subinterval condition fails. To fix ideas, think of a variation of Ellsberg's 2-color, 2-urn example, in which the subject is given imprecise information about the composition of* both *urns, as opposed to the usual example where there is imprecise information about one urn. Each urn has a total of 100 balls, red and/or black. Let E be the event that a red ball is drawn from the urn I which, the subject is told, has between 30 and 70 red balls and let E' be the draw of a red ball from urn II which is known to have between 60 and 80 red balls.*

[19] Elasticity reducing is closely related to star-shapedness. A standard definition of a star-shaped function on $[0,1]$ satisfying $\eta(0) = 0$ is that $x \mapsto \eta(x)/x$ is increasing. In Proposition 3.4 we will define a relation on distributions which is closely related to the star-shaped ordering of Barlow and Proschan (1975).

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Proposition 3.4. *Let $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$. Consider two events, $E, E' \in \mathbf{\Sigma}$. The following are equivalent:*

(i) E is a ($\mathscr{P}$)-more ambiguous (II) event than E';

(ii) There exists an elasticity reducing function $\eta : \{\pi(E)\}_{\pi \in \mathrm{supp}\mu} \to [0,1]$ such that $\pi(E')$ has the same distribution, under μ, as $\eta(\pi(E))$. Specifically,

$$\mu(\{\pi \in \Delta \mid \pi(E') \leq q\}) = \mu(\{\pi \in \Delta \mid \eta(\pi(E)) \leq q\}, q \in [0,1].$$

Consider a probability interval for E with a corresponding event Π_E in Δ. The characterizing condition asserts the existence of a probability interval for E', with a corresponding event $\Pi_{E'}$ in Δ with measure $\mu(\Pi_{E'}) = \mu(\Pi_E)$, such that the probability of the more ambiguous event, E, varies proportionately more than the probability of E', on the respective intervals. Again, evidently, the DM has a less precise belief about the probability of the more ambiguous (II) event.

Remark 3.1. There is a deeper common theme between Propositions 3.4 and 3.3. The revealed preference implications (for the choices used in defining more ambiguous (II) events) would be identical for preferences in $\mathscr{P}^M_{U_1,\Pi}$, if the set $\{(\pi(E), \pi(E')) \mid \pi \in \Pi\}$ were replaced by the convex hull of $(\min \Pi(E'), \min \Pi(E))$ and $(\max \Pi(E'), \max \Pi(E))$. For preferences in $\mathscr{P}^S_{U_1,\mu}$ choices depend on the probability distribution induced over $\{(\pi(E), \pi(E')) \mid \pi \in \Delta\}$ by the measure μ. However, since choices depend only on the marginal probability distributions over the sets $\{\pi(E) \mid \pi \in \Delta\}$ and $\{\pi(E') \mid \pi \in \Delta\}$ we are at liberty to substitute the original probability distribution with any other having the same marginals and the gist of Proposition 3.4 is that we can choose these marginal distributions to be comonotonic in the sense that $\pi(E') = \eta(\pi(E))$ on suppμ for a nondecreasing η. Suppose, for the purpose of illustration, supp$\mu = \Pi$. Then, for both of these revealed preference equivalent transformations of Π, the transformed Π should lie within the graph of an elasticity reducing function. This is illustrated in Figure 1 in Appendix C.

Example 3.2. *We show in this example for smooth ambiguity preferences a more ambiguous (I) event is* not *generally a more ambiguous (II) event. The key intuitive point underlying the construction of the example is that a mean preserving preserving spread may not satisfy single-crossing. Let $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$ and the support of μ consist of four elements, π, π', π'', π'''. Suppose $\pi(E') = \pi'(E') = 0.3, \pi''(E') = \pi'''(E') = 0.7, \pi(E) = 0, \pi'(E) = 0.6, \pi''(E) = 0.6, \pi'''(E) = 0.8$, $\mu(\pi) = \mu(\pi') = \mu(\pi'') = \mu(\pi''') = 0.25$. Then it is easy to confirm that E is a m.a.(I) event than E'. Take the case of linear utility and suppose the DM receives* 1 *if the event occurs and zero if the even does not occur. Let $\phi(u) = \min\{u, 0.7\}$ $\hat{\phi}(u) = \min\{u, 0.65\}$, so $\hat{\phi}$ is an increasing concave transformation of ϕ. It is easy to confirm that $\int \phi(\pi(E'))\, d\mu = \int \phi\left(\pi(E) + p\right) d\mu$ when p = 1/30. However, with this p, $\int \hat{\phi}(\pi(E'))\, d\mu < \int \hat{\phi}\left(\pi(E) + p\right) d\mu$ therefore E is not m.a.(II) than E'.*

4 Characterizing more ambiguous acts

We move on to characterizing more ambiguous acts. The organization and structure of this section follows that of the preceding section giving characterizations in the case of events. We consider, in turn, m.a.(I) and m.a.(II) and for each, characterize the notion for α-MEU and smooth ambiguity preferences. However, to set ideas, we begin with two examples.

Example 4.1. *Consider a model where the return of the an asset is uncertain and ambiguous in the sense that there is uncertainty about the probability distribution governing each random return. We adopt the framework of Hara and Honda (2016) in describing this uncertainty. The uncertainty about return is conceptualized as a consequence of* model uncertainty*: we assume the defining property of a model is that it fixes a probability distribution for each random return, X_i, and the model itself is a random variable, M, where $M_i \in \mathbb{R}$ denotes the mean return of asset i conditional on model M. In this set up, model uncertainty* only *affects the (conditional) mean of the return, not the (conditional) variance, which is identical under each model realization. The conditional mean of the return has a Normal distribution as does the "second order belief" describing the random variable M: $E(X_i \mid M) = M_i \sim N(m_i, \left(\sigma_i^M\right)^2), (X_i \mid M) \sim N(M_i, \sigma_i^2 - \left(\sigma_i^M\right)^2)$ and unconditional distribution of the return is $X_i \sim N\left(m_i, \sigma_i^2\right)$.*

Example 4.2. *We consider an investor with smooth ambiguity preferences with CARA type risk aversion θ, i.e., $u(x) = -exp(-\theta x)$ and overall utility $V(Z) = E\left[\phi\left(Eu(Z)|M\right)\right]$ where $\phi(z) = -(-z)^{\gamma/\theta}$. If $\gamma = \theta$, V is a standard expected utility functional. If $\gamma > \theta$ the decision maker is ambiguity averse. Denote $\eta \equiv \frac{\gamma}{\theta} - 1 = \frac{\gamma-\theta}{\theta}$, which is the coefficient of ambiguity aversion of the decision maker. Given this class of preferences and beliefs about returns described above, the investor will evaluate a unit of asset i (i.e., evaluate a portfolio consisting entirely of a unit of asset i) as:*

$$V(i) = m_i - \frac{\theta}{2}\left(\sigma_i^2\right) - \frac{\gamma-\theta}{2}(\sigma_i^M)^2.$$

Indeed, these preferences coincide with the robust mean variance preferences posited in Maccheroni, Marinacci, and Ruffino (2013).

We may show that Asset 1 is more ambiguous (I) than asset 2 for this class of preferences iff $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$, and $\sigma_1^M > \sigma_2^M$. Ambiguity neutrality implies $\gamma - \theta = 0$ and ambiguity aversion is given by $\gamma - \theta > 0$. Hence, the requirement that $V(1) = V(2)$ for all $\theta > 0$ and $\gamma - \theta = 0$ implies $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$. Given this holds, $V(1) < V(2)$ for all ambiguity averse preferences in the class obtains if and only if $\sigma_1^M > \sigma_2^M$.

On the other hand, since $[V(1) - V(2)]$ is decreasing in η while holding θ constant, we have Asset 1 is more ambiguous (II) than asset 2 for the class of preferences under consideration if and only if $\sigma_1^M > \sigma_2^M$.

4.1 More ambiguous (I)

We characterize m.a.(I) acts for the classes $\mathscr{P}^M_{U_1,\Pi}$ and $\mathscr{P}^S_{U_1,\mu}$. However, the characterizing conditions will refer to u, an arbitrary utility in U_1. We also give sufficient conditions which do not depend on u. For each preference class the sufficient conditions are expressed in two parts. The first, which is common to both, is a 'garbling' condition on the probability distribution over outcomes induced by the acts. The second is a 'balance' preserving condition which ensures indifference of the ambiguity neutral elements of the preference classes.

Definition 4.1. *Let* $\Pi \subset \Delta$ *and* let $f \in \mathscr{F}$. *We say a stochastic kernel $(\pi, C) \mapsto K_\pi(C)$ from* $(\Pi, \mathscr{B}_\Pi)$ *to itself* ***π-garbles*** *act f into act $g \in \mathscr{F}$ if for all $B \in \mathscr{B}_{\boldsymbol{X}}$,*

$$P^g_{\pi'}(B) = \int_\Pi P^f_\pi(B)dK_{\pi'}(\pi), \pi' \in \Pi. \tag{10}$$

We say g is a ***π-garbling*** *of f if there exists a stochastic kernel such that (10) obtains.*

Hence, $P^g_{\pi'}(B)$, $\pi \in \Pi$, probabilities of a set of outcomes under act g lie in a more circumscribed set than those for the same set of outcomes under act f, analogous to condition (8):

$$\text{co}\{P^g_\pi(B) \mid \pi \in \Pi\} \subset \text{co}\{P^f_\pi(B) \mid \pi \in \Pi\}, B \in \mathscr{B}_{\boldsymbol{X}}.$$

As with condition (8), the π-garbling condition makes no reference to a second order probability distribution. We state next a notion of preserving balance that applies to α-MEU preferences.

Definition 4.2. *Let Π be a compact, convex centrally symmetric subset of Δ with center $\pi^\star$, and let $f \in \mathscr{F}$. We say the stochastic kernel $(\pi, C) \mapsto K_\pi(C)$ from $(\Pi, \mathscr{B}_\Pi)$ to itself is (f, Π)-* ***center preserving*** *(or, if clear from the context, simply center preserving) if for all Borel sets $B \in \mathscr{B}_{\boldsymbol{X}}$,*

$$P^f_{\pi^\star}(B) = \int_\Pi P^f_\pi(B) dK_{\pi^\star}(\pi). \tag{11}$$

If there is a center preserving stochastic kernel which π-garbles f into g, we say the π-garbling is center preserving. Then (from substituting $\pi^\star$ into (10)) the acts share the same distribution of outcomes at the belief over states $\pi = \pi^\star$:

$$P^f_{\pi^\star}(B) = P^g_{\pi^\star}(B), B \in \mathscr{B}_{\boldsymbol{X}}. \tag{12}$$

Hence, any ambiguity neutral element in $\mathscr{P}^M_{U_1,\Pi}$ will be indifferent between the acts.
The second notion of preserving balance applies to smooth ambiguity. It is:

Definition 4.3. *Let μ be a probability measure on $(\Delta, \mathscr{B}_\Delta)$. Let K_π be a stochastic kernel $(\pi, C) \mapsto K_\pi(C)$ from $(\Delta, \mathscr{B}_\Delta)$ to $(\Delta, \mathscr{B}')$, where $\mathscr{B}' \subset \mathscr{B}_\Delta$ is a is a sub-sigma algebra. We say K_π is* ***measure–μ preserving*** *(or, if clear from the context, simply measure preserving) if for all $C \in \mathscr{B}'$,*

$$\mu(C) = \int_\Delta K_\pi(C) d\mu(\pi). \tag{13}$$

If there exists a measure-μ preserving stochastic kernel K_π from (Δ, B_Δ) to itself which π-garbles f into g, we say the π-garbling is measure-μ preserving. It is useful to note (from integrating both sides of (10)) that then the acts share the same μ-averaged distribution over outcomes:

$$P^{g,\mu}(B \times \Delta) = \int_\Delta P^g_\pi(B) d\mu(\pi) = \int_\Delta P^f_\pi(B) d\mu(\pi) = P^{f,\mu}(B \times \Delta), B \in \mathscr{B}_{\boldsymbol{X}}. \tag{14}$$

Hence, every ambiguity neutral element in $\mathscr{P}^S_{U_1,\mu}$ is indifferent between the acts.

α-MEU preferences. The following proposition is a natural generalization of the center preserving expansion condition in Proposition 3.1.

Proposition 4.1. *Let Π be a compact, convex centrally symmetric subset of Δ with center $\pi^\star$. Let $f, g \in \hat{\mathscr{F}}$. In the following, (i) and (ii) are equivalent and (iii) implies (i) and (ii).*

(i) f is a $\left(\mathscr{P}^M_{U_1,\Pi}\right)$–more ambiguous (I) act than g.

(ii) For each $u \in U_1$, $\left\{\int u(g) d\pi\right\}_{\pi\in\Pi} \subset \left\{\int u(f) d\pi\right\}_{\pi\in\Pi}$ and $\int u(f) d\pi^\star = \int u(g) d\pi^\star$.

(iii) There exists a center preserving stochastic kernel from $(\Pi, \mathscr{B}_\Pi)$ to itself which π-garbles f into g.

The equivalence condition is that the interval of expected utilities, induced by the given u and π ranging on the given Π acting on the more ambiguous act, contains the one similarly induced via the less ambiguous act with both intervals sharing a common center. The sufficient condition, described by a π-garbling, is independent of u: it is a condition linking the two sets of distributions on outcomes induced by the two acts (and Π). It is sufficient that P^g_π for each π is constructed as a garbling of the set of distributions $\left\{P^f_\pi\right\}_{\pi\in\Pi}$ while ensuring that $P^f_{\pi^\star} = P^g_{\pi^\star}$.

Smooth ambiguity preferences. The next proposition similarly generalizes Proposition 3.2.

Proposition 4.2. *Let $f, g \in \hat{\mathscr{F}}$. Denote $\mathfrak{f}^u(\pi) = \int_X u\, dP^f_\pi$, $\mathfrak{g}^u(\pi) = \int_X u\, dP^g_\pi$. In the following, (i) and (ii) are equivalent and (iii) implies (i) and (ii).*

(i) f is a $\left(\mathscr{P}^S_{U_1,\mu}\right)$—more ambiguous (I) act than g.

(ii) For each $u \in U_1$ there is a measure-μ preserving stochastic kernel $(\pi, C) \mapsto K^u_\pi(C)$ from $(\Delta, \mathscr{B}_\Delta)$ to $(\Delta, \mathfrak{f}^{u-1}(\mathscr{B}_\Delta))$ for which

$$\mathfrak{g}^u(\pi') = \int_\Delta \mathfrak{f}^u(\pi) d K^u_{\pi'}(\pi), \pi' \in \Delta$$

(iii) There is a measure-μ preserving stochastic kernel $(\pi, C) \mapsto K_\pi(C)$ from $(\Delta, \mathscr{B}_\Delta)$ to itself which π-garbles f into g.

The action of the stochastic kernel in condition (ii) implies

$$\mathrm{co}\{\mathfrak{g}^u(\pi) \mid \pi \in \mathrm{supp}\mu\} \subset \mathrm{co}\{\mathfrak{f}^u(\pi) \mid \pi \in \mathrm{supp}\mu\}, \tag{15}$$

with the two convex hulls having the same (μ-weighted) barycenter (analogous to condition (ii) in Proposition 4.1). As in the previous proposition, the sufficient condition is described by a measure preserving π-garbling which does not refer to a particular $u \in U_1$. Here, it is sufficient that P^g_π for each π is constructed as a garbling of the set of distributions $\left\{P^f_\pi\right\}_{\pi\in\mathrm{supp}\mu}$ while ensuring that the acts induce the same μ-averaged distribution over outcomes.

4.2 More ambiguous (II)

α-MEU preferences.

Proposition 4.3. *Let $f, g \in \hat{\mathscr{F}}_J$. The following are equivalent:*

(i) f is a $\left(\mathscr{P}^M_{U_1,\Pi}\right)$-more ambiguous (II) act than g;

(ii)For each $u \in U_1$ and $p \in \mathbb{R}$, $\left|p\right| \le |J|$, the following single crossing property holds:

$$\min_{\pi\in\Pi} \int u(f+p) d\pi > \min_{\pi\in\Pi} \int u(g) d\pi \;\Rightarrow \max_{\pi\in\Pi} \int u(f+p) d\pi \ge \max_{\pi\in\Pi} \int u(g) d\pi.$$

Two kinds of cases satisfy the single crossing property in (ii). First, a somewhat trivial case is when g is such that expected utility of g for every $\pi \in \Pi$ is either, equal to the utility corresponding to the best outcome or, equal to the utility corresponding to the worst outcome. In the second case, f is $\left(\mathscr{P}^M_{U_1,\Pi}\right)$-m.a.(II) than g if and only if, given any $u \in U_1$, there exists a translation p such that, the interval of expected utilities for g is contained in the corresponding interval for $f+p$:

$$\left[\min_{\pi\in\Pi}\int u(g)d\pi, \max_{\pi\in\Pi}\int u(g)d\pi\right] \subset \left[\min_{\pi\in\Pi}\int u((f+p))d\pi, \max_{\pi\in\Pi}\int u((f+p))d\pi\right].$$

Note, the latter case will hold if the set of lotteries on outcomes induced by g and $\pi\in\Pi$ is contained in the set of lotteries on outcomes induced by f and $\pi\in\Pi$. This gives us a sufficient condition for constructing a more ambiguous act from a given act and one that holds independently of $u\in U_1$.

Smooth ambiguity preferences.

Proposition 4.4. *Let $f, g \in \hat{\mathscr{F}}_{\boldsymbol{J}}$. The following are equivalent:*

(i) f is a $\left(\mathscr{P}^S_{U_1,\mu}\right)$-more ambiguous (II) act than g ;

(ii) For each $u\in U_1$ and $p\in\mathbb{R}$, $\left|p\right|\le|\boldsymbol{J}|$, the following single crossing property holds: For each $v_1 < v_2$,

$$\mu\left(\left\{\pi\in\Delta \mid \int u(f+p)d\pi \le v_1\right\}\right) < \mu\left(\left\{\pi\in\Delta \mid \int u(g)d\pi \le v_1\right\}\right)$$
$$\Rightarrow \mu\left(\left\{\pi\in\Delta \mid \int u(f+p)d\pi \le v_2\right\}\right) \le \mu\left(\left\{\pi\in\Delta \mid \int u(g)d\pi \le v_2\right\}\right).$$

Hence, an act f is $\left(\mathscr{P}^S_{U_1,\mu}\right)$-m.a.(II) than g if and only if the probability distribution on expected utilities induced by every translation of f, given u and μ, crosses the distribution so induced by g at most once and if so, from above. So, while m.a.(I) is characterized by a mean preserving spread condition, here it is a single crossing condition showing, essentially, that the expected utility of the more ambiguous act, considered as a function of π, has a steeper gradient, as the following example illustrates.

Example 4.3. *Let acts f, g both be constant on an event E and on the complement of E, E^c, with $f(E^c) = f_1 < g(E^c) = g_1 < g(E) = g_2 < f(E) = f_2$. Hence, for $p\in\mathbb{R}$, $\int u(f+p)d\pi$ and $\int u(g)d\pi$ are both increasing affine functions of $\pi(E)$. If $u\left(f_1+p\right)\ge u\left(g_1\right)$ then $\int u(f+p)d\pi \ge \int u(g)d\pi$ for all $\pi\in\Delta$ and the condition of Proposition 4.4 holds. Similarly, if $u\left(f_2+p\right)\ge u\left(g_2\right)$. Suppose therefore that $u\left(f_1+p\right) < u\left(g_1\right)$, $u\left(f_2+p\right) > u\left(g_2\right)$, so considered as a function of $\pi(E)$, $\int u(f+p)d\pi$ has a larger gradient than does $\int u(g)d\pi$. The condition $\mu\left(\left\{\pi\in\Delta \mid \int u(f+p)d\pi\le v_1\right\}\right) < \mu\left(\left\{\pi\in\Delta \mid \int u(g)d\pi \le v_1\right\}\right)$ implies that if $\int u(g)d\pi = v_1$, then $\int u(f+p)d\pi \ge v_1$. Moreover for any $\pi\in\Delta$ with $\int u(g)d\pi > v_1$, i.e. those with a larger $\pi(E)$, we have $\int u(g)d\pi \le \int u(f+p)d\pi$. Hence, for any $v_2 > v_1$, $\mu\left(\left\{\pi\in\Delta \mid \int u(f+p)d\pi \le v_2\right\}\right) \le \mu\left(\left\{\pi\in\Delta \mid \int u(g)d\pi \le v_2\right\}\right)$, as required.*

In our next example, like in examples 4.1 and 4.2, we consider a model where the return of the an asset is uncertain and can be ambiguous in the sense that there is uncertainty about the probability distribution governing each random return and model uncertainty *only* affects the (conditional) mean of the return, not the (conditional) variance, which is identical under each model realization. Furthermore, the return conditional on model realization has a Normal distribution, $(X_i \mid M) \sim N(M_i, \sigma_i)$ and $E(M) = m_i$. However, unlike in those examples, here we allow the "second order belief" describing the random variable M to be *not* Normal. In the example, we characterize more ambiguous (II) for this class of preferences.

Example 4.4. *In this case, writing $\phi(u) = \phi(-e^{-\alpha u})$, we have Asset* 1 *is more ambiguous (II) than asset 2 for the class of preferences under consideration if and only if* $\int \phi\left(m_1 - p - \alpha\frac{\sigma_1^2}{2}\right) d\pi(m_i) = \int \varphi\left(m_2 - \alpha\frac{\sigma_2^2}{2}\right) d\pi(m_i)$ *implies* $\int \hat{\phi}\left(m_1 - p - \alpha\frac{\sigma_1^2}{2}\right) d\pi(m_i) \leq \int \hat{\phi}\left(m_2 - \alpha\frac{\sigma_2^2}{2}\right) d\pi(m_i)$ *whenever ϕ is increasing and $\hat{\phi}$ is an increasing concave transformation of ϕ. By Theorem 2 of Landsberger and Meilijson (1994), this is equivalent to M_1 is more Bickel-Lehman dispersed than M_2. In this case, with non-Gaussian second-order belief, more ambiguous (II) proves to be more tractable than more ambiguous (I).*

5 Adding belief comonotonicity

5.1 When events are belief comonotone

5.1.1 Definition and behavioral meaning

As discussed in Section 1 belief comonotonicity describes a sense in which we may think of a bet on event E and a bet on event E' as 'similar'.

Definition 5.1. *A pair of **events** $E, E' \in \Sigma$ is **belief** **comonotone** on $\Pi \subset \Delta$ if for all $\pi_1, \pi_2 \in \Pi$, $(\pi_1(E) - \pi_2(E))(\pi_1(E') - \pi_2(E')) \geq 0$.*

Belief comonotonicity for a pair of events $E, E' \in \mathbf{\Sigma}$ imposes, or rather requires, a linear order $\leqq_{E,E'}$ on the set of probability measures $\Pi \subset \Delta$. Equivalently, for $\{\pi(E) \mid \pi \in \Pi\}$ and $\{\pi(E') \mid \pi \in \Pi\}$, there exists an nondecreasing ζ such that $\pi(E) = \zeta(\pi(E'))$. Ambiguity about an event E, the uncertainty about its probability, is described by the variation in $\pi(E)$. Ambiguities about two belief comonotone events are qualitatively similar in the sense that they are described by two comonotone variables: when one variable has a relatively high realization, so does the other.[20]
That ambiguity affects two belief comonotone events in a qualitatively similar way is evident in the manner they are viewed by preferences not neutral to ambiguity: they cannot be used to mutually hedge ambiguity. Fix two acts and consider DMs indifferent between them. A key distinguishing feature of DMs not neutral to ambiguity is that they are not indifferent between one of these acts and a strategy that randomly mixes between the the acts, since such mixing may hedge ambiguity. The next proposition shows that belief comonotonicity (on Π) of a pair of events is characterized, in preference terms, by the property that bets on these events will be considered to offer no mutual hedging possibilities by *every* α-MEU DM with belief $\Pi' \subseteq \Pi$.

Proposition 5.1. *Fix Π, a compact, convex subset of Δ. Let $x, y \in \mathbf{X}$, with $x > y$. Let $E_1, E_2 \in \mathbf{\Sigma}$. The following are equivalent:*

(i) The pair of events E_1, E_2 is belief comonotone on Π.

(ii) For each compact, convex $\Pi' \subseteq \Pi$, and $\lambda \in [0,1]$, if $\succeq \in \mathscr{P}^M_{U_1,\Pi'}$, $z \in \mathbf{X}$, $x(E_1)y \sim z(E_2)y$, then $x(E_1)y \sim z(E_2)y \sim f_\lambda$ where $f_\lambda \in \mathscr{F}$ is an act which pays according to $x(E_1)y$ or $z(E_2)y$ with probabilities λ and $1-\lambda$ respectively.

[20]One may construct examples of belief comonotone events with even a state space containing as few as three states. Let $\mathbf{S} = \{s_i\}_{i=1}^3$, $\pi(\{s_1\}) \in [p,q] \subset [0,1]$, $p \neq q$, $\pi(\{s_2\}) = k\pi(\{s_1\})$, with $kq \leq 1-q$. Then $\{s_1\}$ and $\{s_2\}$ are belief comonotone and the set of probabilities on $\mathbf{S}$ is centrally symmetric.

This result may be contrasted with Theorem 2 of Klibanoff (2001) which, in our language establishes, that mixing between indifferent acts never generates a strictly preferred act for any MEU preference with given $u \in U_1$, if and only if there exist $a \geq 0$ and $b \in \mathbb{R}$ such that either $u(f(s)) = a\,u(g(s)) + b$ or $u(g(s)) = a\,u(f(s)) + b$. Klibanoff's condition translated into the events case covered in Proposition 5.1 is extremely restrictive—it implies the events are identical (providing only that u is strictly nondecreasing). Note the important distinction that Klibanoff's result restricts preferences by fixing a $u \in U_1$ but allowing **any** compact convex $\Pi \in \Delta$, whereas we fix Π but admit any $u \in U_1$.

Remark 5.1. *Belief comonotonicity has a related preference implication for smooth ambiguity preferences:* Let $E, E' \in \mathbf{\Sigma}$ be belief comonotone on suppμ. Suppose $E'' \in \mathbf{\Sigma}$, satisfies $xE'y \sim xE''y$ for all ambiguity averse $\succeq \in \mathscr{P}^S_{U_1,\mu}$, and for some $x, y \in \boldsymbol{X}$. Then, for all ambiguity averse $\succeq \in \mathscr{P}^S_{U_1,\mu}$,

$$\lambda\big(xEy\big) + (1-\lambda)\big(xE'y\big) \preceq \lambda\big(xEy\big) + (1-\lambda)\big(xE''y\big), \lambda \in [0,1]. \tag{16}$$

Whereas Proposition 5.1 shows that belief comonotone events eliminate hedging possibilities for preferences in $\mathscr{P}^M_{U_1,\Pi}$, Remark 5.1 shows that while hedging possibilities are not necessarily eliminated for preferences in $\mathscr{P}^S_{U_1,\mu}$, they are minimized.

5.1.2 Characterizing m.a.(II) events

The main role of belief comonotonicity is in relating acts to events, as will become clear. However, it also leads to a relatively strong conclusion in the case of m.a.(II) events: the characterizing conditions for the two classes of preferences, $\mathscr{P}^M_{U_1,\Pi}$ and $\mathscr{P}^S_{U_1,\mu}$ collapse to, essentially, the same condition—that the probability of the more ambiguous event is more elastic. A notable feature of the condition is that the second order belief, μ, does not matter beyond the determination of its support.

Proposition 5.2. *Either let $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$ or, let $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$, where Π is compact, convex. Suppose, the pair of events E, E' is belief comonotone, on* suppμ *in the case $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$, and on* Π *in the case $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$. Then, in the following, (i) is equivalent to (iii), and (ii) is equivalent to (iii):*

(i) In the $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$ case: there is an elasticity reducing function $\eta : \Pi(E) \to [0,1]$ such that $\pi(E') = \eta(\pi(E))$ on Π.

(ii) In the $\mathscr{P}^S_{U_1,\mu}$ case: there is an elasticity reducing function $\eta :$ supp$\mu \to [0,1]$ such that $\pi(E') = \eta(\pi(E))$ almost everywhere on suppμ.

(iii) E is a ($\mathscr{P}$)-more ambiguous (II) event than E'.

Note the difference between the characterizing condition (ii) here and that in Proposition 3.4. Here, $\pi(E')$ is the same as $\eta(\pi(E))$, a.e., whereas in the other proposition $\pi(E')$ and $\eta(\pi(E))$ have the *same distribution*, under μ. Remark 3.1 explains the intuition underlying the result.

5.2 When acts are belief comonotone

It is not only more compelling conceptually but also likely to be more useful in applications to have characterizations of more ambiguous acts which are not inextricably linked to the

DM's attitude to risk, i.e., which are independent of the particular $u \in U_1$ entering the decision makers preference representation. Of course, in general, we can simply require the characterizing conditions in propositions in the preceding subsection to hold for all $u \in U_1$, as is done in Proposition 4.4, for instance. However, by specializing to the case of acts which satisfy a natural extension of the idea of belief comonotonicity of events, much more simple, powerful characterizations obtain than are available in the general case. These characterizations are expressed in terms which satisfy the criterion of not being linked to attitudes to risk.

5.2.1 Definition and relation to belief comonotonicity of events

First, we extend the notion of belief comonotonicity to acts. Bets on events are *binary* acts, hence so long as utility is nondecreasing, choice of a particular utility would not affect the ordering over Π. In the case of general acts, however, the utility function matters for how the set Π is ordered. Extending the notion to acts is essentially a matter of incorporating this extra consideration.

Definition 5.2. *Given a class of utilities U, a collection of* ***acts*** *$A \subset \mathscr{F}$ is* ***belief comonotone*** *on $\Pi \subset \Delta$ if Π can be placed in linear order $\leqq_U$ such that for each $\pi_1, \pi_2 \in \Pi$, $\pi_1 \leqq_U \pi_2$ implies for each $u \in U$*

$$\int_S u(f)d\pi_1 \leq \int_S u(f)d\pi_2 \text{ for each act } f \in A. \tag{17}$$

A collection of bets on different events combine to form simple acts. Conversely, we will find it useful to decompose acts into a collection of bets on events. In the following proposition, we characterize the relation between belief comonotonicity of events and acts by showing that belief comonotone acts may be decomposed into belief comonotone events.

Notation. Given $f \in \hat{\mathscr{F}}$, let $E^f_x \equiv \{s \in \boldsymbol{S} : f(s) \leq x\}$, $x \in \boldsymbol{X}$, denote the event such that the outcomes under f are worse than x; a *worse-outcome event.*

Proposition 5.3. *Let $f, g \in \hat{\mathscr{F}}$. Fix a set $\Pi \subset \Delta$. The following are equivalent:*
(i) Each pair of events $(E^h_x, E^{h'}_{x'})$, $h, h' \in \{f, g\}$, $x, x' \in \boldsymbol{X}$ is belief comonotone on Π.
(ii) The pair of acts f, g is belief comonotone on Π for the class of utilities U_1.

The proposition shows, for the class of nondecreasing utilities, belief comonotonicity of a pair of acts is equivalent to belief comonotonicity of worse-outcome events under the acts. Taken together with Proposition 5.1 (and Remark 5.1), this shows that if two acts are belief comonotone their respective worse-outcome events are affected by ambiguity in a qualitatively similar way, in that DMs behave as if such events do not combine to hedge ambiguity well.

Remark 5.2. A pair $f, g \in \hat{\mathscr{F}}$ is belief comonotone on Π given $u \in U_1$, if and only if the families of probability distributions $\{P^f_\pi | \pi \in \Pi\}$ and $\{P^g_\pi | \pi \in \Pi\}$ are ordered similarly by (first-order) stochastic dominance.

5.2.2 Relating more ambiguous acts to more ambiguous events

Suppose that the collection of events that combine to form a particular act is more ambiguous than those that do the same for another act. It seems natural to ask, under what condition is this necessary and sufficient to make the first act more ambiguous than the second act? More precisely, is there a condition which ensures that f is more ambiguous than g if and only if every worse-outcome event under f, E_x^f, is more ambiguous than E_x^g? The next proposition answers the question for the case of m.a.(I).

Proposition 5.4. *Either let $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$ or, let $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$, where Π is compact, convex and centrally symmetric. Furthermore, given the class of utilities in U_1, suppose that in the case $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$ the pair of acts $f, g \in \hat{\mathscr{F}}$ is belief comonotone on* suppμ *and in the case $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$ is belief comonotone on Π. Then the following are equivalent:*

(i) For each $x \in \boldsymbol{X}$, E_x^f is a $(\mathscr{P})$-$m.a$ (I) event than E_x^g;

(ii) f is $(\mathscr{P})$-$m.a$ (I) than g.

Remark 5.3. Inspection of the proof will reveal immediately that f is a $(\mathscr{P}^S_{U_1,\mu})$-$m.a.(I)$ act than g *implies* for each $x \in X$, E_x^f is a $(\mathscr{P}^S_{U_1,\mu})$-$m.a.(I)$ event than E_x^f, for any pair of acts f, g, not necessarily ones that are belief comonotone. The same can be shown for $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$.

For an intuition behind the preceding result consider the following. Suppose for each $x \in \boldsymbol{X}$, $\pi\left(E_x^f\right)$ fluctuates more with different choices of $\pi \in \Pi$ than does $\pi\left(E_x^g\right)$. Given belief comonotonicity these fluctuations are aligned across the different values of x—they resonate. This resonance ensures that greater fluctuation at the individual event level translates to greater fluctuation at the aggregate level, i.e., at the level of the act f. Without belief comonotonicity. as π ranges on Π, the fluctuations at the level of individual events would be out of sync and therefore may mutually hedge each other. Hence, greater fluctuation at the individual event level may not necessarily aggregate into a greater fluctuation at the act level. This facilitating role of belief comonotonicity is quite general: it applies to *both* preference classes $\mathscr{P}^M_{U_1,\Pi}$ and $\mathscr{P}^S_{U_1,\mu}$. We now turn to the case of more ambiguous (II).

Proposition 5.5. *Suppose* supp$\mu = \Pi$ *is compact convex and given the class of utilities U_1 the pair of acts $f, g \in \hat{\mathscr{F}}_{\boldsymbol{J}}$ is belief comonotone on Π. Let $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$ or $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$. The following are equivalent:*

(i) For each $x \in \boldsymbol{J}$, E_x^f is a $(\mathscr{P})$-m.a.(II) event than E_x^g. Also, for every $\succeq \in \mathscr{P}$, for each distinct $v, w \in \boldsymbol{J}$, $\exists \succeq^*, \succeq^{**} \in \mathscr{P}$ each of which is either more or less ambiguity averse than $\succeq$, such that $v(E_x^f)w \succ^* v\left(E_x^g\right)w$ and $v\left(E_x^g\right)w \succ^{**} v(E_x^f)w$.

(ii) f is a $(\mathscr{P})$-$m.a.(II)$ act than g. Also, for every $\succeq \in \mathscr{P}$, $\exists \succeq^*, \succeq^{**} \in \mathscr{P}$ each of which is either more or less ambiguity averse than $\succeq$, such that $f \succ^* g$ and $g \succ^{**} f$.

Remark 5.4. Condition (i) in Proposition 5.5 is akin to E_x^f being more ambiguous than E_x^g in the sense of the Gul and Pesendorfer (2014). See the discussion in Section 1.

Hence, given a qualification, belief comonotonicity also enables the m.a.(II) relation between worse outcome events to aggregate to the corresponding m.a.(II) relation between acts. The qualification (in condition (i)) is that neither is E_x^f more likely than E_x^g for *every* $\pi \in \Pi$, nor is it less likely than E_x^g for *every* $\pi \in \Pi$. Hence, we may say, given Π, that E_x^f is neither unambiguously more, nor unambiguously less, likely than E_x^g. The single crossing property, fundamental to m.a.(II), does not aggregate, in general: a convex combination of two functions, each of which satisfies single crossing, does not generally satisfy single crossing. The role of the qualification is that it allows single crossing to be preserved in the aggregate, in this context. An analogous qualification applies to condition (ii): that it is *not* the case there is complete agreement in the preference class, irrespective of ambiguity attitude, as to which of the two acts being compared is preferred. Hence, neither act stochastically dominates the other for all $\pi \in \Pi$, and so again, in this sense, neither act is unambiguously preferred to the other. Thus, the qualification restricts attention to cases where ambiguity attitude *can* affect the *direction* of preference between the pair of events/acts being compared, given beliefs.

The rest of this subsection presents characterizations of more ambiguous acts for each of the two preference families given belief comonotonicity. The characterizations are simple and powerful in that they do not refer to particular utilities; only to distribution(s) over outcomes. The clue to understanding how these obtain is in the two results just presented. In general, characterizations of *more ambiguous* involve reference to particular utilities since the notion is based on more ambiguity averse, a relation which orders preferences only when they share the same risk attitude (Remark 2.1). In the case of events though, because we require (by definition) the more ambiguous property to hold for an entire range of *binary* acts (outcomes ranging over an arbitrary interval on the real line) obtaining characterizations by restricting to a particular (non-constant) utility function is as general as leaving the utility unrestricted. Thus, characterization of more ambiguous events do not refer to particular utilities. Belief comonotonicity is the condition under which more ambiguity for events aggregates up to acts. Hence, that is also the condition under which the characterizations of acts don't involve risk attitudes.

5.2.3 Characterizing m.a.(I) acts

If, for the class of utilities U_1, a pair of acts $f, g \in \hat{\mathscr{F}}$ is belief comonotone on $\Pi \subset \Delta$, then Π admits a parameterization in the real line, a point we now elaborate. By Remark 5.2, $\{P^f_\pi | \pi \in \Pi\}$ and $\{P^g_\pi | \pi \in \Pi\}$ are ordered by first-order stochastic dominance. It follows[21], that $\int f\,d\pi = \int f\,d\pi'$ implies $P^f_\pi = P^f_{\pi'}$. Similarly, $\int g\,d\pi = \int g\,d\pi'$ implies $P^g_\pi = P^g_{\pi'}$. Furthermore, for all $\pi, \pi' \in \Pi$, $\int \frac{f+g}{2}\,d\pi = \int \frac{f+g}{2}\,d\pi' \Rightarrow P^f_\pi = P^f_{\pi'}$ *and* $P^g_\pi = P^g_{\pi'}$ in which case, since the acts f, g have identical distributions of outcomes whether π or π' obtains, π and π' may be classed as equivalent. This means it is admissible to carry out the following reparameterization of Π. Allowing each $\pi \in \Pi$ to be represented as $\rho(\pi)$, where $\rho(\pi) = \int \frac{f+g}{2}\,d\pi$, we can define two derived probability measures on $\mathbb{R}$, one for each class of preferences $\mathscr{P}^S_{U_1,\mu}$ and $\mathscr{P}^M_{U_1,\Pi}$, as follows.

For preferences in $\mathscr{P}^S_{U_1,\mu}$, we define for $B \in \mathscr{B}_{\mathbb{R}}$, $\widehat{\mu}(B) \equiv \mu\left(\rho^{-1}(B)\right)$. Hence, the probability measure $P^{f,\mu}$ on $(\boldsymbol{X} \times \Delta, \mathscr{B}_{\boldsymbol{X}} \times \mathscr{B}_\Delta)$ defined in equation (5) may be replaced by the probability measure $P^{f,\widehat{\mu}}$ on $(\boldsymbol{X} \times \mathbb{R}, \mathscr{B}_{\boldsymbol{X}} \times \mathscr{B}_{\mathbb{R}})$ defined by $P^{f,\widehat{\mu}}(B \times D) = P^{f,\mu}(B \times \rho^{-1}(D))$, $B \in \mathscr{B}_{\boldsymbol{X}}$, $D \in \mathscr{B}_{\mathbb{R}}$. Where, noting $P^f_\pi(B)$ is constant on $\rho^{-1}(r)$, and choosing $\pi(r)$ as any selection from $\rho^{-1}(r)$, we have

$$P^{f,\widehat{\mu}}(B \times D) = \int_D P^f_{\pi(r)}(B)\,d\widehat{\mu}(r), B \in \mathscr{B}_{\boldsymbol{X}}, D \in \mathscr{B}_{\mathbb{R}}. \tag{18}$$

Similarly for $P^{g,\widehat{\mu}}$. Note, $P^{f,\widehat{\mu}}$ has two marginal probability measures: one on outcomes given by, $P^{f,\widehat{\mu}}(B \times \mathbb{R}) = \int_\Delta P^f_\pi(B)d\mu$, $B \in \mathscr{B}_{\boldsymbol{X}}$ and the other is the measure $\widehat{\mu}$ defined on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, representing belief.

For preferences in $\mathscr{P}^M_{U_1,\Pi}$, although the DM's preference does not specify a probability distribution over Δ, for the purposes of comparison with the $\mathscr{P}^S_{U_1,\mu}$ case, we nevertheless find it convenient to construct such a distribution. To this end, note the centrally symmetric belief Π associated with $\mathscr{P}^M_{U_1,\Pi}$ determines the pair $\{\underline{r}, \overline{r}\} = \left\{\min_{\pi\in\Pi} \rho(\pi), \max_{\pi\in\Pi} \rho(\pi)\right\}$

[21]This is a standard result. See, e.g., Theorem 1.2.9 in Müller and Stoyan (2002).

and let $\widehat{m}$ (reminiscent of the M in $\mathscr{P}^M_{U_1,\Pi}$) be uniform on this pair. Since Π is centrally symmetric, $\pi(\underline{r}),\pi(\overline{r})$ may be selected so that $\frac{\pi(\underline{r})+\pi(\overline{r})}{2}=\pi^*$, with this selection we define

$$P^{f,\widehat{m}}(B\times D)=\int_D P^f_{\pi(r)}(B)\,d\widehat{m}(r), B\in\mathscr{B}_{\boldsymbol{X}}, D\in\mathscr{B}_{\mathbb{R}}.$$

Note, in particular, that the marginal distribution of outcomes is $P^f_{\pi^*}(B)$. This follows from the mixture linearity of $P^f_\pi(B)$ in π, which we observed holds when Π is convex. The probability measures $P^{f,\widehat{\mu}}$, $P^{f,\widehat{m}}$ on $(\boldsymbol{X}\times\mathbb{R}, \mathscr{B}_{\boldsymbol{X}}\times\mathscr{B}_{\mathbb{R}})$ have associated distribution functions which we economize on notation by writing

$$P^{f,\widehat{\mu}}(x,pi)\equiv P^{f,\widehat{\mu}}\big(\{\xi\in\boldsymbol{X}|\xi\le x\}\times\{\zeta\in\mathbb{R}|\zeta\le pi\}\big)=\int_{r\le pi} P^f_{\pi(r)}(x)\,d\widehat{\mu}(r),(x,pi)\in\mathbb{R}^2.$$

Similarly for $P^{f,\widehat{m}}$.

α-MEU preferences. As a corollary of Propositions 5.4 and 3.1 we have the following characterization. To make the connection notice, (19) is equivalently stated as $\Pi\big(E^g_x\big)\subset\Pi\big(E^f_x\big)$, $x\in\boldsymbol{X}$.

Proposition 5.6. *Let Π be compact, convex and centrally symmetric with center $\pi^\star$. Suppose the pair of acts $f,g\in\hat{\mathscr{F}}$ is belief comonotone on Π for the class of utilities U_1. The following are equivalent:*

(i) f is a $\left(\mathscr{P}^M_{U_1,\Pi}\right)$-more ambiguous (I) act than g;

(ii)$P^f_{\pi^\star}(B)=P^g_{\pi^\star}(B), B\in\mathscr{B}_{\boldsymbol{X}}$, and

$$\left\{P^g_\pi(x)\mid\pi\in\Pi\right\}\subset\left\{P^f_\pi(x)\mid\pi\in\Pi\right\},\ x\in\mathbb{R}. \tag{19}$$

(iii) The distributions $P^{f,\widehat{m}}$ and $P^{g,\widehat{m}}$ have identical marginal distributions and

$$P^{f,\widehat{m}}(x,pi)\ge P^{g,\widehat{m}}(x,pi)\ on\ \mathbb{R}^2.$$

Smooth ambiguity preferences. Given a class of smooth ambiguity preferences $\mathscr{P}^S_{U_1,\mu}$, and acts $f,g\in\hat{\mathscr{F}}$, $P^{f,\widehat{\mu}}$ and $P^{g,\widehat{\mu}}$ have marginal probability measures over outcomes which represent the beliefs of the ambiguity neutral elements of $\mathscr{P}^S_{U_1,\mu}$ and will be equal if these elements are indifferent between the two acts. We have already noted that $P^{f,\widehat{\mu}}$ and $P^{g,\widehat{\mu}}$ also have the other marginal, $\widehat{\mu}$, in common. Hence, $P^{g,\widehat{\mu}}$ and $P^{f,\widehat{\mu}}$ share the same marginals.

From Proposition 5.4, f is a $(\mathscr{P})$-more ambiguous (I) act than g if and only if for each $x\in\boldsymbol{X}$, E^f_x is a $(\mathscr{P})$-$m.a\ (I)$ event than E^g_x. Hence, for each $x\in\boldsymbol{X}$, $\pi(E^f_x)=P^f_\pi(x)$ is 'riskier' than $\pi(E^g_x)=P^g_\pi(x)$, under the probability measure μ in the sense that for all concave φ, $\int\varphi(P^f_\pi(x))d\mu\le\int\varphi(P^g_\pi(x))d\mu$. Recall the classic majorization result of Hardy, Littlewood, and Pólya (1952) which states that when the n-vectors x and y are both arranged in non-decreasing order, $\frac{1}{n}\sum_{i=1}^n\varphi(x_i)\le\frac{1}{n}\sum_{i=1}^n\varphi(y_i)$ for all concave φ if and only if $x_1\ge y_1$, $x_1+x_2\ge y_1+y_2$... and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$. A natural generalization (Theorem 250 of Hardy, Littlewood, and Pólya (1952)) applicable to our question is

$$\int_{(-\infty,z]} P^f_{\pi(r)}(x)d\widehat{\mu}(r)\ge\int_{(-\infty,z]} P^g_{\pi(r)}(x)d\widehat{\mu}(r),\ x\in\boldsymbol{X}, z\in\mathbb{R}.$$

These observations lead to the following characterization.

Proposition 5.7. *Suppose the pair of acts $f, g \in \hat{\mathscr{F}}$ is belief comonotone on* suppμ *for the class of utilities U_1. Then, the following are equivalent:*

(i) f is a $\left(\mathscr{P}^S_{U_1,\mu}\right)$-more ambiguous (I) act than g;

(ii) The distributions $P^{f,\hat{\mu}}$ and $P^{g,\hat{\mu}}$ have identical marginals and

$$P^{f,\hat{\mu}}(x, pi) \geq P^{g,\hat{\mu}}(x, pi) \text{ on } \boldsymbol{X} \times \mathbb{R}. \tag{20}$$

The fact that f m.a.(I) g implies $P^{g,\hat{\mu}}$ and $P^{f,\hat{\mu}}$ have the same marginals means the relation m.a.(I) can be represented by a comparison of the *copulas*[22] of $P^{g,\hat{\mu}}$ and $P^{f,\hat{\mu}}$. A copula corresponding to $P^{f,\hat{\mu}}$ for act f is a function $C^{f,\hat{\mu}} : [0,1]^2 \to [0,1]$ satisfying

$$C^{f,\hat{\mu}}(P^{f,\hat{\mu}}(x,\infty), P^{f,\hat{\mu}}(\infty, pi)) = P^{f,\hat{\mu}}(x, pi).$$

Similarly for act g. Hence, condition (ii) of the proposition can equivalently be stated as: condition (14) together with

$$C^{f,\hat{\mu}} \geq C^{g,\hat{\mu}} \text{ on } [0,1]^2. \tag{21}$$

This condition is discussed in the statistics literature in many places. For instance, Tchen (1980) calls it *concordance*. This is very natural in our context since it implies, for instance, that conditioning on the "event" $\{\pi' \in \Delta \mid \pi' \leqq_{U_1} \pi\}$ for some given $\pi \in \Pi$ makes the conditional distribution of outcomes worse by first-order stochastic dominance for the more ambiguous act than the less ambiguous act. Therefore, a variation in π affects the probability distribution on outcomes more strongly in the case of the more ambiguous act. An exactly analogous interpretation applies to condition (iii) in Proposition 5.6 so the characterizations for *both* preference classes can be interpreted in a unified way via a condition on copulas.

Finally, before moving on to characterize m.a.(II) acts it will be convenient to note an alternative way of expressing the conditions of Proposition 5.6, which also provides an intuitive sufficient condition for Proposition 5.7.

Remark 5.5. Let, $f, g \in \hat{\mathscr{F}}$ be belief comonotone on some $\Pi \subset \Delta$, given U_1. Consider the condition

$$P^f_{\pi_1}(x) - P^g_{\pi_1}(x) \geq P^f_{\pi_2}(x) - P^g_{\pi_2}(x) \tag{22}$$

whenever $\pi_1, \pi_2 \in \Pi$, $\pi_1 \leqq_{U_1} \pi_2$. Equivalently, $\int u\,dP^f_\pi(x) - \int u\,dP^g_\pi(x)$ is increasing in $\leqq_{U_1}$ order on Π for all $u \in U_1$. If Π is compact, convex and centrally symmetric, and condition (12) holds, then condition (22) is equivalent to condition (ii) of Proposition 5.6. Moreover, let $\Pi = \text{supp}\mu$ and suppose the distributions $P^{f,\hat{\mu}}$ and $P^{g,\hat{\mu}}$ have identical marginals, then condition (22) implies (20) in Proposition 5.7.

Our next example, shows how Proposition 5.7 can be applied to explain a debated preference pattern in Machina's "reflection" example (Machina (2009)).

Example 5.1. *We follow Nau's reformulation of the example (Nau (2014)), where he discusses the acts $f_k, k = 1,2,3,4$, shown in Table 1, with payoffs contingent on states $s_{mn}, m = 1,2$, $n = 1,2$.*[23] *All that is objectively known about probabilities is that the events $\{s_{11}, s_{12}\}$ and $\{s_{21}, s_{22}\}$ each occur with a probability equal to 0.5. Finally, let $y > x > 0$.*

[22]The *copula C* of a random vector (Z_1, Z_2) with cdf $F_{Z_1,Z_2}(z_1,z_2)$ and marginal cdfs $F_{Z_1}(z_1), F_{Z_2}(z_2)$ satisfies $F_{Z_1,Z_2}(z_1,z_2) = C(F_{Z_1}(z_1), F_{Z_2}(z_2))$. By Sklar's theorem (Sklar (1959)), the copula is unique if the marginal distributions are atomless. Otherwise the copula is uniquely defined at points of continuity of the marginal distributions.

[23]The act f_k is called f_{k+4} in Nau (2014), $k = 1,2,3,4$.

Table 1: The acts

	s_{11}	s_{12}	s_{21}	s_{22}
f_1	y	$x+y$	y	0
f_2	y	y	$x+y$	0
f_3	0	$x+y$	y	y
f_4	0	y	$x+y$	y

	s_{11}	s_{12}	s_{21}	s_{22}
π_1	0.5	0	0	0.5
π_2	0	0.5	0	0.5
π_3	0.5	0	0.5	0
π_4	0	0.5	0.5	0

Table 2: Belief on the state space

Recent literature (e.g., Baillon, L'Haridon, and Placido, 2011) has seen quite an intense debate about about what would be "natural" preferences of ambiguity averse DMs for the pairs f_1, f_2 and f_3, f_4. Clearly, from the symmetry, the ranking between f_1 and f_2 should be the same as (or "reflected in") that between f_4 and f_3; the debate is about the rank of f_1 with respect to f_2. The opposing contentions as to which pair of acts is more "exposed to ambiguity" are argued at an intuitive level without reference to a set of formal, general principles defining when to consider an act more exposed to ambiguity. The discussion here will give a justification for the preference pattern $f_1 \succ f_2$ and $f_3 \prec f_4$, based on the principles of more ambiguous (I).

To simplify the exposition and analysis, we consider DMs with beliefs given by $\Pi = \{\pi_i\}_{i=1}^4$ where the π_i are as shown in Table 2, and let $\mu(\pi_i)$ be uniform. Evidently, the pair $\{f_1, f_2\}$ is belief comonotone on Π for increasing utilities, with $\pi_1 \leqq_{U_1} \pi_2 \leqq_{U_1} \pi_3 \leqq_{U_1} \pi_4$. One may verify, using Table 3, that $P^{f_2,\mu} \geq P^{f_1,\mu}$ and that the marginals are identical.[24] Hence, by Proposition 5.7, we have that for the class of smooth ambiguity preferences with increasing u and ϕ, and μ as above, f_2 is more ambiguous (I) than f_1. Notice, conditioning on the "event" $Z(\pi_2) = \{\pi' \in \Pi \mid \pi' \leqq_{U_1} \pi_2\}$, the conditional (cumulative) distribution on outcomes induced by $f_k, k = 1,2$, is shown in the matrix corresponding to f_k in Table 3 by dividing the numbers in the column for π_2 by $\mu(Z(\pi_2)) = 1/2$. We see the conditional distribution induced by f_2 is (strictly) worse than that induced by f_1, by first-order stochastic dominance. Hence, in a sense, the distributions conditional on $Z(\pi_2)$ and $Z(\pi_3)$ induced by f_2 are further apart than those induced by f_1. Thus the insight our theory and the characterization brings to this debate is that a variation in π affects the probability distributions on outcomes more strongly in the case of f_2, the more ambiguous act.

5.2.4 Characterizing m.a.(II) acts

α-MEU and smooth ambiguity As in Proposition 5.2, the characterizing condition for m.a.(II) is essentially the same for both families, α-MEU and smooth ambiguity.

[24]The pair $\{f_3, f_4\}$ is also belief comonotone on Π for increasing utilities, *but with* $\pi_1 \leqq_{U_1} \pi_3 \leqq_{U_1} \pi_2 \leqq_{U_1} \pi_4$. One may similarly verify $P^{f_3,\mu} \geq P^{f_4,\mu}$, so that f_3 is more ambiguous (I) than f_4.

f_1					f_2				
	π_1	π_2	π_3	π_4		π_1	π_2	π_3	π_4
0	1/8	1/4	1/4	1/4	0	1/8	1/4	1/4	1/4
y	1/4	3/8	5/8	3/4	y	1/4	4/8	5/8	3/4
$x+y$	1/4	4/8	3/4	1	$x+y$	1/4	4/8	3/4	1

Table 3: Joint distributions, $P^{f_k,\mu}$, for acts f_1 and f_2.

Proposition 5.8. *Suppose* $\operatorname{supp}\mu = \Pi$ *is compact, convex and that given the class of utilities* U_1*, the pair of acts* $f, g \in \hat{\mathscr{F}}_{\boldsymbol{J}}$ *is belief comonotone on* Π*. Then the following are equivalent:*

(i) f *is a* $(\mathscr{P}^S_{U_1,\mu}) - m.a.(II)$ *act than* g*;*

(ii) f *is a* $(\mathscr{P}^M_{U_1,\Pi}) - m.a.(II)$ *act than* g*.*

(iii) For each $\pi_1 \leqq_{U_1} \pi_2$ *from* Π*, and* $p \in \mathbb{R}$ *with* $|p| \leq |\boldsymbol{J}|$*, there exist* $\lambda_1, \lambda_2 \geq 0$*,* $\lambda_1 + \lambda_2 = 1$ *such that for all* $x \in \boldsymbol{J}$*,*

$$\lambda_1(1 - P^{f+p}_{\pi_1}(x)) + \lambda_2 P^{f+p}_{\pi_2}(x) \leq \lambda_1(1 - P^g_{\pi_1}(x)) + \lambda_2 P^g_{\pi_2}(x). \tag{23}$$

It follows from Proposition 5.5 that for condition (iii) to obtain it is sufficient that $P^f_{\pi_1}$ is stochastically dominated by $P^g_{\pi_1}$ and $P^g_{\pi_2}$ is stochastically dominated by $P^f_{\pi_2}$. This gives us a way of constructing a pair of acts related by m.a.(II) that works for both α-MEU and smooth ambiguity and is independent of $u \in U_1$. Evidently, in this case, the distribution of outcomes under act f is, in a very strong way, more affected by the change from π_1 to π_2 than is the distribution of outcomes under act g.

Remark 5.6. The condition, $\pi_1, \pi_2 \in \Pi$ and $\pi_1 \leqq_{U_1} \pi_2$ implies $P^f_{\pi_2}(x) - P^g_{\pi_2}(x) \leq 0 \leq P^f_{\pi_1}(x) - P^g_{\pi_1}(x)$ for all $x \in \boldsymbol{X}$, is sufficient for condition (iii) of Proposition 5.8.

Remark 5.7. The proof of Proposition 5.8 shows that condition (iii) implies (i) without the requirement of convexity of $\operatorname{supp}\mu$.

6 Illustrative applications

We next give an illustrative analysis of a portfolio choice question involving one sure and one uncertain asset and finally of a consumption-saving decision problem where the return from saving is uncertain. The financial instruments are modeled explicitly as lotteries on outcomes induced by the agent's subjective belief. The instrument with certain return is a degenerate lottery. The instrument with uncertain return is modeled as a set of lotteries, with an associated second order belief if such a belief is admitted by the agent's preferences. The question addressed is the comparative static of the agent's portfolio decision when a given uncertain instrument is replaced by one which is more ambiguous. Since the characterizations of more ambiguous are simply conditions on lotteries (on outcomes), they immediately suggest the way the question may be formulated for analysis.

Our theory has directly addressed the question of more ambiguous relations on acts and on events, but *not* on beliefs. However, it is also natural to ask, for instance, how the optimal portfolio choice is affected if the agent's *belief* becomes more ambiguous? Consider the analogous question in a model with a *subjective* expected utility agent. How is the optimal portfolio choice affected if the agent's *subjective* belief becomes more risky? If we take "subjective belief" to mean the agent's (prior) belief on the state space, the question appears to be ill posed since, generally, the state space is not ordered in the way the outcome space is: a given change in the prior may cause the distribution of outcomes induced by one act f to become riskier while simultaneously causing the distribution of outcomes induced by another act g to become *less* risky. A better posed question, therefore, is to ask how the optimal choice is affected if the agent's subjective belief (on states) changes such that the probability distribution *on outcomes* induced by the uncertain asset is made riskier? The answer to this question is evidently the same as the answer to the question as to how optimal portfolio weights change going from one uncertain asset f to a *different* but riskier asset g, holding subjective belief *constant.* Thus, put this way, we can answer the question on the basis of the theory of increasing *risk*, formulated entirely in terms of (the induced) *lotteries* as, e.g., in Rothschild and Stiglitz.

Back to ambiguity, consider the comparative static exercise for, say, an α-MEU agent with belief Π, of replacing one asset f with a more ambiguous asset g with corresponding induced sets of distributions $(P_\pi^f)_{\pi\in\Pi}$ and $(P_\pi^g)_{\pi\in\Pi}$. As in the SEU case, this exercise may be reinterpreted as showing the comparative static effect of a change in belief, from Π to Π', such that the induced set of distributions of outcomes of a *given* (uncertain) asset changes from $(P_\pi^f)_{\pi\in\Pi}$ to $(P_\pi^f)_{\pi\in\Pi'} = (P_\pi^g)_{\pi\in\Pi}$. That is, the set of distributions induced by the belief change is the same as that of a more ambiguous asset under unchanged belief and hence the analysis of this paper applies.

From this perspective, our representations in terms of stochastic kernels are particularly useful in the case of m.a.(I). Suppose, for instance, that there is a stochastic kernel from $(\Delta, \mathscr{B}_\Delta)$ to itself which maps the centrally symmetric set Π into the centrally symmetric set Π'. Then, since $\Pi(E) \subset \Pi'(E)$ for all $E \in \boldsymbol{\Sigma}$,we have by Proposition 3.1 if for some $\Pi(E') = \Pi'(E)$, E is m.a.(I) E'. There is, therefore, for α-MEU, a clear sense in which the action of the stochastic kernel makes *all* events more ambiguous (I). With modification, the same observation holds for smooth ambiguity preferences and it holds for acts as well as events.

The perspective of looking at the question in terms of changes in (induced) sets of lotteries may also be fruitfully applied to questions involving m.a.(II) comparisons. The idea of m.a.(II) extends naturally and straightforwardly to a model of "objective ambiguity," where we consider preferences over given *sets* of lotteries, abstracting away from a framework of a state space, acts and subjective beliefs. Two prominent examples of such preferences are those considered in Olszewski (2007) and in Ahn (2008). For such a model of preferences, given two sets of lotteries, $\mathscr{L}_f$ and $\mathscr{L}_g$, we say $\mathscr{L}_f$ is a more ambiguous set of lotteries than $\mathscr{L}_g$ if the more ambiguity averse preference requires more compensation to give up $\mathscr{L}_g$ for $\mathscr{L}_f$. One may think of Olszewski's and Ahn's preferences as presenting objective analogs of, respectively, the α-MEU, and smooth ambiguity, in much the same way as von Neuman-Morgenstern theory stands in relation to SEU. Characterizations of the more ambiguous for these, respective, classes of preferences follow as corollaries of Propositions 4.3 and 4.4 (see Appendix A.5 for details). For instance, sets of lotteries ordered by set inclusion is also so ordered by this notion of more ambiguous for Olszewski's class preferences. Appendix A.5 also gives an analog of Proposition 5.8 which (following Remark 5.6) allows one to construct a sufficient condition on a pair of sets of lotteries under which one is a more ambiguous set of lotteries than another for *both* Olszewski's and Ahn's preferences and all increasing utilities.

Therefore, we may rephrase the comparative static question of more ambiguous formally in terms of direct comparison of (induced) *sets of lotteries* and answer the question on the basis of the ideas of m.a.(I) and m.a.(II). This is analogous to appealing to the theory of riskier lotteries to answer comparative questions involving choices becoming riskier owing to implicit changes in subjective beliefs.

6.1 Comparative statics of portfolio choice with more ambiguous (I)

A natural test-bed for the applicability of the more ambiguous characterizations is the standard one risky asset one safe asset portfolio problem analyzed by Arrow (1965). In our setting, the uncertain asset is one whose return embodies not only risk, but also ambiguity. The safe asset has neither risk nor ambiguity.

Let an act $f_1 = f \in \widehat{\mathscr{F}}$ correspond to investing wholly in the uncertain asset and the constant act $f_0 \in \widehat{\mathscr{F}}$, represent investing wholly in the safe asset. The DM's objective is to select a portfolio share θ for the uncertain asset, in order to maximize the ex ante evaluation of her final wealth position. If initial wealth is w_0, the final wealth is determined by $w_1 = w_0(\theta f_1 + (1-\theta) f_0)$. We assume no short-selling and that $w_1(s) \in \boldsymbol{X}$ for all $s \in \boldsymbol{S}, 0 \leq \theta \leq 1$. Normalizing the DM's utility so that $w_0 = 1, f_0 = 0$, the program for α-MEU preferences can be written

$$\max_{\theta\in[0,1]}\left(\alpha \min_{\pi\in\Pi} \int_{\boldsymbol{X}} u(\theta x) dP_\pi^f(x) + (1-\alpha) \max_{\pi\in\Pi} \int_{\boldsymbol{X}} u(\theta x) dP_\pi^f(x)\right), \tag{24}$$

and for smooth ambiguity preferences it is

$$\max_{\theta\in[0,1]} \int_\Delta \phi\left(\int_{\boldsymbol{X}} u(\theta x) dP_\pi^f(x)\right) d\mu(\pi). \tag{25}$$

Let $\theta^*\big((\Pi,\alpha,u);f\big)$ and $\theta^*\big((\mu,\phi,u);f\big)$ denote, respectively, the solution to each program. We suppose u is strictly concave in both cases, and that ϕ is strictly concave in the second. It follows that program (25) is concave in $\theta \in \mathbb{R}$ and strictly so in non-degenerate cases. The presence of the $\max_{\pi\in\Pi}$ operator in program (24) means that concavity is not in general assured. However, the belief comonotonicity condition stated in Proposition 6.1 implies that the program *is* concave. Hence, under these conditions, the maxima in both programs are uniquely attained.

We do not aim here to mirror the exhaustive study of the portfolio comparative statics problem which has been carried out for risk with expected utility preferences.[25] It is, however, convenient to adopt an assumption on preferences motivated by that literature. It is known for expected utility DMs that a first-order stochastic dominance improvement in the return of the risky asset will never lead to a smaller portfolio share of the risky asset only if the DM's preferences satisfies auxiliary conditions sufficient to imply that in terms of the normalized utility, $(x,\theta) \mapsto u(\theta x)$ is supermodular on $[0,1] \times \boldsymbol{X}$. It suffices if the DM's utility (not necessarily normalized) has a coefficient of relative risk aversion bounded below unity.[26] Our first result, for α-MEU preferences, gives a suite of conditions sufficient to ensure that the portfolio share of the uncertain asset does not increase as we move to a more ambiguous (I) uncertain asset.

Proposition 6.1. *Let $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$, where Π is a compact, convex centrally symmetric subset of Δ. Suppose given U_1, the pair of acts $f, g \in \hat{\mathscr{F}}$ is belief comonotone on Π. Suppose u is strictly*

[25] See, e.g., Gollier (2011) for a study of the comparative statics of more ambiguity averse in the standard portfolio choice problem.

[26] These and other conditions are comprehensively discussed in Sections 4.5 and 7.2 of Gollier (2001).

concave and for the normalized utility, $(x,\theta)\mapsto u(\theta x)$ *is supermodular and that* $\alpha \geq 0.5$. *If act* f *is* $(\mathscr{P})$*-more ambiguous (I) than act* g *then* $\theta^*\big((\Pi,\alpha,u);f\big)\leq\theta^*\big((\Pi,\alpha,u);g\big)$.

The next proposition finds sufficient conditions for the comparative static to hold for smooth ambiguity preferences. The key conditions are similar to those invoked for the result for α-MEU. However, the proof is more delicate and requires auxiliary assumptions on ϕ, specifically $-\phi''/\phi'$ is nonincreasing.

Proposition 6.2. *Let* $\mathscr{P}=\mathscr{P}^S_{U_1,\mu}$ *and suppose given* U_1, *the pair of acts* $f,g\in\hat{\mathscr{F}}$ *is belief comonotone on* $\operatorname{supp}\mu$. *Suppose* u *strictly concave and for the normalized utility,* $(x,\theta)\mapsto u(\theta x)$ *is supermodular and that* ϕ *is concave and such that* $\frac{-\phi''}{\phi'}$ *is nonincreasing. If act* f *is* $(\mathscr{P})$*-more ambiguous (I) than act* g *and also satisfies condition* (22) *of Remark 5.5 then* $\theta^*\big((\mu,\phi,u);f\big)\leq\theta^*\big((\Pi,\phi,u);g\big)$.

Remark 6.1. An examination of the proof will show that an alternative to the condition that $\frac{-\phi''}{\phi'}$ is nonincreasing, is $\frac{-\phi''}{\phi'}$ is nondecreasing with $\phi'''\leq 0$. This admits the class of quadratic ϕ.

6.2 Comparative statics of savings with m.a.(II)

To illustrate comparative statics using m.a.(II), we consider the following simple savings problem. The agent lives for two periods, has initial known wealth y_1 and will receive uncertain income Y_2 in period 2 generated by an act $f+a\in\hat{\mathscr{F}}$. A DM with α-MEU preferences, has utility given by

$$V^{f,u}_{\mathscr{M}}(a)\equiv u(y_1-a)+\alpha\min_{\pi\in\Pi}\int u(y_2)dP^{f+a}_{\pi}(y_2)+(1-\alpha)\max_{\pi\in\Pi}\int u(y_2)dP^{f+a}_{\pi}(y_2). \tag{26}$$

If the DM has smooth ambiguity preferences, the utility is

$$V^{f,u}_{\mathscr{S}}(a)\equiv u(y_1-a)+\phi^{-1}\left(\int\phi\left(\int u\,dP^{f+a}_{\pi}\right)d\mu(\pi)\right). \tag{27}$$

The problem is to choose savings $a\in\mathbb{R}$ to maximize (26) or (27). We are interested in investigating the impact on savings of a compensated increase in uncertainty, specifically when g is replaced by f, with $f(\mathscr{P})$-$m.a.(II)$ g, while maintaining the DM's standard of living at the initial level of savings. Hence, we compare the cases, according to which preference family applies, in which f satisfies:

$$V^{f,u}_{\mathscr{M}}(a^g_{\mathscr{M}}) = V^{g,u}_{\mathscr{M}}(a^g_{\mathscr{M}}),\ a^g_{\mathscr{M}}\in\arg\max_{a\in\mathbb{R}}V^{g,u}_{\mathscr{M}}(a), \tag{28}$$

$$V^{f,u}_{\mathscr{S}}(a^g_{\mathscr{S}}) = V^{g,u}_{\mathscr{S}}(a^g_{\mathscr{S}}),\ a^g_{\mathscr{S}}\in\arg\max_{a\in\mathbb{R}}V^{g,u}_{\mathscr{S}}(a). \tag{29}$$

The assumptions in the following proposition will imply that $\underset{a\in\mathbb{R}}{\operatorname{argmax}}V^{f,u}_{\mathscr{M}}(a)$ and $\underset{a\in\mathbb{R}}{\operatorname{argmax}}V^{f,u}_{\mathscr{S}}(a)$ are uniquely attained.

Proposition 6.3. *Suppose* $u\in U_1$ *is concave and CARA.*

(a) Let Π *be convex compact. Suppose the pair of acts* $f,g\in\hat{\mathscr{F}}$ *is belief comonotone on* $\Pi\subset\Delta$, *satisfies (28) and* f *is* $\left(\mathscr{P}^M_{U_1,\Pi}\right)$*—more ambiguous (II) than* g. *Then* $\arg\max_{a\in\mathbb{R}}V^{f,u}_{\mathscr{M}}(a)=\arg\max_{a\in\mathbb{R}}V^{g,u}_{\mathscr{M}}(a)$.

(b) Suppose the pair of acts $f, g \in \hat{\mathscr{F}}$ is belief comonotone on $\text{supp}\mu$*, satisfies (29) and f is $\left(\mathscr{P}^S_{U_1,\mu}\right)$−more ambiguous (II) than g. Suppose ϕ is concave and $-\phi''/\phi'$ is decreasing concave. Then* $\arg\max_{a\in\mathbb{R}} V^{f,u}_{\mathscr{S}}(a) \leq \arg\max_{a\in\mathbb{R}} V^{g,u}_{\mathscr{S}}(a)$.

The reason the increasing ambiguity leaves savings unchanged for preferences in $\mathscr{P}^M_{U_1,\Pi}$, is because with CARA preferences the marginal utility u' is an affine transformation of u. Together with belief comonotonicity, compensation implies the first-order condition and therefore optimal savings is unchanged. For preferences in $\mathscr{P}^S_{U_1,\mu}$, the situation is more complicated. The marginal utility enters the first-order condition after being weighted by ϕ' so there is an interaction created by ambiguity aversion which generally affects savings.

7 Concluding remarks

The characterizations of more ambiguous (I) and (II) events make precise two senses in which the event probability of a more ambiguous events varies more as π, the probability measure on the state space, varies. The characterizations of more ambiguous acts demonstrated the sense in which an expected utility evaluation of a more ambiguous act is more sensitive to which specific π is applied in computing the expectation. In the sense the word was used in the introduction, it is less *robust* to ambiguity.

An important auxiliary finding was in characterizing the central role of belief comonotonicity. It is a natural restriction because it rules out the 'hedging', which otherwise undermines the intuitive connection between more ambiguity of an act and more ambiguity of its worse-outcome events. The role it plays works quite generally; it is neither restricted to a particular family of preferences and nor to one particular notion of more ambiguous.

Finally, our discussion and the illustrative applications in Section 6, indicated that the ideas and characterizations may be used to formulate questions involving more ambiguous in a variety of applied contexts.

Apart from applications, an immediate, obvious direction for future research suggested by the analyses here is towards obtaining characterizations for classes of preferences beyond α-MEU and Smooth Ambiguity. The characterization strategy employed in the paper should extend to preference classes where representations in a particular class share an evident, common, belief element while allowing for parametric variation in risk and ambiguity attitudes. For instance, a class of Variational Preferences (Maccheroni, Marinacci, and Rustichini (2006)) or of Vector Expected Utility (VEU) (Siniscalchi (2009)) with a given baseline prior belief.[27] Implementing more ambiguous (I) would require, additionally, a class which includes a sub-class of ambiguity neutral preferences. To illustrate the feasibility of extending our characterization strategy to such classes, in Appendix A.6 we characterize more ambiguous (I) events for the class of ambiguity averse VEU preferences with a given, common, baseline prior P, and $u \in U_1$. This class is also a sub-class of Variational preferences.

[27] Another example is the "Contraction" preferences model of Gajdos, Hayashi, Tallon, and Vergnaud (2008). It should be possible to obtain characterization for this case by mimicking the proof strategies for the α-MEU case and characterizations obtained would be very analogous.

A Appendix

A.1 Proofs of results in Section 3

Proof. [Proof of Proposition 3.1]

Let $u \in U_1$. Since $\Pi(E)$ is a compact interval and $\big[u(x)\pi(E)+u(y)(1-\pi(E))\big]$ is linear in $\pi(E)$, $\min_{\pi\in\Pi}\big[u(x)\pi(E)+u(y)(1-\pi(E))\big]$ and $\max_{\pi\in\Pi}\big[u(x)\pi(E)+u(y)(1-\pi(E))\big]$ are attained at the two extreme points of $\Pi(E)$. For an ambiguity neutral element of the preference class with $\alpha=\frac{1}{2}$, $u\in U_1$, this implies

$$\begin{aligned}V_{\Pi,\frac{1}{2},u}(xEy) &= \frac{1}{2}\min_{\pi\in\Pi}\big[u(x)\pi(E)+u(y)(1-\pi(E))\big]+\frac{1}{2}\max_{\pi\in\Pi}\big[u(x)\pi(E)+u(y)(1-\pi(E))\big]\\ &= u(x)\left(\frac{1}{2}\min\Pi(E)+\frac{1}{2}\max\Pi(E)\right)+u(y)\left(1-\left(\frac{1}{2}\min\Pi(E)+\frac{1}{2}\max\Pi(E)\right)\right)\\ &= u(x)(\pi^{\star}(E))+u(y)(1-\pi^{*}(E)).\end{aligned}$$

Similarly, $V_{\Pi,\frac{1}{2},u}(xE'y) = u(x)(\pi^{\star}(E'))+u(y)(1-\pi^{\star}(E'))$. This establishes that if $\pi^{\star}(E)=\pi^{\star}(E')$ all ambiguity elements of the preference class $\mathscr{P}^{M}_{U_1,\Pi}$ are indifferent between xEy and $xE'y$. Choosing $x,y\in\boldsymbol{X}$ and $u\in U_1$ such that $u(x)\neq u(y)$ shows the condition also to be necessary.

Using again the fact that $\Pi(E)$ and $\Pi(E')$ are compact intervals, it follows that the condition $\Pi(E)\subset\Pi(E')$ is equivalent to the condition $\min\Pi(E)\le\min\Pi(E')\le\max\Pi(E')\le\max\Pi(E)$. Using this, it is straightforward to show (given $\pi^{\star}(E)=\pi^{\star}(E')$) that all preferences in the class $\mathscr{P}$ which are more ambiguity averse than than the ambiguity neutral element $\big(\Pi,\frac{1}{2},u\big)$, that is elements of $\mathscr{P}$ with (Π,α,u), $\alpha\ge\frac{1}{2}$, weakly prefer $xE'y$ to xEy.

To see this, suppose $u(x)>u(y)$, $\alpha>\frac{1}{2}$, then

$$\begin{aligned}V_{\Pi,\alpha,u}(xEy) &= \alpha\big[u(x)\min\Pi(E)+u(y)(1-\min\Pi(E))\big]\\ &\quad+(1-\alpha)\big[u(x)\max\Pi(E)+u(y)(1-\max\Pi(E)))\big].\end{aligned}$$

It follows that,

$$V_{\Pi,\alpha,u}(xEy)-V_{\Pi,\frac{1}{2},u}(xEy)=(\frac{1}{2}-\alpha)\Big(\Big[\big(u(x)-u(y)\big)(\max\Pi(E)-\min\Pi(E))\Big]\Big).$$

Similarly, for E'. Since, $V_{\Pi,\frac{1}{2},u}(xEy)=V_{\Pi,\frac{1}{2},u}((xE'y)$, it follows that $V_{\Pi,\alpha,u}(xEy)<V_{\Pi,\alpha,u}(xE'y)$ if and only if $((\max\Pi(E)-\min\Pi(E))<(\max\Pi(E')-\min\Pi(E')))$. If $u(x)<u(y)$, the proof proceeds in the same way, $u(x)=u(y)$ is trivial. Likewise all preferences with $\alpha<\frac{1}{2}$ weakly prefer xEy to $xE'y$. This establishes the equivalence of conditions (i) and (ii) of the proposition. □

Proof. [Proof of Proposition 3.2]

Let $u\in U_1$, $x,y\in\boldsymbol{X}$. Setting $a=u(y)$, $b=u(x)-u(y)$, we can write

$$V_{\mu,\phi,u}(xEy)=\sum_{i=1}^{m}\phi\,(a+b\pi_i(E))\mu_i,\; V_{\mu,\phi,u}(xE'y)=\sum_{i=1}^{m}\phi\,\big(a+b\pi_i(E')\big)\mu_i.$$

Hence, (i) implies that

$$\sum_{i=1}^{m} \phi\left(a+b\pi_i(E)\right)\mu_i \leq \sum_{i=1}^{m} \phi\left(a+b\pi_i(E')\right)\mu_i,$$

for all concave nondecreasing $\phi : u(\boldsymbol{X}) \to \mathbb{R}$. The inequality is required to hold with equality when ϕ is affine, corresponding to the case of ambiguity neutral preferences. If $b=0$, there is nothing to prove, hence suppose $b>0$ ($b<0$ leads to an equivalent argument). Let $A=\{z\in\mathbb{R} \mid a+bz\in u(\boldsymbol{X})\}\neq\emptyset$. The condition can be stated equivalently as:

$$\sum \pi_i(E)\mu_i = \sum \pi_i(E')\mu_i, \ \sum \phi\left(\pi_i(E)\right)\mu_i \leq \sum \phi\left(\pi_i(E')\right)\mu_i \tag{30}$$

for all nondecreasing concave $\phi : A \to \mathbb{R}$. Equivalently, $\sum \varphi\left(\pi_i(E)\right)\mu_i \geq \sum \varphi\left(\pi_i(E')\right)\mu_i$ for all convex $\phi : A \to \mathbb{R}$. The result follows from Sherman's extension of Hardy, Littlewood, and Polya (1929) (*Sherman (1951)*) which for the reader's convenience we reproduce below as Theorem A.1. To translate, set $\varsigma(v_i)=\vartheta(w_i)=\mu_i$, $m=n, v_i=\pi_i(E), w_i=\pi_i(E'), k_{ij}=\frac{p_{ij}\mu_i}{\mu_j}$. □

Theorem A.1. *Let W be a real vector space. Let ς,ϑ be measures on W with finite supports, $\{v_1,...,v_n\}$, $\{w_1,...,w_n\}$, respectively, then*

$$\sum\nolimits_i \varsigma(v_i)\varphi(v_i) \geq \sum\nolimits_i \vartheta(w_i)\varphi(w_i)$$

for all convex φ if and only if

$$\vartheta(w_j)w_j = \sum\nolimits_i p_{ij}\varsigma(v_i)v_i$$

where

$$p_{ij}\geq 0, \vartheta(w_j)=\sum\nolimits_i p_{ij}\varsigma(v_i), \sum\nolimits_j p_{ij}=1.$$

Lemma A.1. *Suppose E, E' is belief comonotone on $\Pi\subset\Delta$. The following statements are equivalent.*

(a)$\pi(E')$ is an elasticity reducing transformation of $\pi(E)$.

(b) For each $x,y\in\boldsymbol{J}$, $p\in\mathbb{R}$ with $\left|p\right|<|\boldsymbol{J}|$, following implication is true. For all $u\in U_1$, $\pi_1\preceq\pi_2$ (i.e. $\pi_1(E)\leq\pi_2(E)$ and $\pi_1(E')\leq\pi_2(E')$)

$$\pi_1(E)u(x+p)+\left(1-\pi_1(E)\right)u(y+p) > \pi_1(E')u(x)+\left(1-\pi_1(E')\right)u(y) \tag{31}$$

implies

$$\pi_2(E)u(x+p)+\left(1-\pi_2(E)\right)u(y+p) \geq \pi_2(E')u(x)+\left(1-\pi_2(E')\right)u(y). \tag{32}$$

Hence, E' less elastic with respect to ambiguity than E is understood to be the precise condition which makes expected utilities more variable in the single crossing sense of condition (b) of the Lemma.

Proof. The implication in condition (b) is equivalent to

$$\pi_1(E) > \pi_1(E')k_1 + k_2 \tag{33}$$

implies

$$\pi_2(E) > \pi_2(E')k_1 + k_2$$

where $k_1 = \frac{u(x)-u(y)}{u(x+p)-u(y+p)}$, $k_2 = \frac{u(y)-u(y+p)}{u(x+p)-u(y+p)}$. Take the case $x > y$. If (a) holds, $\pi(E') = \eta(\pi(E))$ with $\eta : \Pi(E) \to [0,1]$ an elasticity reducing function, therefore with $\pi_1(E) = \lambda\pi_2(E)$, $0 \leq \lambda \leq 1$, $\eta(\lambda\pi_2(E))) \geq \lambda\eta(\pi_2(E))$. Making these substitutions, the implication in (b) becomes equivalent to

$$\lambda\pi_2(E) > \eta(\lambda\pi_2(E))k_1 + k_2$$

implies

$$\pi_2(E) \geq \eta(\pi_2(E))k_1 + k_2.$$

Using $\eta(\lambda\pi_2(E))) \geq \lambda\eta(\pi_2(E))$, it is seen that a sufficient condition for this implication to hold is that $k_2 \geq 0$, i.e. if $u(y) - u(y+p) \geq 0$, equivalently $p \leq 0$. The argument establishing it for $p \geq 0$ proceeds in exactly the same way, but instead of collecting terms as in (33), we write inequality (33) as

$$1 - \pi_1(E) < \left(1 - \pi_1(E')\right)k_3 + k_4 \tag{34}$$

where $k_3 = \frac{u(y)-u(x)}{u(y+p)-u(x+p)} > 0$, $k_4 = \frac{u(x)-u(x+p)}{u(y+p)-u(x+p)}$, which is required to imply

$$1 - \pi_2(E) < \left(1 - \pi_2(E')\right)k_3 + k_4.$$

And, we make the substitution $1-\pi_2(E) = \lambda(1-\pi_1(E))$, equivalently $1-\lambda+\lambda\pi_1(E) = \pi_2(E)$, $0 \leq \lambda \leq 1$.

$$1 - \pi_1(E) < \left(1 - \eta(\pi_1(E))\right)k_3 + k_4,$$

which is required to imply

$$\lambda(1 - \pi_1(E)) \leq \left(1 - \eta(1 - \lambda + \lambda\pi_1(E))\right)k_3 + k_4.$$

The inequality corresponding to $\eta(\lambda\pi_2(E))) \geq \lambda\eta(\pi_2(E))$ for complementary events which is implied by η elasticity reducing becomes $1-\eta(1-\lambda+\lambda\pi_1(E))) \geq \lambda\left(1-\eta(\pi_1(E))\right)$, so as above, using this it is seen that a sufficient condition for this implication to hold is that $k_4 > 0$. Noting that $u(y+p)-u(x+p) < 0$, this occurs when $u(x)-u(x+p) \leq 0$, equivalently if $p \leq 0$. This establishes the desired conclusion for the case $x > y$. Since condition (a) is symmetric with respect to the event E and its complement, taking $x > y$ is clearly immaterial.

To establish $(b) \Rightarrow (a)$. Assume not (a). Specifically, that there are $\pi_1, \pi_2 \in \Pi$, $\pi_2(E) > \pi_1(E), \pi_2(E') \geq \pi_1(E')$, $0 < \lambda < 1$ such that

$$\pi_1(E') = \lambda\pi_1(E), \quad \pi_2(E') > \lambda\pi_2(E).$$

Hence, for some $b > 0$ sufficiently small

$$\pi_1(E') < \lambda\pi_1(E) - b, \pi_2(E') > \lambda\pi_2(E) - b.$$

Choosing some $x, y \in \boldsymbol{J}$ with $x > y$, select $p \in \mathbb{R}, \left|p\right| < |\boldsymbol{J}|$, $u \in U_1$ such that $\lambda = \frac{u(x+p)-u(y+p)}{u(x)-u(y)}$, $b = \frac{u(y)-u(y+p)}{u(x)-u(y)}$, this contradicts the implication (31) to (32). Conducting the analogous exercise for complementary events completes the proof. □

Definition A.1. *Let $\mathscr{P}$ be a class of preferences over $\mathscr{F}$ strictly partially ordered by ($\mathscr{P}$)-more ambiguity averse. Given events $E, E' \in \boldsymbol{\Sigma}$, we say E is a ($\mathscr{P}$)-ALTERNATIVE **more ambiguous(II) event** than E' if, for $\succeq_A, \succeq_B \in \mathscr{P}$, $x, y, p, q \in \boldsymbol{X}$, with $x > y$,*

$$xE'y \succeq_A (\succ_A)\, pEy \Rightarrow xE'y \succeq_B (\succ_B)\, pEy$$

and

$$x(\neg E')y \succeq_A (\succ_A) q(\neg E)y \Rightarrow x(\neg E')y \succeq_B (\succ_B)\, q(\neg E)y,$$

whenever $\succeq_B$ is ($\mathscr{P}$)-more ambiguity averse than $\succeq_A$.

Proposition A.1. *Either let $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$ or, let $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$, with Π is a compact, convex subset of Δ. Consider two events, $E, E' \in \boldsymbol{\Sigma}$. The following are equivalent:*

(i) E is a ($\mathscr{P}$)-ALTERNATIVE more ambiguous (II) event than E';

(ii) The act xEy is a ($\mathscr{P}$)-more ambiguous (II) act than $xE'y$ for each $x, y \in \boldsymbol{J} \subset \boldsymbol{X}$.

Proof. A construction similar to that in the proof of Lemma A.1 establishes the equivalence of ($\mathscr{P}$)*-ALTERNATIVE **more ambiguous(II) event*** and $\pi(E')$ is an elasticity reducing transformation of $\pi(E)$. (The construction essentially involves setting the quantities k_2 and k_4 in the proof equal to zero.) This establishes the equivalence when the events are belief comonotone on some set $\Pi \subset \Delta$. Reference to the proofs of Propositions 3.3 and 3.4 makes clear that this condition also suffices in the general case. □

Proof. [Proof of Proposition 3.3]

Choices between bets on events E and E' by preferences in $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$ depend on the beliefs Π only through the set $\{(\pi(E), \pi(E')) \mid \pi \in \Pi\} \subset [0,1]^2$. Indeed, they only depend on this set through the extremes $\min_{\pi\in\Pi}\pi(E)$, $\min_{\pi\in\Pi}\pi(E')$, $\max_{\pi\in\Pi}\pi(E)$, $\max_{\pi\in\Pi}\pi(E')$. Choices are, therefore unchanged if Π is replaced by Π' the convex hull of

$$\{(\min_{\pi\in\Pi}\pi(E), \min_{\pi\in\Pi}\pi(E')), (\max_{\pi\in\Pi}\pi(E), \max_{\pi\in\Pi}\pi(E'))\} \subset [0,1]^2.$$

By this construction, E, E' are belief comonotone on Π'. With these observations in place, we may apply Lemma A.1. Condition (b) of the Lemma is necessary and sufficient for E to be a $\mathscr{P}^M_{U_1,\Pi'}$-m.a.(II) than E'. To see this, let $x, y \in \boldsymbol{J}$, $p \in \mathbb{R}$ with $\left|p\right| < |\boldsymbol{J}|$, $x > y$ and fix $u \in U_1$. Condition (b) is a single crossing condition on expected utilities. If the expected utilities do not cross for some p, there is nothing to prove since the implication holds trivially. Suppose therefore given some p that there is a crossing as specified. This is easily seen to imply $\max\Pi'(E) - \min\Pi'(E) > \max\Pi'(E') - \min\Pi'(E')$ otherwise the crossing would have the wrong sign, i.e. be from positive to negative, rather than from negative to positive. Also, therefore, $\max\Pi'(E) > \max\Pi'(E) \geq \min\Pi'(E') > \min\Pi'(E)$ otherwise there would be no crossing. For preferences in $\mathscr{P}^M_{\{u\},\Pi'}$, this implies there is some preference with ambiguity aversion parameter α which is indifferent between the bets, but those more ambiguity averse prefer the bet on E' and those less ambiguity averse prefer the bet on E. Hence, condition (a) of Lemma A.1 for the belief Π' holds, equivalently given convexity of Π' condition (ii) of Proposition 3.3 holds. □

The following Lemma is known. The sufficiency part is implicit in e.g. Karlin and Novikoff (1963), or see e.g. Gollier (2001, Chapter 4) for a more explicit discussion.

Lemma A.2. *Let F and G be distribution functions with supports in an interval $\boldsymbol{I} \subset \mathbb{R}$. The following two conditions are equivalent.*

(a) $\int_{\boldsymbol{I}} \upsilon_A dG \geq (>) \int_{\boldsymbol{I}} \upsilon_A dF \Rightarrow \int_{\boldsymbol{I}} \upsilon_B dG \geq (>) \int_{\boldsymbol{I}} \upsilon_B dF$ for all integrable nondecreasing functions υ_A, $\upsilon_B : \boldsymbol{I} \to \mathbb{R}$ with υ_A more concave than υ_B (υ_A is a continuous concave transformation of υ_B).

(b) Single crossing. *$\mathbb{R}$ can be partitioned into two intervals (one of which may be null), $\mathbb{R} = \boldsymbol{I}_1 \cup \boldsymbol{I}_2$, $\boldsymbol{I}_1 < \boldsymbol{I}_2$ such that $F \geq G$ on $\boldsymbol{I}_1$, $F \leq G$ on $\boldsymbol{I}_2$. Equivalently, for all $y_1, y_2 \in \mathbb{R}$, $y_1 < y_2$, we have $F(y_1) < G(y_1) \Rightarrow F(y_2) \leq G(y_2)$.*

Proof. $(b) \Rightarrow (a)$. Let the random variable X have cdf F and Y have cdf G, denote the cdf of $\upsilon_A(X)$, by F_A and $\upsilon_A(Y)$ by G_A. Denote $G_A - F_A = H_A$. If (a) holds then, equivalently, $\int_{\upsilon_A(\boldsymbol{I})} v\, dH_A(v) \geq (>) 0 \Rightarrow \int_{\upsilon_A(\boldsymbol{I})} \varphi(v) dH_A(v) \geq (>) 0$ whenever φ is nondecreasing concave. Integration by parts gives the implication $\int_{\upsilon_A(\boldsymbol{I})} H_A(v) dv \leq (<)\ 0 \Rightarrow \int_{\upsilon_A(\boldsymbol{I})} H_A(v) d\varphi(v) \leq (<) 0$ and since φ is absolutely continuous, we may write

$$\int_{\upsilon_A(\boldsymbol{I})} H_A(v) dv \leq (<) 0 \Rightarrow \int_{\upsilon_A(\boldsymbol{I})} H_A(v) \varphi'(v) dv \leq (<) 0 \tag{35}$$

for some $\varphi' > 0$ on the interior of $\upsilon_A(\boldsymbol{I})$. If $H_A(v)$ has a uniform sign, there is stochastic dominance and (a) holds trivially. If (b) holds, there can only be one change of sign, which is from negative to positive. Suppose therefore, $H_A(v) \leq 0$ for $v \leq v'$ and $H_A(v) \geq 0$ for $v \geq v'$ for some $v' \in I$, then $\int_{\upsilon_A(\boldsymbol{I})} H_A(v) \big(\varphi'(v') - \varphi'(v)\big) dv \geq 0$, so $\varphi'(v') \int_{\upsilon_A(\boldsymbol{I})} H_A(v) dv \geq \int_{\upsilon_A(\boldsymbol{I})} H_A(v) \varphi'(v) dv$. Therefore, (35) holds, as required.

$(a) \Rightarrow (b)$. If there are $x_1 < x_2 \in \boldsymbol{I}$ with $F(x_1) > G(x_1)$ and $F(x_2) < G(x_2)$ then with $\upsilon_A(x)$ defined to equal 0 on $x < x_1$, 1 on $x_1 \leq x < x_2$, $1 + B$ on $x \geq x_2$, with $B = \frac{F(x_1) - G(x_1)}{G(x_2) - F(x_2)}$ one verifies that $\int_{\boldsymbol{I}} \upsilon_A dF = \int_{\boldsymbol{I}} \upsilon_A dG$. However, with $\varphi(v) = \min\{v, 1\}$, $\int_{\boldsymbol{I}} \varphi(\upsilon_A) dF = 1 - F(x_1) < \int_{\boldsymbol{I}} \varphi(\upsilon_A) dG = 1 - G(x_1)$, this contradicts (a). □

Proof. [Proof of Proposition 3.4]

For preferences in $\mathscr{P}^S_{U_1,\mu}$ condition (i) requires that for $x, y \in \boldsymbol{J}$, $p \in \mathbb{R}$ with $\left|p\right| < |\boldsymbol{J}|$,

$$\int \phi_A\big(\pi(E) u\big(x + p\big) + (1 - \pi(E)) u(y + p)\big) d\mu \geq (>) \int \phi_A\big(\pi(E') u(x) + (1 - \pi(E')) u(y)\big) d\mu,$$

implies

$$\int \phi_B\big(\pi(E) u\big(x + p\big) + (1 - \pi(E)) u(y + p)\big) d\mu \geq (>) \int \phi\,\big(\pi(E') u(x) + (1 - \pi(E')) u(y)\big) d\mu$$

whenever $u \in U_1$, $\phi_A : u(\boldsymbol{X}) \to \mathbb{R}$, $\phi_B : u(\boldsymbol{X}) \to \mathbb{R}$, and ϕ_A is more concave than ϕ_B. By Lemma A.2 it is necessary and sufficient that with $F(v) = \mu(\{\pi \mid \pi(E) u\big(x + p\big) + (1 - \pi(E)) u(y + p) \leq v\})$ and $G(v) = \mu(\{\pi \mid \pi(E') u(x) + (1 - \pi(E')) u(y)\})$ that F, G satisfies the single crossing condition described in part (b) of that Lemma. Suppose $x > y$ without loss of generality. With Z a real random variable uniformly distributed on $[0, 1]$, for any distribution function F with inverse $F^{-1}(\xi) = \inf\{\eta \in \mathbb{R}_+ \mid F(\eta) > \xi\}$, $F(x) = \Pr[F^{-1}(Z) \leq x]$. Hence, in our case,

$F^{-1}(Z)$ has the same distribution as does $\pi(E)$ under the probability measure μ. Similarly for $G^{-1}(Z)$ and $\pi(E')$. That is, the required implication may be rewritten as

$$\int_0^1 \phi_A\big(F^{-1}(Z)u\big(x+p\big)+(1-F^{-1}(Z))u(y+p)\big)\,dz$$
$$\geq (>) \int_0^1 \phi_A\big(G^{-1}(z)u(x)+(1-G^{-1}(z))u(y)\big)\,dz,$$

implies

$$\int_0^1 \phi_B\big(F^{-1}(Z)u\big(x+p\big)+(1-F^{-1}(Z))u(y+p)\big)\,dz$$
$$\geq (>) \int_0^1 \phi_B\big(G^{-1}(z)u(x)+(1-G^{-1}(z))u(y)\big)\,dz.$$

Equivalently, Lemma A.2 requires single crossing of $F^{-1}(z)u\big(x+p\big)+(1-F^{-1}(z))u(y+p)$ and $G^{-1}(z)u(x)+(1-G^{-1}(z)u(y)$ as z traverses $[0,1]$, in the sense of condition (b) of that Lemma. Since $F^{-1}(z)$ and $G^{-1}(z)$ are comonotone random variables, application of Lemma A.1 is permissible (admitting the obvious change of notation) therefore there exists an elasticity reducing function η such that $F^{-1}(Z)=\eta(G^{-1}(Z))$. Recalling, that $F^{-1}(Z)$ has the same distribution as $\pi(E)$ as required and $G^{-1}(Z)$ as $\pi(E)$, the result follows. □

A.2 Proofs of results in Section 4

Proof. [Proof of Proposition 4.1] (ii) implies (i): From the preference representation (3),

$$V_{\Pi,\alpha,u}(f)=\alpha\min_{\pi\in\Pi}\int_S u(f)\,d\pi+(1-\alpha)\max_{\pi\in\Pi}\int_S u(f)\,d\pi.$$

It follows from Π is centrally symmetric, that there is $\underline{\pi}^u\in\arg\min_{\pi\in\Pi}\int_S u(f)\,d\pi$ and a $\overline{\pi}^u\in\arg\max_{\pi\in\Pi}\int_S u(f)\,d\pi$ such that $0.5\underline{\pi}^u+0.5\overline{\pi}^u=\pi^\star$. Hence, for all $u\in U_1$, $V_{\Pi,0.5,u}(f)=\int_S u(f)\,d\pi^\star=\int_{\boldsymbol{X}} u\,dP^f_{\pi^*}$. Similarly, $V_{\Pi,0.5,u}(g)=\int_S u(g)\,d\pi^\star=\int_{\boldsymbol{X}} u\,dP^g_{\pi^*}$. This establishes that all ambiguity neutral elements of the preference class are indifferent between f and g. Since $\left\{\int u(f)\,d\pi,\pi\in\Pi\right\}\supset\left\{\int u(g)\,d\pi,\pi\in\Pi\right\}$ evidently implies

$$\max_{\pi\in\Pi}\int u(f)\,d\pi-\min_{\pi\in\Pi}\int u(f)\,d\pi\geq\max_{\pi\in\Pi}\int u(g)\,d\pi-\min_{\pi\in\Pi}\int u(g)\,d\pi$$

it follows immediately from this fact and the above preference representation that $V_{\Pi,\alpha,u}(f)\leq V_{\Pi,\alpha,u}(g)$ for all ambiguity averse elements ($\alpha\geq 0.5$). Similarly, all ambiguity loving preferences prefer f to g. Hence, the conditions of Definition 2.3 apply.

(i) implies (ii): Given Π is a convex centrally symmetric $\left\{\int u(f)\,d\pi,\pi\in\Pi\right\}$ and $\left\{\int u(f)\,d\pi,\right.$ $\left.\pi\in\Pi\right\}$ are convex and centrally symmetric. Moreover, they have the same center. This implies, for each $u\in U_1$ either $\left\{\int u(f)\,d\pi,\pi\in\Pi\right\}\supset\left\{\int u(g)\,d\pi,\pi\in\Pi\right\}$ or $\left\{\int u(f)\,d\pi,\pi\in\Pi\right\}\subset\left\{\int u(g)\,d\pi,\pi\in\Pi\right\}$. The same argument as given above implies that if for some $u\in U_1$, $\left\{\int u(f)\,d\pi,\pi\in\Pi\right\}\subsetneqq\left\{\int u(g)\,d\pi,\pi\in\Pi\right\}$, then all ambiguity averse preferences in $\mathscr{P}^M_{\{u\},\Pi}$ strictly prefer f to g. Hence, the set inclusion condition is also necessary.

(iii) implies (i): Suppose that g is a center preserving π-garbling of f, then $P^f_{\pi^*} = P^g_{\pi^*}$ (equation (12)). Hence, $V_{\Pi,0.5,u}(f) = V_{\Pi,0.5,u}(g)$ for all the risk neutral elements of $\mathscr{P}^M_{U_1,\Pi}$, i.e., part (i) of Definition 2.3 holds. To establish part (ii) it will suffice to show that if g is a center preserving π-garbling of f,

$$\max_{\pi\in\Pi}\int_{\boldsymbol{X}} u\,dP^f_\pi - \min_{\pi\in\Pi}\int_{\boldsymbol{X}} u\,dP^f_\pi \geq \max_{\pi\in\Pi}\int_{\boldsymbol{X}} u\,dP^g_\pi - \min_{\pi\in\Pi}\int_{\boldsymbol{X}} u\,dP^g_\pi \tag{36}$$

since this implies g is preferred (dispreferred) to f when $\alpha > 0.5$ ($\alpha < 0.5$). Evidently, since the stochastic kernel $(\pi, C) \mapsto K_\pi(C)$ postulated in (10) 'averages' over Π rather than maximizes,

$$\max_{\pi\in\Pi}\int_{\boldsymbol{X}} u\,dP^f_\pi \geq \int_\Delta\left[\int_{\boldsymbol{X}} u\,dP^f_\pi\right]dK_{\pi'}(\pi), \pi' \in \Pi. \tag{37}$$

Maximizing over the right hand side of (37) establishes

$$\max_{\pi\in\Pi}\int_{\boldsymbol{X}} u\,dP^f_\pi \geq \max_{\pi'\in\Pi}\int_\Delta\left[\int_{\boldsymbol{X}} u\,dP^f_\pi\right]dK_{\pi'}(\pi).$$

The π-garbling condition (10) and T.16, p.16 of Meyer (1966) implies $\int_\Delta\left[\int_{\boldsymbol{X}} u\,dP^f_\pi\right]dK_{\pi'}(\pi) = \int_{\boldsymbol{X}} u\,dP^g_{\pi'}$. Hence, $\max_{\pi\in\Pi}\int_{\boldsymbol{X}} u\,dP^f_\pi \geq \max_{\pi'\in\Pi}\int_{\boldsymbol{X}} u\,dP^g_{\pi'}$. The same argument applied to min rather than max establishes $\min_{\pi\in\Pi}\int_\Delta u\,dP^f_\pi \leq \min_{\pi\in\Pi}\int_\Delta u\,dP^g_\pi$, hence (36) is established as required. □

Proof. [Proof of Proposition 4.2] (i)⟺(ii):

Denote $\int_{\boldsymbol{S}} u(g)d\pi = \mathfrak{g}^u(\pi), \int_{\boldsymbol{S}} u(f)d\pi = \mathfrak{f}^u(\pi)$. $\mathfrak{f}^u, \mathfrak{g}^u$ are μ-integrable functions. We require for all convex, φ, that $\int\varphi(\mathfrak{f}^u)d\mu \geq \int\varphi(\mathfrak{g}^u)d\mu$. Equivalently, $\int_{\boldsymbol{Y}}\varphi\,d\mu^{\mathfrak{f}^u} \geq \int_{\boldsymbol{Y}}\varphi\,d\mu^{\mathfrak{g}}$, where $\mu^{\mathfrak{f}^u} = \mu\circ\mathfrak{f}^{u-1}, \mu^{\mathfrak{g}^u} = \mu\circ\mathfrak{g}^{-1}$ and $\boldsymbol{Y}$ is the closed convex hull of $u(\boldsymbol{X})$. By Theorem 2 of Strassen (1965), there exists a *dilation* D, a stochastic kernel from $(\boldsymbol{Y}, \mathscr{B}_Y)$ to itself such that for all $B \in \mathscr{B}_{\boldsymbol{Y}}$,

$$\mu^{\mathfrak{f}^u}(B) = \int_{\boldsymbol{Y}} D_x(B)\,d\mu^{\mathfrak{g}^u}(x) \tag{38}$$

and

$$\int_{\boldsymbol{Y}} y\,dD_x(y) = x, x \in \boldsymbol{Y}. \tag{39}$$

Hence, from (38)

$$\mu(\mathfrak{f}^{u-1}(B)) = \int_{\boldsymbol{Y}} D_x(B)\,d\mu(\mathfrak{g}^{u-1}(x)) = \int_\Delta D_{\mathfrak{g}^u(\pi)}(B)\,d\mu(\pi),\ B \in \mathscr{B}_{\boldsymbol{Y}}.$$

In other words,

$$\mu(C) = \int_\Delta K_\pi(C)\,d\mu(\pi)$$

on the sigma algebra generated by $\mathfrak{f}^u$, where $K^u_\pi(C) = D_{\mathfrak{g}^u(\pi)}(\mathfrak{f}^u(C)), \pi \in \Delta, C \in \mathfrak{f}^u(\mathscr{B}_Y)$. Since, $\int_{\boldsymbol{\Delta}}\mathfrak{f}^u\,dK^u_\pi = \mathfrak{g}^u(\pi)$ by (39), it follows that

$$\int_{\boldsymbol{\Delta}}\left(\int_{\boldsymbol{S}} u(f)d\pi\right)dK^u_{\pi'} = \left(\int_{\boldsymbol{S}} u(g)d\pi\right), \pi' \in \Delta$$

as required.

(iii)⇒(i): If act $g \in \mathscr{F}$ is a μ-measure preserving π-garbling of $f \in \mathscr{F}$, then using (13) we have for concave ϕ

$$\int_\Delta \phi\left(\int_X u\,dP^f_\pi\right)d\mu = \int_\Delta\left[\int_\Delta \phi\left(\int_X u\,dP^f_\pi\right)dK_{\pi'}(\pi)\right]d\mu(\pi')$$
$$\leq \int_\Delta \phi\left(\int_\Delta\left(\int_X u\,dP^f_\pi\right)dK_{\pi'}(\pi)\right)d\mu(\pi')$$
$$= \int_\Delta \phi\left(\int_X u\,d\left(\int_\Delta P^f_\pi\,dK_{\pi'}(\pi)\right)\right)d\mu(\pi') = \int_\Delta \phi\left(\int_X u\,dP^g_\pi\right)d\mu.$$

The inequality is an application of Jensen's inequality. This shows all ambiguity averse preferences in $\mathscr{P}$ weakly prefer g to f. By the same argument all ambiguity seeking preferences weakly prefer f to g and all ambiguity neutral are indifferent. □

Proof. [Proof of Proposition 4.3]

The single crossing property is equivalent to: for each $p \in \mathbb{R}$, $\left|p\right| \leq |\boldsymbol{J}|$, **either** (a) $\int u(g)d\pi \geq \int u(f+p)d\pi$ for all $\pi \in \Pi$ or $\int u(g)d\pi \leq \int u(f+p)d\pi$ for all $\pi \in \Pi$, **or** (b) $\min_{\pi\in\Pi}\int u(g)d\pi > \min_{\pi\in\Pi}\int u(f+p)d\pi$ and $\max_{\pi\in\Pi}\int u(g)d\pi < \max_{\pi\in\Pi}\int u(f+p)d\pi$.

If (a) holds, all preferences in $\mathscr{P}^M_{\{u\},\Pi}$ unanimously prefer either $f+p$ or g and there is nothing to prove. If (b) holds, there is some $\alpha' \in (0,1)$ such that $V_{\Pi,\alpha',u}(f+p) = V_{\Pi,\alpha',u}(g)$ and, moreover $V_{\Pi,\alpha,u}(f+p) < V_{\Pi,\alpha,u}(g)$ for each $1 \geq \alpha > \alpha'$, and $V_{\Pi,\alpha,u}(f+p) > V_{\Pi,\alpha,u}(g)$ for each $\alpha' > \alpha \geq 0$. □

Proof. [Proof of Proposition 4.4]

Straightforward application of Lemma A.2. □

A.3 Proofs of results in Section 5

Proof. [Proof of Proposition 5.1] $(ii) \Rightarrow (i)$.

Suppose (i) does not hold. Then there exist $\pi', \pi'' \in \Pi$ such that $\pi''(E) > \pi'(E)$ and $\pi''(E') < \pi'(E')$. Let $\Pi' = \text{co}(\{\pi', \pi''\})$. Choose a preference from $\mathscr{P}^M_{U_1,\Pi'}$ with $u \in U_1$ such that $u(x) > u(y)$, and with ambiguity preference parameter $\alpha = 1$. For this preference, $xEy \sim zE'y$ implies $z > y$, and, therefore,

$$V_{\Pi',1,u}(xEy) = \pi'(E)u(x) + (1-\pi'(E))u(y)$$
$$= V_{\Pi',1,u}(zE'y) = \pi''(E')u(z) + (1-\pi''(E'))u(y).$$

Whereas

$$\begin{aligned} V_{\Pi',1,u}(f_\lambda) &= \min_{\pi\in\Pi} \lambda\big(\pi(E)u(x) + (1-\pi(E))u(y)\big) + (1-\lambda)\big(\pi(E')u(z) + (1-\pi(E'))u(y)\big) \\ &= \lambda\big(\pi'''(E)u(x) + (1-\pi'''(E))u(y)\big) + (1-\lambda)\big(\pi'''(E')u(z) + (1-\pi'''(E'))u(y)\big) \end{aligned}$$

for some $\pi''' \in \text{co}(\{\pi', \pi''\})$. The indifference condition $xEy \sim zE'y \sim f_\lambda$, requires $V_{\Pi',1,u}(f_\lambda) = \lambda V_{\Pi',1,u}(xEy) + (1-\lambda)V_{\Pi',1,u}(zE'y)$. This in turn requires $\pi'''(E) = \pi'(E)$ and $\pi'''(E') = \pi''(E')$ which is impossible since $\pi''' \in \text{co}co(\{\pi', \pi''\})$, given the initial assumptions on π', π''.

$(i) \Rightarrow (ii)$:

Belief comonotonicity implies that for each $u \in U_1$, and compact, convex $\Pi' \subseteq \Pi$, $x > y$ and $z > y$,

$$\begin{aligned}\min_{\pi\in\Pi'}\left(\int u(f_\lambda)d\pi\right) &= \min_{\pi\in\Pi'}\left(\lambda\int u(xEy)d\pi + (1-\lambda)\int u(zEy)d\pi\right)\\ &= \lambda\min_{\pi\in\Pi'}\left(\int u(xEy)d\pi\right) + (1-\lambda)\min_{\pi\in\Pi'}\left(\int u(zEy)d\pi\right).\end{aligned}$$

Hence, $V_{\Pi',1,u}(xEy) = V_{\Pi',1,u}(zE'y)$ implies $V_{\Pi',1,u}(f_\lambda) = V_{\Pi',1,u}(xEy) = V_{\Pi',1,u}(zEy)$. The same argument obtains with the min operator replaced by max, from which the conclusion is extended to $V_{\Pi',\alpha,u}(f_\lambda) = V_{\Pi',\alpha,u}(xEy) = V_{\Pi',\alpha,u}(zEy)$ as required. □

Proof. [Proof of Remark 5.1]

The preference inequality (16) may be written

$$\int_\Delta \phi\left(\lambda\pi(E) + (1-\lambda)\pi(E')\right)d\mu \le \int_\Delta \phi\left(\lambda\pi(E) + (1-\lambda)\pi(E'')\right)d\mu$$

for all increasing concave ϕ, where to simplify notation and without loss of generality we have normalized so that $u(x) - u(y) = 1$, $u(y) = 0$. If $xE'y \sim xE''y$ for all ambiguity averse $\succeq \in \mathscr{P}^S_{U_1,\mu}$, then one can verify that $\pi(E')$ and $\pi(E'')$ have identical distributions under the measure μ. Given belief comonotonicity, the pair $\pi(E), \pi(E')$ are similarly ordered. Noting that for each $0 \le \lambda \le 1$, the function $(a, b) \mapsto \phi(\lambda a + (1-\lambda)b)$ is supermodular, the result follows from Lorentz (1953) rearrangement inequality. □

Proof. [Proof of Proposition 5.2]

For the $\mathscr{P} = \mathscr{P}^M_{U_1,\Pi}$ case: Belief comonotonicity on Π forces $\Pi_{E,E'} = \{(\pi(E), \pi(E'))|\pi \in \Pi\}$ to be a nondecreasing arc in the unit square, thereby restricting the dimension of $\Pi_{E,E'}$ to be no more than one. Given Π is compact convex, this arc is the convex hull of the two points, $(\min\Pi(E'), \min\Pi(E))$ and $(\max\Pi(E'), \max\Pi(E))$. Hence, the characterizing condition in Proposition 3.3 reduces to condition (i) of Proposition 5.2. For the $\mathscr{P} = \mathscr{P}^S_{U_1,\mu}$ case: Applying Proposition 3.4, gives $\pi(E')$ has the same distribution as $\eta(\pi(E))$, for some elasticity reducing function $\eta : \{\pi(E)|\pi \in \text{supp}\mu\} \to [0,1]$. Since E, E' are belief comonotone on $\text{supp}\mu$, this implies $\pi(E') = \eta(\pi(E))$ for almost all $\pi \in \text{supp}\mu$. □

Proof. [Proof of Proposition 5.3] (ii)⇒(i):

Let $u_x \in U_1$ denote the simple step function, $u_x(x') = 0$ if $x' \le x$, $u_x(x') = 1$ otherwise. The condition $\int u_x(f(s))d\pi_1(s) \le \int u_x(f(s))d\pi_2(s)$ becomes $\pi_1(E^f_x) \ge \pi_2(E^f_x)$, similarly for act g. Hence if condition (ii) of the proposition holds, then $\pi_1(E^f_x) \ge \pi_2(E^f_x)$ and $\pi_1(E^g_x) \ge \pi_2(E^g_x)$ for each $\pi_1, \pi_2 \in \Pi$, $x \in \boldsymbol{X}$.

(i)⇒(ii): suppose (i) holds, this implies there is a linear order $\leqq$ on Π such that $\pi_1 \leqq \pi_2 \Leftrightarrow \pi_1(E^f_x) \ge \pi_2(E^f_x)$ and $\pi_1(E^g_x) \ge \pi_2(E^g_x)$ for each $x \in \boldsymbol{X}$. Since $\pi_i(E^f_x) = P^f_{\pi_i}(x)$, $\pi_i(E^g_x) = P^g_{\pi_i}(x)$, $i = 1,2$, the result follows by a straightforward application of stochastic dominance (e.g. Müller and Stoyan (2002), Theorem 1.2.8). □

Proof. [Proof of Proposition 5.4]

For $\mathscr{P}=\mathscr{P}^M_{U_1,\Pi}$: Recall, by convexity of Π, P^f_π, P^g_π are mixture linear (equation (2) in Section 2.1). Belief comonotonicity for the acts $f,g\in\hat{\mathscr{F}}$ means that for each $x\in\boldsymbol{X}$, $\int u dP^g_\pi$ and $\int u dP^f_\pi$ are both nondecreasing on Π in the linear order $\leqq_{U_1}$. Hence, since Π is compact, there exist top and bottom elements of Π, denoted respectively $\overline{\pi}$ and $\underline{\pi}$ such for all $u\in U_1$, $\pi\in\Pi$, $\int u dP^f_{\underline{\pi}}\le\int u dP^f_\pi\le\int u dP^f_{\overline{\pi}}$ and $\int u dP^g_{\underline{\pi}}\le\int u dP^g_\pi\le\int u dP^g_{\overline{\pi}}$. Hence, $\lambda:\Pi\to[0,1]$, defined by $\pi=_{U_1}\lambda(\pi)\overline{\pi}+(1-\lambda(\pi))\underline{\pi}$, represents $\leqq_{U_1}$ and $P^f_\pi=\lambda(\pi)P^f_{\overline{\pi}}+(1-\lambda(\pi))P^f_{\underline{\pi}}$, $f\in\mathscr{F}$.

(i)$\Leftrightarrow$(ii): Suppose (ii) holds, then for $u\in U_1$,

$$\alpha\int u dP^f_{\underline{\pi}}+(1-\alpha)\int u dP^f_{\overline{\pi}}\ge(\le)\alpha\int u dP^g_{\underline{\pi}}+(1-\alpha)\int u dP^g_{\overline{\pi}} \tag{40}$$

whenever $\alpha\ge(\le)0.5$. That is, it is necessary and sufficient that $\alpha P^f_{\underline{\pi}}+(1-\alpha)P^f_{\overline{\pi}}$ stochastically dominates $\alpha P^g_{\underline{\pi}}+(1-\alpha)P^g_{\overline{\pi}}$ for $\alpha\ge0.5$ and is stochastically dominated by it when $\alpha\le0.5$. Noting that $f,g\in\hat{\mathscr{F}}$ implies $P^f_\pi(x)=\pi(E^f_x)$, $x\in\boldsymbol{X}$, the condition (40) is equivalent to $\alpha\underline{\pi}(E^f_x)+(1-\alpha)\overline{\pi}(E^f_x)\le(\ge)\ \alpha\underline{\pi}(E^g_x)+(1-\alpha)\overline{\pi}(E^g_x)$ whenever $\alpha\ge(\le)\ 0.5$. Since Π is centrally symmetric, choosing $\alpha=0.5$ gives $\pi^*(E^f_x)=\pi^*(E^g_x)$ and $\underline{\pi}(E^f_x)-\overline{\pi}(E^f_x)\ge\underline{\pi}(E^g_x)-\overline{\pi}(E^g_x)$. Equivalently, $\left[\underline{\pi}(E^f_x),\overline{\pi}(E^f_x)\right]=\Pi(E^f_x)\subset\left[\underline{\pi}(E^g_x),\overline{\pi}(E^g_x)\right]=\Pi(E^g_x)$.

For $\mathscr{P}=\mathscr{P}^S_{U_1,\mu}$: $(ii)\Rightarrow(i)$: Suppose condition (ii) holds. Let $u_x\in U_1$ denote the simple step function, satisfying $u_x(y)=1$ if $y>x$ and $u_x(y)=0$ if $y\le x$. Hence, $\int_{\boldsymbol{S}}u_x(f)d\pi=\left(1-\pi\left(E^f_x\right)\right)$, $\int_{\boldsymbol{S}}u_x(g)d\pi=\left(1-\pi\left(E^g_x\right)\right)$. For $(\mathscr{P}^S_{U_1,\mu})-m.a\ (I)$ we require therefore

$$\int\phi\left(1-\pi\left(E^f_x\right)\right)d\mu\le\int\phi\left(1-\pi\left(E^g_x\right)\right)d\mu$$

for all nondecreasing concave ϕ. Since for E^f_x to be a $(\mathscr{P})-m.a\ (I)$ event than E^g_x, it is required that for all $x>y$,

$$\begin{aligned}\int\phi\left(\left(1-\pi\left(E^f_x\right)\right)u(x)+\pi\left(E^f_x\right)u(y)\right)d\mu &\le \int\phi\left(\left(1-\pi\left(E^g_x\right)\right)u(y)+\pi\left(E^g_x\right)u(y)\right)d\mu\\ \int\phi\left(\left(1-\pi\left(E^f_x\right)\right)\left(u(x)-u(y)\right)+u(y)\right)d\mu &\le \int\phi\left(\left(1-\pi\left(E^g_x\right)\right)\left(u(x)-u(y)\right)+u(y)\right)d\mu,\end{aligned}$$

the implication easily follows.

$(i)\Rightarrow(ii)$: The reverse implication utilizes belief comonotonicity. Let $u=\sum_i\gamma_i u_{x_i}$, $\gamma_i>0$, hence, $\int_{\boldsymbol{S}}u(f)d\pi=\int_{\boldsymbol{S}}\sum_i\gamma_i u_{x_i}d\pi=\sum_i\gamma_i\int_{\boldsymbol{S}}u_{x_i}d\pi=\sum_i\gamma_i\left(1-\pi\left(E^f_{x_i}\right)\right)$. Condition (i) implies for each x_i, and each concave ϕ, $\int\phi\left(1-\pi\left(E^f_{x_i}\right)\right)d\mu\ge\int\phi\left(1-\pi\left(E^g_{x_i}\right)\right)d\mu$, and it is known (see, e.g., Landsberger and Meilijson (1994)) that together with belief comonotonicity this implies

$$\int\phi\left(\sum_i\gamma_i\left(1-\pi\left(E^f_{x_i}\right)\right)\right)d\mu\le\int\phi\left(\sum_i\gamma_i\left(1-\pi\left(E^f_{x_i}\right)\right)\right)d\mu.$$

Hence, $\int\phi\left(\int_{\boldsymbol{S}}u(f)d\pi\right)d\mu\ge\int\phi\left(\int_{\boldsymbol{S}}u(g)d\pi\right)d\mu$, for all concave ϕ, for all u of the form $u=\sum_i\gamma_i u_{x_i}$. Any $u\in U_1$ can be uniformly approximated by a sequence of functions of this form with

$$\lim_m\int\phi\left(\int_{\boldsymbol{S}}u_m(h)d\pi\right)d\mu=\int\phi\left(\int_{\boldsymbol{S}}u(h)d\pi\right)d\mu, h=f,g.$$

□

Proof. [Proof of Proposition 5.5]$(ii) \Rightarrow (i)$.
For $x \in \boldsymbol{J}$, and step utilities $u_x : \boldsymbol{X} \to \mathbb{R}$, $u_x(y) = 1$ for $x > y$, 0 otherwise, we have $\int_{\boldsymbol{S}} u_x(f)d\pi = 1 - \pi(E_x^f)$, $\int_{\boldsymbol{S}} u_x(g)d\pi = 1 - \pi(E_x^g)$, $\pi \in \Pi$. Given belief comonotonicity, Propositions 4.3 and 4.4, for respectively the $\alpha - MEU$ and smooth cases, state that f $(\mathscr{P}) - m.a.(II)$ g implies $\int_{\boldsymbol{S}} u_x(f)d\pi - \int_{\boldsymbol{S}} u_x(g)d\pi$ has at most a single sign change from negative to positive as π becomes larger (in the linear order defined by belief comonotonicity). A strict sign change must occur, i.e. $\left(\int_{\boldsymbol{S}} u_x(f)d\pi - \int_{\boldsymbol{S}} u_x(g)d\pi\right) \times \left(\int_{\boldsymbol{S}} u_x(f)d\pi' - \int_{\boldsymbol{S}} u_x(g)d\pi'\right) < 0$ otherwise either $g \succeq f$ for all $\succeq \in \mathscr{P}$ or $g \succeq f$ for all $\succeq \in \mathscr{P}$. Since $\left\{\left(\pi\left(E_x^g\right), \pi(E_x^f)\right) | \pi \in \Pi\right\}$ is a convex subset of $[0,1]^2$, there is an affine elasticity reducing function $\eta : \Pi(E_x^f) \to [0,1]$ such that $\pi(E_x^g) = \eta(\pi(E_x^f))$ on Π. Proposition 5.2 implies E_x^f is a $(\mathscr{P})$–m.a.(II) event than E_x^g as required. The rest of condition (i) is established by observing that for $\succ^* \in \mathscr{P}^M_{U_1,\Pi}$ with $\alpha = 0$, $v\left(E_x^f\right) w \succ^* v\left(E_x^g\right) w$ since $\max_{\pi\in\Pi} \int_{\boldsymbol{S}} u_x(f)d\pi > \max_{\pi\in\Pi} \int_{\boldsymbol{S}} u_x(g)d\pi$. Similarly, for $\succ^{**} \in \mathscr{P}^S_{U_1,\mu}$ with $\alpha = 1$, $v\left(E_x^g\right) w \succ^* v\left(E_x^f\right) w$. For smooth ambiguity preferences, since there are sequence of preferences in $\mathscr{P}^S_{U_1,\mu}$ whose evaluations converge to $\max_{\pi\in\Pi} \int_{\boldsymbol{S}} u_x d\pi$ and $\min_{\pi\in\Pi} \int_{\boldsymbol{S}} u_x(f)d\pi$, the same conclusion holds.

$(i) \Rightarrow (ii)$: Condition (i) implies for each $x \in \boldsymbol{J}$, $\left\{\left(1 - \pi(E_x^g), 1 - \pi(E_x^f)\right) | \pi \in \Pi\right\}$ is a convex subset of $[0,1]^2$ whose interior intersects the leading diagonal, and it satisfies the single crossing property that $1 - \pi_1(E_x^f) \geq (>) 1 - \pi_1(E_x^g) \Rightarrow 1 - \pi_2(E_x^f) \geq (>) 1 - \pi_2(E_x^g)$ for π_2 larger than π_1. It follows from belief comonotonicity that for any $x \in \boldsymbol{X}^m$, $\lambda \in \mathbb{R}^m_+$,

$$\left\{\left(\sum_{i=1}^m \lambda_i \left(1 - \pi(E_{x_i}^g)\right), \sum_{i=1}^m \lambda_i \left(1 - \pi(E_{x_i}^f)\right)\right) | \pi \in \Pi\right\}$$

inherits these properties: it is a convex subset of $[0,1]^2$ whose interior intersects the leading diagonal, and it satisfies the single crossing property. Equivalently, $\left\{\int_{\boldsymbol{S}} u(f)d\pi, \int_{\boldsymbol{S}} u(g)d\pi\right\}_{\pi\in\Pi}$ inherits these properties, where $u = \sum \lambda_i u_{x_i} \in U_1$. This establishes that $\int_{\boldsymbol{S}} u(f)d\pi$, $\int_{\boldsymbol{S}} u(g)d\pi$ satisfy the single crossing property for any $u \in U_1$ which is a positive linear combination of step functions. The result extends to any $u \in U_1$ by standard arguments. Hence, for the special case $p = 0$, $\left\{\int_{\boldsymbol{S}} u(f)d\pi, \int_{\boldsymbol{S}} u(g)d\pi\right\}_{\pi\in\Pi}$ satisfies the single crossing property (ii) of Proposition 4.3 and also the single crossing property (ii) of Proposition 4.4. The proof will be completed, by establishing that this single crossing property is satisfied for each $|p| \in |\boldsymbol{J}|$. To see this note that for any such $p > 0$ the convex set $\left\{\int_{\boldsymbol{S}} u(f+p)d\pi, \int_{\boldsymbol{S}} u(g)d\pi\right\}_{\pi\in\Pi}$ is obtained from the convex set $\left\{\left(\int_{\boldsymbol{S}} u(g)d\pi, \int_{\boldsymbol{S}} u(f)d\pi\right) | \pi \in \Pi\right\}$ by shifting vertically the $\int_{\boldsymbol{S}} u(f)d\pi$ coordinate corresponding to each point $\pi \in \Pi$ by an amount $\int_{\boldsymbol{S}} u(f+p)d\pi - \int_{\boldsymbol{S}} u(f)d\pi \geq 0$, the $\int_{\boldsymbol{S}} u(g)d\pi$ coordinate is evidently unchanged. □

Proof. [Proof of Proposition 5.6]
The proof is in the text. □

Proof. [Proof of Proposition 5.7]
The proof is in the text. □

Proof. [Proof of Proposition 5.8]

To simplify notation, we replace $f+p$ with f throughout this proof, so for expressions involving f, these may be interpreted as holding not only for f, but for all $f+p$ such that $p\in\mathbb{R}$, $\left|p\right|\le|\boldsymbol{J}|$.

$(i)\Leftrightarrow(iii)$, i.e., equivalence for preferences in $\mathscr{P}^{\mathscr{S}}_{U_1,\mu}$:

By Proposition 4.4 it suffices that there exist no $u\in U_1$,and $a_1<a_2$ such that

$$\mu\left(\left\{\pi\in\Pi\mid\int_{\boldsymbol{X}}u\,dP^f_\pi\le a_1\right\}\right)>\mu\left(\left\{\pi\in\Pi\mid\int_{\boldsymbol{X}}u\,dP^g_\pi\le a_1\right\}\right)\Rightarrow \mu\left(\left\{\pi\in\Pi\mid\int_{\boldsymbol{X}}u\,dP^f_\pi\le a_2\right\}\right)\ge\mu\left(\left\{\pi\in\Pi\mid\int_{\boldsymbol{X}}u\,dP^g_\pi\le a_2\right\}\right). \tag{41}$$

This is equivalent to the condition $\pi_1,\pi_2\in\operatorname{supp}\mu$, with $\pi_1\leqq_{U_1}\pi_2$

$$\int_{\boldsymbol{X}}u\,dP^f_{\pi_1}<\int_{\boldsymbol{X}}u\,dP^g_{\pi_1}\Rightarrow\int_{\boldsymbol{X}}u\,dP^f_{\pi_2}\le\int_{\boldsymbol{X}}u\,dP^g_{\pi_2}. \tag{42}$$

To see that (41) implies (42), note that as in the proof of Proposition 5.4, since $\operatorname{supp}\mu$ is a convex, compact set there are top and bottom elements $\overline{\pi}$ and $\underline{\pi}$ such that λ, defined by $\pi=_{U_1}\lambda(\pi)\overline{\pi}+(1-\lambda(\pi))\underline{\pi}$, represents $\leqq_{U_1}$. Evidently, P^f_π and P^g_π vary continuously with λ and the measure $\mu\circ\lambda^{-1}$ has full support on $[0,1]$. Hence, $\pi_1,\pi_2\in\operatorname{supp}\mu$, $\int_{\boldsymbol{X}}u\,dP^f_{\pi_1}>\int_{\boldsymbol{X}}u\,dP^g_{\pi_1}$ imply $\int_{\boldsymbol{X}}u\,dP^f_\pi>\int_{\boldsymbol{X}}u\,dP^g_\pi$ in an open neighborhood of π_1 and, hence,

$$\mu\left(\left\{\pi\in\Pi\mid\int_{\boldsymbol{X}}u\,dP^f_\pi\le a_1\right\}\right)<\mu\left(\left\{\pi\in\Pi\mid\int_{\boldsymbol{X}}u\,dP^g_\pi\le a_1\right\}\right).$$

Similarly, for $\int_{\boldsymbol{X}}u\,dP^f_{\pi_2}<\int_{\boldsymbol{X}}u\,dP^g_{\pi_2}$. That (42) implies (41) is straightforward. Hence, (42)$\Leftrightarrow$(41).

In the usual way, approximating u by $\sum\alpha_i u_{x_i}$ for positive $\alpha_i\ge0$, $x_i\in\boldsymbol{X}$, it is necessary and sufficient for (42) that for all choices of $x_1,x_2,...\in\boldsymbol{X}$ and $\alpha_1,\alpha_2,...\in(0,1)$.

$$\sum\alpha_i\left(P^g_{\pi_1}(x_i)-P^f_{\pi_1}(x_i)\right)>0,\sum\alpha_i\left(P^g_{\pi_2}(x_i)-P^f_{\pi_2}(x_i)\right)<0. \tag{43}$$

The proof is completed by showing that condition (iii) of Proposition 5.8 is equivalent to the negation of the inequalities (43). With

$$(y_1(x),y_2(x))=\left(P^g_{\pi_1}(x)-P^f_{\pi_1}(x),P^g_{\pi_2}(x)-P^f_{\pi_2}(x)\right),$$

negation of (43) asserts the convex hull of $Y=\{(y_1(x),y_2(x))\mid x\in\mathbb{R}\}\subset\mathbb{R}^2$ does not intersect the orthant $\mathbb{O}=\{(y_1,y_2)\in\mathbb{R}^2\mid y_1>0,y_2<0\}$. Equivalently, by the separating hyperplane theorem, there exist $(\lambda_1,\lambda_2)\gneqq0$ such that

$$\lambda_1\left[P^g_{\pi_1}(x)-P^f_{\pi_1}(x)\right]\le\lambda_2\left[P^g_{\pi_2}(x)-P^f_{\pi_2}(x)\right],x\in\boldsymbol{X}.$$

$(i)\Leftrightarrow(ii)$, i.e., equivalence for preferences in $\mathscr{P}^{\mathscr{M}}_{U_1,\Pi}$: By Definition 2.4 and Proposition 4.3, condition (iii) is equivalent to: for each $u\in U_1$,

$$\begin{aligned}&\alpha\int_{\boldsymbol{X}}u\,dP^f_{\underline{\pi}}+(1-\alpha)\int_{\boldsymbol{X}}u\,dP^f_{\overline{\pi}}-\left(\alpha\int_{\boldsymbol{X}}u\,dP^g_{\underline{\pi}}+(1-\alpha)\int_{\boldsymbol{X}}u\,dP^g_{\overline{\pi}}\right)\\&\quad=\alpha\left(\int_{\boldsymbol{X}}u\,dP^f_{\underline{\pi}}(x)-\int_{\boldsymbol{X}}u\,dP^g_{\underline{\pi}}\right)+(1-\alpha)\left(\int_{\boldsymbol{X}}u\,dP^f_{\overline{\pi}}(x)-\int_{\boldsymbol{X}}u\,dP^g_{\overline{\pi}}\right)\equiv\alpha A+(1-\alpha)B\end{aligned}$$

has (at most) single sign change from negative to positive as α increases from 0 to 1. This rules out the configuration $A<0$, $B>0$ but all others are admissible. As in the previous, $\mathscr{P}^{\mathscr{S}}_{U_1,\mu}$, part of the proof, choosing u to be step functions and arbitrary convex combinations of step functions requires that Y be separated from $\mathbb{O}$**.** Also as above, this establishes condition (iii) with $\pi_1=\underline{\pi}$, $\pi_2=\overline{\pi}$. Using the convexity of Π, and the fact that P^f_π,P^g_π are mixture linear, the equivalence is seen to extend to all $\pi_1\leqq_{U_1}\pi_2\in\Pi$. □

A.4 Proofs of results in Section 6

Proof. [Proof of Proposition 6.1]

Belief comonotonicity implies the objective function may be written as

$$\alpha \min_{\pi\in\Pi} \int_{\boldsymbol{X}} u(\theta x) dP_\pi^f(x) + (1-\alpha) \max_{\pi\in\Pi} \int_{\boldsymbol{X}} u(\theta x) dP_\pi^f(x) = \int_{\boldsymbol{X}} u(\theta x) dP^f_{\alpha\underline{\pi}+(1-\alpha)\overline{\pi}}.$$

If $(\theta, x) \mapsto u(\theta x)$ is supermodular on the lattice $[0,1]\times\boldsymbol{X}$ (with the partial order (θ, x) larger than (θ', x') if $\theta \geq \theta'$ and $x \geq x'$), then $(\theta, P) \mapsto \int_{\boldsymbol{X}} u(\theta x) dP(x)$ is easily seen to be supermodular on the lattice $[0,1]\times$(space of probability distributions on $\boldsymbol{X}$) with the partial order $(\theta, P) \geq (\theta', P')$ if $\theta \geq \theta'$ and P first-order stochastically dominates P'. It can be shown that condition (ii) of Proposition 5.6 implies that for $\alpha > \frac{1}{2}$, $P^g_{\alpha\underline{\pi}+(1-\alpha)\overline{\pi}}$ first order stochastically dominates $P^f_{\alpha\underline{\pi}+(1-\alpha)\overline{\pi}}$, conversely for $\alpha < \frac{1}{2}$, $P^f_{\alpha\underline{\pi}+(1-\alpha)\overline{\pi}}$ first order stochastically dominates $P^g_{\alpha\underline{\pi}+(1-\alpha)\overline{\pi}}$. The result follows immediately from standard monotone comparative statics results (see, e.g., Milgrom and Shannon (1994)). □

Remark A.1. In light of Proposition 5.7, the following equivalence is a restatement of Tchen (1980). Let the pair $f, g \in \hat{\mathscr{F}}$ be belief comonotone on suppμ for the class of utilities U_1. Then f m.a.(II) g if and only if for all supermodular functions $\nu: \boldsymbol{X}\times\Delta \to \mathbb{R}$,

$$\int_{\boldsymbol{X}\times\Delta} \nu dP^{f,\mu} \geq \int_{\boldsymbol{X}\times\Delta} \nu dP^{g,\mu}.$$

Proof. [Proof of Proposition 6.2]

At the portfolio share $\theta^* = \theta^*\big((\mu, \phi, u); (P^g_\pi)_{\pi\in\Pi}\big) \geq 0$, the following first-order condition holds

$$\int_{\boldsymbol{X}\times\Delta} \phi'\left(\int_{\boldsymbol{X}} u dP^g_\pi\right) u'(\theta^* x) x dP^{g,\mu} = 0. \tag{44}$$

It suffices for the result that (44) implies $\int_{\boldsymbol{X}\times\Delta} \phi'\big(\int_{\boldsymbol{X}} u dP^f_\pi\big) u'(\theta^* x) x dP^{f,\mu} \leq 0$, since, by concavity the θ satisfying the first-order condition for f must be greater than θ^*. Since $u'(w+\theta^* x)x$ is nondecreasing in x and $\phi'\big(\int_{\boldsymbol{X}} u dP^g_\pi\big)$ is nonincreasing in $\pi \in \Pi$, by concavity and belief comonotonicity, it follows from $P^{f,\mu}$ more ambiguous (I) than $P^{g,\mu}$ (Remark (A.1)) that (44) implies $\int_{\boldsymbol{X}\times\Delta} \phi'\big(\int_{\boldsymbol{X}} u dP^g_\pi\big) u'(\theta^* x) x dP^{f,\mu} \leq 0$. Hence, it suffices to establish the implication

$$\int_{\boldsymbol{X}\times\Delta} \phi'\left(\int_{\boldsymbol{X}} u dP^g_\pi\right) u'(\theta^* x) x dP^{f,\mu} \leq 0 \Rightarrow \int_{\boldsymbol{X}\times\Delta} \phi'\left(\int_{\boldsymbol{X}} u dP^f_\pi\right) u'(\theta^* x) x dP^{f,\mu} \leq 0.$$

This will be achieved by showing that there exists $\lambda \geq 0$ such that

$$\int_{\boldsymbol{X}\times\Delta} \left(\phi'\left(\int_{\boldsymbol{X}} u dP^f_\pi\right) - \lambda\phi'\left(\int_{\boldsymbol{X}} u dP^g_\pi\right)\right) u'(\theta^* x) x dP^{f,\mu} \leq 0. \tag{45}$$

To this end, choose $\lambda = \lambda^* > 0$ so that

$$\int_{\boldsymbol{X}\times\Delta} \left(\phi'\left(\int_{\boldsymbol{X}} u dP^f_\pi\right) - \lambda^*\phi'\left(\int_{\boldsymbol{X}} u dP^g_\pi\right)\right) dP^{f,\mu} = 0. \tag{46}$$

It follows from the assumptions that ϕ' is convex, hence from f m.a.(I) g it follows that

$$\int_{\Delta} \phi'\left(\int_X u dP_\pi^f\right) d\mu \geq \int_{\Delta} \phi'\left(\int_X u dP_\pi^g\right) d\mu,$$

therefore $\lambda^* \geq 1$. Hence, since ϕ' is decreasing, $\phi'\left(\int_X u dP_\pi^f\right) \geq \lambda^* \phi'\left(\int_X u dP_\pi^g\right)$ implies $\int_X u dP_\pi^f \leq \int_X u dP_\pi^g$. Let $\zeta(\pi) = \int_X u dP_\pi^f - \int_X u dP_\pi^g$, ζ is a nondecreasing function by assumption (Remark 5.5). If ϕ' is log-convex, then $\frac{\phi'(\eta-\zeta)}{\phi'(\eta)}$ is nondecreasing in η for all $\zeta < 0$ and, since ϕ' is decreasing, is nondecreasing in ζ. It follows that, for $\pi' \geqq \pi$,

$$\frac{\phi'\left(\int_X u dP_\pi^f - \zeta(\pi)\right)}{\phi'\left(\int_X u dP_\pi^f\right)} \leq \frac{\phi'\left(\int_X u dP_{\pi'}^f - \zeta(\pi)\right)}{\phi'\left(\int_X u dP_{\pi'}^f\right)} \leq \frac{\phi'\left(\int_X u dP_{\pi'}^f - \zeta(\pi')\right)}{\phi'\left(\int_X u dP_{\pi'}^f\right)}.$$

Hence,

$$\left[\phi'\left(\int_X u dP_\pi^f\right) - \lambda^* \phi'\left(\int_X u dP_\pi^g\right)\right] = \left[\frac{\phi'\left(\int_X u dP_\pi^f\right)}{\phi'\left(\int_X u dP_\pi^f - \zeta(\pi)\right)} - \lambda^*\right] \phi'\left(\int_X u dP_\pi^g\right)$$

has at most a single sign change which is from positive to negative if one occurs. The rest of the proof is standard. Since $\pi \mapsto \int_X u'(\theta^* x) x dP_\pi^f$ is nondecreasing, there exists a $k \in \mathbb{R}$ such that

$$\int_{\Delta} \left(\phi'\left(\int_X u dP_\pi^f\right) - \lambda^* \phi'\left(\int_X u dP_\pi^g\right)\right)\left(\int_X u'(w + \theta^* x) x dP_\pi^f - k\right) d\mu \leq 0.$$

Using (46) this is easily seen to imply (45) as required. □

Proof. [Proof of Proposition 6.3]

For the case of α-MEU preferences, one obtains, by the envelope theorem, the first order condition

$$u'(y_1 - a) = \alpha \int u'(y_2 + a) dP_{\underline{\pi}}^f(y_2) + (1-\alpha) \int u'(y_2 + a) dP_{\overline{\pi}}^f(y_2)), \tag{47}$$

this condition uniquely determines the optimum given u is strictly concave. For the CARA case, compensation

$$\alpha \int u dP_{\underline{\pi}}^f + (1-\alpha) \int u dP_{\overline{\pi}}^f = \alpha \int u dP_{\underline{\pi}}^g + (1-\alpha) \int u dP_{\overline{\pi}}^g \tag{48}$$

means the first order condition is unchanged

$$\alpha \int u' dP_{\underline{\pi}}^f + (1-\alpha) \int u' dP_{\overline{\pi}}^f = \alpha \int u' dP_{\underline{\pi}}^g + (1-\alpha) \int u' dP_{\overline{\pi}}^g.$$

As noted in Gierlinger and Gollier (2008), Hardy, Littlewood, and Pólya (1952)'s generalization of Minkowski's inequality, that the generalized mean $v \mapsto \mathfrak{M}_\varphi(v) = \varphi^{-1}\left(\int \varphi(v(\pi)) d\mu(\pi)\right)$ is concave if $x \mapsto -\varphi'(x)/\varphi''(x)$ is a concave function. Hence, with $v_1(\pi) = \int u(y_2 + s_1) dP_\pi^f(y_2)$, $v_2(\pi) = \int u(y_2 + s_2) dP_\pi^f(y_2)$, $v_\lambda(\pi) = \int u(y_2 + \lambda s_1 + (1-\lambda) s_2) dP_\pi^f(y_2)$ we have $\lambda \mathfrak{M}_\phi(v_1) + (1-$

$\lambda)\mathfrak{M}_\phi(v_2) \le \mathfrak{M}_\phi(\lambda v_1 + (1-\lambda)v_2)$. It follows that if u is strictly concave, $a \mapsto V^f_{\mathscr{S}}(a)$ is strictly concave. The first-order conditions are (having set the optimal savings $a = 0$ without loss of generality) are

$$u'(y_1) = \frac{\int \phi'\left(\int u\, dP^f_\pi\right)\left(\int u'\, dP^f_\pi\right) d\mu}{\phi'\left(\phi^{-1}\left(\int \phi\left(\int u\, dP^f_\pi\right) d\mu\right)\right)}.$$

We wish to show that for an m.a.(II) compensated increase in ambiguity

$$u'(y_1) \le \frac{\int \phi'\left(\int u\, dP^g_\pi\right)\left(\int u'\, dP^g_\pi\right) d\mu}{\phi'\left(\phi^{-1}\left(\int \phi\left(\int u\, dP^g_\pi\right) d\mu\right)\right)}$$

since, a strict inequality will require a reduction in savings to restore the first order constraint. That is, we wish to establish

$$\int \phi\left(U_f\right) d\mu = \int \phi\left(U_g\right) d\mu \Rightarrow \int \phi'\left(U_f\right) U'_f d\mu \ge \int \phi'\left(U_g\right) U'_g d\mu. \tag{49}$$

where $U_f = \int u\, dP^f_\pi$, $U'_f = \int u'\, dP^f_\pi$ and $U_g = \int u\, dP^g_\pi$, $U'_g = \int u'\, dP^g_\pi$. Using CARA, this becomes

$$\int \phi\left(U_f\right) d\mu = \int \phi\left(U_g\right) d\mu \Rightarrow \int \phi'\left(U_f\right) U_f d\mu \le \int \phi'\left(U_g\right) U_g d\mu.$$

Note that the function $\widetilde{\phi}(U) = \phi'(U)U$ is a nondecreasing concave transformation of ϕ on $\mathbb{R}_{-.}$ Nondecreasing is immediate, the concavity part can be seen from the fact that the ratio of derivatives

$$\frac{\phi''(U)U + \phi'(U)}{\phi'(U)} = \frac{-\phi''(U)}{\phi'(U)}(-U) + 1$$

is the product of two positive decreasing function and therefore decreasing. The result now follows since a compensated increase in m.a.(II) satisfying the equality condition in (49) implies that $\int \widetilde{\phi}\left(U_f\right) d\mu \le \int \widetilde{\phi}\left(U_g\right) d\mu$ for any $\widetilde{\phi}$ which is a concave transformation of ϕ. □

A.5 More ambiguous sets of lotteries

Recall a lottery is, formally, an element of $\boldsymbol{L}$, the set of distributions over $\boldsymbol{X}$ with finite supports. Let $2^{\boldsymbol{L}}$ denote the set of all subsets of $\boldsymbol{L}$ and $\mathscr{L} \equiv 2^{\boldsymbol{L}} \backslash \emptyset$, $\mathscr{L}_{\boldsymbol{J}} \equiv 2^{\boldsymbol{L_J}} \backslash \emptyset$. Let $\succeq^{\mathscr{L}}$ be a preference relation on $\mathscr{L}$.

Definition A.2. Let $\mathscr{P}^{\mathscr{L}}$ be a class of preferences over $\mathscr{L}$. Let $\succeq_A^{\mathscr{L}}, \succeq_B^{\mathscr{L}} \in \mathscr{P}^{\mathscr{L}}$. We say $\succeq_B^{\mathscr{L}}$ is ($\mathscr{P}^{\mathscr{L}}$)-**more objective ambiguity averse** than $\succeq_A^{\mathscr{L}}$ if, for all $l \in \boldsymbol{L}$ and $\mathscr{L}_f \in \mathscr{L}$, $\mathscr{L}_f \succeq_B^{\mathscr{L}} \{l\} \Rightarrow \mathscr{L}_f \succeq_A^{\mathscr{L}} \{l\}$ and $\mathscr{L}_f \preceq_A^{\mathscr{L}} \{l\} \Rightarrow \mathscr{L}_f \preceq_B^{\mathscr{L}} \{l\}$.

Notation. Given $\mathscr{L}_f \in \mathscr{L}$ and $x \in \boldsymbol{X}$, $\mathscr{L}_f + \delta_x$ denotes the set of lotteries $\left\{l + \delta_x \mid l \in \mathscr{L}_f\right\}$.

Definition A.3. *Let $\mathscr{P}^{\mathscr{L}}$ be a class of preferences over $\mathscr{L}$. We say $\mathscr{P}^{\mathscr{L}}$ is **strictly partially ordered** by ($\mathscr{P}^{\mathscr{L}}$)-**more objective ambiguity averse** if for each $\succeq^{\mathscr{L}} \in \mathscr{P}^{\mathscr{L}}$ there exists $\tilde{\succeq}^{\mathscr{L}} \in \mathscr{P}^{\mathscr{L}}$, $\succeq^{\mathscr{L}} \neq \tilde{\succeq}^{\mathscr{L}}$, such that $\succeq^{\mathscr{L}}$ is ($\mathscr{P}^{\mathscr{L}}$)-more objective ambiguity averse than $\tilde{\succeq}^{\mathscr{L}}$ or $\tilde{\succeq}^{\mathscr{L}}$ is ($\mathscr{P}^{\mathscr{L}}$)-more objective ambiguity averse than $\succeq^{\mathscr{L}}$.*

The following is the analog of more ambiguous (II), applied to preferences over sets of lotteries.

Definition A.4. *Let $\mathscr{P}^{\mathscr{L}}$ be a class of preferences over $\mathscr{L}$ strictly partially ordered by ($\mathscr{P}^{\mathscr{L}}$)-more objective ambiguity averse. Given sets of lotteries $\mathscr{L}_f, \mathscr{L}_g \in \mathscr{L}_{\boldsymbol{J}}$, we say $\mathscr{L}_f$ is a ($\mathscr{P}^{\mathscr{L}}$)-**more ambiguous set of lotteries** than $\mathscr{L}_g$ if for all $p \in \mathbb{R}$ with $\left|p\right| \leq |\boldsymbol{J}|$, $\mathscr{L}_g \succeq_A^{\mathscr{L}} \left(\mathscr{L}_f + \delta_p\right) \Rightarrow \mathscr{L}_g \succeq_B^{\mathscr{L}} \left(\mathscr{L}_f + \delta_p\right)$, and $\mathscr{L}_g \succ_A^{e\mathscr{L}} \left(\mathscr{L}_f + \delta_p\right) \Rightarrow \mathscr{L}_g \succ_B^{\mathscr{L}} \left(\mathscr{L}_f + \delta_p\right)$, whenever $\succeq_B^{\mathscr{L}}$ is ($\mathscr{P}^{\mathscr{L}}$)-more objective ambiguity averse than $\succeq_A^{\mathscr{L}}$.*

We do not suggest an analog for more ambiguous (I) applied to preferences over sets of lotteries. While it may appear such an extension is straightforward, a problem is the literature does not identify candidate definitions of ambiguity neutral, in general, for such classes of preferences. The difficulty follows from the fact that the setting makes no reference to a state space (and subjective beliefs).[28]

Olszewski (2007) characterizes a class of preferences over sets of lotteries wherein the DM evaluates a set of lotteries $\mathscr{L}_f \in \mathscr{L}$ according to the rule:

$$V_{\alpha,u}^{\mathscr{L}}(\mathscr{L}_f) = \alpha\left(\min_{l \in \mathscr{L}_f} Eu(l)\right) + (1-\alpha)\left(\max_{l \in \mathscr{L}_f} Eu(l)\right), \tag{50}$$

where $Eu(l) \equiv \int u(x)\,dl(x)$ is the expected utility of lottery l. The parameter α can be interpreted as a measure of the DM's attitude to objective ambiguity: if α_i corresponds to a DM $i = A, B$, then DM A is more objective ambiguity averse than DM B if and only if $\alpha_A \geq \alpha_B$.

Ahn (2008) characterizes another class of preferences over sets of lotteries, proposing the following evaluation for a set of lotteries $\mathscr{L}_f \in \mathscr{L}$:

$$V_{m,\phi,u}^{\mathscr{L}}(\mathscr{L}_f) = \frac{\int \phi(Eu(l))\,dm}{m\left(\mathscr{L}_f\right)} \tag{51}$$

$$\equiv \int \phi(Eu(l))\,dm_{|\mathscr{L}_f} \tag{52}$$

[28]Note, Olszewski (2007) and Ahn (2008) only define and characterize comparative ambiguity aversion (and not absolute ambiguity aversion).

where $Eu(l) \equiv \int u(x)\,dl(x)$ is the expected utility of lottery l, m is a probability measure on $\mathscr{L}$, and ϕ is an increasing transformation applied to u. This DM considers all of the relevant lotteries in $\mathscr{L}_f$ when making his decision, with their relative consideration fixed across sets by a measure over all lotteries. Her attitude to ambiguity is, in part, captured by the transformation ϕ. In particular, $V^{\mathscr{L}}_{m,\phi_A,u}$ is more objective ambiguity averse than $V^{\mathscr{L}}_{m,\phi_B,u}$ if and only if $\phi_A = h \circ \phi_B$, where $h : \phi_B(u(\boldsymbol{X})) \to \mathbb{R}$ is concave.

The following two results provide characterizations of more ambiguous sets of lotteries for preferences proposed by Olszweski and Ahn. The statements are exact analogs of those for Propositions 4.3 and 4.4; the proofs obtain as straightforward adaptations of the proofs for those propositions, respectively.

Proposition A.2. *Let $\mathscr{L}_f, \mathscr{L}_g \in \mathscr{L}_{\boldsymbol{J}}$ and set $\mathscr{P}^{\mathscr{L}} = \left\{V^{\mathscr{L}}_{\alpha,u}\right\}_{\alpha\in[0,1],u\in U_1}$. The following are equivalent:*

(i) $\mathscr{L}_f$ is a $\left(\mathscr{P}^{\mathscr{L}}\right)$-more ambiguous set of lotteries than $\mathscr{L}_g$;

(ii) For each $u \in U_1$ and $p \in \mathbb{R}$, $\left|p\right| \leq |\boldsymbol{J}|$, the following single crossing property holds:

$$\left(\min_{l\in\mathscr{L}_f} Eu\left(l+\delta_p\right)\right) > \min_{l\in\mathscr{L}_g} Eu(l) \;\Rightarrow \max_{l\in\mathscr{L}_f} Eu\left(l+\delta_p\right) \geq \max_{l\in\mathscr{L}_g} Eu(l).$$

.

Proposition A.3. *Let $\mathscr{L}_f, \mathscr{L}_g \in \mathscr{L}_{\boldsymbol{J}}$ and set $\mathscr{P}^{\mathscr{L}} = \left\{V^{\mathscr{L}}_{m,\phi,u}\right\}_{\phi\in\Phi_1(u),u\in U_1}$. The following are equivalent:*

(i) $\mathscr{L}_f$ is a $\left(\mathscr{P}^{\mathscr{L}}\right)$-more ambiguous set of lotteries than $\mathscr{L}_g$;

(ii) For each $u \in U_1$ and $p \in \mathbb{R}$, $\left|p\right| \leq |\boldsymbol{J}|$, the following single crossing property holds: For each $\nu_1 < \nu_2$,

$$m_{|\mathscr{L}_f}\left(\left\{l \in \mathscr{L}_f \mid \int u\left(l+\delta_p\right) \leq \nu_1\right\}\right) < m_{|\mathscr{L}_g}\left(\left\{l \in \mathscr{L}_g \mid \int u\left(l+\delta_p\right) \leq \nu_1\right\}\right)$$
$$\Rightarrow m_{|\mathscr{L}_f}\left(\left\{l \in \mathscr{L}_f \mid \int u\left(l+\delta_p\right) \leq \nu_2\right\}\right) \leq m_{|\mathscr{L}_g}\left(\left\{l \in \mathscr{L}_g \mid \int u\left(l+\delta_p\right) \leq \nu_2\right\}\right).$$

The following proposition involves sets of lotteries induced by acts. Given $\pi \in \Delta$, any act $f \in \mathscr{F}$ induces a corresponding lottery $P^f_\pi(x)$, a probability distribution over outcomes. Given a set $\Pi \subseteq \Delta$, denote the set of lotteries induced by f and Π by $\mathscr{L}_{f,\Pi} \equiv \left\{P^f_\pi\right\}_{\pi\in\Pi} \in \mathscr{L}$. Consider a probability measure $\mu : \mathscr{B}_\Delta \longrightarrow [0,1]$, with $\text{supp}\mu = \Pi \subseteq \Delta$. Since $\pi \longmapsto P^f_\pi$ is a $\mathscr{B}_\Delta$ measurable function, the measure μ induces, uniquely, a probability measure on $\mathscr{L}_{f,\Pi}$ which we denote as $m_{f,\mu}$. Finally, given a pair of acts f,g, suppose m is a measure on $\Delta(\Delta(\boldsymbol{X}))$ (or, equivalently, on $\mathscr{L}$) such that $m_{|\mathscr{L}_{f,\Pi}} = m_{f,\mu}$ and $m_{|\mathscr{L}_{g,\Pi}} = m_{g,\mu}$.

Proposition A.4. *Consider a probability measure $\mu : \mathscr{B}_\Delta \longrightarrow [0,1]$ such that $\text{supp}\mu = \Pi \subseteq \Delta$ is compact, convex and that given the class of utilities U_1, the pair of acts $f,g \in \hat{\mathscr{F}}_{\boldsymbol{J}}$ is belief comonotone on Π. Then the following are equivalent:*

(i) For each $\pi_1 \leqq_{U_1} \pi_2$ from Π, and $p \in \mathbb{R}$ with $\left|p\right| \leq |\boldsymbol{J}|$, there exist $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$ such that for all $x \in \boldsymbol{J}$,

$$\lambda_1(1 - P^{f+p}_{\pi_1}(x)) + \lambda_2 P^{f+p}_{\pi_2}(x) \leq \lambda_1(1 - P^g_{\pi_1}(x)) + \lambda_2 P^g_{\pi_2}(x).$$

(ii) $\mathscr{L}_{f,\Pi}$ is a $\left(\mathscr{P}^{\mathscr{L}}\right)$-more ambiguous set of lotteries than $\mathscr{L}_{g,\Pi}$ where $\mathscr{P}^{\mathscr{L}} = \left\{V^{\mathscr{L}}_{\alpha,u}\right\}_{\alpha\in[0,1],u\in U_1}$.

(iii) $\mathscr{L}_{f,\Pi}$ is a $\left(\mathscr{P}^{\mathscr{L}}\right)$-more ambiguous set of lotteries than $\mathscr{L}_{g,\Pi}$ where $\mathscr{P}^{\mathscr{L}} = \left\{V^{\mathscr{L}}_{m,\phi,u}\right\}_{\phi\in\Phi_1(u),u\in U_1}$.

A.6 Vector Expected Utility

The Vector Expected Utility model (Siniscalchi 2009) is represented as,

$$V(f) = \int_S u(f)dP + A\left(\int_S \zeta u(f)dP\right),$$

where, $\zeta = (\zeta_1, \ldots, \zeta_n)$ is is vector random variable, or adjustment factor, that satisfies $\int_S \zeta dP = 0$. The function $A : \mathbb{R} \to \mathbb{R}$ is constrained to satisfy $A(0) = 0, A(-x) = A(x),\ x \in \mathbb{R}^n$. We will consider the class of VEU preferences, $\mathscr{V}(P, U_1)$ with a given, common, baseline prior P, and $u \in U_1$. We wish to characterize more ambiguous (I) events for this preference class.

We will work with the equivalent representation noted by Siniscalchi (2009, p.808), in which the $n-$dimensional adjustment factor ζ can be represented by the vector of signed measures $m = (m_1, \ldots, m_n)$, $m : \mathscr{B}(S) \to \mathbb{R}^n$, where for each $i = 1, \ldots, n$, m_i is absolutely continuous with respect to P and has Radon-Nikodym derivative $\frac{dm_i}{dP} = \zeta_i$, and $m_i(\boldsymbol{S}) = 0$.

A VEU preference is ambiguity averse according to Axiom 9 of Siniscalchi (2009) if and only if A is concave. Such a preference is ambiguity neutral iff $A \equiv 0$. Consider two acts f and g such that $\int_S u(f)dP = \int_S u(g)dP$ but $V(f) \leq V(g)$ for all ambiguity averse preferences in $\mathscr{V}(P, U_1)$. We evidently require

$$A\left(\int_S u(f)dm\right) \leq A\left(\int_S u(g)dm\right).$$

Let the act f represent a bet on the event E and g denote a bet of the same monetary value on the event E'. Without loss of generality we may assume the bet pays 1 if the event occurs and 0 if not.

Proposition A.5. *Consider the class of VEU preferences, $\mathscr{V}(P, U_1)$ with n-dimensional adjustment factor $\zeta = \frac{dm}{dP}$, $A : \mathbb{R}^n \to \mathbb{R}$, $A(x) = A(-x)$, A concave. An event E is ($\mathscr{V}$)-more ambiguous (I) than event E' if and only if the following conditions hold:*

(a) $P(E) = P(E')$ and

(b)$m(E') = \beta\, m(E)$, for some scalar $-1 \leq \beta \leq 1$.

Proof. Sufficiency. Condition (a) implies all ambiguity neutral preferences in the class are indifferent between the two gambles. Conditions (a) and (b) together imply that $m(E')$ is in the convex hull of $m(E)$ and $-(m(E)$. Hence, by linearity $\int u(g)dm = (u(1) - u(0))\, m(E')$ is in the convex hull of $\int u(f)dm$ and $-\int u(f)dm$. Since $A(\int u(f)dm = A(-\int u(f)dm$ and A is concave, it follows that $A(\int u(f)dm)) \leq A(\int u(g)dm$. This establishes that all ambiguity averse members of the preference class prefer the bet on E' as required.

Necessity. Condition (a) is necessary for all ambiguity neutral members of the preference class to be indifferent between the two gambles. Suppose therefore that (a) holds. Choose the utility function u so the utility from winning is unity and losing is zero. It suffices to show that if (b) fails to hold, there is a function A satisfying the conditions of the proposition such that $A(m(E)) < A(m(E')$. To this end, let $b_i \in \mathbb{R}^n$, $i = 1, ..., n-1$ chosen so that $\{b, ..., b_{n-1}\}$ a basis for the $n-1$ dimensional subspace orthogonal to $m(E)$. Define $A^\perp : [0,1]^n \to \mathbb{R}$, $A^\perp(x) = min\{b_1 \cdot x, -b_1 \cdot x, \ldots, b_{n-1} \cdot x, -b_{n-1} \cdot x\}$. $A^\perp$ is the pointwise minimum of a family of linear functions and so is concave. It also evidently satisfies $A^\perp(x) = A^\perp(-x)$. Moreover, by construction, $A^\perp(m(E)) = 0$ and $A^\perp(x) < 0$ for all x not collinear with $m(E)$. It follows that if E' is less ambiguous than E, $m(E)$ and $m(E')$ are collinear. Choosing some strictly concave A establishes that $m(E')$ must lie in that part of this linear set which is the convex hull of $m(E)$ and $-m(E)$. □

In order to better relate this result with our discussion of other representations, we define Q as the vector valued measure $Q = (Q_1, \ldots, Q_n) = (P + m_1, \ldots, P + m_n)$. Note that for each $i = 1, \ldots, n$, $\int_{\boldsymbol{S}} dQ_i = 1$. The following corollary considers the special case, of Q, when the $(Q_i)_{i=1,\ldots,n}$ is a family of probability measures on $\boldsymbol{S}$.

Corollary A.1. *Consider the class of VEU preferences, $\mathscr{V}(P, U_1)$ with n-dimensional adjustment factor $\zeta = \frac{dm}{dP}$, $A : \mathbb{R}^n \to \mathbb{R}$, $A(x) = A(-x)$, A concave. Let $Q_i = P + m_i$, $i = 1, \ldots, n$. Furthermore, suppose $(Q_i)_{i=1,\ldots,n}$ is a family of probability measures on $\boldsymbol{S}$. An event E is ($\mathscr{V}$)-more ambiguous (I) than event E' if and only if the following conditions hold:*
(a) $P(E) = P(E')$ and
(b)$Q_i(E') = (1-\beta)P(E) + \beta Q_i(E)$, $i = 1, \ldots, n$ for some $-1 \leq \beta \leq 1$.

B Appendix: On the non-uniqueness of representation of α–MEU preferences

It is well known, especially since Siniscalchi (2006), that α–MEU preferences do not necessarily have a unique representation.[29] One may query if the more ambiguous relation identified by a characterization obtained in one of the Propositions is affected by the particular representation (of the referenced preference class) chosen in the arguments. However, that is not the case; the characterization for more ambiguous derived depends on the ordinal properties of the referenced preference class itself, rather than any particular representation chosen for it.

Our definitions of more ambiguous are purely in terms of ordinal properties of the referenced preference class: an event (or an act) is more ambiguous than another if any DMs (one of whom has to ambiguity neutral in case of m.a. (I)) related by more ambiguity averse relation, order the acts in certain ways. Ambiguity neutrality and more ambiguity averse are defined using ordinal properties: a preference is ambiguity neutral irrespective of of the representation we may choose for it, and two preferences remain related by more ambiguity averse in the same way irrespective of the preference representation. Our propositions characterize these definitions and hence, a pair of acts/events identified as being related by more ambiguous in a characterization satisfies the ordinal properties required in the definition of more ambiguous. Since, alternative preference representations preserve ordinal properties, the identification of more ambiguous relation does not turn on a representation chosen for the referenced preference class. In following, we illustrate this general point for the cases of Propositions 3.1 and 4.1, which deal with m.a.(I) for α–MEU preferences.

B.1 Events

We show that the more ambiguous (I) relation between events identified in in Proposition 3.1 remains unchanged when we use a generic MEU representation for the reference class of preferences as an alternative to the representation used in the Proposition.

[29] While one does not obtain unique representation in general, there are important special cases where appropriate restrictions on the state space yields beliefs to be uniquely identified. For example, Gul and Pesendorfer (2014) axiomatize a model which has as a special case α–MEU preferences that are represented by a set of priors corresponding to the core of a belief function. See also, Klibanoff, Mukerji, and Seo (2014).

Given a centrally symmetric Π, with center π^*, a preference from the class $\mathscr{P}^M_{U_1,\Pi}$ is represented by

$$V_{\Pi,\alpha,u} = \alpha \min_{\pi\in\Pi} \int u d\pi + (1-\alpha)\max_{\pi\in\Pi} \int u d\pi. \tag{53}$$

If $\alpha > \frac{1}{2}$, then given central symmetry, since the minimizing and maximizing π's must average to π^* it is easy to show that

$$\alpha \min_{\pi\in\Pi} \int u d\pi + (1-\alpha)\max_{\pi\in\Pi} \int u d\pi = \min_{\pi\in\Pi_\alpha} \int u d\pi,$$

where $\Pi \supseteq \Pi_\alpha = \{\pi \in \Pi \mid \pi = (1-\lambda)\pi^* + \lambda\pi', \pi' \in \Pi\}$ with $\lambda = 2\alpha - 1$. A similar alternative representation holds in the ambiguity seeking case $\left(\alpha < \frac{1}{2}\right)$,

$$\alpha \min_{\pi\in\Pi} \int u d\pi + (1-\alpha)\max_{\pi\in\Pi} \int u d\pi = \max_{\pi\in\Pi_\alpha} \int u d\pi,$$

where $\Pi \supseteq \Pi_\alpha = \{\pi \in \Pi \mid \pi = \lambda\pi^* + (1-\lambda)\pi', \pi' \in \Pi\}$ with $\lambda = 2\alpha$. Hence, the preference represented by $V_{\Pi,\alpha,u}$ has the alternative representation $V_{\Pi_\alpha,1,u}$. In either case, as α ranges over the interval $[0,1]$ from 0 to 1 , the DM becomes more ambiguity averse, via the linear relationship between α and λ. We may say that for the preference $V_{\Pi_\alpha,1,u}$, α indexes ambiguity attitude *because* $V_{\Pi_\alpha,1,u}$ has the alternative representation $V_{\Pi,\alpha,u}$ (for which interpretation, we appeal to Ghirardato, Maccheroni, and Marinacci (2004), Proposition 12).

Note that Π_α is simply a radial contraction of Π about its center π^*. Moreover, $\lim_{\alpha\to 1}\Pi_\alpha = \lim_{\alpha\to 0}\Pi_\alpha = \Pi$, $\lim_{\alpha\to\frac{1}{2}}\Pi_\alpha = \{\pi^*\}$. Evidently, therefore, in the notation of Proposition 3.1 of the paper, $\Pi(E') \subset \Pi(E)$ and $\pi^*(E') = \pi^*(E)$ is equivalent to $\Pi_\alpha(E') \subset \Pi_\alpha(E)$ for all $0 \le \alpha \le 1$ and $\pi^*_\alpha(E') = \pi^*_\alpha(E)$. That is, E, E' satisfy centered expansion conditions $\Pi(E') \subset \Pi(E)$ and $\pi^*(E') = \pi^*(E)$ if and only if they satisfy $\Pi_\alpha(E') \subset \Pi_\alpha(E)$ for all $0 \le \alpha \le 1$ and $\pi^*_\alpha(E') = \pi^*_\alpha(E)$. Hence, a pair of events is characterized to be ordered in a particular way by more ambiguous (I) using the representation $V_{\Pi,\alpha,u}$ if and only if the pair is characterized to be ordered in the same way by more ambiguous (I) using the representation $V_{\Pi,\alpha,u}$.

B.2 Acts

Similarly for acts. Consider Proposition 4.1, this is stated in terms of preference classes but the proof used the representation (53). It is easy to see how the same (of course) Proposition emerges if instead we use the above alternative representation of preferences. Proposition 4.1 gives as a(n equivalent) condition for f to be more ambiguous than g that $\left\{\int u(f)d\pi\right\}_{\pi\in\Pi}$, the interval of expected utilities of act f as π ranges over Π contains the equivalent interval for the act g and have the same centers. Suppose instead we represent preferences in the alternative formulation, in terms of Π_α, the interval of expected utilities $\left\{\int u(f)d\pi\right\}_{\pi\in\Pi_\alpha}$ of act f as π ranges over Π_α is completely determined by the interval $\left\{\int u(f)d\pi\right\}_{\pi\in\Pi}$. Specifically, for the ambiguity averse case $\alpha > 1/2$, by the definition of Π_α, $\left\{\int u(f)d\pi\right\}_{\pi\in\Pi_\alpha} = 2(1-\alpha)\left\{\int u(f)d\pi^*\right\} + (2\alpha-1)\left\{\int u(f)d\pi\right\}_{\pi\in\Pi}$. Hence,

$$\left\{\int u(g)d\pi\right\}_{\pi\in\Pi} \subset \left\{\int u(f)d\pi\right\}_{\pi\in\Pi} \Longleftrightarrow \left\{\int u(g)d\pi\right\}_{\pi\in\Pi_\alpha} \subset \left\{\int u(f)d\pi\right\}_{\pi\in\Pi_\alpha},$$

in which case, of course, $min_{\pi\in\Pi_\alpha} \int u(g)d\pi \ge min_{\pi\in\Pi_\alpha} \int u(f)d\pi$. So, all ambiguity averse DMs prefer g to f as before. Similarly, ambiguity seekers prefer f to g and ambiguity neutral DM's are indifferent. Thus, we recover our definition of more ambiguous (I). That is, a pair of acts is characterized to be ordered in a particular way by more ambiguous (I) using the representation $V_{\Pi,\alpha,u}$ if and only if the pair is characterized to be ordered in the same way by more ambiguous (I) using the representation $V_{\Pi,\alpha,u}$.

C Figure

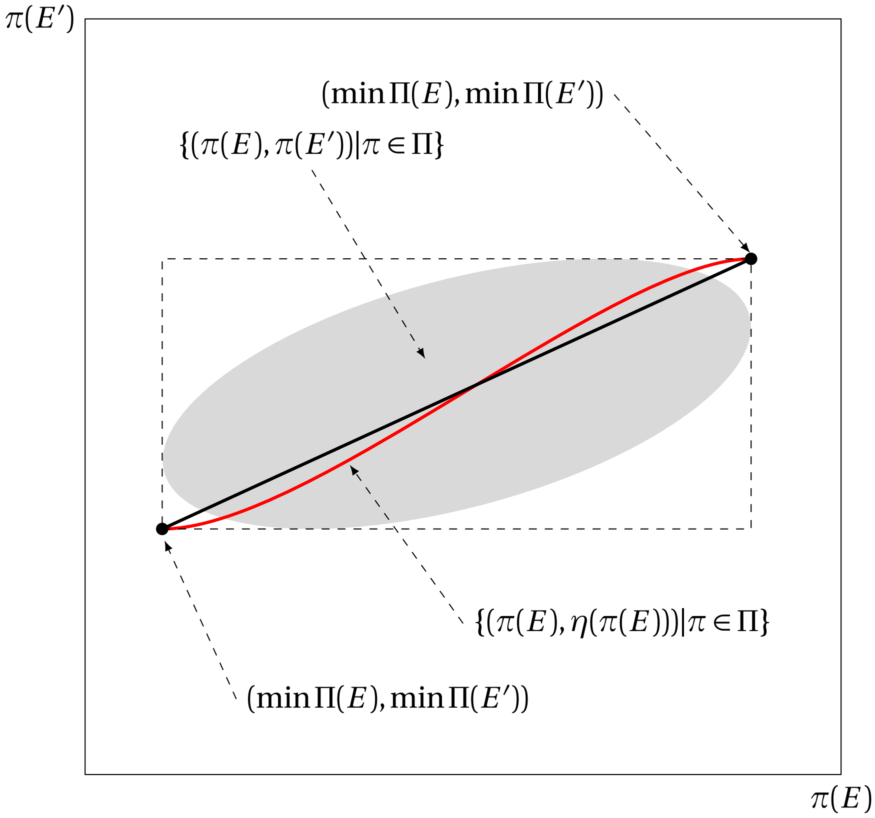

Figure 1: Illustration of E m.a.(II) event than E' for α-MEU and smooth ambiguity.

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