Doctoral Thesis

Characteristic Polynomials of Random Matrices and Quantum Chaotic Scattering

by

André Nock

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School of Mathematical Sciences
Queen Mary University of London
United Kingdom

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Statement of Originality

I, André Nock, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications:
The content of Chapter 3 and corresponding appendices are based on

- Y. V. Fyodorov, B. A. Khoruzhenko, and A. Nock. Universal K-matrix distribution in $\beta = 2$ Ensembles of Random Matrices. *Journal of Physics A: Mathematical and Theoretical* **46**, 262001 (2013).

All three authors worked closely together on the project.
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Abstract

Scattering is a fundamental phenomenon in physics, e.g. large parts of the knowledge about quantum systems stem from scattering experiments. A scattering process can be completely characterized by its K-matrix, also known as the "Wigner reaction matrix" in nuclear scattering or "impedance matrix" in the electromagnetic wave scattering. For chaotic quantum systems it can be modelled within the framework of Random Matrix Theory (RMT), where either the K-matrix itself or its underlying Hamiltonian is taken as a random matrix. These two approaches are believed to lead to the same results due to a universality conjecture by P. Brouwer, which is equivalent to the claim that the probability distribution of K, for a broad class of invariant ensembles of random Hermitian matrices H, converges to a matrix Cauchy distribution in the limit of large matrix dimension of H. For unitarily invariant ensembles, this conjecture will be proved in the thesis by explicit calculation, utilising results about ensemble averages of characteristic polynomials. This thesis furthermore analyses various characteristics of the K-matrix such as the distribution of a diagonal element at the spectral edge or the distribution of an off-diagonal element in the bulk of the spectrum. For the latter it is necessary to know correlation functions involving products and ratios of half-integer powers of characteristic polynomials of random matrices for the Gaussian Orthogonal Ensemble (GOE), which is an interesting and important topic in itself, as they frequently arise in various other applications of RMT to physics of quantum chaotic systems, and beyond. A larger part of the thesis is dedicated to provide an explicit evaluation of the large-N limits of a few non-trivial objects of that sort within a variant of the supersymmetry formalism, and via a related but different method.

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1. Introduction

Scattering is a fundamental phenomenon in physics, where large parts of the knowledge about quantum systems stem from scattering experiments. This dates back to the early 20th century, a famous example being Rutherford's gold foil experiment which discovered that atoms are built of a nucleus orbited by electrons. Also today a lot of new insights are gained by scattering experiments, the discovery of the Higgs-boson being the latest most famous example. In addition also in classical wave systems scattering plays an important role, e.g. for electromagnetic or elastodynamic waves. At the heart of such a scattering process is the *scattering matrix* (S-matrix), which relates incoming and outgoing waves or particles, therefore completely characterising the process. An equivalent description of scattering is via the so-called K-matrix (also known as the "Wigner reaction matrix" in nuclear scattering or "impedance matrix" in the electromagnetic wave scattering), a quantity closely related to the S-matrix.

This thesis is concerned with *chaotic scattering*, i.e. a scattering process where a slight change of the parameters describing it changes the outcome in an irregular way, which has attracted both experimental and theoretical interest over the last decades, see e.g. [1–5]. In these cases, a statistical description is most appropriate, and it turns out that the theory of *random matrices* provides a powerful tool to model chaotic scattering processes. To that end, one can either model the S-matrix (or equivalently the K-matrix) as a random matrix, or alternatively the Hamiltonian underlying the scattering process and infer statistical quantities like distributions or correlation functions [5–11].

The thesis is organised as follows: In the remainder of this chapter, all the prerequisites necessary to model quantum chaotic scattering using Random Matrix Theory will be introduced. The main aim is to make it self-contained, such that any scientist with a background in mathematics or physics will be able to understand it without further background reading. We start with an introduction to Random Matrix Theory

in Section 1.1. Then, in Section 1.2, after a short detour into the theory of classical chaos, we introduce the ideas of Quantum Chaos and how they tie with the theory of random matrices. Such prepared, we finally introduce the ideas of quantum chaotic scattering in Section 1.3.

In Chapter 2 we explain two mathematical tools which will be needed throughout the thesis. These are the saddle-point approximation in Section 2.1, and the supersymmetry method in Section 2.2. The main focus is not on mathematical rigour, but on explaining how these methods work and can be applied to problems.

Chapters 3, 4 and 5 form the main body of the thesis. In Chapter 3, which is based on [12], we show that the K-matrix for chaotic scattering is Cauchy-distributed, starting from a Hamiltonian description, where the Hamiltonian H is taken as a large unitarily invariant random matrix (which corresponds to systems with broken time-reversal symmetry). We thereby prove the equivalence of the two different methods of modeling directly the K-matrix or its underlying Hamiltonian H as a random matrix, extending earlier results of P. Brouwer [13] who showed this equivalence for the case of Cauchy-distributed H. This is done by explicit calculation, utilising results about ensemble averages of products and ratios of characteristic polynomials of random matrices. We encounter that for the case where H is an orthogonally invariant random matrix (which correspond to systems with preserved time-reversal symmetry) it is necessary to know such objects not only for integer, but also half-integer powers of characteristic polynomials. Such objects are an interesting and important topic in itself, arising also in various other applications of Random Matrix Theory. Hence Chapter 4, which is based on [14], is dedicated to provide explicit evaluations of the large-N limits of a few non-trivial objects of that sort for the case of Gaussian distributed random matrices. Finally we analyse various other characteristics of the K-matrix in Chapter 5 such as the distribution of a diagonal element at the spectral edge or the distribution of an off-diagonal element in the bulk of the spectrum. We conclude the thesis in Chapter 6, with a summary of the results and an outlook into open problems. Some technical details and calculations are deferred into the Appendix A.

1.1. Random Matrix Theory

Random Matrix Theory (RMT) is a rich topic that evolved from the humble attempt to describe level statistics of compound nuclei to an indispensible tool in physics and beyond. Due to its universal features it has also become an interesting topic in the mathematical community, without any direct applications in mind. Section 1.1.1 shall serve as a short summary of the origins of Random Matrix Theory and how this theory evolved. In Section 1.1.2 we review the so-called "classical ensembles", i.e. those originally introduced in Random Matrix Theory, which also in the present days play an overwhelming role. There are many good textbooks and review articles about Random Matrix Theory, e.g. [15–19].

1.1.1. Historical Overview and Main Ideas

The main concern of Random Matrix Theory is to understand the spectral properties of a matrix whose entries are drawn randomly from a given probability distribution. Its origins can be traced back to the work of Wishart in 1928 [20] in the context of biostatistics. However, the real foundation of the field is usually attributed to Wigner's work in the 1950's [21, 22], motivated by nuclear physics applications, and further development by Dyson [23], Mehta [24] and co-workers.

Wigner's original idea is along the following lines: Suppose we want to study properties of the energy levels E_j[1] of a large compound nucleus.[2] In principle these are completely determined via the Schrödinger equation $H\psi_j = E_j\psi_j$, where ψ_j is the j-th eigenstate of the system corresponding to the energy level E_j and H is the Hamiltonian describing the system. However, a compound nucleus is an object for which it is too complicated to write down a Hamiltonian, and even if it was possible the resulting Schrödinger equation would be too complicated to be solved. On the other hand, the complicated interactions make the problem amenable to a statistical description. It is natural to expect that many compound nuclei share similar statistical spectral prop-

[1] In general, the spectrum consist of a continuum and a – usually infinite – number of discrete energy levels. Here only the latter are considered.

[2] In a nuclear reaction, the incident particle and the target nucleus can form an intermediate state where they become indistinguishable from each other, the energy of the incident particle being shared among all nucleons of the system. Such an excited, quasi-bound nucleus is called compound nucleus.

erties (we call the collection of all such systems an "ensemble"). This is in analogy to the theory of statistical mechanics, where a lot of different microscopic states (e.g. positions and velocities of particles in a gas) lead to the same macroscopic properties (e.g. temperature, pressure). It is clear that such an approach is limited to explore universal features, but cannot reproduce any system specific properties.

In general, H is a linear operator living in an infinite-dimensional Hilbert space. However, if we truncate H at a large but finite number N of energy eigenvalues, it can, after choosing an appropriate basis, be represented as an $N \times N$ matrix. In the spirit of the above discussed statistical description, we then take the elements of H to be random variables, owing to the global symmetries of H. Hence the nucleus is modelled as a kind of "black box" where the real Hamiltonian is replaced by a random one. Taking the ensemble average over all such matrices H of the spectral property in question should then yield the answer for a typical representative of the corresponding nuclei ensemble. Such an ensemble average of a quantity $F(H, x_1, x_2, \dots)$, where the x_j (e.g. energy, quantum numbers) are fixed parameters can be computed as

$$\langle F(H, x_1, x_2, \dots)\rangle_H = \int \mathrm{d}H\, \mathcal{P}(H) F(H, x_1, x_2, \dots), \tag{1.1}$$

where $\mathcal{P}(H)$ is the underlying joint probability distribution of the random matrix H and the measure $\mathrm{d}H$ denotes the product of the differentials of all independent matrix elements. The notation on the left-hand side, using angular brackets $\langle\cdot\rangle_H$ shall here and henceforth denote the ensemble average over H as defined by the right-hand side. One should note that in an experiment, the data usually comes from a single system rather than from an ensemble of systems described by different Hamiltonians, and the ensemble average is replaced by the running average over the spectrum. Therefore one needs the notion of *ergodicity*, i.e. equality of these two averages for almost all representants of the ensemble in the limit of large matrices (in analogy to statistical mechanics where the ensemble average equals the average over time). It can be shown that this is indeed the case [18, 25].

The remaining question is how to choose the underlying probability distribution. Since the random matrix H represents a Hamiltonian having real eigenvalues, we require it to be Hermitian. Based on Wigner's work [22] Dyson showed [23] via group theoretical analysis that further symmetry restrictions can be boiled down to three

different symmetry classes labelled by the *Dyson index* β. These are ensembles of random matrices where the joint probability distribution remains invariant under *orthogonal* ($\beta = 1$), *unitary* ($\beta = 2$) or *unitary symplectic* ($\beta = 4$) transformations respectively. The unitary ensemble (UE) applies to systems with broken time-reversal invariance (e.g. by strong applied magnetic field) irrespective of the behaviour under spin rotation. In that case the matrices are complex Hermitian. The orthogonal ensemble (OE) applies to time-reversal invariant systems with rotational symmetry as well as to systems with broken rotational symmetry and integer spin. The matrices are real symmetric. The symplectic ensemble (SE) applies to systems with preserved time-reversal invariance with half-integer spin and broken rotational symmetry. It is described by Hermitian self-dual matrices. The entries of those matrices are quaternions that can be expressed via the Pauli spin matrices.

In addition, one might require the independent matrix entries to be statistically independent, which can be interpreted as a minimum knowledge requirement. This means it should be possible to write the joint probability distribution $\mathcal{P}(H)$ as a product of probability distributions for each single independent matrix element. It turns out [15] that this requirement, together with the above described invariance under a certain transformation, leads for all three symmetry classes to a unique joint probability distribution given by $\mathcal{P}(H) = \exp(-a \operatorname{Tr} H^2 + b \operatorname{Tr} H + c)$. Without loss of generality, the parameter b can be chosen as zero since this only amounts to a shift in the zero of energy. The parameter a determines the variance and the parameter c fixes the normalisation. We will discuss these so called *Gaussian ensembles* in further detail in the following section.

From a computational point of view, working with these Gaussian ensembles is very convenient because of their nice properties such that often explicit calculations can be performed. However, from a conceptional point of view they have some drawbacks. The independence under orthogonal, unitary or symplectic transformations is a necessary requirement. This is because when representing the Hamiltonian as a matrix one has to choose a basis. The physics, however, should be independent of the particular choice which translates to the condition that the joint probability should be invariant under one of the above mentioned transformations. On the other hand, the requirement of independent matrix elements being statistically independent does not stem from first principles. Hence other non-Gaussian ensembles which are

invariant under said transformations are a possible choice as well. Indeed we will see in the next section that the eigenvalue density of the Gaussian ensemble has compact support (see Eq. (1.9)) and thus on this global scale is not an appropriate model for a physical system, where the level density is supposed to increase. In addition to that there is a class of problems (e.g. scattering problems, see Section 1.3) where the basic objects are unitary matrices. To address both the problems of a non-trivial profile for the density of states and the need of ensembles where the matrices are unitary, Dyson introduced his so called *circular ensembles* [23], which we will discuss at the end of Section 1.1.2. One can show [15] that the circular and Gaussian ensembles are equivalent in the sense that their local spectral fluctuations in the limit $N \to \infty$ share the same properties. This observation lead to an early *universality conjecture* which claims that large random matrices have the same local spectral fluctuations if they belong to the same symmetry class, independent of the underlying probability distribution. This conjecture further justifies both why a compound nucleus could be described by a random matrix and why it usually suffices to work with Gaussian ensembles.

Starting from the '80s, Random Matrix Theory, which up to this point was motivated mainly by applications to nuclear physics, attracted considerable attention in a more widespread theoretical physics community. On the one hand, this was driven by advances in the field of Quantum Chaos, in particular the Bohigas-Gianoni-Schmit conjecture which establishes a connection between RMT and Quantum Chaos. We will explore this link in Section 1.2. On the other hand the supersymmetry approach developed by Efetov [26,27] in the context of disordered systems and adapted to RMT by Verbaarschot, Weidenmüller and Zirnbauer [8], provided both a powerful technical tool and a link between RMT and the theory of disordered systems. In Section 2.2 we will illustrate how this approach works.

Up to the late '80s, Random Matrix Theory was mainly – with some notable exceptions[3] – studied by theoretical physicists. This changed considerable in the '90s when Random Matrix Theory started to attract interest in the mathematical community as well, mainly driven by its conjectured universality features. To this end one should

[3] E.g. the work of Marchenko and Pastur on the spectrum of random covariance matrices [28] or the Montgomery conjecture [29], which establishes a link between RMT and number theory by observing similarities between the spectrum of GUE matrices and the zeros of the Riemann-zeta function.

note that one can distinguish two broad classes of random matrix ensembles. The *invariant matrix ensembles*, and the *Wigner matrix ensembles*, which are ensembles of matrices whose entries are independently distributed. The Gaussian ensembles are the only ensembles that belong to both classes. For important contributions and a mathematical rigorous treatment on universality of invariant ensembles see [30, 31] and references therein, and for the universality features of Wigner matrices the recent books [32, 33] and references therein.

Nowadays Random Matrix Theory is a vast and vibrant interdisciplinary research area, not only with numerous applications in physics (going far beyond those touched in this thesis, like e.g. quantum chromodynamics), but also a wide range of other disciplines like wireless communication theory in engineering, the study of financial markets or number theory (see e.g. the various chapters of Part III in [17]).

1.1.2. Gaussian and Circular Ensembles

In this section we introduce the "classical ensembles", which are the Gaussian and circular ensembles introduced by Wigner, Dyson and Mehta. The main focus will be on the Gaussian ensembles.

Gaussian ensembles

The three Gaussian ensembles are the Gaussian Unitary Ensemble (GUE, $\beta = 2$), the Gaussian Orthogonal Ensemble (GOE, $\beta = 1$) and the Gaussian Symplectic Ensemble (GSE, $\beta = 4$). As discussed in the previous section, their joint probability distribution is given by

$$\mathcal{P}(H) = C_{\mathrm{G}\beta\mathrm{E}} \exp\left(-\frac{\beta N}{4J^2} \operatorname{Tr} H^2\right). \tag{1.2}$$

The parameter J determines the variance

$$\langle H_{ij} H_{kl} \rangle_H = \frac{J^2}{\beta N} \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{1.3}$$

where δ_{ij} is the Kronecker delta. The particular form of the prefactor $\frac{\beta N}{4J^2}$ is arbitrary at the moment, but will become clear later (see footnote 8). $C_{\mathrm{G}\beta\mathrm{E}}$ is the normalisation constant, which is different for the three different ensembles (as indicated by the

short-hand notation GβE for the GOE, GUE and GSE, respectively). It ensures that $\int \mathrm{d}H\, \mathcal{P}(H) = 1$. The random matrix H is complex Hermitian (GUE), real symmetric (GOE) or Hermitian self-dual (GSE). This means the number of independent real parameters is N^2 for the GUE, $N(N+1)/2$ for the GOE and $N(2N-1)$ for the GSE.[4]

Due to the cyclic invariance of the trace it is obvious that the joint probability (1.2) is invariant under orthogonal, unitary or symplectic transformation and with a bit more effort one can also show the invariance of the measure $\mathrm{d}H$ [25]. Furthermore it is easy to check that the independent matrix elements are statistically independent, e.g. for the GOE one gets $\mathrm{Tr}\, H^2 = \sum_j H_{jj}^2 + 2\sum_{j<k} H_{jk}^2$ and hence the individual matrix elements are Gaussian distributed with $\mathcal{P}(H_{jj}) \propto \exp\left(-\frac{\beta N}{4J^2} H_{jj}^2\right)$ on the diagonal and $\mathcal{P}(H_{jk}) \propto \exp\left(-\frac{\beta N}{2J^2} H_{jk}^2\right)$ for off-diagonal matrix elements. Similar arguments lead to independently Gaussian distributed matrix elements in the two other cases as well. Showing that the Gaussian ensembles are the only ensembles having these two properties requires some additional effort [15].

Since $\int \mathrm{d}H\, \mathcal{P}(H)$ separates into a product of Gaussian integrals, the normalisation constants are easy to compute and given by

$$C_{\mathrm{G}\beta\mathrm{E}} = 2^{-N/2} \left(\frac{2^{\beta-2} N}{\pi J^2}\right)^{\frac{N}{4}(\beta N + 2 - \beta)}, \tag{1.4}$$

with the measure for the GUE chosen as $\mathrm{d}H = \prod_{j=1}^{N} \mathrm{d}H_{jj} \prod_{j<k}^{N} \mathrm{d}(\mathrm{Re}\, H_{jk}) \mathrm{d}(\mathrm{Im}\, H_{jk})$ (equivalently one could choose H_{jk} and H_{jk}^* as independent variables with measure $\frac{\mathrm{d}H_{jk}\mathrm{d}H_{jk}^*}{2i}$) and analogously for the GSE as the product of differentials of the different quaternion parts.

A natural question to ask is what the Gaussian distribution for H implies for the distribution of the eigenvalues λ_j, $j = 1 \ldots N$.[5] To that end we note that H can be diagonalised by $H = U\Lambda U^{-1}$, where U is an orthogonal, unitary or unitary symplectic matrix, respectively, and Λ is the diagonal matrix of eigenvalues. Due to the invariance of $\mathcal{P}(H)$ under such transformation, it will only depend on the

[4] Note that the matrix dimension is $N \times N$ for the GOE and GUE, but $2N \times 2N$ for the GSE if the quaternion entries are expressed via the 2×2 Pauli matrices.

[5] For the GSE H has $2N$ eigenvalues, however, they are doubly degenerate, which is called Kramer's degeneracy. With $\lambda_1 \ldots \lambda_N$ we denote the set of distinct eigenvalues.

eigenvalues. Transformation of the measure is a bit more involved and given by

$$\mathrm{d}H = \prod_{1\leq j<k\leq N} |\lambda_j - \lambda_k|^\beta \,\mathrm{d}\Lambda \,\mathrm{d}\mu(U), \tag{1.5}$$

where $\mathrm{d}\Lambda = \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_N$ and $\mathrm{d}\mu(U)$ is the part of the measure which only depends on the eigenvectors. It turns out that this part is the invariant Haar measure on the group $O(N)$, $U(N)$ or $Sp(2N)$, respectively. For a derivation of (1.5) in the $\beta = 2$ case see the comment below Eq. (A.10) in Appendix A.1, where the measure for a different object is calculated, but the computation can be easily adapted for the diagonalisation of H. Very similar arguments as in A.1 lead to the measures for the cases $\beta = 1, 4$. The product

$$\Delta\{\lambda\} = \prod_{1\leq j<k\leq N} (\lambda_j - \lambda_k) = (-)^{N(N-1)/2} \det\left(\lambda_j^{k-1}\right)_{j,k=1\ldots N} \tag{1.6}$$

is called the *Vandermonde determinant* and shall here and henceforth be denoted by the symbol on the left-hand side. An easy argument why the above identity is true is as follows: First we note that both expressions are homogeneous polynomials (i.e. polynomials whose non-zero terms all have the same degree) of degree $N(N-1)/2$ (for the first expression this follows simply from the number of terms in the product; for the second it follows from the definition of the determinant: Each term will be of the form $\lambda_{i_1}^0 \lambda_{i_2}^1 \ldots \lambda_{i_N}^{N-1}$, where $[i_1, \ldots, i_N]$ is some permutation of $[1, \ldots, N]$). Furthermore both expressions vanish whenever $\lambda_j = \lambda_k$ (this is obvious for the first expression; for the second expression two rows of the matrix become identical, which makes the determinant vanish). This implies that both expressions are equal up to a constant factor. Comparing the term $\lambda_2 \lambda_3^2 \ldots \lambda_N^{N-1}$ in both expressions yields the factor $(-)^{N(N-1)/2}$.

Integration over the eigenvalues and eigenvectors separates[6] and the eigenvectors can be integrated out, leaving us with the *joint probability for the eigenvalues*

$$\mathcal{P}(\Lambda) = C_{\Lambda,\mathrm{G}\beta\mathrm{E}} |\Delta\{\Lambda\}|^\beta \exp\left(-\frac{\beta N}{4J^2} \sum_{j=1}^N \lambda_j^2\right). \tag{1.7}$$

[6] Notice that this implies that the eigenvalues are independent from the eigenvectors, with the matrix of N orthonormal eigenvectors being uniformly distributed over the Haar's measure $O(N)$, $U(N)$ or $Sp(2N)$, respectively.

We observe that the eigenvalues are highly correlated via the Vandermonde determinant. More specifically they show a level repulsion (the probability of two eigenvalues coming close to each other being small and exactly zero for any $\lambda_j = \lambda_k$) which is linear, quadratic or quartic for $\beta = 1, 2, 4$, respectively. The normalisation constant $C_{\Lambda,\mathrm{G}\beta\mathrm{E}}$ is given by the inverse of the Selberg integral [15]

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}\lambda_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}\lambda_N \, |\Delta\{\Lambda\}|^\beta \exp\left(-\frac{\beta N}{4J^2}\sum_{j=1}^{N}\lambda_j^2\right) \\ = (2\pi)^{N/2}\left(\frac{\beta N}{2J^2}\right)^{-\frac{N}{4}(\beta N-\beta+2)} \prod_{j=1}^{N}\frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)}.\end{aligned} \tag{1.8}$$

Another important question is how the eigenvalues are distributed globally. As explained the physically most interesting case is for large matrix dimension N. It turns out that in the limit $N \to \infty$, the level density is given for all three ensembles by the *semicircular law*[7]

$$\rho(\lambda) = \begin{cases} \sqrt{4J^2 - \lambda^2}/(2\pi J^2) & \text{for } |\lambda| < 2J, \\ 0 & \text{for } |\lambda| > 2J. \end{cases} \tag{1.9}$$

Besides Wigner's original approach, there are nowadays various methods to calculate this quantity. We will re-derive (1.9) in Section 2.2, where the calculation of the level density serves us as an example to illustrate the so called supersymmetry method. The semicircular law suggests that in the limit $N \to \infty$, the eigenvalues are distributed on a compact support, i.e. between $-2J$ and $2J$ with square-root singularities at the edges.[8] This result is clearly not what one would expect for the level density of a physical system, where usually the number of energy levels in a certain interval increases with higher energy. However, the global level density is very system specific, so in general one could not expect it to be captured by Random Matrix Theory.[9]

[7] Wigner initially proved [21] the semicircular law for the case of symmetric Bernoulli matrices, i.e. matrices where the random entries are either $+1$ or -1 and later realised that the result holds more generally.

[8] Here the scaling of the eigenvalues becomes important. The particular choice of the prefactor $\frac{\beta N}{4J^2}$ in the joint probability distribution ensures that the semicircular law becomes independent of β and N. Otherwise the support would grow with $\sqrt{N}$.

[9] On the other hand, there are other RMT ensembles (Wishart, Jacobi, chiral etc.) whose level densities do describe unversal features of some physical observables.

Instead, Random Matrix Theory should be able to describe local properties, i.e. spectral fluctuations. Such quantities are for example the k-point correlation functions, which are the functions that measure the probability to find eigenvalues of H around each of the positions $x_1 \dots x_k$. They can in principle be found by taking the joint probability of the eigenvalues (1.7) and integrating out the last $N-k$ variables.[10] A quantity which is closely related to the two-point correlation function is the *nearest neighbour distribution* $\mathcal{P}(s)$ which measures the probability to find two adjacent eigenvalues at distance s. Wigner proposed (originally only for the case $\beta = 1$) that the nearest neighbour distribution should be given by [18]

$$\mathcal{P}(s) = a_\beta s^\beta \exp\left(-b_\beta s^2\right), \tag{1.10}$$

where the constants can be fixed by normalisation and e.g. the requirement that on average two levels are at distance one. For the case $\beta = 1$ this leads for example to $a_1 = \pi/2$, $b_1 = \pi/4$. This distribution is known as *Wigner surmise*. It is exact for the case $N = 2$, but not correct in the limit $N \to \infty$. However, the Wigner surmise is very close to the exact solution, with the maximal error being less than two percent [25].

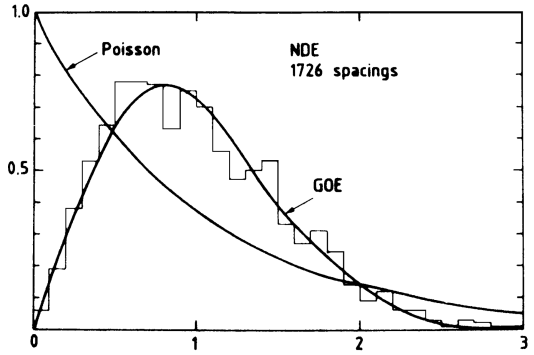

Figure 1.1.: Nearest neighbour distribution for the levels of compound nuclei. In addition the RMT result and the exponential distribution are shown. Taken from [18], originally appeared in [35].

To compare random matrix results with data from physics experiments, one needs to rescale the spectra by the local mean level spacing (given by $d(\lambda) = [N\rho(\lambda)]^{-1}$) such that on average two levels are at distance one, a procedure referred to as *unfolding*.

[10] A standard approach along these lines is the method of orthogonal polynomials [15]. For a derivation of the k-point correlation functions using the supersymmetry method see [34].

Figure 1.1 shows such a comparison of the nearest neighbour distribution for the GOE and the levels of compound nuclei (called "Nuclear Data Ensemble" (NDE) comprising 1726 spacings). In addition the exponential distribution is shown, which is the result for uncorrelated levels. Notice that for the latter case it is very likely for two eigenvalues to be close together which is in contrast to the observed level repulsion in the RMT- and compound nuclei cases.

Spectral edge

The $N \to \infty$ limit of the level density $\rho(\lambda)$ for the Gaussian ensembles, given by the semicircular law (1.9), has a square-root singularity at $\lambda = 2J$ (and $\lambda = -2J$) called the *spectral edge*. For large but finite N, the semicircle is still a very good approximation as long as one does not come to close to this edge. This regime is called the *bulk of the spectrum*. Close to the edge finite-N effects make for the singularity to be smoothed out. While the mean level spacing in the bulk is of order N^{-1}, it is of order $N^{-2/3}$ close to the edge.[11] This suggests the correct large-N limit in this regime, called the *edge scaling limit* is given by considering $N \to \infty$ while $\lambda = 2J + \xi N^{-2/3}$, where ξ is of order unity. In this limit, the level density for the GUE is given by

$$N^{1/3}\rho(2J + \xi N^{-2/3}) = \mathrm{Ai}'(\xi)^2 - \mathrm{Ai}(\xi)\,\mathrm{Ai}''(\xi) = \mathrm{Ai}'(\xi)^2 - \xi\,\mathrm{Ai}(\xi)^2, \tag{1.11}$$

where $\mathrm{Ai}(\xi)$ denotes the Airy function, we used its property $\mathrm{Ai}''(\xi) = \xi\,\mathrm{Ai}(\xi)$. Using results from Section 5.5, we derive Eq. (1.11) in Appendix A.9. Figure 1.2 shows the level density at the edge of the GUE spectrum. It has oscillations left of the edge at $2J$ and decays exponentially on the right (compare also with Eq. (2.18)). This means for large but finite N there are indeed only very few eigenvalues which lie outside of the support of the semicircle. Because of this fact the edge is also called a *soft edge*, in contrast to a *hard edge* which cannot be penetrated; e.g. for a distribution with strictly positive eigenvalues, zero would be a hard edge.

Note that Eq. (1.11) is only valid for the GUE, but e.g. for the GOE one has instead

$$N^{1/3}\rho(2J + \xi N^{-2/3}) = \mathrm{Ai}'(\xi)^2 - \xi\,\mathrm{Ai}(\xi)^2 + \frac{1}{2}\,\mathrm{Ai}(\xi)\left(1 - \int_\xi^\infty \mathrm{d}\eta\ \mathrm{Ai}(\eta)\right). \tag{1.12}$$

[11] This is a consequence of the square-root singularity. More generally the scaling is given by $N^{-1/(1+\alpha)}$ if the level density, close to the edge at λ_{edge}, behaves as $\rho(\lambda) \propto |\lambda - \lambda_{\mathrm{edge}}|^\alpha$.

This means while the level density is the same for all Gaussian ensembles in the bulk of the spectrum, it differs at the edge.

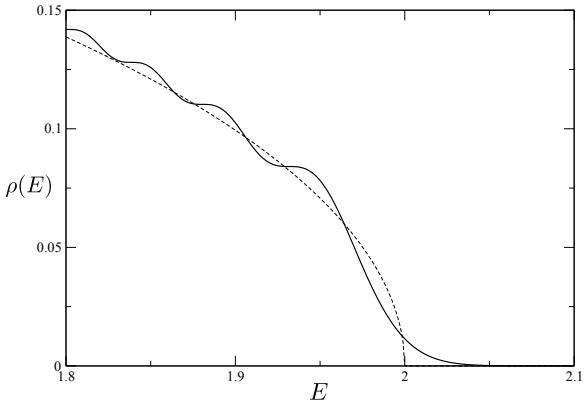

Figure 1.2.: The level density (1.11) at the edge of the GUE spectrum (with $J = 1$) for $N = 200$ compared with the semicircular distribution (1.9) (dashed line).

Circular ensembles

The circular ensembles introduced by Dyson [23] are defined as unitary random matrices which are invariant under orthogonal, unitary or symplectic transformation and have a *uniform level density*. Analogous to the Gaussian case we distinguish between symmetric unitary matrices ($\beta = 1$, COE), arbitrary unitary matrices ($\beta = 2$, CUE) and self-dual unitary quaternion matrices ($\beta = 4$, CSE). Since these matrices are unitary, their eigenvalues will be of the form $\exp(i\theta_j)$, where the θ_j, $j = 1 \dots N$ are the eigenphases. Their joint distribution is given by

$$\mathcal{P}(\theta_1, \dots, \theta_N) = C_{\mathrm{C}\beta\mathrm{E}} \prod_{1 \le j < k \le N} \left| e^{i\phi_j} - e^{i\phi_k} \right|^\beta = C_{\mathrm{C}\beta\mathrm{E}} \prod_{1 \le j < k \le N} \left| 2 \sin \frac{\theta_j - \theta_k}{2} \right|^\beta, \quad (1.13)$$

where $C_{\mathrm{C}\beta\mathrm{E}} = \frac{\Gamma(1+\beta/2)^N}{(2\pi)^N \Gamma(1+\beta N/2)}$. Notice that $\mathcal{P}(\theta_1, \dots, \theta_N)$, in contrast to the Gaussian ensembles, depends only on the differences $\theta_j - \theta_k$. As required, this yields a constant level density valid for any N,

$$\rho(\theta) = \frac{1}{2\pi}, \quad (1.14)$$

which means the eigenvalues are distributed uniformly on the unit circle in the complex plane. In the limit $N \to \infty$, the circular ensembles have the same local spectral fluctuation properties as the Gaussian ensembles [15].

1.2. Quantum Chaos

Classical mechanics distinguishes between two very different types of motion. On the one hand there is the *regular motion* of *integrable systems*, e.g. the harmonic oscillator, a simple pendulum or the two-body (Kepler) problem. On the other hand there is *chaotic motion* of *non-integrable systems*, e.g. a double pendulum or the three-body (or more generally n-body) problem. While there are precise measure theoretical definitions of chaos, for our purposes it is sufficient to say that a system is chaotic if the distance between two trajectories which were initially close together grows exponentially in time. The system is hence very sensitive to initial conditions.

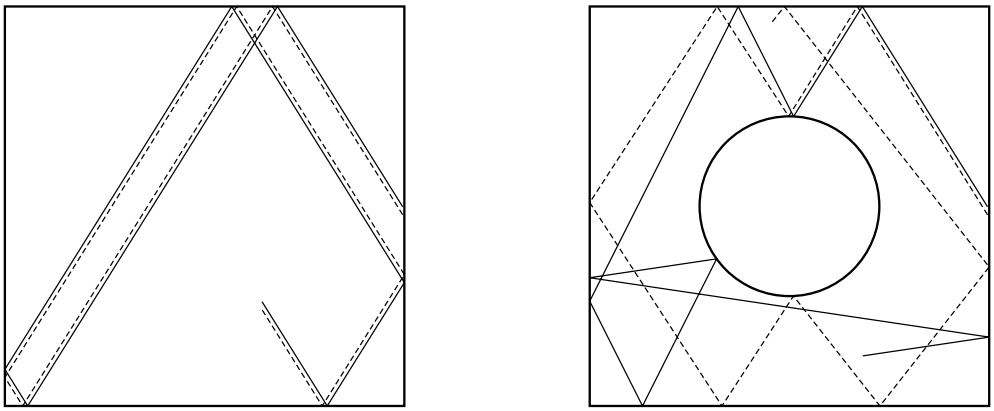

Figure 1.3.: Comparison of the classical square- and Sinai-billiard. The left shows a square cavity. It is is regular, two trajectories that are close together stay close all the time. The right shows the classical Sinai billiard. It is chaotic, two trajectories that start close together look completely different after a few collisions with the boundaries.

A very nice toy model for studying chaotic behaviour are *billiards*, which are two-dimensional cavities in which a classical particle moves freely, i.e. in a straight line with reflection at the boundary. Mathematically this can be described by the Hamilton formalism with potential zero inside of the cavity and infinity at its boundary. Figure 1.3 shows a square shaped billiard and the so called Sinai-billiard which is also square shaped but with a disk removed from its centre. For each billiard, two different trajectories which were initially close together are shown. While for the square shaped billiard the two trajectories stay close together all the time, they become completely different after a few collisions with the boundaries in the Sinai billiard. This shows

that the square shaped billiard exhibits regular motion[12] whereas the Sinai billiard exhibits chaotic motion.[13] A system is integrable if it has as many independent constants of motion, which must be in involution to each other, as it has degrees of freedom. In a Hamiltonian system, the energy is always conserved and hence a constant of motion. For billiards it is simply the kinetic energy of the particle given by $E = \frac{p_x^2+p_y^2}{2m}$, where m is the mass of the particle and p_x, p_y are the momenta in x- and y-direction, respectively. In the square-shaped billiard also $|p_x|$ and $|p_y|$, i.e. the modulus of each component of the momentum, are conserved, hence it has two independent constants of motion and is thus integrable, showing regular motion. The Sinai billiard, which shows chaotic behaviour, has no further constants of motion apart from the energy and is thus non-integrable.

The correspondence principle ensures that the behaviour of a classical system emerges from quantum mechanics in the limit of large quantum numbers (formally $\hbar \to 0$). In quantum mechanics, however, the Heisenberg principle forbids to know both a particles position and momentum at the same time. Hence the notion of trajectories becomes meaningless and it is a priori not clear how chaotic behaviour manifests in quantum mechanics. Another idea could be to look at the "distance" of two wave functions, $\delta\psi(t) = \psi_2(t) - \psi_1(t)$. However, its modulus is constant under time evolution, $|\delta\psi(t)| = |\delta\psi(0)|$, due to the linearity of the Schrödinger equation. This shows that a simple transfer of the concepts of classical chaos to quantum mechanics is not possible, although chaotic behaviour has to be founded in quantum mechanics due to the correspondence principle. *Quantum Chaos* is the name of the branch of physics which tries to discover this correspondence. Textbooks on this topic are e.g. [25, 36, 37], the latter also containing a treatment of classical chaos.

One powerful approach to Quantum Chaos is the *semiclassical approach*, which analyses quantum systems in the above mentioned limit of large quantum numbers ($\hbar \to 0$), the so called semiclassical limit. An important early result in this direction is Gutzwiller's trace formula which is the semiclassical approximation of the density of states for a quantum chaotic system [37, 38]. The main ingredients to this formula are the periodic orbits of the classical systems which shows a direct correspondence

[12] In this example the distance between the two trajectories is constant. However, also any system where the distance only grows sub-exponential is in our definition regular.

[13] Note, however, that the exponential growth of the system is limited by the system size. The chaotic behaviour develops well before such effects take place.

between the classical and the quantum world. However, while this approach shows how classical chaotic structures emerge from quantum mechanics, it is not clear how chaotic behaviour manifests on the quantum level.

It turns out that the correct way to classify quantum chaotic behaviour is via the spectral correlations of quantum systems. A good toy model are again billiards. A *quantum billiard* is defined analogous to its classical counterpart, i.e. a quantum particle confined to a two-dimensional region which has potential zero inside and potential infinity at its boundary. Mathematically this means the eigenstates ψ_n (where $n \in \mathbb{N}$ labels the eigenstates) of the system can be described by the stationary Schrödinger equation for a free particle $-\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi_n(x,y) = E_n\psi_n(x,y)$, with the condition that $\psi_n(x,y)$ vanishes at the boundary (Dirichlet boundary conditions). Here $\hbar$ is the reduced Planck constant, m the mass of the quantum particle, E_n the energy level corresponding to the n-th eigenstate and x, y the (Cartesian) position coordinates. Note that this follows immediately via quantisation of the Hamilton function $\frac{1}{2m}(p_x^2+p_y^2)$ for the classical billiard. For example the quantum Sinai billiard yields, after unfolding the spectrum (i.e. rescaling it such that the mean level spacing is unity), the nearest neighbour distribution as shown in Figure 1.4.[14] Remarkably this seems to be the same distribution which was found for the levels of compound nuclei, compare with Figure 1.1, and which can be modeled within the framework of Random Matrix Theory. This observation led Bohigas, Giannoni and Schmit to formulate their famous (BGS) conjecture

> "Spectra of time-reversal-invariant systems whose classical analogs are K systems show the same fluctuation properties as predicted by GOE." [39]

K-systems are strongly mixing classical systems that show a high degree of chaos in the sense of exponential growth of the distance between two trajectories. The Sinai billiard is an example of such a K-system where the exponential growth is achieved through the convex boundary. Notice that the conjecture goes beyond two-point correlations like the nearest neighbour distribution and claims that all fluctuation properties, e.g. also higher k-point correlation functions, can be described in the framework of Random Matrix Theory. Although this conjecture has been verified in

[14] The billiard used to produce the figure is a desymmetrised version of the Sinai billiard, its shape is shown in Figure 1.4 as well.

a vast amount of experiments and there are "heuristic proofs" by physicists [40,41], no rigorous mathematical proof has been found so far. Note, however, that considerable progress in this direction has been made for the case of quantum graphs[15] [42,43].

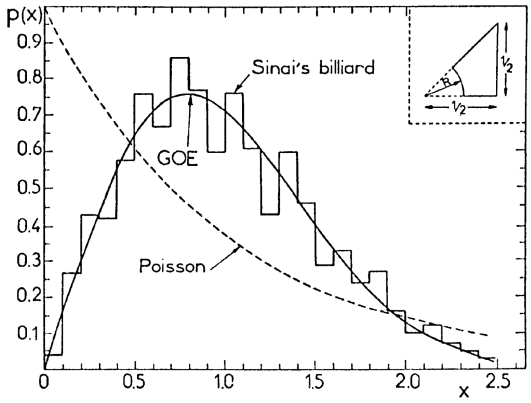

Figure 1.4.: Nearest neighbour distribution for the levels of the Sinai quantum billiard (shape shown in picture). In addition the RMT result and the exponential distribution are shown. Taken from [18], originally appeared in [39].

The reason this is a good way to classify chaotic behaviour of quantum systems is because the spectral correlations for quantum systems whose classical counterparts are integrable are very different from the GOE statistics. This is summarised in the Berry-Tabor conjecture [44] which states that spectra for those systems show usually Poisson statistics, i.e. their levels are uncorrelated. Figure 1.5 shows the nearest neighbour distribution for the levels of a rectangular quantum billiard with incommensurable ratio of the sides[16], which can be well described by the exponential distribution $\mathcal{P}(s) = \exp(-s)$. Notice, however, that there are notable exceptions already known to Berry and Tabor, e.g. the one-dimensional harmonic oscillator has a uniform distribution of eigenvalues, and hence its nearest neighbour distribution in the unfolded spectrum is a delta-peak at $s = 1$.

The BGS conjecture established a deep link between the previously disconnected fields of Quantum Chaos and Random Matrix Theory. However, it should be noted

[15] A quantum graph is a set of vertices which are connected by bonds (or edges) of assigned lengths, the whole graph being equipped with a Hamilton operator.

[16] For commensurable ratio of sides the spacings will be integer multiples of each other, however still exponentially distributed with $\mathcal{P}(n) \propto \exp(-na)$, a being a constant depending on the side lengths.

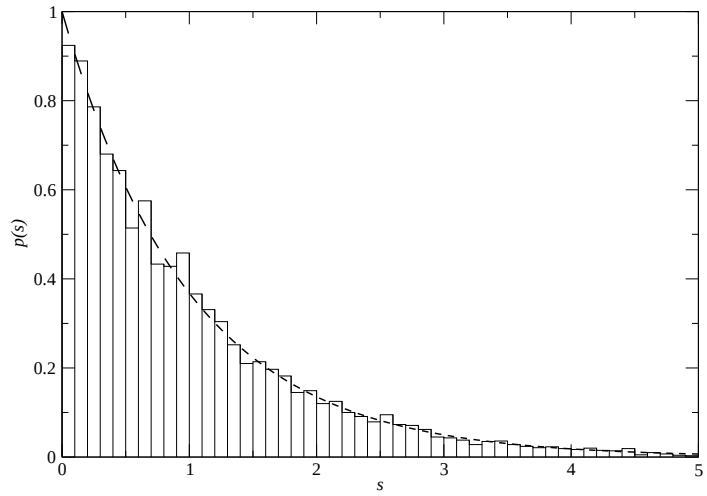

Figure 1.5.: Nearest neighbour distribution for the rectangular quantum billiard with side ratio $\sqrt{2}$. The first 10000 levels were considered. In addition the exponential distribution is shown (dashed line).

that applications of Random Matrix Theory go beyond the BGS conjecture: We have already seen that the levels of compound nuclei can be well described using random matrices. However, a nucleus is a many particle system, with the particles being indistinguishable from each other and possessing spin which cannot be neglected in the description of the interactions. These phenomena are of a purely quantum nature and do not exist in classical systems. Therefore the compound nucleus does not have a classical counterpart, yet its levels also show the same fluctuation properties as the GOE. In this sense we can also classify the compound nucleus as "quantum chaotic", although the BGS-conjecture does not apply. More generally one can expect such quantum chaotic behaviour in many systems where the interactions are very complicated, and thus Random Matrix Theory is capable of describing the fluctuation properties of a vast amount of quantum systems.

In addition also some other systems, which are not quantum at all, can be described successfully by random matrices. An example is a two-dimensional electromagnetic cavity. Electromagnetic waves inside such a cavity can be described by the Helmholtz equation $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) A(x,y) = -k^2 A(x,y)$, where k is the wave-number and $A(x,y)$ is the amplitude, which vanishes on the boundary. This equation is mathematically equivalent to the Schrödinger equation and hence the electromagnetic wave in a cavity

is equivalent to a quantum billiard. Indeed it is possible to model quantum billiards experimentally with microwave cavities [36, 45], which has huge advantages such as having a macroscopic experimental set-up and being able to collect more data via measurements as would be possible for quantum systems.[17] More generally many wave systems are also amenable to a description via random matrices, another example being the modes of elastodynamic (acoustic) waves in aluminium blocks [46] or quartz crystals [47], where the underlying wave equations are very different from the Schrödinger equation.

1.3. Quantum Chaotic Scattering

So far all considerations have been for *closed* systems (levels of compound nuclei, quantum billiards etc.). However, one can also consider their *open* counterparts, i.e. such a system coupled to the environment. A very important physical phenomenon that arises for such open systems is *scattering*.

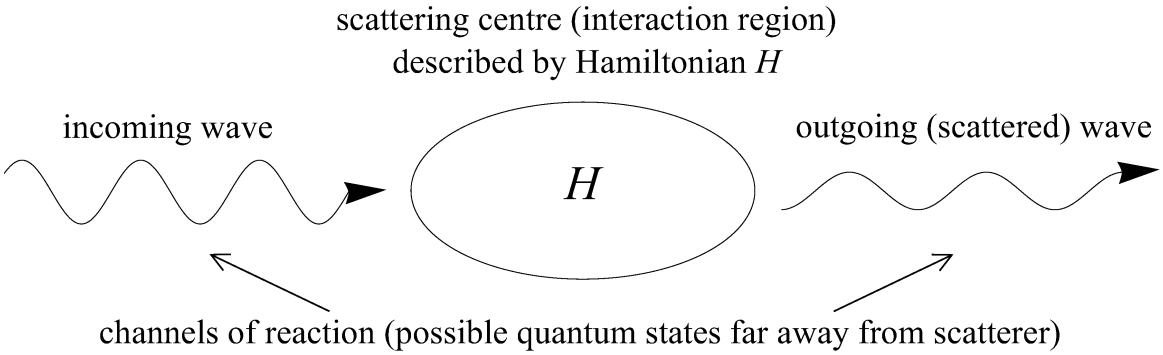

Figure 1.6.: Set-up of a generic scattering process.

Figure 1.6 shows a set-up of such a (not necessarily chaotic) scattering process. We identify the formely closed system, which can be described by the Hamiltonian H, as *interaction region* or *scattering centre* and assume that outside of this region interaction is absent. This means a quantum particle or wave outside of the interaction region exhibits a free motion. Upon entering the interaction region, the particle or

[17] In quantum experiments one is usually only able to measure cross-sections which does not allow to determine modulus and phase separately. In a microwave experiment this does not pose a problem.

wave will get scattered and may thereafter (again far away from the interaction region) be in a different state than before. We call the set of all possible states a particle can exhibit asymptotically far away from the interaction region *channels of reaction*. In the quantum mechanical setting these states are the wave functions (probability amplitudes), characterised by the energy E and a set of quantum numbers. For electromagnetic or acoustic waves, they are the electromagnetic fields or displacement vectors, respectively.

This is the most generic set-up of a scattering process where a scatterer (compact interaction region) can be identified. It describes a wide range of physical systems, the scatterer could e.g. be a nucleus, atom or molecule, a mesoscopic ballistic device (quantum dot), microwave cavity etc. The number and nature of the channels of reaction is determined by the physical system under consideration. In nuclear scattering for example these channels are the particles into which a compound nucleus can decay. Figure 1.7 shows a schematic view of such a system with five different channels. Another example is a quantum dot as shown in Figure 1.8, where the channels are transverse modes in the leads attached to the dot, their number being determined by the geometry of the lead.

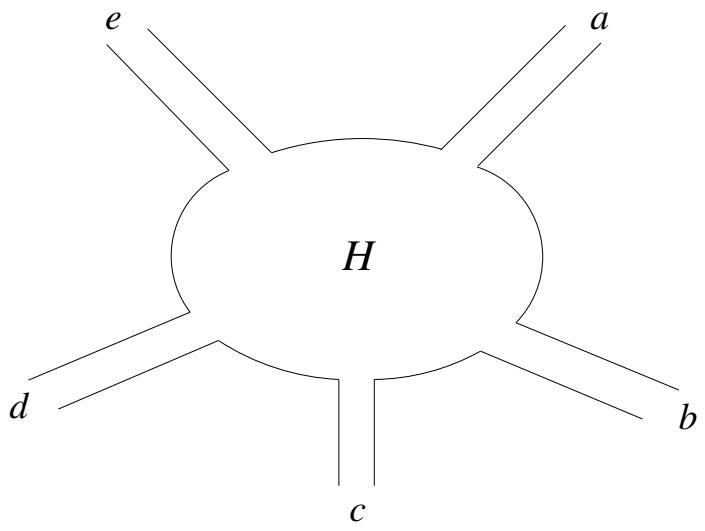

Figure 1.7.: Schematic view of a general scattering problem with five different channels of reaction (labeled $a, b, \dots$). They are connected via a compact interaction region described by a Hamiltonian H. In nuclear scattering for example these channels could be the particles into which the compound nucleus can decay, e.g. neutron, proton, electron, photon and alpha particle.

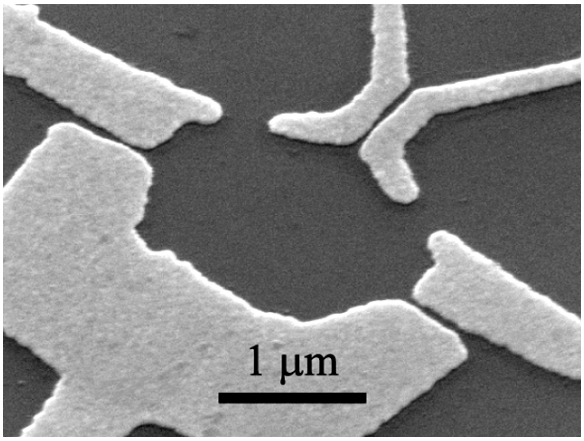

Figure 1.8.: "Quantum dot defined by 5 metallic gates fabricated on the surface of a GaAs based heterostructure, in which a two-dimensional electron gas recides." Taken from http://pages.unibas.ch/phys-meso/Pictures/pictures.html (2012)

It should be noted that qualitatively there are two different scattering mechanisms: On the one hand are immediate responses, the so-called *direct processes*, on the other hand are delayed responses, which are due to the formation of *resonances*, i.e. long-living intermediate states. The first one happens on a much shorter time-scale than the second one. In nuclear scattering, an example for a direct process could be when a particle upon hitting the nucleus breaks one of its neutrons free. This can be interpreted as a coupling of two different channels without interacting with the scattering centre. In contrast to this a resonance, i.e. an excited intermediate state, can form when a particle gets absorbed by the nucleus. After a much longer time-scale the nucleus goes back to its initial state by emitting another particle.

One major simplification we will assume throughout the thesis is the absence of absorption, i.e. there are no internal losses. Note, however, that we briefly discuss in the Conclusions 6 how one can introduce absorption into the model.

1.3.1. Scattering Matrix (S-Matrix)

At the heart of the scattering problem lies the scattering matrix (S-matrix). Assuming at a given energy E there are M channels of reaction, we collect the amplitudes of incoming waves in the vector $\boldsymbol{\psi}^{(in)} = (\psi_1^{(in)}, \ldots, \psi_M^{(in)})^T$ and the amplitudes of outgoing (scattered) waves in the vector $\boldsymbol{\psi}^{(out)} = (\psi_1^{(out)}, \ldots, \psi_M^{(out)})^T$. The scattering

matrix S is then defined as the $M \times M$ matrix which relates incoming and outgoing waves, $\boldsymbol{\psi}^{(out)} = S\boldsymbol{\psi}^{(in)}$. It therefore completely characterises the scattering process. Owing to the flux conservation requirement the S-matrix is unitary in nature [48], i.e. $S^\dagger S = SS^\dagger = \mathbb{1}_M$.

An easy example is the one-dimensional two-channel case, i.e. S is a 2×2 matrix. Particles on the left of the scatterer (which we assume to be centered around $x = 0$) can then be described by plane waves, $\psi_L(k, x) = A(k)\exp(ikx) + B(k)\exp(-ikx)$, $x \ll 0$ similarly for particles on the right $\psi_R(k, x) = C(k)\exp(ikx) + D(k)\exp(-ikx)$, $x \gg 0$ where $k = \sqrt{2mE}/\hbar$ is the wave vector (m being the mass of the particle) and A, B, C, D are the wave amplitudes. Clearly the terms proportional to $A(k)$ and $D(k)$ are incoming waves, whereas the other two are outgoing waves. Hence

$$\boldsymbol{\psi}_{out} = \begin{bmatrix} B(k) \\ C(k) \end{bmatrix} = \begin{bmatrix} S_{11}(k) & S_{12}(k) \\ S_{21}(k) & S_{22}(k) \end{bmatrix} \begin{bmatrix} A(k) \\ D(k) \end{bmatrix} = S\boldsymbol{\psi}_{in}, \tag{1.15}$$

the elements of S completely describing the scattering process. Moreover the probability flux in a one-dimensional system is given by $J = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$. This implies for the flux on the left $J_L = \frac{\hbar k}{m}(|A|^2 - |B|^2)$ and on the right $J_R = \frac{\hbar k}{m}(|C|^2 - |D|^2)$. Due to conservation of flux we require those two to be equal. $J_L = J_R$ is then equivalent to unitarity of S,

$$\begin{aligned} J_L = J_R &\Leftrightarrow |A|^2 + |D|^2 = |B|^2 + |C|^2 \\ &\Leftrightarrow \boldsymbol{\psi}_{in}^\dagger \boldsymbol{\psi}_{in} = \boldsymbol{\psi}_{out}^\dagger \boldsymbol{\psi}_{out} = \boldsymbol{\psi}_{in}^\dagger S^\dagger S \boldsymbol{\psi}_{in} \Leftrightarrow S^\dagger S = \mathbb{1}_2. \end{aligned} \tag{1.16}$$

For the special case of incoming waves from the left only, $D(k) \equiv 0$, the unitarity condition implies that S takes the form $S(k) = \begin{bmatrix} r(k) & t(k) \\ t(k) & r(k) \end{bmatrix}$ with r and t obeying the relations $|r|^2 + |t|^2 = 1$ and $rt^* + r^*t = 0$, hence $|r \pm t|^2 = 1$. From the form of ψ_L and ψ_R it becomes clear that $r(k)$ and $t(k)$ describe reflexion and transmission of the incoming wave, respectively.

For chaotic scattering, e.g. when the scattering centre is a quantum chaotic system as described in the previous section, a slight change of the parameters of incoming waves or the scattering centre changes the behaviour of the S-matrix characteristics in an irregular way. Therefore it seems most appropriate to find a statistical description

for the scattering process, i.e. to describe the S-matrix and related quantities in terms of distributions and correlation functions. Like for the description of closed quantum chaotic systems, there are two standard approaches in this direction, the semiclassical and the stochastic approach. The former relies on representing the S-matrix elements in terms of a sum over the classical periodic orbits, starting with the genuine microscopic Hamiltonian representing the system, while the latter in contrast models the Hamiltonian or the S-matrix itself as a stochastic quantity using Random Matrix Theory, justified by the universality conjectures. Both approaches have their advantages and drawbacks: the semiclassical approach for example works only well for the case of many open channels. This, however, does not cover all the important cases since we can have physical systems with only a few open channels. The stochastic approach on the other hand is limited to explore only universal aspects, leaving aside system specific properties. The comparison between these two approaches has been discussed at length in [49]. As indicated above there are two different approaches to adopt the stochastic route, which will be described in more detail in the following.

1.3.2. Hamiltonian Approach (Heidelberg Approach)

The Hamiltonian or Heidelberg approach introduces stochasticity on the level of the Hamiltonian describing the scattering centre. It was developed by Verbaarschot, Weidenmüller and Zirnbauer in [8] (see also the review article [50] for earlier work leading to [8]). For simplicity we restrict to the case of *resonance scattering*, i.e. we neglect any direct processes. This is equivalent to the claim that the scattering matrix is diagonal on average, $\langle S_{ab} \rangle = \delta_{ab} \langle S_{aa} \rangle$ [8]. This restriction is justified due to [51], where the authors show that for any scattering matrix which is non-diagonal on average one can find a new scattering matrix $USU^\dagger$ (U being a unitary transformation) which is diagonal on average and thus shares the same fluctuation properties as an S-matrix without direct processes.

The first step is to find an analytical expression which relates the scattering matrix with the system Hamiltonian, starting from a microscopic description. As suggested by the general setting described previously and illustrated by Figures 1.6 and 1.7, we identify an interaction region associated with a Hamiltonian H coupled to M channels of reaction. This Hamiltonian possesses a discrete set of orthonormal eigenstates

(bound states) $|n\rangle$, $n = 1, \ldots, N$. The number of eigenstates N is usually very large and has eventually to be taken to infinity. The channels of reaction, on the other hand, can be described by a continous set of functions $|c, E\rangle$, $c = 1 \ldots M$, where E is the total energy of the system. They satisfy the orthonormality condition $\langle a, E_1 | b, E_2 \rangle = \delta_{ab}\, \delta(E_1 - E_2)$. The number M of channels depends on the system under consideration and can have wide range from very few to very many channels (but it is always assumed that M stays finite with $M \ll N$). The full Hamiltonian $\mathcal{H}$ depicting the scattering process can then be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V}. \tag{1.17}$$

Here $\mathcal{H}_0$ describes the part of the Hamiltonian which is present without any interaction between the internal states of the system Hamiltonian H and states of the channels,

$$\mathcal{H}_0 = \sum_{n,m=1}^{N} |n\rangle H_{nm} \langle m| + \sum_{c=1}^{M} \int_{\epsilon_c}^{\infty} \mathrm{d}E\, |c, E\rangle E \langle c, E|, \tag{1.18}$$

where ϵ_c is the threshold energy of channel c. Only for an energy $E > \epsilon_c$ wave propagation is possible (we say the channel is "open"). Note that any direct interaction between different channels has been neglected as discussed above, thus rendering the second term diagonal in c. $\mathcal{V}$ represents the interaction part,

$$\mathcal{V} = \sum_{n=1}^{N} \sum_{c=1}^{M} \left(|n\rangle \int \mathrm{d}E\, W_{nc} \langle c, E| + \text{herm. conj.} \right). \tag{1.19}$$

The *coupling amplitudes* W_{nc} describe how the N bound states of H are coupled to the M channels. They are arranged in the $N \times M$ *channel matrix* W, composed of the N-dimensional *channel vectors* $\boldsymbol{w}_c$, $c = 1 \ldots M$ describing the interaction between channel c and the bound states (which are then to be understood as resonances).

The scattering matrix associated with the full Hamiltonian $\mathcal{H}$ can be worked out using standard techniques, i.e. to write down the Lippmann-Schwinger equations for the incoming and outcoming waves (which are equivalent to the Schrödinger equation but more suitable for scattering problems) and use them to work out the scattering matrix which is given as the inner product $S_{ab} = \langle \psi_a^{(out)} | \psi_b^{(in)} \rangle$, $a, b = 1 \ldots M$. This

formal derivation shall not be pursued here, it can e.g. be found in [48]. An illuminating derivation for a special case (a single lead attached to a cavity) is explained in [6]. Assuming the channel matrix W is energy-independent[18], it turns out the S-matrix can be expressed as[19]

$$S(E) = \frac{\mathbb{1}_M - iK(E)}{\mathbb{1}_M + iK(E)}, \qquad K(E) = W^\dagger (E - H)^{-1} W, \tag{1.20}$$

i.e. as the Cayley transform of the so called K-matrix, also known as the "Wigner reaction matrix" in nuclear scattering or "impedance matrix" in the electromagnetic wave scattering. Due to this one-to-one correspondence, also the K-matrix is well suited to characterise a scattering problem. In the main body of the thesis we will mainly work with the K-matrix instead of the S-matrix. Notice that K is Hermitian, ensuring unitarity of S. Expanding (1.20) in a Taylor series and rearranging terms, another representation of S frequently used in applications can be found as

$$S(E) = \mathbb{1}_M - 2iW^\dagger (E - H + iWW^\dagger)^{-1} W. \tag{1.21}$$

This representation shows that the open system can be described by the effective (non-Hermitian) Hamiltonian $\mathcal{H}_{\text{eff}} = H - iWW^\dagger$ (e.g. its eigenvalues are singularities of S and hence correspond to the resonance states of the system). For vanishing coupling between the channels and the centre, $W = 0$, the scattering matrix becomes unity (K vanishes), in accordance with the neglection of any direct processes.

So far Eqs. (1.20) and (1.21) are valid for any scattering process where an interaction region can be identified, within the discussed limitations (absence of direct processes, absence of absorption, energy-independence of W). To study quantum chaos-induced fluctuations of S one replaces now H by an $N \times N$ random matrix in the spirit of the Bohigas-Giannoni-Schmit conjecture[20] and performs an ensemble average, usually for the case $N \gg M$, i.e. in the limit of large number of resonances with finite number of channels M.

[18] This is usually a fair assumptions since W typically varies only very slowly with energy far away from the threshold [5].

[19] For notational convenience we write $(E-H)^{-1}$ instead of $(E\mathbb{1}_N - H)^{-1}$. This convention, omitting identity matrices where appropriate, will be used throughout the thesis.

[20] This means usually H is taken from the GOE or GUE according to time-reversal symmetry being preserved or broken, respectively, and from the GSE if spin becomes important.

At first sight it seems with the channel matrix W a large amount of parameters has been introduced. This, however is not the case. As previously mentioned, the absence of direct processes is equivalent to the requirement that S is diagonal on average, $\langle S_{ab}\rangle = \delta_{ab}\langle S_{aa}\rangle$ [8]. Moreover orthogonal or unitary invariance of the underlying joint probability distribution (e.g. for GOE, GUE) implies that moments of S can only depend on $\boldsymbol{w}_a^\dagger \boldsymbol{w}_b$, $a, b = 1 \ldots M$ [8, 49], the $\boldsymbol{w}_c$ being the channel vectors, which in the absence of direct processes can be chosen as mutually orthogonal [6, 8, 49], such that

$$\boldsymbol{w}_a^\dagger \boldsymbol{w}_b = \gamma_a \delta_{ab}. \tag{1.22}$$

The γ_c, $c = 1 \ldots M$ are called *coupling coefficients*. Indeed it can be shown [8] that with condition (1.22) the ensemble averaged S-matrix is diagonal and for H taken from the GOE or GUE given by

$$\langle S_{ab}\rangle = \delta_{ab}\frac{1 - \gamma_a g(E)}{1 + \gamma_a g(E)}, \tag{1.23}$$

where $g(E) = (iE + \sqrt{4J^2 - E^2})/(2J^2)$.

The relation (1.22) implies that the input parameters of the model are not the huge number $(N \cdot M)$ of coupling amplitudes W_{nc}, but merely the M coupling coefficients γ_c. In general their values can be any positive real numbers. However, it is more convenient to characterise the strength of the coupling not by the γ_c, but instead by the *transmission coefficients* defined by $T_c = 1 - |\langle S_{cc}\rangle|^2$. Using Eq. (1.23) they can be written as

$$T_c = \frac{2\gamma_c\sqrt{4J^2 - E^2}}{J^2 + \gamma_c\sqrt{4J^2 - E^2} + \gamma_c^2}. \tag{1.24}$$

Due to their definition the transmission coefficients T_c have values between 0 and 1. They measure the part of the flux in channel c that is not directly reflected back but spends a significant time inside the interaction region [6, 49]. This means the cases $T_c = 1$ and $T_c \ll 1$ correspond to a perfectly open or almost closed channel c, respectively. As one can see from Eq. (1.24), very small as well as very large γ_c-values result in small transmission coefficients. At given energy E the largest value of T_c is obtained for $\gamma_c = J$, while the perfect coupling $T_c = 1$ is only reached if in addition also $E = 0$. The mean of the scattering matrix vanishes in this case.

An important special case is that of *equivalent channels*, $\gamma_c \equiv \gamma$ for all $c = 1 \dots M$. For this case the orthogonality relation (1.22) simplifies to

$$W^\dagger W = \gamma \mathbb{1}_M. \tag{1.25}$$

We shall refer to the model described by Eqs. (1.22) or (1.25) as *"Fixed Amplitude Model"*.

Another model which ensures diagonal $\langle S \rangle$, originally suggested in [52], is the *"Random Amplitude Model"*. In this model the channel vectors $\boldsymbol{w}_c$ are considered to be Gaussian random vectors with joint probability defined via

$$\langle \boldsymbol{w}_a \rangle = 0, \quad \langle \boldsymbol{w}_a^\dagger \boldsymbol{w}_b \rangle = \gamma_a \delta_{ab}. \tag{1.26}$$

For the special case of equivalent channels this simplifies to

$$\langle W \rangle = 0, \quad \langle W^\dagger W \rangle = \gamma \mathbb{1}_M, \tag{1.27}$$

which implies for the joint probability of W (with $\beta = 1$ and real W_{nc} for systems with preserved time-reversal invariance and $\beta = 2$ and complex W_{nc} for systems with broken time reversal invariance)

$$\mathcal{P}(W) = \left(\frac{N}{2\pi\gamma} \right)^{\beta MN/2} \exp\left(-\frac{\beta N}{2\gamma} \operatorname{Tr} W^\dagger W \right). \tag{1.28}$$

Both models are expected to lead to the same results in the limit $N \to \infty$ as long as the number of channels M remains fixed. Such an equivalence was explicitly verified in [53] for particular scattering characteristics (Wigner delay times), but is expected to hold generally.

1.3.3. Maximum Entropy Approach (Mexico Approach)

The maximum entropy or Mexico approach was pioneered in [7, 54, 55]. It introduces stochasticity on the level of the S-matrix itself and thus avoids introducing a Hamiltonian altogether. The requirements for S are unitarity (due to flux conservation), causality (i.e. S needs to be analytic in the upper half-plane) and (if relevant) time-

reversal invariance imposed on S. In [56] was shown that causality is enforced by the so-called *analyticity-ergodicity requirement* $\overline{S}^p = \langle S \rangle^p = \langle S^p \rangle$ for all $p \in \mathbb{N}$, or equivalently the *reproducing property*

$$f(\overline{S}) = \int \mathrm{d}\mu(S) \mathcal{P}_{\overline{S}}(S) f(S) \tag{1.29}$$

for any analytic function $f(S)$. $\overline{S}$ denotes the energy-averaged S-matrix which is usually determined by experiment and a free parameter in this model. Due to ergodicity this quantity is equal to the ensemble-averaged S-matrix denoted by $\langle S \rangle$. The right-hand side of Eq. (1.29) is the ensemble average of $f(S)$ (i.e. $\langle f(S) \rangle$), where $\mathrm{d}\mu(S)$ is the Haar measure and $\mathcal{P}_{\overline{S}}(S)$ is the distribution of S. For the one-channel case $M = 1$, the above mentioned requirements determine this distribution uniquely [7] and it is given by $\mathcal{P}_{\overline{S}}(S) = \frac{1}{2\pi}(1 - |\overline{S}|^2)/|1 - \overline{S}^* S|^2$. In an electrostatic context Eq. (1.29) is known as Poisson's formula which for $M = 1$ is the solution of finding the potential on the unit disc from the values it takes on the unit circle. Accordingly $\mathcal{P}_{\overline{S}}(S)$ is referred to as *Poisson kernel*.

For $M > 1$ the unitarity, causality and (if relevant) time-reversal invariance requirements are not sufficient to determine the distribution of S uniquely. However, if in addition one also requires the entropy of the system to be maximal (which can be understood as a minimal information assumption), the probability density of S is uniquely determined by [7]

$$\mathcal{P}_{\overline{S}}(S) = \frac{1}{C_\beta} \left| \frac{\det[\mathbb{1}_M - \overline{S}^\dagger \overline{S}]}{\det[\mathbb{1}_M - \overline{S}^\dagger S]^2} \right|^{(\beta M + 2 - \beta)/2}, \tag{1.30}$$

where $\beta = 1, 2, 4$ is the Dyson index related to the underlying symmetries w.r.t. time reversal as discussed in Section 1.1.1 and C_β is a normalisation constant. Eq. (1.30) is a generalisation of the previously discussed distribution for $M = 1$ to arbitrary M and is hence also referred to as *Poisson kernel*. It was first discussed in the mathematical literature by Hua [57].

The only free parameter of the model is the mean $\overline{S}$ which hence encodes all the information for the given scattering problem, like the coupling of the channels to the scattering centre or the total energy of the system. Statistical properties of scattering

observables for fixed energy can then be inferred from the corresponding Poisson kernel distribution.

Notice that this method does not require S to be diagonal on average as opposed to the Hamiltonian approach, and thus can also account for direct processes which are then encoded in the non-diagonality of $\overline{S}$. The 'perfect coupling' case is characterised by a vanishing mean, $\overline{S} = 0$. In this case the density (1.30) becomes constant, which implies that S belongs to one of the circular ensembles introduced at the end of Section 1.1.2.

While this method is very successful in the statistical description of scattering characteristics at fixed energy [4], it cannot be used to study the statistics of fluctuations of the scattering observables over an energy interval comparable with a typical separation between resonances. For this matter, or to study how properties of S depend on other external parameters like the transmission coefficients, the Heidelberg approach is most appropriate.

In Section 1.3.2, the Hermitian K-matrix was introduced, which can formally be defined as the inverse Cayley transform of S, see Eq. (1.20). This relation enables us to infer a probability density of K from the Poisson kernel (1.30). It turns out that S being distributed according to the Poisson kernel with equivalent channels (i.e. $\overline{S}$ being proportional to the unit matrix) is equivalent to K being distributed according to the (matrix) *Cauchy distribution*

$$\mathcal{P}(K) = C_{\beta,M} \lambda^{\frac{M}{2}(\beta M+2-\beta)} \det(\lambda^2 + (K-\epsilon)^2)^{-\frac{1}{2}(\beta M+2-\beta)}, \tag{1.31}$$

where the width λ and the mean ϵ are two real parameters determined by the mean $\overline{S}$ of the scattering matrix and vice versa. $C_{\beta,M}$ is a normalisation constant independent of λ and ϵ. Appendix A.1 shows the derivation of formula (1.31) exemplary for the case $\beta = 2$. The perfect coupling case is obtained for the choice $\epsilon = 0$ and $\lambda = 1$.

2. Mathematical Tools

This chapter is aimed to introduce two very useful mathematical tools which will be used at various places throughout the thesis. The main purpose is to make the thesis self-contained. Section 2.1 explains the saddle-point approximation for integrals whose integrand depends on a large parameter (which in the thesis most frequently will be the size N of the random matrix). Section 2.2 will introduce supersymmetry which will later be used to perform random matrix ensemble averages.

2.1. The Saddle-Point Approximation

Very often one encounters integrals of the form

$$I = \int_C \mathrm{d}z\, f(z) \exp(x\phi(z)), \tag{2.1}$$

where C is some contour in the complex z-plane, $f(z)$ and $\phi(z)$ are complex-valued analytic functions and x is a large real parameter (in our applications $x \equiv N$ will always be the dimension of the random matrix H and hence a natural number). Often exact evaluation of these integrals is not feasible, however, since N is a large parameter one is usually interested in the large-N asymptotics of (2.1). A very useful method to derive them is the saddle-point approximation (also known as method of steepest descent) which will be explained in this section. First we look at an easier case of so called Laplace integrals. Then we come back to the asymptotic expansion of (2.1).

2.1.1. Laplace's Method

A Laplace integral is an integral of the form

$$I_L = \int_a^b \mathrm{d}t\, f(t) \exp(x\phi(t)), \tag{2.2}$$

where $f(t)$ and $\phi(t)$ are real continuous functions. We further assume that $\phi(t)$ attains its maximum at a unique point t_0 in the interval (a, b) and that $f(t_0) \neq 0$. The main idea of Laplace's method is that for large x, only points in the neighbourhood of t_0 will contribute to the asymptotic expansion of I_L. We will illustrate the method by considering the asymptotics of the modified Bessel function $I_n(x)$, $n \in \mathbb{Z}$ for large $x \in \mathbb{R}$. Our starting point is an integral representation for $I_n(x)$,

$$I_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t\, \cos(nt) \exp(x \cos t). \tag{2.3}$$

We can identify $f(t) = \cos(nt)$ and $\phi(t) = \cos(t)$. Moreover $\phi(t)$ has a global maximum at $t_0 = 0$. Laplace's method now proceeds in three steps:

1. Approximate the integral by neglecting its tails We replace the original range of integration $(-\pi, \pi)$ by a small neighbourhood around $t_0 = 0$,

$$I_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t\, \cos(nt) \exp(x \cos t) \approx \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \mathrm{d}t\, \cos(nt) \exp(x \cos t). \tag{2.4}$$

Let us estimate the error of this approximation: The right tail of the integral is given by the range (ϵ, π) and the integrand assumes its maximum at ϵ. We can thus estimate the integral by

$$\begin{aligned} &\left| \frac{1}{2\pi} \int_{\epsilon}^{\pi} \mathrm{d}t\, \cos(nt) \exp(x \cos t) \right| < \frac{1}{2\pi} \int_{\epsilon}^{\pi} \mathrm{d}t\, \exp(x \cos t) \\ &< \frac{\pi - \epsilon}{2\pi} \exp(x \cos \epsilon) \xrightarrow{\epsilon \to 0} \frac{1}{2} \exp\left[x \left(1 - \frac{\epsilon^2}{2} + \mathcal{O}(\epsilon^4) \right) \right], \end{aligned} \tag{2.5}$$

where we used the Taylor expansion of the cosine. Due to symmetry, the left tail has the same estimate.

2. Approximate the integrand by a Gaussian Let us substitute $t = t_0 + y/\sqrt{x}$. Then the exponent is given by

$$x\phi\left(\frac{y}{\sqrt{x}}\right) = x\cos\left(\frac{y}{\sqrt{x}}\right) = x - \frac{y^2}{2} + \mathcal{O}\left(\frac{y^4}{x}\right) \tag{2.6}$$

where we used again the Taylor expansion of the cosine. Note that also for general $\phi(t)$ the expansion has never a linear term because $\phi'(t) = 0$ due to t_0 being a maximum. Now $\mathcal{O}\left(\frac{y^4}{\sqrt{x}}\right)$ will be small for large x as long as y is smaller than $\mathcal{O}(x^{1/4})$. If this condition is fulfilled, we may also approximate $f(y/\sqrt{x}) = \cos(ny/\sqrt{x}) = 1+\mathcal{O}(y^2/x)$. For $\epsilon = x^{-\alpha}$, since the integral boundaries after the transformation are given by $\pm\sqrt{x}\epsilon$, this means we require $\alpha > 1/4$. However, at the same time we want the error made in step 1 to be small which is only the case if we choose $\alpha < 1/2$. For example we may choose $\epsilon = x^{-1/3}$ in (2.4). According to (2.5), the error made in step 1 is then given by $\exp(x - \frac{1}{2}x^{1/3} + \mathcal{O}(x^{-1/3}))$. Moreover this choice will ensure that the approximation (2.6) is correct on the whole integration range and thus

$$I_n(x) = \frac{\exp(x)}{2\pi\sqrt{x}}\left(\int_{-x^{1/6}}^{x^{1/6}} \mathrm{d}y\, \exp\left(-\frac{y^2}{2}\right) + \mathcal{O}(x^{-1/3})\right). \tag{2.7}$$

Note that the main error of order $\mathcal{O}(x^{-1/3})$ comes from approximating the integrand by a Gaussian, whereas the error from neglecting the tails is only $\mathcal{O}(e^{-x^{1/3}})$ and hence exponentially smaller.

3. Complete the tails of the Gaussian integral The last step is to extend the range of the integral to $\pm\infty$. This is valid since its tails are again exponentially small. For any $a > 0$ we have

$$\int_a^\infty \mathrm{d}y\, e^{-y^2/2} = \int_0^\infty \mathrm{d}y\, e^{-(a+y)^2/2} \leq e^{-a^2/2}\int_0^\infty \mathrm{d}y\, e^{-y^2/2} = \sqrt{\frac{\pi}{2}}e^{-a^2/2}, \tag{2.8}$$

and due to symmetry the same estimate holds for the left tail. In our example this means the error is of order $\mathcal{O}(e^{-x^{1/3}})$. Hence we have

$$I_n(x) \approx \frac{\exp(x)}{2\pi\sqrt{x}}\int_{-\infty}^{\infty} \mathrm{d}y\, \exp\left(-\frac{y^2}{2}\right) = \frac{\exp(x)}{\sqrt{2\pi x}}. \tag{2.9}$$

The above result is the lowest order approximation of $I_n(x)$, which turns out to be independent of n.

Note that the errors in steps 1 and 3 are exponentially smaller than the error from step 2. Hence we can obtain higher order corrections by improving the second step and expanding both $f(t)$ and $\phi(t)$ further. To that extend we use

$$\begin{aligned}\cos\left(\frac{ny}{\sqrt{x}}\right)\exp\left(x\cos\left(\frac{y}{\sqrt{x}}\right)\right) &\approx \left(1-\frac{n^2y^2}{2x}\right)\exp\left(x-\frac{y^2}{2}+\frac{y^4}{24x}\right)\\ &\approx \left(1-\frac{n^2y^2}{2x}\right)\exp\left(x-\frac{y^2}{2}\right)\left(1+\frac{y^4}{24x}\right). \end{aligned} \tag{2.10}$$

We can then compute

$$I_n(x) \approx \frac{\exp(x)}{2\pi\sqrt{x}}\int_{-\infty}^{\infty}\mathrm{d}y\,\exp\left(-\frac{y^2}{2}\right)\left(1-\frac{n^2y^2}{2x}+\frac{y^4}{24x}\right) = \frac{\exp(x)}{\sqrt{2\pi x}}\left(1-\frac{4n^2-1}{8x}\right), \tag{2.11}$$

where we used $\int_{-\infty}^{+\infty}\mathrm{d}y\,y^{2m}e^{-y^2/2} = \sqrt{2\pi}\frac{(2m)!}{2^m m!}$. Expanding $f(t)$ and $\phi(t)$ to even higher order, more terms of the asymptotic expansion can be obtained.

General formula for Laplace integrals Following the three steps for general $f(t)$ and $\phi(t)$, where t_0 is the unique maximum of $\phi(t)$ in the interval (a,b), one can derive the formula

$$I_L = \int_a^b \mathrm{d}t\, f(t)\exp(x\phi(t)) \approx \sqrt{\frac{2\pi}{x|\phi''(t_0)|}}f(t_0)\exp(x\phi(t_0)),\ \text{as } x\to\infty. \tag{2.12}$$

Note that one can show that it is always possible to find an ϵ such that the approximation in step 2 is valid and the tails from step 1 and 3 are exponentially smaller than the error in step 2 [58]. In the remainder of the thesis we hence will not be as precise as in the introductory example when performing Laplace's method, and simply assume that such an ϵ indeed exists without explicitly determining it or estimating the errors of the tails.

Obtaining the next higher order correction to (2.12), it does not suffice to expand $f(t)$ and $\phi(t)$ to the next higher order, instead one needs to expand $f(t)$ to second and $\phi(t)$ to fourth order. This is necessary to pick up all contributions to the next higher order.

There can be a few obstacles when performing Laplace's method which shall be shortly summarised in the following.

- *The function $\phi(t)$ has no maximum on the interval (a, b).* In this case the maximum t_0 will be assumed at the end points a or b. Let us assume the maximum is at a, and $\phi'(a) \neq 0$. Then one necessarily has $\phi'(a) < 0$. One can approximate $\phi(t)$ to first order around a and extend the upper integration limit to infinity, thus reducing the problem to

$$I_L = \int_a^b \mathrm{d}t\, f(t) e^{x\phi(t)} \approx f(a) e^{x\phi(a)} \int_a^\infty \mathrm{d}t\, e^{x\phi'(a)(t-a)} = -\frac{f(a) e^{x\phi(a)}}{x\phi'(a)}. \tag{2.13}$$

  If the maximum is assumed at b, a similar result can be obtained as in (2.13) with $a \to b$ and no minus-sign in front of the fraction. If the maximum happens to be at one of the end-points, but ϕ' vanishes there, the usual result (2.12) holds but multiplied with $1/2$ because only one of the integral limits is extended to infinity.

- *The function $\phi(t)$ has more than one maximum on the interval (a, b).* For large x, the integrand will then have distinct peaks at the maxima of $\phi(t)$, and consequently main contributions to the integral will come from (distinct) neighbourhoods around these maxima. One can apply Laplace's method to each maximum separately and the final result will be a sum over all maxima. Note that some maxima might give only sub-dominant contributions. It hence might be beneficial to determine which maxima contribute to the lowest order approximation before applying the method.

- *The function $f(t)$ vanishes at the maximum, $f(t_0) = 0$.* Then the lowest order approximation is not given by (2.12) (this contribution vanishes in that case) but by a contribution obtained expanding $f(t)$ and $\phi(t)$ to a higher order. As explained above the next contribution is obtained expanding $f(t)$ to second and $\phi(t)$ to fourth order. However, it turns out that for $f(t_0) = 0$ it suffices to expand $\phi(t)$ to third order (but note that in general it does not suffice to expand $f(t)$ alone). As long as at least one of the two $f'(t_0)$ or $f''(t_0)$ are non-vanishing, this will yield the lowest order approximation for this case. If also the first two

derivatives of $f(t)$ vanish at the maximum, one has to expand to higher orders in $f(t)$ and $\phi(t)$ accordingly.

- *The second derivative of $\phi(t)$ vanishes at the maximum, $\phi''(t_0) = 0$.* In this case we expand $\phi(t)$ to the next non-vanishing order around t_0, i.e. in general $\phi(t) \approx \phi(t_0) + \frac{1}{p!}(t-t_0)^p \phi^{(p)}(t_0)$, where $p > 2$ is the first non-vanishing power. Since t_0 is a maximum necessarily p is even and $\phi^{(p)}(t_0) < 0$. The resulting integral will not be of Gaussian form any more, but instead (we substitute $t = t_0 + yx^{-1/p}$)

$$\begin{aligned} I_L \approx & \frac{f(t_0)e^{x\phi(t_0)}}{x^{1/p}} \int_{-\infty}^{\infty} \mathrm{d}y \, \exp\left(\frac{\phi^{(p)}(t_0)}{p!} y^p\right) \\ &= 2f(t_0)\exp(x\phi(t_0))\frac{\Gamma(1/p)}{p}\left(\frac{p!}{x|\phi^{(p)}(t_0)|}\right)^{1/p}, \end{aligned} \tag{2.14}$$

where we used $\int_{-\infty}^{\infty} \mathrm{d}y \, \exp(-y^p) = 2\Gamma(1/p)/p$.

2.1.2. The Saddle-Point Method

The saddle-point method is an extension of Laplace's method for complex integrals given in Eq. (2.1). $f(z)$ being complex does not pose any significant difference from before as one could always view the problem as the sum of two integrals, one containing the real part of $f(z)$ and the other containing its imaginary part.

The exponential will be of the form $\exp(x \operatorname{Re}\phi(z) + ix \operatorname{Im}\phi(z))$, and in the spirit of Laplace's method the main contribution to the integral should come from the neighbourhood of the maximum of $\operatorname{Re}\phi(z)$ along the curve C. Let $z = u + iv$. If z_0 is a maximum of $\operatorname{Re}\phi(z)$ along the curve C, it must fulfil the conditions $\frac{\partial \operatorname{Re}\phi(z)}{\partial u}\big|_{z_0} = \frac{\partial \operatorname{Re}\phi(z)}{\partial v}\big|_{z_0} = 0$. Since $\phi(z)$ is assumed to be analytic it fulfils the Cauchy-Riemann equations which implies that also the partial derivatives of $\operatorname{Im}\phi(z)$ vanish at z_0, and combining these conditions means $\phi'(z) = 0$. Such a point is called a saddle point of $\phi(z)$. Figure 2.1 shows a typical picture of a saddle point and its vicinity. Of course the initial path of integration C needs not to pass this point at all. However, because of the analyticity of $f(z)$ and $\phi(z)$, the contour of integration can be deformed to any path as long as it does not cross any singularities of the integrand. One should hence deform the contour in such a way that it crosses or at least comes close to a saddle point of $\phi(z)$.

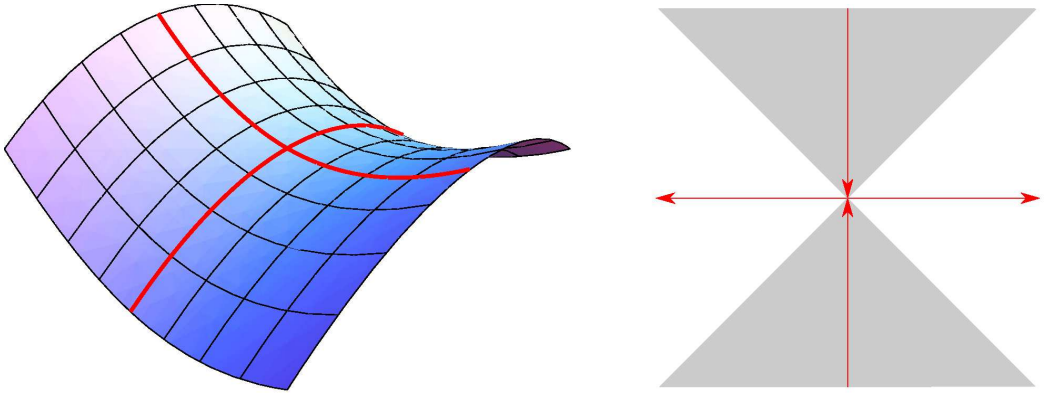

Figure 2.1.: Local region of a simple saddle point. The picture on the left shows the function $\mathrm{Re}\,\phi(u+iv)$ close to its saddle point. The red curves are the steepest paths which intersect at the saddle point, black curves are less steep paths. The picture on the right is a schematic view. Grey areas are above the saddle point, white areas below. The arrows show the direction of steepest descent.

Another important remark is called for: The term $\exp(ix\,\mathrm{Im}\,\phi(z))$ makes, in general, the integrand oscillate rapidly for large x, and such oscillations might cancel each other out, making it a priori not clear if the main contribution really comes from the neighbourhood of the saddle point. It is hence desirable to deform the contour in such a way that $\mathrm{Im}\,\phi(z) = const.$ in the vicinity of the saddle point. Such a contour is also called steepest descent contour (or steepest ascent contour, depending on the direction), because $\mathrm{Re}\,\phi(z)$ will change most rapidly along such contours. A saddle point is characterised as the intersection of at least two steepest contours (more specifically if $\phi^{(p)}(z)$ is the first non-vanishing derivative of $\phi(z)$, then $2p$ steepest contours meet at the saddle point). Steepest contours cannot intersect anywhere else. Figuratively speaking it is clear that if you stand at the flank of a mountain, there is only one steepest path. If you are standing on a saddle point like in Figure 2.1, you are at the minimum of a steepest path and at the same time at the maximum of another steepest path.

Note that an analytic function cannot have any maxima or minima: Let $\phi(z) = \rho(z) + i\psi(z)$, $\rho, \psi \in \mathbb{R}$ be an analytic function and $z = u + iv$, $u, v \in \mathbb{R}$. If z_0 is an extremum of $\rho(u+iv)$, the determinant of its Hessian matrix, given by $\rho_{uu}\rho_{vv} - \rho_{uv}^2$, needs to be positive which implies $\rho_{uu}\rho_{vv} > 0$ (the subscripts denote partial derivatives w.r.t. the given variable). However, since $\phi(z)$ is analytic, it fulfils the Cauchy-

Riemann equations $\rho_u = \psi_v$, $\rho_v = -\psi_u$. Taking the derivative w.r.t. u on the first equation and w.r.t. v on the second equation we see that the Cauchy-Riemann equations imply $\rho_{uu}\rho_{vv} = -\psi_{uv}^2 \leq 0$ for all $u, v \in \mathbb{R}$, and hence only saddle points are possible. The same argument also applies to $\psi(z)$.

We are now able to formulate a method to approximate an integral of the form (2.1). This method is called "saddle point method" or "method of steepest descent" and has two steps:

1. Deform the initial contour C to a new contour C' which passes through (or at least comes close to) a saddle point z_0 of the function $\phi(z)$, characterised by $\phi'(z_0) = 0$, in (approximately) the direction of steepest descent. By this we mean the direction where $\phi(z)$ has stationary phase, $\operatorname{Im}\phi(z) = 0$, and $\operatorname{Re}\phi(z)$ assumes its maximum at z_0.

2. Perform Laplace's method on the integral. This step is possible because after restricting the range of integration to a small neighbourhood around the saddle point z_0, the phase will be (approximately) constant and hence the integral can be treated like a Laplace integral (although z is complex, the z-dependent part of the exponent, given by $x \operatorname{Re}\phi(z)$, is real and for complex $f(z)$ integration can always be split up into two integrals containing $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$, respectively).

Note that it is especially not necessary to deform the contour to a steepest contour globally. This is true because as long as one ensures that the contour is approximately in the direction of steepest decent close to the saddle point, only a small region around the saddle point will contribute to the integral and tails can be completely neglected.

Multidimensional saddle-point analysis The saddle-point method can be extended to multidimensional integrals, where $f(\boldsymbol{z})$ and $\phi(\boldsymbol{z})$ are then functions taking complex vectors as their argument and $\boldsymbol{z}$ is the vector comprising the (complex) integration variables. Saddle points are then given by the condition that the gradient of ϕ vanishes, $\nabla\phi(\boldsymbol{z}) = 0$. However, in the course of the thesis we will only encounter a much simpler version where ϕ factorises, $\phi(\boldsymbol{z}) = \prod_{j=1}^{k} \phi_j(z_j)$, and hence the saddle points are given by the conditions $\phi_j'(z) = 0$ for all j. Moreover also the measure of

integration factorises, i.e. the integrals encountered take the form

$$I = \int_{C_1} \mathrm{d}z_1 \ \ldots \int_{C_k} \mathrm{d}z_k \, f(z_1, \ldots, z_k) \prod_{j=1}^{k} \exp(x\phi_j(z_j)). \tag{2.15}$$

If also f factorised, the problem would reduce to perform a saddle-point analysis on each of the k integrals. However, also for non-factorising f we can apply the saddle-point method successively for each variable z_j: Assume each of the functions $\phi_j(z_j)$ gives rise to a saddle point z_j^0 in $\mathbb{C}$. First we deform the contour C_1 to a contour which goes through the saddle point z_1^0 in the direction of steepest descent. Then integration over z_1 will be dominated by the neighbourhood of z_1^0, and we can replace $\phi_1(z_1)$ by its second order Taylor expansion and $f(z_1, \ldots, z_k)$ by $f(z_1^0, z_2, \ldots, z_k)$. We proceed with all following integrations in the same manner, such that in the end $f(z_1, \ldots, z_k)$ has been replaced by the constant $f(z_1^0, \ldots, z_k^0)$, and all functions $\phi_j(z_j)$ by their second order Taylor expansions. The problem reduces to compute k one-dimensional Gaussian integrals which can be done easily.

Example: Airy function We want to conclude discussion of the saddle-point method by considering two examples, the first being the asymptotics for large positive x of the Airy function $\mathrm{Ai}(x)$. Our starting point is the integral representation

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t \, \exp\left(i\left(\frac{t^3}{3} + xt)\right)\right) = \frac{\sqrt{x}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t \, \exp\left(ix^{3/2}\left(\frac{t^3}{3} + t)\right)\right), \tag{2.16}$$

where we substituted $t \to \sqrt{x}t$ to make the integral amenable for the saddle-point method. We identify $\phi(t) = t^3/3 + t$, with the saddle points given by $t_{1,2} = \pm i$. We interpret now t as a complex variable and compute the steepest paths through the saddle points. with $t = u + iv$ we have $\mathrm{Im}\,\phi(u + iv) = u(u^2/3 - v^2 + 1)$. Stationary phase curves through the saddle points are hence given by $u = 0$ (the imaginary axis) and $v = \pm\sqrt{u^2/3 + 1}$. We further note that $\mathrm{Re}\,\phi(\pm i + \epsilon) = -2/3 \mp \epsilon^2$, i.e. $+i$ is a maximum of $\mathrm{Re}\,\phi$ on the path $v = \sqrt{u^2/3 + 1}$, whereas $-i$ is a minimum on the path $v = -\sqrt{u^2/3 + 1}$. We hence want to deform the contour to go through $+i$. We choose the contour which is parallel to the real axis and goes through $+i$ (and hence tangent to the steepest descent contour, ensuring we approach the saddle point from

the correct direction). Since the integrand has no poles, deforming the contour does not pose a problem. We split the contour into three parts: The paths from $-\infty$ to $-\infty + i$, from $-\infty + i$ to $\infty + i$ and from $\infty + i$ to ∞. The two integrals parallel to the imaginary axis are zero, as can be seen by the following chain,

$$\left|\int_a^{a+i} \mathrm{d}t\, e^{ix^{3/2}(t^3/3+t)}\right| = \left|\int_0^1 \mathrm{d}t\, e^{ix^{3/2}((a+it)^3/3+a+it)}\right| \leq \int_0^1 \mathrm{d}t\, e^{x^{3/2}(-t-a^2t+t^3/3)}$$
$$\leq \int_0^1 \mathrm{d}t\, e^{-x^{3/2}a^2 t} = \frac{1-e^{-x^{3/2}a^2}}{x^{3/2}a^2} \to 0 \quad (a \to \pm\infty). \tag{2.17}$$

The remaining integral can now be treated using Laplace's method. Substituting $t = i + yx^{-3/4}$, and expanding the exponent to second order yields

$$\mathrm{Ai}(x) \approx \frac{1}{2\pi x^{1/4}} \int_{-\infty}^{\infty} \mathrm{d}y\, \exp\left(-\frac{2}{3}x^{3/2} - y^2\right) = \frac{\exp\left(-\frac{2}{3}x^{3/2}\right)}{2\sqrt{\pi}x^{1/4}} \quad (x \to \infty). \tag{2.18}$$

Example: Hermite polynomial The Hermite polynomial $H_{N-k}(\sqrt{N}z)$, where both the order and the argument are large (but of different order in N) will be encountered at various points throughout the thesis. Hence we want to provide its large-N asymptotics at this point. Its integral representation is given by (see e.g. 8.951 in [59])

$$H_N(z) = \frac{i^N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}q\, q^N \exp[-\tfrac{1}{2}(q+iz)^2]. \tag{2.19}$$

The first step is to rescale $q \to \sqrt{N}q$ and write $q^N = \exp(N \ln q)$, such that

$$H_{N-k}(\sqrt{N}z) = \frac{i^{N-k}}{\sqrt{2\pi}} N^{\frac{N-k+1}{2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{q^k} \exp[-\tfrac{N}{2}(q+iz)^2 + N \ln q]. \tag{2.20}$$

We identify $f(q) = q^{-k}$ and $\phi(q) = -\frac{1}{2}(q+iz)^2 + \ln q$. This implies that there are two saddle points at $q_\pm = \frac{1}{2}(-iz \pm \sqrt{4-z^2})$ (we only consider the case $|z| < 2$). The simplest possible contours of integration are those where either real or imaginary part vanish. Hence it would be convenient to deform the contour of integration like in the last example, but going through the point $-iz/2$. Note that both saddle points are on this contour. It is of course not a steepest descent contour, neither can one expect that it is tangent to such a contour at the saddle points. Nevertheless the saddle-point method is usually quite robust and yields even correct results when only

reaching the saddle point in approximately the correct direction. We simply assume here the chosen contour is "good enough" to give correct results. As in the previous case one can furthermore show that the two integrals parallel to the imaginary axis at $\pm\infty$ are zero. Then we can follow the usual procedure of evaluating $f(q)$ at the saddle point, expanding $\phi(q)$ to second order around the saddle point and computing the resulting Gaussian integral. Notice that we have to do this for each saddle point separately, the result being the sum of both contributions. Those two contributions are then given by

$$I_\pm = i^{N-k} N^{\frac{N-k}{2}} q_\pm^{N-k} \exp[-\tfrac{N}{2}(q_\pm + iz)^2] \left(1 + \frac{1}{q_\pm^2}\right)^{-1/2}. \tag{2.21}$$

Inserting the definition of $q_\pm$ and using the easy to check relations $q_\pm + iz = q_\pm^*$ and $\left(1 + \frac{1}{q_\pm^2}\right)^{-1} = \pm\frac{q_\pm}{\sqrt{4-z^2}}$ one thus arrives at

$$H_{N-k}(\sqrt{N}z) \approx \frac{\exp[-\frac{N}{2} + \frac{N}{4}z^2]}{\sqrt{2}(4-z^2)^{1/4}} \left(\frac{i\sqrt{N}}{2}\right)^{N-k} \left\{\tilde{A}^*(k,N,z) + (-1)^{N-k}\tilde{A}(k,N,z)\right\}, \tag{2.22}$$

where we defined

$$\tilde{A}(k,N,z) = (\sqrt{4-z^2} + iz)^{N-k+1/2} \exp[\tfrac{N}{4} iz\sqrt{4-z^2}]. \tag{2.23}$$

This is the correct large-N asymptotics, which can be checked for example by numerics. With some more effort the saddle-point analysis can certainly be done with full mathematical rigour, however, for the purposes of this thesis a non-rigorous treatment as above usually suffices. Thus in the remainder of the thesis we will only concentrate on explaining the gross structures of the saddle-point analysis which yield the correct results without going into much more detail.

2.2. The Supersymmetry Method

The name "supersymmetry" originates from particle physics. A supersymmetric theory is a theory where every boson has a fermionic partner and vice versa. The photon for example would have a fermionic partner called photino. In this theory bosons are described by a set of commuting variables, while fermions are described by a set of anticommuting or Grassmann variables.

Although mathematically the same, the use of supersymmetry in random matrix theory is entirely different. The variables do not represent bosons or fermions any more, instead they are just bookkeeping devices which have no physical meaning, although the nomenclature of certain quantities sometimes refers to the concepts of particle physics. This use of supersymmetry was first introduced by Efetov [26,27]. A more recent review of supersymmetry in random matrix theory can be found in [60].

The aim of this chapter is to present how supersymmetry is utilised to tackle problems in random matrix theory, however, no completeness or mathematical rigour is claimed. For a more mathematical treatment the reader is referred to the book of Berezin [61].

Grassmann variables are introduced in Section 2.2.1, whereas in Section 2.2.2 supervectors and supermatrices as well as invariants like the supertrace and superdeterminant are presented. Section 2.2.3 finally defines integrals over Grassmann variables, with special focus on Gaussian integrals. We also discuss how to change integration variables and the boundary contributions that may arise, known as Efetov-Wegner terms. In Section 2.2.4 is finally shown, how supersymmetry can be applied to random matrix theory. Application is explained via a toy model and three different methods are presented, known as Hubbard-Stratonovich transformation, Superbosonization and a "hybrid method" of these two methods. Parts of this section are taken from [9].

2.2.1. Grassmann Variables

Let us introduce variables $\zeta_i, i = 1 \dots N$ which obey the relation

$$\zeta_i \zeta_j = -\zeta_j \zeta_i. \tag{2.24}$$

These variables are called anticommuting or Grassmann variables. They are nilpotent, meaning that the square or any higher power vanishes. This can easily be seen from the above equation as we get $\zeta_i^2 = -\zeta_i^2$ for $i = j$. This relation can only be satisfied if

$$\zeta_i^2 = 0. \tag{2.25}$$

Unlike commuting variables, we cannot represent anticommuting variables as numbers, thus they are purely formal objects. It is convenient to define an operation of complex conjugation for them by

$$(\zeta_i)^* = \zeta_i^*, \tag{2.26}$$

$$(\zeta_i^*)^* = \zeta_i^{**} = -\zeta_i, \tag{2.27}$$

$$(\zeta_i \zeta_j)^* = \zeta_i^* \zeta_j^*. \tag{2.28}$$

ζ_i^* is again an anticommuting variable and independent from ζ_i in the same sense as an ordinary complex variable and its complex conjugate are independent. The definitions (2.27) and (2.28) ensure that $\zeta_i^* \zeta_i$ remains invariant under complex conjugation,

$$(\zeta_i^* \zeta_i)^* = \zeta_i^{**} \zeta_i^* = -\zeta_i \zeta_i^* = \zeta_i^* \zeta_i. \tag{2.29}$$

Another possible definition is to drop the minus sign in (2.27) and to reverse the order of the Grassmann variables in (2.28).

Note that Grassmann variables commute with ordinary variables, but also a product of an even number of Grassmann variables is a nilpotent commuting variable,

$$(\zeta_i \zeta_j)\zeta_k = \zeta_i \zeta_j \zeta_k = -\zeta_i \zeta_k \zeta_j = +\zeta_k \zeta_i \zeta_j = \zeta_k (\zeta_i \zeta_j). \tag{2.30}$$

Similar to the definition of matrix functions we can write functions of anticommuting variables as power series. Due to $\zeta_i^2 = 0$, every series gives a finite polynomial. For example we have

$$\exp(a \zeta_i^* \zeta_i) = 1 + a \zeta_i^* \zeta_i. \tag{2.31}$$

More generally speaking a function comprising n anticommuting variables can always be expressed as a sum of at most 2^n terms, starting from a term that contains no Grassmannians up to a term that contains all n of them.

2.2.2. Objects and Invariants in Superspace

We now consider objects which comprise both commuting and anticommuting variables. A supervector consists of n_B ordinary commuting variables z_j and n_F anticommuting variables ζ_j. It can be written as

$$\boldsymbol{\psi} = \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\zeta} \end{bmatrix}, \tag{2.32}$$

where $\boldsymbol{z}$ denotes a vector with commuting entries z_j, $j = 1 \dots n_B$, and $\boldsymbol{\zeta}$ denotes a vector with anticommuting entries ζ_j, $j = 1 \dots n_F$. In general, $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ can have different dimensions. The adjoint of a supervector is defined in the usual way by complex conjugation of the entries and transposition.

In ordinary space, a matrix acts as linear transformation between vectors. In the same sense we can define an $(n_B + n_F) \times (n_B + n_F)$ supermatrix σ as linear transformation between $(n_B + n_F)$-component supervectors, $\boldsymbol{\psi}' = \sigma\boldsymbol{\psi}$, where n_B and n_F are the number of commuting and anticommuting variables, respectively. For the multiplication of two supermatrices the usual matrix multiplication is employed. We require that $\boldsymbol{\psi}'$ has the same structure as $\boldsymbol{\psi}$, i.e. the first n_B components have to be commuting, and the following n_F have to be anticommuting. Decomposing σ into block matrices,

$$\sigma = \begin{bmatrix} a_1 & \mu_1 \\ \mu_2 & a_2 \end{bmatrix}, \tag{2.33}$$

the multiplication yields

$$\sigma\boldsymbol{\psi} = \begin{bmatrix} a_1 & \mu_1 \\ \mu_2 & a_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\zeta} \end{bmatrix} = \begin{bmatrix} a_1\boldsymbol{z} + \mu_1\boldsymbol{\zeta} \\ \mu_2\boldsymbol{z} + a_2\boldsymbol{\zeta} \end{bmatrix}. \tag{2.34}$$

We see now that the above requirement implies that a_1 and a_2 are $n_B \times n_B$ and $n_F \times n_F$ matrices with commuting entries and μ_1 and μ_2 are rectangular matrices of matching dimension with anticommuting entries. Note that the commuting entries can also be composed of pairs and higher even combinations of anticommuting variables.

The transpose and the adjoint of a supermatrix are defined as

$$\sigma^T = \begin{bmatrix} a_1^T & \mu_2^T \\ -\mu_1^T & a_2^T \end{bmatrix}, \quad \sigma^\dagger = (\sigma^T)^*. \tag{2.35}$$

The minus sign in the definition ensures that the familiar relation $(\sigma_1\sigma_2)^T = \sigma_2^T\sigma_1^T$ also holds for supermatrices. One should notice that in general $(\sigma^T)^T \neq \sigma$, however, $(\sigma^\dagger)^\dagger = \sigma$ is always true. A supermatrix is Hermitian if $\sigma^\dagger = \sigma$. Although we cannot define the inverse of a Grassmann variable in a meaningful way, a quadratic supermatrix σ can have an inverse. Moreover we can also have unitary supermatrices with $\sigma^{-1} = \sigma^\dagger$.

In ordinary space trace and determinant are invariants of a matrix. The corresponding quantities for supermatrices are called supertrace and superdeterminant (sometimes also graded trace and graded determinant). If we use again the notation with block matrices (2.33), the supertrace is defined by

$$\operatorname{Str}\sigma = \operatorname{Tr} a_1 - \operatorname{Tr} a_2. \tag{2.36}$$

This definition ensures the cyclic invariance of the supertrace $\operatorname{Str}\sigma_1\sigma_2 = \operatorname{Str}\sigma_2\sigma_1$. Since trace and determinant of an ordinary matrix M are related by $\ln\det M = \operatorname{Tr}\ln M$, we want to define the superdeterminant in such a way that

$$\ln\operatorname{sdet}\sigma = \operatorname{Str}\,\ln\sigma \tag{2.37}$$

also holds. This ensures that we have the familiar properties $\operatorname{sdet}\sigma^T = \operatorname{sdet}\sigma$ and $\operatorname{sdet}\sigma_1\sigma_2 = \operatorname{sdet}\sigma_1\operatorname{sdet}\sigma_2$. In terms of the block matrices this implies (see ref. [27])

$$\operatorname{sdet}\sigma = \frac{\det(a_1 - \mu_1 a_2^{-1}\mu_2)}{\det a_2} = \frac{\det a_1}{\det(a_2 - \mu_2 a_1^{-1}\mu_1)}. \tag{2.38}$$

As one can see the superdeterminant is only well defined for $\det a_2 \neq 0$. Instead of "str" and "sdet" the symbols "trg" and "detg" (for graded trace and graded determinant) are also frequently used.

pq-notation

The above introduced structure of a supermatrix is called block- or boson-fermion-notation (*bf*-notation). a_1 is the boson-boson block, a_2 the fermion-fermion block, μ_1 and μ_2 the boson-fermion and fermion-boson block, where the nomenclature refers to its origin in particle physics. Sometimes, however, it is more convenient to use another notation, the so called *pq*- or [1,2]-notation. Consider an $(2n_B + 2n_F) \times (2n_B + 2n_F)$ supermatrix σ consisting of sixteen block matrices $a_j, b_j, \mu_j, \nu_j,\ j = 1 \ldots 4$. The a_j comprise the boson-boson block and are of dimension $(n_B \times n_B)$, the b_j comprise the fermion-fermion block and are of dimension $(n_F \times n_F)$ and the μ_j, ν_j comprise the boson-fermion and fermion boson block and are of matching dimension. In *pq*-notation this supermatrix takes the form

$$\sigma_{bf} = \begin{bmatrix} a_1 & a_2 & \mu_1 & \mu_2 \\ a_3 & a_4 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & b_1 & b_2 \\ \nu_3 & \nu_4 & b_3 & b_4 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 & \mu_1 & a_2 & \mu_2 \\ \nu_1 & b_1 & \nu_2 & b_2 \\ a_3 & \mu_3 & a_4 & \mu_4 \\ \nu_3 & b_3 & \nu_4 & b_4 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \sigma_{pq}. \tag{2.39}$$

where the σ_{ij} are now $(n_B+n_F)\times(n_B+n_F)$ supermatrices in standard block notation as in Eq. (2.33). For the supertrace we get the rule $\mathrm{Str}\,\sigma_{pq} = \mathrm{Str}\,\sigma_{11} + \mathrm{Str}\,\sigma_{22}$. Changing from *bf*- to *pq*-notation and vice versa is especially useful when supermatrices become block-diagonal in one of the notations.

2.2.3. Integration

Berezin integral

We cannot integrate over Grassmann variables in the sense of a Riemann integral because they have no representation as numbers. Thus, an integration over anti-commuting variables can only be a formal definition. Since there are no powers of anticommuting variables, it suffices to use the two definitions

$$\int \mathrm{d}\zeta_i \ = 0, \quad \int \mathrm{d}\zeta_i\, \zeta_j = \delta_{ij}. \tag{2.40}$$

This is referred to as Berezin integral [61]. The formally introduced differentials $\mathrm{d}\zeta_i$ are assumed to be anticommuting with all initial Grassmann variables and with themselves. Changing the order of integration therefore might give an additional overall minus sign.

To integrate a function of anticommuting variables, one just needs to expand it into a power series. Due to the definition (2.40) the integration over all terms, which do not contain all Grassmann variables involved in the integration, yields zero. Thus only the highest order coefficient contributes to the integral. For example we have

$$\begin{aligned}\int \mathrm{d}\zeta_i \int \mathrm{d}\zeta_i^* \, \exp(a\zeta_i^*\zeta_i) &= \int \mathrm{d}\zeta_i \int \mathrm{d}\zeta_i^* \, (1 + a\zeta_i^*\zeta_i) \\ &= a \int \mathrm{d}\zeta_i \left(\int \mathrm{d}\zeta_i^* \zeta_i^* \right) \zeta_i = a. \end{aligned} \tag{2.41}$$

It is also possible to think of Berezin integrals as derivatives since it is very natural to define a derivative as

$$\frac{\partial \zeta_i}{\partial \zeta_j} = \delta_{ij}, \tag{2.42}$$

where one has to be careful with signs and has to distinguish if the derivative operator acts from the left or from the right onto a Grassmann variable. Comparing the definitions of integration and differentiation we see that they coincide. However, for our purposes it is more useful to think of the definition (2.40) as integral because it follows similar rules when we change variables. In particular, when we change a single Grassmann variable $a\zeta_i = \chi_i$, the differential has to change accordingly by $\mathrm{d}\zeta_i = a \, \mathrm{d}\chi_i$. For changing a vector with anticommuting entries $A\boldsymbol{\zeta} = \boldsymbol{\chi}$ this rule extends to $\mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n = \det A \; \mathrm{d}\chi_1 \ldots \mathrm{d}\chi_n$.

An integral over the components of a supervector or a supermatrix contains both integrals over commuting and anticommuting variables. We may also in this case change variables. For an arbitrary transformation of supervectors $\boldsymbol{\chi}^T = \boldsymbol{\chi}^T(\boldsymbol{\psi}) = [\boldsymbol{y}^T(\boldsymbol{z}, \boldsymbol{\zeta}) \; \boldsymbol{\eta}^T(\boldsymbol{z}, \boldsymbol{\zeta})]$ we have

$$\mathrm{d}\boldsymbol{\chi} = \mathrm{sdet} \begin{bmatrix} \partial \boldsymbol{y}/\partial \boldsymbol{z}^T & \partial \boldsymbol{y}/\partial \boldsymbol{\zeta}^T \\ \partial \boldsymbol{\eta}/\partial \boldsymbol{z}^T & \partial \boldsymbol{\eta}/\partial \boldsymbol{\zeta}^T \end{bmatrix} \mathrm{d}\boldsymbol{\psi} \tag{2.43}$$

with $\mathrm{d}\boldsymbol{\chi} = \mathrm{d}y_1 \ldots \mathrm{d}y_n \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n$ and $\mathrm{d}\boldsymbol{\psi} = \mathrm{d}z_1 \ldots \mathrm{d}z_n \mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n$. The Jacobian in superspace is also referred to as Berezinian.

Gaussian integrals over Grassmannians

The most important Berezin integrals in view of the supersymmetry method are Gaussian integrals over Grassmann variables given by

$$I = \int \mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}^\dagger \exp\left(\boldsymbol{\zeta}^\dagger A\boldsymbol{\zeta} + \boldsymbol{\mu}^\dagger\boldsymbol{\zeta} + \boldsymbol{\zeta}^\dagger\boldsymbol{\nu}\right), \tag{2.44}$$

where $\boldsymbol{\zeta}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are vectors with N anticommuting entries, A is an $N \times N$ invertible matrix with commuting entries, and $\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}^\dagger = \mathrm{d}\zeta_1\mathrm{d}\zeta_1^* \ldots \mathrm{d}\zeta_N\mathrm{d}\zeta_N^*$.
Since we can treat $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}^\dagger$ as independent variables, we may transform $A\boldsymbol{\zeta} = \boldsymbol{\eta}$ but leave $\boldsymbol{\zeta}^\dagger$ as it is. The measure transforms accordingly to $\mathrm{d}\boldsymbol{\zeta} = \det A\ \mathrm{d}\boldsymbol{\eta}$ such that

$$\begin{aligned} I &= \det A \int \mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\boldsymbol{\zeta}^\dagger \exp\left(\boldsymbol{\zeta}^\dagger\boldsymbol{\eta} + \boldsymbol{\mu}^\dagger A^{-1}\boldsymbol{\eta} + \boldsymbol{\zeta}^\dagger\boldsymbol{\nu}\right) \\ &= \det A \prod_{j=1}^{N} \int \mathrm{d}\eta_j \mathrm{d}\zeta_j^* \exp\left(\zeta_j^\dagger\eta_j + (\boldsymbol{\mu}^\dagger A^{-1})_j\eta_j + \zeta_j^\dagger\nu_j\right) \\ &= \det A \prod_{j=1}^{N} \int \mathrm{d}\eta_j \mathrm{d}\zeta_j^* (1 + \zeta_j^*\eta_j)(1 + (\boldsymbol{\mu}^\dagger A^{-1})_j\eta_j)(1 + \zeta_j^*\nu_j). \end{aligned} \tag{2.45}$$

In the last step we expanded the exponential function as in (2.31). The solution of the integral is just the highest order coefficient, thus

$$I = \det A \prod_{j=1}^{N} \left(1 - (\boldsymbol{\mu}^\dagger A^{-1})_j\nu_j\right). \tag{2.46}$$

Writing the term in the bracket again as an exponential, we finally get the identity

$$\int \mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}^\dagger \exp\left(\boldsymbol{\zeta}^\dagger A\boldsymbol{\zeta} + \boldsymbol{\mu}^\dagger\boldsymbol{\zeta} + \boldsymbol{\zeta}^\dagger\boldsymbol{\nu}\right) = \det A\ \exp\left(-\boldsymbol{\mu}^\dagger A^{-1}\boldsymbol{\nu}\right). \tag{2.47}$$

Note the similarity to a Gaussian integral over complex commuting variables,

$$\int \mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{z}^\dagger \exp\left(-\boldsymbol{z}^\dagger A\boldsymbol{z} + \boldsymbol{b}^\dagger\boldsymbol{z} + \boldsymbol{z}^\dagger\boldsymbol{c}\right) = \det{}^{-1} A\ \exp\left(\boldsymbol{b}^\dagger A^{-1}\boldsymbol{c}\right), \tag{2.48}$$

where $\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{z}^\dagger = \prod_{j=1}^{N} \mathrm{d}(\mathrm{Re}\, z_j)\mathrm{d}(\mathrm{Im}\, z_j)/\pi$ and A has to be positive definite to ensure convergence. The main difference is that in this case the inverse determinant occurs.

Another type of Gaussian integral is one that does not comprise the complex conjugate Grassmannians,

$$I = \int \mathrm{d}\boldsymbol{\zeta}\, \exp\left(\boldsymbol{\zeta}^T A \boldsymbol{\zeta} + \boldsymbol{\mu}^T \boldsymbol{\zeta}\right), \tag{2.49}$$

where $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are vectors with N anticommuting entries, A is an $N \times N$ matrix with commuting entries and $\mathrm{d}\boldsymbol{\zeta} = \mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_N$. To calculate this integral we look at its square and write

$$I^2 = \int \mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\eta}\, \exp\left[\boldsymbol{\zeta}^T A \boldsymbol{\zeta} + \boldsymbol{\eta}^T A \boldsymbol{\eta} + \boldsymbol{\mu}^T (\boldsymbol{\zeta} + \boldsymbol{\eta})\right] \tag{2.50}$$

$$= \int \mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\eta}\, \exp\left[\sum_{j,k} A_{jk}(\zeta_j \zeta_k + \eta_j \eta_k) + \mu_j(\zeta_j + \mu_j)\right]. \tag{2.51}$$

We introduce new variables

$$\zeta_j = \frac{1}{\sqrt{2}}(\theta_j + \theta_j^*), \quad \eta_j = \frac{i}{\sqrt{2}}(\theta_j - \theta_j^*). \tag{2.52}$$

This implies $\mathrm{d}\zeta_j \mathrm{d}\eta_j = i\mathrm{d}\theta_j \mathrm{d}\theta_j^*$, and altogether we have

$$\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\eta} = i^N (-1)^{N(N-1)/2} \mathrm{d}\theta_1 \mathrm{d}\theta_1^* \ldots \mathrm{d}\theta_N \mathrm{d}\theta_N^* = i^{(N^2)} \mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\theta}^\dagger, \tag{2.53}$$

where the measure is now defined as for the Gaussian integral over complex Grassmann variables. The factor $(-1)^{N(N-1)/2}$ comes from changing the order of the differentials. The integral then transforms to

$$I^2 = i^{(N^2)} \int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\theta}^\dagger \exp\left[\boldsymbol{\theta}^\dagger (A - A^T)\boldsymbol{\theta} + e^{i\pi/4}\boldsymbol{\mu}^T\boldsymbol{\theta} - e^{-i\pi/4}\boldsymbol{\theta}^\dagger\boldsymbol{\mu}\right]. \tag{2.54}$$

This integral is now of the form that one can apply the identity (2.47) and after taking the square root on both sides one gets

$$\int \mathrm{d}\boldsymbol{\zeta}\, \exp\left(\boldsymbol{\zeta}^T A \boldsymbol{\zeta} + \boldsymbol{\mu}^T \boldsymbol{\zeta}\right) = \sqrt{\det\left(A - A^T\right)} \exp\left[\frac{1}{2}\boldsymbol{\mu}^T (A - A^T)^{-1} \boldsymbol{\mu}\right], \tag{2.55}$$

where we used that $\det(A - A^T)$ vanishes for odd N and $i^{(N^2)} = 1$ for even N. The expression in front of the exponential is called *Pfaffian*, $\mathrm{pf}(A - A^T)^2 = \det(A - A^T)$.

Efetov-Wegner terms

One should notice that changes of variables in superspace can lead to boundary contributions which have no analogue in ordinary space, they are referred to as Efetov-Wegner terms. In order to discuss them we consider the example

$$I = \int_0^\infty \mathrm{d}y \int \mathrm{d}\zeta \int \mathrm{d}\zeta^* \, 2y \exp\left(-y^2 + \zeta^*\zeta\right). \tag{2.56}$$

With Eq. (2.41) the integral over the anticommuting variables yields a factor of unity, while the integral over the commuting variables gives unity as well and thus $I = 1$. Now we want to change the variables to

$$y = r(1 + \tfrac{1}{2}\chi^*\chi), \quad \zeta = r\chi, \quad \zeta^* = r\chi^*, \quad r \in (0, \infty). \tag{2.57}$$

The Berezinian of this transformation can be calculated to $1/(yr)$ with (2.43). Together with

$$-y^2 + \zeta^*\zeta = -r^2(1 + \chi^*\chi) + r^2\chi^*\chi = r^2 \tag{2.58}$$

the integral transforms to

$$I = 2\int_0^\infty \mathrm{d}r \int \mathrm{d}\chi \int \mathrm{d}\chi^* \, \frac{\exp(-r^2)}{r}. \tag{2.59}$$

At first glance one might think that $I = 0$ since the integrand does not depend on the Grassmann variables any more. This can of course not be correct since we know that the integral should evaluate to $I = 1$. If we look closer at the integrand we also observe that it has a singularity at $r = 0$ and thus the integral over the commuting variable does not converge. Hence the whole integral yields an indefinite expression of the form "$0 \cdot \infty$".

In order to circumvent this problem and get a meaningful result for the integration we do the calculation again, but exclude an infinitesimal sphere with radius $\epsilon > 0$ around the singularity at zero,

$$I_\epsilon = \int_0^\infty \mathrm{d}y \int \mathrm{d}\zeta \int \mathrm{d}\zeta^* \, 2y \exp\left(-y^2 + \zeta^*\zeta\right) \Theta(y - \epsilon), \tag{2.60}$$

where Θ denotes the Heaviside function which is 1 for positive and 0 for negative arguments. This approach was followed for a more general case in [62]. At the end of the calculation we have to take the limit $\epsilon \to 0$. Doing the transformation now yields

$$I_\epsilon = 2\int_0^\infty \mathrm{d}r \int \mathrm{d}\chi \int \mathrm{d}\chi^* \, \frac{\exp(-r^2)}{r} \Theta(r(1+\tfrac{1}{2}\chi^*\chi) - \epsilon). \tag{2.61}$$

Since the argument of the Θ-function includes Grassmann variables, its series terminates after the first order and we get

$$\Theta(r(1+\tfrac{1}{2}\chi^*\chi) - \epsilon) = \Theta(r-\epsilon) + \frac{r}{2}\chi^*\chi\delta(r-\epsilon). \tag{2.62}$$

This yields for the integral

$$I_\epsilon = 2\int_\epsilon^\infty \mathrm{d}r \int \mathrm{d}\chi \int \mathrm{d}\chi^* \, \frac{e^{-r^2}}{r} + \int_0^\infty \mathrm{d}r \int \mathrm{d}\chi \int \mathrm{d}\chi^* \, e^{-r^2}\chi^*\chi \, \delta(r-\epsilon). \tag{2.63}$$

Hence the appearance of the δ-function ensures that we get an additional contribution at the boundary $r = \epsilon$. The first integral is now definitely zero for any $\epsilon > 0$ since the integration over r yields then a finite value, while the integration over the Grassmann variables yields zero. The second integral still contains the anticommuting variables and thus we get the desired result

$$I_\epsilon = \exp\left(\epsilon^2\right) \overset{\epsilon\to 0}{\longrightarrow} 1. \tag{2.64}$$

This example shows that one has to be careful when changing variables in superspace since additional contributions from the boundary of the integration can arise. A more rigorous treatment of this topic and methods how one can in general calculate these Efetov-Wegner terms can be found in [63].

2.2.4. Supersymmetry in Random Matrix Theory

In order to illustrate how supersymmetry is applied to random matrix theory we use the level density $\rho(E) = (1/N)\langle \mathrm{Tr}\, \delta(E-H)\rangle_H$ as a toy model. As usual the angular brackets denote the random matrix ensemble average. First we note that the level density can be expressed as the imaginary part of the trace of the resolvent $(E-H)^{-1}$,

$$\rho(E) = \frac{1}{\pi N}\left\langle \mathrm{Im}\,\mathrm{Tr}(E-H-i\epsilon)^{-1}\Big|_{\epsilon\to 0}\right\rangle_H . \tag{2.65}$$

The above identity can be shown easily by computing the imaginary part of $\mathrm{Tr}(E-H-i\epsilon)^{-1}$ and then taking the limit $\epsilon \to 0$,

$$\frac{1}{\pi N}\sum_j \lim_{\epsilon\to 0}\mathrm{Im}(E-E_j-i\epsilon)^{-1} = \frac{1}{\pi N}\sum_j \lim_{\epsilon\to 0}\frac{\epsilon}{(E-E_j)^2+\epsilon^2} = \frac{1}{N}\sum_j \delta(E-E_j), \tag{2.66}$$

where E_j are the eigenvalues of the Hamiltonian H. Equation (2.65) on the other hand can be expressed via a generating function $Z(E^-, s)$ (we abbreviate $E^- = E - i\epsilon$; the limit $\epsilon \to 0$ is implied and will from here not be written explicitly),

$$\rho(E) = \frac{1}{\pi N}\,\mathrm{Im}\left[\frac{\mathrm{d}}{\mathrm{d}s}Z(E^-,s)\Big|_{s=0}\right], \quad Z(E^-,s) = \left\langle \frac{\det(E^- - H)}{\det(E^- - s - H)}\right\rangle_H . \tag{2.67}$$

To show the validity of this equation one can go to the eigenvalues of H and perform the derivative.

Now to calculate the level density, or equivalently its generating function, for a given random matrix ensemble, say the GUE with joint probability distribution $\mathcal{P}(H) = C_{\mathrm{GUE}}\exp(-\frac{N}{2J^2}\,\mathrm{Tr}\,H^2)$ (see Eq. (1.2)), we need to perform the ensemble average in Eq. (2.67), i.e. calculate the integral

$$Z(E^-,s) = \int \mathrm{d}H\,\mathcal{P}(H)\frac{\det(E^- - H)}{\det(E^- - s - H)}, \tag{2.68}$$

where $\mathrm{d}H$ is the flat measure comprising all independent variables of H, whose number for an $N \times N$ Hermitian matrix is N^2. Moreover one is usually interested in the limit $N \to \infty$. Performing this integral is hence highly non-trivial. However, one possible

trick to compute it is to rewrite it into a supersymmetric model. Therefore we use the two identities (2.48) and (2.47) to replace the determinants with Gaussian integrals over N commuting and N anticommuting variables, respectively,

$$\begin{aligned} Z(E^-, s) = \int \mathrm{d}H\, \mathcal{P}(H) \int \mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{z}^\dagger \exp\left(-i\boldsymbol{z}^\dagger(E^- - s - H)\boldsymbol{z}\right) \\ \times \int \mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}^\dagger \exp\left(+i\boldsymbol{\zeta}^\dagger(E^- - H)\boldsymbol{\zeta}\right). \end{aligned} \tag{2.69}$$

Note that we had to introduce the imaginary unit in the integral over the commuting variables in order to ensure convergence (we assume $\epsilon > 0$). Integrals over Grassmann variables never have convergence problems, however, we also added the imaginary unit here for symmetry reasons.

The H-dependent part of the exponent in (2.69) is given by

$$i\boldsymbol{z}^\dagger H\boldsymbol{z} - i\boldsymbol{\zeta}^\dagger H\boldsymbol{\zeta} = i\,\mathrm{Tr}\, H(\boldsymbol{z}\otimes\boldsymbol{z}^\dagger + \boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^\dagger) = i\,\mathrm{Tr}\, HA, \tag{2.70}$$

where we introduced the Hermitian $N \times N$ matrix A given by the term in brackets. For Gaussian joint probability density, the ensemble average becomes now trivial,

$$\langle e^{i\,\mathrm{Tr}\, HA}\rangle_{\mathrm{GUE}} = C_{\mathrm{GUE}} \int \mathrm{d}H\, \exp\left(-\frac{N}{2J^2}\,\mathrm{Tr}\, H^2 + i\,\mathrm{Tr}\, HA\right) = \exp\left(-\frac{J^2}{2N}\,\mathrm{Tr}\, A^2\right). \tag{2.71}$$

One way to show the validity of this formula is by explicitly writing the traces as sum over the matrix elements. This yields just a product of Gaussian integrals. After performing them and rewriting the product of exponentials back to a trace one gets the right-hand result. Note that Eq. (2.71) yields $\frac{1}{4}\,\mathrm{Tr}(A + A^T)^2$ instead of $\mathrm{Tr}\, A^2$ if the ensemble average is taken for the GOE instead (see Eq. (1.2)).

The simplification of the ensemble average by introducing Gaussian integrals over commuting and Grassmann variables is at the heart of the supersymmetry method. It is especially tailored for Gaussian matrix ensembles. However, for more general joint probability density function $\mathcal{P}(H)$, the expression $\langle e^{i\,\mathrm{Tr}\, HA}\rangle_H$ can be viewed as a generalised Fourier transform for matrices and effort has been made using this fact to generalise the supersymmetry method also for non-Gaussian matrix ensembles [64–68].

After performing the ensemble average, we are left with

$$Z(E^-, s) = \int \mathrm{d}\boldsymbol{\psi}\mathrm{d}\boldsymbol{\psi}^\dagger \exp\left(-i\boldsymbol{\psi}^\dagger[\mathcal{M}\otimes \mathbb{1}_N]\boldsymbol{\psi}\right) \exp\left(-\frac{J^2}{2N}\operatorname{Tr} A^2\right), \tag{2.72}$$

where we introduced the supervector $\boldsymbol{\psi}^T = [\boldsymbol{z}^T, \boldsymbol{\zeta}^T]$ and the 2×2 matrix $\mathcal{M} = \mathrm{diag}(E^- - s, -E^-)$. Hence we have rewritten our original problem into a supersymmetric model involving the supervector $\boldsymbol{\psi}$. This is already a remarkable result, as we reduced the degrees of freedom in our statistical model from N^2 independent variables of a Hermitian $N\times N$ matrix to the $4N$ independent variables of the supervector $\boldsymbol{\psi}$ ($\boldsymbol{\psi}$ comprises N complex commuting and N complex anticommuting variables).

Note that the level density is a fairly simple object, and usually more complicated supersymmetric models arise. E.g. for the k-point correlation function, which is the function that measures the probability to find eigenvalues of H around each of the positions $x_1 \dots x_k$, one would need to introduce k N-dimensional vectors $\boldsymbol{z}_1, \dots, \boldsymbol{z}_k$ comprising complex commuting variables and the same number of vectors comprising Grassmann variables. One could combine them into one $2kN$ dimensional vector $\boldsymbol{\psi}$ (the level density is obtain for the choice $k = 1$). In even more general cases the number of $\boldsymbol{z}$-vectors could be different from the number of $\boldsymbol{\zeta}$-vectors.

The main concern is now to perform the remaining integrations. There exist various methods to proceed from here an we will present three different ones in the following.

Hubbard-Stratonovich transformation

There is a crucial duality between ordinary and superspace [64], given by the identity $\operatorname{Tr} A^m = \operatorname{Str}(QL)^m$, $m \in \mathbb{N}$, where A is the $N\times N$ matrix defined in (2.70), $L = \mathrm{diag}(1,-1)$ and Q is the 2×2 Hermitian supermatrix

$$Q = \begin{bmatrix} \boldsymbol{z}^\dagger\boldsymbol{z} & \boldsymbol{z}^\dagger\boldsymbol{\zeta} \\ -\boldsymbol{\zeta}^\dagger\boldsymbol{z} & -\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta} \end{bmatrix}. \tag{2.73}$$

While for the 2×2 case it is still possible to show the duality relation by explicit calculation of the trace and supertrace, a more succinct way is as follows [64]: First introduce a matrix B whose columns are the integration vectors, $B = \begin{bmatrix} \boldsymbol{z} & \boldsymbol{\zeta} \end{bmatrix}$. Note that this is not a standard supermatrix of the form (2.33) introduced in Sec-

tion 2.2.2. However, the composition $B^\dagger B$ is such a standard supermatrix. To ensure that it has the correct properties under transposition and conjugation, compare Eq. (2.35), we need to define the complex conjugate of B with an extra minus sign, $B^\dagger = \begin{bmatrix} +\boldsymbol{z}^\dagger \\ -\boldsymbol{\zeta}^\dagger \end{bmatrix}$. Furthermore define the metric $L = \mathrm{diag}(1, -1)$. Then the matrix A can be written as $A = BLB^\dagger$, and the trace of its m-th power takes accordingly the form $\mathrm{Tr}\, BL(B^\dagger BL)^{m-1}B^\dagger$. We define $Q = B^\dagger B$ (this is the same definition as (2.73)) such that $\mathrm{Tr}\, A^m = \mathrm{Tr}\, BL(QL)^{m-1}B^\dagger$. As mentioned above Q is a standard supersymmetric matrix. If B was an ordinary matrix one could now use the cyclic invariance of the trace. However, due to the presence of the Grassmann variables, one needs to be a bit more careful. To that end we write $(QL)^{m-1}$ in block notation, $(QL)^{m-1} = \begin{bmatrix} a & \eta \\ \eta^\dagger & b \end{bmatrix}$. Performing the trace then yields $\mathrm{Tr}\, BL(QL)^{m-1}B^\dagger = \mathrm{Tr}(\boldsymbol{z}a\boldsymbol{z}^\dagger - \boldsymbol{z}\eta\boldsymbol{\zeta}^\dagger - \boldsymbol{\zeta}\eta^\dagger\boldsymbol{z}^\dagger + \boldsymbol{\zeta}b\boldsymbol{\zeta}^\dagger)$. For the terms in this expression we can use the cyclic invariance of the trace and the right-hand side becomes $\mathrm{Tr}(\boldsymbol{z}^\dagger\boldsymbol{z}a - \boldsymbol{z}^\dagger\boldsymbol{\zeta}\eta^\dagger) - \mathrm{Tr}(-\boldsymbol{\zeta}^\dagger\boldsymbol{z}\eta + \boldsymbol{\zeta}^\dagger\boldsymbol{\zeta}b) = \mathrm{Str}\,(QL(QL)^{m-1})$ (compare with the definition of the supertrace (2.36)) which proves the claim $\mathrm{Tr}\, A^m = \mathrm{Str}\,(QL)^m$.

This proof can be generalised easily for cases involving $k_B + k_F$ integration vectors where B becomes an $N \times (k_B + k_F)$ matrix. The supermatrix $Q = B^\dagger B$ will be accordingly a Hermitian supermatrix with upper left block of size $k_B \times k_B$, lower right block of size $k_F \times k_F$ and the other two blocks of matching size (compare with Eq. (2.105)). E.g. for the the k-point correlation function, Q would be a $2k \times 2k$ Hermitian supermatrix. L is a diagonal matrix of same size as Q, where each entry on the diagonal is either $+1$ or -1. The duality relation is true for any possible L of that form.

The duality is also applicable to the orthogonal case, although we have to deal with $\mathrm{Tr}(A+A^T)^2$ instead. In our example we have $A+A^T = \boldsymbol{z}\otimes\boldsymbol{z}^\dagger+\boldsymbol{z}^*\otimes\boldsymbol{z}^T+\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^\dagger-\boldsymbol{\zeta}^*\otimes\boldsymbol{\zeta}^T$. Note the minus sign in the last term since $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}^*$ anticommute. We can get the same expression if we take the Hermitian case for $k = 2$ (Two-point correlation function) given by $A = \boldsymbol{z}_1 \otimes \boldsymbol{z}_1^\dagger + \boldsymbol{z}_2 \otimes \boldsymbol{z}_2^\dagger + \boldsymbol{\zeta}_1 \otimes \boldsymbol{\zeta}_1^\dagger + \boldsymbol{\zeta}_2 \otimes \boldsymbol{\zeta}_2^\dagger$ and set $\boldsymbol{z}_2 = \boldsymbol{z}_1^*$ and $\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_1^*$ (such that $\boldsymbol{\zeta}_2 \otimes \boldsymbol{\zeta}_2^\dagger = -\boldsymbol{\zeta}_1^* \otimes \boldsymbol{\zeta}_1^T$ where we used $(\boldsymbol{\zeta}^*)^\dagger = -\boldsymbol{\zeta}^T$ which follows from definition (2.27)). Hence Q will be a 4×4 Hermitian supermatrix for the level density and more generally a $4k \times 4k$ Hermitian supermatrix for the k-point correlation function. Note

that besides Hermiticity Q will have a few more symmetries in the orthogonal case: Its first two diagonal elements are the same ($\boldsymbol{z}^\dagger\boldsymbol{z} = \boldsymbol{z}^T\boldsymbol{z}^*$) and similar the following two diagonal elements ($-\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta} = \boldsymbol{\zeta}^T\boldsymbol{\zeta}^*$), two entries in its lower right block will be zero ($\boldsymbol{\zeta}^T\boldsymbol{\zeta} = \boldsymbol{\zeta}^\dagger\boldsymbol{\zeta}^* = 0$ due to nilpotency of Grassmann variables) and in the off-diagonal block we have $Q_{13} = \boldsymbol{z}^\dagger\boldsymbol{\zeta} = Q_{24}^*$ and $Q_{14} = \boldsymbol{z}^\dagger\boldsymbol{\zeta}^* = Q_{23}^*$.

Coming back to the unitary case we use the duality relation to replace $\operatorname{Tr} A^2$ by $\operatorname{Str}(QL)^2$ in our supersymmetric model for $Z(E^-, s)$ (2.72). Furthermore we use the following identity, which is known as Hubbard-Stratonovich transformation [26],

$$\int \mathrm{d}\sigma \, \exp\left(-r \operatorname{Str}\sigma^2 + i \operatorname{Str}\sigma QL\right) = \exp\left(-\frac{1}{4r}\operatorname{Str}(QL)^2\right), \quad \operatorname{Re}(r) > 0, \tag{2.74}$$

where σ is a 2×2 supermatrix of the form $\sigma = \begin{bmatrix} x & \eta^* \\ \eta & iy \end{bmatrix}$, $x, y \in \mathbb{R}$, i.e. it has the same symmetry as Q apart from the imaginary unit in its lower right entry. Because of the minus sign in the definition of the supertrace (2.36) this so called Wick rotation [60] is needed to render the y-integration convergent. The identity (2.74) is also valid for Hermitian supermatrices Q of larger size or those which one obtains for the GOE, and any diagonal L having ± 1 on its diagonal. In all cases σ will have the same symmetries as Q apart from the lower right block which will be multiplied by the imaginary unit. The identity (2.74) can again be viewed as a generalised Fourier transformation, this time in superspace.

The term $\operatorname{Str}\sigma QL$ can now be expressed in terms of the supervector $\boldsymbol{\psi}$ introduced in (2.72), $\operatorname{Str}\sigma QL = \boldsymbol{\psi}^\dagger(L\sigma \otimes \mathbb{1}_N)\boldsymbol{\psi}$. This is also valid for the orthogonal case if one defines $\boldsymbol{\psi}^T = [\boldsymbol{z}^T, \boldsymbol{z}^\dagger, \boldsymbol{\zeta}^T, \boldsymbol{\zeta}^\dagger]$. Hence the Hubbard-Stratonovich transformation allows us to write Eq. (2.72) as

$$\begin{aligned} Z(E^-, s) = &\int \mathrm{d}\sigma \, \exp\left(-\frac{N}{2J^2}\operatorname{Str}\sigma^2\right) \\ &\times \int \mathrm{d}\boldsymbol{\psi}\mathrm{d}\boldsymbol{\psi}^\dagger \exp\left(-i\boldsymbol{\psi}^\dagger[(\mathcal{M} - L\sigma)\otimes \mathbb{1}_N]\boldsymbol{\psi}\right). \end{aligned} \tag{2.75}$$

Integration over the supervector can now be performed using $\int \mathrm{d}\boldsymbol{\psi}\mathrm{d}\boldsymbol{\psi}^\dagger \exp(i\boldsymbol{\psi}^\dagger\Sigma\boldsymbol{\psi}) = \operatorname{sdet}^{-1}\Sigma$ (a similar identity exists for the orthogonal case, $\int \mathrm{d}\boldsymbol{\psi}\mathrm{d}\boldsymbol{\psi}^\dagger \exp(i\boldsymbol{\psi}^\dagger\Sigma\boldsymbol{\psi}) = \operatorname{sdet}^{-1/2}\Sigma$. One gets a different result because not all entries of $\boldsymbol{\psi}$ are independent

from each other), which is valid for any dimension of $\boldsymbol{\psi}$. This identity can be shown representing the supermatrix Σ by its blocks, writing $\boldsymbol{\psi}^T = [\boldsymbol{z}^T, \boldsymbol{\zeta}^T]$ and then performing the Gaussian integrals over the anticommuting vector $\boldsymbol{\zeta}$ and the commuting vector $\boldsymbol{z}$ successively. The final result is then given by

$$Z(E^-, s) = (-1)^N \int \mathrm{d}\sigma \, \exp\left(-\frac{N}{2J^2}\,\mathrm{Str}\,\sigma^2\right) \mathrm{sdet}^{\,-N}\left[\sigma - \mathrm{diag}(E^- - s, E^-)\right], \quad (2.76)$$

where we used $\mathrm{sdet}\,(L\sigma - \mathcal{M}) = \mathrm{sdet}\,L\,\mathrm{sdet}\,(\sigma - L\mathcal{M})$ and $\mathrm{sdet}\,L = -1$.

Mapping the problem to a supermatrix-integral, we were able to reduce the degree of freedoms from N^2 integrations to just 4, that is the entries of the 2×2 supermatrix σ. Two of these integrations are over real commuting variables and two are over anticommuting variables. Moreover N is now an explicit parameter in the integrand, which makes it possible to analyse large-N asymptotics of $Z(E^-, s)$, e.g. by means of the saddle-point approximation (see previous section).

This is still valid for supersymmetric models of higher dimension, the k-point correlation function would for example be expressed by an integral over a $2k \times 2k$ Hermitian[21] supermatrix. For the orthogonal case supermatrices will be twice as large, however, due to further symmetries the number of independent variables reduces to $8k^2$, half of them commuting and half anticommuting (e.g. for the level density the integral will be over an 8×8 Hermitian supermatrix where the first two diagonal entries and the last two diagonal entries are identical, respectively, the two remaining entries in the lower right block are zero, and the off-diagonal blocks are only composed of two different Grassmann variables and their complex conjugates instead of four. This reduces the number to two real and one complex commuting variables and two complex Grassmann variables, accounting for eight integrations in total).

Equation (2.76) can be seen as the final exact result for arbitrary N. If one wishes one could also integrate the remaining Grassmann variables out to obtain the generating function of the level density in terms of an ordinary two-fold integral. From there one could also proceed to take the large-N limit. However, the symmetries of (2.76) make it easier to perform a saddle-point analysis at the level of the supermatrix σ.

[21] Strictly speaking σ is not Hermitian since its lower right block is multiplied by i. However, other than its lower diagonal entries becoming negative on complex conjugation σ has all properties of a Hermitian supermatrix.

The first step is to determine the N-dependence of the integrand. As explained in Section 2.2.2, the superdeterminant can be defined via $\operatorname{sdet}\sigma = \exp(\operatorname{Str}\ln\sigma)$. We use this to rewrite the superdeterminant

$$\operatorname{sdet}\,(\sigma_E + \operatorname{diag}(s,0))^{-N} = \exp\left\{-N\left[\operatorname{Str}\,\ln\sigma_E + \operatorname{Str}\,\ln(\mathbb{1}_2 + \sigma_E^{-1}\operatorname{diag}(s,0))\right]\right\}, \tag{2.77}$$

where we abbreviated $\sigma_E = \sigma - E^-\mathbb{1}_2$. The variable s is the source variable, i.e. in the end we have to take the derivative with respect to s at $s = 0$. This means we are free to choose s to be of order $1/N$, such that Ns is of order unity. Then we can expand the second term in the above equation in terms of $1/N$ which yields for the generating function

$$Z(E^-,s) \approx \int \mathrm{d}\sigma\,\exp\left\{-N\left[\frac{1}{2J^2}\operatorname{Str}\sigma^2 + \operatorname{Str}\,\ln\sigma_E\right] - Ns\operatorname{Str}\sigma_E^{-1}\operatorname{diag}(1,0)\right\}, \tag{2.78}$$

where we neglected all terms of order $\mathcal{O}(1/N)$ or higher. The above integral is of the form applicable to the saddle-point method, but with an integral over a supermatrix instead.[22] The idea of finding saddle points can be generalised to such a (super)matrix-integral. The condition is that the (super)matrix differential of the $\mathcal{O}(N)$ part vanishes, which in our case is given by $\mathrm{d}\left(\frac{1}{2J^2}\sigma^2 + \ln\sigma_E\right) = \left(\frac{1}{J^2}\sigma + \sigma_E^{-1}\right)\mathrm{d}\sigma$. Hence we require the term in brackets on the right-hand side to vanish, which gives the saddle-point condition

$$\frac{1}{J^2}\sigma + \sigma_E^{-1} = 0. \tag{2.79}$$

Let us first look for a diagonal solution. Then the saddle-point condition applies to both individual entries on the diagonal with solutions $\frac{1}{2}\left(E^- \pm \sqrt{(E^-)^2 - 4J^2}\right)$. Let us first assume $|E| < 2J$. Then we have in principle four possible diagonal solutions, corresponding to the two choices we have for each diagonal entry.

To determine which of these saddle points we have to choose recall that for a diagonal saddle-point solution $\sigma_D = \operatorname{diag}(\sigma_1^D, \sigma_2^D)$ one has $\operatorname{sdet}\,[\sigma_D - E^-\mathbb{1}_2 + \operatorname{diag}(s,0)]^{-1} =$

[22] Although the saddle-point method in this supersymmetric form is a standard tool used in such calculations, it has not been justified with full mathematical rigour. A more rigorous procedure is to integrate out the Grassmann variables at this point and then deal with the resulting expression in a controlled way. In this way the saddle-point method can be justified a posteriori. Such a strategy was for example employed in [69, 70].

$\frac{\sigma_2^D - E^-}{\sigma_1^D - E^- + s}$. Now recall that E^- carries a small negative imaginary unit, which implies the integrand in (2.76) has a pole at $\sigma_1^D = E - i\epsilon - s$, i.e. in the lower half-plane. We need to deform the contour of integration in such a way that we reach the saddle point without crossing this pole. This is not possible if the saddle-point lies in the lower half plane, such that we have to choose $\sigma_1^D = \frac{1}{2}\left(E^- + i\sqrt{4J^2 - (E^-)^2}\right)$, i.e. with positive imaginary part. For σ_2^D on the other hand, both solutions can be reached since there are no poles of the integrand, and the two diagonal saddle-point solutions are hence given by

$$\sigma_D = \frac{E^- \mathbb{1}_2 + i\sqrt{4J^2 - (E^-)^2}\,\mathrm{diag}(1, \pm 1)}{2}. \tag{2.80}$$

Following the method developed in Section 2.1, the main contribution to the integral should now come from these saddle points and small fluctuations $\delta\sigma$ around them, hence we deform the contours of integration through the saddle point and let $\sigma = \sigma_D + \delta\sigma$. As discussed in Section 2.1, it then suffices to evaluate the term of order unity in the integrand directly at the saddle point σ_D and expand the term of order N in the exponential to second order in $\delta\sigma$, where the first order vanishes due to the saddle-point condition (2.79). This procedure yields

$$\begin{aligned} Z(E^-, s) \approx \mathrm{sdet}\, \sigma_{D,E}^{-N} \exp\left[-\frac{N}{2J^2}\,\mathrm{Str}\,\sigma_D^2 + \frac{Ns}{J^2}\,\mathrm{Str}\,\sigma_D\,\mathrm{diag}(1,0)\right] \\ \times \int \mathrm{d}(\delta\sigma)\, \exp\left\{-\frac{N}{2J^4}\,\mathrm{Str}\,\left[J^2\delta\sigma^2 - (\sigma_D\delta\sigma)^2\right]\right\}. \end{aligned} \tag{2.81}$$

In terms of the entries of $\delta\sigma$ (which has the same structure as σ), the supertrace in the integrand is given by

$$\left[J^2 - (\sigma_1^D)^2\right](\delta x)^2 + \left[J^2 - (\sigma_2^D)^2\right](\delta y)^2 + 2\left[J^2 - \sigma_1^D\sigma_2^D\right]\delta\eta^*\delta\eta. \tag{2.82}$$

Assume now we had chosen σ_2^D different from σ_1^D. Then the term in front of the Grassmann variables vanishes

$$J^2 - \sigma_1^D\sigma_2^D = J^2 - \tfrac{1}{4}(E^- + i\sqrt{4J^2 - (E^-)^2})(E^- - i\sqrt{4J^2 - (E^-)^2}) = 0. \tag{2.83}$$

Consequently the integrand does not depend on any Grassmann variables at all, and

hence the integral and thus the whole expression vanishes when performing Grassmann integration. This shows that the contribution for different entries on the diagonal vanishes. For the remaining contribution $\sigma_2^D = \sigma_1^D$ we notice that $\operatorname{Str} \sigma_D^2 = 0$ and $\operatorname{sdet} \sigma_{D,E} = 1$. Furthermore the exponent in the integrand simplifies to

$$-\frac{N}{2J^4}\left[J^2 - (\sigma_1^D)^2\right]\left[(\delta x)^2 + 2\delta\eta^*\delta\eta + (\delta y)^2\right]. \tag{2.84}$$

Integration over the Grassmann variables yields precisely the inverse of the integration over the ordinary variables, such that the integral over $\delta\sigma$ evaluates to unity. The final solution for the case $|E| < 2J$ is hence given by the term of order unity, evaluated at the saddle point $\sigma_D = \frac{E^- + i\sqrt{4J^2-(E^-)^2}}{2}\mathbb{1}_2$,

$$Z(E^-, s) \approx \exp\left[\frac{Ns}{J^2}\operatorname{Str}\sigma_D \operatorname{diag}(1,0)\right] = \exp\left[\frac{Ns}{2J^2}\left(E^- + i\sqrt{4J^2-(E^-)^2}\right)\right]. \tag{2.85}$$

The second case $|E| > 2J$ can be treated along the same lines. The only difficulty is to determine in which half-plane the saddle points are. To that end we expand the saddle-point solution for small ϵ (since the limit $\epsilon \to 0$ has to be taken in the end it can be taken arbitrarily small) and get $2\sigma_1^D = E \pm \sqrt{E^2 - 4J^2} - i\epsilon\left(1 \pm \frac{E}{\sqrt{E^2-4J^2}}\right)$. To be able to deform the contour we need to choose the saddle point with imaginary part $> -\epsilon$ (which is where the pole of the integrand is). Hence for $E < -2J$ we choose the plus-solution and for $E > 2J$ the minus-solution. All other steps are the same as before, including the fact that the choice σ_2^D different from σ_1^D gives a vanishing contribution. Thus the final solution for the generating function is

$$Z(E^-, s) \approx \begin{cases} \exp\left[\frac{Ns}{2J^2}\left(E^- + i\sqrt{4J^2-(E^-)^2}\right)\right] & \text{for } |E| < 2J, \\ \exp\left[\frac{Ns}{2J^2}\left(E^- - \operatorname{sgn}(E)\sqrt{(E^-)^2-4J^2}\right)\right] & \text{for } |E| > 2J. \end{cases} \tag{2.86}$$

Notice that the restriction to diagonal solutions is not entirely correct: In principle any supermatrix of the form $T^{-1}\sigma_D T$ solves the saddle-point equation. For the case where σ_D is proportional to the unit matrix, the solution becomes independent of T and σ_D is indeed the only solution. In the other case we would have to deal with a manifold of solutions $\sigma_G = \frac{1}{2}(E^- + i\sqrt{4J^2 - E^2}T^{-1}\operatorname{diag}(1,-1)T)$. However, as we have seen the diagonal solution ($T = \mathbb{1}_2$) does not contribute to the saddle-point solution, and this

phenomenon carries over to the transformed solutions, such that the diagonal solution proportional to the unit matrix really is the only relevant contribution. While the non-diagonal solutions do not play a role here, they are of crucial importance for other problems, e.g. for the two-point correlation function it turns out [60] that one has to choose the imaginary increments on opposite sites of the real line which results in a continuous saddle-point manifold contributing to the integral. Integrating out the fluctuations $\delta\sigma$ usually does not pose a problem and one is left with an integral over this super-manifold, the so-called nonlinear-sigma model [26, 27]. We will encounter such a model in Chapter 4.4.2.

With the asymptotic result (2.86) for its generating function at hand it is now easy to compute the level density using Eq. (2.67), where the limit $\epsilon \to 0$ is also implied. Notice that the resulting expression for $|E| > 2J$ will be proportional to ϵ and hence vanish in this limit. For the case $|E| < 2J$, on the other hand, one gets $\rho(E) = \frac{\sqrt{4J^2-E^2}}{2\pi J^2}$. Hence we have shown that the level density of the GUE in the limit $N \to \infty$ is given by Wigner's semicircle law (1.9).

"Hybrid method"

This method uses the Hubbard-Stratonovich transformation only for the fermionic part, but a formula related to Superbosonization (explained in the following section) for the bosonic part. There is no standard name for this method in the literature which is why we call it the "hybrid method". We start again from (2.72). However, instead of going to the Q-supermatrix we simply perform the trace

$$\operatorname{Tr} A^2 = \operatorname{Tr}(\boldsymbol{z} \otimes \boldsymbol{z}^\dagger + \boldsymbol{\zeta} \otimes \boldsymbol{\zeta}^\dagger)^2 = (\boldsymbol{z}^\dagger \boldsymbol{z})^2 - (\boldsymbol{\zeta}^\dagger \boldsymbol{\zeta})^2 - 2\boldsymbol{\zeta}^\dagger (\boldsymbol{z} \otimes \boldsymbol{z}^\dagger)\boldsymbol{\zeta}. \tag{2.87}$$

Our first aim is now to perform the Grassmann integration, however, the term $(\boldsymbol{\zeta}^\dagger \boldsymbol{\zeta})^2$ prevents us from doing so. To deal with it we do a Hubbard-Stratonovich transformation for the Grassmann variables only,

$$\exp\left(\frac{J^2}{2N}(\boldsymbol{\zeta}^\dagger \boldsymbol{\zeta})^2\right) = \sqrt{\frac{N}{2\pi J^2}} \int_{-\infty}^{\infty} \mathrm{d}q \, \exp\left(-\frac{N}{2J^2} q^2 + q \, \boldsymbol{\zeta}^\dagger \boldsymbol{\zeta}\right). \tag{2.88}$$

For supersymmetric models which involve more vectors comprising Grassmann variables (e.g. k for the k-point correlation function), the trace takes the form

$$\operatorname{Tr} A^2 = \operatorname{Tr} Z^2 - \sum_{j,l=1}^{k} \boldsymbol{\zeta}_j^\dagger \boldsymbol{\zeta}_l \boldsymbol{\zeta}_l^\dagger \boldsymbol{\zeta}_j - \sum_{j=1}^{k} \boldsymbol{\zeta}_j^\dagger Z \boldsymbol{\zeta}_j = \operatorname{Tr} Z^2 - \operatorname{Tr} Q_F^2 - \sum_{j=1}^{k} \boldsymbol{\zeta}_j^\dagger Z \boldsymbol{\zeta}_j, \quad (2.89)$$

where we abbreviated $Z = \boldsymbol{z}_1 \otimes \boldsymbol{z}_1^\dagger + \cdots + \boldsymbol{z}_k \otimes \boldsymbol{z}_k^\dagger$ (Note that this is for the k-point correlation function, but in general the number of vectors $\boldsymbol{z}_j$ could be different from the number of vectors $\boldsymbol{\zeta}_j$) and Q_F is the $k \times k$ matrix having elements $q_{ij} = \boldsymbol{\zeta}_i^\dagger \boldsymbol{\zeta}_j$. For this case we may use the Hubbard-Stratonovich transformation

$$\exp\left(\frac{J^2}{2N} Q_F^2\right) = \left(\frac{N^k}{2\pi^k J^{2k}}\right)^{k/2} \int \mathrm{d}\hat{Q}_F \, \exp\left(-\frac{N}{2J^2} \operatorname{Tr} \hat{Q}_F^2 + \operatorname{Tr} \hat{Q}_F Q_F\right), \quad (2.90)$$

where $\hat{Q}_F$ is a Hermitian $k \times k$ matrix whose entries are commuting variables, and furthermore we can rewrite $\operatorname{Tr} \hat{Q}_F Q_F = \sum_j \boldsymbol{\zeta}_j^\dagger \hat{Q}_F \boldsymbol{\zeta}_j$. As usual, the orthogonal case is a bit more involved. The trace for the level density is e.g. given by (2.89) with $k = 2$ and $\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_1^*$ (the definition of Z has to change accordingly). Consequently Q_F will be a 2×2 matrix with $q_{11} = q_{22} = \boldsymbol{\zeta}^\dagger \boldsymbol{\zeta}$ and $q_{12} = \boldsymbol{\zeta}^\dagger \boldsymbol{\zeta}^* = 0 = \boldsymbol{\zeta}^T \boldsymbol{\zeta} = q_{21}$. More generally, if we decompose the $2k \times 2k$ matrix Q_F into $k \times k$ blocks, $Q_F = \begin{bmatrix} Q_F^{11} & Q_F^{12} \\ Q_F^{21} & Q_F^{22} \end{bmatrix}$, then we have $Q_F^{11} = Q_F^{22}$ and $Q_F^{12} = (Q_F^{21})^\dagger$. Furthermore Q_F^{11} will be Hermitian, giving rise to k^2 independent variables, whereas Q_F^{12} will be antisymmetric with complex entries, giving rise to $k^2 - k$ independent variables. Hence the total number of independent variables in the $2k \times 2k$ matrix Q_F for the orthogonal case is given by $k(2k-1)$.

Coming back to our original problem of calculating the level density, the exponent of (2.72) is now bilinear in the Grassmann variables and hence integration can be performed,

$$\int \mathrm{d}\boldsymbol{\zeta} \mathrm{d}\boldsymbol{\zeta}^\dagger \exp\left(\boldsymbol{\zeta}^\dagger (iE^- + q + \frac{J^2}{N} \boldsymbol{z} \otimes \boldsymbol{z}^\dagger) \boldsymbol{\zeta}\right) = \det\left(iE^- + q + \frac{J^2}{N} \boldsymbol{z} \otimes \boldsymbol{z}^\dagger\right). \quad (2.91)$$

Using Sylvester's theorem $\det(\mathbb{1} + AB) = \det(\mathbb{1} + BA)$ we are able to perform the

determinant and obtain

$$\det\left(iE^- + q + \frac{J^2}{N}\boldsymbol{z}\otimes\boldsymbol{z}^\dagger\right) = (iE^- + q)^{N-1}\left(iE^- + q + \frac{J^2}{N}\boldsymbol{z}^\dagger\boldsymbol{z}\right). \tag{2.92}$$

We furthermore change the variable $iE^- + q \to q$ which leaves us with

$$\begin{aligned} Z(E^-, s) =& \sqrt{\frac{N}{2\pi J^2}}\int \mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{z}^\dagger \exp\left(-\frac{J^2}{2N}(\boldsymbol{z}^\dagger\boldsymbol{z})^2 - i(E^- - s)\boldsymbol{z}^\dagger\boldsymbol{z}\right) \\ &\times \int_{-\infty}^{\infty}\mathrm{d}q\, q^{N-1}\exp\left(-\frac{N}{2J^2}(q - iE^-)^2\right)\left(q + \frac{J^2}{N}\boldsymbol{z}^\dagger\boldsymbol{z}\right). \end{aligned} \tag{2.93}$$

Note that at this point all N Grassmann integrations have been performed and we are left with $2N+1$ integrals over ordinary variables. Furthermore the integrand only depends on $\boldsymbol{z}^\dagger\boldsymbol{z} = \sum_j |z_j|^2$, which allows us to go to polar coordinates (note that $\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{z}^\dagger = \prod_{j=1}^N \mathrm{d}(\mathrm{Re}\, z_j)\mathrm{d}(\mathrm{Im}\, z_j)/\pi$),

$$\int \mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{z}^\dagger f(\boldsymbol{z}^\dagger\boldsymbol{z}) = \frac{S_{2N-1}}{\pi^N}\int_0^\infty r^{2N-1} f(r^2) = \frac{S_{2N-1}}{2}\left(\frac{N}{\pi J^2}\right)^N \int_0^\infty R^{N-1} f(\tfrac{N}{J^2}R), \tag{2.94}$$

where S_{2N-1} is the surface of the $2N$-sphere given by $S_{2N-1} = 2\pi^N/\Gamma(N)$, such that

$$\begin{aligned} Z(E^-, s) =& \sqrt{\frac{N}{2\pi J^2}}\left(\frac{N}{J^2}\right)^N \frac{1}{\Gamma(N)}\int_{-\infty}^{\infty}\mathrm{d}q\, q^{N-1}\exp\left(-\frac{N}{2J^2}(q - iE^-)^2\right) \\ &\times \int_0^\infty \mathrm{d}R\, R^{N-1}\exp\left(-\frac{N}{2J^2}R^2 - \frac{iN}{J^2}(E^- - s)R\right)(q + R). \end{aligned} \tag{2.95}$$

Like with the Hubbard-Stratonovich transformation method, we managed to reduce the problem to a small number of integrations left, N being an explicit parameter of the integrand, allowing for calculations of large-N asymptotics. Here the problem was reduced to a two-fold integral over commuting variables, whereas the previous method led to an integral over a 2×2 supermatrix. Note however that one could easily integrate the Grassmann variables in (2.76) and obtain a two-fold integral over ordinary variables as well (although the representation of $Z(E^-, s)$ will be still different, e.g. both integrals will be along the full real axis as opposed to Eq. (2.95) where one of the integrals is only along the half axis).

Equation (2.95) could serve as a starting point to obtain large-N asymptotics. However, in this case one can make further progress for finite N. To that end we notice that rescaling $q \to \sqrt{\frac{J^2}{N}}q$ one can write the q-integral in terms of the Hermite polynomial, Eq. (2.19). If we also rescale $R \to \sqrt{\frac{J^2}{N}}R$ and further define the function

$$F_N(z) = i^{N-1} \int_0^\infty \mathrm{d}R\, R^N \exp[-\tfrac{1}{2}R^2 + i\,\mathrm{Im}(\mathrm{sgn}\, z) z R], \tag{2.96}$$

which is the Cauchy (or Stieltjes) transform

$$F_N(z) = -\frac{(\mathrm{sgn}\,\mathrm{Im}\, z)^{N-1}}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathrm{d}y\, \frac{H_N(y)\exp(-\frac{1}{2}y^2)}{z-y} \tag{2.97}$$

of the Hermite polynomial [71][23], the generating function can be written compactly in terms of a determinant (we also use $H_N(-z) = (-1)^N H_N(z)$ and $\mathrm{Im}\,\mathrm{sgn}\, z = -1$ since E^- carries a small negative imaginary part),

$$Z(E^-, s) = \frac{1}{\Gamma(N)} \det \begin{bmatrix} H_{N-1}\left(\frac{\sqrt{N}}{J}E^-\right) & F_{N-1}\left(\frac{\sqrt{N}}{J}(E^- - s)\right) \\ H_N\left(\frac{\sqrt{N}}{J}E^-\right) & F_N\left(\frac{\sqrt{N}}{J}(E^- - s)\right). \end{bmatrix} \tag{2.98}$$

The derivative of Z with respect to s can now be done easily since it amounts to taking the derivative of F_N which is given by $F_N'(z) = \mathrm{Im}(\mathrm{sgn}\, z) F_{N+1}(z)$ as can be seen from its definition (2.96). Hence with Eq. (2.67) the exact solution for the level density is given by

$$\rho(E) = \frac{1}{\Gamma(N)\sqrt{N}\pi J}\,\mathrm{Im}\det \begin{bmatrix} H_{N-1}\left(\frac{\sqrt{N}}{J}E\right) & F_N\left(\frac{\sqrt{N}}{J}E\right) \\ H_N\left(\frac{\sqrt{N}}{J}E\right) & F_{N+1}\left(\frac{\sqrt{N}}{J}E\right) \end{bmatrix}. \tag{2.99}$$

Notice that the Hermite polynomials are real but their Cauchy transforms are in general complex.

Large-N asymptotics are now easily obtained considering the asymptotic forms of the functions contained in the determinant. Asymptotics of the Hermite polynomial have been calculated in the previous Section 2.1.2 for $|E| < 2J$ and are given by Eqs.

[23] We choose an unconventional prefactor in the definition of the transform which simplifies the result.

(2.22,2.23). Along the same lines one can obtain large-N asymptotics for $F_{N-k}(\sqrt{N}z)$ (equivalently one could also just directly take the Cauchy transform of the asymptotic expression for $H_{N-k}(\sqrt{N}z)$) which is given by

$$F_{N-k}(\sqrt{N}z) \approx \frac{\sqrt{\pi}\exp[-\frac{N}{2}-\frac{N}{4}z^2]}{i(4-z^2)^{1/4}}\left(\frac{i\sqrt{N}}{2}\right)^{N-k}\tilde{A}(k,N,\mathrm{Im}(\mathrm{sgn}\, z)z), \tag{2.100}$$

where $\tilde{A}(k,N,z)$ is defined as in the case for the Hermite polynomial, Eq. (2.23). Entries of the determinant have now a very similar structure which causes a lot of cancellation and yields

$$\rho(E) \approx \frac{1}{\Gamma(N)\sqrt{2\pi}J^2}e^{-N}N^{N-1/2}\,\mathrm{Im}(E+i\sqrt{4J^2-E^2}). \tag{2.101}$$

Since we consider large N-asymptotics, one should also express the Γ-function by its large-N asymptotics $\Gamma(N) \approx \sqrt{2\pi}e^{-N}N^{N-1/2}$ (Stirling's formula). This cancels any N-dependence of $\rho(E)$ and after taking the imaginary part one gets precisely the same semicircular law $\rho(E) = \frac{\sqrt{4J^2-E^2}}{2\pi J^2}$, $|E| < 2J$, as with the Hubbard-Stratonovich transformation method.

The remaining question is how one can generalise the above used method. For more complicated supersymmetric models, bilinearising the exponent in the Grassmann variables using the Hubbard-Stratonovich transformation (2.90) and then performing the Grassmann integration, one ends up with an expression similar to (2.93), i.e. an expression which only involves integration over commuting vectors $\boldsymbol{z}_j$, $j = 1 \dots k$. The integrand will turn out to depend only on the $k \times k$ matrix Q_B, where elements of Q_B are given by $q_{ij} = \boldsymbol{z}_i^\dagger \boldsymbol{z}_j$ (e.g. $\mathrm{Tr}\, Z^2$ from Eq. (2.89) is given by $\mathrm{Tr}\, Z^2 = \mathrm{Tr}\, Q_B^2$). The main idea of this supersymmetric method is now to generalise the step going from $\boldsymbol{z}^\dagger\boldsymbol{z}$ to polar coordinates in (2.94). Assuming that each vector $\boldsymbol{z}_j$ is of dimension N, this generalisation is given by the formula

$$\int \mathrm{d}\boldsymbol{z}_1\mathrm{d}\boldsymbol{z}_1^\dagger \dots \int \mathrm{d}\boldsymbol{z}_k\mathrm{d}\boldsymbol{z}_k^\dagger\, f(Q_B) = \frac{2\pi^{-k(k-1)/2}}{\prod_{j=1}^k (N-j)!}\int_{\hat{Q}_B>0}\mathrm{d}\hat{Q}_B\ \det(\hat{Q}_B)^{N-k} f(\hat{Q}_B), \tag{2.102}$$

where the integration on the right-hand side goes over the manifold of Hermitian positive definite $k \times k$ matrices $\hat{Q}_B$. This formula was derived heuristically in [72] and

not much later proved in [73] (see Theorem I). A similar formula holds for vectors with real entries (such that Q_B becomes real symmetric) and is given by

$$\int \mathrm{d}\boldsymbol{x}_1 \ldots \int \mathrm{d}\boldsymbol{x}_k \, f(Q_B) = \frac{\pi^{-k(k-1)/4}}{\prod_{j=0}^{k-1} \Gamma(\frac{N-j}{2})} \int_{\hat{Q}_B > 0} \mathrm{d}\hat{Q}_B \, \det(\hat{Q}_B)^{(N-k-1)/2} f(\hat{Q}_B), \quad (2.103)$$

where the integration now goes over the manifold of real symmetric positive definite $k \times k$ matrices $\hat{Q}_B$. This was proved in the same paper [73] (see Theorem Ia in Appendix D).

In summary, assuming the original supersymmetric model had k_B vectors comprising N complex commuting variables and k_F vectors comprising N complex Grassmann variables, this method replaces integration over these vectors (that is $2N(k_B + k_F)$ independent variables) with integration over a Hermitian $k_F \times k_F$ matrix $\hat{Q}_F$ and a positive definite Hermitian $k_B \times k_B$ matrix Q_B (i.e. $k_B^2 + k_F^2$ independent variables). Note that the Hubbard-Stratonovich transformation method left us with twice the number of integrations, however, half of them over commuting variables, the other half over anticommuting ones, i.e. the number of ordinary integrations is the same in both methods.

An advantage of this method is that the Grassmann variables are integrated out at an early stage and hence one has not to worry about obstacles given by their presence like finding the correct integration-supermanifold when doing a saddle-point approximation or the occurrence of Efetov-Wegner terms. Since ordinary and Grassmann integrations are treated separately, this method can also be better suited if one has to deal with "incomplete supersymmetry", i.e. when the number of commuting variables differs from the number of Grassmann variables. On the other hand the Hubbard-Stratonovich transformation method usually gives a more compact and elegant expression in superspace, especially if one has "full supersymmetry", i.e. the same number of commuting and anticommuting variables, and employing symmetries of the supermatrix σ can lead to a simpler analysis of the model. However, there are exceptions to this, e.g. we have seen that the "hybrid method" revealed a nice determinantal structure of the level density (and its generating function) which was not obtained from the Hubbard-Stratonovich transformation method.

Superbosonization

The last method which will be presented here is called "Superbosonization" [66–68]. It is a supersymmetric extension of the formulas (2.102) and (2.103), based on the formula

$$\int \mathrm{d}\boldsymbol{z}_1 \mathrm{d}\boldsymbol{z}_1^\dagger \ldots \int \mathrm{d}\boldsymbol{z}_{k_B} \mathrm{d}\boldsymbol{z}_{k_B}^\dagger \int \mathrm{d}\boldsymbol{\zeta}_1 \mathrm{d}\boldsymbol{\zeta}_1^\dagger \ldots \int \mathrm{d}\boldsymbol{\zeta}_{k_F} \mathrm{d}\boldsymbol{\zeta}_{k_F}^\dagger \, f(Q) \propto \int \mathrm{d}\hat{Q} \, \mathrm{sdet}\, \hat{Q}^{N+k_F-k_B} f(\hat{Q}), \tag{2.104}$$

where on the left-hand side the complex commuting vectors $\boldsymbol{z}_j$, $j = 1 \ldots k_B$ and complex anticommuting vectors $\boldsymbol{\zeta}_j$, $j = 1 \ldots k_F$ enter the function $f(Q)$ only via the supermatrix

$$Q = \begin{bmatrix} \boldsymbol{z}_1^\dagger \boldsymbol{z}_1 & \ldots & \boldsymbol{z}_{k_B}^\dagger \boldsymbol{z}_1 & \boldsymbol{\zeta}_1^\dagger \boldsymbol{z}_1 & \ldots & \boldsymbol{\zeta}_{k_F}^\dagger \boldsymbol{z}_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{z}_1^\dagger \boldsymbol{z}_{k_B} & \ldots & \boldsymbol{z}_{k_B}^\dagger \boldsymbol{z}_{k_B} & \boldsymbol{\zeta}_1^\dagger \boldsymbol{z}_{k_B} & \ldots & \boldsymbol{\zeta}_{k_F}^\dagger \boldsymbol{z}_{k_B} \\ \boldsymbol{z}_1^\dagger \boldsymbol{\zeta}_1 & \ldots & \boldsymbol{z}_{k_B}^\dagger \boldsymbol{\zeta}_1 & \boldsymbol{\zeta}_1^\dagger \boldsymbol{\zeta}_1 & \ldots & \boldsymbol{\zeta}_{k_F}^\dagger \boldsymbol{\zeta}_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{z}_1^\dagger \boldsymbol{\zeta}_{k_F} & \ldots & \boldsymbol{z}_{k_B}^\dagger \boldsymbol{\zeta}_{k_F} & \boldsymbol{\zeta}_1^\dagger \boldsymbol{\zeta}_{k_F} & \ldots & \boldsymbol{\zeta}_{k_F}^\dagger \boldsymbol{\zeta}_{k_F} \end{bmatrix}. \tag{2.105}$$

Integration on the right-hand side goes over the $(k_B + k_F) \times (k_B + k_F)$ Hermitian supermatrix $\hat{Q}$, with its upper left block being positive definite. A similar formula can be derived for the case of real variables (replace † in the definition of Q by ^{T}),

$$\int \mathrm{d}\boldsymbol{z}_1 \ldots \int \mathrm{d}\boldsymbol{z}_{k_B} \int \mathrm{d}\boldsymbol{\zeta}_1 \ldots \int \mathrm{d}\boldsymbol{\zeta}_{k_F} f(Q) \propto \int \mathrm{d}\hat{Q} \, \mathrm{sdet}\, \hat{Q}^{(N+k_F-k_B-1)/2} f(\hat{Q}). \tag{2.106}$$

Here $\hat{Q}$ has positive definite real symmetric upper left block, and skew-symmetric unitary lower right block.

Let us try to apply the superbosonization formula (2.104) to our toy model (2.72). First we need to rewrite the integrand such that it only depends on Q. Note that the definition of Q, Eq. (2.105), is the same as the earlier definition from the Hubbard-Stratonovich transformation method (2.73). Hence we may replace $\mathrm{Tr}\, A^2 = \mathrm{Str}\, Q^2$ in (2.72). Also the bilinear form in (2.72) can be expressed as a function of Q via $\boldsymbol{\psi}^\dagger (\mathcal{M} \otimes \mathbb{1}_N) \boldsymbol{\psi} = \mathrm{Str}\,(\mathcal{M} Q\, \mathrm{diag}(1, -1))$. Note that one has to insert the matrix

$\mathrm{diag}(1,-1)$ to cancel the minus-sign in the definition of the supertrace (2.36). Hence we can apply the superbosonization formula (2.104),

$$\begin{aligned} Z(E^-,s) &= \int \mathrm{d}\boldsymbol{\psi}\mathrm{d}\boldsymbol{\psi}^\dagger \exp\left(-i\,\mathrm{Str}\left[\mathrm{diag}(E^- - s, E^-)Q\right]\right) \exp\left(-\frac{J^2}{2N}\,\mathrm{Str}\,Q^2\right) \\ &\propto \int \mathrm{d}\hat{Q}\,\mathrm{sdet}^{\,N}\hat{Q} \exp\left(-\frac{N}{2J^2}\,\mathrm{Str}\,\hat{Q}^2 - \frac{iN}{J^2}\,\mathrm{Str}\left[\mathrm{diag}(E^- - s, E^-)\hat{Q}\right]\right), \end{aligned} \tag{2.107}$$

where we also changed $\hat{Q} \to \frac{N}{J^2}\hat{Q}$. The integration is now over the Hermitian 2×2 supermatrix $\hat{Q} = \begin{bmatrix} x & \eta^* \\ \eta & y \end{bmatrix}$, where $x \in (0,\infty)$ and y unimodular complex are commuting variables and η, η^* are Grassmann variables. Similar to the Hubbard-Stratonovich transformation method, Superbosonization reduces the problem to an integral over a 2×2 (or more generally $(k_B+k_F)\times(k_B+k_F)$) Hermitian supermatrix, which is also very well suited for large-N approximations. Compare the result obtained via Superbosonization, Eq. (2.107), to the result obtain via the Hubbard Stratonovich transformation, Eq. (2.76). They are equivalent [74], however, this equivalence is not obvious at all. The main difference is that (2.76) contains the inverse of the superdeterminant. Note also that the range of integration is different.

Large-N asymptotics of (2.107) can be computed following the same steps as for the Hubbard-Stratonovich method. The N-dependent part of the exponent is here given by $-\frac{N}{2J^2}\,\mathrm{Str}\,\hat{Q}^2 + N\,\mathrm{Str}\,\ln\hat{Q} - \frac{iN}{J^2}E^-\,\mathrm{Str}\,\hat{Q}$ and the term of order unity by $\exp[\frac{iNs}{J^2}\,\mathrm{Str}\,\hat{Q}\,\mathrm{diag}(1,0)]$. This leads to a different saddle-point condition as for the Hubbard-Stratonovich method,

$$\frac{1}{J^2}\hat{Q} + \hat{Q}^{-1} - \frac{i}{J^2}E^- = 0. \tag{2.108}$$

However, for a diagonal solution $\hat{Q}_D = \mathrm{diag}(\hat{Q}_1^D, \hat{Q}_2^D)$ this implies entries should be of the form $-\frac{1}{2}\left(iE \pm \sqrt{4J^2 - E^2}\right)$. These are precisely the saddle points obtained from the other method multiplied by $-i$. Also the term of order unity is the same apart from an extra i in the exponent, which will cancel with the $-i$ from the saddle points, such that the final solution, which is essentially the term of order unity evaluated at

the correct saddle point solution, will be the same.[24]

This concludes the discussion of the supersymmetric methods used in random matrix theory. We have seen that it is a very powerful tool to perform RMT calculations. Depending on the type of problem, one of the methods might be more suited than another.

[24] All other considerations like the choice of the correct solution are similar, apart from the range of integration of the upper left entry x in $\hat{Q}$ being strictly positive. This means the contour of integration needs to be deformed in a different way, and subsequently its lower integration range extended to $-\infty$ which can be justified as described in Section 2.1.

3. Universality of K-Matrix Distribution for $\beta = 2$

In this chapter we show that the distribution of the K-matrix for a quantum chaotic scattering problem is universal in the sense that it is always distributed according to a matrix-Cauchy distribution, irregardless of the joint probability density of the large random matrix which describes the underlying Hamiltonian of the system. This furthermore proves that the two different RMT-approaches in chaotic scattering, known as "Heidelberg" and "Mexico" approach are equivalent. This work is published in [12].

In the first section we motivate the problem. In the next section the characteristic function of the K-matrix for the random amplitude model is calculated for arbitrary dimension N of the random matrix describing the Hamiltonian, and its large N limit is taken which turns out to be universal. Section 3.3 then proves that K is Cauchy-distributed by first calculating the characteristic function of a Cauchy distribution and then comparing it to the characteristic function from Section 3.2. The last section of this chapter will show that the same expression for the characteristic function of the K-matrix can also be obtained starting with the fixed amplitude model and thus the corresponding distribution will be also Cauchy-distributed.

3.1. Motivation

In the introductory Chapter 1 we presented two different approaches to deal with scattering problems within the framework of random matrix theory. The Heidelberg approach described in Section 1.3.2 introduces stochasticity at the level of the underlying Hamiltonian H in (1.20), while the Mexico approach described in Section 1.3.3 introduces it on the level of the S-matrix itself, thus deriving the Poisson kernel (1.30) (or equivalently the matrix-Cauchy distribution (1.31) for the K-matrix).

These two random matrix approaches look very different in their formulation, yet they are meant to describe precisely the same object, the S-matrix for a chaotic system. The consistency therefore requires that the Poisson kernel distribution (1.30) for S must also follow from the law of distribution of H entering the relation (1.20). Surprisingly, a direct verification of such a correspondence turns out to be a rather challenging task. The challenge here is that the two objects are related via the resolvent-like K-matrix, and to convert the law of distribution of H into that of the resolvent is not at all trivial. A very elegant indirect way round this problem was discovered by P. Brouwer [13] who proposed to choose H from the Cauchy ensemble of random matrices with density $\mathcal{P}(H) \propto \det\left[\lambda^2 + (H-\epsilon)^2\right]^{-(\beta N+2-\beta)/2}$, where λ, ϵ are two real parameters. The main advantage of such a choice is that the resolvent $(E-H)^{-1}$ is Cauchy-distributed as well albeit with modified parameters and, moreover, diagonal blocks of Cauchy matrices have again closely related distributions. Using these facts Brouwer indeed was able to demonstrate the validity of the Poisson kernel for such a choice of H for all values of $\beta = 1, 2, 4$. He then showed that in the large-N limit the eigenvalue correlation functions in the Cauchy ensemble (called "Lorentzian" ensemble by Brouwer) have the standard Dyson form, and conjectured that such equivalence of eigenvalue correlation functions should be enough to ensure the same S-matrix distribution is to be shared by all representatives of the corresponding universality class. Although such conjecture sounds very natural, the particular mechanism by which the generic spectral properties of H are translated into universality of the probability density of the K-matrix and then $\mathcal{P}(S)$ remained unclear. To the best of knowledge no further attempts to verify universality of the S-matrix distribution were undertaken in the literature apart from (i) the simplest case $M = 1$ and $H \in$ GUE considered in [6] and (ii) the recent work [75] which however concentrated on the universality of two-point spectral correlations of the individual S-matrix entries rather than on the one-point matrix distribution.

In this thesis chapter, under fairly generic assumptions on H belonging to a unitary ensemble, it is verified that the law of distribution of the K-matrix is Cauchy with distribution (1.31) and its parameters λ and ϵ are related to the strength of the coupling amplitudes W and the density of states of the underlying matrix H as well as details of its (invariant) distribution. Since S- and K-matrix are related via the Cayley transformation (1.20), we thereby also establish the universality of the Poisson

distribution for the S-matrix. For simplicity we restrict to the condition of equivalent coupling to continuum in all scattering channels, but note that the method can easily be extended to arbitrary coupling constants.

3.2. Characteristic Function $\mathcal{F}(X)$ of the K-Matrix

The first step in showing the equivalence of the two approaches is to compute an expression for the characteristic function of the K-matrix given by $K = W^\dagger(E - H)^{-1}W$ (see Eq. (1.20)). Its distribution is given by

$$\mathcal{P}(K) = \left\langle \delta\left(K - W^\dagger(E-H)^{-1}W\right)\right\rangle, \tag{3.1}$$

where δ denotes the (matrix valued) Dirac Delta function, and the angular brackets denote the average over all random variables, i.e. all independent elements of the random matrix H, and in the random amplitude model also over the independent entries of W. The characteristic function is then consequently given by

$$\mathcal{F}_{\beta,N}(X) = \left\langle \exp\left(-\frac{i\beta}{2}\operatorname{Tr} XW^\dagger(E-H)^{-1}W\right)\right\rangle, \tag{3.2}$$

where the matrix X has the same dimensions and symmetries as the matrix K. In terms of the characteristic function, the probability density function of K can be obtain by

$$\mathcal{P}(K) = \left(\frac{\beta}{4\pi}\right)^{\frac{\beta M}{2}(M+2-\beta)} \int \mathrm{d}X\, \mathcal{F}_{\beta,N}(X) \exp\left(\frac{i\beta}{2}\operatorname{Tr} XK\right). \tag{3.3}$$

We now employ the random amplitude model and perform the averaging over the coupling matrix W in (3.2) for $\beta = 1, 2$,

$$\int \mathrm{d}W\, \mathcal{P}(W) \exp\left(-\frac{i\beta}{2}\operatorname{Tr} XW^\dagger(E-H)^{-1}W\right). \tag{3.4}$$

The probability density function of $\mathcal{P}(W)$ is Gaussian (see Eq. (1.28)) and $\mathrm{d}W$ stands for the appropriately normalized Lebesgue measure on the space of complex or real $N \times M$ matrices W. In order to compute the average we first diagonalise $X =$

$T\ \mathrm{diag}(x_1, \ldots, x_M)\, T^{-1}$ where T is orthogonal for $\beta = 1$ and unitary for $\beta = 2$. Then we change $(WT) \to W$ and exploit that $W^\dagger W$ and the measure $\mathrm{d}W$ are invariant with respect of such a transformation. This leaves us with the same expression (3.4), where now X is replaced by a diagonal matrix containing its eigenvalues $x_1, \ldots, x_M$. This allows us to rewrite the trace in the following form

$$\mathrm{Tr}\left[\mathrm{diag}(x_1, \ldots, x_M) W^\dagger (E - H)^{-1} W\right] = \sum_c x_c \boldsymbol{w}_c^\dagger (E - H)^{-1} \boldsymbol{w}_c, \tag{3.5}$$

where the $\boldsymbol{w}_c$ are the column vectors of W (channel vectors). In terms of these vectors the probability density function becomes $\mathcal{P}(W) = (\frac{N}{2\pi\gamma})^{\beta MN/2} \exp(-\frac{\beta N}{2\gamma} \sum_c \boldsymbol{w}_c^\dagger \boldsymbol{w}_c)$, and hence the ensemble average amounts to perform the Gaussian integral

$$\begin{aligned} &\left(\frac{N}{2\pi\gamma}\right)^{\beta MN/2} \int \mathrm{d}W\ \exp\left[-\frac{\beta}{2} \sum_c \boldsymbol{w}_c^\dagger \left(\frac{N}{\gamma} + i x_c (E - H)^{-1}\right) \boldsymbol{w}_c\right] \\ &= \prod_{c=1}^{M} \det\left(\mathbb{1}_N + \frac{i\gamma x_c}{N} (E - H)^{-1}\right)^{-\frac{\beta}{2}}. \end{aligned} \tag{3.6}$$

At the next step we bring the characteristic function $\mathcal{F}_{\beta,N}(X)$ to the following form:

$$\mathcal{F}_{\beta,N}(X) = \left\langle \prod_{c=1}^{M} \frac{\det(E - H)^{\frac{\beta}{2}}\, \mathrm{sgn}\det(E - H)^{(2-\beta)\Theta(-x_c)}}{\det(E + i\gamma x_c/N - H)^{\frac{\beta}{2}}} \right\rangle_H, \tag{3.7}$$

where the angular brackets now stand for the averaging over the $N \times N$ matrices H. $\Theta(-x_c)$ is the Heaviside-Theta function which is 1 for positive arguments and 0 otherwise. This means for negative eigenvalues of X we get additional sgn det factors in the $\beta = 1$ case. These are due to the branch cut in the complex plane when taking square-roots. Appendix A.2 has a more detailed analysis how these factors come about.

The above relation is *exact* in the random amplitude model (1.27,1.28) for any choice of N and M. We will show in Section 3.4 that for $\beta = 2$ the same equation (3.7) is valid asymptotically in the fixed amplitude model (1.25) in the limit $N \gg M$ provided the probability density of H is rotationally invariant.

3.2.1. Large N Limit in the Random Amplitude Model

For $\beta = 2$ and $\beta = 4$ the object in the right-hand side of (3.7) is well-studied in the random matrix theory and formulas have been derived for finite N as well as for the case $N \to \infty$, using various methods, e.g. orthogonal polynomials or supersymmetry [71, 76–78]. For $\beta = 1$, results for integer powers have been calculated, but unfortunately no result of comparable generality seems to be known for the products of *square roots* of the characteristic polynomials. We will derive such formulas for small M in Chapter 4. For now, we consider in full generality only the case of Hermitian ensembles with $\beta = 2$.

For this purpose, the formula (2.14) from [76] appears to be most useful for our goals. Namely, for $N \times N$ matrices H distributed according to an invariant ensemble density with polynomial potential V,

$$\mathcal{P}(H) \propto \exp\left[-N \operatorname{Tr} V(H)\right], \quad V(H) = \sum_{l=0}^{p} c_l H^{2l}, \quad c_p > 0, \tag{3.8}$$

the following universal relation holds asymptotically[25]:

$$\begin{aligned} &\lim_{N\to\infty} \left\langle \prod_{c=1}^{M} \frac{\det(E + \eta_c/(N\rho(E)) - H)}{\det(E + \zeta_c/(N\rho(E)) - H)} \right\rangle_H \\ &= (-)^{\frac{M(M-1)}{2}} \exp\left(-\pi\alpha_E \sum_{c=1}^{M} (\zeta_c - \eta_c)\right) \frac{\Delta\{\zeta,\eta\}}{\Delta^2\{\zeta\}\Delta^2\{\eta\}} \det(\mathcal{S}(\zeta_n - \eta_m))_{m,n=1\ldots M}, \end{aligned} \tag{3.9}$$

where $\Delta\{\eta\} = \prod_{m<n}(\eta_m - \eta_n)$ denotes the Vandermonde determinant, $\Delta\{\zeta,\eta\} = \Delta\{\zeta\}\Delta\{\eta\}\prod_{m,n}(\zeta_m - \eta_n)$, and

$$\mathcal{S}(\zeta - \eta) = \frac{\exp(i\pi \operatorname{sgn}(\operatorname{Im}\zeta)(\zeta - \eta))}{\zeta - \eta}, \quad \alpha_E = \frac{V'(E)}{2\pi\rho(E)}. \tag{3.10}$$

$\rho(E)$ is the large-N limit of the mean eigenvalue density of H at point E inside the support of $\rho(E)$ (so that $\rho(E) > 0$). An analogous result for averaged products of

[25] We restrict ourselves to the polynomial potentials in (3.8) for the notational convenience. The asymptotic relation (3.9), and as a consequence the final result holds for invariant ensembles of random matrices under fairly general conditions on the matrix measure, see the recent paper [79]

ratios of characteristic polynomials with $\beta = 4$ is also known [77], but has a more complex structure, with Pfaffians replacing determinants.

With this asymptotic relation in hand, one can evaluate the characteristic function (3.7) of the K-matrix for $\beta = 2$ in the limit $N \to \infty$ and M fixed. Our case is a special limit of (3.9), where all $\eta_c \to 0$. This limit, however, cannot be performed trivially, e.g. the Vandermonde determinant $\Delta\{\eta\}$ featured in the denominator of Eq. (3.9) vanishes in this case.

We start the calculation noting that

$$\lim_{\eta_1 \dots \eta_M \to 0} \frac{\Delta\{\zeta, \eta\}}{\Delta^2\{\zeta\}\Delta\{\eta\}} = \frac{(\zeta_1 \times \ldots \times \zeta_M)^M}{\Delta\{\zeta\}}, \tag{3.11}$$

and hence Eq. (3.9) becomes

$$\begin{aligned} \lim_{N\to\infty} \left\langle \prod_{c=1}^{M} \frac{\det(E-H)}{\det(E+\zeta_c/(N\rho(E))-H)} \right\rangle_H &= (-)^{\frac{M(M-1)}{2}} \exp\left(-\pi\alpha_E \sum_{c=1}^{M} \zeta_c\right) \\ \times \frac{(\zeta_1 \times \ldots \times \zeta_M)^M}{\Delta\{\zeta\}} \lim_{\eta_1 \dots \eta_M \to 0} \frac{1}{\Delta\{\eta\}} &\det(\mathcal{S}(\zeta_n - \eta_m))_{m,n=1\dots M}. \end{aligned} \tag{3.12}$$

We now perform the limits successively, starting with η_M. This leaves us with the expression

$$\lim_{\eta_1 \dots \eta_{M-1} \to 0} \frac{1}{\Delta_{M-1}\{\eta\}\eta_1 \dots \eta_{M-1}} \det(\mathcal{S}(\zeta_n - \eta_m))_{m,n=1\dots M}\Big|_{\eta_M=0}, \tag{3.13}$$

where $\Delta_{M-1}\{\eta\}$ denotes now the Vandermonde determinant for the variables η_1 to η_{M-1}. For the next limit $\eta_{M-1} \to 0$ naively replacing η_{M-1} just results in an indeterminate expression, as the denominator would become zero and the determinant as well (its two last rows become the same). Instead we use l'Hospital's rule, which states that $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x)$, provided both $f(x)$ and $g(x)$ tend to 0 for $x \to a$. The derivative of the denominator can be performed easily and is given by

$$\frac{\partial}{\partial\eta_{M-1}} \Delta_{M-1}\{\eta\}\, \eta_1 \dots \eta_{M-1}\Big|_{\eta_{M-1}=0} = \Delta_{M-2}\{\eta\}\, \eta_1^2 \dots \eta_{M-2}^2. \tag{3.14}$$

One way to define the determinant of an $M \times M$ matrix A is via the Leibniz formula

$$\det A = \sum_{i_1,\ldots,i_M=1}^{M} \epsilon_{i_1\ldots i_M} a_{1,i_1} \ldots a_{M,i_M}, \tag{3.15}$$

where $\epsilon_{i_1\ldots i_M}$ denotes the total antisymmetric Levi-Civita symbol which is 1 or -1 if the sequence $i_1, \ldots, i_M$ is an even or odd permutation of $1, \ldots, M$, respectively, and is 0 otherwise (i.e. when any two i's have the same value). From this formula, using the product rule, it is easy to see that the derivative of $\det A$ is given by

$$(\det A(x))' = \sum_{m=1}^{M} \left(\sum_{i_1,\ldots,i_M=1}^{M} \epsilon_{i_1\ldots i_M} a'_{m,i_m}(x) \prod_{n\neq m} a_{n,i_n}(x) \right). \tag{3.16}$$

The term in brackets is nothing else than the determinant of the matrix A where all entries in the m-th row (or equivalently m-th column, since $\det A = \det A^T$) have been differentiated; to get the overall derivative one has to sum over all rows (or columns). In our case, however, only the second last row depends on the variable η_{M-1}, and thus only one term in the m-summation of Eq. (3.16) is non-vanishing. Hence, after performing the $\eta_{M-1} \to 0$ limit, Eq. (3.13) becomes

$$\begin{aligned} &\lim_{\eta_1\ldots\eta_{M-2}\to 0} \frac{1}{\Delta_{M-2}\{\eta\}\eta_1^2\ldots\eta_{M-2}^2} \\ &\times \det \begin{pmatrix} \mathcal{S}(\zeta_1-\eta_1) & \ldots & \mathcal{S}(\zeta_M-\eta_1) \\ \vdots & \ddots & \vdots \\ \mathcal{S}(\zeta_1-\eta_{M-2}) & \ldots & \mathcal{S}(\zeta_M-\eta_{M-2}) \\ \frac{\partial}{\partial\eta_{M-1}}\mathcal{S}(\zeta_1-\eta_{M-1})\Big|_{\eta_{M-1}=0} & \ldots & \frac{\partial}{\partial\eta_{M-1}}\mathcal{S}(\zeta_M-\eta_{M-1})\Big|_{\eta_{M-1}=0} \\ \mathcal{S}(\zeta_1-0) & \ldots & \mathcal{S}(\zeta_M-0) \end{pmatrix}. \end{aligned} \tag{3.17}$$

Next we proceed with taking the limit $\eta_{M-2} \to 0$. We observe that after using l'Hospital's rule once the expression remains indeterminate (now the second last and third last rows will be the same) and hence it needs to be applied twice. This essentially amounts to adjusting the denominator (it becomes $2\Delta_{M-3}\{\eta\}\,\eta_1^3\ldots\eta_{M-3}^3$) and to replace $\mathcal{S}$ in the third last row of (3.17) by its second derivative with respect to η_{M-2}, evaluated at $\eta_{M-2} = 0$. We may proceed in the same manner with

all other limits, always applying l'Hospital's rule one time more than in the previous step. After m steps, the denominator will have taken the form $2!\,3!\ldots(m-1)!\,\Delta_{M-m}\{\eta\}\,\eta_1^m\ldots\eta_{M-m}^m$, and the last m rows of the matrix have been adjusted accordingly. Carrying on with this procedure until the last limit $\eta_1 \to 0$ is taken, we hence arrive at the result

$$\lim_{\eta_1\ldots\eta_M\to 0} \frac{\det(\mathcal{S}(\zeta_n-\eta_m))_{m,n=1\ldots M}}{\Delta\{\eta\}} = \left(\prod_{m=0}^{M-1}\frac{1}{m!}\right)\det(\tilde{f}_{M-m}(\zeta_n))_{m,n=1\ldots M}, \tag{3.18}$$

where we have defined $\tilde{f}_m(\zeta) = \frac{\partial^m}{\partial\eta^m}\mathcal{S}(\zeta-\eta)\big|_{\eta=0}$. Recalling the definition of $\mathcal{S}(\zeta-\eta)$, Eq. (3.10), we can perform the derivatives and get

$$\begin{aligned}\tilde{f}_m(\zeta) &= \frac{\partial^m}{\partial\eta^m}\frac{\exp(i\pi\,\mathrm{sgn}(\mathrm{Im}\,\zeta)(\zeta-\eta))}{\zeta-\eta}\bigg|_{\eta=0}\\ &= \exp(i\pi\,\mathrm{sgn}(\mathrm{Im}\,\zeta)\zeta)\sum_{l=0}^{m}\frac{m!}{(m-l)!}[-i\pi\,\mathrm{sgn}(\mathrm{Im}\,\zeta)]^{m-l}\zeta^{-l-1}.\end{aligned} \tag{3.19}$$

Now we insert Eq. (3.18) into (3.12) and simplify the result further: We absorb the prefactor $(\zeta_1\times\ldots\times\zeta_M)^M$ into the determinant by multiplying all entries in the first column with ζ_1^M, all in the second with ζ_2^M and so forth. In the same fashion we absorb the prefactor $\left(\prod_{m=0}^{M-1}\frac{1}{m!}\right)$, this time by multiplying all entries in the first row by $1/(M-1)!$, all in the second by $1/(M-2)!$ etc. This means we change the definition of $\tilde{f}_m(\zeta)$ to (we also change the summation index $m-l\to l$)

$$\hat{f}_{M-m}(\zeta) = \exp(i\pi\,\mathrm{sgn}(\mathrm{Im}\,\zeta)\zeta)\,\zeta^{m-1}\sum_{l=0}^{M-m}\frac{1}{l!}[-i\pi\,\mathrm{sgn}(\mathrm{Im}\,\zeta)\zeta]^l. \tag{3.20}$$

At last we go back to the original variables $\zeta_n = i\gamma x_n\rho(E)$ (see Eq. (3.7)). This introduces the factor $(i\gamma\rho)^{m-1}$ in $\hat{f}_{M-m}$, which cancels with the factor introduced by $\Delta\{i\gamma\rho X\} = (i\gamma\rho)^{M(M-1)/2}\Delta\{X\}$, which gives the final result

$$\mathcal{F}_{\beta=2}(X) = (-)^{M(M-1)/2}\,\frac{\exp\left(-\frac{i}{2}\gamma V'(E)\,\mathrm{Tr}\,X\right)}{\Delta\{X\}}\det(f_{M-m}(x_n))_{m,n=1\ldots M}, \tag{3.21}$$

with

$$f_{M-m}(x) = \exp(-\pi\gamma\rho(E)|x|)\, x^{m-1} \sum_{l=0}^{M-m} \frac{1}{l!} |\pi\gamma\rho(E)x|^l, \tag{3.22}$$

where $\mathcal{F}_{\beta=2}(X)$ is the large-N limit for the $\beta = 2$ case of Eq. (3.7). This is the first main result of this chapter and is valid for any random matrix ensemble with unitarily invariant distribution (3.8). Recalling that $\mathcal{F}_{\beta=2}(X)$ is the characteristic function of the K-matrix, the next and final step is to show that it is the characteristic function of a matrix Cauchy distribution.

3.3. Proof that K is Cauchy-Distributed

The way in which we will prove that K is Cauchy-distributed is as follows: In a first step we will calculate the characteristic function for the matrix Cauchy distribution $\mathcal{P}(K) \propto \det[\lambda^2 + (K - \epsilon)^2]^{-M}$. The second step is to show that this characteristic function is equal to the characteristic function of the K-matrix given in Eq. (3.21). Since the characteristic function uniquely determines the law of distribution one could then conclude that indeed K is matrix Cauchy-distributed.

3.3.1. Characteristic Function $\mathcal{G}(X)$ of a Matrix Cauchy Distribution

The characteristic function of $\mathcal{P}(K) = C_M \lambda^{M^2} \det[\lambda^2 + (K - \epsilon)^2]^{-M}$ (with C_M a constant independent of λ and ϵ) is given by

$$\mathcal{G}(X) = C_M \lambda^{M^2} \int \frac{e^{-i \operatorname{Tr} KX}\, \mathrm{d}K}{\det\left[\lambda^2 + (K - \epsilon)^2\right]^M}. \tag{3.23}$$

The very first step to calculate this integral is to shift $K \to K + \epsilon \mathbb{1}_M$. This shift does not change the measure and produces a factor of $\exp(-i\epsilon \operatorname{Tr} X)$. Next we diagonalise $K = UkU^{-1}$, where k is now a diagonal matrix comprising the eigenvalues of K, and U is a unitary matrix, $U^{-1} = U^\dagger$, because K is Hermitian. The measure changes accordingly to $\mathrm{d}K = \Delta^2\{k\}\, \mathrm{d}k\, \mathrm{d}\mu(U)$, where $\Delta\{k\}$ is again the Vandermonde determinant and $\mathrm{d}\mu(U)$ is the Haar measure on the unitary group. For a description how to derive this result see the paragraph after Eq. (A.10) in Appendix A.1. This leaves

us with

$$\mathcal{G}(X) = C_M \lambda^{M^2} \, e^{-i\epsilon \operatorname{Tr} X} \int \mathrm{d}k \, \frac{\Delta^2\{k\}}{\prod_{l=1}^{M} (\lambda^2 + k_l^2)^M} \int \mathrm{d}\mu(U) \, e^{-i \operatorname{Tr} UkU^\dagger X}. \tag{3.24}$$

The second integral in the above equation is the famous Itzykson-Zuber-Harish-Chandra (IZHC) integral, which can be evaluated exactly in terms of a determinant [80–82],

$$\int \mathrm{d}\mu(U) \, e^{-i \operatorname{Tr} UkU^\dagger X} = i^{M(M-1)/2} \left(\prod_{n=1}^{M-1} n! \right) \frac{\det[\exp(-ik_m x_n)]_{m,n=1\ldots M}}{\Delta\{k\}\Delta\{X\}}, \tag{3.25}$$

where the x_n are the eigenvalues of the Hermitian matrix X. Thus $\mathcal{G}$ is given by

$$\mathcal{G}(X) = \frac{\tilde{C}_{M,\lambda} \, e^{-i\epsilon \operatorname{Tr} X}}{\Delta\{X\}} \int \mathrm{d}k \, \frac{\Delta\{k\}}{\prod_{j=1}^{M} (\lambda^2 + k_j^2)^M} \det[\exp(-ik_m x_n)]_{m,n=1\ldots M}, \tag{3.26}$$

where $\tilde{C}_{M,\lambda} = i^{M(M-1)/2} \lambda^{M^2} \left(\prod_{n=1}^{M-1} n! \right) C_M$.

To make further progress we note that $\Delta\{k\}$ can be expressed as a determinant as well (see Eq. (1.6)), and make use of another identity, which is called Andréief-de-Bruijn identity (see e.g. [83]),

$$\begin{aligned} &\frac{1}{M!} \int \left(\prod_{m=1}^{M} \mathrm{d}x_m \, h(x_m) \right) \det[f_m(x_n)]_{m,n=1\ldots M} \det[g_m(x_n)]_{m,n=1\ldots M} \\ &= \det \left[\int \mathrm{d}x \, h(x) f_m(x) g_n(x) \right]_{m,n=1\ldots M}. \end{aligned} \tag{3.27}$$

Since the proof of this statement is fairly simple we will present it here: First we use the Leibniz formula to express the determinants, so the left-hand side becomes

$$\frac{1}{M!} \sum_{\sigma,\tau \in S_M} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \int \prod_{l=1}^{M} \mathrm{d}x_l \, h(x_l) f_{\sigma_l}(x_l) g_{\tau_l}(x_l), \tag{3.28}$$

where the sums are computed over all permutations σ and τ of the set $\{1, \ldots, M\}$.

Next we may rearrange terms in the product and change $\sigma = \alpha\tau$, which gives

$$\frac{1}{M!}\sum_{\alpha,\tau\in S_M}\operatorname{sgn}(\alpha)\prod_{l=1}^{M}\int \mathrm{d}x\, h(x) f_{\alpha\tau_l}(x) g_{\tau_l}(x). \tag{3.29}$$

Now we rewrite this expression again into a determinant,

$$\frac{1}{M!}\sum_{\tau\in S_M}\det\left[\int \mathrm{d}x\, h(x) f_{\tau_m}(x) g_{\tau_n}(x)\right]_{m,n=1\ldots M}. \tag{3.30}$$

The remaining sum comprises $M!$ terms (the number of permutations of τ). However, every term in the sum is identical, because the matrices generated by the sum will be identical up to a permutation of their rows and columns. The determinant remains invariant under such permutations, provided their number is even (which is here always the case since the same number of rows and columns are permuted). Performing the remaining sum hence yields the right-hand side of Eq. (3.27) which completes the proof.

Applying the identity (3.27) to our problem yields

$$\mathcal{G}(X) = \frac{\hat{C}_{M,\lambda}\, e^{-i\epsilon \operatorname{Tr} X}}{\Delta\{X\}}\det\left[(-i)^m\int_{-\infty}^{\infty}\mathrm{d}k\,\frac{k^{m-1}\exp(-ikx_n)}{(\lambda^2+k^2)^M}\right]_{m,n=1\ldots M}, \tag{3.31}$$

where we absorbed the factor $(-i)^{M(M-1)/2}$ into the determinant and redefined $\hat{C}_{M,\lambda} = \lambda^{M^2}\left(\prod_{n=1}^{M} n!\right) C_M$. The next step is to evaluate the integral featured in the determinant. First we notice than we can express the function featured in the m-th row as the $(m-1)$-th derivative of the function in the first row ($m = 1$) since

$$\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\int_{-\infty}^{\infty}\mathrm{d}k\,\frac{\exp(-ikx)}{(\lambda^2+k^2)^M} = (-i)^{m-1}\int_{-\infty}^{\infty}\mathrm{d}k\,\frac{k^{m-1}\exp(-ikx)}{(\lambda^2+k^2)^M}. \tag{3.32}$$

This means performing the integral for general m amounts to evaluate it for the case $m = 1$, where it is given by

$$g_{M,\lambda}(x) = \frac{1}{c_{M,\lambda}}\int_{-\infty}^{\infty} dk\,\frac{e^{-ikx}}{(\lambda^2+k^2)^M} = \sqrt{\tfrac{2}{\pi}}|\lambda x|^{M-\frac{1}{2}}K_{M-\frac{1}{2}}(|\lambda x|), \tag{3.33}$$

where $K_\nu(x)$ is the modified Bessel (Macdonald) function and the constant is given by $c_{M,\lambda} = \frac{\pi}{\lambda^{2M-1}2^{M-1}(M-1)!}$. In particular, for $M = 1$ we have $g_{1,\lambda}(x) = e^{-\lambda|x|}$. For higher M we may use the formula (see e.g. 8.468 in [59])

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{l=0}^{n}\frac{(n+l)!}{l!(n-l)!(2z)^l}, \quad n - \text{integer}. \tag{3.34}$$

Hence the characteristic function of a matrix Cauchy distribution with parameters λ and ϵ is given by

$$\mathcal{G}(X) = \frac{\hat{C}_{M,\lambda}\, e^{-i\epsilon \operatorname{Tr} X}}{\Delta\{X\}} c_{M,\lambda}^{M} \det\left[g_{M,\lambda}^{(m-1)}(x_n)\right]_{m,n=1\ldots M}, \tag{3.35}$$

where

$$g_{M,\lambda}(x) = e^{-\lambda|x|}\sum_{l=0}^{M-1}\frac{(M-1+l)!}{l!(M-1-l)!2^l}|\lambda x|^{M-1-l}, \quad g_{M,\lambda}^{(m)}(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}g_{M,\lambda}(x). \tag{3.36}$$

Hence the functions comprising the determinantal expression are essentially polynomials of order $M-1$ in $|\lambda x|$, multiplied by $e^{-\lambda|x|}$.

The last step to get a closed expression for $\mathcal{G}(X)$ is to find a general formula for the m-th derivative of $g_{M,\lambda}(x)$. We employ a rule for the derivative of a Bessel function,

$$\frac{\mathrm{d}}{\mathrm{d}x}K_\alpha(x) = -(K_{\alpha-1}(x) + \frac{\alpha}{x}K_\alpha(x)), \tag{3.37}$$

to compute the first derivative of $g_M(x) = \sqrt{\frac{2}{\pi}}|\lambda x|^{M-1/2}K_{M-1/2}(|\lambda x|)$ (for brevity we omit λ-dependence in the notation), which is hence given by

$$g_M'(x) = \sqrt{\tfrac{2}{\pi}}\lambda^2 x|\lambda x|^{M-\frac{3}{2}}K_{M-\frac{3}{2}}(|\lambda x|) = -\lambda^2 x g_{M-1}(x), \tag{3.38}$$

where we first used the product rule together with Eq. (3.37), and then identified $g_{M-1}(x)$ in the solution. This gives us a recursive relation to compute higher derivatives, and inductively one gets

$$g_M^{(m)}(x) = \lambda^m\sum_{l=0}^{\lfloor m/2\rfloor}\frac{m!(-1)^{m-l}}{l!(m-2l)!2^l}(\lambda x)^{m-2l}g_{M-m+l}(x), \tag{3.39}$$

where $\lfloor\cdot\rfloor$ denotes the floor-function. A proof of this equation can be found in Appendix A.3.

Instead of derivatives of $g_M(x)$, the determinant features now sums of the form (3.39). However, these sums are still unhandy as each entry in the m-th row is composed of $\lfloor m/2\rfloor + 1$ different functions $g_j(x)$, $M - m \leq j \leq M - m + \lfloor m/2\rfloor$. It would be desirable that each entry only depends on a single function $g_j(x)$. In particular we want to show that we can bring the determinant to the form

$$\det\left[g_M^{(m-1)}(x_n)\right]_{m,n=1\ldots M} = C_{M,\lambda}\det\left[x_n^{m-1}g_{M-m+1}(x_n)\right]_{m,n=1\ldots M}, \tag{3.40}$$

where $C_{M,\lambda}$ is some proportionality constant independent of the x_n.

In order to achieve this goal, we first employ yet another recurrence relation of the Bessel functions, $K_{\alpha+1}(x) = \frac{2\alpha}{x}K_\alpha(x) + K_{\alpha-1}(x)$. Using this relation on $g_M(x)$ yields

$$\begin{aligned} g_M(x) &= \sqrt{\tfrac{2}{\pi}}|\lambda x|^{M-1/2}\left(\frac{2M-3}{|\lambda x|}K_{M-3/2}(|\lambda x|) + K_{M-5/2}(|\lambda x|)\right) \\ &= (2M-3)g_{M-1}(x) + (\lambda x)^2 g_{M-2}(x). \end{aligned} \tag{3.41}$$

We claim that this relation implies that for every $i, j \in \mathbb{N}$ with $i \leq j$ the following equation holds,

$$(\lambda x)^i g_{M-j}(x) = \sum_{l=i}^{2j-i} a_{M,i,j,l}\,(\lambda x)^l g_{M-l}(x), \tag{3.42}$$

where $a_{M,i,j,l}$ are some coefficients independent of x and λ. More specifically, since $g_{M-j}(x)$ is an even function, the coefficients vanish for all even (odd) l when i is odd (even). Eq. (3.42) can be shown as follows: We first apply the relation (3.41) on $g_{M-j}(x)$, which gives an expression featuring g_{M-j+1} and g_{M-j-1}, viz, $g_{M-j} = \frac{1}{2(M-j+1)-3}(g_{M-j+1} - (\lambda x)^2 g_{M-j-1})$. Now in this expression we replace again both functions using the same relation (3.41). Repeating this procedure for s steps we will end up with the expression

$$g_{M-j}(x) = \sum_{l=0}^{s} a_l\,(\lambda x)^{2l} g_{M-j+s-2l}(x), \tag{3.43}$$

where we omit dependence of the coefficient on other variables but l for brevity. This

relation can be shown most easily by induction: For the first step, $s = 1$, the above formula yields $a_0 g_{M-j+1}(x) + a_1(\lambda x)^2 g_{M-j-1}(x)$ which is of the correct form. For the $s+1$ step we replace $g_{M-j+s-2l}(x)$ in Eq. (3.43) using (3.41), rearrange the sums and redefine the coefficients,

$$\begin{aligned} g_{M-j}(x) &= \sum_{l=0}^{s} a_l(\lambda x)^{2l} b_l \left[g_{M-j+s+1-2l}(x) + (\lambda x)^2 g_{M-j+s-1-2l}(x) \right] \\ &= \sum_{l=0}^{s} a_l b_l (\lambda x)^{2l} g_{M-j+s+1-2l}(x) + \sum_{l=1}^{s+1} a_{l-1} b_{l-1} (\lambda x)^{2l} g_{M-j+s+1-2l}(x) \\ &= \sum_{l=0}^{s+1} \tilde{a}_l (\lambda x)^{2l} g_{M-j+s+1-2l}, \quad \tilde{a}_0 = a_0 b_0, \tilde{a}_{s+1} = a_s b_s, \tilde{a}_l = a_l b_l + a_{l-1} b_{l-1}, \end{aligned} \tag{3.44}$$

where $b_l = 1/(2(M-j+s+1-2l)-3)$, and thus Eq. (3.43) follows by induction. After performing $s = j - i$ steps, Eq. (3.43) takes the form

$$g_{M-j}(x) = \sum_{l=0}^{j-i} a_l \, (\lambda x)^{2l} g_{M-i-2l}(x). \tag{3.45}$$

Multiplying this equation by $(\lambda x)^i$ and then changing $2l + i \to l$ finally yields Eq. (3.42).

Let us now look at the m-th row (we start to count the rows from 0 onwards) of the matrix in (3.35). For simplicity we choose m to be even, but the same argument presented applies also to odd m. Then the n-th entry of this row, according to Eq. (3.39) is the sum

$$\lambda^m \sum_{l=0}^{m/2} c_{m,l} (\lambda x_n)^{2l} g_{M-m/2-l}(x_n), \tag{3.46}$$

where we have changed $l \to m/2 - l$ compared to (3.39) and abbreviated the coefficients with $c_{m,l}$. Now let us assume that all entries in the j rows above the m-th row ($j < m$) are already of the desired form $x_n^{j-1} g_{M-j+1}(x_n)$ given in (3.40). Now we may add to the m-th row a multiple of the zeroth row, a multiple of the second row and so forth without changing the value of the determinant (this is one of the basic properties a determinant has). More specifically we may add those rows together such

that

$$\sum_{l=0}^{m-2} a_l \, (\lambda x)^l g_{M-l}(x) = g_{M-m/2+1}(x), \tag{3.47}$$

i.e. we choose the a_l such that the sum fulfils Eq. (3.42) with $i = 0$ and $j = m/2 - 1$. Hence we can add any multiple of $g_{M-m/2+1}(x_n)$ to each entry of the m-th row without changing the determinant. The multiple we choose is $-\frac{c_{m,0}\lambda^m}{2(M-m/2+1)-3} g_{M-m/2+1}(x_n)$, because with Eq. (3.41) we can then recast the $l = 0$ term from (3.46) into the form

$$\begin{aligned} & c_{m,0}\lambda^m \left(g_{M-m/2}(x_n) - \frac{1}{2(M-m/2+1)-3} g_{M-m/2+1}(x_n) \right) \\ & = -\frac{c_{m,0}\lambda^m}{2(M-m/2+1)-3} (\lambda x_n)^2 g_{M-m/2-1}(x_n). \end{aligned} \tag{3.48}$$

We see that this is – up to the prefactor – the $l = 1$ term of (3.46), i.e. we can combine the old $l = 1$ term with the rewritten $l = 0$ term. This only changes the factor $c_{m,1}$ in the expression (3.46), however, by this procedure we have effectively removed the $l = 0$ term. With the same arguments we can now remove the $l = 1$ term by adding an appropriate multiple of $(\lambda x_n)^2 g_{M-m/2}(x_n)$ to each entry in the m-th row. This is again possible since we can combine the previous rows such that Eq. (3.42) is fulfilled with $i = 2$ and $j = m/2$. More generally we may add any multiple of the form $(\lambda x_n)^{2l} g_{M-m/2-l+1}(x_n)$ with $0 \leq l \leq (m-2)/2$ to the m-th row, and thus we can successively eliminate every term from the sum in (3.46) but the last one, which is proportional to $(\lambda x_n)^m g_{M-m}(x_n)$. However, this is exactly the desired form for the m-th row, compare with Eq. (3.40). Now all terms above the $(m+1)$-th row are of the form given in (3.40), thus we can use the same method again to bring this row, and successively all other rows below, into the desired form. Since the zeroth and first row of our matrix are already of the form given in (3.40) from the beginning, we can apply the procedure given above starting from the second row onwards.

The last missing piece is the proportionality constant $C_{M,\lambda}$ in (3.40). It is not feasible to track it when applying the above described procedure, hence we use a different method. First we note that every matrix entry in the n-th column has the common factor $\exp(-\lambda x_n)$ on both sides of Eq. (3.40). Hence by multiplying both sides of the equation with $\exp\left(\lambda \sum_{n=1}^{M} x_n\right)$ we can eliminate this dependence. This essentially means that we redefine $\tilde{g}_m(x) = \exp(\lambda x) g_m(x)$. Now let us take the

derivative $\frac{\partial}{\partial x_2}\frac{\partial^2}{\partial x_3^2}\dots\frac{\partial^{M-1}}{\partial x_M^{M-1}}$ on both sides and then evaluate the ensuing expression for all $x_n = 0$. Let us start with the left-hand side. Taking this derivative amounts to differentiate every entry in the second column once, every entry in the third column twice and so forth (see Eq. (3.16) and the paragraph below it). However, recall from definition (3.36) that $\tilde{g}_M(x)$ is a polynomial of degree $M-1$, and hence $\tilde{g}_M^{(M+k)}(x) = 0$ for $k \geq 0$. This means the determinant on the left-hand side takes a quasi-triangular form

$$\det \begin{bmatrix} \tilde{g}_M(0) & \tilde{g}'_M(0) & \dots & \tilde{g}_M^{(M-1)}(0) \\ \tilde{g}'_M(0) & \tilde{g}''_M(0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{g}_M^{(M-2)}(0) & \tilde{g}_M^{(M-2)}(0) & \dots & 0 \\ \tilde{g}_M^{(M-1)}(0) & 0 & \dots & 0 \end{bmatrix}. \tag{3.49}$$

By interchanging the first with the last column, the second with the second last and so forth we can bring it to an upper triangular form. This interchanging yields a factor of $(-)^{M(M-1)/2}$. The value of the determinant is then given by the product of its diagonal and hence the left-hand side becomes

$$(-)^{M(M-1)/2}\left[\tilde{g}_M^{(M-1)}(0)\right]^M. \tag{3.50}$$

Next we take the same derivative on the right-hand side. Again taking the derivative amounts to differentiate every entry in the n-th column $n-1$ times and then evaluate at $x_n = 0$. The entries in the first column are of the form $x_1^{m-1}\tilde{g}_{M-m+1}(x_1)$, where $1 \leq m \leq M$ denotes the row. When evaluated at 0, only the first entry ($m = 1$) is non-vanishing. Similar in the second column, if we differentiate once and then evaluate at 0, only the first two entries will be non vanishing. More generally, the n-th column will take the form

$$[\tilde{g}_M^{(n-1)}(0), \tfrac{(n-1)!}{(n-2)!}\,\tilde{g}_{M-1}^{(n-2)}(0), \tfrac{(n-1)!}{(n-3)!}\,\tilde{g}_{M-2}^{(n-3)}(0), \dots, (n-1)!\,\tilde{g}_{M-n+1}(0), 0, \dots, 0]^T. \tag{3.51}$$

Hence the ensuing determinantal expression will feature a triangular matrix, where the diagonal entries are given by $(m-1)!\,\tilde{g}_{M-m+1}(0)$, and hence the determinant

evaluates to $\prod_{m=1}^{M}(m-1)!\,\tilde{g}_m(0)$. Thus we have transformed Eq. (3.40) into

$$(-)^{M(M-1)/2}\left[\tilde{g}_M^{(M-1)}(0)\right]^M = C_{M,\lambda}\prod_{m=1}^{M}(m-1)!\,\tilde{g}_m(0). \tag{3.52}$$

Recall that $\tilde{g}_M(x)$ is a polynomial of degree $M-1$ given by $e^{\lambda x}$ times (3.36). Hence its $(M-1)$-th derivative is given by $\lambda^{M-1}(M-1)!$, where we used that the highest order coefficient is given by λ^{M-1}. On the other hand, when evaluated at 0, only the zeroth-order term of $g_m(x)$ survives and thus $\tilde{g}_m(0) = \frac{(2m-2)!}{(m-1)!2^{m-1}}$. This enables us to calculate the constant in (3.40), and hence we have shown

$$\begin{aligned}\det\left[g_{M,\lambda}^{(m-1)}(x_n)\right]_{m,n=1\ldots M} =&(-2\lambda^2)^{\frac{M(M-1)}{2}}\left(\prod_{m=1}^{M}\frac{(M-1)!}{(2m-2)!}\right)\\ &\times\det\left[x_n^{m-1}g_{M-m+1}(x_n)\right]_{m,n=1\ldots M}.\end{aligned} \tag{3.53}$$

With Eq. (3.35), collecting and combining all constants, the characteristic function of a matrix Cauchy distribution is thus given by

$$\mathcal{G}(X) = \left(\prod_{m=0}^{M-1}\frac{2^m\,m!}{(2m)!}\right)\frac{\exp(-i\epsilon\,\mathrm{Tr}\,X)}{\Delta\{X\}}\det\left[x_n^{m-1}g_{M-m+1}(x_n)\right]_{m,n=1\ldots M}, \tag{3.54}$$

where the combination involved in the m-th row in the right-hand side is given explicitly by

$$x^{m-1}g_{M-m+1}(x) = e^{-\lambda|x|}x^{m-1}\sum_{l=0}^{M-m}\frac{(2M-2m-l)!}{l!(M-m-l)!2^{M-m-l}}|\lambda x|^l. \tag{3.55}$$

Compare this result with the characteristic function $\mathcal{F}(X)$ given in Eq. (3.21). We observe a striking similarity if we identify $\lambda = \pi\gamma\rho(E)$ and $\epsilon = \gamma V'(E)/2$ (recall that λ and ϵ are the two parameters which determine centre and width of the Cauchy distribution). In this case, they have a similar X-dependence, though the coefficients of the terms in the sum are still different. In fact this similarity can be further exploited to show that the determinants in the expressions for $\mathcal{F}(X)$ and $\mathcal{G}(X)$ are proportional to each other, thus verifying the claim that K is Cauchy distributed. This will be done in the next section.

3.3.2. Proof that $\mathcal{G}(X) = \mathcal{F}(X)$

We start our demonstration with bringing the first row of the determinant in Eq. (3.21) to the form coinciding with the first row of the determinant in (3.54). These rows feature the functions $f_{M-1}(x)$ (see Eq. (3.22)) and $g_M(x)$ (see Eq. (3.55)), respectively, given by

$$f_{M-1}(x) = e^{-\lambda|x|}\sum_{l=0}^{M-1}\frac{1}{l!}|\lambda x|^l, \quad g_M(x) = e^{-\lambda|x|}\sum_{l=0}^{M-1}\frac{(2M-2-l)!}{l!(M-1-l)!2^{M-1-l}}|\lambda x|^l, \tag{3.56}$$

where we chose $\lambda = \pi\gamma\rho(E)$. The zeroth and the first order coefficients of $f_{M-1}(x)$ are both equal to unity, and the two corresponding coefficients in the expression for $g_M(x)$ are equal as well (but different from unity) since

$$a_0 = \frac{(2M-2)!}{(M-1)!2^{M-1}} = \frac{2(M-1)(2M-3)!}{(M-1)!2^{M-1}} = \frac{(2M-3)!}{(M-2)!2^{M-2}} = a_1. \tag{3.57}$$

Hence we can safely change those coefficients in $f_{M-1}(x)$ to the coefficients in $g_M(x)$ as such a change gives rise to a constant proportionality factor for the determinant. Next we may consider adding subsequent rows to the first row as this does not change the value of the determinant. Note in particular that the zeroth and first order coefficient of $f_{M-3}(x)$ in Eq. (3.22) are vanishing. Hence we may add a multiple of the third row to the first row without changing the already adjusted coefficients of $f_{M-1}(x)$. More generally the first $n-1$ coefficients in $f_{M-n}(x)$, n odd (see Eq. (3.22)) are vanishing. This means we can add that particular function to $f_{M-1}(x)$ without changing its first $n-1$ coefficients. Hence, assuming that we can adjust two coefficients at once with each step, we can successively add odd rows until all coefficients are adjusted.

The main observation is that the adjustment of both the coefficients a_{2n} and a_{2n+1}, given that all previous coefficients are already adjusted, can indeed be done simultaneously by adding the $(2n+1)$-th row multiplied with the factor

$$c_n = (-1)^n\frac{(2M-2n-2)!}{n!(M-n-1)!2^{M-1}}. \tag{3.58}$$

In other words this means that the $(2n)$-th and $(2n+1)$-th coefficient of the sum $\sum_{l=0}^{n} c_l f_{M-(2l+1)}(x)$ (which is what becomes of the function $f_{M-1}(x)$ after multiplying

it with c_0 and adding multiples of all functions featured in the odd rows up to $2n+1$, choosing the multiplication factors according to (3.58)) should be equal to the corresponding coefficients of $g_M(x)$. Hence, for our procedure to work we need to verify, for any integer $0 \leq n \leq \lfloor \frac{M-1}{2} \rfloor$, the following identity:

$$\sum_{l=0}^{n} (-1)^l \frac{(2M-2l-2)!}{l!(M-l-1)!2^{M-1}} \frac{1}{(2n+\delta-2l)!} = \frac{(2M-2n-\delta-2)!}{(2n+\delta)!(M-2n-1)!2^{M-2n-\delta-1}}, \tag{3.59}$$

with $\delta = 0$ (corresponding to the $(2n)$-th order coefficient) or $\delta = 1$ (corresponding to the $(2n+1)$-th order coefficient). Both equations can be conveniently combined into a single relation:

$$\sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l \binom{2\mathcal{M}-2l}{m-2l} \binom{\mathcal{M}}{l} = 2^m \binom{\mathcal{M}}{m}, \tag{3.60}$$

where $\mathcal{M} = M-1$ and $m = 2n$ or $m = 2n+1$. To verify (3.60) we first note that we can identify $\binom{2\mathcal{M}-2l}{m-2l}$ as the coefficient of z^{2m-2l} in $(1+z)^{2\mathcal{M}-2l}$. Cauchy's theorem then implies

$$\binom{2\mathcal{M}-2l}{m-2l} = \frac{1}{2\pi i} \oint_\gamma dz \, \frac{(1+z)^{2\mathcal{M}-2l}}{z^{m-2l+1}}, \tag{3.61}$$

where γ is a small loop around the origin. Let us denote the left-hand side of (3.60) by $S_{\mathcal{M},m}$. Inserting (3.61) we get

$$\begin{aligned} S_{\mathcal{M},m} &= \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l \frac{1}{2\pi i} \oint_\gamma dz \, \frac{(1+z)^{2\mathcal{M}-2l}}{z^{m-2l+1}} \binom{\mathcal{M}}{l} \\ &= \frac{1}{2\pi i} \oint_\gamma dz \, \frac{(1+z)^{2\mathcal{M}}}{z^{m+1}} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l \left(\frac{z}{1+z} \right)^{2l} \binom{\mathcal{M}}{l}. \end{aligned} \tag{3.62}$$

To perform the sum, note that terms with $l > \lfloor m/2 \rfloor$ do not contribute because the integrand in (3.61) becomes analytic for that case (so the integral becomes zero). This means we can sum all the way up to $\mathcal{M}$ (remember that m is related to one specific row in an $(\mathcal{M}+1) \times (\mathcal{M}+1)$ matrix and thus always $\lfloor m/2 \rfloor < \mathcal{M}$) without changing

the integral. The sum then follows from the standard binomial formula,

$$S_{\mathcal{M},m} = \frac{1}{2\pi i}\oint_\gamma dz\,\frac{(1+z)^{2\mathcal{M}}}{z^{m+1}}\left(1-\frac{z^2}{(1+z)^2}\right)^{\mathcal{M}}$$
$$= \frac{1}{2\pi i}\oint_\gamma dz\,\frac{(1+z)^{2\mathcal{M}}}{z^{m+1}}\left(\frac{1+2z}{(1+z)^2}\right)^{\mathcal{M}} = \frac{1}{2\pi i}\oint_\gamma dz\,\frac{(1+2z)^{\mathcal{M}}}{z^{m+1}}. \tag{3.63}$$

The last integral is the coefficient of z^m in $(1+2z)^{\mathcal{M}}$, which is precisely the right-hand side of (3.60).

We thus have shown that the identity (3.59) is valid for any integer $0 \leq n < M-1$ (this follows from the condition $\lfloor m/2 \rfloor < \mathcal{M}$) and conclude that it is indeed possible to transform $f_{M-1}(x)$ into $g_M(x)$ by adding multiples of all odd rows to the first row. Note that in each step two coefficients get adjusted simultaneously (with the exception of the last step in the case of odd M, where only the highest order coefficient gets adjusted), and this is precisely the mechanism ensuring the whole procedure being functional. Had it not been for that property, we would be only able to change half of the coefficients to the required form, since adding even rows to odd rows or vice versa is meaningless due to their rather different structure (Functions in the odd rows are of the form $e^{-\lambda|x|}$ times a polynomial in $|x|$. Function in even rows on the other hand take the form $e^{-\lambda|x|}$ times $\operatorname{sgn} x$ times a polynomial in $|x|$).

All remaining odd rows as well as all even rows can be treated by exactly the same procedure. This is because $f_{M-n}(x)$, for any n, has the same coefficients as $f_{M-1}(x)$ but shifted, such that the $(n-1+j)$-th coefficient of $f_{M-n}(x)$ is the same as the j-th coefficient of $f_{M-1}(x)$ for all $0 \leq j \leq M-n$. The first $n-2$ coefficients of $f_{M-n}(x)$ are all vanishing. The same applies for the coefficients of $x^{m-1}g_{M-m+1}(x)$, and hence any odd (even) row can be adjusted by adding all subsequent odd (even) rows according to the above described procedure.

Note also that as the very last row contains on both sides the function $e^{-\lambda|x|}x^{n-1}$ the coincidence is ensured automatically.

We have thus shown that the determinant featured in the expression for $\mathcal{F}_{\beta=2}(X)$, Eq. (3.21), can be transformed to the determinant featured in the expression for $\mathcal{G}(X)$, Eq. (3.54) and hence both functions are proportional for the appropriate choice of λ and ϵ in $\mathcal{G}(X)$. Moreover normalisation ensures $\mathcal{F}(0) = \mathcal{G}(0) = 1$ and hence we have

shown

$$\mathcal{F}_{\beta=2}(X) = \mathcal{G}(X), \tag{3.64}$$

if we choose $\lambda = \pi\gamma\rho(E)$ and $\epsilon = \gamma V'(E)/2$. Equation (3.64) and its implications constitute the main result of this chapter.

Recall that $\mathcal{G}(X)$ is the characteristic function of a matrix Cauchy distribution with free parameters λ and ϵ, while $\mathcal{F}_{\beta=2}(X)$ is the characteristic function of the distribution of the K-matrix with equivalent channels characterised by the channel factor γ and energy E, obtained via the Hamiltonian (Heidelberg) approach, where the joint probability density of the Hamiltonian is of the form $\mathcal{P}(H) \propto \exp\left[-N \operatorname{Tr} V(H)\right]$ with $V(H)$ an even polynomial.

Since the characteristic function uniquely determines the law of distribution, we can conclude that the distribution of the K-matrix (1.20) converges in the limit $N \to \infty$ to the matrix Cauchy distribution with density $\mathcal{P}_{\beta=2}(K)$ (1.31) having mean $\epsilon = \gamma V'(E)/2$ and width $\lambda = \pi\gamma\rho(E)$. With Eq. (A.16), this corresponds to the Poisson kernel distribution (1.30) for the S-matrix with mean

$$\overline{S}_{ij} = \frac{1 - \gamma[\pi\rho(E) + iV'(E)/2]}{1 + \gamma[\pi\rho(E) + iV'(E)/2]}\delta_{ij}, \tag{3.65}$$

in complete agreement with the mean found in [8] for the Gaussian case, compare with Eq. (1.21) for $V(H) = H^2/(2J^2)$ and $\rho(E)$ given by the semicircle (1.9). The case of perfect coupling (where $\overline{S}_{ij} = 0$) is then obtained for $V'(E_{max}) = 0$ and $\pi\gamma\rho(E_{max}) = 1$, where E_{max} denotes the point where $\rho(E)$ has its maximum. Thus indeed, the Poisson kernel distribution for the S-matrix is universal in the random amplitude model (1.27,1.28) in that it does not depend on the choice of the random matrix ensemble for the underlying matrix H.

3.4. Universality in the Fixed Amplitude Model

So far all our previous considerations have been made for the random amplitude model (1.27,1.28). Finally, we would like to demonstrate that the fixed amplitude model (1.25) yields the same universal behaviour of the K-matrix in the limit $N \to \infty$.

Let us again consider the characteristic function given in Eq. (3.2), however, now the average is only over the random matrix H since the coupling matrices are fixed. We diagonalise $H = U\Lambda U^\dagger$ and use the invariance of the trace under cyclic permutation, such that

$$\mathcal{F}_{\beta=2,N}(X) = \left\langle \exp(-i \operatorname{Tr} \Gamma_x U R_\Lambda U^\dagger) \right\rangle_H , \tag{3.66}$$

with $\Gamma_x = WXW^\dagger$ and $R_\Lambda = (E - \Lambda)^{-1}$. Here U is the unitary matrix of eigenvectors of H and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ stands for the diagonal matrix of the corresponding eigenvalues. The averaging over H then can be performed in two steps, the first step being the averaging over the Haar measure on the unitary group $U(N)$. As this is again a special case of the IZHC integral (see Eq. (3.25)) it can be done explicitly. The important new feature however is that the $N \times N$ matrix Γ_x is of a reduced rank, with its $M \ll N$ non-zero eigenvalues coinciding with the eigenvalues γx_c, $c = 1, \ldots, M$ of the matrix $XW^\dagger W = \gamma X$ (here we used the fixed amplitude condition (1.25)), the rest of its $N - M$ eigenvalues being exactly zero. At the same time the resolvent matrix R_Λ is of full rank N. The problem of performing the IZHC integral for two matrices of different rank can be most efficiently done by employing equation (A4) of the Appendix A in the paper [84] (which is in fact closely related to the so-called duality IZHC relation, see equation (17.3.8) in [82]). In our case it takes the form:

$$\left\langle \exp\left(-i \operatorname{Tr} \Gamma_x U R_\Lambda U^\dagger\right) \right\rangle_U \propto \frac{\det X^{M-N}}{\Delta\{X\}} \int_{\mathcal{C}_\Gamma} \Delta\{Y\} \prod_{c=1}^{M} \frac{e^{-i\gamma x_c y_c}}{\det\left(y_c - R_\Lambda\right)} \, \mathrm{d}y_1 \ldots \mathrm{d}y_M, \tag{3.67}$$

where the integration goes over the complex variables $y_1, \ldots, y_M$ along contours parallel to the real axis such that $\operatorname{sgn}(\operatorname{Im} y_c) = \operatorname{sgn}(x_c)$.

Now we should perform the next step of the ensemble average over the eigenvalues Λ of H entering via the resolvent R_Λ. After rescaling $y_c \to N y_c$ and a simple rearranging in the integrand we can see that the eigenvalue-averaged right-hand side of (3.67) is

proportional to (with proportionality constant $N^{M(M+1)/2-N}$)

$$\frac{\det X^{M-N}}{\Delta\{X\}} \int_{\mathcal{C}_\Gamma} \Delta\{Y\} e^{-N\sum_c (i\gamma x_c y_c + \ln y_c)} \left\langle \prod_{c=1}^{M} \frac{\det(E-\Lambda)}{\det\left(E - \frac{1}{Ny_c} - \Lambda\right)} \right\rangle_\Lambda dy_1 \dots y_M. \quad (3.68)$$

In the limit $N \to \infty$ the integrals over y_c can be straightforwardly evaluated by the saddle-point method developed in Section 2.1. For this method to work we need the expected value in the integrand to be of order unity, because then the saddle-point values would be given by the requirement that the derivative of $i\gamma x_c y_c + \ln y_c$ vanishes, hence $y_c^{(s.p)} = \frac{i}{\gamma x_c}$. However, this is justified as Eq. (3.9) ensures that said expected value in the integrand tends for $N \to \infty$ to a well-defined limit of the order of unity along contours in the vicinity of the chosen saddle point. Moreover the singularities of the integrand are given by $y_c = (N(E-\lambda_j))^{-1}$ and hence lie all on the real axis. The original contour of integration is parallel to the real axis with $\mathrm{sgn}(\mathrm{Im}\, y_c) = \mathrm{sgn}(x_c)$. This means the contour and the saddle point lie in the same half plane (upper half plane if $x_c > 0$, lower half plane if $x_c < 0$) and hence we can deform the original contours without crossing any singularities of the integrand. For each y_c-integration we shift the contour to a parallel of the real axis going through the saddle point $\frac{i}{\gamma x_c}$, evaluate the part of the integrand which is of order unity at the saddle point, expand the exponential to second order around the saddle point and perform the Gaussian integrals. This yields for the $N \to \infty$ approximation of the integral in (3.68)

$$\Delta\{Y^{(s.p)}\} \left\langle \prod_{c=1}^{M} \frac{\det(E-\Lambda)}{\det(E + i\gamma x_c/N - \Lambda)} \right\rangle_\Lambda \prod_{c=1}^{M} \left[\sqrt{2\pi} N^{-1/2}\, i^{-N} e^N (\gamma x_c)^{N-1}\right]. \quad (3.69)$$

The Vandermonde determinant evaluated at the saddle point is given by $\Delta\{Y^{(s.p)}\} = (i/\gamma)^{M(M-1)/2}\Delta\{1/X\}$, and furthermore

$$\Delta\left\{\frac{1}{X}\right\} = \prod_{i<j}\left(\frac{1}{x_i} - \frac{1}{x_j}\right) = \Delta\{X\}\prod_{i<j}\frac{1}{x_i x_j} = \Delta\{X\}\prod_{c=1}^{M}\frac{1}{x_c^{M-1}} = \Delta\{X\}\det X^{1-M}. \quad (3.70)$$

Together with $\prod_{c=1}^{M} x_c^{N-1} = \det(X)^{N-1}$, which is a term that arises from the Gaussian fluctuations around the saddle point, one gets the prefactor $\det(X)^{N-1}\Delta\{Y^{(s.p)}\} = \Delta\{X\}\det X^{N-M}$, which cancels precisely the term in front of the integral in (3.68).

Moreover, since the expected value in (3.69) is invariant under unitary rotation, we may replace Λ by H and taking all these facts together we hence get for the fixed amplitude model with $M \ll N$

$$\mathcal{F}_{\beta=2,N\to\infty}(X) = \left\langle \prod_{c=1}^{M} \frac{\det(E-H)}{\det(E+i\gamma x_c/N - H)} \right\rangle_H . \tag{3.71}$$

This is precisely the expression (3.7) which was exact in the random amplitude model. Hence one can conclude that also in the fixed amplitude model (1.25) the K-matrix in the limit $N \to \infty$ has the Cauchy distribution with density (1.31).

This result has an interesting corollary. If the $\boldsymbol{w}_c$ are chosen to be the first M columns of the $N \times N$ identity matrix, then $W^\dagger (E-H)^{-1} W$ is nothing else as the $M \times M$ block of the resolvent $(E-H)^{-1}$. Therefore for invariant ensembles of Hermitian random matrices H, finite blocks of the resolvent of H are Cauchy-distributed in the limit of large matrix dimension.

4. Random Matrix Averages Involving Half-Integer Powers

The goal of this chapter is to systematically evaluate large-N asymptotics of random matrix averages of the form

$$\mathcal{C}_{k,l}(\mu_{F1},\ldots,\mu_{Fk};\mu_{B1},\ldots,\mu_{Bl}) = \left\langle \frac{\det(\mu_{F1}-H)\ldots\det(\mu_{Fk}-H)}{\det(\mu_{B1}-H)^{1/2}\ldots\det(\mu_{Bl}-H)^{1/2}} \right\rangle_{\mathrm{GOE}} \tag{4.1}$$

where μ_{Fi}, $i=1,\ldots,k$ and μ_{Bj}, $j=1,\ldots,l$ are sets of complex parameters. This work is published in [14]. For technical reasons we consider *integer* powers in the numerator and *half-integer* powers in the denominator, but note that the correlation functions involving products of square roots of the characteristic polynomials in the numerator can be always reduced to the above form by multiplying and dividing both the numerator and the denominator with the same corresponding factors.

For similar objects involving only integer powers it has been discovered [71, 76–78, 85–87] that they show a determinantal ($\beta=2$) or Pfaffian ($\beta=1,4$) structure.[26] Although there are reasons to suspect that the correlation functions (4.1) may have a nice mathematical structure even for finite N as well, we are not able to reveal such structures beyond the simplest case $k=1$, $l=1$, see Section 4.2 and in particular (4.21) below. Instead we are mainly concentrating on the large-N limit of a few simplest, yet non-trivial examples of the correlation function of the type (4.1).

As it should be clear from the list of examples which – along with some further motivation for the problem – we give in Section 4.1, the most physically interesting

[26] Compare also with Section 3.2.1 where this has been used to compute the characteristic function of the probability density of the K-matrix for $\beta=2$. See in particular Eq. (3.9) which shows the determinantal structure of the universal large-N limit of such a correlation function. For finite N the correlation function can be expressed as a determinant of orthogonal polynomials and their Cauchy transforms.

(bulk) scaling regime in the large-N limit arises when all spectral parameters are close to some value $E \in (-2J, 2J)$ by a distance of the order of the mean spacing between neighbouring eigenvalues in the bulk, i.e. $\mathcal{O}(J/N)$. Correspondingly we define the scaled version of the correlation function as

$$\mathcal{C}_{k,l}^{(\mathrm{bulk})}(\omega_{F1}, \ldots, \omega_{Fk}; \omega_{B1}, \ldots, \omega_{Bl}) \approx \left\langle \frac{\det(E + i\omega_{F1}/N - H) \ldots \det(E + i\omega_{Fk}/N - H)}{\det(E + i\omega_{B1}/N - H)^{1/2} \ldots \det(E + i\omega_{Bl}/N - H)^{1/2}} \right\rangle_{\mathrm{GOE},\, N\to\infty}, \tag{4.2}$$

where the approximate equality sign above should be understood in the sense of extracting the leading asymptotic dependence on the parameters when $N \to \infty$.

After deriving a determinantal structure for $\mathcal{C}_{1,1}$ in Section 4.2, we consider correlation functions with two square roots in the denominator, and with one or two characteristic polynomials in the numerator, that is $\mathcal{C}_{1,2}(\mu_{F1}; \mu_{B1}, \mu_{B2})$ (Section 4.3) and $\mathcal{C}_{2,2}(\mu_{F1}, \mu_{F2}; \mu_{B1}, \mu_{B2})$ (Section 4.4). The results for their large-N asymptotics are given in Eqs. (4.45) and (4.67) for $\mathcal{C}_{1,2}^{(\mathrm{bulk})}$ and in Eqs. (4.131) and (4.135) for $\mathcal{C}_{2,2}^{(\mathrm{bulk})}$. We then try to compute the correlation function involving four square roots in the denominator, and two determinants in the numerator, that is $\mathcal{C}_{2,4}$, in Section 4.5. While Section 4.5.1 serves only to illustrate the problems which arise when trying to treat the general case, preventing us from getting a meaningful result, Section 4.5.2 shows the derivation of the special case

$$\mathcal{C}_{2,4}^{(\mathrm{bulk})}(0; 0, \omega_{B1}, -\omega_{B1}, \omega_{B2}, -\omega_{B2}) \approx \left\langle \frac{\det H^2}{\det(H^2 + \frac{\omega_{B1}^2}{N^2})^{1/2} \det(H^2 + \frac{\omega_{B2}^2}{N^2})^{1/2}} \right\rangle_{\mathrm{GOE},\, N\to\infty}, \tag{4.3}$$

which leads to the result given in (4.186), and for the case $\omega_{B2} \equiv 0$ further simplifies to the result in (4.188). These objects are already rich enough to provide answers for quantities arising in applications of random matrices in the field of Quantum Chaos in closed and open (scattering) systems. Such relations are shown in Section 4.6.

4.1. Motivation

To explain the origin of interest in the correlation functions (4.1) we start with recalling that the phenomenon of Quantum Chaos attracted considerable theoretical and experimental interest for more than three decades and remains one of the areas where applications of Random Matrix Theory are most fruitful and successful [18]. As explained in Chapter 1, the applications are based on the famous Bohigas-Giannoni-Schmit conjecture [39] and the universality of many random matrix properties. As a consequence one of the common strategies for predicting universal observables of quantum chaotic systems has been expressing them in terms of resolvents of underlying Hamiltonians, then replacing the actual Hamiltonians by random matrices taken from analytically tractable (usually, Gaussian) ensembles of $N \times N$ random matrices.

The characteristic functions of the probability densities of the observables under consideration can be frequently computed explicitly by appropriate ensemble averages. Note that the eigenvalues of unitary ($\beta = 2$), orthogonal ($\beta = 1$) or symplectic ($\beta = 4$) ensembles are independent of the eigenvectors, with the matrix of N orthonormal eigenvectors being uniformly distributed over the Haar's measure of the Unitary $U(N)$, Orthogonal $O(N)$ or Symplectic $Sp(2N)$ group, correspondingly. To that end it is natural to evaluate the corresponding characteristic functions by performing first the ensemble average over the eigenvectors. For the $\beta = 2$ case the average can be frequently done exactly for any N by employing the Itzykson-Zuber-Harish-Chandra [80,81] formula[27], which is not yet available for $\beta = 1, 4$ group averages. Nevertheless, one is able to perform the eigenvector averages in the limit $N \gg 1$ by using a heuristic idea (going back to [88]) that the set of eigenvectors essentially behaves for $N \gg 1$ as if their components were independent, identically distributed Gaussian variables with mean zero and variance $1/N$. One can rigorously justify this procedure if only a number $n \ll N^{1/2}$ of eigenvectors is involved in the set, see e.g. [89], but in general a rigorous justification of such a step requires some non-trivial estimates on the resolvents. The heuristic procedure is widely employed in Theoretical Physics for RMT applications to Quantum Chaos using the properties of the standard Gaussian integrals over complex or real variables. In this way the analysis of many distributions of practical interest is reduced to correlation functions

[27] The derivation of (3.71) in Section 3.4 is an example for such a calculation.

of products and ratios involving *integer* (for $\beta = 2, 4$) or *half-integer* (for $\beta = 1$) powers of characteristic polynomials of random matrices.

Similar averages arise if one is interested in statistics of the matrix elements of the resolvents computed in the basis of random Gaussian vectors, as it is frequently done in applications to scattering systems with Quantum Chaos, and the derivation of Eq. (3.7) in the previous chapter is an example which yields precisely a correlation function involving integer powers for $\beta = 2$ and half-integer powers for $\beta = 1$.

For those and other reasons averages of products and ratios of powers of characteristic polynomials of random matrices attracted much interest over the years. When only *integer* powers are involved in the average the corresponding theory was developed for $\beta = 2$ in [71, 76, 86] and extended to $\beta = 1, 4$ in [77]. The case of half-integer powers for $\beta = 1$ remains however outstanding, despite the fact that it is most relevant for an overwhelming majority of experiments in Quantum Chaos due to the preserved time-reversal invariance of the underlying Hamiltonians.

Additional interest to this type of averages gives the fact that they are closely related to the problem of evaluating averages of quantities involving *absolute values* of characteristic polynomials due to the relation

$$|\det(E - H)| = \lim_{\epsilon \to 0} \det(E - H + i\epsilon/N)^{1/2} \det(E - H - i\epsilon/N)^{1/2} \tag{4.4}$$

valid for matrices H with real eigenvalues. Such averages emerge, for example, when studying the statistics of the so-called "level curvatures" in quantum chaotic systems [90, 91], see Eq. (4.6) below, as well as in the problem of counting the number of stationary points of random Gaussian surfaces, see [92, 93].

4.1.1. Examples of Such Averages in Physics Problems

To support the above picture we describe below explicitly a few examples of relations between the characteristic functions of the physical observables of interest in quantum chaotic systems which can be related to particular instances of the correlation function (4.1). The list is almost certainly not exhaustive, but hopefully representative.

- **LDoS distribution.** One of the first examples of that sort which is worth mentioning is related to the statistics of the local density of states (LDoS)

$\rho(x;E,\eta)$ at a point x of a quantum system with energy levels broadening η due to a uniform absorption in the sample. Mathematically the LDoS is defined in terms of the diagonal matrix element of the resolvent as $\rho(x;E,\eta) = \frac{1}{\pi}\,\mathrm{Im}\langle x|(E-\frac{i\eta}{N}-H)^{-1}|x\rangle$, and one is interested in understanding the statistics of the LDoS assuming a random matrix GOE Hamiltonian H of size $N\times N$, with the parameter η being fixed when $N\to\infty$. The Laplace transform for the probability density $\mathcal{P}(\rho)$ of the LDoS can be expressed in the large-N limit as [94]

$$\int_0^\infty \mathrm{d}\rho\,\mathcal{P}(\rho)\exp(-s\rho) = \left\langle \frac{\det\left[(E-H)^2+\frac{\eta^2}{N^2}\right]^{1/2}}{\det\left[(E-H)^2+\frac{\eta^2}{N^2}+\frac{\eta s}{N}\right]^{1/2}}\right\rangle_{\mathrm{GOE},\,N\to\infty} . \qquad (4.5)$$

Evaluation of the above random matrix average (which in our notation is a particular case of $\mathcal{C}^{(\mathrm{bulk})}_{2,4}$) attempted in [94] resulted in a quite impractical 5-fold integral, and to this end remains an outstanding RMT problem. Note however that the density $\mathcal{P}(\rho)$ has been found via a different route avoiding (4.5) as a sum of two-fold integrals in [5, 95].

- **Probability distribution of "level curvatures".** Consider a perturbation $\mathcal{H} = H + \alpha V$ of the Hamiltonian H where α is a control parameter and V is a real symmetric matrix. "Level curvatures" are defined as second derivatives of the eigenvalues $E_n(\alpha)$ of $\mathcal{H}$ (interpreted as energy levels of a quantum-chaotic system) with respect to the external parameter α. They can be expressed in terms of the eigenvalues λ_n and eigenvectors $|n\rangle$ of the unperturbed Hamiltonian H: $C_n = \frac{\partial^2 E_n(\alpha)}{\partial\alpha^2} = \sum_{m\neq n}\frac{\langle n|V|m\rangle^2}{\lambda_n-\lambda_m}$. Assuming the perturbation V to be taken as well from the GOE one can show that the probability density $\mathcal{P}_E(c) = \frac{1}{\bar\rho(E)}\left\langle\sum_{n=1}^N \delta(c-C_n)\delta(E-\lambda_n)\right\rangle$ of the level curvatures for GOE matrices H with eigenvalues λ_n and mean density of eigenvalues $\bar\rho(E)$ can be represented as [90, 91]

$$\mathcal{P}_E(c) \propto \int_{-\infty}^{+\infty}\mathrm{d}\omega\,\exp(i\omega c)\left\langle\frac{|\det(E-H)|\det(E-H)^{1/2}}{\det(E+\frac{i\omega}{N}-H)^{1/2}}\right\rangle_{\mathrm{GOE},\,N\to\infty}, \qquad (4.6)$$

where the required random matrix average in the right-hand side was independently evaluated by several alternative methods in [90, 91].

- **Statistics of S-matrix poles.** Various questions related to the statistics of quantum chaotic resonances (poles of the scattering matrix in the complex energy plane [2]) in the regime of a weakly open scattering system can be related to evaluation of the averages

$$\left\langle \frac{\det H^2}{\det(H^2+\frac{\omega^2}{N^2})^{1/2}} \right\rangle_{\mathrm{GOE},\,N\to\infty} \quad \text{and} \quad \left\langle \det\left(H^2+\frac{\omega^2}{N^2}\right)^{1/2} \right\rangle_{\mathrm{GOE},\,N\to\infty} \tag{4.7}$$

  where ω is considered as N-independent parameter. The first of these averages features in the statistics of resonance widths change under influence of a small perturbation of the Hamiltonian $H \to H + \alpha V$ akin to that considered above for the level curvature case. Such change reflects the intrinsic non-orthogonality of the associated resonance eigenfunctions [96]. Another manifestation of the same non-orthogonality is the statistics of the so-called Petermann factor which again can be related to random matrix averages involving half-integer powers of characteristic polynomials, see [97]. The second average in (4.7) arose in a recent attempt of clarifying the statistics of resonance widths beyond the standard first-order perturbation theory, see [98].

- **Statistics of Wigner K-matrix.** In the previous chapter we have shown that the characteristic function of the distribution of the K-matrix for $\beta = 1, 2$ is given by Eq. (3.7). For $\beta = 2$ this is a ratio of products of integer powers of characteristic polynomials and we were able to show that this object leads to the universal probability density $\mathcal{P}(K)$ given in (1.31). For $\beta = 1$, on the other hand, Eq. (3.7) is a correlation function involving half-integer powers of characteristic polynomials which can be brought to the form (4.1).[28]

  Another example is to consider the probability density $\mathcal{P}(K_{ab})$ of the individual off-diagonal entries $K_{a\neq b}$ for $\beta = 1$. This will be done in the next chapter, and the Fourier transformed $\mathcal{P}(K_{ab})$, given in Eq. (5.30), is again of the form (4.1), in particular a special case of $\mathcal{C}_{2,4}$. Note that the quantities K_{ab} are of direct experimental relevance and can be measured in microwave experiments as they are related to the real part of the electromagnetic impedance [99, 100].

[28] The term $\operatorname{sgn}\det(E-H)$ can be expressed as $\det(E-H)/|\det(E-H)|$ where the absolute value can be related to half-integer powers via Eq. (4.4).

- A particular type of the correlation functions (4.2) was investigated in [101] where it has been shown that for any integer $k > 0$ and fixed real δ holds [29]

$$\begin{aligned}&\left\langle \frac{1}{\det^{k/2}(i\delta/N - H)\det^{k/2}(-i\delta/N - H)} \right\rangle_{\mathrm{GOE},\, N\to\infty} \\ &\propto e^{k\delta} \int_1^\infty \frac{d\lambda_1 e^{-\delta\lambda_1}}{\sqrt{\lambda_1^2 - 1}} \ldots \int_1^\infty \frac{d\lambda_k e^{-\delta\lambda_k}}{\sqrt{\lambda_k^2 - 1}} \prod_{i<j}^{k} |\lambda_i - \lambda_j|.\end{aligned} \tag{4.8}$$

  In our notation this is a special case of correlation function $\mathcal{C}_{0,2k}^{(\mathrm{bulk})}$ at $E = 0$, where half of its arguments are δ and the other half are $-\delta$.

4.2. Calculation of Correlation Function $\mathcal{C}_{1,1}$

We start by calculating the simplest correlation function of that form

$$\mathcal{C}_{1,1} = \left\langle \frac{\det(\mu_F - H)}{\det(\mu_B - H)^{1/2}} \right\rangle_{\mathrm{GOE}}. \tag{4.9}$$

At present the only systematic method for evaluating such an ensemble average seems to be the supersymmetric formalism as described in Section 2.2. Hence we start by replacing the characteristic polynomials by Gaussian integrals. There is, however, an important difference compared to the examples considered in Section 2.2 given by the presence of the square root in the denominator. Instead of introducing N complex commuting variables, we introduce N real commuting variables x_j and make use of

$$\int \mathrm{d}\boldsymbol{x}\, \exp(i\boldsymbol{x}^T A \boldsymbol{x}) = e^{i\pi N/4} \det A^{-1/2}, \tag{4.10}$$

where $\boldsymbol{x}$ is an N dimensional vector comprising the real commuting variables x_j, $\mathrm{d}\boldsymbol{x} = \prod_{j=1}^N \mathrm{d}x_j/\sqrt{\pi}$ and A is a symmetric matrix. For the numerator we introduce a vector $\boldsymbol{\zeta}$ comprising N complex Grassmann variables ζ_j and use (2.47) as described

[29] Note also that an ensemble average closely related to the left-hand side of (4.8) was evaluated explicitly in [102], with the general circular $\beta-$ensemble replacing the GOE. The result was expressed for all $\beta > 0$ and all integer $N \geq 1$ in terms of a certain generalised hypergeometric function. The $\delta \to 0$ asymptotics for large $N \gg 1$ of the latter function does agree with the one following from the right-hand side of (4.8).

in Section 2.2.4. In that way the correlation function $\mathcal{C}_{1,1}$ can be represented by

$$\mathcal{C}_{1,1} = \left\langle c \int \mathrm{d}\boldsymbol{x} \, \exp\left(is\boldsymbol{x}^T(\mu_B - H)\boldsymbol{x}\right) \int \mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}^\dagger \exp\left(-i\boldsymbol{\zeta}^\dagger(\mu_F - H)\boldsymbol{\zeta}\right) \right\rangle_{\mathrm{GOE}} . \quad (4.11)$$

Here we introduced $s = \mathrm{sgn}(\mathrm{Im}\,\mu_B)$ which is necessary to make the $\boldsymbol{x}$ integration convergent. The Grassmann integration has no convergence problems, however, for convenience we introduced a minus sign in the exponent. The constant is given by $c = i^N \exp[-i\pi sN/4]$.

Performing the ensemble average can now be done easily (see Eq. (2.71) and the following paragraph) and we are left with

$$\mathcal{C}_{1,1} = c \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}^\dagger \exp\left[i(s\mu_B\boldsymbol{x}^T\boldsymbol{x} - \mu_F\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta})\right] \exp\left[-\frac{J^2}{4N}\,\mathrm{Tr}(A + A^T)^2\right], \quad (4.12)$$

with $A = s\,\boldsymbol{x}\otimes\boldsymbol{x}^T + \boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^\dagger$. From this point there are different options to proceed, we choose to follow the route described in subsection "Hybrid method" of Section 2.2.4.

The trace in Eq. (4.12) is given by

$$\mathrm{Tr}(A + A^T)^2 = 4(\boldsymbol{x}^T\boldsymbol{x})^2 - 2(\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta})^2 - 8s\boldsymbol{\zeta}^\dagger(\boldsymbol{x}\otimes\boldsymbol{x}^T)\boldsymbol{\zeta}, \quad (4.13)$$

where we used $s^2 = 1$, $\boldsymbol{\zeta}^T\boldsymbol{\zeta}^* = -\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta}$, $\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta}^* = \boldsymbol{\zeta}^T\boldsymbol{\zeta} = 0$ and $\boldsymbol{\zeta}^T X\boldsymbol{\zeta}^* = -\boldsymbol{\zeta}^\dagger X\boldsymbol{\zeta}$, which is valid for symmetric matrices X. Next we use Eq. (2.88),

$$\exp\left(\frac{J^2}{2N}(\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta})^2\right) = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{2\pi}} \exp\left(-\frac{q^2}{2} - \frac{Jq}{\sqrt{N}}\,\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta}\right), \quad (4.14)$$

to bilinearise the term $(\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta})^2$, such that the Grassmann integration can be performed,

$$\begin{aligned}
&\int \mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}^\dagger \exp\left[\boldsymbol{\zeta}^\dagger\left(\frac{Jq}{\sqrt{N}} - i\mu_F + \frac{2sJ^2}{N}\boldsymbol{x}\otimes\boldsymbol{x}^T\right)\boldsymbol{\zeta}\right] \\
&= \det\left[-\frac{Jq}{\sqrt{N}} - i\mu_F + \frac{2sJ^2}{N}\boldsymbol{x}\otimes\boldsymbol{x}^T\right] \\
&= \left(-\frac{J}{\sqrt{N}}\right)^N \left(q + \frac{i\sqrt{N}\mu_F}{J}\right)^{N-1} \left(q + \frac{i\sqrt{N}\mu_F}{J} - \frac{2sJ}{\sqrt{N}}\boldsymbol{x}^T\boldsymbol{x}\right). \quad (4.15)
\end{aligned}$$

Substituting $q + i\sqrt{N}\mu_F/J \to q$ hence yields for the correlation function (we may shift the contour of integration back to the real axis without changing the integral)

$$\begin{aligned}\mathcal{C}_{1,1} =& c\left(-\frac{J}{\sqrt{N}}\right)^N \int_{-\infty}^{\infty} \mathrm{d}q\, \frac{q^{N-1}}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(q - i\sqrt{N}\mu_F/J\right)^2\right] \\ &\times \int \mathrm{d}\boldsymbol{x}\, \exp\left[-\frac{J^2}{N}(\boldsymbol{x}^T\boldsymbol{x})^2 + is\mu_B \boldsymbol{x}^T\boldsymbol{x}\right]\left(q - \frac{2sJ}{\sqrt{N}}\boldsymbol{x}^T\boldsymbol{x}\right).\end{aligned} \tag{4.16}$$

Next we go to polar coordinates as described in Eq. (2.94), but keeping in mind that $\boldsymbol{x}$ is a real vector,

$$\int \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x}^T\boldsymbol{x}) = \frac{S_{N-1}}{\pi^{N/2}} \int_0^\infty \mathrm{d}r\, r^{N-1} f(r^2) = \frac{S_{N-1}}{2}\left(\frac{N}{2\pi^2 J^2}\right)^{N/4} \int_0^\infty \mathrm{d}R\, R^{N/2-1} f(\tfrac{\sqrt{N}}{\sqrt{2}J}R), \tag{4.17}$$

with S_{N-1} being the surface of the N-sphere given by $S_{N-1} = 2\pi^{N/2}\Gamma(N/2)$, such that

$$\begin{aligned}\mathcal{C}_{1,1} =& \left(\frac{J^2}{2N}\right)^{N/4} \frac{(-i)^N \exp[-i\pi sN/4]}{\Gamma(N/2)} \int_{-\infty}^{\infty} \mathrm{d}q\, \frac{q^{N-1}}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(q - i\sqrt{N}\mu_F/J\right)^2\right] \\ &\times \int_0^\infty \mathrm{d}R\, R^{N/2-1} \exp\left[-\frac{1}{2}\left(R^2 - \frac{is\sqrt{2N}\mu_B}{J}R\right)\right](q - s\sqrt{2}R).\end{aligned} \tag{4.18}$$

Similar to the example considered in Section 2.2.4, the q-integration can be expressed in terms of Hermite polynomials $H_N(z)$, using their integral representation (2.19) and $H_N(-z) = (-1)^N H_N(z)$ such that

$$\begin{aligned}\mathcal{C}_{1,1} =& \left(\frac{J^2}{2N}\right)^{N/4} \frac{\exp[-i\pi sN/4]}{\Gamma(N/2)} \\ &\times \left\{ H_N(\sqrt{N}\mu_F/J) \int_0^\infty \mathrm{d}R\, R^{N/2-1} \exp\left[-\frac{1}{2}\left(R^2 - \frac{is\sqrt{2N}\mu_B}{J}R\right)\right]\right. \\ &\left. + i\sqrt{2}sH_{N-1}(\sqrt{N}\mu_F/J) \int_0^\infty \mathrm{d}R\, R^{N/2} \exp\left[-\frac{1}{2}\left(R^2 - \frac{is\sqrt{2N}\mu_B}{J}R\right)\right]\right\}.\end{aligned} \tag{4.19}$$

Furthermore we also introduce the Cauchy (or Stieltjes) transform, but for the notational convenience with a slightly different prefactor than in Section 2.2.4 (compare Eq. (2.96))

$$F_N(z) = (-\sqrt{2}i\,\mathrm{sgn}(\mathrm{Im}\,z))^N \int_0^\infty \mathrm{d}R\, R^N \exp\left[-\frac{1}{2}\left(R^2 - 2i\,\mathrm{sgn}(\mathrm{Im}\,z) z R\right)\right], \quad (4.20)$$

such that $\mathcal{C}_{1,1}$ can be written in terms of a determinant, and we get as final result the rather compact expression

$$\mathcal{C}_{1,1} = \left(\frac{J^2}{4N}\right)^{N/4} \frac{\sqrt{2}i\,\mathrm{sgn}(\mathrm{Im}\,\mu_B)}{\Gamma(N/2)} \det \begin{bmatrix} H_{N-1}(\frac{\sqrt{N}}{J}\mu_F) & F_{N/2-1}(\frac{\sqrt{N}}{\sqrt{2}J}\mu_B) \\ H_N(\frac{\sqrt{N}}{J}\mu_F) & F_{N/2}(\frac{\sqrt{N}}{\sqrt{2}J}\mu_B) \end{bmatrix}. \quad (4.21)$$

Notice the similarity between this result and the generating function for the level density of the GUE, Eq. (2.98). While this generating function was a ratio of two characteristic polynomials, $\mathcal{C}_{1,1}$ is a ratio with the square-root of a characteristic polynomial in the denominator. This is apparently reflected in the result by the fact that the Cauchy transforms of the Hermite polynomials (which can be associated with the bosonic part of the supersymmetric model, i.e. the part coming from the denominator) have to be evaluated at $N/2$ instead of N.

4.3. Calculation of Correlation Function $\mathcal{C}_{1,2}$

The next correlation function we want to calculate is

$$\mathcal{C}_{1,2} = \left\langle \frac{\det(\mu_F - H)}{\det(\mu_{B1} - H)^{1/2} \det(\mu_{B2} - H)^{1/2}} \right\rangle_{\mathrm{GOE}}. \quad (4.22)$$

The result is obtained following two different approaches.

4.3.1. Hybrid Method

This is the same approach as for the calculation of $\mathcal{C}_{1,1}$ in the previous section, i.e. the first step is to replace the characteristic polynomials by Gaussian integrals over commuting and anticommuting vectors. In contrast to the calculation of $\mathcal{C}_{1,1}$, we have

to introduce two N-dimensional real vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, such that

$$\mathcal{C}_{1,2} = (s_1 s_2)^{N/2} \Bigg\langle \int \mathrm{d}\boldsymbol{x}_1 \, \exp\left(i s_1 \boldsymbol{x}_1^T (\mu_{B1} - H)\boldsymbol{x}_1\right) \int \mathrm{d}\boldsymbol{x}_2 \, \exp\left(i s_2 \boldsymbol{x}_2^T (\mu_{B2} - H)\boldsymbol{x}_2\right) \\ \times \int \mathrm{d}\boldsymbol{\zeta} \mathrm{d}\boldsymbol{\zeta}^\dagger \, \exp\left(-i \boldsymbol{\zeta}^\dagger (\mu_F - H)\boldsymbol{\zeta}\right) \Bigg\rangle_{\mathrm{GOE}} . \tag{4.23}$$

Again, to ensure convergence, we have to introduce sign factors in the integrals over commuting variables, where $s_1 = \mathrm{sgn}(\mathrm{Im}\,\mu_{B1})$ and $s_2 = \mathrm{sgn}(\mathrm{Im}\,\mu_{B2})$.

In complete analogy to the previous case the ensemble average can be performed and

$$\mathcal{C}_{1,2} = (s_1 s_2)^{N/2} \int \mathrm{d}\boldsymbol{x}_1 \int \mathrm{d}\boldsymbol{x}_2 \int \mathrm{d}\boldsymbol{\zeta} \mathrm{d}\boldsymbol{\zeta}^\dagger \exp\left[-\frac{J^2}{4N} \mathrm{Tr}(A + A^T)^2\right] \\ \times \exp\left[i(s_1 \mu_{B1} \boldsymbol{x}_1^T \boldsymbol{x}_1 + s_2 \mu_{B2} \boldsymbol{x}_2^T \boldsymbol{x}_2 - \mu_F \boldsymbol{\zeta}^\dagger \boldsymbol{\zeta})\right] , \tag{4.24}$$

where now $A = s_1 \, \boldsymbol{x}_1 \otimes \boldsymbol{x}_1^T + s_2 \, \boldsymbol{x}_2 \otimes \boldsymbol{x}_2^T + \boldsymbol{\zeta} \otimes \boldsymbol{\zeta}^\dagger$. This implies

$$\mathrm{Tr}(A + A^T)^2 = 4 \, \mathrm{Tr}(s_1 \boldsymbol{x}_1 \otimes \boldsymbol{x}_1^T + s_2 \boldsymbol{x}_2 \otimes \boldsymbol{x}_2^T)^2 \\ - 2(\boldsymbol{\zeta}^\dagger \boldsymbol{\zeta})^2 - 8 \boldsymbol{\zeta}^\dagger (s_1 \boldsymbol{x}_1 \otimes \boldsymbol{x}_1^T + s_2 \boldsymbol{x}_2 \otimes \boldsymbol{x}_2^T) \boldsymbol{\zeta} . \tag{4.25}$$

The $\boldsymbol{\zeta}$-dependence is the same as in the previous case (the only difference being that the matrix in the bilinear form in (4.25) is $s_1 \, \boldsymbol{x}_1 \otimes \boldsymbol{x}_1^T + s_2 \, \boldsymbol{x}_2 \otimes \boldsymbol{x}_2^T$ instead of $s \, \boldsymbol{x} \otimes \boldsymbol{x}^T$). This means the $\boldsymbol{\zeta}$ integration can be performed introducing a Gaussian integral over q, Eq. (4.14), and yields a similar determinant as in (4.15), given by

$$\left(-\frac{J}{\sqrt{N}}\right)^N \det \left[q + \frac{i\sqrt{N}\mu_F}{J} - \frac{2J}{\sqrt{N}} \left(s_1 \boldsymbol{x}_1 \otimes \boldsymbol{x}_1^T + s_2 \boldsymbol{x}_2 \otimes \boldsymbol{x}_2^T\right)\right] . \tag{4.26}$$

Introducing the matrices

$$Q_B = \begin{bmatrix} \boldsymbol{x}_1^T \boldsymbol{x}_1 & \boldsymbol{x}_1^T \boldsymbol{x}_2 \\ \boldsymbol{x}_2^T \boldsymbol{x}_1 & \boldsymbol{x}_2^T \boldsymbol{x}_2 \end{bmatrix} , \quad L = \begin{bmatrix} \mathrm{sgn}(\mathrm{Im}\,\mu_{B1}) & 0 \\ 0 & \mathrm{sgn}(\mathrm{Im}\,\mu_{B2}) \end{bmatrix} , \tag{4.27}$$

and using Sylvester's identity $\det(\mathbb{1} + AB) = \det(\mathbb{1} + BA)$, this determinant can be rewritten as

$$\begin{aligned}&\det\left[q + \frac{i\sqrt{N}\mu_F}{J} - \frac{2J}{\sqrt{N}}\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2\end{bmatrix} L \begin{bmatrix}\boldsymbol{x}_1^T \\ \boldsymbol{x}_2^T\end{bmatrix}\right] \\ &\quad = \left(q + \frac{i\sqrt{N}\mu_F}{J}\right)^{N-2} \det\left(q + \frac{i\sqrt{N}\mu_F}{J} - \frac{2J}{\sqrt{N}} Q_B L\right). \end{aligned} \tag{4.28}$$

Furthermore the terms in the exponent of the expression for $\mathcal{C}_{1,2}$ which depend on $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ can be written in terms of Q_B as well since $\mathrm{Tr}(s_1\boldsymbol{x}_1 \otimes \boldsymbol{x}_1^T + s_2\boldsymbol{x}_2 \otimes \boldsymbol{x}_2^T)^2 = \mathrm{Tr}(Q_B L)^2$ and $s_1\mu_{B1}\boldsymbol{x}_1^T\boldsymbol{x}_1 + s_2\mu_{B2}\boldsymbol{x}_2^T\boldsymbol{x}_2 = \mathrm{Tr}[Q_B L\,\mathrm{diag}(\mu_{B1}, \mu_{B2})]$. Hence the correlation function becomes (shifting $q + \frac{i\sqrt{N}\mu_F}{J} \to q$ as in the previous case)

$$\begin{aligned}\mathcal{C}_{1,2} \propto \int_{-\infty}^{\infty} \mathrm{d}q\, q^{N-2} \exp\left[-\frac{1}{2}\left(q - i\sqrt{N}\mu_F/J\right)^2\right] \int \mathrm{d}\boldsymbol{x}_1 \int \mathrm{d}\boldsymbol{x}_2\, \det\left(q - \frac{2J}{\sqrt{N}} Q_B L\right) \\ \times \exp\left[-\frac{J^2}{N}\,\mathrm{Tr}(Q_B L)^2 + i\,\mathrm{Tr}[Q_B L\,\mathrm{diag}(\mu_{B1}, \mu_{B2})]\right],\end{aligned} \tag{4.29}$$

where the proportionality factor is given by $\frac{(-1)^N}{\sqrt{2\pi}}(s_1 s_2 J^2/N)^{N/2}$.

This form of the correlation function, depending only on the matrix Q_B, allows us to employ the identity (2.103) discussed in Section 2.2.4. This means we can replace the integration over the two N-dimensional vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ by an integral over a positive definite real symmetric 2×2 matrix $\hat{Q}_B$, such that

$$\begin{aligned}\mathcal{C}_{1,2} \propto \int_{-\infty}^{\infty} \mathrm{d}q\, q^{N-2} \exp\left[-\frac{1}{2}\left(q - i\sqrt{N}\mu_F/J\right)^2\right] \int_{\hat{Q}_B > 0} \mathrm{d}\hat{Q}_B\, (\det \hat{Q}_B)^{\frac{N-3}{2}} \\ \times \det\left(q - \frac{2J}{\sqrt{N}}\hat{Q}_B L\right) \exp\left[-\frac{J^2}{N}\,\mathrm{Tr}(\hat{Q}_B L)^2 + i\,\mathrm{Tr}[\hat{Q}_B L\,\mathrm{diag}(\mu_{B1}, \mu_{B2})]\right].\end{aligned} \tag{4.30}$$

The proportionality factor changes accordingly to $(-1)^N(\frac{s_1 s_2 J^2}{N})^{N/2}\frac{2^{N-5/2}}{\pi^{3/2}\Gamma(N-1)}$. Finally

we rescale $q \to \sqrt{N}q/J$ and $\hat{Q}_B \to N\hat{Q}_B/(2J^2)$ and obtain

$$\begin{aligned}\mathcal{C}_{1,2} =& C_N \int_{-\infty}^{\infty} \mathrm{d}q\, q^{N-2} \exp\left[-\frac{N}{2J^2}\left(q - i\mu_F\right)^2\right] \int_{\hat{Q}_B>0} \mathrm{d}\hat{Q}_B \left(\det \hat{Q}_B\right)^{\frac{N-3}{2}} \\ & \times \det(q - \hat{Q}_B L) \exp\left[-\frac{N}{4J^2}\operatorname{Tr}(\hat{Q}_B L)^2 + \frac{iN}{2J^2}\operatorname{Tr}[\hat{Q}_B L \operatorname{diag}(\mu_{B1}, \mu_{B2})]\right],\end{aligned} \tag{4.31}$$

with proportionality factor $C_N = \left(\frac{N}{J^2}\right)^{N+1/2} \frac{(-1)^N (s_1 s_2)^{N/2}}{4\pi\sqrt{2\pi}\Gamma(N-1)}$. This is one first important result in calculating this correlation function since we managed to reduce the problem to a 4-fold integral (q integration and the three independent variables of $\hat{Q}_B$). Next we want to perform as many of the remaining integrations as possible, i.e. simplify the integration over $\hat{Q}_B$.

As the integrand in (4.31) actually depends on the combination $\hat{Q}_B L$ we change the integration from $\hat{Q}_B$ to $\hat{Q}_B L$. The Jacobian of this transformation is unity. Recall that the matrix $L = \operatorname{diag}(\operatorname{sgn}(\operatorname{Im}\mu_{B1}), \operatorname{sgn}(\operatorname{Im}\mu_{B2}))$ reflects the signs of μ_{B1} and μ_{B2} and this fact will play now a crucial role. If μ_{B1} and μ_{B2} are of the same sign, L is proportional to the identity and hence $\hat{Q}_B L$ is still positive (or negative if $L = (-1, -1)$) definite real symmetric and can be diagonalized by an orthogonal transformation $\hat{Q}_B L = \pm O \operatorname{diag}(p_1, p_2) O^T$. If, however, the signs are different (we may assume for definiteness $\operatorname{Im}\mu_{B1} > 0$ and $\operatorname{Im}\mu_{B2} < 0$), then the matrix $\hat{Q}_B L$ will have an underlying hyperbolic symmetry and can be parametrised as [72, 101]

$$\hat{Q}_B L = T \operatorname{diag}(p_1, -p_2) T^{-1}, \quad T = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}, \tag{4.32}$$

where $p_1, p_2 > 0$ and $\theta \in (-\infty, \infty)$. This will eventually lead to quite different expressions for the correlation function $\mathcal{C}_{1,2}$, we will start with the case that the signs are the same.

Large-N limit for the same-sign case

Assume the imaginary parts of μ_{B1} and μ_{B2} have the same sign. We denote this sign by s. Then $\hat{Q}_B L$ is real symmetric and we can diagonalise $\hat{Q}_B L = s\, O \operatorname{diag}(p_1, p_2) O^T$, the measure changes accordingly to $|p_1 - p_2|\, \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}\mu(O)$. Since $\hat{Q}_B$ is positive definite,

its two eigenvalues have to be positive, $p_1, p_2 > 0$. This yields for the correlation function

$$\begin{aligned}\mathcal{C}_{1,2} =& C_N \int_{-\infty}^{\infty} \mathrm{d}q\, q^{N-2} \exp\left[-\frac{N}{2J^2}\left(q - i\mu_F\right)^2\right] \int_0^{\infty} \mathrm{d}p_1 \int_0^{\infty} \mathrm{d}p_2\, (p_1 p_2)^{\frac{N-3}{2}} \\ & \times |p_1 - p_2|(q - sp_1)(q - sp_2) \exp\left[-\frac{N}{4J^2}(p_1^2 + p_2^2)\right] \\ & \times \int \mathrm{d}\mu(O)\, \exp\left[\frac{isN}{2J^2} \operatorname{Tr} O \operatorname{diag}(p_1, p_2) O^T \operatorname{diag}(\mu_{B1}, \mu_{B2})\right].\end{aligned} \tag{4.33}$$

The integral over the orthogonal group in the third line is of the Itzykson-Zuber-Harish-Chandra type (see Eq. (3.25)). However, unlike to the unitary case, there is no general formula to calculate that type of integral for the orthogonal group. Nevertheless we can make progress in the 2×2 case by explicit parametrisation $O = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$, where $\phi \in (0, \pi/2)$. The trace featured in the exponential then becomes

$$\frac{1}{2}[(\mu_{B1} + \mu_{B2})(p_1 + p_2) + (\mu_{B1} - \mu_{B2})(p_1 - p_2)\cos(2\phi)]. \tag{4.34}$$

The ϕ-integration can now be performed explicitly and expressed as a Bessel function using $\int_0^{\pi/2} \mathrm{d}\phi\, \exp\left[\pm ix\cos(2\phi)\right] = \frac{\pi}{2} J_0(x)$, such that the group integral becomes

$$\begin{aligned}&\int \mathrm{d}\mu(O)\, \exp\left[\frac{isN}{2J^2} \operatorname{Tr} O \operatorname{diag}(p_1, p_2) O^T \operatorname{diag}(\mu_{B1}, \mu_{B2})\right] \\ &\quad = \frac{\pi}{2} \exp\left[\frac{iN}{4J^2}(\mu_{B1} + \mu_{B2})(p_1 + p_2)\right] J_0\left[\frac{N(\mu_{B1} - \mu_{B2})(p_1 - p_2)}{4J^2}\right].\end{aligned} \tag{4.35}$$

Equation (4.33), together with (4.35) states the final result of this subsection for arbitrary N. Unfortunately, unlike the correlation function $\mathcal{C}_{1,1}$, it is not obvious if this expression can be written as a determinant or Pfaffian. We proceed from here taking the large-N limit in the previously discussed scaling which is most relevant to the physical problems presented in Section 4.1.

Correspondingly we scale $\mu_F = E + i\omega_F/N$, and accordingly for μ_{B1} and μ_{B2}, with

$|E| < 2J$, such that the correlation function becomes

$$\begin{aligned}\mathcal{C}_{1,2}^{(\mathrm{bulk})} =& \frac{\pi C_N}{2}\int_{-\infty}^{\infty}\mathrm{d}q\, q^{N-2}\exp\left[-\frac{N}{2J^2}\left(q-iE+\frac{\omega_F}{N}\right)^2\right]\int_0^\infty \mathrm{d}p_1\int_0^\infty \mathrm{d}p_2 \\ &\times\exp\left[-\frac{N}{4J^2}(p_1^2+p_2^2)+\frac{s(p_1+p_2)}{4J^2}(2iNE-(\omega_{B1}+\omega_{B2}))\right] \\ &\times(p_1p_2)^{\frac{N-3}{2}} I_0\left(\frac{\omega_{B1}-\omega_{B2}}{4J^2}(p_1-p_2)\right)|p_1-p_2|(q-sp_1)(q-sp_2),\end{aligned} \tag{4.36}$$

where we used $J_0(ix) = I_0(x)$. Note that $s = \mathrm{sgn}(\mathrm{Im}\,\mu_{B1}) = \mathrm{sgn}\,\omega_{B1} = \mathrm{sgn}\,\omega_{B2}$. The integral is of the form (2.15) and hence we can proceed from here taking the large-N limit by performing a saddle-point analysis. We rewrite $q^N = N\exp(\ln q)$ and $(p_1p_2)^{N/2} = \exp(\frac{N}{2}(\ln p_1 + \ln p_2))$ and assume that all parameters $E, J, \omega_F, \omega_{B1}, \omega_{B2}$ are of order unity. The saddle points are then given by

$$p_{1,\pm}^{SP} = p_{2,\pm}^{SP} = \frac{isE \pm \sqrt{4J^2-E^2}}{2}, \quad q_\pm^{SP} = \frac{iE \pm \sqrt{4J^2-E^2}}{2}. \tag{4.37}$$

Since $p_1, p_2 > 0$ we want to deform the contours of integration through the saddle points with positive real part, i.e. for both p_1 and p_2 we deform the contour to a line along the imaginary axis going from 0 to $isE/2$ and a line parallel to the real axis going from $isE/2$ to $\infty + isE/2$. The saddle points $p_{1,-}^{SP}$ and $p_{2,-}^{SP}$ do not play a role, whereas the saddle points $p_{1,+}^{SP}$ and $p_{2,+}^{SP}$, lie on the curves going from $isE/2$ to $\infty + isE/2$. The curves from 0 to $isE/2$ are far away from any saddle point and hence their contributions are negligible. For q we deform the contour to a line parallel to the real axis through the point $iE/2$. Note that both saddle points q_+^{SP} and q_-^{SP} lie on this curve. However, the term $(q - sp_1)(q - sp_2)$, evaluated at the saddle points, takes the form $(q^{SP} - sp_+^{SP})^2$. For the case $s = 1$, this term vanishes if $q^{SP} \equiv q_+^{SP}$. This suggests that the contribution from q_+^{SP} is sub-dominant and hence negligible. Likewise in the case $s = -1$, the term vanishes for q_-^{SP}, and thus in both cases only one of the two saddle points contributes to the lowest order approximation. In terms of s, the relevant saddle point is hence given by (for brevity we drop the superscript "SP") $q_s = \frac{1}{2}(iE - s\sqrt{4J^2-E^2})$. The saddle points for p_1 and p_2 can be expressed in terms of q_s via $p_{1,+}^{SP} = p_{2,+}^{SP} = -sq_s^*$.

Next we shift the integration variables by those saddle points to $q = q_s + \eta$, $p_{1,2} = -sq_s^* + \xi_{1,2}$, such that the correlation function becomes

$$\begin{aligned}\mathcal{C}_{1,2}^{(\mathrm{bulk})} \approx \frac{\pi C_N}{2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta}{(q_s+\eta)^2} \exp\left[-\frac{N}{2J^2}\left(q_s^* + \eta + \frac{\omega_F}{N}\right)^2 + N\ln(q_s+\eta)\right] \\ \times \int_{-\Delta}^{\infty}\mathrm{d}\xi_1 \int_{-\Delta}^{\infty}\mathrm{d}\xi_2 \, \frac{|\xi_1-\xi_2|}{[(\xi_1 - sq_s^*)(\xi_2 - sq_s^*)]^{3/2}} I_0\left[\frac{\omega_{B1}-\omega_{B2}}{4J^2}(\xi_1-\xi_2)\right] \\ \times \prod_{j=1}^{2} \exp\left[-\frac{N}{4J^2}(\xi_j - sq_s^*)^2 + \frac{iNE}{2J^2}(s\xi_j - q_s^*) + \frac{N}{2}\ln(\xi_j - sq_s^*)\right] \\ \times \prod_{j=1}^{2} \exp\left[-\frac{\omega_{B1}+\omega_{B2}}{4J^2}(s\xi_j - q_s^*)\right](\eta - s\xi_j + q_s + q_s^*),\end{aligned} \tag{4.38}$$

where in the first exponent we used $q_s - iE = q_s^*$ and abbreviated $\Delta = \sqrt{4J^2 - E^2}$ for the lower limits of the $\xi_{1,2}$ integrations. Note that the above expression is already an approximation since we neglected higher order contributions from the $\xi_{1,2}$-integrations along the curve from 0 to $isE/2$.

The next step in the saddle-point analysis would be to replace the non-N dependent term by its zeroth order approximation and expand the N-depended terms in the exponent to second order around the saddle points (or in our case around 0 since we shifted the integration variables accordingly, thereby neglecting the sub-dominant contribution from the other saddle point along the q-integration). However, in this case the zeroth order of the non-N dependent term vanishes due to $|\xi_1 - \xi_2|$. Hence we need to expand the integrand to a higher order as discussed in Section 2.1. However, it will turn out that only the N independent term needs to be modified, while it suffices to expand the exponents to second order in η and $\xi_{1,2}$, respectively. For η one gets

$$\begin{aligned}&\exp\left[-\frac{N}{2J^2}\left(q_s^* + \eta + \frac{\omega_F}{N}\right)^2 + N\ln(q_s+\eta)\right] = \\ &q_s^N \exp\left[-\frac{N(q_s^*)^2}{2J^2} - \frac{q_s^*\omega_F}{J^2} + \mathcal{O}\left(\frac{1}{N}\right)\right] \exp\left[-\frac{N}{2}\left(\frac{1}{J^2} + \frac{1}{q_s^2}\right)\eta^2 + N\mathcal{O}(\eta^3)\right],\end{aligned} \tag{4.39}$$

and accordingly for $\xi_j,\ j=1,2$,

$$\begin{aligned}&\exp\left[-\frac{N}{4J^2}(\xi_j-sq_s^*)^2+\frac{iNE}{2J^2}(s\xi_j-q_s^*)+\frac{N}{2}\ln(\xi_j-sq_s^*)\right]=(-sq_s^*)^{N/2}\\&\times\exp\left[-\frac{N}{4J^2}((q_s^*)^2+2iEq_s^*)\right]\exp\left[-\frac{N}{4}\left(\frac{1}{J^2}+\frac{1}{(q_s^*)^2}\right)\xi_j^2+N\mathcal{O}(\xi_j^3)\right].\end{aligned}\tag{4.40}$$

Notice that the terms $q_s^N(-sq_s^*)^N=(-sJ^2)^N$ and $\exp[-\frac{N(q_s^*)^2}{2J^2}]\exp[-\frac{N}{2J^2}((q_s^*)^2+2iEq_s^*)]=\exp[-\frac{Nq_s^*}{J^2}(q_s^*+iE)]=e^{-N}$ are independent of E and can be absorbed into the proportionality factor. Following the saddle-point method, we replace the integrand by the approximations above and extend the lower limits of the $\xi_{1,2}$-integrations to $-\infty$. Finally let us also rescale $\sqrt{N}\eta\to\eta,\ \sqrt{N}\xi_{1,2}\to\xi_{1,2}$, such that

$$\begin{aligned}\mathcal{C}_{1,2}^{(\mathrm{bulk})}\approx&\tilde{C}\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta}{(q_s+\frac{\eta}{\sqrt{N}})^2}\exp\left[-\frac{1}{2}\left(\frac{1}{J^2}+\frac{1}{q_s^2}\right)\eta^2+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right]\\&\times\int_{-\infty}^{\infty}\mathrm{d}\xi_1\int_{-\infty}^{\infty}\mathrm{d}\xi_2\,\frac{|\xi_1-\xi_2|I_0\left[\frac{\omega_{B1}-\omega_{B2}}{4\sqrt{N}J^2}(\xi_1-\xi_2)\right]}{[(\frac{\xi_1}{\sqrt{N}}-sq_s^*)(\frac{\xi_2}{\sqrt{N}}-sq_s^*)]^{3/2}}\\&\times\prod_{j=1}^{2}\exp\left[-\frac{1}{4}\left(\frac{1}{J^2}+\frac{1}{(q_s^*)^2}\right)\xi_j^2+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right]\exp\left[-\frac{q_s^*\omega_F}{J^2}+\mathcal{O}\left(\frac{1}{N}\right)\right]\\&\times\prod_{j=1}^{2}\exp\left[-\frac{\omega_{B1}+\omega_{B2}}{4J^2}\left(\frac{s\xi_j}{\sqrt{N}}-q_s^*\right)\right]\left(\frac{\eta-s\xi_j}{\sqrt{N}}+q_s+q_s^*\right),\end{aligned}\tag{4.41}$$

where $\tilde{C}=\frac{\pi}{2N^2}(\frac{-sJ^2}{e})^NC_N\approx\frac{1}{16\pi J}$. Here we used Stirling's formula to approximate the Γ-function in C_N by $\Gamma(N-1)\approx\sqrt{2\pi}N^{N-3/2}e^{-N}$ which shows that $\tilde{C}$ is – at least approximately – independent of N. In lowest order we neglect all terms of order $\mathcal{O}(1/\sqrt{N})$ and higher in the integrand,

$$\begin{aligned}\mathcal{C}_{1,2}^{(\mathrm{bulk})}=&\tilde{C}\frac{(q_s+q_s^*)^2}{q_s^2(-sq_s^*)^3}\exp\left[-\frac{q_s^*}{2J^2}(2\omega_F-(\omega_{B1}+\omega_{B2}))\right]\\&\times\int_{-\infty}^{\infty}\mathrm{d}\eta\,\exp\left[-\frac{1}{2}\left(\frac{1}{J^2}+\frac{1}{q_s^2}\right)\eta^2\right]\\&\times\int_{-\infty}^{\infty}\mathrm{d}\xi_1\int_{-\infty}^{\infty}\mathrm{d}\xi_2\,|\xi_1-\xi_2|\prod_{j=1}^{2}\exp\left[-\frac{1}{4}\left(\frac{1}{J^2}+\frac{1}{(q_s^*)^2}\right)(\xi_1^2+\xi_2^2)\right].\end{aligned}\tag{4.42}$$

The η-integration can now be performed easily as it is a simple Gaussian integral which yields the factor $\sqrt{2\pi}\left(\frac{1}{J^2}+\frac{1}{q_s^2}\right)^{-1/2}$, whereas the integration over ξ_1 and ξ_2 can be performed by doing the transformation $(\xi_1+\xi_2)/2 = R$, $\xi_1-\xi_2 = r$. The Jacobian of this is unity and

$$\int_{-\infty}^{\infty}\mathrm{d}\xi_1\int_{-\infty}^{\infty}\mathrm{d}\xi_2\, e^{-a(\xi_1^2+\xi_2^2)}|\xi_1-\xi_2| = \int_{-\infty}^{\infty}\mathrm{d}R\, e^{-2aR^2}\int_0^{\infty}\mathrm{d}r\, e^{-\frac{a}{2}r^2} r = \frac{2}{a}\sqrt{\frac{\pi}{2a}}. \quad (4.43)$$

To calculate the q_s-dependent prefactor (which could depend on E) we make use of its definition $q_s = \frac{1}{2}(iE - s\Delta)$, with $\Delta = \sqrt{4J^2-E^2}$. This implies $4(\frac{1}{J^2}+\frac{1}{(q_s^*)^2})^{-1} = \frac{8J^2(q_s^*)^2}{\Delta}(\Delta+isE)^{-1}$ and hence $4\pi(\frac{1}{J^2}+\frac{1}{(q_s^*)^2})^{-1/2}(\frac{J}{J^2}+\frac{1}{q_s^2})^{-1/2} = 4\pi J^3/\Delta$. Moreover, with $q_s^* q_s = J^2$, one gets $\frac{(q_s+q_s^*)^2}{q_s^2(-sq_s^*)} = \frac{-4s\,\mathrm{Re}^2\, q_s}{J^2 q_s} = \frac{2\Delta^2}{J^2(\Delta-isE)}$. Multiplying all three factors together finally yields the term $64\pi J^3(\Delta+isE)^{-1}(\Delta-isE)^{-1} = 16\pi J$ which is independent of E and moreover yields unity when multiplied with $\tilde{C}$.

The final result for the large-N limit of the correlation function

$$\mathcal{C}_{1,2}^{(\mathrm{bulk})} \approx \left\langle \frac{\det(E+\frac{i\omega_F}{N}-H)}{\det(E+\frac{i\omega_{B1}}{N}-H)^{1/2}\det(E+\frac{i\omega_{B2}}{N}-H)^{1/2}}\right\rangle_{\mathrm{GOE},\,N\to\infty} \quad (4.44)$$

in the case $\operatorname{sgn}\omega_{B1} = \operatorname{sgn}\omega_{B2}$ is thus given by

$$\begin{aligned}&\mathcal{C}_{1,2}^{(\mathrm{bulk},\,\operatorname{sgn}\omega_{B1}=\operatorname{sgn}\omega_{B2})}(\omega_{F1};\omega_{B1},\omega_{B2}) \approx \\ &\quad \exp\left[\frac{2\omega_{F1}-\omega_{B1}-\omega_{B2}}{4J^2}(iE+\operatorname{sgn}\omega_B\sqrt{4J^2-E^2})\right].\end{aligned} \quad (4.45)$$

Note that the prefactor of the exponential being unity is actually necessary due to normalisation. We can see this by taking the limit $\omega_F, \omega_{B1}, \omega_{B2} \to 0$ in (4.44). In this limit the correlation function itself becomes unity. Equation (4.45) is the first major result of this section. The case of different signs, $\operatorname{sgn}\omega_{B1} \neq \operatorname{sgn}\omega_{B2}$ remains to be done.

Large-N limit for the different-sign case

We go back to Eq. (4.31) and assume for definiteness $\mathrm{Im}\,\mu_{B1} > 0$ and $\mathrm{Im}\,\mu_{B2} < 0$. Then the matrix $\hat{Q}_B L$ can be parametrised by (4.32) with $p_1, p_2 > 0$ and $\theta \in (-\infty, \infty)$.

The measure of such parametrisation is given by $\mathrm{d}\hat{Q}_B = (p_1 + p_2)\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}\theta$. Note that the absolute value in the measure, which was necessary in the previous case, can be dropped here since $p_1 + p_2 > 0$.

The next step is to express the $\hat{Q}_B L$ dependent terms in (4.31) by these new variables. All but the term $\mathrm{Tr}[\hat{Q}_B L\,\mathrm{diag}(\mu_{B1}, \mu_{B2})]$ (which is again of the IZHC-type, this time for the group of matrices T) depend only on the eigenvalues p_1, p_2 and are hence similar to the previous case with $p_2 \to -p_2$. The term depending on the matrix T, parametrised by the variable θ as in (4.32), becomes

$$\frac{1}{2}[(\mu_{B1} + \mu_{B2})(p_1 - p_2) + (\mu_{B1} - \mu_{B2})(p_1 + p_2)\cosh(2\theta)]. \tag{4.46}$$

Comparing this with the corresponding term from the integration over the orthogonal group, Eq. (4.34), we see that it is similar with the replacements $p_2 \to -p_2$ and $\cos(2\phi) \to \cosh(2\theta)$. The correlation function is then given by Eq. (4.33) with the aforementioned replacements (and $s = 1$ since we fixed the sign of the imaginary part of μ_{B1} to be positive). The rescaled version is accordingly given by Eq. (4.36) with the replacements $p_2 \to -p_2$, $I_0 \to K_0$ and $s = 1$, where K_0 is the modified Bessel function of second kind (MacDonald function). This factor comes from integration over θ, for the rescaled version given by

$$\int_{-\infty}^{\infty} \mathrm{d}\theta\, \exp\left[-\frac{(\mu_{B1} - \mu_{B2})(p_1 + p_2)}{4J^2}\cosh(2\theta)\right] = K_0\left[\frac{(\mu_{B1} - \mu_{B2})(p_1 + p_2)}{4J^2}\right]. \tag{4.47}$$

So far the two cases are quite similar. The main difference comes from the saddle-point structure.

The saddle points are now given by (we already restrict to the case of positive real part for the p_1 and p_2 saddle points as discussed in the previous case)

$$\begin{aligned} p_1^{SP} &= \frac{iE + \sqrt{4J^2 - E^2}}{2},\ p_2^{SP} = \frac{-iE + \sqrt{4J^2 - E^2}}{2}, \\ q_\pm^{SP} &= \frac{iE \pm \sqrt{4J^2 - E^2}}{2}. \end{aligned} \tag{4.48}$$

The difference to the previous case (4.37) is that p_1^{SP} and p_2^{SP} are not equal, but complex conjugates of each other. We deform the contours of integration as before, i.e.

for q a parallel to the real axis through iE and for p_1 and p_2 along the imaginary axis from 0 to $+iE$ and $-iE$, respectively (which are sub-dominant and can be neglected) and a parallel to the real axis from $\pm iE$ to $\infty \pm iE$. Again, both saddle points of q lie on the chosen contour. In the same-sign case, a term of the integrand vanished at one of these saddle points, but not at the other, making one saddle point contribution sub-dominant. In the different-sign case, however, the corresponding term $(q-p_1)(q+p_2)$ vanishes for both since $q_+^{SP} = p_1^{SP}$ and $q_-^{SP} = -p_2^{SP}$. This suggests that both saddle points yield a contribution to the lowest order approximation. The result will hence be the sum of both contributions, $\mathcal{C}_{1,2}^{(\mathrm{bulk})} = \mathcal{C}_+ + \mathcal{C}_-$.

To perform the saddle-point approximation we shift the variables by their respective saddle points (and for brevity drop the "SP" superscript), $q = q_\pm + \eta$, $p_1 = q_+ + \xi_1$ and $p_2 = -q_- + \xi_2$. Then we expand the N-dependent exponent to third order (it will turn out that the second order is not sufficient to get all relevant contributions). The expansion for η is given by

$$\begin{aligned}&\exp\left[-\frac{N}{2J^2}\left(-q_\mp + \eta + \frac{\omega_F}{N}\right)^2 + N\ln(q_\pm+\eta)\right] = q_\pm^N \exp\left[-\frac{Nq_\mp^2}{2J^2} + \frac{q_\mp \omega_F}{J^2}\right] \\ &\times \exp\left[-\frac{N}{2}\left(\frac{1}{J^2}+\frac{1}{q_\pm^2}\right)\eta^2 + \frac{N}{3q_\pm^3}\eta^3 - N\mathcal{O}(\eta^4) + \mathcal{O}\left(\frac{1}{N}\right)\right],\end{aligned} \tag{4.49}$$

where we used $q_\pm - iE = -q_\mp$. The expansion for ξ_1 evaluates to

$$\begin{aligned}&\exp\left[-\frac{N}{4J^2}(q_+ + \xi_1)^2 + \frac{iNE}{2J^2}(q_+ + \xi_1) + \frac{N}{2}\ln(q_+ + \xi_1)\right] = q_+^{N/2} \\ &\times \exp\left[-\frac{Nq_+^2}{4J^2} + \frac{iNEq_+}{2J^2}\right]\exp\left[-\frac{N}{4}\left(\frac{1}{J^2}+\frac{1}{q_+^2}\right)\xi_1^2 + \frac{N}{6q_+^3}\xi_1^3 - N\mathcal{O}(\xi_1^4)\right],\end{aligned} \tag{4.50}$$

and analogously for ξ_2 one gets

$$\begin{aligned}&\exp\left[-\frac{N}{4J^2}(-q_- + \xi_2)^2 - \frac{iNE}{2J^2}(-q_- + \xi_2) + \frac{N}{2}\ln(-q_- + \xi_2)\right] = (-q_-)^{N/2} \\ &\times \exp\left[-\frac{N}{4J^2}q_-^2 + \frac{iNE}{2J^2}q_-\right]\exp\left[-\frac{N}{4}\left(\frac{1}{J^2}+\frac{1}{q_-^2}\right)\xi_2^2 - \frac{N}{6q_-^3}\xi_2^3 - N\mathcal{O}(\xi_2^4)\right].\end{aligned} \tag{4.51}$$

The factors in these expansions which do not depend on the integration variables can be multiplied together and, using the definition of $q_\pm$, yield $J^N q_\pm^N \exp(-N \pm$

$\frac{NiE}{4J^2}\sqrt{4J^2-E^2}+\omega_F q_\mp/J^2$). Analogously to the previous case we use the above expansions in the integrand, change variables $\eta \to \eta/\sqrt{N}$, $\xi_j \to \xi_j/\sqrt{N}$, $j=1,2$ and extend the lower limits of the ξ-integrations to negative infinity. This yields

$$\begin{aligned}\mathcal{C}_\pm \approx A_\pm &\int_{-\infty}^{\infty} \frac{\mathrm{d}\eta}{(q_\pm+\frac{\eta}{\sqrt{N}})^2} \exp\left[-\frac{1}{2}\left(\frac{1}{J^2}+\frac{1}{q_\pm^2}\right)\eta^2+\frac{1}{3\sqrt{N}q_\pm^3}\eta^3+\mathcal{O}\left(\frac{1}{N}\right)\right] \\ &\times \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi_1}{(q_+ + \frac{\xi_1}{\sqrt{N}})^{3/2}} \exp\left[-\frac{1}{4}\left(\frac{1}{J^2}+\frac{1}{q_+^2}\right)\xi_1^2+\frac{1}{6\sqrt{N}q_+^3}\xi_1^3\right] \\ &\times \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi_2}{(-q_- + \frac{\xi_2}{\sqrt{N}})^{3/2}} \exp\left[-\frac{1}{4}\left(\frac{1}{J^2}+\frac{1}{q_-^2}\right)\xi_2^2-\frac{1}{6\sqrt{N}q_-^3}\xi_2^3\right] \\ &\times \exp\left[-\frac{\omega_{B1}+\omega_{B2}}{4\sqrt{N}J^2}(\xi_1-\xi_2)\right] K_0\left[\frac{\omega_{B1}-\omega_{B2}}{4J^2}\left(q_+-q_-+\frac{\xi_1+\xi_2}{\sqrt{N}}\right)\right] \\ &\times \left(q_+-q_-+\frac{\xi_1+\xi_2}{\sqrt{N}}\right)\left(q_\pm-q_++\frac{\eta-\xi_1}{\sqrt{N}}\right)\left(q_\pm-q_-+\frac{\eta+\xi_2}{\sqrt{N}}\right),\end{aligned} \tag{4.52}$$

where

$$A_\pm = \frac{C_N J^N e^{-N}}{N^{3/2}} q_\pm^N \exp\left(\pm\frac{NiE}{4J^2}\sqrt{4J^2-E^2}+\frac{\omega_F q_\mp}{J^2}-\frac{iE}{4J^2}(\omega_{B1}+\omega_{B2})\right). \tag{4.53}$$

The above equations combine expansions around both saddle points q_+ and q_-. As discussed above both of them contribute and the correlation function itself is given by the sum $\mathcal{C}_{1,2}^{(\mathrm{bulk})} = \mathcal{C}_+ + \mathcal{C}_-$.

In the previous case we could neglect all terms of order $1/\sqrt{N}$ or higher. However the second bracket in the last line of Eq. (4.52) vanishes for $\mathcal{C}_+$ if the N-dependent term is neglected. Likewise the last bracket vanishes for $\mathcal{C}_-$. Hence we cannot neglect those terms and all other terms of the same order. This is another major difference to the case of same signs.

To simplify the large-N analysis, we expand the last two brackets of Eq. (4.52) and rewrite it to

$$\begin{aligned}\mathcal{C}_\pm \approx\ & A_\pm \int_{-\infty}^{\infty}\mathrm{d}\eta\, e^{-b_\pm\eta^2}\int_{-\infty}^{\infty}\mathrm{d}\xi_1\, e^{-b_+\xi_1^2/2}\int_{-\infty}^{\infty}\mathrm{d}\xi_2\, e^{-b_-\xi_2^2/2} f\left(\frac{\eta}{\sqrt{N}},\frac{\xi_1}{\sqrt{N}},\frac{\xi_2}{\sqrt{N}}\right) \\ &\times\left[\frac{1}{\sqrt{N}}((q_\pm-q_+)(\eta+\xi_2)+(q_\pm-q_-)(\eta-\xi_1))+\frac{1}{N}(\eta-\xi_1)(\eta+\xi_2)\right],\end{aligned} \tag{4.54}$$

where we used $(q_\pm - q_+)(q_\pm - q_-) = 0$. The new introduced variables $b_\pm$ and the function f can be read off comparing Eq. (4.54) with Eq. (4.52). If we neglect the term proportional to $1/N$ in (4.54) and replace $f\left(\frac{\eta}{\sqrt{N}}, \frac{\xi_1}{\sqrt{N}}, \frac{\xi_2}{\sqrt{N}}\right)$ with $f(0,0,0)$, the resulting integrals all vanish since $\int_{-\infty}^{\infty} \mathrm{d}\eta \int_{-\infty}^{\infty} \mathrm{d}\xi \, \exp(-a\eta^2 - b\xi^2)(\eta \pm \xi) = 0$. In conclusion we have to go one order higher, i.e. to the term $\propto N^{-1}$. This order is the sum of two terms: The first term in the second line of (4.54) times the term in the series expansion of f which is proportional to $N^{-1/2}$ on the one hand and the second term in the second line of (4.54) times the zero-th order term of f on the other hand.

Let us start with the latter contribution: The zero-th order term of f is given by

$$f(0,0,0) = \frac{q_+ - q_-}{q_\pm^2(-q_+q_-)^{3/2}} K_0\left[\frac{\omega_{B1} - \omega_{B2}}{4J^2}(q_+ - q_-)\right]. \tag{4.55}$$

Noting that $(\eta - \xi_1)(\eta + \xi_2) = \eta^2 + \eta(\xi_2 - \xi_1) - \xi_1\xi_2$ the integration reduces to

$$\int_{-\infty}^{\infty} \mathrm{d}\eta \, e^{-b_\pm \eta^2} \eta^2 \int_{-\infty}^{\infty} \mathrm{d}\xi_1 \, e^{-b_+ \xi_1^2/2} \int_{-\infty}^{\infty} \mathrm{d}\xi_2 \, e^{-b_2 \xi_2^2/2} = \frac{\pi^{3/2}}{b_\pm^{3/2}(b_+b_-)^{1/2}} = \frac{\pi^{3/2}}{b_\pm^2\sqrt{b_\mp}}, \tag{4.56}$$

where we used that the integration over odd powers in η or $\xi_{1,2}$ vanishes. Hence the contribution is given by

$$\frac{\pi^{3/2} A_\pm}{N b_\pm^2 \sqrt{b_\mp}} \frac{q_+ - q_-}{q_\pm^2(-q_+q_-)^{3/2}} K_0\left[\frac{\omega_{B1} - \omega_{B2}}{4J^2}(q_+ - q_-)\right]. \tag{4.57}$$

Obtaining the other contribution is more difficult since one needs to expand f to order $N^{-1/2}$. Now it also becomes evident why we had to expand the terms in the exponentials to third order, because the terms proportional to η^3 (and $\xi_{1,2}^3$ respectively) contribute to this as well. The expansions of the various functions contained in f are given by

$$\exp\left(\frac{ax^3}{\sqrt{N}}\right) = 1 + \frac{ax^3}{\sqrt{N}} + \mathcal{O}(N^{-1}), \tag{4.58}$$

$$\frac{1}{(\frac{x}{\sqrt{N}} + a)^m} = \frac{1}{a^m}\left(1 - \frac{mx}{\sqrt{N}a}\right) + \mathcal{O}(N^{-1}), \tag{4.59}$$

$$K_0\left(a\left(b + \frac{x}{\sqrt{N}}\right)\right) = K_0(ab) - \frac{ax}{\sqrt{N}} K_1(ab) + \mathcal{O}(N^{-1}). \tag{4.60}$$

With these formulas we are able to determine the term proportional to $N^{-1/2}$ in the expansion of f itself. We refrain from presenting it here. It needs to be multiplied with $(\eta - \xi_2)$ and $(\eta + \xi_1)$ respectively (the first term of the second line in (4.54)). The result is a polynomial in the integration variables, where in the different terms each of the variables appears as a power between zero and four. Integration over odd powers vanishes and for even powers we may use

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \exp(-ax^2) x^{2n} = \frac{(2n-1)!!}{(2a)^n} \sqrt{\frac{\pi}{a}}. \tag{4.61}$$

The result after integration is given by

$$\begin{aligned} &\frac{\pm\pi^{3/2}(q_+ - q_-)A_\pm}{N b_\pm^2 \sqrt{b_\mp} q_\pm^2 (-q_+q_-)^{3/2}} \Bigg\{ \pm \frac{q_+ - q_-}{2J^2} (\omega_{B1} - \omega_{B2}) K_1 \left[\frac{\omega_{B1} - \omega_{B2}}{4J^2} (q_+ - q_-) \right] \\ &+ \left[\mp 2 + (q_+ - q_-) \left(\frac{1}{q_\pm} - \frac{1}{2q_\pm^3 b_\pm} - \frac{\omega_F}{J^2} + \frac{\omega_{B1} + \omega_{B2}}{2J^2} \right) \right] \\ &\quad \times K_0 \left[\frac{\omega_{B1} - \omega_{B2}}{4J^2} (q_+ - q_-) \right] \Bigg\}. \end{aligned} \tag{4.62}$$

Using $b_\pm = \frac{1}{2} \left(\frac{1}{J^2} + \frac{1}{q_\pm^2} \right)$ the term $(q_+ - q_-)(\frac{1}{q_\pm} - \frac{1}{2q_\pm^3 b_\pm})$ in the second line of Eq. (4.62) simplifies to positive or negative unity,

$$(q_+ - q_-) \left(\frac{1}{q_\pm} - \frac{1}{2q_\pm^3 b_\pm} \right) = \frac{(q_+ - q_-) q_\pm}{q_\pm^2 + J^2} = \pm 1. \tag{4.63}$$

This, together with the contribution (4.57), which we need to add to (4.62), cancels the ∓ 2 in the second line of Eq. (4.62). The term $\frac{1}{(-q_+q_-)^{3/2}} (\frac{q_+ - q_-}{b_\pm q_\pm})^2$ in front of (4.62) simplifies to a similar expression,

$$\frac{1}{(-q_+q_-)^{3/2}} \left(\frac{q_+ - q_-}{b_\pm q_\pm} \right)^2 = 4J \left(\frac{q_\pm}{q_\pm^2 + J^2} \right)^2 = 4J. \tag{4.64}$$

Furthermore we simplify $q_+ - q_- = \sqrt{4J^2 - E^2} = 2\pi J^2 \rho$, where ρ is the mean eigenvalue density of large GOE matrices in the bulk of the spectrum, see Eq. (1.9). Then

the lowest order approximation of Eq. (4.54) is finally given by

$$\begin{aligned}\mathcal{C}_{\pm} \approx & \frac{2\pi^{3/2}}{NJ}\frac{A_{\pm}}{\sqrt{b_{\mp}}}\Bigg\{(\omega_{B1}-\omega_{B2})K_1\left(\frac{\omega_{B1}-\omega_{B2}}{2}\pi\rho\right) \\ & \pm(\omega_{B1}+\omega_{B2}-2\omega_F)\,K_0\left(\frac{\omega_{B1}-\omega_{B2}}{2}\pi\rho\right)\Bigg\},\end{aligned} \tag{4.65}$$

and the correlation function $\mathcal{C}_{12}^{(\mathrm{bulk})}$ is given by the sum

$$\begin{aligned}\mathcal{C}_{+}+\mathcal{C}_{-} \approx & \frac{2\pi^{3/2}}{NJ}\Bigg\{\left(\frac{A_{+}}{\sqrt{b_{-}}}+\frac{A_{-}}{\sqrt{b_{+}}}\right)(\omega_{B1}-\omega_{B2})K_1\left(\frac{\omega_{B1}-\omega_{B2}}{2}\pi\rho\right) \\ & +\left(\frac{A_{+}}{\sqrt{b_{-}}}-\frac{A_{-}}{\sqrt{b_{+}}}\right)(\omega_{B1}+\omega_{B2}-2\omega_F)\,K_0\left(\frac{\omega_{B1}-\omega_{B2}}{2}\pi\rho\right)\Bigg\}.\end{aligned} \tag{4.66}$$

Finally we use the definition of $A_{\pm}$, Eq. (4.53), and $b_{\pm}^{-1/2}=(\frac{\pm q_{\pm}}{\pi\rho})^{1/2}=\frac{1}{\sqrt{2\pi\rho}}(2\pi J^2\rho\pm iE)^{1/2}$ and take into account that the solution should be invariant under the exchange $\omega_{B1}\leftrightarrow\omega_{B2}$ (i.e. choosing $\omega_{B1}<0$ and $\omega_{B2}>0$) to obtain the final result for the correlation function (4.44) and different signs $\operatorname{sgn}\omega_{B1}\neq\operatorname{sgn}\omega_{B2}$,

$$\begin{aligned}&\mathcal{C}_{1,2}^{(\mathrm{bulk},\ \operatorname{sgn}\omega_{B1}=-\operatorname{sgn}\omega_{B2})}(\omega_{F1};\omega_{B1},\omega_{B2}) \approx \frac{(-i)^N}{\pi\sqrt{2N\rho}(2J)^{N+1}}\,e^{-\frac{iE}{4J^2}(\omega_{B1}+\omega_{B2}-2\omega_{F1})} \\ &\times\Bigg\{[Ae^{-\pi\rho\omega_{F1}}-(-1)^N A^* e^{+\pi\rho\omega_{F1}}](\omega_{B1}+\omega_{B2}-2\omega_{F1})K_0\left(\tfrac{\pi\rho}{2}|\omega_{B1}-\omega_{B2}|\right) \\ &\quad+[Ae^{-\pi\rho\omega_{F1}}+(-1)^N A^* e^{+\pi\rho\omega_{F1}}]|\omega_{B1}-\omega_{B2}|K_1\left(\tfrac{\pi\rho}{2}|\omega_{B1}-\omega_{B2}|\right)\Bigg\}\end{aligned} \tag{4.67}$$

with

$$A(E,N)=(2\pi J^2\rho+iE)^{N-1/2}\,e^{\frac{i\pi N}{2}\rho E}. \tag{4.68}$$

Note that this asymptotic expression shows an interesting "parity effect": it behaves differently depending on whether N is even or odd for arbitrary large values of N.

4.3.2. "Method without Grassmannians"

In this section we want to demonstrate that the same result (4.67) can be obtained following a different route which is inspired by the insightful work of Schomerus, Frahm, Patra and Beenakker [97], who have used a similar approach to obtain the probability distribution of the so-called Petermann factor, a quantity that measures the non-orthogonality of the resonance eigenfunctions of a scattering system. This approach avoids introducing anticommuting variables altogether, hence we will call it the "method without Grassmannians" to distinguish it from the other methods used in this chapter. This section shall also serve as a case study for Section 4.5, where this approach will be the only feasible method.

We start again with the correlation function (4.22). Already employing the bulk scaling $\mu = E + i\omega/N$ for all μ's, we note that the denominator can be rewritten as

$$\det\left[(E-H)^2 - \frac{\omega_{B1}\omega_{B2}}{N^2} + i(E-H)\frac{\omega_{B1}+\omega_{B2}}{N}\right]^{1/2}. \tag{4.69}$$

However, in general this is only valid for $\omega_{B1}\omega_{B2} < 0$. Writing the determinants as products over the eigenvalues λ_j of H, we see that equality of the two expressions translates into validity of the relation $\sqrt{z_{1,j}}\sqrt{z_{2,j}} = \sqrt{z_{1,j}z_{2,j}}$ for all j, with $z_{1,j} = E-\lambda_j+i\omega_{B1}/N$ and $z_{2,j} = E-\lambda_j+i\omega_{B2}/N$. As discussed in Appendix A.2, for complex values $z_{1,j}$ and $z_{2,j}$ this relation is only correct for $|\arg(z_{1,j})+\arg(z_{2,j})| \leq \pi$. To ensure this one has to choose ω_{B1} and ω_{B2} with different signs, such that $\arg(z_{1,j}) \in [0,\pi]$ and $\arg(z_{2,j}) \in [-\pi,0]$ (or vice versa). For same signs the inequality is not necessarily fulfilled depending on the particular value of λ_j. Hence the method presented in this approach is only suited for the case $\omega_{B1}\omega_{B2} < 0$.

We represent (4.69) by a Gaussian integral over a real N-component vector $\boldsymbol{x}$ and hence get

$$\mathcal{C}_{1,2}^{(\mathrm{bulk})} = \int \mathrm{d}\boldsymbol{x}\, \exp\left(\frac{\omega_{B1}\omega_{B2}}{2N^2}\boldsymbol{x}^2\right)\Phi(\boldsymbol{x},\omega_F,\omega_{B1}+\omega_{B2},E), \tag{4.70}$$

where $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\ldots\mathrm{d}x_N$, x_j being the j-th component of $\boldsymbol{x}$, and

$$\Phi = \left\langle \frac{\det(\frac{i\omega_F}{N} - H_E)}{(2\pi)^{N/2}} \exp\left[-\frac{1}{2}\boldsymbol{x}^T\left(H_E^2 - iH_E\frac{\omega_{B1}+\omega_{B2}}{N}\right)\boldsymbol{x}\right]\right\rangle_{\mathrm{GOE},\,N\to\infty}. \tag{4.71}$$

Here we abbreviated $H_E = H - E\mathbb{1}_N$. Note that also the above integral is well-defined only for ω_{B1} and ω_{B2} having different signs, otherwise the term $\omega_{B1}\omega_{B2}/N^2 > 0$ would render the integral divergent. Furthermore it is not possible to fix this problem for ω_{B1} and ω_{B2} having same sign (e.g. by representing the determinant as $\int \mathrm{d}\boldsymbol{x} \exp[i\boldsymbol{x}^T(\dots)\boldsymbol{x})]$) as always one of the three terms $\omega_{B1}\omega_{B2}$, H_E^2 or $H_E(\omega_{B1} + \omega_{B2})/N$ will render the integral divergent (unless H is positive ore negative definite which cannot be assumed here), making the choice $\omega_{B1}\omega_{B2} < 0$ necessary on this level as well.

Next we parametrize the vector $\boldsymbol{x}$ of integration variables as $\boldsymbol{x} = |\boldsymbol{x}|O\boldsymbol{e}_1$, where $\boldsymbol{e}_1 = [1, 0, \dots, 0]$ is an N-dimensional unit vector and O is an orthogonal matrix: $O^{-1} = O^T$. Since both the determinant factor and the GOE probability density $\mathcal{P}(H)$ in (4.71) are invariant under orthogonal transformations $H \to O^T H O$ the matrices O, O^T can be omitted. The result is then Eq. (4.71) with $\boldsymbol{x}$, $\boldsymbol{x}^T$ being replaced by $\boldsymbol{e}_1$, $\boldsymbol{e}_1^T$ and with factor $|\boldsymbol{x}|^2$ in the exponential. The term $\boldsymbol{e}_1^T H_E^2 \boldsymbol{e}_1$ then suggests that it is advantageous to decompose H as

$$H = \begin{bmatrix} H_{11} & \boldsymbol{h}^T \\ \boldsymbol{h} & H_{N-1} \end{bmatrix}, \tag{4.72}$$

where $\boldsymbol{h}$ is a real $N-1$-component vector, H_{N-1} is the $(N-1)\times(N-1)$ sub-block of H and H_{11} is the first element of H. Such decomposition implies

$$H_E^2 = \begin{bmatrix} H_{11,E}^2 + \boldsymbol{h}^T\boldsymbol{h} & \sim \\ \sim & H_{N-1,E}^2 + \boldsymbol{h}\otimes\boldsymbol{h}^T \end{bmatrix}, \tag{4.73}$$

$$\det(\tfrac{i\omega_F}{N} - H_E) = (\tfrac{i\omega_F}{N} - H_{11,E} - \boldsymbol{h}^T(\tfrac{i\omega_F}{N} - H_{N-1,E})^{-1}\boldsymbol{h})\det(\tfrac{i\omega_F}{N} - H_{N-1,E}), \tag{4.74}$$

$$\boldsymbol{e}_1^T\left[H_E^2 - iH_E\frac{\omega_{B1}+\omega_{B2}}{N}\right]\boldsymbol{e}_1 = H_{11,E}^2 + \boldsymbol{h}^T\boldsymbol{h} - \frac{i(\omega_{B1}+\omega_{B2})}{N}H_{11,E}, \tag{4.75}$$

$$\mathcal{P}(H)\mathrm{d}H = C_{\mathrm{GOE}}\, e^{-\frac{N}{4J^2}\operatorname{Tr}H^2}\mathrm{d}H = C_{\mathrm{GOE}}\, e^{-\frac{N}{4J^2}(H_{11}^2 + 2\boldsymbol{h}^T\boldsymbol{h} + \operatorname{Tr}H_{N-1}^2)}\mathrm{d}H_{11}\,\mathrm{d}\boldsymbol{h}\,\mathrm{d}H_{N-1}, \tag{4.76}$$

where C_{GOE} is the normalisation constant ensuring that $\int \mathrm{d}H\,\mathcal{P}(H) = 1$, given by $C_{\mathrm{GOE}} = 2^{-N/2}(\frac{N}{2\pi J^2})^{N(N+1)/4}$ (see Eq. (1.4)). The off-diagonal blocks of H_E^2 are not needed and thus not shown. Taking all the above identities into account, we observe

that Φ splits conveniently up into $\Phi = C_{\mathrm{GOE}}(\Phi_I - \Phi_{II})$, where

$$\Phi_I = \int \frac{\mathrm{d}H_{11}}{\sqrt{2\pi}} (\tfrac{i\omega_F}{N} - H_{11,E}) \exp\left[-\frac{|\boldsymbol{x}|^2}{2} H_{11,E}^2 - \frac{N}{4J^2} H_{11}^2 + \frac{i(\omega_{B1}+\omega_{B2})}{2N} |\boldsymbol{x}|^2 H_{11,E} \right] \\ \times \langle \det(\tfrac{i\omega_F}{N} - H_{N-1,E}) \rangle_{H_{N-1}} \int \frac{\mathrm{d}\boldsymbol{h} \, \exp\left[-\frac{\boldsymbol{h}^T \boldsymbol{h}}{2} (|\boldsymbol{x}|^2 + \frac{N}{J^2}) \right]}{(2\pi)^{(N-1)/2}} \tag{4.77}$$

and

$$\Phi_{II} = \int \frac{\mathrm{d}H_{11}}{\sqrt{2\pi}} \, \exp\left[-\frac{|\boldsymbol{x}|^2}{2} H_{11,E}^2 - \frac{N}{4J^2} H_{11}^2 + \frac{i(\omega_{B1}+\omega_{B2})}{2N} |\boldsymbol{x}|^2 H_{11,E} \right] \\ \times \left\langle \det(\tfrac{i\omega_F}{N} - H_{N-1,E}) \int \frac{\mathrm{d}\boldsymbol{h} \, \exp\left[-\frac{\boldsymbol{h}^T \boldsymbol{h}}{2} (|\boldsymbol{x}|^2 + \frac{N}{J^2}) \right]}{(2\pi)^{(N-1)/2}} \, (\boldsymbol{h}^T (\tfrac{i\omega_F}{N} - H_{N-1,E})^{-1} \boldsymbol{h}) \right\rangle_{H_{N-1}}, \tag{4.78}$$

where the angular bracket now stands for the ensemble average over the $(N-1) \times (N-1)$ GOE matrix H_{N-1}, $\langle \cdot \rangle_{H_{N-1}} = \int \mathrm{d}H_{N-1} \, (\cdot) \exp(-\frac{N}{4J^2} \operatorname{Tr} H_{N-1}^2)$. Note that there is no normalisation factor in this definition. In both terms, integrations over H_{11} and over $\boldsymbol{h}$ factorise and can be performed (for details of the $\boldsymbol{h}$-integration in Φ_{II} see Eq. (A.75) in the appendix). After integrating them out one gets, together with Eq. (4.70),

$$\mathcal{C}_{1,2}^{(\mathrm{bulk})} = C_{\mathrm{GOE}} \int \mathrm{d}\boldsymbol{x} \, \exp\left[-\frac{\frac{NE^2|\boldsymbol{x}|^2}{J^2} + \frac{iE|\boldsymbol{x}|^2(\omega_{B1}+\omega_{B2})}{J^2} + \frac{(\omega_{B1}+\omega_{B2})^2 |\boldsymbol{x}|^4}{2N^2}}{4(|\boldsymbol{x}|^2 + \frac{N}{2J^2})} \right] \\ \times \frac{\left(\frac{i\omega_F}{N} + \frac{\frac{EN}{2J^2} - \frac{i(\omega_{B1}+\omega_{B2})|\boldsymbol{x}|^2}{2N}}{|\boldsymbol{x}|^2 + \frac{N}{2J^2}} \right) I_1 - \frac{I_2}{|\boldsymbol{x}|^2 + \frac{N}{J^2}}}{(|\boldsymbol{x}|^2 + \frac{N}{J^2})^{\frac{N-1}{2}} (|\boldsymbol{x}|^2 + \frac{N}{2J^2})^{1/2}} \exp\left[\frac{\omega_{B1}\omega_{B2}}{2N^2} |\boldsymbol{x}|^2 \right], \tag{4.79}$$

where the two short-hand notations $I_1 = \langle \det(E + \frac{i\omega_F}{N} - H_{N-1}) \rangle_{H_{N-1}}$ as well as $I_2 = \left\langle \det(E + \frac{i\omega_F}{N} - H_{N-1}) \, \operatorname{Tr}(E + \frac{i\omega_F}{N} - H_{N-1})^{-1} \right\rangle_{H_{N-1}}$ have been introduced.

Setting the problem of calculating those two quantities aside for the moment, it remains to perform the $\boldsymbol{x}$-integration for which it is advantageous to introduce rescaled polar coordinates, such that $|\boldsymbol{x}|^2 = N^2 R$. The problem then reduces to performing a

single integral

$$\int \mathrm{d}\boldsymbol{x}\, f(|\boldsymbol{x}|^2) = \frac{S_{N-1}}{2} N^N \int_0^\infty \mathrm{d}R\, R^{\frac{N}{2}-1} f(N^2 R), \tag{4.80}$$

where S_{N-1} is the surface area of the N-sphere, $S_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$. Applying the above identity to (4.79) yields

$$\begin{aligned} \mathcal{C}_{1,2}^{(\mathrm{bulk})} =& C_N \int_0^\infty \frac{\mathrm{d}R}{R} \exp\left[-\frac{\frac{NE^2 + iE(\omega_{B1}+\omega_{B2})}{4J^2} + \frac{(\omega_{B1}+\omega_{B2})^2 R}{8}}{1+\frac{1}{2NJ^2R}} \right] \\ & \times \frac{\left(i\omega_F + \frac{\frac{E}{2J^2R} - \frac{i(\omega_{B1}+\omega_{B2})}{2}}{1+\frac{1}{2NJ^2R}} \right) I_1 - \frac{1}{NR(1+\frac{1}{NJ^2R})} I_2}{(1+\frac{1}{NJ^2R})^{(N-1)/2} (1+\frac{1}{2NJ^2R})^{1/2}} \exp\left[\frac{\omega_{B1}\omega_{B2}}{2} R \right], \end{aligned} \tag{4.81}$$

where the constant is given by $C_N = \frac{1}{2N} S_{N-1} C_{\mathrm{GOE}}$.

So far the computation was exact. The next step is to take the limit $N \to \infty$, such that $1/(NJ^2R) \ll 1$ and $(1+\frac{1}{NJ^2R})^{(N-1)/2} \to \exp(1/(2J^2R))$. Note further that

$$-\frac{\frac{NE^2}{4J^2}}{1+\frac{1}{2NJ^2R}} = -\frac{NE^2}{4J^2} + \frac{E^2}{8J^4R} + \mathcal{O}\left(\frac{1}{N}\right). \tag{4.82}$$

Then the leading contribution to the integral can be written as

$$\begin{aligned} \mathcal{C}_{1,2}^{(\mathrm{bulk})} \approx & C_N \exp\left[-\frac{NE^2 + iE(\omega_{B1}+\omega_{B2})}{4J^2} \right] \int_0^\infty \frac{\mathrm{d}R}{R} \exp\left[-\frac{(\omega_{B1}-\omega_{B2})^2 R}{8} - \frac{\pi^2\rho^2}{2R} \right] \\ & \times \left(\left(i\omega_F + \frac{E}{2J^2R} - \frac{i(\omega_{B1}+\omega_{B2})}{2} \right) I_1 - \frac{1}{NR} I_2 \right), \end{aligned} \tag{4.83}$$

where the result was simplified by introducing the GOE level density $\rho = \frac{\sqrt{4J^2-E^2}}{2\pi J^2}$, see Eq. (1.9). Using Stirling's formula to approximate the Gamma-function, the proportionality factor simplifies to $C_N \approx \frac{1}{2\sqrt{\pi N}} (\frac{\pi e}{N})^{N/2} (\frac{N}{2\pi J^2})^{N(N+1)/4}$. In order to evaluate the remaining integral we use the identity (see e.g. 3.471.12 in [59])

$$\int_0^\infty dx\, x^{\nu-1} \exp\left(-\frac{A}{x} - Bx \right) = 2 \left(\frac{A}{B} \right)^{\nu/2} K_\nu(2\sqrt{AB}), \tag{4.84}$$

where K_ν denotes the modified Bessel function of second kind (MacDonald function). We apply the above identity with parameters $\nu = 0$ and $\nu = -1$ to Eq. (4.83) and use $K_{-1}(x) = K_1(x)$ to finally obtain

$$\begin{aligned}\mathcal{C}_{1,2}^{(\mathrm{bulk})} \approx & C_N \exp\left[-\frac{NE^2 + iE(\omega_{B1}+\omega_{B2})}{4J^2}\right] \\ & \times \left\{ \frac{|\omega_{B1}-\omega_{B2}|}{\pi\rho}\left(\frac{E}{2J^2}I_1 - \frac{1}{N}I_2\right) K_1\left(\frac{\pi\rho|\omega_{B1}-\omega_{B2}|}{2}\right)\right. \\ & \left. - iI_1(\omega_{B1}+\omega_{B2}-2\omega_F) K_0\left(\frac{\pi\rho|\omega_{B1}-\omega_{B2}|}{2}\right)\right\}. \end{aligned} \tag{4.85}$$

The last step is to determine the quantities $I_1 = \langle \det(E + \frac{i\omega_F}{N} - H_{N-1})\rangle_{H_{N-1}}$ and $I_2 = \left\langle \det(E + \frac{i\omega_F}{N} - H_{N-1})\ \mathrm{Tr}(E + \frac{i\omega_F}{N} - H_{N-1})^{-1}\right\rangle_{H_{N-1}}$. However, it actually suffices to calculate the first quantity, because of the following identity (let λ_j, $j = 1, \ldots, N-1$ be the eigenvalues of H_{N-1}):

$$\begin{aligned}\frac{\mathrm{d}I_1}{\mathrm{d}\omega_F} &= \frac{\mathrm{d}}{\mathrm{d}\omega_F}\left\langle \prod_{j=1}^{N-1}\left(E + \tfrac{i\omega_F}{N} - \lambda_j\right)\right\rangle_{H_{N-1}} = \frac{\mathrm{d}}{\mathrm{d}\omega_F}\left\langle \exp\left[\sum_{j=1}^{N-1}\ln\left(E + \tfrac{i\omega_F}{N} - \lambda_j\right)\right]\right\rangle_{H_{N-1}} \\ &= \left\langle \frac{i}{N}\sum_{j=1}^{N-1}\frac{1}{E + \frac{i\omega_F}{N} - \lambda_j}\prod_{j=1}^{N-1}\left(E + \tfrac{i\omega_F}{N} - \lambda_j\right)\right\rangle_{H_{N-1}} = \frac{i}{N}I_2. \end{aligned} \tag{4.86}$$

By integrating out the first row and column of H, we have reduced the problem of finding $\mathcal{C}_{1,2}^{(\mathrm{bulk})}$ to the much simpler problem of calculating the ensemble average of the reduced matrix H_{N-1} of the characteristic polynomial $\det(E + i\omega_F/N - H_{N-1})$. Here we will calculate it using the supersymmetry method established in Section 2.2, but note that one could calculate it as well by different means and avoid supersymmetry altogether.

First we replace the determinant by an integral over an $N-1$ component vector $\boldsymbol{\chi}$ of Grassmann variables,

$$I_1 = \int \mathrm{d}\boldsymbol{\chi}\mathrm{d}\boldsymbol{\chi}^\dagger \exp\left[\left(E + \frac{i\omega_F}{N}\right)\boldsymbol{\chi}^\dagger\boldsymbol{\chi}\right]\left\langle \exp\left[-\boldsymbol{\chi}^\dagger H_{N-1}\boldsymbol{\chi}\right]\right\rangle_{H_{N-1}}. \tag{4.87}$$

The ensemble average $\langle\cdot\rangle_{H_{N-1}} = \int \mathrm{d}H_{N-1}\,(\cdot)\exp(-\frac{N}{4J^2}\,\mathrm{Tr}\,H_{N-1}^2)$ can now easily be

performed and yields (mind that it was defined without a normalisation constant)

$$\left\langle \exp\left[-\boldsymbol{\chi}^\dagger H_{N-1}\boldsymbol{\chi}\right]\right\rangle_{H_{N-1}} = 2^{\frac{N-1}{2}}\left(\frac{2\pi J^2}{N}\right)^{\frac{N(N-1)}{4}} \exp\left[-\frac{J^2}{2N}(\boldsymbol{\chi}^\dagger\boldsymbol{\chi})^2\right]. \tag{4.88}$$

This exponential can now be replaced by a Gaussian integral (see Eq. (2.88)) which yields

$$I_1 = B_N \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{2\pi}} \int \mathrm{d}\boldsymbol{\chi}\mathrm{d}\boldsymbol{\chi}^\dagger \exp\left[-\frac{q^2}{2} + i\boldsymbol{\chi}^\dagger\boldsymbol{\chi}\left(\frac{Jq}{\sqrt{N}} - iE + \frac{\omega_F}{N}\right)\right], \tag{4.89}$$

where B_N is the constant defined via Eq. (4.88). After performing Grassmann integration and shifting $q \to \frac{\sqrt{N}}{J}(iE - \frac{\omega_F}{N}) + q$ we are hence left with

$$\begin{aligned} I_1 &= \left(\frac{iJ}{\sqrt{N}}\right)^{N-1} B_N \int_{-\infty}^{\infty} \frac{\mathrm{d}q\, q^{N-1}}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(q + \frac{i\sqrt{N}E - \omega_F/\sqrt{N}}{J}\right)^2\right] \\ &= \pi^{\frac{N(N-1)}{4}}\left(\frac{2J^2}{N}\right)^{\frac{N(N+1)}{4}-\frac{1}{2}} H_{N-1}\left(\frac{\sqrt{N}E + i\omega_F/\sqrt{N}}{J}\right), \end{aligned} \tag{4.90}$$

where $H_{N-1}(x)$ denotes a Hermite polynomial, see Eq. (2.19) (which should not to be confused with the reduced matrix H_{N-1}). While this result is exact, it is for our purpose advantageous to express I_1 in its large-N limit since the overall result (4.85) for $\mathcal{C}_{1,2}^{(\mathrm{bulk})}$ is already an approximation. To that end we can use the approximate result (2.22) for $H_{N-k}(\sqrt{N}z)$ derived in Section 2.1.2 with $k = 1$ and $z = \frac{E+i\omega_F/N}{J}$,

$$\begin{aligned} I_1 \approx & i^{N-1}\frac{2^{-N/2}J^{N-1/2}}{(4J^2 - E^2)^{1/4}}\left(\frac{2\pi J^2}{N}\right)^{N(N-1)/4} \exp\left[-\frac{N}{2} + \frac{NE^2}{4J^2} + \frac{iE\omega_F}{2J^2}\right] \\ & \times\left\{\tilde{A}^*(1,N,\tfrac{E-i\omega_F/N}{J}) - (-1)^N\tilde{A}(1,N,\tfrac{E+i\omega_F/N}{J})\right\}, \end{aligned} \tag{4.91}$$

where we neglected the term of order $\mathcal{O}(1/N)$ in $(4J^2 - E^2)^{1/4}$ and in the exponential. $\tilde{A}(1,N,z) = (\sqrt{4-z^2} + iz)^{N-1/2}\exp(iNz\sqrt{4-z^2}/4)$ is as defined in (2.23), but has also to be understood with all sub-leading orders being neglected. This needs some further consideration. We expand $\sqrt{4J^2 - (E \pm i\omega_F/N)^2} = 2\pi J^2\rho \mp \frac{iE\omega_F}{2N\pi J^2\rho} + \mathcal{O}(1/N^2)$, where we identified $\sqrt{4J^2 - E^2} = 2\pi J^2\rho$. Furthermore for the

term $(\sqrt{4-z^2}+iz)^{N-1/2}$ we use $(a+b/N)^{N-1/2} \to a^{N-1/2}\exp(ba^{-1})$ for $N \to \infty$. This yields

$$\tilde{A}(1,N,\tfrac{E\pm i\omega_F/N}{J}) \approx J^{1/2-N}(2\pi J^2\rho + iE)^{N-1/2}\exp(\mp\pi\rho\omega_F + iEN\pi\rho/2), \tag{4.92}$$

and hence I_1 is given asymptotically by

$$\begin{aligned} I_1 \approx -\left(\frac{2\pi J^2}{N}\right)^{\frac{N(N-1)}{4}} \frac{(-i)^{N-1}}{2^{\frac{N+1}{2}}J\sqrt{\pi\rho}} \exp\left[-\frac{N}{2}+\frac{NE^2}{4J^2}+\frac{iE\omega_F}{2J^2}\right] \\ \times \left(A(E,N)\exp\left[-\pi\rho\omega_F\right] - (-1)^N A^*(E,N)\exp\left[+\pi\rho\omega_F\right]\right), \end{aligned} \tag{4.93}$$

where we defined $A(E,N) = J^{N-1/2}\tilde{A}(1,N,\frac{E}{J}) = (2\pi J^2\rho+iE)^{N-1/2}\exp\left[\frac{iEN\pi\rho}{2}\right]$ as in the previous approach, Eq. (4.68). From this, I_2 follows immediately via the derivative of the above equation w.r.t. ω_F, Eq. (4.86), and is hence given by

$$\begin{aligned} I_2 \approx & \frac{N}{i}\left(\frac{2\pi J^2}{N}\right)^{\frac{N(N-1)}{4}} \frac{(-i)^{N-1}\sqrt{\pi\rho}}{2^{\frac{N+1}{2}}J} \exp\left[-\frac{N}{2}+\frac{NE^2}{4J^2}+\frac{iE\omega_F}{2J^2}\right] \\ & \times \left(A(E,N)\exp\left[-\pi\rho\omega_F\right] + (-1)^N A^*(E,N)\exp\left[+\pi\rho\omega_F\right]\right) + \frac{NE}{2J^2}I_1. \end{aligned} \tag{4.94}$$

Now we insert these asymptotic results for I_1 and I_2 into Eq. (4.85) and observe some cancellations: The term $\frac{NE}{2J^2}I_1$, featured in the expression for I_2, cancels in the term $\left(\frac{E}{2J^2}I_1 - \frac{1}{N}I_2\right)$. Moreover the term $\exp[\frac{NE^2}{4J^2}]$ featured in I_1 and I_2 cancels with $\exp[-\frac{NE^2}{4J^2}]$ from (4.85). Inserting the remaining terms one gets

$$\begin{aligned} \mathcal{C}_{1,2}^{(\text{bulk, } \operatorname{sgn}\omega_{B1}=-\operatorname{sgn}\omega_{B2})}(\omega_{F1};\omega_{B1},\omega_{B2}) \approx & \frac{(-i)^N}{\pi\sqrt{2N\rho}(2J)^{N+1}}\, e^{-\frac{iE}{4J^2}(\omega_{B1}+\omega_{B2}-2\omega_{F1})} \\ \times \Big\{ & [Ae^{-\pi\rho\omega_{F1}} - (-1)^N A^* e^{+\pi\rho\omega_{F1}}](\omega_{B1}+\omega_{B2}-2\omega_{F1}) K_0\left(\tfrac{\pi\rho}{2}|\omega_{B1}-\omega_{B2}|\right) \\ & + [Ae^{-\pi\rho\omega_{F1}} + (-1)^N A^* e^{+\pi\rho\omega_{F1}}]|\omega_{B1}-\omega_{B2}| K_1\left(\tfrac{\pi\rho}{2}|\omega_{B1}-\omega_{B2}|\right)\Big\}. \end{aligned} \tag{4.95}$$

This is the exact same result as (4.67) which was obtained following a supersymmetric approach. One advantage of the present approach is that the large-N limit can be taken easily, in contrast to the previous approach where one of the main difficulties was performing a quite cumbersome saddle-point analysis. Note, however, that the second approach is not suitable for the case $\operatorname{sgn}\omega_{B1} = \operatorname{sgn}\omega_{B2}$.

4.4. Calculation of Correlation Function $\mathcal{C}_{2,2}$

In this section we want to calculate the correlation function

$$\mathcal{C}_{2,2} = \left\langle \frac{\det(\mu_{F1} - H)\det(\mu_{F2} - H)}{\det(\mu_{B1} - H)^{1/2}\det(\mu_{B2} - H)^{1/2}} \right\rangle_{\text{GOE}}. \tag{4.96}$$

First we choose the same approach as in Section 4.3.1. As a second approach we want to show that also the Hubbard-Stratonovich method described in Section 2.2.4 yields the same answer.

4.4.1. Hybrid Method

The model is in complete analogy to (4.23) but with two integrals over Grassmann vectors $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$, and after performing the ensemble average given by

$$\begin{aligned}\mathcal{C}_{2,2} \propto \int \mathrm{d}\boldsymbol{x}_1 \int \mathrm{d}\boldsymbol{x}_2 \int \mathrm{d}\boldsymbol{\zeta}_1 \mathrm{d}\boldsymbol{\zeta}_1^\dagger \int \mathrm{d}\boldsymbol{\zeta}_2 \mathrm{d}\boldsymbol{\zeta}_2^\dagger \exp\left[i(s_1\mu_{B1}\boldsymbol{x}_1^T\boldsymbol{x}_1 + s_2\mu_{B2}\boldsymbol{x}_2^T\boldsymbol{x}_2)\right] \\ \times \exp\left[-i(\mu_{F1}\boldsymbol{\zeta}_1^\dagger\boldsymbol{\zeta}_1) + \mu_{F2}\boldsymbol{\zeta}_2^\dagger\boldsymbol{\zeta}_2)\right] \exp\left[-\frac{J^2}{4N}\operatorname{Tr}(A + A^T)^2\right],\end{aligned} \tag{4.97}$$

where now $A = s_1\,\boldsymbol{x}_1 \otimes \boldsymbol{x}_1^T + s_2\,\boldsymbol{x}_2 \otimes \boldsymbol{x}_2^T + \boldsymbol{\zeta}_1 \otimes \boldsymbol{\zeta}_1^\dagger + \boldsymbol{\zeta}_2 \otimes \boldsymbol{\zeta}_2^\dagger$ and the proportionality factor is given by $(-i)^N(s_1s_2)^{N/2}$. Introducing the matrices

$$Q_F = \begin{bmatrix} \boldsymbol{\zeta}_1^\dagger\boldsymbol{\zeta}_1 & \boldsymbol{\zeta}_1^\dagger\boldsymbol{\zeta}_2 \\ \boldsymbol{\zeta}_2^\dagger\boldsymbol{\zeta}_1 & \boldsymbol{\zeta}_2^\dagger\boldsymbol{\zeta}_2 \end{bmatrix},\ Q_B = \begin{bmatrix} \boldsymbol{x}_1^T\boldsymbol{x}_1 & \boldsymbol{x}_1^T\boldsymbol{x}_2 \\ \boldsymbol{x}_2^T\boldsymbol{x}_1 & \boldsymbol{x}_2^T\boldsymbol{x}_2 \end{bmatrix},\ L = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}, \tag{4.98}$$

and abbreviating

$$B = s_1\boldsymbol{x}_1 \otimes \boldsymbol{x}_1^T + s_2\boldsymbol{x}_2 \otimes \boldsymbol{x}_2^T \tag{4.99}$$

we can rewrite the trace to

$$\operatorname{Tr}\left(\frac{A + A^T}{2}\right)^2 = 4\operatorname{Tr}(Q_BL)^2 - 2\operatorname{Tr}Q_F^2 + 4\boldsymbol{\zeta}_1^T\boldsymbol{\zeta}_2\boldsymbol{\zeta}_2^\dagger\boldsymbol{\zeta}_1^* - 8\boldsymbol{\zeta}_1^\dagger B\boldsymbol{\zeta}_1 - 8\boldsymbol{\zeta}_2^\dagger B\boldsymbol{\zeta}_2. \tag{4.100}$$

Note that if we set $\boldsymbol{\zeta}_2 \equiv 0$ we recover the corresponding expression (4.25) for $\mathcal{C}_{1,2}$ (such that Q_F becomes scalar).

In this previous case we bilinearised the terms involving Grassmann variables by introducing a Gaussian integral. In analogy to this we perform here a Hubbard-Stratonovich transformation as described in (2.90),

$$\exp\left(\frac{J^2}{2N}Q_F^2\right) = \frac{N^2}{2\pi^2 J^4}\int \mathrm{d}\hat{Q}_F\, \exp\left(-\frac{N}{2J^2}\operatorname{Tr}\hat{Q}_F^2 + \operatorname{Tr}\hat{Q}_F Q_F\right), \tag{4.101}$$

where $\hat{Q}_F$ is a Hermitian 2×2 matrix whose entries are commuting variables, and furthermore we can rewrite

$$\operatorname{Tr}\hat{Q}_F Q_F = \begin{bmatrix}\boldsymbol{\zeta}_1^\dagger & \boldsymbol{\zeta}_2^\dagger\end{bmatrix}(\hat{Q}_F\otimes \mathbb{1}_N)\begin{bmatrix}\boldsymbol{\zeta}_1\\ \boldsymbol{\zeta}_2\end{bmatrix}. \tag{4.102}$$

The term $4\boldsymbol{\zeta}_1^T\boldsymbol{\zeta}_2\boldsymbol{\zeta}_2^\dagger\boldsymbol{\zeta}_1^*$ is a new feature which was not present to the previous case, however, we can bilinearise it as well using the identity

$$\exp(-v_1 v_2) = \frac{1}{\pi}\int \mathrm{d}^2u\, \exp(-u^*u - i(uv_1 + u^*v_2)), \tag{4.103}$$

where $\mathrm{d}^2u = \mathrm{d}(\operatorname{Re}u)\,\mathrm{d}(\operatorname{Im}u)$. This yields in our case (rescaling $u\to\frac{\sqrt{N}}{J}u$)

$$\exp\left(-\frac{J^2}{N}\boldsymbol{\zeta}_1^T\boldsymbol{\zeta}_2\boldsymbol{\zeta}_2^\dagger\boldsymbol{\zeta}_1^*\right) = \frac{N}{\pi J^2}\int \mathrm{d}^2u\, \exp\left[-\frac{N}{J^2}u^*u - i(u\boldsymbol{\zeta}_1^\dagger\boldsymbol{\zeta}_2^* + u^*\boldsymbol{\zeta}_2^T\boldsymbol{\zeta}_1)\right]. \tag{4.104}$$

Introduce now the vector $\boldsymbol{\zeta}^T = \begin{bmatrix}\boldsymbol{\zeta}_1^\dagger & \boldsymbol{\zeta}_1^T & \boldsymbol{\zeta}_2^\dagger & \boldsymbol{\zeta}_2^T\end{bmatrix}$, then we may rewrite and perform the integral over the Grassmann variables,

$$\begin{aligned}&\int \mathrm{d}\boldsymbol{\zeta}_1\mathrm{d}\boldsymbol{\zeta}_1^\dagger\int \mathrm{d}\boldsymbol{\zeta}_2\mathrm{d}\boldsymbol{\zeta}_2^\dagger \exp\left\{\begin{bmatrix}\boldsymbol{\zeta}_1^\dagger & \boldsymbol{\zeta}_2^\dagger\end{bmatrix}(\hat{Q}_F\otimes\mathbb{1}_N)\begin{bmatrix}\boldsymbol{\zeta}_1\\ \boldsymbol{\zeta}_2\end{bmatrix} - i(u\boldsymbol{\zeta}_1^\dagger\boldsymbol{\zeta}_2^* + u^*\boldsymbol{\zeta}_2^T\boldsymbol{\zeta}_1)\right\}\\ &\times\exp\left[\sum_{j=1}^{2}\boldsymbol{\zeta}_j^\dagger(\tfrac{2J^2}{N}B - i\mu_{Fj}\mathbb{1}_N)\boldsymbol{\zeta}_j\right] = \int \mathrm{d}\boldsymbol{\zeta}\, \exp(\tfrac{1}{2}\boldsymbol{\zeta}^T\mathcal{M}\boldsymbol{\zeta}) = \sqrt{\det\mathcal{M}},\end{aligned} \tag{4.105}$$

where we used identity (2.55) and the fact that $\mathcal{M}$ is skew-symmetric ($\mathcal{M}^T = -\mathcal{M}$).

It is explicitly given by

$$\mathcal{M} = \begin{bmatrix} 0 & +A_1 & -iu\mathbb{1}_N & +q_{12}^*\mathbb{1}_N \\ -A_1 & 0 & -q_{12}\mathbb{1}_N & +iu^*\mathbb{1}_N \\ +iu\mathbb{1}_N & +q_{12}\mathbb{1}_N & 0 & +A_2 \\ -q_{12}^*\mathbb{1}_N & -iu^*\mathbb{1}_N & -A_2 & 0 \end{bmatrix}, \tag{4.106}$$

where the q_{ij} are the entries of the matrix $\hat{Q}_F$ and

$$A_j = q_{jj}\mathbb{1}_N - i\mu_{Fj}\mathbb{1}_N + \frac{2J^2}{N}B. \tag{4.107}$$

Recall that B, defined in (4.99), is real symmetric and has only two non-vanishing eigenvalues which are equal to the eigenvalues of $Q_B L$. Hence we can diagonalise $B = O^T \operatorname{diag}(\lambda_B^{(1)}, \lambda_B^{(2)}, 0, \ldots, 0)O$, and furthermore $\det \mathcal{M}$ is independent of the orthogonal matrix O and splits into the product

$$\det \mathcal{M} = \det \mathcal{M}_0^{N-2} \det \mathcal{M}_1(\lambda_B^{(1)}) \det \mathcal{M}_1(\lambda_B^{(2)}). \tag{4.108}$$

$\mathcal{M}_0$ is a 4×4 matrix given by Eq. (4.106) with $B \equiv 0$ and $\mathbb{1}_N \equiv 1$. Its structure suggests that its determinant is given by

$$\begin{aligned} \det \mathcal{M}_0 &= [(q_{11} - i\mu_{F1})(q_{22} - i\mu_{F2}) - q_{12}^* q_{12} - u^* u]^2 \\ &= \left[\det(\hat{Q}_F - i \operatorname{diag}(\mu_{F1}, \mu_{F2})) - u^* u\right]^2. \end{aligned} \tag{4.109}$$

$\mathcal{M}_1(\lambda_B^{(1,2)})$ is of the same structure as $\mathcal{M}_0$ with $q_{jj} \to q_{jj} + \frac{2J^2}{N}\lambda_B^{(1,2)}$ and hence

$$\begin{aligned} \det \mathcal{M}_1(\lambda_B^{(1,2)}) = \Bigg[&\det(\hat{Q}_F - i \operatorname{diag}(\mu_{F1}, \mu_{F2})) - u^* u + \frac{4J^4}{N^2}\lambda_B^{(1,2)} \\ &+ \frac{2J^2}{N}\lambda_B^{(1,2)} \operatorname{Tr}(\hat{Q}_F - i \operatorname{diag}(\mu_{F1}, \mu_{F2}))\Bigg]^2. \end{aligned} \tag{4.110}$$

These results conclude the integration over the Grassmann variables.

For the integration over the commuting vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ we can proceed as we did for the previous correlation function, using the integration theorem (2.103), i.e. replacing integration over the vectors with an integral over a positive definite real

symmetric 2×2 matrix $\hat{Q}_B$. After shifting $\hat{Q}_F \to \hat{Q}_F + i\,\mathrm{diag}(\mu_{F1},\mu_{F2})$ and rescaling $\hat{Q}_B \to \frac{N}{2J^2}\hat{Q}_B$ the correlation function is then given by

$$\begin{aligned}\mathcal{C}_{2,2} \propto \int \mathrm{d}\hat{Q}_B\,(\det \hat{Q}_B)^{\frac{N-3}{2}} \exp\left[-\frac{N}{4J^2}\,\mathrm{Tr}(\hat{Q}_B L)^2 + \frac{iN}{2J^2}\,\mathrm{Tr}\,\hat{Q}_B L\,\mathrm{diag}(\mu_{B1},\mu_{B2})\right] \\ \times \int \mathrm{d}\hat{Q}_F \int \mathrm{d}^2u\,\exp\left[-\frac{N}{2J^2}\,\mathrm{Tr}(\hat{Q}_F - i\,\mathrm{diag}(\mu_{F1},\mu_{F2}))^2 - \frac{N}{J^2}u^*u\right] \\ \times \left[\det \hat{Q}_F - u^*u\right]^{N-2} \prod_{j=1}^{2}\left[\det \hat{Q}_F - u^*u + (\lambda_B^{(j)})^2 + \lambda_B^{(j)}\,\mathrm{Tr}\,\hat{Q}_F\right],\end{aligned} \tag{4.111}$$

where $\lambda_B^{(1,2)}$ are now to be understood as the eigenvalues of $\hat{Q}_B L$. The proportionality factor is $\frac{(-i)^N (s_1 s_2)^{N/2}}{8\pi^{3/2}\Gamma(N-1)}\left(\frac{N}{\pi J^2}\right)^{N+3}$.

We managed to reduce the model to an integral over the positive definite real symmetric 2×2 matrix $\hat{Q}_B$ (three independent variables), the Hermitian 2×2 matrix $\hat{Q}_F$ (four independent variables) and the complex variable u (two independent variables), i.e. we are left with a total of nine integrations. Compare this to the corresponding equation for the previous correlation function, Eq. (4.31), where in addition to the same matrix $\hat{Q}_B$ only one more integration was present. Hence including one more determinant in the numerator accounts for five more integrations at this point. We proceed to integrate out as many of these variables as possible.

Integration over $\hat{Q}_F$

We start by diagonalising $\hat{Q}_F = U\,\mathrm{diag}(q_{F1},q_{F2})U^\dagger$, where U is unitary since $\hat{Q}_F$ is Hermitian. The measure changes accordingly to $\mathrm{d}\hat{Q}_F = (q_{F1}-q_{F2})^2 \mathrm{d}q_{F1}\,\mathrm{d}q_{F2}\,\mathrm{d}\mu(U)$. Almost all terms in (4.111) are invariant under such transformation, the only exception being the term $\mathrm{Tr}(\hat{Q}_F - i\,\mathrm{diag}(\mu_{F1},\mu_{F2})^2)$ in the second line of (4.111) which becomes

$$\mathrm{Tr}\,\hat{Q}_F^2 - 2i\,\mathrm{Tr}\,U\,\mathrm{diag}(q_{F1},q_{F2})U^\dagger\,\mathrm{diag}(\mu_{F1},\mu_{F2}) - \mathrm{Tr}\,\mathrm{diag}(\mu_{F1}^2,\mu_{F2}^2). \tag{4.112}$$

However, the integral over the unitary group is now of the Ityzkson-Zuber-Harish-Chandra type [80–82] and can be performed exactly (see Eq. (3.25)),

$$\begin{aligned}&\int \mathrm{d}\mu(U) \exp\left[\frac{iN}{J^2}\operatorname{Tr} U \operatorname{diag}(q_{F1}, q_{F2}) U^\dagger \operatorname{diag}(\mu_{F1}, \mu_{F2})\right] \\ &= \frac{J^2}{iN} \frac{\exp\left[\frac{iN}{J^2}(q_{F1}\mu_{F1} + q_{F2}\mu_{F2})\right] - \exp\left[\frac{iN}{J^2}(q_{F1}\mu_{F2} + q_{F2}\mu_{F1})\right]}{(q_{F1} - q_{F2})(\mu_{F1} - \mu_{F2})}.\end{aligned} \tag{4.113}$$

Next let us introduce the matrix

$$R = \begin{bmatrix} q_{F1} & u^* \\ u & q_{F2} \end{bmatrix}. \tag{4.114}$$

The advantage is that we can identify $\det \hat{Q}_F - u^* u = \det R$, $\operatorname{Tr} \hat{Q}_F = \operatorname{Tr} R$, $\operatorname{Tr} \hat{Q}_F^2 + 2u^* u = \operatorname{Tr} R^2$. In terms of R (we identify further $q_{F1} \equiv R_{11}$ and $q_{F2} \equiv R_{22}$) the correlation function (4.111) becomes

$$\begin{aligned}\mathcal{C}_{2,2} \propto &\int \mathrm{d}\hat{Q}_B \, (\det \hat{Q}_B)^{\frac{N-3}{2}} \exp\left[-\frac{N}{4J^2}\operatorname{Tr}(\hat{Q}_B L)^2 + \frac{iN}{2J^2}\operatorname{Tr}\hat{Q}_B L \operatorname{diag}(\mu_{B1}, \mu_{B2})\right] \\ &\times \int \mathrm{d}R \, (R_{11} - R_{22}) \det R^{N-2} \exp\left[-\frac{N}{2J^2}(\operatorname{Tr} R^2 - \mu_{F1}^2 - \mu_{F2}^2)\right] \\ &\times \prod_{j=1}^{2}\left[\det R + \lambda_B^{(j)} \operatorname{Tr} R + (\lambda_B^{(j)})^2\right] \\ &\times \frac{\exp\left[\frac{iN}{J^2}(R_{11}\mu_{F1} + R_{22}\mu_{F2})\right] - \exp\left[\frac{iN}{J^2}(R_{11}\mu_{F2} + R_{22}\mu_{F1})\right]}{(\mu_{F1} - \mu_{F2})}.\end{aligned} \tag{4.115}$$

Now R is Hermitian as well and thus we can diagonalise it via $R = U_2 \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} U_2^\dagger$. The measure becomes accordingly $\mathrm{d}R = (r_1 - r_2)^2 \mathrm{d}r_1 \, \mathrm{d}r_2 \, \mathrm{d}\mu(U_2)$. However, the integrand is no longer of a form where one can apply the IZHC formula, instead we parametrise U_2, being unitary, as

$$U_2 = \begin{bmatrix} e^{+i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \ \phi \in [0, 2\pi], \ \theta \in [0, \pi/2]. \tag{4.116}$$

This implies

$$\mathrm{d}\mu(U_2) = \sin(2\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi, \quad R_{jj} = \tfrac{1}{2}(r_1 + r_2) \pm \tfrac{1}{2}(r_1 - r_2)\cos(2\theta), \tag{4.117}$$

where the plus-sign is chosen for $j = 1$ and the minus-sign for $j = 2$. Furthermore

$$\begin{aligned} &\exp\left[\frac{iN}{J^2}(R_{11}\mu_{F1} + R_{22}\mu_{F2})\right] - \exp\left[\frac{iN}{J^2}(R_{11}\mu_{F2} + R_{22}\mu_{F1})\right] \\ &= 2i\exp\left[\frac{iN}{2J^2}(r_1 + r_2)(\mu_{F1} + \mu_{F2})\right]\sin\left[\frac{N}{2J^2}(r_1 - r_2)(\mu_{F1} - \mu_{F2})\cos(2\theta)\right]. \end{aligned} \tag{4.118}$$

All other terms in the integrand are traces or determinants of R and thus independent of U_2. Furthermore the whole integrand is independent of ϕ and thus integration over ϕ just yields a constant factor. The θ-integration amounts to perform the integral

$$2\int_0^{\pi/2}\mathrm{d}\theta\,\sin(2\theta)\cos(2\theta)\sin\left(\frac{N(r_1 - r_2)(\mu_{F1} - \mu_{F1})}{2J^2}\cos(2\theta)\right). \tag{4.119}$$

We change $\cos(2\theta) = x$, and hence $\mathrm{d}x = 2\sin(2\theta)\mathrm{d}\theta$. Then the integral simplifies to

$$\int_0^1 \mathrm{d}x\,x\sin\left(\frac{N(r_1 - r_2)(\mu_{F1} - \mu_{F2})x}{2J^2}\right). \tag{4.120}$$

This can be solved using integration by parts and yields the result

$$\begin{aligned} &-\left[\frac{2J^2}{N(r_1 - r_2)(\mu_{F1} - \mu_{F2})}\cos\left(\frac{N(r_1 - r_2)(\mu_{F1} - \mu_{F2})}{2J^2}\right)\right. \\ &\left.-\frac{4J^4}{N^2(r_1 - r_2)^2(\mu_{F1} - \mu_{F2})^2}\sin\left(\frac{N(r_1 - r_2)(\mu_{F1} - \mu_{F2})}{2J^2}\right)\right]. \end{aligned} \tag{4.121}$$

Taking all the above equations into account, the correlation function simplifies to

$$\begin{aligned}\mathcal{C}_{2,2} =& C_N \int \mathrm{d}\hat{Q}_B \,(\det \hat{Q}_B)^{\frac{N-3}{2}} \exp\left[-\frac{N}{4J^2}\operatorname{Tr}(\hat{Q}_B L)^2 + \frac{iN}{2J^2}\operatorname{Tr}\hat{Q}_B L \operatorname{diag}(\mu_{B1},\mu_{B2})\right] \\ &\times \int_{-\infty}^{\infty}\mathrm{d}r_1 \int_{-\infty}^{\infty}\mathrm{d}r_2 \exp\left[-\frac{N}{2J^2}(r_1^2+r_2^2 - i(r_1+r_2)(\mu_{F1}+\mu_{F2}))\right] \\ &\times (r_1 r_2)^{N-2}(r_1-r_2)(r_1+\lambda_B^{(1)})(r_2+\lambda_B^{(1)})(r_1+\lambda_B^{(2)})(r_2+\lambda_B^{(2)}) \\ &\times \frac{1}{(\mu_{F1}-\mu_{F2})^3}\exp\left[\frac{N}{2J^2}(\mu_{F1}^2+\mu_{F2}^2)\right] \\ &\left[(\mu_{F1}-\mu_{F2})(r_1-r_2)\cos\left(\frac{N(r_1-r_2)(\mu_{F1}-\mu_{F2})}{2J^2}\right)\right. \\ &\left.-\frac{2J^2}{N}\sin\left(\frac{N(r_1-r_2)(\mu_{F1}-\mu_{F2})}{2J^2}\right)\right],\end{aligned} \tag{4.122}$$

where $C_N = \frac{i^N (s_1 s_2)^{N/2}}{2\pi^{5/2}\Gamma(N-1)}\left(-\frac{N}{\pi J^2}\right)^{N+1}$. We managed to perform four more integrals, leaving us with integration over the two variables r_1 and r_2 and the three independent variables of $\hat{Q}_B$. Since $\hat{Q}_B$ did not change from the previous case in Section 4.3.1 we can use the same method to perform the remaining integrations. As we have already seen it is of crucial difference if the imaginary parts of μ_{B1} and μ_{B2} have same or different signs.

Large-N limit for the same-sign case

We use the same parametrisation for $\hat{Q}_B$ as in Section 4.3.1 (see the paragraph above (4.33)). The eigenvalues of $\hat{Q}_B L$ in the new coordinates are $\lambda_B^{(1)} \equiv sp_1$, $\lambda_B^{(2)} \equiv sp_2$, where $s = \operatorname{sgn}\operatorname{Im}\mu_{B1} = \operatorname{sgn}\operatorname{Im}\mu_{B2}$. After performing the ϕ-integration, which is the same as in Section 4.3.1 and hence yields again a Bessel function, the correlation

function is then given by

$$\begin{aligned}\mathcal{C}_{2,2} =& \frac{\pi C_N}{2} \frac{\exp\left[\frac{N}{2J^2}(\mu_{F1}^2+\mu_{F2}^2)\right]}{(\mu_{F1}-\mu_{F2})^3} \int_0^\infty \mathrm{d}p_1 \int_0^\infty \mathrm{d}p_2\, |p_1-p_2| (p_1 p_2)^{\frac{N-3}{2}} \\ & \times \exp\left[-\frac{N}{4J^2}(p_1^2+p_2^2) + \frac{isN}{4J^2}(\mu_{B1}+\mu_{B2})(p_1+p_2)\right] \\ & \times \int_{-\infty}^\infty \mathrm{d}r_1 \int_{-\infty}^\infty \mathrm{d}r_2 \exp\left[-\frac{N}{2J^2}(r_1^2+r_2^2 - i(r_1+r_2)(\mu_{F1}+\mu_{F2}))\right] \\ & \times (r_1 r_2)^{N-2}(r_1-r_2)(r_1+sp_1)(r_2+sp_1)(r_1+sp_2)(r_2+sp_2) \\ & \left[(\mu_{F1}-\mu_{F2})(r_1-r_2)\cos\left(\frac{N(r_1-r_2)(\mu_{F1}-\mu_{F2})}{2J^2}\right)\right. \\ & \left. -\frac{2J^2}{N}\sin\left(\frac{N(r_1-r_2)(\mu_{F1}-\mu_{F2})}{2J^2}\right)\right] J_0\left[\frac{sN}{4J^2}(\mu_{B1}-\mu_{B2})(p_1-p_2)\right].\end{aligned} \tag{4.123}$$

So far this is exact. Now we employ again the bulk scaling $\mu = E + i\omega/N$ for all μ's and do a saddle-point analysis. The saddle points are given by

$$p_1^{SP} = p_2^{SP} = \frac{isE + \sqrt{4J^2 - E^2}}{2}, \quad r_{1,2}^{SP} = r_\pm = \frac{-iE \pm \sqrt{4J^2-E^2}}{2}. \tag{4.124}$$

The saddle points for the p's are the same as for $\mathcal{C}_{1,2}$ and we already restricted to those with positive real part. We deform the contour through them in the same way as before, see the paragraph below (4.37). For r_1 we deform the contour to a parallel to the real axis through $-iE/2$. Note that two saddle points are on that line. For r_2 we choose the same contour. This means we get four contributions in total, from each possible combination of r-saddle-points. However, the term $(r_1 - r_2)$, evaluated at the saddle points, vanishes if one chooses $r_1^{SP} = r_2^{SP}$. This means two out of the four contributions are sub-dominant and hence negligible. Moreover notice that the integrand is invariant under the exchange $r_1 \leftrightarrow r_2$. This suggests that the remaining two contributions are identical and hence it suffices to consider just one of them. Thus we shift $r_1 = r_+ + \eta_1/\sqrt{N}$, $r_2 = r_- + \eta_1/\sqrt{N}$, $p_{1,2} = p_s + \xi_{1,2}/\sqrt{N}$, where we abbreviated $p_s = p_1^{SP} = p_2^{SP}$. The next steps are analogous to those in Section 4.3.1, i.e. we expand the terms in the exponentials of the integrand to second order in the η's and ξ's, extend the lower limits of the ξ-integrations to negative infinity and

neglect all terms of order $\mathcal{O}(N^{-1/2})$ and higher. This procedure yields

$$\begin{aligned}\mathcal{C}_{2,2}^{(\mathrm{bulk})} \propto & \exp\left[-\frac{sp_s}{2J^2}(\omega_{B1}+\omega_{B2})\right]\exp\left[-\frac{1}{2J^2}(r_+ + r_-)(\omega_{F1}+\omega_{F2})\right] \\ & \times\left[(\omega_{F1}-\omega_{F2})(r_+-r_-)\cosh\left(\frac{(r_+-r_-)(\omega_{F1}-\omega_{F2})}{2J^2}\right)\right. \\ & \left.-2J^2\sinh\left(\frac{(r_+-r_-)(\omega_{F1}-\omega_{F2})}{2J^2}\right)\right]\frac{\exp\left[\frac{iE}{J^2}(\omega_{F1}+\omega_{F2})\right]}{(\omega_{F1}-\omega_{F2})^3} \\ & \times\int_{-\infty}^{\infty}\mathrm{d}\xi_1\int_{-\infty}^{\infty}\mathrm{d}\xi_2\,\exp\left[-\frac{1}{4}\left(\frac{1}{J^2}+\frac{1}{p_s^2}\right)(\xi_1^2+\xi_2^2)\right] \\ & \times\int_{-\infty}^{\infty}\mathrm{d}\eta_1\int_{-\infty}^{\infty}\mathrm{d}\eta_2\,\exp\left[-\frac{1}{2}\left(\frac{1}{J^2}+\frac{1}{r_+^2}\right)\eta_1^2-\frac{1}{2}\left(\frac{1}{J^2}+\frac{1}{r_+^2}\right)\eta_2^2\right] \\ & \times|\xi_1-\xi_2|(\eta_{2,1}+s\xi_1)(\eta_{2,1}+s\xi_2),\end{aligned} \tag{4.125}$$

where in the last line η_2 is chosen for $s=1$ and η_1 for $s=-1$. This comes from the fact that $p_{s=1}=-r_-$ and $p_{s=-1}=r_+$ and hence in the case $s=1$ the saddle points add up to zero in the terms r_2+p_1 and r_2+p_2, while in the case $s=-1$ this happens for the terms r_1-p_1 and r_1-p_2. Note that for simplicity we also neglected any terms which depend only on E but not the ω's, hence the overall proportionality factor depends on N, J and E. The remaining integrations are all convergent and yield a factor which does not depend on the ω's either and hence we absorb it into the factor as well. We simplify the remaining expression using $r_+ + r_- = -iE$, $r_+ - r_- = \sqrt{4J^2-E^2} = 2\pi J^2\rho$ and $sp_s = \frac{iE+s\sqrt{4J^2-E^2}}{2} = \frac{iE}{2}+s\pi J^2\rho$ and get

$$\begin{aligned}\mathcal{C}_{2,2}^{(\mathrm{bulk})} \approx & f(N,E)\exp\left[-\frac{1}{2}(\omega_{B1}+\omega_{B2})\left(\frac{iE}{2J^2}+s\pi\rho\right)\right]\exp\left[\frac{iE}{2J^2}(\omega_{F1}+\omega_{F2})\right] \\ & \times\frac{(\omega_{F1}-\omega_{F2})\pi\rho\cosh\left[\pi\rho(\omega_{F1}-\omega_{F2})\right]-\sinh\left[\pi\rho(\omega_{F1}-\omega_{F2})\right]}{(\omega_{F1}-\omega_{F2})^3}.\end{aligned} \tag{4.126}$$

Since we did not track the factor when performing the saddle-point analysis, we still need to determine the function $f(N,E)$. We consider the limit where all ω are 0. The above expression becomes

$$\lim_{\omega' s\to 0}\mathcal{C}_{2,2}^{(\mathrm{bulk})} = f(N,E)\lim_{\omega\to 0}\frac{\pi\rho\omega\cosh(\pi\rho\omega)-\sinh(\pi\rho\omega)}{\omega^3} = \frac{(\pi\rho)^3}{3}f(N,E). \tag{4.127}$$

The correlation function

$$\mathcal{C}_{2,2}^{(\text{bulk})} \approx \left\langle \frac{\det(E - \frac{i\omega_{F1}}{N} - H)\det(E - \frac{i\omega_{F2}}{N} - H)}{\det(E - \frac{i\omega_{B1}}{N} - H)^{1/2}\det(E - \frac{i\omega_{B2}}{N} - H)^{1/2}} \right\rangle_{\text{GOE},\, N\to\infty} \tag{4.128}$$

on the other hand becomes in this limit

$$\lim_{\omega' s\to 0} \mathcal{C}_{2,2}^{(\text{bulk})} = \langle \det(E - H)\rangle_{\text{GOE},\, N\to\infty}\,. \tag{4.129}$$

The factor is hence given by the large-N limit of the ensemble averaged characteristic polynomial, $f(N,E) = \frac{3}{(\pi\rho)^3}\langle \det(E-H)\rangle_{N\to\infty}$. We have already computed this quantity in Section 4.3.2, where we have shown that its exact solution is a Hermite polynomial, Eq. (4.90). However, this was for the reduced matrix H_{N-1} and with a different prefactor. Adjusting for these facts the ensemble averaged characteristic polynomial is given by the Hermite polynomial

$$\langle \det(E - H)\rangle_{\text{GOE},\, N} = \left(\frac{J}{\sqrt{N}}\right)^N H_N\left(\frac{\sqrt{N}}{J}E\right). \tag{4.130}$$

Then the final result for the large-N limit of the correlation function (4.128) with same signs $\operatorname{sgn}\omega_{B1} = \operatorname{sgn}\omega_{B2}$ is

$$\begin{aligned}
&\mathcal{C}_{2,2}^{(\text{bulk},\ \operatorname{sgn}\omega_{B1}=\operatorname{sgn}\omega_{B2})}(\omega_{F1},\omega_{F2};\omega_{B1},\omega_{B2}) \approx \\
&\left(\frac{J}{\sqrt{N}}\right)^N \frac{3\tilde{H}_N\left(\frac{\sqrt{N}E}{J}\right)}{[\pi\rho(\omega_{F1}-\omega_{F2})]^3} e^{\frac{iE(\omega_{F1}+\omega_{F2})}{2J^2}} e^{-\frac{iE(\omega_{B1}+\omega_{B2})}{4J^2}} e^{-\frac{\pi\rho(|\omega_{B1}|+|\omega_{B2}|)}{2}} \\
&\times\left[\pi\rho(\omega_{F1}-\omega_{F2})\cosh\left(\pi\rho(\omega_{F1}-\omega_{F2})\right) - \sinh\left(\pi\rho(\omega_{F1}-\omega_{F2})\right)\right],
\end{aligned} \tag{4.131}$$

where $\tilde{H}_N\left(\frac{\sqrt{N}E}{J}\right) = \frac{(i\sqrt{N}/2)^N}{2\sqrt{J}\sqrt{\pi\rho}} e^{-N/2}\, e^{\frac{N}{4J^2}E^2}\left\{\tilde{A}^*(0,N,\frac{E}{J}) + (-1)^N\tilde{A}(0,N,\frac{E}{J})\right\}$ is the appropriate large-N asymptotic of the N-th Hermite polynomial, as calculated in Section 2.1.2, Eq. (2.19), with $\tilde{A}(0,N,\frac{E}{J}) = \frac{1}{J^{N+1/2}}(2\pi J^2\rho + iE)^{N+1/2}\exp\left[\frac{iEN\pi\rho}{2}\right]$ as given in Eq. (2.23).

Large-N limit in the different-sign case

For $\mathrm{sgn}(\mathrm{Im}\,\mu_{B1}) \neq \mathrm{sgn}(\mathrm{Im}\,\mu_{B2})$ the matrix $\hat{Q}_B L$ has hyperbolic symmetry and we parametrise it as in Section 4.3.1 (for definiteness we choose again $\mathrm{sgn}(\mathrm{Im}\,\mu_{B1}) = 1$ and $\mathrm{sgn}(\mathrm{Im}\,\mu_{B2}) = -1$). In complete analogy to this previous case the correlation function is then given by the corresponding expression for same signs, Eq. (4.123), with $s = 1$ and the replacements (after employing the bulk scaling) $p_2 \to -p_2$ and $I_0 \to K_0$. We proceed from there performing the saddle-point analysis, where the saddle points, restricting the ones for $p_{1,2}$ to positive real part, are now given by

$$\begin{aligned} p_1^{SP} &= \frac{iE + \sqrt{4J^2 - E^2}}{2},\ p_2^{SP} = \frac{-iE + \sqrt{4J^2 - E^2}}{2}, \\ r_{1,2}^{SP} &= r_\pm = \frac{-iE \pm \sqrt{4J^2 - E^2}}{2}. \end{aligned} \tag{4.132}$$

The saddle points for the p's are the same as for $\mathcal{C}_{1,2}$ and we deform the contours in the same way, as described in the paragraph below (4.48). The saddle points for the r's, on the other hand, are the same as for the same-sign case of $\mathcal{C}_{2,2}$. Moreover we can apply the same argumentation why only one of the possible four contributions needs to be considered, see the paragraph below Eq. (4.124). Note that this is different from the calculation of $\mathcal{C}_{1,2}$, where two saddle points where contributing to the lowest order approximation.

We proceed as usual shifting the integration variables by the saddle points and rescaling them with $1/\sqrt{N}$. However, as in the different-sign case for $\mathcal{C}_{1,2}$, it does not suffice to expand the exponentials to second order and neglect all terms of order $\mathcal{O}(N^{-1/2})$ and higher. This is because the integrand features the term $(r_2 + p_1)(r_1 - p_2)$ where the saddle points cancel each other out, i.e. this term becomes $(\eta_2 + \xi_1)(\eta_1 - \xi_2)$ in the new variables. The only other dependence of the integrand on the integration variables would be the Gaussian factors $\exp(-a\eta_1^2)$ etc. However, these integrals vanish, $\int \mathrm{d}\eta_2 \int \mathrm{d}\xi_1 \exp(-a\eta_2^2 - b\xi_1^2)(\eta_2 + \xi_1) = 0$ and likewise for the η_1- and ξ_2-integration. Hence we expand the exponentials to third order in the integration variables, and then every term to order $\mathcal{O}(N^{-1/2})$. The integrand then takes the form

$$(\eta_2+\xi_1)(\eta_1-\xi_2)(c_1\xi_1+c_2\xi_2+c_3\eta_1+c_4\eta_2+c_5\xi_1^3+c_6\xi_2^3+c_7\eta_1^3+c_8\eta_2^3) \\ \times\exp(-a_1\eta_1^2-a_2\eta_2^2-b_1\xi_1^2-b_2\xi_2^2), \tag{4.133}$$

where the c_i are independent of the integration variables. The presence of the first two factors is explained above. The cubic part in the third factor comes from expanding the terms $e^{\frac{a}{\sqrt{N}}\xi_1^3}$ etc. to first order. The linear terms come from expanding all other parts of the integrand to first order. Integration over this again vanishes, because when expanding it, each term contains odd powers of at least one of the integration variables. Thus we have to go even one order higher, which means proportional to $1/N$. Such terms can arise through two different means: Either by multiplication of two terms proportional to $1/\sqrt{N}$, or through the next higher expansion terms of the factors in the integrand. However, most of the latter case yields again only terms where odd powers are present (replace in the last bracket of the above equation ξ_i by ξ_i^2, ξ_i^3 by ξ_i^4 and the same for the η's). Only the second-order terms of the Bessel function K_0 and of the hyperbolic functions give rise to even powers in the integration variables and thus contribute.

All other relevant contributions come from multiplying two terms proportional to $1/\sqrt{N}$, thus we do not need to expand the other functions further. This especially means that the exponentials do not have to be expanded to fourth order in the integration variables. Moreover, only multiplication of two terms which yield only even powers in all integration variables are relevant. Collecting all those contributing terms is quite tedious, but manageable. When one is done with this all that remains is performing the integrals

$$\int_{-\infty}^{\infty}\mathrm{d}x\, e^{-ax^2}x^{2n}=\frac{(2n-1)!!}{(2a)^n}\sqrt{\frac{\pi}{a}}, \qquad n=0,1,2. \tag{4.134}$$

The final result of this procedure is

$$\begin{aligned}
&\mathcal{C}_{2,2}^{(\text{bulk, } \operatorname{sgn}\omega_{B1}=-\operatorname{sgn}\omega_{B2})}(\omega_{F1},\omega_{F2};\omega_{B1},\omega_{B2}) \approx \\
&\sqrt{\frac{2N}{\pi}}\frac{J^{N+1}e^{-N/2}}{(\omega_{F1}-\omega_{F2})^3}\exp\left[\frac{N}{4J^2}E^2+\frac{iE(\omega_{F1}+\omega_{F2})}{2J^2}-\frac{iE(\omega_{B1}+\omega_{B2})}{4J^2}\right] \\
&\Big\{[(\omega_{F1}+\omega_{F2})(\omega_{B1}+\omega_{B2})-2\omega_{F1}\omega_{F2}-2\omega_{B1}\omega_{B2}]K_0\left(\tfrac{\pi\rho}{2}|\omega_{B1}-\omega_{B2}|\right) \\
&\quad\times[\pi\rho(\omega_{F1}-\omega_{F2})\cosh\left(\pi\rho(\omega_{F1}-\omega_{F2})\right)-\sinh\left(\pi\rho(\omega_{F1}-\omega_{F2})\right)] \\
&\quad+\pi\rho(\omega_{F1}-\omega_{F2})^2|\omega_{B1}-\omega_{B2}|\sinh\left(\pi\rho(\omega_{F1}-\omega_{F2})\right)K_1\left(\tfrac{\pi\rho}{2}|\omega_{B1}-\omega_{B2}|\right)\Big\},
\end{aligned} \tag{4.135}$$

where we also took into account that the result should remain invariant under exchanging ω_{B1} and ω_{B2}. In contrast to the result for $\mathcal{C}_{1,2}^{(\text{bulk, } \operatorname{sgn}\omega_{B1}=-\operatorname{sgn}\omega_{B2})}$, Eq. (4.67), the parity of N plays no role for the large-N behaviour of this correlation function.

4.4.2. Hubbard-Stratonovich Method

We want to illustrate how the same results (4.131) and (4.135) can be computed using the Hubbard-Stratonovich method. The first steps, introducing Gaussian integrals and performing the ensemble average is the very same as in the previous approach, the result given in (4.97). However, instead of calculating the trace we make now use of the duality relation between ordinary space and superspace [64] as explained in Section 2.2.4 and replace the trace over the $N \times N$ matrix $A + A^T$ by the supertrace of a 6×6 supermatrix Q. For maximal symmetry it is advantageous to rescale the Grassmann vectors $\boldsymbol{\zeta}_1 \to \sqrt{2}\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2 \to \sqrt{2}\boldsymbol{\zeta}_2$. In the first approach there is no advantage because the Grassmann variables are integrated out at an early stage of the calculation. Then for our case the duality relation becomes $\operatorname{Tr}\left(\frac{A+A^T}{2}\right)^2 = \operatorname{Str}(QL)^2$ with the metric

$$L = \operatorname{diag}(s_1, s_2, 1, 1, 1, 1). \tag{4.136}$$

Q can be explicitly constructed via $Q = B^\dagger B$, with $B = \begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{\zeta}_1 & \boldsymbol{\zeta}_1^* & \boldsymbol{\zeta}_2 & \boldsymbol{\zeta}_2^*\end{bmatrix}$. Recall the definition of the complex conjugate of B, where the last $4N$ rows (i.e. the rows containing the Grassmann variables) get an extra minus sign to ensure that $(B^\dagger)^\dagger = B$.

The next step is to perform the Hubbard-Stratonovich transformation, Eq. (2.74), and replace

$$\exp\left(-\frac{J^2}{N}\operatorname{Str}(QL)^2\right) \propto \int \mathrm{d}\sigma\, \exp\left(-\tfrac{N}{4J^2}\operatorname{Str}\sigma^2 + i\boldsymbol{\psi}^\dagger(L\sigma\otimes\mathbb{1}_N)\boldsymbol{\psi}\right), \tag{4.137}$$

where $\boldsymbol{\psi}$ is the supervector comprising the vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_1^*$, $\boldsymbol{\zeta}_2$ and $\boldsymbol{\zeta}_2^*$, in this order. For simplicity we do not track the proportionality factor. σ is explicitly given by

$$\sigma = \begin{bmatrix} a_1 & b & \eta_1 & \eta_1^* & \eta_2 & \eta_2^* \\ b & a_2 & \eta_3 & \eta_3^* & \eta_4 & \eta_4^* \\ -\eta_1^* & -\eta_3^* & ix_1 & 0 & iz & iw^* \\ \eta_1 & \eta_3 & 0 & ix_1 & -iw & iz^* \\ -\eta_2^* & -\eta_4^* & iz^* & -iw^* & ix_2 & 0 \\ \eta_2 & \eta_4 & iw & iz & 0 & ix_2 \end{bmatrix}. \tag{4.138}$$

a_1, a_2, b, x_1 and x_2 are real commuting variables, w and z are complex commuting variables and all η's are complex anticommuting variables. This means σ comprises 9 independent commuting variables (3 from the Boson-Boson (upper left) and 6 from the Fermion-Fermion (lower right) block) and 8 independent Grassmann variables. $\mathrm{d}\sigma$ is the flat measure on all these variables, i.e. the product of their differentials. σ has the same structure as Q, with its lower right block multiplied by i (due to convergence requirements, see the paragraph below (2.74)). The remaining term in the exponential (coming from the H-independent part of $\mathcal{C}_{2,2}$) is given by $i(s_1\mu_{B1}\boldsymbol{x}_1^T\boldsymbol{x}_1 + s_2\mu_{B2}\boldsymbol{x}_2^T\boldsymbol{x}_2 + 2\mu_{F1}\boldsymbol{\zeta}_1^\dagger\boldsymbol{\zeta}_1 + 2\mu_{F2}\boldsymbol{\zeta}_2^\dagger\boldsymbol{\zeta}_2)$. We use $\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta} = \boldsymbol{\zeta}^T\boldsymbol{\zeta}^*$ to symmetrize this expression and rewrite it to $i\boldsymbol{\psi}^\dagger(L\mathcal{M}\otimes\mathbb{1}_N)\boldsymbol{\psi}$, where $\mathcal{M} = \operatorname{diag}(\mu_{B1},\mu_{B2},\mu_{F1},\mu_{F1},\mu_{F2},\mu_{F2})$. Integration over the supervector can now be performed and yields

$$\int \mathrm{d}\boldsymbol{\psi}\, \exp\{i\boldsymbol{\psi}^\dagger[L(\sigma+\mathcal{M})\otimes 1_N]\boldsymbol{\psi}\} = (s_1s_2)^{N/2}\operatorname{sdet}(\sigma+\mathcal{M})^{-N/2}, \tag{4.139}$$

where we used $\operatorname{sdet} L = \frac{\det\operatorname{diag}(s_1,s_2)}{\det\mathbb{1}_4} = s_1s_2$. The correlation function itself becomes

$$\mathcal{C}_{2,2} \propto \int \mathrm{d}\sigma\, \exp\left(-\frac{N}{4J^2}\operatorname{Str}\sigma^2\right)\operatorname{sdet}(\sigma+\mathcal{M})^{-N/2}. \tag{4.140}$$

We thus expressed the correlation function by a supersymmetric model which involves integration over the $(2+4)\times(2+4)$ supermatrix σ having $9+8$ independent variables. In principle one could try to integrate out some of these variables[30], however, it is much more advantageous to make use of the symmetry of Eq. (4.140) by performing the saddle-point analysis at the next step.

Saddle-point analysis

Let us for simplicity consider the case $E = \mu_{F1} = \mu_{F2} = 0$. Defining the matrix $\Omega = \mathrm{diag}(\omega_{B1}, \omega_{B2}, 0, 0, 0, 0)$, $\mathcal{M}$ is then given by $\mathcal{M} = \frac{i}{N}\Omega$. The first step is to determine the N-dependence of the integrand. First we rewrite the superdeterminant using $\mathrm{sdet}\,\sigma = \exp(\,\mathrm{Str}\,\ln\sigma)$,

$$\mathrm{sdet}\,\left(\sigma + \tfrac{i}{N}\Omega\right)^{-N/2} = \exp\left\{-\tfrac{N}{2}\left[\,\mathrm{Str}\,\ln\sigma + \,\mathrm{Str}\,\ln(\mathbb{1}_6 + \tfrac{i}{N}\sigma^{-1}\Omega)\right]\right\}. \tag{4.141}$$

The second term can now be expanded in $1/N$, which yields for the correlation function

$$\mathcal{C}_{2,2}^{(\mathrm{bulk})} \propto \int \mathrm{d}\sigma\, \exp\left\{-\frac{N}{2}\left[\frac{1}{2J^2}\,\mathrm{Str}\,\sigma^2 + \,\mathrm{Str}\,\ln\sigma\right] - \frac{i}{2}\,\mathrm{Str}\,(\sigma^{-1}\Omega) + \mathcal{O}(1/N)\right\}, \tag{4.142}$$

where we neglect all terms of order $\mathcal{O}(1/N)$ or higher. The integral is now of a form where we can apply the saddle-point method as described in the Hubbard-Stratonovich subsection of Section 2.2.4. Hence we require the matrix differential $\mathrm{d}\left(\frac{1}{2J^2}\sigma^2 + \ln\sigma\right) = \left(\frac{1}{J^2}\sigma + \sigma^{-1}\right)\mathrm{d}\sigma$ to vanish, which gives the saddle-point condition

$$\frac{1}{J^2}\sigma + \sigma^{-1} = 0. \tag{4.143}$$

First we look for a diagonal solution. Then the saddle-point condition applies to each individual entry on the diagonal with solutions $\pm iJ$. To determine which of these saddle points we have to choose recall that for a diagonal solution σ_D one has $\mathrm{sdet}\,(\sigma_D + \frac{i}{N}\Omega)^{-1} = \frac{\det(\sigma_D + \frac{i}{N}\Omega)_{FF}}{\det(\sigma_D + \frac{i}{N}\Omega)_{BB}}$, where the subscripts denote the boson-boson and fermion-fermion block of the supermatrix, respectively. This implies the integrand in

[30] E.g. integrating out all the Grassmann variables should yield an expression which is equivalent to Eq. (4.111), however, this is by no means obvious.

(4.140) has poles at $\sigma_{BB}^{\text{pole}} = -\frac{i}{N}\Omega_{BB}$. We need to deform the contours of integration in such a way that we reach the saddle points without crossing any of these poles. The first entry of $\sigma_{BB}^{\text{pole}}$ lies in the lower half-plane if $\omega_{B1} > 0$ (i.e. $s_1 = 1$). Correspondingly we choose the saddle point $+iJ$ which lies in the upper half-plane. If $\omega_{B1} < 0$ (i.e. $s_1 = -1$), the pole is in the upper half-plane, and we choose the saddle point $-iJ$. The same argument applies to the second entry. For the entries in the fermionic block, we are free to choose any of the two saddle points.

The saddle-point condition (4.143) is not only fulfilled for σ_D, but in principle for any transformation $\sigma_G = T^{-1}\sigma_D T$, which obeys the symmetries of σ and yields convergent integrals. Now the form of σ_D becomes important. In the cases $\operatorname{sgn}\omega_{B1} = \operatorname{sgn}\omega_{B2}$, one can choose the entries for the fermionic block that σ_D becomes proportional to the unit matrix, $\sigma_D = \operatorname{sgn}\omega_{B1} iJ\mathbb{1}_6$. For this case one hence gets $\sigma_G = T^{-1}\sigma_D T = \sigma_D$ and thus σ_D will be the only saddle point. For $\operatorname{sgn}\omega_{B1} \neq \operatorname{sgn}\omega_{B2}$, we cannot make this choice. Instead we choose σ_D such that it becomes proportional to the metric L, Eq. (4.136), such that $\sigma_D = \operatorname{sgn}\omega_{B1} iJL$ with $L = \operatorname{diag}(+1,-1,+1,+1,+1,+1)$. This implies there will be a continuous manifold of solutions $\sigma_G = T^{-1}\sigma_D T$ to the saddle-point condition (4.143). These two very different cases reflect our earlier findings that the signs of ω_{B1} and ω_{B2} play an important role.

We start with the easier case $\operatorname{sgn}\omega_{B1} = \operatorname{sgn}\omega_{B2} = s$, let $\sigma = \sigma_D + \delta\sigma = iJs\mathbb{1}_6 + \delta\sigma$ and shift the contours of integration accordingly. As usual we evaluate the N-independent part of the integrand at the saddle point and expand the N-dependent part in the exponent to second order in $\delta\sigma$. This yields

$$C_{2,2}^{(\text{bulk})} \propto \exp\left(\frac{N}{4}\operatorname{Str}\mathbb{1}_6 - \frac{s}{2J}\operatorname{Str}\Omega\right)\operatorname{sdet}(iJs\mathbb{1}_6)^{-N/2}\int \mathrm{d}(\delta\sigma)\,\exp\left[-\frac{N}{2J^2}\operatorname{Str}(\delta\sigma)^2\right]. \tag{4.144}$$

The supertrace of a matrix proportional to the unit matrix vanishes and the superdeterminant becomes unity (compare with the definitions (2.36) and (2.38)). The remaining $\delta\sigma$-integral gives a factor independent of ω_{B1} and ω_{B2}. Hence the final result is given as

$$C_{2,2}^{(\text{bulk})} \propto \exp\left(-\frac{s}{2J}\operatorname{Str}\Omega\right) \propto \exp\left[-\frac{1}{2J}(|\omega_{B1}| + |\omega_{B2}|)\right]. \tag{4.145}$$

Compare this result with Eq. (4.131), which was the corresponding result for same

signs from the previous approach. For $E = 0$, its dependence on ω_{B1} and ω_{B2} is exactly the same as in (4.145), and hence both approaches yield the same result[31] for same signs of ω_{B1}, ω_{B2} and $E = \omega_{F1} = \omega_{F2} = 0$.

The case $\operatorname{sgn}\omega_{B1} = -\operatorname{sgn}\omega_{B2} = s$ is much more involved. Owing to symmetry and convergence arguments it can be shown [8] that the solution to (4.143) is the manifold given by all rotations T of σ_D which obey the rule $T^\dagger L T = L$. At this point it is convenient to go from bf-notation to pq-notation as explained in Section 2.2.2, such that $L_{pq} = \operatorname{diag}(\mathbb{1}_3, k)$ with $k = (-1, +1, +1)$. For the notational convenience we drop any subscript indicating the chosen notation, as for the remainder of this section we adopt this pq-notation for all supermatrices such as σ, T, etc. In particular Ω becomes $\Omega = \operatorname{diag}(\omega_{B1}, 0, 0, \omega_{B2}, 0, 0)$. T can be decomposed [8] into $T = RT_0$, where $R = \operatorname{diag}(R_1, R_2)$ is block-diagonal and commutes with L. Then the saddle-point manifold becomes independent of R, $\sigma_G = isJT_0^{-1}LT_0$, and is hence not determined by the full group of rotations T but merely a subgroup.

To perform the saddle-point method, we expand σ around the saddle-point manifold, $\sigma = \sigma_G + \delta\sigma$, where $\delta\sigma$ are small deviations from this manifold having same properties as σ, in particular they can be written as $\delta\sigma = T^{-1}\delta P_0 T$ where P_0 is diagonal. In the literature, the variables σ_G are referred to as Goldstone modes, whereas the variables of $\delta\sigma$ are referred to as massive modes [103]. Evaluating the N-dependent part in the exponent for this choice of σ one gets

$$-\frac{N}{2}\left[\frac{1}{2J^2}\operatorname{Str}\sigma_G^2 + \operatorname{Str}\ln\sigma_G + \frac{1}{2J^2}\operatorname{Str}(\delta\sigma)^2 - \frac{1}{2}\operatorname{Str}(\sigma_G^{-1}\delta\sigma)^2\right]. \tag{4.146}$$

With $\sigma_G = isJT_0^{-1}LT_0$, the first two terms in the square bracket become independent of T_0. The first term vanishes and the second amounts to $\operatorname{sdet}(isJL)^{-N/2} = J^N$. To simplify the last two terms, we use the decomposition $\delta\sigma = T_0^{-1}R^{-1}\delta P_0 R T_0$, and since R is block-diagonal and δP_0 is diagonal, the expression $R^{-1}\delta P_0 R$ is block-diagonal as

[31] Strictly speaking Eq. (4.145) is not the full result since one is free to choose the sign in the last four entries of σ_D and also other choices need to be taken into account. In our case these other choices will only adjust the proportionality factor. However, it is of crucial importance if ω_{F1} and ω_{F2} are not zero. The σ_D chosen above would only yield an exponential dependence on $\omega_{F1} + \omega_{F2}$, but would not give the hyperbolic functions from (4.131). They are only obtained for the choice $\sigma_D = iJ\operatorname{diag}(1, 1, 1, 1, -1, -1)$, and it turns out that the contribution from $\sigma_D = iJ\mathbb{1}_6$ is sub-leading.

well and we define $\delta P = \mathrm{diag}(\delta P_1, \delta P_2) = R^{-1}\delta P_0 R$. Then the remaining two terms in the square bracket become $\mathrm{Str}\,(\delta P)^2 + \frac{1}{2J}\,\mathrm{Str}\,(\delta P)^2$, where for the second term we used that L and δP commute. This shows that the N-dependent part of the integrand, in the limit $N \to \infty$, does not depend on the Goldstone modes σ_G at all. On the other hand, it suffices to evaluate the N-independent part at the saddle points, which consequently does not depend on the massive modes. Hence Goldstone and massive modes are completely decoupled. Moreover integration over the massive modes yields a constant[32]. The correlation function is hence given by an integral over the saddle-point manifold,

$$\mathcal{C}_{2,2}^{(\mathrm{bulk})} \propto \int \mathrm{d}\mu(T_0)\, \exp\left\{-\frac{s}{2J}\,\mathrm{Str}\,(T_0^{-1}LT_0\Omega)\right\}. \tag{4.147}$$

This is the so called zero-dimensional non-linear σ model [26, 27]. Before we did the saddle point approximation, we had to deal with an integral over the 9 commuting and 8 anticommuting independent variables of σ. After mapping it onto the nonlinear σ model, almost half of the variables (the massive modes) are integrated out and we are just left with an integral over the 5 commuting and 4 anticommuting independent variables of T_0, that is the Goldstone modes.

The main problem is now to find a proper parametrisation of the matrix T_0, or equivalently for the supermatrix $\mathcal{Q}$ which we define as $\mathcal{Q} = -iT_0^{-1}LT_0$. This is done in Appendix A.4.1, and explicitly given by Eqs. (A.42a)–(A.42f). In this parametrisation, the supertrace featured in (4.147) becomes

$$i\,\mathrm{Str}\,\mathcal{Q}\Omega = (\omega_{B1} - \omega_{B2})\lambda_1 + 2(\lambda_0 - \lambda_1)(\omega_{B1}\alpha_1^*\alpha_1 + \omega_{B2}\alpha_2^*\alpha_2), \tag{4.148}$$

which can be shown by explicit calculation. $\lambda_1 \in (1, \infty)$ and $\lambda_0 \in (-1, 1)$ are real commuting variables and α_1, α_2 are complex anticommuting variables. The next difficulty is to compute the Jacobian of the chosen parametrisation, which is performed in Appendix A.4.2. It turns out that the Jacobian differs significantly from the usual

[32] This is a general feature of the method. If the number of independent commuting and Grassmann variables is the same, which is usually the case for δP, an integral of the form $\int \mathrm{d}(\delta P)\, e^{-r(\delta P)^2}$ will be independent of r.

standard expressions as it still depends on the Grassmann variables, and is given by

$$\mathcal{J} = \frac{(1+\alpha_1^*\alpha_1)(1-\alpha_2^*\alpha_2)}{2^4(1+m^2+r^2+s^2)^2}\frac{1-\lambda_0^2}{\sqrt{\lambda_1^2-1}(\lambda_0-\lambda_1)^2}. \tag{4.149}$$

The only dependence on the three real parameters $m, r, s \in \mathbb{R}$ comes from the Jacobian, and integrating them out just yields a constant. The Grassmann integration is hence given by the term proportional to $\alpha_1^*\alpha_1\alpha_2^*\alpha_2$ in

$$(1-\tfrac{s}{J}(\lambda_0-\lambda_1)\omega_{B1}\alpha_1^*\alpha_1)(1-\tfrac{s}{J}(\lambda_0-\lambda_1)\omega_{B2}\alpha_2^*\alpha_2)(1+\alpha_1^*\alpha_1)(1-\alpha_2^*\alpha_2), \tag{4.150}$$

where the first two factors come from expanding the exponentials involving Grassmann variables and the last two from the Jacobian. After Grassmann integration the correlation function thus takes the form

$$\begin{aligned}\mathcal{C}_{2,2}^{(\mathrm{bulk})} \propto \int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \mathrm{d}\lambda_0 \, \frac{1-\lambda_0^2}{\sqrt{\lambda_1^2-1}(\lambda_0-\lambda_1)^2} \exp\left(-\frac{|\omega_{B1}-\omega_{B2}|}{2J}\lambda_1\right) \\ \times \left[1-\tfrac{s}{J}\omega_{B1}(\lambda_0-\lambda_1)\right]\left[1+\tfrac{s}{J}\omega_{B2}(\lambda_0-\lambda_1)\right] + \text{boundary terms.}\end{aligned} \tag{4.151}$$

At this point a caveat is necessary. As explained in Section 2.2.3, one can get additional boundary terms called Efetov-Wegner terms when changing variables in superspace, which we have done by introducing the chosen parametrisation for $\mathcal{Q}$. More specifically, these terms only occur, when the term which contains no Grassmann variables diverges. In our case this term is given by

$$\int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \mathrm{d}\lambda_0 \, \frac{1-\lambda_0^2}{\sqrt{\lambda_1^2-1}(\lambda_0-\lambda_1)^2} \exp\left(-\frac{|\omega_{B1}-\omega_{B2}|}{2J}\lambda_1\right). \tag{4.152}$$

The integrand clearly diverges when $\lambda_1 \to 1$. However, it is a priori not clear if the integral itself diverges. To investigate this, we note that the λ_0-integration can be performed explicitly and yields

$$4\int_1^\infty \mathrm{d}\lambda_1 \, \frac{1}{\sqrt{\lambda_1^2-1}}(-1+\lambda_1 \operatorname{arcoth}\lambda_1) \exp\left(-\frac{|\omega_{B1}-\omega_{B2}|}{2J}\lambda_1\right). \tag{4.153}$$

The integral splits into two terms, the first of the form $\int_1^\infty \mathrm{d}\lambda_1 \, \exp(-a\lambda_1)/\sqrt{\lambda_1^2-1}$

yields the Bessel function $K_0(a)$[33]. For the second one we change $\lambda_1 = 1 + x$, such that one gets

$$-4K_0\left(\frac{|\omega_{B1}-\omega_{B2}|}{2J}\right) + 4\int_0^\infty \mathrm{d}x\, \frac{(1+x)\,\mathrm{arcoth}(1+x)}{\sqrt{x}\sqrt{2+x}} \exp\left[-\frac{|\omega_{B1}-\omega_{B2}|}{2J}(1+x)\right]. \tag{4.154}$$

Now the integrand diverges for $x \to 0$. For small x, we can approximate it, using $\mathrm{arcoth}(1+x) = \frac{1}{2}(\ln 2 - \ln x) + \mathcal{O}(x)$, to $\frac{1}{2\sqrt{2x}}(\ln 2 - \ln x)\exp(-\frac{|\omega_{B1}-\omega_{B2}|}{2J})$. This suggests that the integrand diverges as $\frac{a}{\sqrt{x}} + \frac{b\ln x}{\sqrt{x}}$ for $x \to 0$, where a and b are constants. Such a divergence is still integrable which implies that the integral (4.152) converges and hence there are no Efetov-Wegner terms in Eq. (4.151).

The last step is to evaluate the remaining integrals in (4.151). Expanding the brackets in the second line, the integral splits into three terms. In one of these terms the expression $(\lambda_0 - \lambda_1)^2$ vanishes, and the two integrals separate and can be performed,

$$\int_1^\infty \frac{\mathrm{d}\lambda_1}{\sqrt{\lambda_1^2-1}} \exp\left(-\frac{|\omega_{B1}-\omega_{B2}|}{2J}\lambda_1\right) \int_{-1}^1 \mathrm{d}\lambda_0\,(1-\lambda_0^2) = \frac{4}{3}K_0\left(\frac{|\omega_{B1}-\omega_{B2}|}{2J}\right). \tag{4.155}$$

In the other two terms, λ_0-integration can be done as well, and one gets

$$\begin{aligned}\mathcal{C}_{2,2}^{(\mathrm{bulk})} \propto &-\frac{4\omega_{B1}\omega_{B2}}{3J^2}K_0\left(\frac{|\omega_{B1}-\omega_{B2}|}{2J}\right) + \int_1^\infty \mathrm{d}\lambda_1\, \frac{1}{\sqrt{\lambda_1^2-1}} \exp\left(-\frac{|\omega_{B1}-\omega_{B2}|}{2J}\lambda_1\right) \\ &\times 4\left[-1 + \tfrac{|\omega_{B1}+\omega_{B2}|}{2J}\lambda_1 + \left(\tfrac{|\omega_{B1}-\omega_{B2}|}{2J} + \lambda_1 - \tfrac{|\omega_{B1}-\omega_{B2}}{2J}|\lambda_1^2\right)\mathrm{arcoth}\,\lambda_1\right].\end{aligned} \tag{4.156}$$

The first term in the second line yields upon integration again the Bessel function K_0, whereas the second term yields K_1, such that

$$\begin{aligned}\mathcal{C}_{2,2}^{(\mathrm{bulk})} \propto &-\left(\frac{\omega_{B1}\omega_{B2}}{3J^2}+1\right)K_0\left(\frac{|\omega_{B1}-\omega_{B2}|}{2J}\right) + \frac{|\omega_{B1}+\omega_{B2}|}{2J}K_1\left(\frac{|\omega_{B1}-\omega_{B2}|}{2J}\right) \\ &+ \int_1^\infty \mathrm{d}\lambda_1\, \frac{\mathrm{arcoth}\,\lambda_1}{\sqrt{\lambda_1^2-1}} \exp\left(-\frac{|\omega_{B1}-\omega_{B2}|}{2J}\lambda_1\right)\left(\tfrac{|\omega_{B1}-\omega_{B2}|}{2J} + \lambda_1 - \tfrac{|\omega_{B1}-\omega_{B2}|}{2J}|\lambda_1^2\right).\end{aligned} \tag{4.157}$$

[33]Substitute e.g. $\lambda_1 = \cosh\theta$ to bring the integral to the form in (4.47).

There does not seem to be a simple way to evaluate the remaining integration. However, compare this result with the one from the previous approach, Eq. (4.135). For $E = \omega_{F1} = \omega_{F2} = 0$, the expression in (4.135) will be proportional to the first line of (4.157) apart from the "+1" in the bracket. This suggests that the remaining integration should cancel with this term and hence we claim

$$\int_1^\infty \mathrm{d}\lambda_1 \, \frac{\operatorname{arcoth} \lambda_1}{\sqrt{\lambda_1^2 - 1}} \exp\left(-a\lambda_1\right) \left(a + \lambda_1 - a\lambda_1^2\right) = K_0(a). \tag{4.158}$$

If this claim is true, the results we obtained from the previous approach and from the Hubbard-Stratonovich method will coincide. We prove the claim in Appendix A.5.

The calculations of this section show that it is possible to obtain the same results as in the previous approaches. However, the main complexity of the previous approach, which was the saddle-point analysis due to its vanishing integrand, is here shifted to finding a parametrisation of the saddle-point manifold and evaluating the remaining integrals over λ_0 and λ_1. Eq. (4.157) shows that this approach leads indeed to some integrals which are hard to evaluate. In fact the choice $\omega_{F1} = \omega_{F2} = 0$ made the λ_0-integration in the above calculation trivial. For non-vanishing ω_{F1} and ω_{F2}, also the λ_0-integration becomes more difficult and the equivalence of the integral representation obtained from this approach and the result (4.135) from the previous approach is not obvious at all. Therefore the previous approach seems to be more suitable for this kind of problem.

4.5. Calculation of Correlation Function $\mathcal{C}_{2,4}$

The last correlation function we want to compute is $\mathcal{C}_{2,4}$, i.e. a correlation function with a product of two characteristic polynomials in the numerator and four square roots of characteristic polynomials in the denominator. Following a supersymmetric approach like in Sections 4.3.1 and 4.4 one runs into problems preventing us from finding a solution. Nevertheless the approach will be discussed here to show how these problems arise. Confronted with these difficulties we then follow the method described in Section 4.3.2 which will yield an answer for a special case.

4.5.1. Hybrid Method

We start again as usual by representing the correlation function by Gaussian integrals, this time two integrals over complex anticommuting vectors and four integrals over real commuting vectors. This allows us then to perform the ensemble average and we end up with a representation similar to that from the case $\mathcal{C}_{2,2}$, Eq. (4.97), only that we have two additional integrations and exponentials $\int \mathrm{d}\boldsymbol{x}_3 \int \mathrm{d}\boldsymbol{x}_4 \exp(is_3\mu_{B3}\boldsymbol{x}_3 + is_4\mu_{B4}\boldsymbol{x}_4)$ and in the definition of A we have accordingly the two additional terms $+s_3\boldsymbol{x}_3\otimes\boldsymbol{x}_3^T + s_4\boldsymbol{x}_4\otimes\boldsymbol{x}_4^T$, where $s_3 = \operatorname{sgn}\operatorname{Im}(\mu_{B3})$ and $s_4 = \operatorname{sgn}\operatorname{Im}(\mu_{B4})$. This suggests that $\operatorname{Tr}\left(\frac{A+A^T}{2}\right)^2$ can be written in the same form as before,

$$\operatorname{Tr}\left(\frac{A+A^T}{2}\right)^2 = 4\operatorname{Tr}(Q_B L)^2 - 2\operatorname{Tr} Q_F^2 + 4\boldsymbol{\zeta}_1^T\boldsymbol{\zeta}_2\boldsymbol{\zeta}_2^\dagger\boldsymbol{\zeta}_1^* - 8\boldsymbol{\zeta}_1^\dagger B\boldsymbol{\zeta}_1 - 8\boldsymbol{\zeta}_2^\dagger B\boldsymbol{\zeta}_2, \quad (4.159)$$

but now with Q_B being a real symmetric 4×4 matrix having entries $(Q_B)_{jk} = \boldsymbol{x}_j^T\boldsymbol{x}_k$, $L = \operatorname{diag}(s_1, s_2, s_3, s_4)$ and $B = \sum_{j=1}^4 s_j\boldsymbol{x}_j\otimes\boldsymbol{x}_j^T$. Q_F is the same 2×2 matrix as before. In particular the whole integrand has the same dependence on the Grassmann variables as for $\mathcal{C}_{2,2}$ and we can use its result for the Grassmann integration, given as the square root of the product

$$\det\mathcal{M}_0^{N-4}\prod_{j=1}^4\det\mathcal{M}_1(\lambda_B^{(j)}), \quad (4.160)$$

where $\mathcal{M}_0$ and $\mathcal{M}_1(\lambda_B^{(j)})$ are given in Eqs. (4.109) and (4.110), respectively. $\lambda_B^{(j)}$, $j = 1\ldots 4$ are the eigenvalues of $Q_B L$. Notice the difference from Eq. (4.108) due to the fact that $Q_B L$ is now a 4×4 matrix. For Q_B we use again formula (2.103) and thus replace integration over the four real vectors $\boldsymbol{x}_j$ by an integral over a 4×4 positive definite real symmetric matrix $\hat{Q}_B$. The result looks similar to Eq. (4.111),

$$\begin{aligned}\mathcal{C}_{2,4} \propto &\int \mathrm{d}\hat{Q}_B\,(\det\hat{Q}_B)^{\frac{N-5}{2}}\exp\left[-\frac{N}{4J^2}\operatorname{Tr}(\hat{Q}_B L)^2 + \frac{iN}{2J^2}\operatorname{Tr}\hat{Q}_B L M_B\right]\\ &\times\int\mathrm{d}\hat{Q}_F\int\mathrm{d}^2u\,\exp\left[-\frac{N}{2J^2}\operatorname{Tr}(\hat{Q}_F - i\operatorname{diag}(\mu_{F1},\mu_{F2}))^2 - \frac{N}{J^2}u^*u\right]\\ &\times\left[\det\hat{Q}_F - u^*u\right]^{N-4}\prod_{j=1}^4\left[\det\hat{Q}_F - u^*u + (\lambda_B^{(j)})^2 + \lambda_B^{(j)}\operatorname{Tr}\hat{Q}_F\right],\end{aligned} \quad (4.161)$$

where we abbreviated $M_B = \mathrm{diag}(\mu_{B1}, \mu_{B3}, \mu_{B3}, \mu_{B4})$. Simplifying the $\hat{Q}_F$-integration is in complete analogy to the previous case and the result is given by Eq. (4.122), where we have to replace $(\det \hat{Q}_B)^{\frac{N-3}{2}}$ by $(\det \hat{Q}_B)^{\frac{N-5}{2}}$ and $\mathrm{diag}(\mu_{B1}, \mu_{B2})$ by M_B in the first line as well as the entire third line by $(r_1 r_2)^{N-4} \prod_{j=1}^{4} (r_1 + \lambda_B^{(j)})(r_2 + \lambda_B^{(j)})$.

In contrast to the calculation of $\mathcal{C}_{1,2}$ and $\mathcal{C}_{2,2}$ we have now in principle three different cases to consider: All the signs of the imaginary parts of the μ_B's are equal, three signs are equal and one sign is different and two at a time have same signs. Here we will only consider the latter case, i.e. we will choose L, which is the diagonal matrix having the signs as entries, to be of the form $L = \mathrm{diag}(+1, +1, -1, -1)$. Then we can block diagonalise

$$\hat{Q}_B = T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} T^{-1}, \tag{4.162}$$

where P_1 is a 2×2 real symmetric matrix with eigenvalues $p_1 = \lambda_B^{(1)} > 0$, $p_2 = \lambda_B^{(2)} > 0$ and P_2 is a 2×2 real symmetric matrix with eigenvalues $p_3 = -\lambda_B^{(3)} > 0$, $p_4 = -\lambda_B^{(4)} > 0$. We can diagonalise $P_1 = O_1 \,\mathrm{diag}(p_1, p_2) O_1^T$ and $P_2 = O_2 \,\mathrm{diag}(p_3, p_4) O_2^T$ and absorb the diagonalising matrices into T and T^{-1} thus defining new matrices $\tilde{T}$, $\tilde{T}^{-1}$. The measure is given by $\mathrm{d}\hat{Q}_B = (p_1+p_3)(p_1+p_4)(p_2+p_3)(p_2+p_4)\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3\,\mathrm{d}p_4\,\mathrm{d}\tilde{T}$. Next we employ the bulk scaling $\mu = E + i\omega/N$ and for simplicity furthermore restrict to the case $E = \omega_{F1} = \omega_{F2} = 0$ as their presence is not important for the following discussion. For notational convenience we also change variables $p_1 \to -p_1$, $p_2 \to -p_2$ and relabel $r_1 \equiv p_5$, $r_2 \equiv p_6$. Then the correlation function simplifies to

$$\begin{aligned} \mathcal{C}_{2,4}^{(\mathrm{bulk})} \propto & \int_{-\infty}^{0} \mathrm{d}p_1 \int_{-\infty}^{0} \mathrm{d}p_2 \int_{0}^{\infty} \mathrm{d}p_3 \int_{0}^{\infty} \mathrm{d}p_4 \prod_{j=1}^{4} \exp\left(-\frac{N}{4J^2} p_j^2 + \frac{N}{2} \ln p_j \right) \\ & \int_{-\infty}^{\infty} \mathrm{d}p_5 \int_{-\infty}^{\infty} \mathrm{d}p_6 \, \exp\left(-\frac{N}{2J^2}(p_5^2 + p_6^2) + N(\ln p_5 + \ln p_6) \right) \\ & \times \frac{(p_5 - p_6)^3}{(p_1 - p_2)(p_3 - p_4)(p_5 p_6)^4 (p_1 p_2 p_3 p_4 p_5)^{5/2}} \prod_{i<j}^{6} (p_i - p_j) \\ & \times \int \mathrm{d}\tilde{T} \, \exp\left(\frac{1}{2J^2} \,\mathrm{Tr}\, \tilde{T} P \tilde{T}^{-1} L \Omega \right), \end{aligned} \tag{4.163}$$

with $\Omega = \mathrm{diag}(\omega_{B1}, \omega_{B2}, \omega_{B3}, \omega_{B4})$ and $P = \mathrm{diag}(p_1, p_2, p_3, p_4)$. An explicit parametri-

sation of the matrix $\tilde{T}$ is given by

$$\tilde{T} = \begin{bmatrix} O_L^T & 0 \\ 0 & O_R^T \end{bmatrix} \begin{bmatrix} \cosh\hat{\psi}/2 & \sinh\hat{\psi}/2 \\ \sinh\hat{\psi}/2 & \cosh\hat{\psi}/2 \end{bmatrix} \begin{bmatrix} O_L & 0 \\ 0 & O_R \end{bmatrix} \begin{bmatrix} O_1 & 0 \\ 0 & O_2 \end{bmatrix}, \tag{4.164}$$

$$\mathrm{d}\tilde{T} = |\cosh\psi_1 - \cosh\psi_2|\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2\,\mathrm{d}O_L\,\mathrm{d}O_R\,\mathrm{d}O_1\,\mathrm{d}O_2, \tag{4.165}$$

where $\hat{\psi} = \mathrm{diag}(\psi_1, \psi_2)$ and O_L, O_R are 2×2 orthogonal matrices. In the previous cases we were always able to compute the ensuing group integral over the diagonalising matrices (which in the hyperbolic case were parametrised by a single variable θ and yielded the Bessel function K_0, see Eq. (4.47)). Here, however, we need six variables to parametrise the diagonalising matrices (ψ_1, ψ_2 and one variable for each of the four 2×2 orthogonal matrices O_L, O_R, O_1 and O_2. It does not seem possible to integrate out those variables[34].

Instead we could try to perform a direct saddle-point analysis for large N along the same lines as before with the group integral unevaluated. Determining the saddle points is straightforward and all of them are given by $p_j^{SP} = \pm J$, $j = 1 \ldots 6$. They lie on the real axis so we do not need to shift the contour and furthermore we can restrict to the saddle point with negative sign for p_1 and p_2 and with positive sign for p_3 and p_4 because of the integration range. Due to the factor $(p_5 - p_6)^4$ the contribution of the case $p_5^{SP} = p_6^{SP}$ is sub-dominant and can be neglected, and because the integrand is invariant under the exchange of p_5 and p_6 we can restrict here to the saddle point with positive sign for p_5 and negative sign for p_6. In summary we have the saddle-point structure $p_6^{SP} = -J$ and all other saddle points $p_j^{SP} = +J$, $j = 1, \ldots, 5$.

Now one could in principle proceed as before, i.e. expand all functions contained in the integrand around the saddle up to a certain order, only collect terms where all integration variables appear as even powers and then perform the integrals of the type $\int_{-\infty}^{+\infty} e^{-ax^2} x^{2n}$. However, it turns out that not only the zero-th and first, but also the second order of the integrand expansion in fluctuations around the above discussed saddle points are vanishing at the saddle points. Expanding to an even higher order and collecting all relevant terms with the group integrals still present is extremely

[34] Compare to the same-sign case, where the corresponding integral is an IZHC-integral over the group of orthogonal 4×4 matrices, which cannot be evaluated either due to the lack of a general formula (as opposed to the unitary group, Eq. (3.25))

tedious and does not seem like a viable option. Hence Eq. (4.163) can be seen as the final solution for arbitrary N following this approach, but however impractical for determining large-N asymptotics. Note that other supersymmetric methods like the Hubbard-Stratonovich transformation method or Superbosonization would run into similar problems. These methods would yield a model involving integration over a 16×16 supermatrix Q which after diagonalisation would yield similar group integrals which cannot be evaluated.

4.5.2. "Method without Grassmannians"

At the moment, the only viable option to proceed for the correlation function $\mathcal{C}_{2,4}$ seems to be the approach used in Section 4.3.2 and inspired by the work of Schomerus, Frahm, Patra and Beenakker [97]. We want to focus on the main difficulties which come from the denominator as seen in the previous approach. Hence for simplicity we only discuss the case $E = \omega_{F1} = \omega_{F2} = 0$. Then, with the bulk-scaling, we can write the denominator of the correlation function $\mathcal{C}_{2,4}^{(\mathrm{bulk})}$ as

$$\det\left(H^2 - \frac{\omega_{B1}\omega_{B3}}{N^2} - iH\frac{\omega_{B1}+\omega_{B2}}{N}\right)^{1/2} \det\left(H^2 - \frac{\omega_{B2}\omega_{B4}}{N^2} - iH\frac{\omega_{B3}+\omega_{B4}}{N}\right)^{1/2}. \tag{4.166}$$

As discussed in Section 4.3.2 this replacement is only valid if we restrict to $\omega_{B1}\omega_{B3} < 0$ and $\omega_{B2}\omega_{B4} < 0$. Hence this approach is only applicable to the case where two of the parameters have positive sign and two of them have negative sign. We go one step further and choose $\omega_{B3} = -\omega_{B1}$ and $\omega_{B4} = -\omega_{B2}$. The advantage of such choice is that the terms linear in H vanish, which makes the calculation considerably easier (note however that this is no requirement to make the approach functional). We will now replace these two determinants by Gaussian integrals over two N-component real vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, such that the correlation function $\mathcal{C}_{2,4}^{(\mathrm{bulk})}(0,0;\omega_{B1},-\omega_{B1},\omega_{B2},-\omega_{B2})$ can be represented by (for brevity we omit the arguments of this special case of $\mathcal{C}_{2,4}^{(\mathrm{bulk})}$ in all following equations)

$$\mathcal{C}_{2,4}^{(\mathrm{bulk})} = \frac{1}{(2\pi)^N}\int \mathrm{d}\boldsymbol{x}_1 \int \mathrm{d}\boldsymbol{x}_2 \, \exp\left[-\frac{1}{2N^2}(\omega_{B1}^2 \boldsymbol{x}_1^T\boldsymbol{x}_1 + \omega_{B2}^2 \boldsymbol{x}_2^T\boldsymbol{x}_2)\right] \Psi(\boldsymbol{x}_1,\boldsymbol{x}_2), \tag{4.167}$$

where

$$\Psi(\boldsymbol{x}_1,\boldsymbol{x}_2)=\left\langle \det H^2 \exp\left(-\frac{1}{2}\operatorname{Tr} H^2 Q\right)\right\rangle_{\mathrm{GOE}},\quad Q=\boldsymbol{x}_1\otimes\boldsymbol{x}_1^T+\boldsymbol{x}_2\otimes\boldsymbol{x}_2^T. \tag{4.168}$$

In contrast to a single vector $\boldsymbol{x}$ in the previous case we now have to deal with two real vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, which we can conveniently combine into the matrix Q. Such a rank-two $N\times N$ matrix has two non-zero eigenvalues which we call q_1 and q_2, all other $N-2$ eigenvalues being identically zero. Being real symmetric Q can be diagonalised by an orthogonal transformation: $Q=O\,\mathrm{diag}(q_1,q_2,0,\ldots,0)O^T$ and the orthogonal matrices can be omitted from the integrand since $\det H^2$ and the joint probability $\mathcal{P}(H)$ are invariant under such transformation. Owing to this structure we can conveniently decompose H into its upper left 2×2 block, its lower right $(N-2)\times(N-2)$ block H_{N-2} and the two ensuing off-diagonal blocks. One of these off-diagonal blocks will be of dimension $(N-2)\times 2$ and we can express it in terms of two $N-2$ dimensional vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ as $[\boldsymbol{h}_1,\boldsymbol{h}_2]$. The other off-diagonal block will be its transpose. Note that this decomposition is slightly different from the previous case, Eq. (4.72), due to Q having two non-zero eigenvalues instead of just one. In this decomposition we get,

$$\begin{aligned}\Psi=&\int_{-\infty}^{\infty}\mathrm{d}H_{11}\,e^{-(\frac{N}{4J^2}+\frac{q_1}{2})H_{11}^2}\int_{-\infty}^{\infty}\mathrm{d}H_{22}\,e^{-(\frac{N}{4J^2}+\frac{q_2}{2})H_{22}^2}\int_{-\infty}^{\infty}\mathrm{d}H_{12}\,e^{-(\frac{N}{2J^2}+\frac{q_1+q_2}{2})H_{12}^2}\\
&\times\int\mathrm{d}\boldsymbol{h}_1\,e^{-(\frac{N}{2J^2}+\frac{q_1}{2})\boldsymbol{h}_1^T\boldsymbol{h}_1}\int\mathrm{d}\boldsymbol{h}_2\,e^{-(\frac{N}{2J^2}+\frac{q_2}{2})\boldsymbol{h}_2^T\boldsymbol{h}_2}\int\mathrm{d}H_{N-2}\,e^{-\frac{N}{4J^2}\operatorname{Tr}H_{N-2}^2}\\
&\times\left[(H_{11}-y_{11})(H_{22}-y_{22})-(H_{12}-y_{12})^2\right]^2\det H_{N-2}^2,\end{aligned} \tag{4.169}$$

where we abbreviated $y_{kl}=\boldsymbol{h}_k^T H_{N-2}^{-1}\boldsymbol{h}_l$. Integration over the three entries H_{11}, H_{12} and H_{22} of the upper left 2×2 block of H can be performed and yields

$$\begin{aligned}\Psi=&\int\mathrm{d}\boldsymbol{h}_1\,e^{-(\frac{N}{2J^2}+\frac{q_1}{2})\boldsymbol{h}_1^T\boldsymbol{h}_1}\int\mathrm{d}\boldsymbol{h}_2\,e^{-(\frac{N}{2J^2}+\frac{q_2}{2})\boldsymbol{h}_2^T\boldsymbol{h}_2}\int\mathrm{d}H_{N-2}\,e^{-\frac{N}{4J^2}\operatorname{Tr}H_{N-2}^2}\\
&\times\sqrt{\frac{(2\pi)^3}{a_1a_2a_3}}\left[\left(y_{11}^2+\frac{1}{a_1}\right)\left(y_{22}^2+\frac{1}{a_2}\right)+2y_{11}y_{22}\left(y_{12}^2+\frac{1}{a_3}\right)\right.\\
&\left.+\left(y_{12}^4+\frac{6y_{12}^2}{a_1+a_2}+\frac{3}{(a_1+a_2)^2}\right)\right]\det H_{N-2}^2,\end{aligned} \tag{4.170}$$

where we introduced $a_1 = \frac{N}{2J^2} + q_1$ and $a_2 = \frac{N}{2J^2} + q_2$. The next step is to perform integration over $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$. This is worked out in Appendix A.6 and the final result is

$$\begin{aligned}\Psi = \frac{(2\pi)^{3/2}}{\sqrt{a_1 a_2 (a_1+a_2)}} \left(\frac{4\pi^2}{c_1 c_2}\right)^{N/2-1} \Bigg\langle \det H_{N-2}^2 \Bigg\{ \frac{1}{a_1 a_2} + \frac{3}{(a_1+a_2)^2} \\ + \left(\frac{1}{a_2 c_1^2} + \frac{1}{a_1 c_2^2}\right) [2\,\mathrm{Tr}\, A^2 + (\mathrm{Tr}\, A)^2] + \frac{2}{c_1 c_2 (a_1+a_2)} [3\,\mathrm{Tr}\, A^2 - (\mathrm{Tr}\, A)^2] \\ + \frac{1}{c_1^2 c_2^2} [(\mathrm{Tr}\, A)^4 - 8\,\mathrm{Tr}\, A\,\mathrm{Tr}\, A^3 + 7(\mathrm{Tr}\, A^2)^2 + 2(\mathrm{Tr}\, A)^2\,\mathrm{Tr}\, A^2 - 2\,\mathrm{Tr}\, A^4] \Bigg\} \Bigg\rangle_{H_{N-2}},\end{aligned} \tag{4.171}$$

where we used the notations

$$A = H_{N-2}^{-1}, \quad a_{1,2} = q_{1,2} + \frac{N}{2J^2}, \quad c_{1,2} = q_{1,2} + \frac{N}{J^2}. \tag{4.172}$$

The result now reduces to performing ensemble averages over expressions $\det H_{N-2}^2$ multiplied with various powers of traces of the inverse matrices H_{N-2}^{-k} for a few instances of positive integers k. In the previous case the only expression of this kind was of the form $\mathrm{Tr}(x\mathbb{1} - H)^{-1} \det(x\mathbb{1} - H)$ which we could replace by the derivative of $\det(x\mathbb{1} - H)$ w.r.t. x, see Eq. (4.86). Here we can do the same but with derivatives of correlation functions of two characteristic polynomials. To that end we use the two identities

$$\det H_{N-2}^2 \left[(\mathrm{Tr}\, H_{N-2}^{-1})^2 - \mathrm{Tr}\, H_{N-2}^{-2}\right] = \lim_{\xi_1,\xi_2 \to 0} \frac{\partial^2}{\partial \xi_1^2} [\det(H_{N-2} - \xi_1) \det(H_{N-2} - \xi_2)], \tag{4.173}$$

$$\det H_{N-2}^2 (\mathrm{Tr}\, H_{N-2}^{-1})^2 = \lim_{\xi_1,\xi_2 \to 0} \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \left[\det(H_{N-2} - \xi_1) \det(H_{N-2} - \xi_2)\right], \tag{4.174}$$

which follow immediately from $\frac{\mathrm{d}}{\mathrm{d}x} \det(H - x) = \mathrm{Tr}(H - x)^{-1} \det(H - x)$ and $\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{Tr}(H - x)^{-k} = k\,\mathrm{Tr}(H - x)^{-(k+1)}$. They allow us to express any linear combination of the two terms $\det H_{N-2}^2 (\mathrm{Tr}\, H_{N-2}^{-1})^2$ and $\det H_{N-2}^2\,\mathrm{Tr}\, H_{N-2}^{-2}$ as derivatives of a correlation function, in particular the two terms in the second line of Eq. (4.171). For the third

line, which contains powers of fourth order, we use instead the identities (with A as defined in (4.172)),

$$\begin{aligned}\lim_{\xi_1,\xi_2\to 0}\frac{\partial^4}{\partial\xi_1^4}\det(H_{N-1}-\xi_1)\det(H_{N-1}-\xi_2)&=\det H_{N-2}^2\\ \times[(\operatorname{Tr}A^4)-6\operatorname{Tr}A^4+3(\operatorname{Tr}A^2)^2+8\operatorname{Tr}A\operatorname{Tr}A^3-6(\operatorname{Tr}A)^2\operatorname{Tr}A^2],\end{aligned}\tag{4.175}$$

$$\begin{aligned}\lim_{\xi_1,\xi_2\to 0}\frac{\partial^4}{\partial\xi_1^3\partial\xi_2}\det(H_{N-1}-\xi_1)\det(H_{N-1}-\xi_2)&\\ =\det H_{N-2}^2\operatorname{Tr}A((\operatorname{Tr}A)^3+2\operatorname{Tr}A^3+3\operatorname{Tr}A\operatorname{Tr}A^2),&\end{aligned}\tag{4.176}$$

$$\begin{aligned}\lim_{\xi_1,\xi_2\to 0}\frac{\partial^4}{\partial\xi_1^2\partial\xi_2^2}\det(H_{N-1}-\xi_1)\det(H_{N-1}-\xi_2)&\\ =\det H_{N-2}^2((\operatorname{Tr}A)^2-\operatorname{Tr}A^2)((\operatorname{Tr}A)^2-\operatorname{Tr}A^2).&\end{aligned}\tag{4.177}$$

We have only three identities which comprise five different terms $\operatorname{Tr}A^4$, $\operatorname{Tr}A^3\operatorname{Tr}A$, $(\operatorname{Tr}A^2)^2$, $\operatorname{Tr}A^2(\operatorname{Tr}A)^2$ and $(\operatorname{Tr}A)^4$. This means we cannot express any possible linear combinations of those five terms by derivatives. Nevertheless it turns out that the combination in question, given by the third line in Eq. (4.171), can in fact be expressed by the left-hand sides of the three identities above via

$$\lim_{\xi_1,\xi_2\to 0}\left[\left(\frac{1}{3}\frac{\partial^4}{\partial\xi_1^4}+6\frac{\partial^4}{\partial\xi_1^2\partial\xi_2^2}-\frac{16}{3}\frac{\partial^4}{\partial\xi_1^3\partial\xi_2}\right)\det(H_{N-1}-\xi_1)\det(H_{N-1}-\xi_2)\right].\tag{4.178}$$

As a result for the object featuring in (4.168) we have:

$$\Psi(q_1,q_2)=\lim_{\xi_1,\xi_2\to 0}\mathcal{D}_{\xi_1,\xi_2}(q_1,q_2)\langle\det(H_{N-2}-\xi_1)\det(H_{N-2}-\xi_2)\rangle_{\mathrm{GOE},\,N-2},\tag{4.179}$$

where we interchanged the differentiation with the ensemble averaging. The differential operator $\mathcal{D}_{\xi_1,\xi_2}(q_1,q_2)$ is explicitly given by

$$\begin{aligned}\mathcal{D}_{\xi_1,\xi_2}(q_1,q_2)=&\frac{(2\pi)^{3/2}}{\sqrt{a_1a_2(a_1+a_2)}}\left(\frac{4\pi^2}{c_1c_2}\right)^{\frac{N}{2}-1}\left\{\left(\frac{1}{a_2c_1^2}+\frac{1}{a_1c_2^2}\right)\left(3\frac{\partial^2}{\partial\xi_1\partial\xi_2}-2\frac{\partial^2}{\partial\xi_1^2}\right)\right.\\ &+\frac{2}{c_1c_2(a_1+a_2)}\left(2\frac{\partial^2}{\partial\xi_1\partial\xi_2}-3\frac{\partial^2}{\partial\xi_1^2}\right)+\left(\frac{1}{a_1a_2}+\frac{3}{(a_1+a_2)^2}\right)\\ &\left.+\frac{1}{3c_1^2c_2^2}\left(\frac{\partial^4}{\partial\xi_1^4}+18\frac{\partial^4}{\partial\xi_1^2\partial\xi_2^2}-16\frac{\partial^4}{\partial\xi_1^3\partial\xi_2}\right)\right\}.\end{aligned}\tag{4.180}$$

The problem of calculating the ensemble average in $\Psi(q_1, q_2)$ reduced to calculate the correlation function $\mathcal{C}_{2,0} = \langle \det(H_{N-2} - \xi_1) \det(H_{N-2} - \xi_2) \rangle_{\mathrm{GOE},\, N-2}$. The large-$N$ asymptotics of this correlation function are known (see e.g. [104]), but for completeness we will re-derive them here.

The easiest way is to utilize our results from Section 4.4. The correlation function considered there also had two characteristic polynomials in the numerator. Hence we can take the result for exact N, Eq. (4.122) and delete all terms which were due to the characteristic polynomials in the denominator, i.e. $\hat{Q}_B$-integration and all terms of the integrand involving $\hat{Q}_B$. Furthermore we need to adjust for the fact that the ensemble average is over the reduced matrix H_{N-2} with joint probability $\mathcal{P}(H_{N-2}) \propto \exp(-\frac{N}{4J^2} \operatorname{Tr} H_{N-2}^2)$. This means the factor $(r_1 r_2)^N$ has to be replaced by $(r_1 r_2)^{N-2}$. We further rescale $\xi_{1,2} = \frac{i}{N} \omega_F$ and finally get

$$\begin{aligned} \mathcal{C}_{2,0}^{(\mathrm{bulk})} \propto & \int_{-\infty}^{\infty} \mathrm{d}r_1 \int_{-\infty}^{\infty} \mathrm{d}r_2 \, \exp\left[-\frac{N}{2J^2}(r_1^2 + r_2^2) - \frac{1}{2J^2}(r_1 + r_2)(\omega_{F1} + \omega_{F2}) \right] \\ & \times \frac{(r_1 r_2)^{N-2}(r_1 - r_2)}{(\omega_{F1} - \omega_{F2})^3} \exp\left[-\frac{1}{2NJ^2}(\omega_{F1}^2 + \omega_{F2}^2) \right] \\ & \times \left\{ \frac{(\omega_{F1} - \omega_{F2})(r_1 - r_2)}{2J^2} \cosh\left[\frac{(r_1 - r_2)(\omega_{F1} - \omega_{F2})}{2J^2} \right] \right. \\ & \left. - \sinh\left[\frac{(r_1 - r_2)(\omega_{F1} - \omega_{F2})}{2J^2} \right] \right\}. \end{aligned} \tag{4.181}$$

The saddle points are given by $r_1^{SP} = \pm J$ and $r_2^{SP} = \pm J$. For the same reasons as in the $\mathcal{C}_{2,2}$-calculation we can restrict to the choice $r_1^{SP} = +J$, $r_2^{SP} = -J$, because the contribution where both saddle points are equal is negligible due to the factor $r_1 - r_2$ and the case with reversed signs yields the same contribution. In contrast to the previous cases the integrand does not vanish at these saddle points, making the saddle-point analysis considerably easier, such that the lowest order solution is given by the integrand evaluated at the saddle points (the Gaussian integrals over the fluctuations around the saddle points yield only a constant)

$$\mathcal{C}_{2,0}^{(\mathrm{bulk})}(\omega_{F1}, \omega_{F2}) \propto \frac{\frac{\omega_{F1} - \omega_{F2}}{J} \cosh\left(\frac{\omega_{F1} - \omega_{F2}}{J}\right) - \sinh\left(\frac{\omega_{F1} - \omega_{F2}}{J}\right)}{(\omega_{F1} - \omega_{F2})^3}. \tag{4.182}$$

This result is in complete agreement with [104], where it has been shown that the

correlation function $\mathcal{C}_{2,0}^{(\mathrm{bulk})}$ for large N is given asymptotically by the above equation not only for the GOE, but any real symmetric Wigner matrix (i.e. a random matrix where the entries are identically independently distributed).

The asymptotic result for $\mathcal{C}_{2,0}^{(\mathrm{bulk})}(\omega_{F1}, \omega_{F2})$ only depends on the difference $r = \omega_{F1} - \omega_{F2}$. This means we can express the partial derivatives featured in (4.180) by this difference as $N^2 \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial \xi_1^2} = -\frac{\partial^2}{\partial \xi_1 \partial \xi_2}$ and $N^4 \frac{\partial^4}{\partial r^4} = \frac{\partial^4}{\partial \xi_1^4} = \frac{\partial^4}{\partial \xi_1^2 \partial \xi_2^2} = -\frac{\partial^4}{\partial \xi_1^3 \partial \xi_2}$, and the differential operator simplifies to

$$\begin{aligned}\mathcal{D}_r(q_1, q_2) =& \frac{(2\pi)^{3/2}}{\sqrt{a_1 a_2 (a_1 + a_2)}} \left(\frac{4\pi^2}{c_1 c_2}\right)^{\frac{N}{2}-1} \left\{ \frac{1}{a_1 a_2} + \frac{3}{(a_1 + a_2)^2} \right. \\ &\left. + 5N^2 \left(\frac{1}{a_2 c_1^2} + \frac{1}{a_1 c_2^2} + \frac{2}{c_1 c_2 (a_1 + a_2)} \right) \frac{\partial^2}{\partial r^2} + \frac{35 N^4}{3 c_1^2 c_2^2} \frac{\partial^4}{\partial r^4} \right\}.\end{aligned} \tag{4.183}$$

Calculating the remaining two derivatives and taking the limit $r \to 0$ is straightforward and Ψ is finally given by

$$\begin{aligned}\Psi(q_1, q_2) \propto& \frac{1}{\sqrt{a_1 a_2 (a_1 + a_2)}} \left(\frac{1}{c_1 c_2}\right)^{\frac{N}{2}-1} \left\{ \left(\frac{1}{a_1 a_2} + \frac{3}{(a_1 + a_2)^2} \right) \right. \\ &\left. + \frac{N^2}{J^2} \left(\frac{1}{a_2 c_1^2} + \frac{1}{a_1 c_2^2} + \frac{2}{c_1 c_2 (a_1 + a_2)} \right) + \frac{N^4}{J^4 c_1^2 c_2^2} \right\}.\end{aligned} \tag{4.184}$$

We do not longer keep track of the overall proportionality factor (which can depend on N and J) as it can easily be restored at the end of the calculation.

It remains to perform the integrals over $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in Eq. (4.167). The quantity $\Psi(q_1, q_2)$ depends only on the two non-vanishing eigenvalues of $Q = \boldsymbol{x}_1 \otimes \boldsymbol{x}_1^T + \boldsymbol{x}_2 \otimes \boldsymbol{x}_2^T$. However, this are also the eigenvalues of the 2×2 matrix $\tilde{Q} = \begin{bmatrix} |\boldsymbol{x}_1|^2 & \boldsymbol{x}_1^T \boldsymbol{x}_2 \\ \boldsymbol{x}_1^T \boldsymbol{x}_2 & |\boldsymbol{x}_2|^2 \end{bmatrix}$. Hence we introduce new integration variables $\hat{q}_1$ and $\hat{q}_2$ by first using the integration theorem (2.103) and thus replacing integration over the vectors by an integral over a positive definite real symmetric 2×2 matrix $\hat{Q}$ and then diagonalising $\hat{Q} = O \operatorname{diag}(\hat{q}_1, \hat{q}_2) O^T$.

Then the correlation function becomes

$$\begin{aligned}\mathcal{C}_{2,4}^{(\mathrm{bulk})} \propto & \int_0^\infty \mathrm{d}\hat{q}_1 \int_0^\infty \mathrm{d}\hat{q}_2\, (\hat{q}_1\hat{q}_2)^{\frac{N-3}{2}} |\hat{q}_1 - \hat{q}_2| \Psi(\hat{q}_1, \hat{q}_2) \\ & \times \int \mathrm{d}\mu(O)\, \exp\left[-\frac{1}{2N^2} \operatorname{Tr} \operatorname{diag}(\omega_{B1}^2, \omega_{B_2}^2) O \operatorname{diag}(\hat{q}_1, \hat{q}_2) O^T \right].\end{aligned} \tag{4.185}$$

The integral over the orthogonal group in the second line is again of the Itzykson-Zuber-Harish-Chandra type with similar solution as in (4.35), i.e. an exponential times a Bessel function.

The above result is exact for arbitrary N. The next step is to perform the large-N limit. First we rescale $\hat{q}_1 \to \frac{N^2}{J^2}\hat{q}_1$ and $\hat{q}_2 \to \frac{N^2}{J^2}\hat{q}_2$. The parameters contained in Ψ, Eq. (4.184), and defined in Eq. (4.172) change accordingly to $a_{1,2} = \frac{N^2}{J^2}(\hat{q}_{1,2} + \frac{1}{2N})$, $c_{1,2} = \frac{N^2}{J^2}(\hat{q}_{1,2} + \frac{1}{N})$. The term $(c_1c_2)^{-N/2}$ contained in (4.184) then becomes in the large-N limit $(c_1c_2)^{-N/2} \approx (\frac{J}{N})^{2N}(q_1q_2)^{-N/2} \exp(-\frac{1}{2q_1} - \frac{1}{2q_2})$. In all other terms we can neglect the terms proportional to $\frac{1}{N}$, i.e. we replace $a_1 \approx \frac{N^2}{J^2}\hat{q}_1$, $a_2 \approx \frac{N^2}{J^2}\hat{q}_2$, $c_1 \approx \frac{N^2}{J^2}\hat{q}_1$ and $c_2 \approx \frac{N^2}{J^2}\hat{q}_2$. Thus, in the large-$N$ limit we get the result (for brevity we omit the hats on $\hat{q}_1$ and $\hat{q}_2$),

$$\begin{aligned}\mathcal{C}_{2,4}^{(\mathrm{bulk})} \approx & C \int_0^\infty \mathrm{d}q_1 \int_0^\infty \mathrm{d}q_2\, \frac{|q_1 - q_2|}{(q_1q_2)\sqrt{q_1+q_2}} I_0\left[\frac{(\omega_{B1}^2 - \omega_{B2}^2)(q_1 - q_2)}{4J^2} \right] \\ & \times \exp\left[-\frac{1}{2}\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{(\omega_{B1}^2 + \omega_{B2}^2)(q_1+q_2)}{2J^2} \right) \right] \\ & \times \left\{ \frac{(1+q_1)(1+q_2)}{q_1^2 q_2^2} + \frac{3}{(q_1+q_2)^2} + \frac{2}{q_1q_2(q_1+q_2)} \right\},\end{aligned} \tag{4.186}$$

with some proportionality factor C. This can be seen as the final result of this approach, i.e. we were able to represent the correlation function

$$\mathcal{C}_{2,4}^{(\mathrm{bulk})}(0,0;\omega_{B1},-\omega_{B1},\omega_{B2},-\omega_{B2}) \approx \left\langle \frac{\det H^2}{\det(H^2 + \frac{\omega_{B1}^2}{N^2})^{1/2} \det(H^2 + \frac{\omega_{B2}^2}{N^2})^{1/2}} \right\rangle_{\mathrm{GOE}, N\to\infty} \tag{4.187}$$

by the above two-fold integral. To determine C notice that in the limit $\omega_{B1} \to 0$ and $\omega_{B2} \to 0$ the correlation function (4.187) becomes unity. Also the integral representation (4.186) does not depend on N or J in this limit, and hence the proportionality factor is just a real number given by the integral (4.186) where $\omega_{B1} = \omega_{B2} = 0$. The

calculations performed in Appendix A.7 show that $C = \frac{1}{12\sqrt{2\pi}}$.

Due to insights from Section 5.2 in the next chapter, where it is shown that the probability distribution of an off-diagonal matrix element of the K-matrix is related to $\lim_{\epsilon\to 0} \mathcal{C}_{2,4}^{(\mathrm{bulk})}(0,0;\epsilon,-\epsilon,x,-x)$ (see Eq. (5.30)), it turns out that the rather complicated looking integral (4.186), for the case $\omega_{B2} \equiv 0$, is actually given by

$$\lim_{\epsilon\to 0} \mathcal{C}_{2,4}^{(\mathrm{bulk})}(0,0;\epsilon,-\epsilon,x,-x) = \frac{2}{\pi}\left(\frac{|x|}{J}K_0(|x|/J) + \int_{|x|/J}^{\infty} dy\, K_0(y)\right). \tag{4.188}$$

A proof of this can be found in Appendix A.7.

4.6. Discussion of the Results

Let us now discuss a few special cases of the correlation functions, motivated by applications mentioned in Section 4.1.

Probability distribution of "level curvatures". The characteristic function of the "level curvatures", Eq. (4.6) can be represented as a special limit of $\mathcal{C}_{2,2}^{(\mathrm{bulk})}$. To that end we represent $|\det(E-H)|\det(E-H)^{1/2} = \lim_{\epsilon\to 0}\det(E+i\epsilon/N-H)\det(E-i\epsilon/N-H)^{1/2}$ using Eq. (4.4), and then multiply the numerator and denominator by $\det(E-i\epsilon/N-H)^{1/2}$ to get rid of the remaining square root in the numerator. This procedure yields

$$\begin{aligned}&\left\langle \frac{|\det(E-H)|\det(E-H)^{1/2}}{\det(E+i\omega/N-H)^{1/2}}\right\rangle_{\mathrm{GOE},\,N\to\infty}\\ &= \lim_{\epsilon\to 0}\mathcal{C}_{2,2}^{(\mathrm{bulk})}(\epsilon,-\epsilon;-\epsilon,\omega) \propto \exp\left(-\frac{iE}{4J^2}\omega\right)|\omega| K_1\left(\tfrac{\sqrt{4J^2-E^2}}{4J^2}|\omega|\right).\end{aligned} \tag{4.189}$$

The Fourier transform of this result (for brevity we choose $E=0$, $J=1$) yields the curvature distribution,

$$\mathcal{P}(c) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega|\omega| K_1\left(\tfrac{1}{2}|\omega|\right)\exp(i\omega c) = (1+4c^2)^{-3/2}, \tag{4.190}$$

which coincides with the expression found in earlier works by alternative methods [90, 91]. The validity of (4.189) was tested by direct numerical simulations of GOE

matrices of a moderate size, see Figure 4.1.

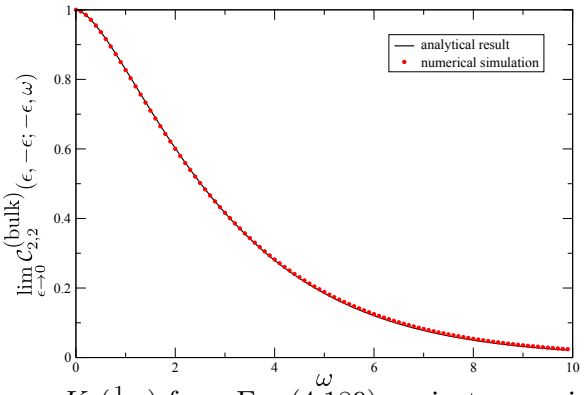

Figure 4.1.: The function $\omega K_1(\frac{1}{2}\omega)$ from Eq. (4.189) against numerical results of the corresponding correlation function, obtained from a sample of 40000 GOE-matrices of size 80×80.

Statistics of S-matrix poles. The two averages featuring in Eq. (4.7) can be recovered as special cases from $\mathcal{C}^{(\mathrm{bulk})}_{2,2}$ and are for the choice $J = 1$ given by

$$\begin{aligned} \left\langle \frac{\det^2 H}{\det^{1/2}(H^2 + \frac{\omega^2}{N^2})} \right\rangle_{\mathrm{GOE},\, N\to\infty} &= \mathcal{C}^{(\mathrm{bulk})}_{2,2}(0,0;\omega,-\omega) \\ &\approx 2\sqrt{\frac{2N}{\pi}} e^{-N/2} \left[\frac{\omega^2}{3} K_0\left(|\omega|\right) + |\omega| K_1\left(|\omega|\right) \right], \end{aligned} \tag{4.191}$$

$$\begin{aligned} \left\langle \det(H^2 + \tfrac{\omega^2}{N^2})^{1/2} \right\rangle_{\mathrm{GOE},\, N\to\infty} &= \mathcal{C}^{(\mathrm{bulk})}_{2,2}(\omega,-\omega;\omega,-\omega) \\ &\approx \sqrt{\frac{2N}{\pi}} e^{-N/2} \left[\left(\cosh(2\omega) - \frac{\sinh(2\omega)}{2\omega} \right) K_0(|\omega|) + \sinh(2|\omega|) K_1(|\omega|) \right]. \end{aligned} \tag{4.192}$$

The above formulas have been already presented in [96, 98], with derivation relegated to [14], which is the paper this chapter is based on. A comparison of these results with numerical simulations of the correlation functions is shown in Figure 4.2.

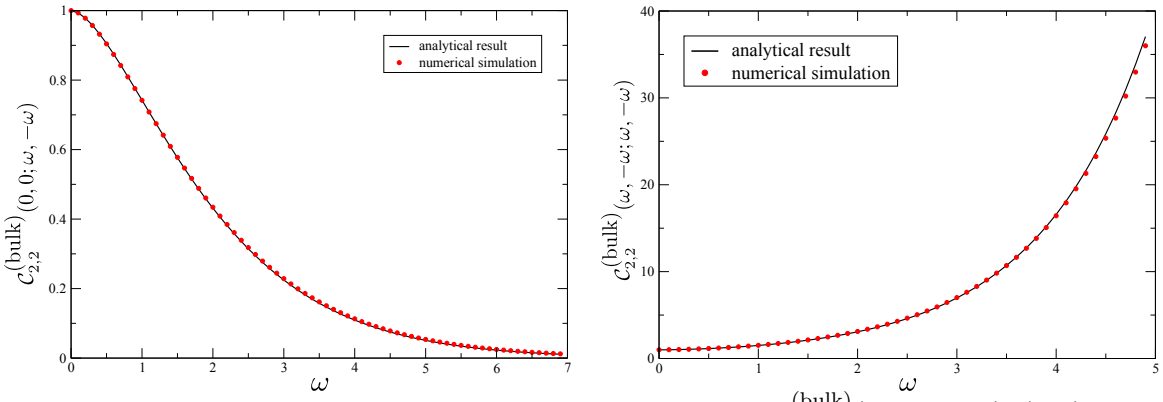

Figure 4.2.: Analytic results for the correlation functions $\mathcal{C}^{(\mathrm{bulk})}_{2,2}(0,0;\omega,-\omega)$ (left) from Eq. (4.191) and $\mathcal{C}^{(\mathrm{bulk})}_{2,2}(\omega,-\omega;\omega,-\omega)$ (right) from Eq. (4.192) (with $J=1$) against numerical results obtained from a sample of 40000 GOE-matrices of size 80×80.

Statistics of Wigner K-matrix. The $M=2$ case of Eq. (3.7) for $\beta=1$ features the correlation function

$$\mathcal{F}_{\beta=1,N}(X)=\left\langle \frac{\det(E-H)\operatorname{sgn}\det(E-H)^{\Theta(-x_1x_2)}}{\det{}^{1/2}(E+\frac{i\gamma_1x_1}{N}-H)\det{}^{1/2}(E+\frac{i\gamma_2x_2}{N}-H)}\right\rangle_H . \tag{4.193}$$

Assume that $x_1x_2>0$ so that $\Theta(-x_1x_2)=0$ and the sign-factor is immaterial. Then we can use formula (4.45) and the correlation function for H belonging to the GOE and in the limit $N\to\infty$ takes the form of

$$\mathcal{C}^{(\mathrm{bulk})}_{1,2}(0;\gamma_1x_1,\gamma_2x_2)\approx e^{\frac{-\gamma_1x_1-\gamma_2x_2}{4J^2}(iE+\operatorname{sgn}x_1\sqrt{4J^2-E^2})}, \tag{4.194}$$

which simplifies even further to $e^{\frac{-|x_1|-|x_2|}{2J}}$ for the "perfect coupling" case $E=0$, $\gamma_1=\gamma_2=1$. In the opposite case $x_1x_2<0$ on the other hand the correlation function takes the form

$$\left\langle \frac{|\det(E-H)|}{\det{}^{1/2}(E+\frac{i\gamma_1x_1}{N}-H)\det{}^{1/2}(E+\frac{i\gamma_2x_2}{N}-H)}\right\rangle_{\mathrm{GOE}}, \tag{4.195}$$

which is a special case of $\mathcal{C}^{(\mathrm{bulk})}_{2,4}$. To see this one can shift the absolute value to the denominator using $|\det(E-H)|=\det(E-H)^2/|\det(E-H)|$ and then represent the absolute value as product of two square roots of characteristic polynomials using (4.4),

such that (4.195) is given by $\lim_{\epsilon\to 0}\mathcal{C}_{2,4}^{(\mathrm{bulk})}(0,0;\epsilon,-\epsilon,\gamma_1 x_1,\gamma_2 x_2)$. In the particular case $\gamma_1 x_1 = -\gamma_2 x_2 \equiv \gamma x$, the above expression assumes the form (4.186) with solution given in (4.188). While a full proof that K is distributed according to the Cauchy distribution, Eq. (1.31), requires the knowledge of the above expression for arbitrary values of x_1 and x_2, one can show that our partial results for $\gamma_1 x_1 = -\gamma_2 x_2 \equiv \gamma x$ are indeed consistent with Eq. (1.31). This is shown in Appendix A.8, where we also utilise results from Chapter 5.

In the same chapter we also discuss the implications of our result for $\mathcal{C}_{2,4}^{(\mathrm{bulk})}$ on the probability distribution of an off-diagonal matrix element of the K-matrix (see e.g. Figure 5.1).

Absolute value of the GOE characteristic polynomial. One can obtain the absolute value $|\det(E-H)|$ as special case from $\mathcal{C}_{2,2}^{(\mathrm{bulk})}$. If we choose $\omega_{B1} = -\omega_{B2} = \epsilon$, the denominator becomes $|\det(E-H)|$ in the limit $\epsilon \to 0$ according to Eq. (4.4), and combined with the numerator (with $\omega_{F1} = \omega_{F2} = 0$) yields $|\det(E-H)|$. With Eq. (4.135) we then get asymptotically

$$\langle|\det(E-H)|\rangle_{\mathrm{GOE},\,N\to\infty} = \lim_{\epsilon\to 0}\mathcal{C}_{2,2}^{(\mathrm{bulk})}(\epsilon,\epsilon;\epsilon,-\epsilon) \approx 2\rho\sqrt{\frac{2N}{\pi}}J^{N+1}e^{-N/2}e^{\frac{N}{4J^2}E^2}. \quad (4.196)$$

This is in complete agreement with [92], where the result has been found using a different representation of $\langle|\det(E-H)|\rangle_{\mathrm{GOE},\,N\to\infty}$ in terms of just a ratio of two characteristic polynomials and employing the supersymmetry method. Note, however, that a derivation along the lines we computed this quantity is suggested in [92] as well.

Sign of the GOE characteristic polynomial. Finally we notice that an interesting special case of $\mathcal{C}_{1,2}^{(\mathrm{bulk})}$ is the average of the sign of the GOE characteristic polynomial. This follows from the same arguments that lead to the absolute value from $\mathcal{C}_{2,2}^{(\mathrm{bulk})}$. With Eq. (4.67) this quantity is given asymptotically by

$$\begin{aligned}\langle\mathrm{sgn}\det(E-H)\rangle_{\mathrm{GOE},\,N\to\infty} &= \lim_{\epsilon\to 0}\mathcal{C}_{1,2}^{(\mathrm{bulk})}(0;\epsilon,-\epsilon)\\ &\approx \frac{2J^2(-i/(2J))^N}{\sqrt{\pi N}(4J^2-E^2)^{3/4}}[A(E,N)+(-1)^N A^*(E,N)],\end{aligned} \quad (4.197)$$

where $A(E,N)$ is defined in Eq. (4.68). Due to the "parity effect" of Eq. (4.67), the average of the sign of the GOE characteristic polynomial is an even or odd function in E if N is even or odd, respectively. In particular this average vanishes for $E=0$ if N is odd and for even N one gets $\langle \operatorname{sgn}\det H\rangle_{\mathrm{GOE},\,N\text{ even},\,N\to\infty} = (-1)^{N/2}\sqrt{\pi N}$. A comparison of the asymptotic result with numerical simulations for E in the bulk of the spectrum is shown in Figure 4.3.

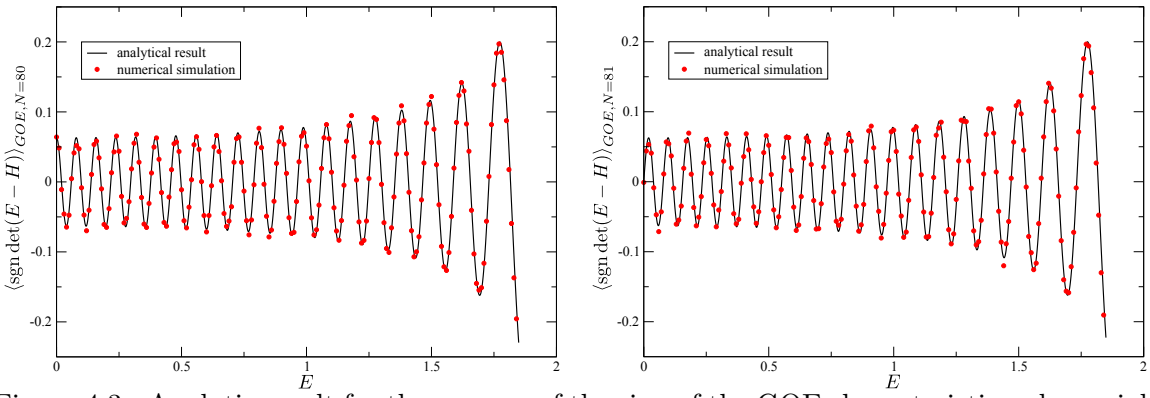

Figure 4.3.: Analytic result for the average of the sign of the GOE characteristic polynomial (with $J=1$) against numerical results obtained from a sample of 40000 GOE-matrices of size 80×80 (left) and 81×81 (right). Notice that the analytic result is not valid close to the edge of the spectrum $E=2$.

For a single realisation of H, the graph of $\operatorname{sgn}\det(E-H)$ will be jumping between $+1$ and -1, where each jump occurs when E equals an eigenvalue of H. Hence such jumps will happen more often where the level density is high. Therefore the ensemble averaged sign of the characteristic polynomial directly reflects local fluctuations of the density of states. Surprisingly, the absolute value of the determinant does not reveal such a structure, but rather an exponential dependence on E, see Eq. (4.196). This means for the absolute value other characteristics than just its zeros play a role for the ensemble averaged quantity. This can be explained as follows: For fixed E, the quantity $\operatorname{sgn}\det(E-H)$ will be $+1$ or -1 if the number of eigenvalues $\lambda_j > E$ is even or odd, respectively. Hence for large N, the ensemble average $\langle \operatorname{sgn}\det(E-H)\rangle$ will be completely determined by the local eigenvalue fluctuations around E. For $|\det(E-H)|$, on the other hand, the distance of the eigenvalues from E becomes important. Consequently the ensemble average $\langle \operatorname{sgn}\det(E-H)\rangle$ will also depend on higher k-point correlation functions.

5. Characteristics of the K-Matrix

In this chapter we want to collect some more K-matrix characteristics, e.g. the probability distribution of a single K-matrix element K_{ab}. These characteristics can usually be obtained in two ways, following either a Hamiltonian description or starting from the claim that the K-matrix as a whole is distributed according to the matrix-Cauchy distribution (1.31). For the most parts of the present chapter we illustrate how to get the K-matrix characteristics within both approaches.

In Chapter 3 it was proved that these two approaches are equivalent for a broad class of unitarily invariant matrices. Consequently one would expect them to yield the same answers for the K-matrix characteristics calculated in the following sections. This is indeed the case which can be seen as further verification of the proof in Chapter 3. For the orthogonal case $\beta = 1$ no proof of similar generality exists. In [13] it was shown that a Hamiltonian description with H belonging to an orthogonal matrix-Cauchy ensemble can be used to derive the Poisson kernel distribution (1.30) for the S-matrix which in turn implies the Cauchy distribution (1.31) for the K-matrix. Furthermore we derived partial results for the two-channel case $M = 2$ and H belonging to the GOE in Chapter 4, see in particular Section 4.6. The fact that both approaches always yield the same results for the GOE in the following sections further suggests their equivalence is true beyond the case of Cauchy-distributed H.

When following the Hamiltonian approach one needs to specify how the coupling matrix W is chosen. In the present chapter we only consider the random amplitude model (1.27,1.28). Howewever, as discussed in the introduction, another model frequently used is the one of fixed amplitudes (1.25). For the $\beta = 2$ case it was shown in Section 3.4 that both models yield the same characteristic function for the distribution of the K-matrix as long as M stays finite for $N \to \infty$. In the same sense all the K-matrix characteristics derived in this chapter are also valid for the fixed amplitude model. While for the statistics of the K-matrix for a GOE Hamiltonian H no such

proof is present, the two choices of the coupling W are nevertheless expected to yield the same answers for $N \gg M$. This is e.g. justified by [53], where such an equivalence has been explicitly verified for some specific S-matrix characteristics.

The probability distribution of a single K-matrix element K_{ab} is of direct experimental relevance and can be measured in microwave experiments as these elements are related to the real part of the electromagnetic impedance [99, 100]. It turns out that entries on the diagonal are distributed differently than off-diagonal ones. For real E in the bulk of the spectrum the statistics of the diagonal entries K_{aa} is long known to be given by the same Cauchy distribution for all $\beta = 1, 2, 4$, see e.g. [6, 105] and was recently shown to hold for a broad class of Wigner matrices in [106]. We will rederive this result for $\beta = 1, 2$ in Section 5.1. The probability density $\mathcal{P}(K_{ab})$ for $a \neq b$ will be found in Section 5.2 for both the unitary and (Gaussian) orthogonal case. The GOE case was also presented in [14].[35]

The distribution of $\operatorname{Tr} K$, i.e. of the sum of all diagonal entries, is calculated in Section 5.3. It turns out that it behaves as if the diagonal entries are independent of each other, i.e. their sum is again Cauchy-distributed and independent of β. However, one would expect the diagonal entries to be highly correlated, and in Section 5.4 it is shown that this is indeed the case by explicitly calculating the correlation function of two diagonal entries of the K-matrix.

The last problem we consider is specific to the choice of the Gaussian ensemble whose level density in the limit $N \to \infty$ is given by the semicircular law (1.9), i.e. has compact support with square root singularity at the spectral edge. As described in Section 1.1.2, spectral properties in the vicinity of this edge behave differently than their counterparts in the bulk of the spectrum. Hence one would expect the distribution of a diagonal entry of the K-matrix to differ from a Cauchy distribution. This is shown in Section 5.5 where such a distribution is explicitly calculated. For technical reason we restrict to the GUE case. On the other hand one should retrieve back the Cauchy distribution in the bulk limit, i.e. taking E from the edge back into the bulk. Appendix A.10 shows that this is indeed the case.

[35] These findings are related to the distribution of the off-diagonal entries $S_{a\neq b}$ of the scattering matrix S which is also experimentally relevant [10,107] and has been calculated in [9,11]. However, for $\beta = 1$ it remains a challenge to extract the statistics of $K_{a\neq b}$ from it in a manageable form.

5.1. Distribution of Diagonal Entries in the Bulk

The first K-matrix characteristic we consider is the distribution of a single entry on its diagonal, which will be calculated utilising results from the previous chapters.

A single diagonal element of the K-matrix is given by

$$K_{aa} = \boldsymbol{w}_a^\dagger (E - H)^{-1} \boldsymbol{w}_a, \tag{5.1}$$

where $\boldsymbol{w}_a$ is the a-th column of the coupling matrix W. Consequently its distribution is given by $\mathcal{P}(K_{aa}) = \left\langle \delta(K_{aa} - \boldsymbol{w}_a^\dagger (E-H)^{-1}\boldsymbol{w}_a)\right\rangle$, and the characteristic function by the Fourier transform $\mathcal{R}(x) = \left\langle \exp(-\frac{i}{2}\beta x \boldsymbol{w}_a^\dagger (E-H)^{-1}\boldsymbol{w}_a)\right\rangle$, where the angular brackets stand for the average over all random variables and the limit $N \to \infty$ is implied. This is the same expression as in Eq. (3.2) for the choice $M = 1$, i.e. a diagonal element K_{aa} of the K-matrix for arbitrary M is distributed like the K-"matrix" (which is then a scalar) for the single channel case $M = 1$. In Chapter 3 we have shown that in the unitary case for any $M \ll N$ the K-matrix is Cauchy-distributed according to (1.31). In particular for $M = 1$ this implies that the distribution of a diagonal element of the K-matrix for $\beta = 2$ is given by

$$\mathcal{P}(K_{aa}) = \frac{\lambda/\pi}{\lambda^2 + (K_{aa} - \epsilon)^2} = \frac{\rho(E)/\gamma_a}{\pi^2\rho(E)^2 + (K_{aa}/\gamma_a - V'(E)/2)^2}, \tag{5.2}$$

where we used the known dependence of λ and ϵ on the coupling coefficient γ_a, potential V of the joint probability density $\mathcal{P}(H) \propto \exp(-\frac{N\beta}{2}\operatorname{Tr} V(H))$ and the mean eigenvalue density $\rho(E)$ (see the paragraph above Eq. (3.65)).

For the GOE case, $\beta = 1$, the characteristic function $\mathcal{R}(x)$ is given by (3.7) with $M = 1$,

$$\mathcal{R}(x) = \left\langle \frac{\det(E-H)^{1/2} \operatorname{sgn}\det(E-H)^{\Theta(-x)}}{\det(E + i\gamma_a x/N - H)^{1/2}} \right\rangle_{\mathrm{GOE},\, N\to\infty}. \tag{5.3}$$

The next step is to rewrite this expression such that one of the formulas calculated in Chapter 4 is applicable. To that end we introduce an imaginary increment in the numerator and rewrite

$$\mathcal{R}(x) = \left\langle \frac{\det(E + i\epsilon/N - H) \operatorname{sgn}\det(E-H)^{\Theta(-x)\Theta(\epsilon)}}{\det(E + i\epsilon/N - H)^{1/2} \det(E + i\gamma_a x/N - H)^{1/2}} \right\rangle_{\mathrm{GOE},\, N\to\infty}, \tag{5.4}$$

where the limit $\epsilon \to 0$ is implied and the extra factor $\Theta(\epsilon)$ (which is unity for $\epsilon \geq 0$ and zero otherwise) comes from careful consideration of the square-root discontinuity at the branch cut, in analogy to Appendix A.2. Choosing ϵ to be of the same sign as x, the product $\Theta(-x)\Theta(\epsilon)$ vanishes and the problem reduces to the special case $\mathcal{C}_{1,2}^{(\text{bulk, } \operatorname{sgn}\epsilon=\operatorname{sgn} x\gamma_a)}(\epsilon;\epsilon,x\gamma_a)$ (compare with Eq. (4.44)), with the result given in Eq. (4.45). Taking the limit $\epsilon \to 0$, the characteristic function is hence given by

$$\mathcal{R}(x) = \exp\left(-\frac{i\gamma_a E}{4J^2}x - \frac{\gamma_a\sqrt{4J^2-E^2}}{4J^2}|x|\right) = \exp\left(-\frac{i\gamma_a V'(E)}{4}x - \frac{\gamma_a \pi\rho(E)}{2}|x|\right), \tag{5.5}$$

where in the second step we used that for the GOE the mean level density is given by the semicircular law, Eq. (1.9), and the potential is $V(H) = H^2/(2J^2)$ (compare with Eq. (1.2)). The probability distribution $\mathcal{P}(K_{aa})$ is obtained by taking the inverse Fourier transform of Eq. (5.5), which yields the exact same Cauchy distribution as in (5.2). Notice, however, that we have shown validity of (5.2) for $\beta = 1$ only in the Gaussian case, but for $\beta = 2$ for any even polynomial $V(H)$.

Starting from the claim of the Cauchy distribution (1.31) for K yields the same distribution (5.2) for $\beta = 1$ and $M = 1$. Hence we have also proved the equivalence of both approaches for $M = 1$ and H belonging to the GOE.

5.2. Distribution of Off-Diagonal Entries in the Bulk

In order to complete the picture we now want to calculate the distribution of an off-diagonal K-matrix entry in the bulk of the spectrum. For $\beta = 1$, K is real symmetric and hence K_{ab} is real. For $\beta = 2$, however, K is complex Hermitian and thus K_{ab} is in general complex. Hence for this case we will derive the joint-probability density of the real and imaginary part of K_{ab}. The distributions will be calculated following both the Poisson kernel and Hamiltonian approach. The $\beta = 1$ case was also presented in [14].

5.2.1. Poisson Kernel Approach

In this approach we start with the distribution of the K-matrix, i.e. the joint probability density of all independent matrix elements, given by the Cauchy distribution

(1.31) which follows from the Poisson kernel (1.30). To obtain the distribution of an off-diagonal K-matrix element one then needs to integrate out all other matrix elements. However, any subblock of the K-matrix will be distributed by the same distribution (1.31), with M being replaced by the size of the subblock. This means to compute the distribution of the element K_{12}, one does not need to consider the full distribution but merely the one for its 2×2 subblock. Furthermore the rotational invariance of $\mathcal{P}(K)$ ensures that any off-diagonal element K_{ab} will have the same distribution as K_{12}. Hence, without loss of generality, we may choose $M = 2$ in (1.31) such that we only need to integrate out the two diagonal elements of the 2×2 K-matrix, viz.,

$$\mathcal{P}_\beta(K_{ab}) \propto \int_{-\infty}^{\infty} \mathrm{d}K_{aa} \int_{-\infty}^{\infty} \mathrm{d}K_{bb}\, \det[\lambda^2 + (K-\epsilon)^2]^{-(1+\beta/2)}. \tag{5.6}$$

For $\beta = 1$ this yields the distribution of an off-diagonal K-matrix element, for $\beta = 2$ the joint distribution of its real and imaginary parts. The first step is to shift $K_{aa} - \epsilon \to K_{aa}$ and similar for K_{bb}. This shows that the solution will be independent of ϵ. Furthermore rescaling $K_{aa} \to \lambda K_{aa}$ and similar for K_{bb} yields

$$\mathcal{P}_\beta(K_{ab}) \propto \int_{-\infty}^{\infty} \mathrm{d}K_{aa} \int_{-\infty}^{\infty} \mathrm{d}K_{bb}\, \det \left\{ \mathbb{1}_2 + \begin{bmatrix} K_{aa} & K_{ab}^*/\lambda \\ K_{ab}/\lambda & K_{bb} \end{bmatrix}^2 \right\}^{-(1+\beta/2)}. \tag{5.7}$$

This shows that it suffices to consider the case of perfect coupling, $\lambda = 1$, $\epsilon = 0$, since the more general case can be obtained easily by replacing $K_{ab} \to K_{ab}/\lambda$ and adjusting the normalisation constant accordingly.

For this perfect coupling case, the Cauchy distribution (1.31) evaluates to

$$\mathcal{P}(K) \propto [(1+K_{aa}^2)(1+K_{bb}^2) + 2|K_{ab}|^2(1 - K_{aa}K_{bb}) + |K_{ab}|^4]^{-(1+\beta/2)}. \tag{5.8}$$

We observe that it only depends on the modulus $|K_{ab}|$. To obtain its distribution we start with integrating out the variable K_{bb}. The integrand is of the form

$$(uK_{bb}^2 + vK_{bb} + w)^{-(1+\beta/2)} = \left[u\left(K_{bb} + \frac{v}{2u^2}\right)^2 - \frac{v^2}{4u} + w \right]^{-(1+\beta/2)} \tag{5.9}$$

with $u = 1 + K_{aa}^2,\ v = -2K_{aa}|K_{ab}|^2,\ w = 1 + K_{aa}^2 + 2|K_{ab}|^2 + |K_{ab}|^4$. Now we change variables $\sqrt{\frac{u}{D}}(K_{bb} + \frac{v}{2u^2}) \to K_{bb}$ where we denoted $D = w - \frac{v^2}{4u} = \frac{(1+K_{aa}^2+|K_{ab}|^2)^2}{1+K_{aa}^2} > 0$ such that the joint probability of K_{aa} and $|K_{ab}|$ evaluates to

$$\mathcal{P}_\beta(K_{aa}, |K_{ab}|) \propto \frac{1}{\sqrt{u} D^{(\beta+1)/2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}K_{bb}}{(1 + K_{bb}^2)^{1+\beta/2}} \propto \frac{(1 + K_{aa}^2)^{\beta/2}}{(1 + K_{aa}^2 + |K_{ab}|^2)^{\beta+1}}, \quad (5.10)$$

where we used the definitions of u, D, and that the integral over K_{bb} yields now a factor independent of K_{aa} and $|K_{ab}|$.

Unitary Symmetry The last step is to integrate out K_{aa} in (5.10). For $\beta = 2$ the integrand is a ratio of polynomials and can be solved using standard techniques (partial fraction decomposition) with the result

$$\mathcal{P}_{\beta=2}(|K_{ab}|) = \frac{\lambda(4\lambda^2 + |K_{ab}|^2)}{4\pi(\lambda^2 + |K_{ab}|^2)^{5/2}}, \quad (5.11)$$

where K_{ab} was replaced by K_{ab}/λ to obtain the result for arbitrary λ as discussed above and the proportionality constant was chosen such that normalisation is guaranteed, $2\pi \int_0^\infty \mathrm{d}|K_{ab}|\, |K_{ab}| \mathcal{P}_{\beta=2}(|K_{ab}|) = 1$. The joint probability of the real and the imaginary part of an off-diagonal K-matrix element K_{ab} is at the same time the probability of its modulus. This implies that the phase of K_{ab} is uniformly distributed and that real and imaginary part have the same distribution. It can be obtained from integrating out the imaginary (or equivalently real) part of K_{ab} in (5.11). In order to do this we do the substitution $\operatorname{Im} K_{ab} = \frac{x\sqrt{\lambda^2 + \operatorname{Re}^2 K_{ab}}}{\sqrt{1-x^2}}$ and use that the integrand is even to cast the distribution into the form

$$\mathcal{P}_{\beta=2}(\operatorname{Re} K_{ab}) = \frac{\lambda}{2\pi(\lambda^2 + \operatorname{Re}^2 K_{ab})^2} \int_0^1 \mathrm{d}x\, (4\lambda^2 + \operatorname{Re}^2 K_{ab} - 3\lambda^2 x^2). \quad (5.12)$$

This can be integrated trivially and yields the solution

$$\mathcal{P}_{\beta=2}(\operatorname{Re} K_{ab}) = \frac{\lambda(3\lambda^2 + \operatorname{Re}^2 K_{ab})}{2\pi(\lambda^2 + \operatorname{Re}^2 K_{ab})^2}. \quad (5.13)$$

As discussed above, the imaginary part of K_{ab} has the same distribution (5.13).

Orthogonal Symmetry To integrate out K_{aa} we change variables $K_{aa} = \frac{y}{\kappa}\sqrt{\frac{1}{1-y^2/\kappa^2}}$, with $\kappa = \frac{K_{ab}}{\sqrt{1+K_{ab}^2}}$. As the integrand is even the integral then transforms to

$$\mathcal{P}_{\beta=1}(K_{ab}) \propto \int_{-\infty}^{\infty} \mathrm{d}K_{aa}\, \frac{\sqrt{1+K_{aa}^2}}{(1+K_{aa}^2+K_{ab}^2)^2} \propto \frac{1}{K_{ab}(1+K_{ab}^2)^{3/2}} \int_0^{\kappa} \frac{\mathrm{d}y}{(1-y^2)^2}. \quad (5.14)$$

The integration on the right-hand side can easily be performed as $(1-y^2)^{-2} = \frac{1}{4}\left(\frac{1}{(1+y)^2} + \frac{1}{(1-y)^2} + \frac{2}{1-y^2}\right)$ such that

$$\int_0^{\kappa} \frac{\mathrm{d}y}{(1-y^2)^2} = \frac{1}{4}\int_{-\kappa}^{\kappa} \frac{\mathrm{d}y}{(1+y)^2} + \frac{1}{2}\int_0^{\kappa} \frac{\mathrm{d}y}{1-y^2} = \frac{1}{2}\left(\frac{\kappa}{1-\kappa^2} + \operatorname{artanh}\kappa\right). \quad (5.15)$$

In this way we arrive at the probability density for K_{ab} in the form

$$\mathcal{P}_{\beta=1}(K_{ab}) \propto \frac{1}{K_{ab}(1+K_{ab}^2)^{3/2}}\left(\frac{\kappa(K_{ab})}{1-\kappa^2(K_{ab})} + \operatorname{artanh}\kappa(K_{ab})\right). \quad (5.16)$$

Finally we reinsert $\kappa(K_{ab}) = \frac{K_{ab}}{\sqrt{1+K_{ab}^2}}$, employ the identity $\operatorname{artanh}\left(\frac{x}{\sqrt{1+x^2}}\right) = \operatorname{arsinh} x$, and change $K_{ab} \to K_{ab}/\lambda$ which gives us the distribution of an off-diagonal K-matrix element for arbitrary λ as

$$\mathcal{P}_{\beta=1}(K_{ab}) = \frac{2\lambda}{\pi^2(\lambda^2+K_{ab}^2)}\left(1 + \frac{\lambda^2 \operatorname{arsinh}(K_{ab}/\lambda)}{K_{ab}\sqrt{\lambda^2+K_{ab}^2}}\right), \quad (5.17)$$

where the proportionality constant was chosen such that $\int_{-\infty}^{\infty}\mathrm{d}K_{ab}\,\mathcal{P}_{\beta=1}(K_{ab}) = 1$.

5.2.2. Hamiltonian Approach

In terms of the channel vectors $\boldsymbol{w}_a$, $\boldsymbol{w}_b$ and the resolvent $\Gamma = (E-H)^{-1}$, an off-diagonal element of the K-matrix is given by

$$K_{ab} = \boldsymbol{w}_a^{\dagger}\Gamma\boldsymbol{w}_b, \quad a \neq b. \quad (5.18)$$

For $\beta = 2$, this quantity is complex in general with real and imaginary part given by

$$\operatorname{Re} K_{ab} = \frac{K_{ab} + K_{ab}^*}{2} = \frac{1}{2}\left(\boldsymbol{w}_a^\dagger \Gamma \boldsymbol{w}_b + \boldsymbol{w}_b^\dagger \Gamma \boldsymbol{w}_a\right), \tag{5.19}$$

$$\operatorname{Im} K_{ab} = \frac{K_{ab} - K_{ab}^*}{2i} = \frac{1}{2i}\left(\boldsymbol{w}_a^\dagger \Gamma \boldsymbol{w}_b - \boldsymbol{w}_b^\dagger \Gamma \boldsymbol{w}_a\right), \tag{5.20}$$

where we used the Hermiticity $\Gamma^\dagger = \Gamma$. The characteristic function of the joint-probability density is thus

$$\begin{aligned}\mathcal{R}_{\beta=2}(x,y) &= \langle \exp\left(-ix \operatorname{Re} K_{ab} - iy \operatorname{Im} K_{ab}\right)\rangle = \left\langle \exp\left[-\frac{i}{2}(z^* K_{ab} + z K_{ab}^*)\right]\right\rangle \\ &= \left\langle \exp\left[-\frac{i}{2} z^* \boldsymbol{w}_a^\dagger \Gamma \boldsymbol{w}_b + z w_b^\dagger \Gamma \boldsymbol{w}_a\right]\right\rangle,\end{aligned} \tag{5.21}$$

where we introduced the complex Fourier-variable $z = x + iy$. As usual, the ensemble average is over all random variables, i.e. over H and, employing the random amplitude model (1.26), over the complex vectors $\boldsymbol{w}_a$ and $\boldsymbol{w}_b$, with implied limit $N \to \infty$.

For $\beta = 1$, those vectors can be chosen as real and Γ is real symmetric, such that $K_{ab} = \boldsymbol{w}_a^\dagger \Gamma \boldsymbol{w}_b = \frac{1}{2}\left(\boldsymbol{w}_a^\dagger \Gamma \boldsymbol{w}_b + \boldsymbol{w}_b^\dagger \Gamma \boldsymbol{w}_a\right)$. Then the characteristic function $\mathcal{R}_{\beta=1}(z) = \langle \exp(-\frac{iz}{2} K_{ab})\rangle$, where now z is real, takes the same form as the second line of (5.21) with factor $\frac{i}{4}$ instead of $\frac{i}{2}$ and $z = z^*$.

In order to perform the average over the vectors $\boldsymbol{w}_a$ and $\boldsymbol{w}_b$ with probability density $\mathcal{P}(W) \propto \exp\left(-\frac{\beta N}{2\gamma_c} \boldsymbol{w}_c^\dagger \boldsymbol{w}_c\right)$ we rewrite the exponent such that

$$\mathcal{R}_\beta(x,y) = C \left\langle \int \mathrm{d}^\beta \boldsymbol{w}_a \int \mathrm{d}^\beta \boldsymbol{w}_b \exp\left(-\frac{\beta}{2}\begin{bmatrix}\boldsymbol{w}_a^\dagger & \boldsymbol{w}_b^\dagger\end{bmatrix}\begin{bmatrix}\frac{N}{\gamma_a}\mathbb{1}_N & \frac{iz^*}{2}\Gamma \\ \frac{iz}{2}\Gamma & \frac{N}{\gamma_B}\mathbb{1}_N\end{bmatrix}\begin{bmatrix}\boldsymbol{w}_a \\ \boldsymbol{w}_b\end{bmatrix}\right)\right\rangle_H \tag{5.22}$$

with $\mathrm{d}^2\boldsymbol{w}_c = \prod_{j=1}^N \mathrm{d}w_{c,j}\mathrm{d}w_{c,j}^*$ and normalisation constant $C = \left(\frac{N^2}{4\pi^2\gamma_a\gamma_b}\right)^{\beta N/2}$. In the $2N \times 2N$ matrix, the diagonal blocks come from the Gaussian probability density of $\boldsymbol{w}_a$ and $\boldsymbol{w}_b$, whereas the off-diagonal blocks come from the expression for K_{ab} and K_{ab}^*, respectively. Performing the integrals over the channel vectors then yields

$$\mathcal{R}_\beta(x,y) = C(2\pi)^{\beta N}\left\langle \det\begin{bmatrix}\frac{N}{\gamma_a}\mathbb{1}_N & \frac{iz^*}{2}\Gamma \\ \frac{iz}{2}\Gamma & \frac{N}{\gamma_B}\mathbb{1}_N\end{bmatrix}^{-\beta/2}\right\rangle_H = \left\langle \det\left(\mathbb{1}_N + \frac{\gamma_a\gamma_b|z|^2}{4N^2}\Gamma^2\right)^{-\beta/2}\right\rangle_H \tag{5.23}$$

where we used the identity for block matrices $\det(A_{ij})_{i,j=1,2} = \det A_{11} \det(A_{22} - A_{21}A_{11}^{-1}A_{12})$. Rewriting the determinantal expression one finally gets

$$\mathcal{R}_\beta(z) = \left\langle \frac{|\det(E-H)|^\beta}{\det[\frac{\gamma_a\gamma_b|z|^2}{4N^2} + (E-H)^2]^{\beta/2}} \right\rangle_{H,\,N\to\infty}. \tag{5.24}$$

We now proceed to calculate this correlation function separately for the unitary and orthogonal case.

Unitary Symmetry

In the unitary case, we rewrite the denominator $\det[\frac{\gamma_a\gamma_b|z|^2}{4N^2} + (E-H)^2] = \det(E + \frac{i\sqrt{\gamma_a\gamma_b}|z|}{2N} - H)\det(E - \frac{i\sqrt{\gamma_a\gamma_b}|z|}{2N} - H)$. This shows that (5.24) for $\beta = 2$ takes a form similar to the characteristic function for the whole K-matrix, Eq. (3.2), with $M = 2$ and the replacements $\gamma x_1 \to \sqrt{\gamma_a\gamma_b}|z|/2$ and $\gamma x_2 \to -\sqrt{\gamma_a\gamma_b}|z|/2$. The large-$N$ limit of this correlation function was calculated in Section 3.2.1 and is given by Eqs. (3.21) and (3.22). For $M = 2$ the ensuing 2×2 determinant can be performed easily and the result reads

$$\mathcal{R}_{\beta=2}(z) = \exp(-\pi\rho\sqrt{\gamma_a\gamma_b}|z|)\left(1 + \frac{\pi\rho\sqrt{\gamma_a\gamma_b}}{2}|z|\right), \quad z = x + iy. \tag{5.25}$$

Note that the characteristic function of the distribution of the real or imaginary part alone can be obtained by setting $y \equiv 0$ or $x \equiv 0$, respectively (compare with (5.21)), which yields the same characteristic function in both cases in agreement with the solution from the Poisson kernel approach in the previous section.

The joint probability is obtained by inverse Fourier transform with respect to both variables x and y, viz.,

$$\mathcal{P}_{\beta=2}(\operatorname{Re} K_{ab}, \operatorname{Im} K_{ab}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \exp(i \operatorname{Re}(z^* K_{ab}))\mathcal{R}_{\beta=2}(|z|). \tag{5.26}$$

Since $\mathcal{R}_{\beta=2}(z)$ depends only on the modulus $|z|$, it is advantageous to go to polar

coordinates, $z = r \exp\phi$ such that

$$\begin{aligned}\mathcal{P}_{\beta=2}(\mathrm{Re}\, K_{ab}, \mathrm{Im}\, K_{ab}) &= \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}r\, r\mathcal{R}_{\beta=2}(r)\int_0^{2\pi}\mathrm{d}\phi\, \exp[ir\,\mathrm{Re}(\exp(-i\phi)K_{ab})] \\ &= \frac{1}{2\pi}\int_0^\infty \mathrm{d}r\, r\mathcal{R}_{\beta=2}(r)J_0(r|K_{ab}|), \end{aligned} \tag{5.27}$$

where in the second line we performed the ϕ-integration using $\int_0^{2\pi}\mathrm{d}\phi\, \exp(iA\cos\phi + iB\sin\phi) = 2\pi J_0(\sqrt{A^2+B^2})$. The last step is the r-integration. To perform it we abbreviate $u = \pi\rho\sqrt{\gamma_a\gamma_b}$, introduce a differential operator and use 6.623.2 in [59] to perform the remaining integral,

$$\begin{aligned}\mathcal{P}_{\beta=2}(\mathrm{Re}\, K_{ab}, \mathrm{Im}\, K_{ab}) &= \frac{1}{2\pi}\left(1 - \frac{u}{2}\frac{\mathrm{d}}{\mathrm{d}u}\right)\int_0^\infty \mathrm{d}r\, r\exp(-ur)J_0(|K_{ab}|r) \\ &= \left(1 - \frac{u}{2}\frac{\mathrm{d}}{\mathrm{d}u}\right)\frac{u/(2\pi)}{(u^2+|K_{ab}|^2)^{3/2}}. \end{aligned} \tag{5.28}$$

Applying the differential operator and replacing u yields the final solution

$$\mathcal{P}_{\beta=2}(\mathrm{Re}\, K_{ab}, \mathrm{Im}\, K_{ab}) = \frac{\rho(E)\sqrt{\gamma_a\gamma_b}(4\pi^2\rho(E)^2\gamma_a\gamma_b) + |K_{ab}|^2}{4(\pi^2\rho(E)^2\gamma_a\gamma_b + |K_{ab}|^2)^{5/2}}. \tag{5.29}$$

Compare this solution with the one obtained from the Poisson kernel approach, Eq. (5.11). Both are identical if we identify $\lambda = \pi\rho(E)\sqrt{\gamma_a\gamma_b}$ in agreement with the findings of Chapter 3 where we showed that both approaches are equivalent in general with $\lambda = \pi\gamma\rho(E)$ (for equivalent channels $\gamma_a = \gamma_b = \gamma$). Note that this approach allows for two different ways to obtain the distribution of just $\mathrm{Re}\, K_{ab}$ or $\mathrm{Im}\, K_{ab}$. In addition to integrating out one variable from the joint distribution as was done for the Poisson kernel approach we could also set $y = 0$ in the formula (5.25) for the characteristic function and perform the inverse Fourier transform with respect to x. This also yields the distribution given in (5.13).

Orthogonal Symmetry

For simplicity, we only consider $E = 0$ in the $\beta = 1$ case. Then we shift the absolute value featured in the numerator of (5.24) to the denominator using (4.4), such that

$$\mathcal{R}_{\beta=1}(x) = \lim_{\epsilon\to 0} \left\langle \frac{\det H^2}{\det(H^2 + \frac{x^2}{N^2})^{1/2} \det(H^2 + \frac{\epsilon^2}{N^2})^{1/2}} \right\rangle_{\mathrm{GOE},\, N\to\infty} . \tag{5.30}$$

We see that this is precisely the special case $\mathcal{C}_{2,4}^{(\mathrm{bulk})}(0, 0; \epsilon, -\epsilon, \sqrt{\gamma_a\gamma_b}x/2, -\sqrt{\gamma_a\gamma_b}x/2)$, compare with Eq. (4.187), which was calculated for Gaussian probability density in the limit $N \to \infty$ in Section 4.5.2. For arbitrary ϵ it is given by the rather complicated two-fold integral (4.186), however, in the limit $\epsilon \to 0$ it simplifies to the expression (4.188), which is shown in Appendix A.7.[36] Hence for the GOE, the characteristic function is given by

$$\mathcal{R}_{\beta=1}(x) = \frac{2}{\pi}\left(|\tilde{x}|K_0(|\tilde{x}|) + \int_{|\tilde{x}|}^{\infty} dy\, K_0(y)\right), \tag{5.31}$$

with $\tilde{x} = \sqrt{\gamma_a\gamma_b}x/(2J)$. We compare this function for $\sqrt{\gamma_a\gamma_b}/(2J) = 1$ with a numerical simulation of the characteristic function $\langle \exp(-\frac{ix}{2}K_{ab})\rangle$ in Figure 5.1.

To obtain the probability distribution we take the inverse Fourier transform

$$\mathcal{P}_{\beta=1}(K_{ab}) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}x\, \exp(ixK_{ab})\left(|\tilde{x}|K_0(|\tilde{x}|) + \int_{|\tilde{x}|}^{\infty} dy\, K_0(y)\right). \tag{5.32}$$

To perform the x-integration we use the integral representation of the Bessel function $K_0(x) = \int_0^\infty \mathrm{d}\phi\, \exp(-x\cosh\phi)$, which further implies the y-integral can be written as $\int_0^\infty \mathrm{d}\phi\, \exp(-|\tilde{x}|\cosh\phi)/\cosh\phi$, and thus

$$\begin{aligned}\mathcal{P}(K_{ab}) &= \frac{1}{\pi^2}\int_0^\infty \mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}x \left(|\tilde{x}| + \frac{1}{\cosh\phi}\right)\exp\left(ixK_{ab} - |\tilde{x}|\cosh\phi\right) \\ &= \frac{1}{\pi^2}\int_0^\infty \mathrm{d}\phi \left(\frac{1}{\cosh\phi} - \frac{\mathrm{d}}{\mathrm{d}(\cosh\phi)}\right)\int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(ixK_{ab} - |\tilde{x}|\cosh\phi\right). \end{aligned} \tag{5.33}$$

[36] When first confronted with this problem it was not clear if the complicated expression (4.186) could be simplified. The form (4.188) was obtained by computing the Fourier transform of the distribution $\mathcal{P}(K_{ab})$ obtained by the Poisson kernel approach given in (5.17). The claim that this approach is equivalent to the Hamiltonian approach gave then the idea for Appendix A.7.

The x-integration can now be identified with the Fourier transform of the function $\exp(-u|x|)$ with solution $\frac{2u}{u^2+K_{ab}^2}$ where $u = \frac{\sqrt{\gamma_a\gamma_b}}{2J}\cosh\phi$. After applying the differential operator we are hence left with

$$\mathcal{P}_{\beta=1}(K_{ab}) = \frac{(\gamma_a\gamma_b)^{3/2}}{2\pi^2 J^3}\int_0^\infty \mathrm{d}\phi\, \frac{\cosh^2\phi}{(K_{ab}^2 + \frac{\gamma_a\gamma_b}{4J^2}\cosh^2\phi)^2}. \tag{5.34}$$

Next we substitute $\sinh\phi = y$, such that

$$\mathcal{P}_{\beta=1}(K_{ab}) = \frac{8J}{\pi^2\sqrt{\gamma_a\gamma_b}}\int_0^\infty \mathrm{d}y\, \frac{\sqrt{1+y^2}}{(\frac{4J^2}{\gamma_a\gamma_b}K_{ab}^2 + 1 + y^2)^2}. \tag{5.35}$$

For $\gamma_a\gamma_b = 4J^2$ this is the same integration as we had to perform in the previous section, compare with Eq. (5.14). This implies we get the same solution as via the Poisson kernel approach, Eq. (5.17), with $\lambda = \sqrt{\gamma_a\gamma_b}/(2J)$ which again suggests that both approaches are also equivalent in the $\beta = 1$ case. Figure 5.1 shows a comparison of our result for $\lambda = 1$ with a numerical simulation.

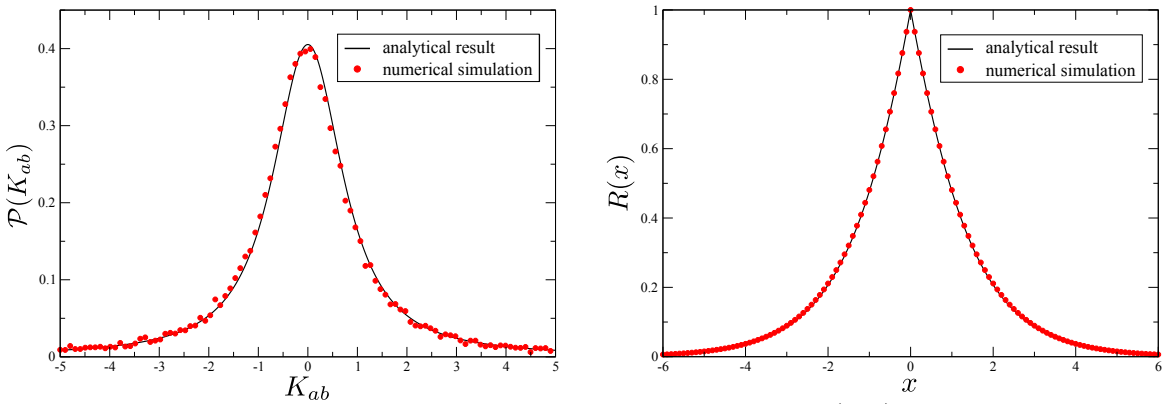

Figure 5.1.: Distribution of an off-diagonal K-matrix element K_{ab} (left) and its characteristic function (right). The numerical results were obtained from samples of 40000 GOE-matrices of size 80×80.

5.3. Distribution of Tr K

The distribution of the trace of K can be written as

$$\mathcal{P}(k) = \langle \delta(k - \operatorname{Tr} K) \rangle_{W,H} = \langle \delta(k - \operatorname{Tr} W^\dagger (E - H)^{-1} W) \rangle_{W,H}, \tag{5.36}$$

Its characteristic function $\mathcal{F}_\beta(x) = \langle \exp(-\frac{i\beta}{2} x \operatorname{Tr} K) \rangle_{W,H}$ is a special case of the characteristic function (3.2) for the K-matrix itself with $X = x\mathbb{1}_M$. This means after performing the W-average it is given by Eq. (3.7) where all x_c are the same, viz.,

$$\mathcal{F}_\beta(x) = \left\langle \frac{\det(E - H)^{M\beta/2} \operatorname{sgn} \det(E - H)^{M(2-\beta)\Theta(-x)}}{\det(E + \frac{i\gamma x}{N} - H)^{M\beta/2}} \right\rangle_{H,\, N\to\infty}. \tag{5.37}$$

For $\beta = 2$ this is the average of a ratio of two characteristic polynomials raised to the power M. Also for $\beta = 1$ and even M the sign-factor vanishes and one is left with a similar average with integer-power $M/2$. These cases are treated in [77], where the authors noticed that for a ratio with same number of determinants in the numerator and denominator, where all signs of the imaginary part of the arguments in the denominator are the same, the limit $N \to \infty$ will be the same for the GOE, GUE and GSE, given by

$$\lim_{N\to\infty} \left\langle \prod_{c=1}^{M} \frac{\det(\alpha_c/N - H)}{\det(\beta_c/N - H)} \right\rangle_{\text{GOE, GUE, GSE}} = \exp\left[\pm \frac{i}{J} \sum_{c=1}^{M} (\alpha_c - \beta_c) \right], \tag{5.38}$$

where the plus-sign or minus-sign are assumed if all imaginary parts in the denominator are negative or positive, respectively. In our case all $\alpha_c \equiv 0$ and all $\beta_c \equiv i\gamma x$ which yields

$$\mathcal{F}_\beta(x) = \exp\left(-\frac{\beta\gamma M}{2J} |x| \right), \quad \beta = 2 \text{ or } \beta = 1 \text{ and } M \text{ even}. \tag{5.39}$$

This implies also the trace of K is Cauchy distributed. Notice that one can easily extend the result to non-equivalent channels by choosing $\beta_c = i\gamma_c x$.

The remaining case for $\beta = 1$ and odd M, which involves square-roots of characteristic polynomials, will be treated using supersymmetry. In the first step we introduce

an imaginary increment ϵ, rewrite the correlation function similar to Eq. (5.4) (notice that this implies we can get rid of the sign-factor by choosing the sign of ϵ to be same as the sign of x) and replace the characteristic polynomials by Gaussian integrals,

$$\mathcal{F}_1 \propto \prod_{c=1}^{M} \int \mathrm{d}\boldsymbol{z}_{1c}\mathrm{d}\boldsymbol{z}_{2c}\, e^{is\boldsymbol{z}_{1c}^\dagger(E+i\gamma x/N-H)\boldsymbol{z}_{1c}+is\boldsymbol{z}_{2c}^\dagger(E+i\epsilon/N-H)\boldsymbol{z}_{2c}} \int \mathrm{d}\boldsymbol{\zeta}_c\mathrm{d}\boldsymbol{\zeta}_c^\dagger\, e^{2is\boldsymbol{\zeta}_c^\dagger(E+i\epsilon/N-H)\boldsymbol{\zeta}_c}, \tag{5.40}$$

where $\boldsymbol{z}_{1c}$ and $\boldsymbol{z}_{2c}$, $c = 1, \ldots, M$ are real commuting vectors and $\boldsymbol{\zeta}_c$ are complex anticommuting vectors. $s \equiv \operatorname{sgn} x = \operatorname{sgn}\epsilon$ ensures convergence of the integration, the limit $\epsilon \to 0$ is implied. The factor $2is$ in the integral over the anticommuting variables was introduced for symmetry reasons. As usual the ensemble average can now be performed easily with the result $\exp\left[-\frac{J^2}{N}\operatorname{Tr}\left(\frac{A+A^T}{2}\right)^2\right]$, where

$$\frac{A+A^T}{2} = \operatorname{sgn} x \sum_{c=1}^{M} (\boldsymbol{z}_{1c}\otimes\boldsymbol{z}_{1c}^T + \boldsymbol{z}_{2c}\otimes\boldsymbol{z}_{2c}^T - \boldsymbol{\zeta}_c\otimes\boldsymbol{\zeta}_c^\dagger + \boldsymbol{\zeta}_c^*\otimes\boldsymbol{\zeta}_c^T). \tag{5.41}$$

From here we proceed with the Hubbard-Stratonovich method, i.e. we replace

$$\exp\left[-\frac{J^2}{N}\operatorname{Tr}\left(\frac{A+A^T}{2}\right)^2\right] = \int \mathrm{d}\sigma\, \exp\left[-\frac{N}{4J^2}\operatorname{Str}\sigma^2 + i\boldsymbol{\psi}^\dagger(\sigma\otimes 1_N)\boldsymbol{\psi}\right], \tag{5.42}$$

where $\boldsymbol{\psi}^T = [\boldsymbol{z}_{11}^T, \ldots, \boldsymbol{z}_{1M}^T, \boldsymbol{z}_{21}^T, \ldots, \boldsymbol{z}_{2M}^T, \boldsymbol{\zeta}_1^T, \ldots, \boldsymbol{\zeta}_M^T, \boldsymbol{\zeta}_1^\dagger, \ldots, \boldsymbol{\zeta}_M^\dagger]$ and σ is a $4M\times 4M$ supermatrix of appropriate symmetry.

The H independent part in the exponential of Eq. (5.40) is given by

$$i\boldsymbol{\psi}^\dagger\left(sE\mathbb{1}_{4M} + \frac{i\gamma}{N}P_{x,\epsilon}\right)\otimes\mathbb{1}_N\boldsymbol{\psi}, \quad P_{x,\epsilon} = \operatorname{diag}(|x|\mathbb{1}_M, |\epsilon|\mathbb{1}_{3M}). \tag{5.43}$$

Hence, after $\boldsymbol{\psi}$-integration, we are left with

$$\mathcal{F}_{\beta=1}(x) \propto \int \mathrm{d}\sigma\, \exp\left(-\frac{N}{4J^2}\operatorname{Str}\sigma^2\right) \operatorname{sdet}\left(\sigma + sE\mathbb{1}_{4M} + \frac{i\gamma}{N}P_{x,\epsilon}\right)^{-N/2}. \tag{5.44}$$

Next we rewrite the superdeterminant using $\operatorname{sdet} A = \exp(\operatorname{Str}\ln A)$ and approximate

the ensuing logarithm to lowest order,

$$\mathcal{F}_{\beta=1}(x) \propto \int \mathrm{d}\sigma \, \exp\left(-\frac{N}{4J^2}\,\mathrm{Str}\,\sigma^2 - \frac{N}{2}\,\mathrm{Str}\,\ln\sigma_E - \frac{i\gamma}{2}\,\mathrm{Str}\,\sigma_E^{-1}P_{x,\epsilon} + \mathcal{O}(1/N)\right), \tag{5.45}$$

where $\sigma_E \equiv \sigma + sE\mathbb{1}_{4M}$. This implies the saddle-point equation $\sigma/J^2 + \sigma_E^{-1} = 0$ with diagonal solution $\sigma_D = (-sE/2 + i\sqrt{4J^2 - E^2}/2)\mathbb{1}_{4M}$. Here we used that the superdeterminant in (5.44) has a pole at $\sigma_{BB} = -sE\mathbb{1}_{2M} - \frac{i\gamma}{N}\,\mathrm{diag}(|x|\mathbb{1}_M, |\epsilon|\mathbb{1}_M)$, i.e. the imaginary part of this pole is negative. To perform the saddle-point approximation we need to deform the contour such that no singularities are crossed, and hence we have to choose the boson-boson part of σ_D with positive imaginary part. For the fermion-fermion part one could in principle choose any metric, however, in the limit $\epsilon \to 0$ only the case where all imaginary parts are positive will contribute to the solution. The argument is similar to the one given in Section 2.2.4, where we encountered a very similar supermatrix integral, compare with Eq. (2.78) and the subsequent discussion. In contrast to Section 4.4.2 we do not encounter a saddle-point manifold here. Instead σ_D is the only contributing solution. This is because it is proportional to the unit matrix. Hence we substitute $\sigma = \sigma_D + \delta\sigma$ in Eq. (5.45) and expand to second order in $\delta\sigma$,

$$\begin{aligned}\mathcal{F}_{\beta=1}(x) \propto &\exp\left(-\frac{N}{4J^2}\,\mathrm{Str}\,\sigma_D^2 - \frac{N}{2}\,\mathrm{Str}\,\ln(\sigma_{D,E}) + \frac{i\gamma}{2J^2}\,\mathrm{Str}\,\sigma_D P_{x,0}\right) \\ &\times \int \mathrm{d}\delta\sigma \, \exp\left(-\frac{N}{4J^2}\,\mathrm{Str}\,\delta\sigma^2 + \frac{N}{4}\,\mathrm{Str}\,(\sigma_{D,E}^{-1}\delta\sigma)^2\right).\end{aligned} \tag{5.46}$$

The first two terms vanish due to the definition of the supertrace (2.36) and the integral yields unity such that the final solution is given by

$$\mathcal{F}_{\beta=1}(x) = \exp\left(\frac{i\gamma}{2J^2}\,\mathrm{Str}\,\sigma_D P_{x,0}\right) = \exp\left(-\frac{\gamma M}{4J^2}(iEx + \sqrt{4J^2 - E^2}|x|)\right), \tag{5.47}$$

where we determined the constant to be unity due to the requirement $\mathcal{F}_{\beta=1}(0) = 1$ which follows from normalisation of the corresponding probability distribution

$\mathcal{P}_{\beta=1}(k)$. It is given by the inverse Fourier transform of $\mathcal{F}_{\beta=1}(x)$

$$\mathcal{P}_{\beta=1}(k) = \frac{\rho(E)^2/(\gamma M)}{\left(\frac{k}{\gamma M} - \frac{E}{2J^2}\right)^2 + \pi^2\rho(E)^2}, \tag{5.48}$$

with level density $\rho(E) = \sqrt{4J^2 - E^2}/(2\pi J^2)$. Notice that the derivation is independent of the parity of M and thus also generalises the earlier findings, Eq. (5.39), for $E \neq 0$. The unitary case can be treated along similar lines and yields the same expression (5.48). This shows that for both the unitary and orthogonal case the trace of the K-matrix is distributed according to the same Cauchy distribution, where for equivalent channels, their number M determines the width of the distribution (otherwise replace γM by the sum over all coupling amplitudes γ_c).

5.4. Correlation of Diagonal Entries in the Bulk

The diagonal entries of the K-matrix show an interesting property: They are Cauchy-distributed, see Section 5.1, but also their sum (the trace of K) is Cauchy-distributed as shown in the last section. This is usually a property shown by independent variables, a linear combination of independently Cauchy-distributed random variables has also a Cauchy distribution; this is called stability (other examples for stable distributions are the Gaussian and Lévy distribution). However, with the K-matrix itself being distributed according to the matrix-Cauchy distribution (1.31), one would expect the diagonal entries to be highly correlated. That this is indeed the case will be shown in this section by explicitly calculating the correlation function of two diagonal entries. For the unitary case this will be done following both the Poisson kernel and the Hamiltonian approach. In the orthogonal case, we only get results via the Poisson kernel approach and briefly discuss the difficulties encountered in the Hamiltonian approach.

5.4.1. Poisson Kernel Approach

Our starting point is again the Cauchy distribution (1.31) for the K-matrix, which follows from the claim that the scattering matrix is distributed according to the

Poisson kernel (1.30). As discussed in Section 5.2.1, it suffices to consider the two-channel case $M = 2$. First we introduce a shifted and rescaled matrix $\tilde{K} = (K - \epsilon)/\lambda$ such that $\mathcal{P}(\tilde{K})$ is given by Eq. (5.8). Then the correlation function of the two diagonal entries can be obtained by integrating out the off-diagonal element K_{ab}. Rescaling this integration variable $K_{ab} \to \lambda K_{ab}$ shows that we may omit the tilde for K_{ab} in $\mathcal{P}(\tilde{K})$. For brevity we also omit it for K_{aa} and K_{bb}, which corresponds to perfect coupling.

Unitary Symmetry Since for $\beta = 2$ the off-diagonal element K_{ab} is complex, we introduce new coordinates $K_{ab} = \sqrt{r}\exp(i\phi)$. Since $\mathcal{P}(K)$ depends only on the modulus $|K_{ab}|$ (see Eq. (5.8)) it will be independent of ϕ such that

$$\mathcal{P}_{\beta=2}(K_{aa}, K_{bb}) \propto \int_0^\infty \mathrm{d}r\, [(1 + K_{aa}^2)(1 + K_{bb}^2) + 2r(1 - K_{aa}K_{bb}) + r^2]^{-2}. \tag{5.49}$$

This integral is of the form $\int_0^\infty \mathrm{d}r\, (r^2 + 2ur + v)^{-2}$. Since the integrand is a ratio of polynomials the integration can be performed easily with solution

$$\mathcal{P}_{\beta=2}(K_{aa}, K_{bb}) \propto \frac{1}{v - u^2}\left[-\frac{u}{2v} + \frac{\pi - 2\arctan(\frac{u}{\sqrt{v-u^2}})}{4\sqrt{v - u^2}}\right], \tag{5.50}$$

where $u = (1 - K_{aa}K_{bb})$, $v = (1 + K_{aa}^2)(1 + K_{bb}^2)$ and hence $v - u^2 = (K_{aa} + K_{bb})^2 > 0$. We further simplify this result using $\arctan\left(\frac{1-xy}{|x+y|}\right) = \pi/2 - \mathrm{sgn}(x + y)(\arctan x + \arctan y)$ and obtain

$$\mathcal{P}_{\beta=2}(K_{aa}, K_{bb}) \propto \frac{1}{(K_{aa} + K_{bb})^2}\left[\frac{K_{aa}K_{bb} - 1}{(1 + K_{aa}^2)(1 + K_{bb}^2)} + \frac{\arctan K_{aa} + \arctan K_{bb}}{K_{aa} + K_{bb}}\right]. \tag{5.51}$$

This is the final result for the perfect coupling case. The previous discussion implies that the more general case can be obtained by replacing $K_{aa} \to (K_{aa} - \epsilon)/\lambda$ and $K_{bb} \to (K_{bb} - \epsilon)/\lambda$.

Orthogonal Symmetry For $\beta = 1$, the correlation function of two diagonal elements is given by integrating (5.8) over the real variable K_{ab}, where we can restrict to positive

K_{ab} since the integrand is even, viz.,

$$\mathcal{P}_{\beta=1}(K_{aa}, K_{bb}) \propto \int_0^\infty \frac{\mathrm{d}K_{ab}}{(u^4 + 2v^2K_{ab}^2 + K_{ab}^4)^{3/2}} = -\frac{1}{2u^3}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^\infty \frac{\mathrm{d}K_{ab}}{\sqrt{u^4 + 2v^2K_{ab}^2 + K_{ab}^4}}, \tag{5.52}$$

with $u^4 = (1 + K_{aa}^2)(1 + K_{bb}^2)$ and $v^2 = K_{aa}K_{bb} - 1$. The solution of this integral can be expressed in terms of an elliptic integral, see e.g. formula 3.165.2 in [59],

$$\int_y^\infty \frac{\mathrm{d}x}{\sqrt{u^4 + 2v^2x^2 + x^4}} = \frac{1}{2u}F(\alpha, r), \quad \alpha = \arccos\frac{y^2 - u^2}{y^2 + u^2},\ r = \frac{\sqrt{u^2 - v^2}}{u\sqrt{2}}, \tag{5.53}$$

with $u^2 > v^2 > -\infty$, $u^2 > 0$, $y \geq 0$. In our case we have $y = 0$, which implies that $\alpha = \arccos(-1) = \pi$. Furthermore we have $u^2 = \sqrt{1 + K_{aa}^2}\sqrt{1 + K_{bb}^2} \geq 1 > 0$. To check the last condition $u^2 > v^2$ let us assume $v^2 > 0$ (otherwise $u^2 > v^2$ is trivially fulfilled). Then we may square the inequality and check if $u^4 > v^4$. This is indeed the case since $u^4 - v^4 = (K_{aa} + K_{bb})^2 > 0$ for all $K_{aa} \neq -K_{bb}$. Note that for $K_{aa} = -K_{bb}$ we necessarily have $v^2 < 0$ and also in that case the condition is fulfilled.

$F(\alpha, r)$ is the incomplete elliptic integral of the first kind, defined as $F(\alpha, r) = \int_0^\alpha d\phi(1 - r^2\sin^2\phi)^{-1/2}$. Note that in our case we can rewrite $F(\pi, r) = 2F(\pi/2, r) \equiv 2K(r)$, where $K(r)$ is the more fundamental complete elliptic integral of the first kind. Thus we get the correlation function in terms of $K(r)$ and its derivative,

$$\mathcal{P}_{\beta=1}(K_{aa}, K_{bb}) \propto -\frac{1}{u^3}\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{1}{u}K(r(u))\right] = -\frac{1}{u^3}\left[-\frac{1}{u^2}K(r) + \frac{r'(u)}{u}K'(r)\right], \tag{5.54}$$

where $r'(u)$ denotes the derivative of $r = \frac{\sqrt{u^2-v^2}}{u\sqrt{2}}$ w.r.t. u, given as $r'(u) = \frac{v^2}{2u^3r}$. For the derivative of the elliptic integral we use the formula (see e.g. 8.123.2 in [59])

$$K'(r) = \frac{E(r)}{r(1 - r^2)} - \frac{K(r)}{r}, \tag{5.55}$$

where $E(r) = \int_0^{\pi/2} d\phi(1 - r^2\sin^2\phi)^{1/2}$ is the complete elliptic integral of the second kind. Hence we get the final result in terms of these two elliptic integrals,

$$\mathcal{P}_{\beta=1}(K_{aa}, K_{bb}) \propto -\frac{1}{u^5}\left[\left(1 + \frac{v^2}{2u^2r^2}\right)K(r) - \frac{v^2}{2u^2r^2(1 - r^2)}E(r)\right], \tag{5.56a}$$

$$r = \frac{\sqrt{u^2 - v^2}}{u\sqrt{2}}, \quad u^4 = (1 + K_{aa}^2)(1 + K_{bb}^2), \quad v^2 = K_{aa}K_{bb} - 1, \tag{5.56b}$$

$$K(r) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2 \phi}}, \quad E(r) = \int_0^{\pi/2} d\phi \sqrt{1 - r^2 \sin^2 \phi}. \tag{5.56c}$$

As for the unitary case one obtains the case of non-perfect coupling by the replacement $K_{aa} \to (K_{aa} - \epsilon)/\lambda$ and $K_{bb} \to (K_{bb} - \epsilon)/\lambda$.

5.4.2. Hamiltonian Approach

As usual we calculate the correlation of two diagonal entries K_{aa} and K_{bb} via its characteristic function,

$$\mathcal{R}_\beta(x_a, x_b) = \left\langle \exp\left(-\frac{i\beta}{2} x_a \boldsymbol{w}_a^\dagger (E - H)^{-1} \boldsymbol{w}_a - \frac{i\beta}{2} x_b \boldsymbol{w}_b^\dagger (E - H)^{-1} \boldsymbol{w}_b \right) \right\rangle, \tag{5.57}$$

which after doing the average over $\boldsymbol{w}_a$ and $\boldsymbol{w}_b$ becomes

$$\mathcal{R}_\beta(x_a, x_b) = \left\langle \frac{\operatorname{sgn}\det(E - H)^{(2-\beta)\Theta(-x_a x_b)} \det(E - H)^\beta}{\det(E - H + i x_a \gamma_a/N)^{\beta/2} \det(E - H + i x_b \gamma_b/N)^{\beta/2}} \right\rangle_{H,\, N\to\infty}. \tag{5.58}$$

Note that the sign of the determinant only plays a role in the $\beta = 1$ case when the signs of x_a and x_b are different.

Unitary Symmetry

For the $\beta = 2$ case this characteristic function reads

$$\mathcal{R}_{\beta=2}(x_a, x_b) = \left\langle \frac{\det(E - H)^2}{\det(E - H + i x_a \gamma_a/N) \det(E - H + i x_b \gamma_b/N)} \right\rangle_{H,\, N\to\infty}, \tag{5.59}$$

which is equivalent to the characteristic function of the whole K-matrix in the two channel case $M = 2$, compare with Eq. (3.2) (with γ replaced by γ_c for arbitrary coupling constants). A solution of the ensemble average in the limit $N \to \infty$ for joint probability density $\mathcal{P}(H) \propto \exp(-N \operatorname{Tr} V(H))$ has been calculated in Section 3.2.1

which in this case evaluates to

$$\begin{aligned}\mathcal{R}_{\beta=2}(x_a, x_b) &= \mathcal{R}(x_a)\mathcal{R}(x_b)\left(1 - \frac{\pi\rho(E)|x_a|x_b - x_a|x_b|}{x_a/\gamma_b - x_b/\gamma_a}\right) \\ &= \mathcal{R}(x_a)\mathcal{R}(x_b)\left(1 - \Theta(-x_a x_b)\frac{2\pi\rho(E)|x_a|x_b}{x_a/\gamma_b - x_b/\gamma_a}\right),\end{aligned} \tag{5.60}$$

where $\mathcal{R}(x_a) = \exp[-\gamma_a(iV'(E)x_a/2 + \pi\rho(E)|x_a|)]$ is the characteristic function of the probability distribution of the diagonal entry x_a and similar for x_b. This means for $x_a x_b > 0$ the characteristic function factorises (this would indicate independence of K_{aa} and K_{bb} if it was also true for all $x_a x_b < 0$) whereas for $x_a x_b < 0$ there is an additional correlation term.

Consequently the correlation of the diagonal entries is of a similar form as its characteristic function,

$$\mathcal{P}_{\beta=2}(K_{aa}, K_{bb}) = \mathcal{P}(K_{aa})\mathcal{P}(K_{bb}) + \mathcal{P}_c(K_{aa}, K_{bb}), \tag{5.61}$$

where $\mathcal{P}(K_{aa})$ and $\mathcal{P}(K_{bb})$ are the probability distributions of K_{aa} and K_{bb}, respectively, which have been calculated in Section 5.1 and are given by the Cauchy distribution (5.2). The correlation term is given by

$$\mathcal{P}_c(K_{aa}, K_{bb}) = -\frac{\rho}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x_a \int_{-\infty}^{\infty}\mathrm{d}x_b\, \mathcal{R}(x_a)\mathcal{R}(x_b)\frac{\Theta(-x_a x_b)|x_a|x_b}{x_a/\gamma_b - x_b/\gamma_a}\, e^{ix_a K_{aa} + ix_b K_{bb}}. \tag{5.62}$$

The term $\Theta(-x_1 x_2)$ ensures that the integration only involves the region where x_a and x_b have different signs. Thus we may split the integral into a sum of two integrals, where we integrate x_a over the positive real numbers and x_b over the negative real numbers and vice versa,

$$\begin{aligned}\mathcal{P}_c(K_{aa}, K_{bb}) = -\frac{\rho}{2\pi}\Bigg(&\int_0^{\infty}\mathrm{d}x_a \int_{-\infty}^{0}\mathrm{d}x_b - \int_{-\infty}^{0}\mathrm{d}x_a \int_0^{\infty}\mathrm{d}x_b\Bigg) \\ &\times \frac{\mathcal{R}(x_a)\mathcal{R}(x_b)x_a x_b}{x_a/\gamma_b - x_b/\gamma_a}\exp(ix_a K_{aa} + ix_b K_{bb}).\end{aligned} \tag{5.63}$$

Now we change variables $\gamma_a x_a \to x_a$, $\gamma_b x_b \to -x_b$ in the first term and $\gamma_a x_a \to -x_a$, $\gamma_b x_b \to x_b$ in the second term. Then both terms will be the complex conjugate of each

other such that we can combine them to

$$\begin{aligned}\mathcal{P}_c(K_{aa},K_{bb}) =& \frac{\rho}{\gamma_a\gamma_b\pi}\int_0^\infty \mathrm{d}x_a \int_0^\infty \mathrm{d}x_b\, \frac{x_a x_b}{x_a+x_b}\exp[-\pi\rho(x_a+x_b)] \\ &\times \cos[x_a K_{aa}/\gamma_a - x_b K_{bb}/\gamma_b - V'(E)(x_a-x_b)/2].\end{aligned} \tag{5.64}$$

To make further progress we introduce new variables $p=\pi\rho(x_a+x_b)$ and $q=\pi\rho(x_a-x_b)$ with $p\in(0,\infty)$ and $q\in(-p,p)$. The Jacobian of this transformation is $1/(2\pi\rho)$. Then the correlation term simplifies to

$$\begin{aligned}\mathcal{P}_c(K_{aa},K_{bb}) = \frac{1}{8\gamma_a\gamma_b\pi^4\rho^2}\int_0^\infty\frac{\mathrm{d}p}{p}\exp(-p)\int_{-p}^{+p}\mathrm{d}q\,(p^2-q^2) \\ \times\cos\left(\frac{p}{2}(\tilde{K}_{aa}-\tilde{K}_{bb})+\frac{q}{2}(\tilde{K}_{aa}+\tilde{K}_{bb})\right),\end{aligned} \tag{5.65}$$

where we introduced the shifted and rescaled variables $\tilde{K}_{aa}=\frac{K_{aa}/\gamma_a-V'/2}{\pi\rho}$ and $\tilde{K}_{bb}=\frac{K_{bb}/\gamma_b-V'/2}{\pi\rho}$. The integral over q is now of the simple form $\int_{-p}^{+p}\mathrm{d}q\,(p^2-q^2)\cos(uq+vp)$. This can be easily integrated using integration by parts twice, yielding the result $\frac{2}{u^3}[\sin(p(u+v))+\sin(p(u-v))]-\frac{2}{u^2}p[\cos(p(u+v))+\cos(p(u-v))]$. Now with $u=(\tilde{K}_{aa}+\tilde{K}_{bb})/2$ and $v=(\tilde{K}_{aa}-\tilde{K}_{bb})/2$ we get therefore

$$\begin{aligned}\mathcal{P}_c(K_{aa},K_{bb}) = \frac{2/(\gamma_a\gamma_b\pi^4\rho^2)}{(\tilde{K}_{aa}+\tilde{K}_{bb})^3}\int_0^\infty\frac{\mathrm{d}p}{p}\exp(-p)\Big\{\big[\sin(p\tilde{K}_{aa})+\sin(p\tilde{K}_{bb})\big] \\ -\frac{p}{2}(\tilde{K}_{aa}+\tilde{K}_{bb})\big[\cos(p\tilde{K}_{aa})+\cos(p\tilde{K}_{bb})\big]\Big\}.\end{aligned} \tag{5.66}$$

The remaining integrals are now either of the form $I_1=\int_0^\infty\mathrm{d}p\,e^{-p}\cos(up)$ or of the form $I_2=\int_0^\infty\frac{\mathrm{d}p}{p}e^{-p}\sin(up)$. The first is given by $I_1=\mathrm{Re}\int_0^\infty\mathrm{d}p\,e^{-(1+iu)p}=\mathrm{Re}(1+iu)^{-1}=(1+u^2)^{-1}$. For I_2 note that $\frac{\mathrm{d}I_2}{\mathrm{d}u}=I_1$ and $I_2(0)=0$, and hence $I_2=\int\mathrm{d}u\,(1+u^2)^{-1}=\arctan u$. Collecting these results we finally end up with

$$\mathcal{P}_{\beta=2}(K_{aa},K_{bb})=\mathcal{P}(K_{aa})\mathcal{P}(K_{bb})+\mathcal{P}_c(K_{aa},K_{bb}), \tag{5.67a}$$

with

$$\mathcal{P}(K_{cc})=\frac{1}{\gamma_c\pi^2\rho(E)(1+\tilde{K}_{cc}^2)},\quad \tilde{K}_{cc}=\frac{1}{\pi\rho(E)}\left(\frac{K_{cc}}{\gamma_c}-\frac{V'(E)}{2}\right),\quad c=a,b, \tag{5.67b}$$

$$\mathcal{P}_c(K_{aa}, K_{bb}) = \frac{2/(\gamma_a\gamma_b\pi^4\rho(E)^2)}{(\tilde{K}_{aa}+\tilde{K}_{bb})^3}\Bigg\{ \arctan\tilde{K}_{aa} + \arctan\tilde{K}_{bb} \\ - \frac{\tilde{K}_{aa}+\tilde{K}_{bb}}{2}\left(\frac{1}{1+\tilde{K}_{aa}^2}+\frac{1}{1+\tilde{K}_{bb}^2}\right)\Bigg\}. \tag{5.67c}$$

To compare this result with the one obtain via the Poisson kernel approach, Eq. (5.51), we can combine the term $\mathcal{P}(K_{aa})\mathcal{P}(K_{bb})$ with the term in the second line of (5.67c). This yields precisely the first term of (5.51) (with prefactor $2/(\gamma_a\gamma_b\pi^4\rho^2)$) which shows that both approaches yield the same result as expected.

Orthogonal Symmetry

For $\beta = 1$ and $x_a x_b > 0$, the characteristic function (5.58) simplifies to

$$\mathcal{R}_{\beta=1}(x_a, x_b) = \left\langle \frac{\det(E-H)}{\det(E-H+ix_a\gamma_a/N)^{1/2}\det(E-H+ix_b\gamma_b/N)^{1/2}} \right\rangle_H. \tag{5.68}$$

In the limit $N\to\infty$ for the GOE, this is $\mathcal{C}_{1,2}^{(\text{bulk}),\ \operatorname{sgn}x_a=\operatorname{sgn}x_b}(0; x_a\gamma_a, x_b\gamma_b)$ in the notation from Chapter 4, with the solution (see Eq. (4.45))

$$\begin{aligned}\mathcal{R}_{\beta=1}^{(x_a x_b>0)}(x_a, x_b) &= \exp\left[-\frac{iE}{4J^2}(\gamma_a x_a+\gamma_b x_b) - \frac{\sqrt{4J^2-E^2}}{4J^2}(\gamma_a|x_a|+\gamma_b|x_b|)\right] \\ &= \mathcal{R}(x_a)\mathcal{R}(x_b),\end{aligned} \tag{5.69}$$

where $\mathcal{R}(x)$ is the characteristic function of the distribution (5.5) of a diagonal entry as calculated in Section 5.1. Similar to the unitary case, the characteristic function factorises for $x_a x_b > 0$, compare with Eq. (5.60).

However, for $x_a x_b < 0$ we have to calculate instead

$$\mathcal{R}_{\beta=1}(x_a, x_b) = \left\langle \frac{|\det(E-H)|}{\det(E-H+ix_a\gamma_a/N)^{1/2}\det(E-H+ix_b\gamma_b/N)^{1/2}} \right\rangle_H, \tag{5.70}$$

i.e. the correlation function $\lim_{\epsilon\to 0}\mathcal{C}_{2,4}(0,0;\epsilon,-\epsilon,x_a\gamma_a,x_b\gamma_b)$ in the notation of Chapter 4. So far we have only been able to calculate this correlation function in the case

$E = 0$ and $\gamma_a x_a = -\gamma_b x_b$ see Section 4.5.2. This partial result is not sufficient here, as we need the function for arbitrary x_a and x_b of different sign to finally perform the inverse Fourier transform. Hence at the moment the Poisson kernel approach seems to be the only viable option. However, the result for $x_a = -x_b \equiv x$ at least shows that the diagonal entries are correlated (which is not clear from the case $x_a x_b > 0$), since $\mathcal{R}_{\beta=1}(x, -x) \neq \mathcal{R}(x)\mathcal{R}(-x)$.

5.5. Distribution of Diagonal Entries at the Edge of the GUE Spectrum

So far we have only considered results for E being in the bulk of the spectrum. However, for the Gaussian ensembles we know that the picture gets different if we come close to the spectral edge at $E = 2J$ (or equivalently $E = -2J$) of the semicircular level density, see Section 1.1.2, in particular Eq. (1.11) for the level density around the edge for $\beta = 2$. Hence it is an interesting question how the bulk result for the distribution of a diagonal entry of the K-matrix, given by the Cauchy distribution (5.2), changes at this edge. For technical reasons we restrict here to the unitary case.

Our starting point is again the characteristic function for the distribution of a diagonal entry given by Eq. (3.7) with $M = 1$. In [76] the authors show this quantity can be expressed in terms of monic orthogonal polynomials $\pi_N(x)$ and their Cauchy transforms $h_N(x) = \frac{1}{2\pi i} \int \mathrm{d}y \, \exp(-NV(y)) \, \pi_N(y)/(y - x)$ as

$$F(\mu, \epsilon) = \left\langle \frac{\det(\mu - H)}{\det(\epsilon - H)} \right\rangle_H = -\frac{2\pi i}{c_{N-1}^2} \left[h_N(\epsilon)\pi_{N-1}(\mu) - h_{N-1}(\epsilon)\pi_N(\mu) \right], \tag{5.71}$$

where in our case $\mu = E$ and $\epsilon = E + i\gamma x/N$. The polynomials are orthogonal with respect to the weight $\exp(-NV(x))$, which means $\int \mathrm{d}x \, \pi_k(x)\pi_m(x) = c_k c_l \delta_{km}$; this also defines the constant in (5.71). The potential $V(x)$ is the same as in the joint probability density for the random matrix H, i.e. for the GUE the weight is Gaussian and the orthogonal polynomials are Hermite polynomials.[37]

[37] In [76] it is shown that the large-N limit of expression (5.71) is universal if E belongs to the bulk of the spectrum. This, together with the fact that for arbitrary M the correlation function has determinantal structure with (5.71) as kernel, yields the universal limit (3.9) used in Chapter 3.

The $N \to \infty$ limit is obtained by replacing the quantites in (5.71) with their appropriate large-N asymptotics. For E in the bulk of the spectrum this will recover the results from Section 5.1. However, as we are interested in the behaviour at the edge of the GUE spectrum, we employ the edge scaling $E = 2 + \xi N^{-2/3}$, where ξ is of order unity (for simplicity we consider the case $J = 1$). The easiest way to compute the large-N asymptotics is to express the Hermite polynomial by its integral representation (2.19) and perform a saddle-point analysis, similar for its Cauchy transform. Such a procedure was done in Appendix B of [108] with the result

$$F\left(2 + \tfrac{\xi}{N^{2/3}}, 2 + \tfrac{\xi+i\omega}{N^{2/3}}\right) \approx -ie^{-iN^{1/3}\omega} \operatorname{sgn}\omega[\mathrm{Ai}(\xi)\,\alpha'(\xi,\omega) - \mathrm{Ai}'(\xi)\,\alpha(\xi,\omega)], \tag{5.72a}$$

$$\begin{aligned}\alpha(\xi,\omega) = &\int_0^\infty \mathrm{d}\tau\, \exp\left(i\operatorname{sgn}\omega(\tau\xi + \tau^3/3) - |\omega|\tau\right) \\ &+ i\operatorname{sgn}\omega \int_0^\infty \mathrm{d}\tau\, \exp\left(\tau\xi - \tau^3/3 + i\omega\tau\right),\end{aligned} \tag{5.72b}$$

where $\mathrm{Ai}(\xi)$ is the Airy function and the dash denotes the derivative w.r.t. ξ. In our case we have $\omega = N^{-1/3}\gamma x$. Notice that this result can be used to compute the level density at the edge of the GUE spectrum. This is done in Appendix A.9 and can be seen as further verification of Eqs. (5.72a,5.72b).

The form of F makes it clear that $\mathcal{P}_\xi(K_{aa})$ cannot have an N-independent limiting distribution. Hence we introduce a new random variable κ, related to K_{aa} via $\kappa = N^{1/3}(1 - K_{aa}/\gamma)$. Its limiting distribution will be N-independent and is given by

$$\mathcal{P}_\xi(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \exp[ix(\omega)K_{aa}(\kappa)] F\left(2 + \tfrac{\xi}{N^{2/3}}, 2 + \tfrac{\xi+i\omega}{N^{2/3}}\right), \tag{5.73}$$

with $x(\omega) = N^{1/3}\omega/\gamma$ and $K_{aa}(\kappa) = \gamma(1 - \kappa N^{-1/3})$. We can split the above integral into a sum of two integrals, $\mathcal{P}_\xi(\kappa) = I_1 + I_2$, where

$$I_1 = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\tau\, [\mathrm{Ai}(\xi)\tau - \mathrm{Ai}'(\xi)]\, e^{\tau\xi - \tau^3/3} \int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{i\omega(\tau-\kappa)}, \tag{5.74}$$

$$I_2 = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\omega\, [\mathrm{Ai}(\xi)\tau + i\operatorname{sgn}\omega\ \mathrm{Ai}'(\xi)]\, e^{i\operatorname{sgn}\omega(\tau\xi + \tau^3/3)}\, e^{-i\omega\kappa - |\omega|\tau}. \tag{5.75}$$

The first integral I_1 is easy to perform since the ω-integration yields a delta-function

such that

$$\begin{aligned} I_1 &= \int_0^\infty \mathrm{d}\tau \, [\mathrm{Ai}(\xi)\tau - \mathrm{Ai}'(\xi)] \exp(\tau\xi - \tau^3/3)\delta(\tau - \kappa) \\ &= \Theta(\kappa)[\mathrm{Ai}(\xi)\kappa - \mathrm{Ai}'(\xi)] \exp\left(\kappa\xi - \frac{1}{3}\kappa^3\right), \end{aligned} \tag{5.76}$$

where $\Theta(\kappa)$ is the Heaviside-Theta function which had to be introduced because of the integration range of τ. To perform I_2 we split the ω-integration into the sum of two integrals, one over positive ω-values and one over negative ω-values. Due to the $\operatorname{sgn}\omega$ terms featured in the integrand these two integrals are then complex conjugates of each other which implies

$$\begin{aligned} I_2 &= \frac{1}{\pi} \operatorname{Re}\left\{ \int_0^\infty \mathrm{d}\tau \, [\mathrm{Ai}(\xi)\tau + i\,\mathrm{Ai}'(\xi)] \, e^{i(\tau\xi + \tau^3/3)} \int_0^\infty \mathrm{d}\omega \, e^{i\omega(i\tau - \kappa)} \right\} \\ &= \frac{1}{\pi} \operatorname{Re}\left\{ \int_0^\infty \mathrm{d}\tau \, [\mathrm{Ai}(\xi)\tau + i\,\mathrm{Ai}'(\xi)] \, e^{i(\tau\xi + \tau^3/3)} \frac{1}{\tau + i\kappa} \right\}, \end{aligned} \tag{5.77}$$

where in the second line the ω-integration was performed. Combining the results for I_1 and I_2 the probability distribution of κ, which is related to a diagonal K-matrix element for the GUE at the edge of the spectrum via $\kappa = N^{1/3}(1 - K_{aa}/\gamma)$ is hence given by

$$\begin{aligned} \mathcal{P}_\xi(\kappa) =& \frac{1}{\pi} \operatorname{Re}\left\{ \int_0^\infty \mathrm{d}\tau \, \exp\left[i\left(\tau\xi + \frac{\tau^3}{3}\right)\right] \frac{\mathrm{Ai}(\xi)\tau + i\,\mathrm{Ai}'(\xi)}{\tau + i\kappa} \right\} \\ &+ \Theta(\kappa)\,[\mathrm{Ai}(\xi)\kappa - \mathrm{Ai}'(\xi)] \exp\left[\kappa\xi - \frac{1}{3}\kappa^3\right]. \end{aligned} \tag{5.78}$$

This is our final result. As expected, the distribution at the edge is not Cauchy, but a more complicated expression. Figure 5.2 shows it examplary for $\xi = -1$. Notice, however, that we expect to retain the Cauchy distribution in the bulk limit $\xi \to -\infty$ of (5.78). This is indeed the case as is shown in Appendix A.10.

We restricted here to the unitary case, however, similar to the level density, which is the same in the bulk but differs at the edge for the GOE and GUE, one can expect the GOE result to be different from (5.78), although both have the same Cauchy distribution (5.2) in the bulk of the spectrum.

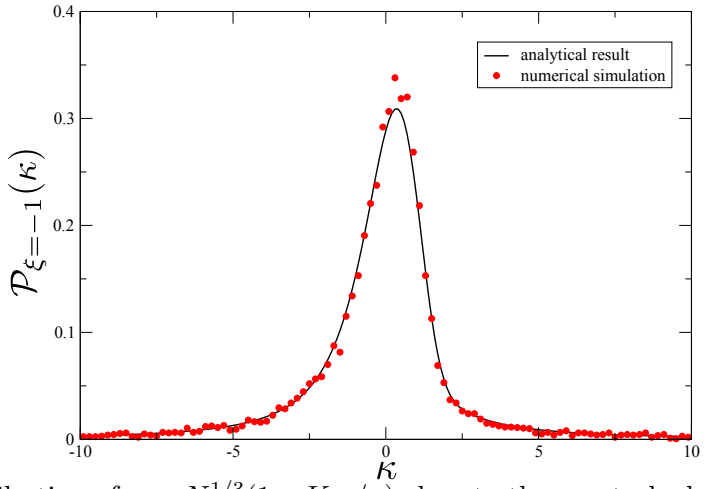

Figure 5.2.: Distribution of $\kappa = N^{1/3}(1 - K_{aa}/\gamma)$ close to the spectral edge $E = 2 + \xi N^{-2/3}$ for $\xi = -1$. The numerical results were obtained from samples of 10000 GUE-matrices of size 2000×2000.

6. Conclusion and Outlook

Conclusion

In Chapter 3 of this thesis we proved that the distribution of the K-matrix, which is closely related to an $M \times M$ diagonal block of the resolvent $(E-H)^{-1}$ (see Eq. (1.20)), is given by the Cauchy distribution (1.31) in the limit of large, complex Hermitian random matrix H with joint probability density (3.8). It hence applies to a broad class of unitarily invariant matrix ensembles and is in this sense universal. This further proves that the two different RMT approaches to quantum chaotic scattering, which rely on either representing the underlying Hamiltonian H (Heidelberg approach, see Section 1.3.2) or the scattering matrix itself (Mexico approach, see Section 1.3.3) as a random matrix are equivalent when H belongs to an ensemble where the proof is applicable.

The main observation in the proof is that the characteristic function of the K-matrix distribution can be expressed as a very general spectral object, the ensemble averaged product of the ratios of powers of characteristic polynomials $\det(\mu - H)$ of random matrices H, see Eq. (3.7). For the case of random coupling amplitudes, this follows immediately from performing the ensemble average over those amplitudes. For fixed coupling amplitudes it follows from performing the ensemble average over the eigenvectors of H in the limit $N \to \infty$. Universality of the K-matrix distribution is then a direct consequence of the universal limit of these spectral objects, which are explicitly given by a determinantal structure (3.21). The same determinantal expression can be obtained for the characteristic function of a generic matrix-Cauchy distribution. Equality of the characteristic functions then implies the universal distribution of the K-matrix is given by the matrix-Cauchy distribution.

For the case of H belonging to an orthogonal ensemble, the characteristic function of the K-matrix distribution takes a very similar form as for the unitary case,

but with square-roots of characteristic polynomials in the denominator. This lead us to start the program of systematic evaluation of correlation functions (4.1) involving half-integer powers of the characteristic polynomials of $N \times N$ GOE matrices in Chapter 4. Motivated further by diverse other applications outlined in the introductory Section 4.1 we mainly concentrated on extracting the asymptotic behaviour of several objects of that type as $N \to \infty$. Our calculations were based on variants of the supersymmetry method or related techniques. The method in a nutshell amounts to replacing the initial average involving the product of k characteristic polynomials divided by l square roots of characteristic polynomials of $N \times N$ GOE matrices H with an average over the sets of $k \times k$ matrices Q_F and $l \times l$ matrices $Q_B > 0$ with Gaussian weights augmented essentially with the factors $\det Q_B$ and $\det Q_F$ raised to powers of order N, see e.g. (4.111). As we are eventually mostly interested in k, l fixed but $N \to \infty$ this replacement is very helpful as it allows to employ saddle-point approximations.

Although it is reasonable to expect that these correlation functions can be expressed via determinantal or Pfaffian structures, even for finite N, we were only able to show such structure for the simplest case $k, l = 1$, given in Eq. (4.21). The results for correlation functions with $k = 1, 2$ and $l = 2$ in the large-N limit are obtained as closed expressions and summarised in the Eqs. (4.45), (4.67) for $k = 1$ and Eqs. (4.131), (4.135) for $k = 2$. Validity of the formulas was tested by considering various special cases for which the answers were already known in the literature and by comparison with direct numerical simulations of GOE matrices of moderate size. This is shown in the Figures 4.1, 4.2 and 4.3. For $k = 2$, $l = 4$ only a special case was considered which is related to the probability distribution of an off-diagonal K-matrix element. Its result is given in Eq. (4.188) and compared with numerics in Figure 5.1.

In Chapter 5 we finally computed several K-matrix characteristics. Where possible we computed the results following both a Hamiltonian approach and starting from the claim that K is matrix-Cauchy distributed. For the unitary case these approaches have to yield the same result due to the proof given in Chapter 3, for the same broad class of unitarily invariant matrix ensembles. For $\beta = 1$ we had to restrict to the case of the Gaussian Orthogonal Ensemble for technical reasons. However, also here the results turn out to be identical to those obtained starting from the matrix-Cauchy distribution. While not a proof, this strongly suggests the claim that K is matrix-

Cauchy distributed if H belongs to the GOE is true, or equivalently the claim of the Poisson kernel for the S-matrix. The fact that P. Brouwer derived the same distribution assuming H itself is matrix-Cauchy distributed leads to believe that it is furthermore true for a broader class of orthogonally invariant random matrices.

The individual results of Chapter 5 are as follows: The distribution of a diagonal element of the K-matrix as well as the sum of all of them (i.e. the trace of K) are Cauchy distributed and independent of β. The correlation of two entries, however, takes a very different form for $\beta = 1$ and $\beta = 2$ and is summarised in Eqs. (5.51) and (5.56). Results on the distribution of an off-diagonal K-matrix element are given in Eq. (5.13) for $\beta = 2$ (in this case K_{ab} is complex with both real and imaginary part having the same distribution) and Eq. (5.17) for $\beta = 1$. Figure 5.1 illustrates the analytic result for orthogonal symmetry and compares it with numerical simulations. Finally we analysed how the Cauchy distribution changes to a more complicated distribution if one considers an energy close to the edge of the GUE spectrum. The result is given in Eq. (5.78).

Outlook

In this thesis we managed to perform all steps of the calculation of correlation functions (4.1) involving half-integer powers of the characteristic polynomials successfully only for relatively small values of k and l, the case of correlation functions with higher k and l remaining an outstanding problem. However, it might be possible that the general case can eventually be treated along similar lines. One reason and guiding principle for a moderate optimism is as follows. An inspection of a somewhat simpler example of $\beta = 2$ shows, see in particular [71], that the success of the method used in this thesis is deeply connected to the existence of the so-called *duality relations* for Gaussian ensembles, see [109] for a better understanding of such dualities. In particular, the Proposition 7 of the latter paper shows that one of such duality relations exists for general Gaussian β-ensembles with $\beta > 0$ for an object involving the ensemble average of the product of the corresponding characteristic polynomials raised to the power $-\beta/2$. For the GOE with $\beta = 1$ that object (see Proposition 2 in [109]) is exactly the particular case of (4.1) with $k = 0$ and arbitrary integer l which makes a contact to the present context, e.g. one can employ such a duality to reproduce the

relation (4.8) in an alternative way. A deeper understanding of connections between the supersymmetric approach and the duality relations for Gaussian ensembles will certainly be helpful in dealing efficiently with asymptotics of (4.1) for arbitrary integer values k and l. The problem of revealing possible Pfaffian-determinant structures behind (4.1) for finite matrix size N remains at the moment completely outstanding. It may well be that the methods of [78, 87] or relations to generalized hypergeometric functions noticed for some particular instances in [102] could be useful for clarifying that issue.

While partial results on the $M = 2$ case and several calculations of the K-matrix characteristics in Chapter 5 seem to support the claim of the K-matrix being Cauchy-distributed when H belongs to the GOE, a general proof for arbitrary M is outstanding. This is closely related with the open problems discussed above as a proof along the same lines as for the unitary case would require evaluation of correlation functions $\mathcal{C}_{2M,4M}$ (compare with Eq. (3.7)). Further complication is given by the fact that the computation of the characteristic function of a matrix-Cauchy distribution (which eventually would have to be shown to be equal with the characteristic function of the K-matrix) relied on integration over the eigenvectors which could be performed employing the Itzykson-Zuber-Harish-Chandra formula. In addition for the case of fixed coupling amplitudes it was necessary to employ such formula at an earlier stage, as the ensemble average over H is then performed in two steps, the first being the average over its eigenvectors. Such a formula is at present not available for the cases $\beta = 1, 4$. For these reasons it seems a proof along the same lines as for $\beta = 2$ is unfeasible. Another open problem is the $\beta = 4$ case which was not considered in this thesis. For the random amplitude model, evaluation of the characteristic function should be straightforward as it would amount to an ensemble average of ratios of characteristic polynomials with integer powers which are known to exhibit Pfaffian structures. However, the lack of an IZHC-type formula prevents to compare these structures with the characteristic function of a Cauchy distribution as was done in the $\beta = 2$ case. For the same reason, also the case of fixed amplitudes has to be treated in a different way.

While we computed most of the K-matrix characteristics in Chapter 5 for both $\beta = 1$ and $\beta = 2$, the distribution of a diagonal K-matrix element at the edge of the GOE spectrum remains an outstanding problem. The calculation for the unitary

case relied on known results for the ensemble average of the ratio of two characteristic polynomials at the spectrum edge. For the GOE one would correspondingly need to know this expression with the characteristic polynomials raised to the power 1/2 instead, which is not yet available. However, there is good reason to believe that it is possible to derive a solution with the help of the supersymmetry approach. In fact the expression one needs to calculate, in the notation of Chapter 4, is the correlation function $\mathcal{C}_{1,2}^{(\mathrm{edge})}(0; x\gamma, \epsilon)$ with $\operatorname{sgn}\epsilon = \operatorname{sgn} x$, in the limit $\epsilon \to 0$ (this is necessary to reproduce the square-rooted characteristic polynomial in the numerator). In Chapter 4 we derived an expression for the correlation function $\mathcal{C}_{1,2}$ for arbitrary spectral parameters in terms of a three-fold integral, see Eq. (4.33). We then employed the bulk-scaling and performed a saddle-point analysis for large N. In the same fashion one could try to employ the edge-scaling and proceed with a saddle-point analysis. However, so far complexity of the ensuing saddle-point structure prevented a successful calculation. The K-matrix characteristics for the symplectic case $\beta = 4$, which was not considered in Chapter 5, remain an outstanding problem. However, it should be possible to treat this case along the same lines as was done for the cases $\beta = 1, 2$.

A major simplification we assumed throughout the thesis is the absence of absorption, i.e. that there are no internal losses. This, however, is almost never given in an experimental set-up where absorption is always present to some extend. This is especially important if one wants to test the theoretical predictions, e.g. the distribution of an off-diagonal K-matrix element, with data obtained from scattering experiments. A slight modification of the scattering model presented in Section 1.3.2 allows to incorporate such absorption: In addition to the M real scattering channels one introduces M_ϕ fictitious channels, each having transmission coefficient T_ϕ, and considers the limit $M_\phi \to \infty$ and $T_\phi \to 0$ in such a way that the product $\gamma_{\mathrm{abs}} = M_\phi T_\phi$ is kept fixed [110–112]. γ_{abs} is the (dimensionless) absorption rate. From the representation of the S-matrix (1.21) one can work out that this model is equivalent to an imaginary shift of the energy $E \to E + i\Delta\gamma_{\mathrm{abs}}/(4\pi)$, Δ being the mean level spacing of the closed system [111]. Choosing γ_{abs} negative, this model also accounts for the case of amplification, e.g. in a laser cavity [97]. It is easy to see that this modification renders S non-unitary (and K non-Hermitian) such that its distribution will no longer be given by the Poisson kernel. This also means K will no longer be Cauchy-distributed.

On the other hand it is – at least formally – easy to incorporate the changes due to absorption into the Hamiltonian approach and such prepared one might be able to compute the K-matrix characteristics of Chapter 5 for arbitrary absorption (or amplification) rate. This can actually done in the two above described ways, either by shifting E into the complex plane or by considering M_ϕ fictitious channels and taking the appropriate limit. E.g. for the distribution of an off-diagonal K-matrix element it should be possible to follow an approach along the lines of [11] where the distribution of an off-diagonal S-matrix element was expressed as a supersymmetric model for arbitrary coupling amplitudes and arbitrary M, N. This makes it possible to add absorbtion into the model at this point by shifting $M \to M + M_\phi$, taking the above described limit and then proceeding with the large-N asymptotics.

A. Appendix

A.1. Deriving $\mathcal{P}(K)$ from the Poisson Kernel

In this appendix we want to show that the Poisson kernel distribution $\mathcal{P}(S)$ for the scattering matrix S implies that the K-matrix is distributed according to a Cauchy distribution. For brevity we restrict here to the case $\beta = 2$, however, similar calculations can be performed for $\beta = 1, 4$ without further complications.

We start by calculating the integration measure induced by the transformation $S = (\mathbb{1}_M - iK)(\mathbb{1}_M + iK)^{-1}$. The first step is to diagonalise $K = UkU^\dagger$ where k is a diagonal matrix and U is unitary since K is Hermitian, and hence $S = U(\mathbb{1}_M - ik)(\mathbb{1}_M + ik)^{-1}U^\dagger$. The measure $\mathrm{d}S$ is then given by

$$\mathrm{d}S = \mathrm{d}U\frac{\mathbb{1}_M - ik}{\mathbb{1}_M + ik}U^{-1} + U\frac{2i}{-i + k^2}\mathrm{d}k\,U^\dagger - U\frac{\mathbb{1}_M - ik}{\mathbb{1}_M + ik}U^\dagger \mathrm{d}U U^\dagger. \tag{A.1}$$

In the last term we used $\mathrm{d}(U^\dagger) = U^\dagger \mathrm{d}U U^\dagger$ which can be verified by taking the differential of the identity $U^\dagger U = \mathbb{1}_M$ which yields $\mathrm{d}(U^\dagger)U + U^\dagger \mathrm{d}U = 0$. Next we introduce $\delta U = U^\dagger \mathrm{d}U$ and abbreviate $\frac{\mathbb{1}_M - ik}{\mathbb{1}_M + ik} = a$, $\frac{2i}{-i+k^2} = b$. Then

$$\mathrm{d}S = U(\delta U\, a + b\,\mathrm{d}k - a\,\delta U)U^\dagger. \tag{A.2}$$

The length element is accordingly given by

$$\begin{aligned}(\mathrm{d}s)^2 = \mathrm{Tr}(\mathrm{d}S\mathrm{d}S^\dagger) = \mathrm{Tr}[&b\,\mathrm{d}k\,b^*\mathrm{d}k + 2aa^*\delta U^2 - 2a\,\delta U\,a^*\delta U \\ &+ (\delta U a - a\,\delta U)b^*\mathrm{d}k + (a^*\delta U - \delta U a^*)b\,\mathrm{d}k],\end{aligned} \tag{A.3}$$

where we used that $\delta U^\dagger = (U^\dagger \mathrm{d}U)^\dagger = \mathrm{d}U^\dagger U = -\delta U$. Now notice that a and a^* are diagonal, and hence the diagonal entries of the commutators $(\delta U a - a\,\delta U)$ and

$(a^*\delta U - \delta U a^*)$ are zero. This means the last two terms in Eq. (A.3) vanish. On the other hand, two of the remaining terms can be combined to

$$\begin{aligned}\mathrm{Tr}(2aa^*\delta U^2 - 2a\,\delta U\,a^*\delta U) &= 2\sum_{mn}(a_m^* a_m - a_m^* a_n)\delta U_{mn}\delta U_{nm} \\ &= \sum_{mn}(a_m - a_n)(a_m^* - a_n^*)\delta U_{mn}\delta U_{nm}, \end{aligned} \tag{A.4}$$

and thus the length element simplifies to

$$(\mathrm{d}s)^2 = \sum_m |b_m|^2(\mathrm{d}k_m)^2 - \sum_{m<n}|a_m - a_n|^2\delta U_{mn}^*\delta U_{mn}, \tag{A.5}$$

where we used again the Anti-Hermicity $\delta U_{nm} = -\delta U_{mn}^*$. This length element defines a Riemannian metric g_{mn} via $(\mathrm{d}s)^2 = \sum_{mn}^M g_{mn}\mathrm{d}q_m\mathrm{d}q_n$, and this metric on the other hand induces the corresponding integration measure with the volume element given by $\mathrm{d}\mu = \sqrt{\det(g_{mn})_{m,n=1}^M}\mathrm{d}q_1\ldots\mathrm{d}q_M$. Hence in our case the measure is given by

$$\mathrm{d}\mu(S(K)) = \prod_{m=1}^M |b_m|\mathrm{d}k_m \prod_{m<n}|a_m - a_n|^2\mathrm{d}\mu(U), \tag{A.6}$$

where $\mathrm{d}\mu(U)$ is the part of the measure which depends only on the U variables (which in fact turns out to be the invariant Haar measure on the group $U(N)$). The factors are given by

$$|b_m|^2 = \left|\frac{2i}{(-i+k_m)^2}\right|^2 = \frac{4}{(1+k_m^2)^2}, \tag{A.7}$$

$$|a_m - a_n|^2 = \left|\frac{1-ik_m}{1+ik_m} - \frac{1-ik_n}{1+ik_n}\right| = \frac{4(k_m-k_n)^2}{(1+k_m^2)(1+k_n^2)}. \tag{A.8}$$

The second line implies

$$\begin{aligned}\prod_{m<n}\frac{4(k_m-k_n)^2}{(1+k_m^2)(1+k_n^2)} &= 2^{M(M-1)}\Delta^2\{k\}\prod_{m<n}\frac{1}{(1+k_m^2)(1+k_n^2)} \\ &= 2^{M(M-1)}\Delta^2\{k\}\prod_{m=1}^M\frac{1}{(1+k_m^2)^{M-1}}, \end{aligned} \tag{A.9}$$

where $\Delta\{k\} = \prod_{m<n}(k_m - k_n)$ is the Vandermonde determinant. The measure is hence given by

$$\mathrm{d}\mu(S(K)) = \frac{2^{M^2}}{\det(\mathbb{1}_M + k^2)^M}\Delta^2(k)\,\mathrm{d}k\,\mathrm{d}\mu(U) = \frac{2^{M^2}}{\det(\mathbb{1}_M + K^2)^M}\mathrm{d}K. \tag{A.10}$$

In the last step we used the well known fact [15] that the induced measure when diagonalising $K = UkU^\dagger$ is given by $\Delta^2(k)\,\mathrm{d}k\,\mathrm{d}\mu(U)$. This can easily be verified by mimicking the above calculation. In fact the only difference is that now $a = k$ and $b = \mathbb{1}_M$, and together with (A.6) this yields the correct measure.

The probability distribution of K is now given by

$$\begin{aligned}\mathcal{P}(K)\mathrm{d}K &= \mathcal{P}(S(K))\mathrm{d}\mu(S(K)) \\ &\propto \left|\det\left(\mathbb{1}_M - \bar{S}^\dagger\frac{\mathbb{1}_M - iK}{\mathbb{1}_M + iK}\right)\right|^{-2M}\det(\mathbb{1}_M + K^2)^{-M}\mathrm{d}K.\end{aligned} \tag{A.11}$$

For the special case $\bar{S}^\dagger = 0$ (perfect coupling) it follows immediately that K is Cauchy distributed, $\mathcal{P}(K) \propto \det(\mathbb{1}_M + K^2)^{-M}$. Now let us assume $\bar{S}^\dagger \neq 0$ and for simplicity we furthermore just consider the case of equivalent channels, hence $\bar{S}^\dagger = R^*\,\mathbb{1}_M$, where the complex parameter R characterises the channel reflection ($|R|^2$ is the reflection coefficient). Note that $\mathcal{P}(K)$ given in (A.11) remains independent under unitary transformation, hence we can replace K by the diagonal matrix k containing its eigenvalues. Then the first determinant simplifies to

$$\begin{aligned}&\left[\det\left(\mathbb{1}_M - R^*\frac{\mathbb{1}_M - ik}{\mathbb{1}_M + ik}\right)\det\left(\mathbb{1}_M - R\frac{\mathbb{1}_M + ik}{\mathbb{1}_M - ik}\right)\right]^{-M} \\ &= \det\left[\mathbb{1}_M + |R|^2\mathbb{1}_M - R^*\frac{(\mathbb{1}_M - ik)^2}{\mathbb{1}_M + k^2} - R\frac{(\mathbb{1}_M + ik)^2}{\mathbb{1}_M + k^2}\right]^{-M}.\end{aligned} \tag{A.12}$$

Multiplying with the second determinant yields

$$\mathcal{P}(k) \propto \det\left[(1 + |R|^2)(\mathbb{1}_M + k^2) - R^*(\mathbb{1}_M - ik)^2 - R(\mathbb{1}_M + ik)^2\right]^{-M}, \tag{A.13}$$

which, after rearranging terms, can be written as a shifted Cauchy distribution,

$$\mathcal{P}(k) \propto \det\left[\left|\tfrac{R-1}{R+1}\right|^2 - \left(\tfrac{2\,\mathrm{Im}\,R}{|R+1|^2}\right)^2 + \left(k + \tfrac{2\,\mathrm{Im}\,R}{|R+1|^2}\right)\right]^{-M}. \tag{A.14}$$

This proves for the unitary case that the Poisson kernel distribution $\mathcal{P}(S)$ implies that K is Cauchy distributed,

$$\mathcal{P}(K) \propto \det[\lambda^2 + (K-\epsilon)^2]^{-M}, \tag{A.15}$$

where the dependence of λ and ϵ on the reflection coefficient can be read off from Eq. (A.14). This implies on the other hand that the reflection coeffient can be expressed as

$$R = \frac{1-\lambda-i\epsilon}{1+\lambda+i\epsilon}. \tag{A.16}$$

Notice that this is in agreement with the solution obtained in [8] for the Gaussian case using the Hamiltonian approach, Eq. (1.23), if one chooses $\lambda = \gamma\sqrt{4J^2-E^2}/(2J^2)$ and $\epsilon = \gamma E/(2J^2)$. In Chapter 3 we show that this can be generalised to $\lambda = \pi\gamma\rho(E)$ and $\epsilon = \gamma V'(E)/2$ for joint probability density $\mathcal{P}(H) \propto \exp(-NV(H))$.

A.2. Calculation of Formula (3.7)

For $\beta = 2$, the equality

$$\det\left(\mathbb{1}_N + \frac{i\gamma x_c}{N}(E-H)^{-1}\right)^{-1} = \frac{\det(E-H)}{\det(E+i\gamma x_c/N - H)} \tag{A.17}$$

is evident, as one just needs to multiply both numerator and denominator by $\det(E-H)$. For $\beta = 1$, however, the formula (3.7) features also a dependence on the sign of the determinant. In this appendix we discuss how this factor comes about.

We can define the (principal) square-root of a complex number $z = re^{i\phi}$ via

$$\sqrt{z} = \sqrt{r}e^{i\phi/2}, \quad -\pi < \phi \leq \pi. \tag{A.18}$$

Note that this definition introduces a branch cut at the non-positive real axis, the

square-root is discontinuous on this cut. The product of two square-roots is then given by

$$\sqrt{z_1}\sqrt{z_2} = \sqrt{r_1 r_2} e^{\frac{i}{2}(\phi_1+\phi_2)}, \quad -2\pi < \phi_1 + \phi_2 \leq 2\pi. \tag{A.19}$$

However, for the square-root of a product we might get an additional phase shift,

$$\sqrt{z_1 z_2} = \sqrt{r_1 r_2} e^{\frac{i}{2}(\phi_1+\phi_2+2\pi n)} = \sqrt{z_1}\sqrt{z_2} e^{i\pi n}, \tag{A.20}$$

where n has to be chosen such that $\phi_1 + \phi_2$ remains in the interval $(-\pi, \pi]$, according to the the definition (A.18), viz,

$$n = \begin{cases} 0 & \text{if} \quad -\pi < \phi_1 + \phi_2 \leq \pi, \\ +1 & \text{if} \quad -2\pi < \phi_1 + \phi_2 \leq -\pi, \\ -1 & \text{if} \quad \pi < \phi_1 + \phi_2 \leq 2\pi. \end{cases} \tag{A.21}$$

This implies we have the rule $\sqrt{z_1 z_2} = \sqrt{z_1}\sqrt{z_2}$ only for $|\phi_1 + \phi_2| \leq \pi$, whereas for the case $\pi < |\phi_1 + \phi_2| \leq 2\pi$ we get an additional minus sign, $\sqrt{z_1 z_2} = -\sqrt{z_1}\sqrt{z_2}$.

When performing the W integration in (3.6), the integrals decouple into $N \cdot M$ Gaussian integrals for each element of W, yielding

$$\prod_{c=1}^{M}\prod_{n=1}^{N}\left(1 + \frac{i\gamma x_c}{N(E-\lambda_n)}\right)^{-\frac{1}{2}} = \prod_{c=1}^{M} \det\left[\mathbb{1}_N + \frac{i\gamma x_c}{N}(E-H)^{-1}\right]^{-1/2}, \tag{A.22}$$

where $\lambda_1 \ldots \lambda_N$ are the eigenvalues of H. Thus here and henceforth, the notation $\det A^{1/2}$ has to be understood as $\det(A^{1/2}) = \prod_n \sqrt{a_n}$, rather than $(\det A)^{1/2} = \sqrt{\prod_n a_n}$. If the a_n are complex, the two expressions are not necessarily the same as seen above.

Let us look at a single factor of (A.22),

$$\left(1 + \frac{i\tilde{x}_c}{E-\lambda_n}\right)^{-1/2} = \sqrt{\frac{E-\lambda_n}{E-\lambda_n+i\tilde{x}_c}}, \tag{A.23}$$

where for brevity we introduced $\tilde{x}_c = \gamma x_c/N$. Since the argument of this square root is complex, we have to investigate the expression $|\arg z_1 + \arg z_2|$, with $z_1 = E - \lambda_n$ and $z_2 = (E - \lambda_n + i\tilde{x}_c)^{-1}$. Only z_2 is complex, whereas z_1 is real. Let us suppose

$z_1 \geq 0$. Then its argument is 0, and $|\arg z_1 + \arg z_2| = |\arg z_2| \leq \pi$, and thus $\sqrt{z_1 z_2} = \sqrt{z_1}\sqrt{z_2}$. For $z_1 < 0$, on the other hand, its argument is π and it is a priori not clear if $|\arg z_1 + \arg z_2|$ is bigger or smaller than π. This, in fact, depends on the sign of the imaginary part of z_2, i.e. the sign of of the eigenvalue x_c of the matrix X. If its sign is positive, z_2 lies in the lower half-plane and hence its argument lies between $-\pi$ and 0 and consequently $|\arg z_1 + \arg z_2| \leq \pi$. For negative sign, however, z_1 is in the upper half-plane with argument between 0 and π, and $|\arg z_1 + \arg z_2| \geq \pi$. Hence in this case, and this case only, we get $\sqrt{z_1 z_2} = -\sqrt{z_1}\sqrt{z_2}$. Combining this into a formula one gets,

$$\sqrt{\frac{E-\lambda_n}{E-\lambda_n+i\tilde{x}_c}} = \frac{\sqrt{E-\lambda_n}}{\sqrt{E-\lambda_n+i\tilde{x}_c}} \times \begin{cases} 1 & \text{for } x_c \geq 0, \\ \operatorname{sgn}(E-\lambda_n) & \text{for } x_c < 0. \end{cases} \tag{A.24}$$

Performing the product over n in the above formula then yields the result

$$\prod_{n=1}^{N} \sqrt{\frac{E-\lambda_n}{E-\lambda_n+i\tilde{x}_c}} = \frac{\det(E-H)^{1/2}}{\det(E-H+i\tilde{x}_c)^{1/2}} \times \begin{cases} 1 & \text{for } x_c \geq 0, \\ \operatorname{sgn}\det(E-H) & \text{for } x_c < 0. \end{cases} \tag{A.25}$$

This shows that for $\beta = 1$, in the presence of negative eigenvalues of X, one has to take into account extra $\operatorname{sgn}\det$ factors.

A.3. Proof of Eq. (3.39)

We want to prove that the m-th derivative of the function $g_M(x)$ defined in Eq. (3.33) is given by

$$g_M^{(m)}(x) = \lambda^m \sum_{l=0}^{\lfloor m/2 \rfloor} \frac{m!(-1)^{m-l}}{l!(m-2l)!2^l} (\lambda x)^{m-2l} g_{M-m+l}(x), \tag{A.26}$$

where $\lfloor \cdot \rfloor$ denotes the floor-function. We prove that equation by induction. It was already shown in the main body that the first derivative is given by $g'_M(x) = -\lambda^2 x g_{M-1}(x)$ hence verifying the $m = 1$ case (note that the case $m = 0$ is fulfilled trivially as well). Now assuming that the relation is true for the first m derivatives,

the $m+1$ derivative is given by

$$\begin{aligned} g_M^{(m+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} g_M^{(m)}(x) = \lambda^m \frac{\mathrm{d}}{\mathrm{d}x} \sum_{l=0}^{\lfloor m/2 \rfloor} \frac{m!(-1)^{m-l}}{l!(m-2l)!2^l} (\lambda x)^{m-2l} g_{M-m+l}(x) \\ &= \sum_{l=0}^{\lfloor m/2 \rfloor} a_{m,l} \lambda^{2m-2l} x^{m-2l-1} \left[(m-2l) g_{M-m+l}(x) - (\lambda x)^2 g_{M-m+l-1}(x) \right], \end{aligned} \tag{A.27}$$

where we abbreviated the prefactors with $a_{m,l}$ and used the formula for $g'_M(x)$. Next we change $l \to l-1$ in the first term which gives

$$\begin{aligned} &\sum_{l=1}^{\lfloor m/2 \rfloor + 1} a_{m,l-1} \lambda^{2m-2l+2} x^{m-2l+1} (m-2l+2) g_{M-m+l-1}(x) \\ &- \sum_{l=0}^{\lfloor m/2 \rfloor} a_{m,l} \lambda^{2m-2l+2} x^{m-2l+1} g_{M-m+l-1}(x). \end{aligned} \tag{A.28}$$

Next we split the $l = \lfloor m/2 \rfloor + 1$ term from the first sum, the $l = 0$ term from the second sum and combine both sums, which leaves us with

$$\begin{aligned} &\lambda^{m+1} \Bigg\{ \sum_{l=1}^{\lfloor m/2 \rfloor} (\lambda x)^{m-2l+1} g_{M-m+l-1}(x) [a_{m,l-1}(m-2l+2) - a_{m,l}] \\ &+ a_{m,\lfloor m/2 \rfloor} (\lambda x)^{m-2\lfloor m/2 \rfloor - 1} (m - 2\lfloor m/2 \rfloor) g_{M-m+\lfloor m/2 \rfloor}(x) \\ &- a_{m,0} (\lambda x)^{m+1} g_{M-(m+1)}(x) \Bigg\}. \end{aligned} \tag{A.29}$$

Now the term in square brackets in the first line is given by

$$\frac{m!(-1)^{m-l-1}}{(l-1)!(m-2l+1)!2^{l-1}} - \frac{m!(-1)^{m-l}}{l!(m-2l)!2^l} = \frac{(m+1)!(-1)^{m+1-l}}{l!2^l(m+1-2l)!} = a_{m+1,l}. \tag{A.30}$$

Furthermore in the third line we can replace $-a_{m,0} = a_{m+1,0}$ which simply follows from the definition of $a_{m,l} = \frac{m!(-1)^{m-l}}{l!(m-2l)!2^l}$. If m is even, the second line vanishes since then $(m - 2\lfloor m/2 \rfloor) = 0$. For odd m on the other hand we have $\lfloor m/2 \rfloor = (m-1)/2$ and the second line simplifies to $a_{m,\frac{m-1}{2}} g_{M-\frac{m+1}{2}}(x)$. Now in this expression we may

replace $a_{m,\frac{m-1}{2}} = a_{m+1,\frac{m+1}{2}}$, because

$$a_{m,\frac{m-1}{2}} = \frac{m!(-1)^{(m+1)/2}}{(\frac{m-1}{2})!2^{(m-1)/2}} = \frac{2(m+1)!(-1)^{(m+1)/2}}{(m+1)(\frac{m-1}{2})!2^{(m+1)/2}} = \frac{(m+1)!(-1)^{(m+1)/2}}{(\frac{m+1}{2})!2^{(m+1)/2}}, \quad \text{(A.31)}$$

where we used $(n+1)(\frac{n-1}{2})! = 2(\frac{n+1}{2})!$ for odd n. Combining all these results we thus get

$$g_M^{(m+1)}(x) = \lambda^{m+1} \sum_{l=0}^{\lfloor (m+1)/2 \rfloor} a_{m+1,l}(\lambda x)^{m+1-2l} g_{M-(m+1)+l}(x), \quad \text{(A.32)}$$

and by induction Eq. (3.39) is true.

A.4. Parametrisation of $\mathcal{Q}$ and its Jacobian

In Section 4.4.2 the supermatrix $\mathcal{Q} = -iT_0^{-1}LT_0$ was introduced, with the metric $L = \mathrm{diag}(+\mathbb{1}_3, k)$ in pq-notation (see Section 2.2.2) and $k = \mathrm{diag}(-1, +1, +1)$. T_0 is completely defined by symmetry and convergence requirements as described in Section 4.4.2, especially it obeys the relation $T_0^\dagger L T_0 = L$. In this appendix we want to give a proper parametrisation of $\mathcal{Q}$ and calculate the Jacobian (or Berezinian) of this parametrisation. In large parts we follow the steps suggested in the paper [8] of Verbaarschot, Weidenmüller and Zirnbauer.

A.4.1. Parametrisation

A standard parametrisation of the matrix T_0 is given by [8]

$$T_0 = \begin{bmatrix} \sqrt{\mathbb{1}_3 + t_{12}t_{21}} & it_{12} \\ -it_{21} & \sqrt{\mathbb{1}_3 + t_{21}t_{12}} \end{bmatrix}, \quad \text{(A.33)}$$

where in our case t_{12} is explicitly given by

$$t_{12} = \begin{bmatrix} a & i\eta_1 & i\eta_1^* \\ -\eta_2^* & iz & iw^* \\ \eta_2 & -iw & iz^* \end{bmatrix}. \quad \text{(A.34)}$$

t_{21} can be obtained from t_{12} via the relation $t_{12}^\dagger = -kt_{21}$. Here a is a real commuting variable, z and w are complex commuting variables and η_1, η_2 are complex Grassmann variables. Thus the number of independent variables in T_0 (and hence also in $\mathcal{Q}$) is five commuting and four anticommuting variables as required (see the paragraph after Eq. (4.147)). Moreover it is easy to check that such parametrisation obeys the required rules. However, this parametrisation is not very practical for further calculations and hence we seek a different parametrisation for the supermatrices t_{12} and t_{21}. In the paper of Verbaarschot, Weidenmüller and Zirnbauer, a similar parametrisation was derived for the case of 4×4 supermatrices t_{12} and t_{21} of similar type as in our case (deleting the second row and second column from $t_{12,\mathrm{VWZ}}$ yields our t_{12} in (A.34)), and $L_{\mathrm{VWZ}} = \mathrm{diag}(+\mathbb{1}_4, -\mathbb{1}_4)$.

The first step is to quasi-diagonalise those two matrices via $t_{12} = u_1^{-1} M_{12} u_2$, $t_{21} = u_2^{-1} M_{21} u_1$. The dependence on the Grassmann variables is completely shifted into the matrices u_1 and u_2 and the quasi-diagonal matrices M_{12} and M_{21} are given by

$$M_{12} = \mathrm{diag}(\mu_1, i\mu_0 U), \quad M_{21} = \mathrm{diag}(\mu_1, i\mu_0 U^\dagger), \tag{A.35}$$

where $\mu_1 > 0$, $\mu_0 \in (0,1)$ are real variables and U is a member of the special unitary group $SU(2)$ which can be explicitly parametrised as [8]

$$U = \frac{1}{\sqrt{1+m^2+r^2+s^2}} \begin{bmatrix} 1+im & -(r+is) \\ r-is & 1-im \end{bmatrix}, \tag{A.36}$$

$m, r, s \in \mathbb{R}$. This is in analogy to the 4×4 case from [8], where the boson-boson block being 2×2 made it necessary to introduce another variable μ_2 such that $M_{12,\mathrm{VWZ}} = \mathrm{diag}(\mu_1, \mu_2, i\mu_0 U)$. Substituting for t_{12} and t_{21} in the upper left block of T_0 one gets

$$\begin{aligned} \sqrt{\mathbb{1}_3 + t_{12}t_{21}} &= \sqrt{\mathbb{1}_3 + u_1^{-1} M_{12} M_{21} u_1} = u_1^{-1} \sqrt{\mathbb{1}_3 + M_{12} M_{21}} u_1 \\ &= u_1^{-1} \sqrt{\mathbb{1}_3 + \mathrm{diag}(\mu_1^2, -\mu_0^2 U U^\dagger)} u_1 = u_1^{-1} M_1 u_1, \end{aligned} \tag{A.37}$$

where in the last step we used $UU^\dagger = \mathbb{1}_3$ and defined

$$M_1 = \mathrm{diag}\left(\sqrt{1+\mu_1^2}, \sqrt{1-\mu_0^2}, \sqrt{1-\mu_0^2} \right). \tag{A.38}$$

Analogously one can show for the lower right block of T_0 that $\sqrt{\mathbb{1}_3 + t_{21}t_{12}} = u_2^{-1}M_1u_2$. Hence with Eq. (A.33) one gets

$$T_0 = \begin{bmatrix} u_1^{-1} & 0 \\ 0 & u_2^{-1} \end{bmatrix} \begin{bmatrix} M_1 & iM_{12} \\ -iM_{21} & M_1 \end{bmatrix} \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \equiv \hat{U}^{-1}M\hat{U}, \tag{A.39}$$

where we defined the 6×6 matrices M and $\hat{U}$. The inverses of u_1 and u_2 can be related to their complex conjugates. To that end we look at $t_{12}^\dagger = u_2^\dagger M_{12}^\dagger (u^{-1})^\dagger = -u_2^\dagger k M_{21}(u^{-1})^\dagger$, where in the last step we used that $M_{12}^\dagger = -kM_{21}$ which can be seen from the definition (A.35). On the other hand this expression has to be equal to $-kt_{21} = -ku_2^{-1}M_{21}u_1$. Comparing the two expressions yields $(u^{-1})^\dagger = u_1$ and $u_2^\dagger k = ku_2^{-1}$, which implies that u_1 is unitary, $u_1^{-1} = u_1^\dagger$, and u_2 obeys $u_2^{-1} = ku_2^\dagger k$.

The inverse of T_0 can now easily be computed via its conjugate transpose, $T_0^{-1} = LT_0^\dagger L$, which follows from rearranging the group property $T_0^\dagger L T_0 = L$. Using further $t_{12}^\dagger = kt_{21}$ one gets

$$T_0^{-1} = \begin{bmatrix} \sqrt{\mathbb{1}_3 + t_{12}t_{21}} & -it_{12} \\ it_{21} & \sqrt{\mathbb{1}_3 + t_{21}t_{12}} \end{bmatrix} = \hat{U}^{-1} \begin{bmatrix} M_1 & -iM_{12} \\ iM_{21} & M_1 \end{bmatrix} \hat{U} \equiv \hat{U}^{-1}M^{-1}\hat{U}. \tag{A.40}$$

This enables us to express $\mathcal{Q} = -iT_0^{-1}LT_0 = -i\hat{U}^{-1}M^{-1}\hat{U}L\hat{U}^{-1}MU$, and using that U and L commute this simplifies to $\mathcal{Q} = -i\hat{U}^{-1}M^{-1}LM\hat{U}$. In terms of the blocks of M this becomes (we use that the blocks commute with each other)

$$\mathcal{Q} = \hat{U}^{-1} \begin{bmatrix} -i(M_1^2 + M_{12}M_{21}) & 2M_1M_{12} \\ 2M_1M_{21} & i(M_1^2 + M_{12}M_{21}) \end{bmatrix} \hat{U}, \tag{A.41a}$$

with the blocks explicitly given by

$$\mp i(M_1^2 + M_{12}M_{21}) = \mp i\,\mathrm{diag}(1 + 2\mu_1^2, (1 - 2\mu_0^2)\mathbb{1}_2), \tag{A.41b}$$

$$2M_1M_{12} = \mathrm{diag}\left(2\mu_1\sqrt{1+\mu_1^2}, 2i\mu_0\sqrt{1-\mu_0^2}U\right), \tag{A.41c}$$

$$2M_1M_{21} = \mathrm{diag}\left(2\mu_1\sqrt{1+\mu_1^2}, 2i\mu_0\sqrt{1-\mu_0^2}U^\dagger\right). \tag{A.41d}$$

The form of these blocks suggests to introduce new variables $\lambda_1 = 1+2\mu_1^2 \in (1,\infty)$ and $\lambda_0 = 1-2\mu_0^2 \in (-1,1)$, such that the diagonal blocks of $\mathcal{Q}$ become $\mp i\,\mathrm{diag}(\lambda_1,\lambda_0,\lambda_0)$ and the off-diagonal blocks of $\mathcal{Q}$ become accordingly $\mathrm{diag}(\sqrt{\lambda_1^2-1}, i\sqrt{1-\lambda_0^2}\,U)$ and $\mathrm{diag}(\sqrt{\lambda_1^2-1}, i\sqrt{1-\lambda_0^2}\,U^\dagger)$, respectively.[38]

The remaining question is how to parametrise the matrix $\hat{U} = \mathrm{diag}(u_1,u_2)$ which contains the anticommuting variables. In [8] it was shown that for the case where u_1 and u_2 are 4×4 matrices, this can be done via $u_{p,\mathrm{VWZ}} = O_p v_p$, $p=1,2$, where O_p is in $O(2)$ and v_p and its inverse can be expressed as $v_p^{\pm 1} = \mathbb{1}_4 \pm i^{p-1}Y_p + \frac{1}{2}i^{2(p-1)}Y_p^2 \pm \frac{1}{2}i^{3(p-1)}Y_p^3 + \frac{3}{8}Y_p^4$, where $Y_p = \begin{bmatrix} 0 & -\xi_p^\dagger \\ \xi_p & 0 \end{bmatrix}$ and $\xi_{p,\mathrm{VWZ}} = \begin{bmatrix} \alpha_p & \beta_p \\ \alpha_p^* & \beta_p^* \end{bmatrix}$. The orthogonal matrices O_1 and O_2 stem from diagonalisation of the boson-boson block of the (real symmetric) 2×2 matrices $t_{12}t_{21}$ and $t_{21}t_{12}$. In our case, however, the corresponding blocks are just scalar, and hence we do not need to introduce any orthogonal matrices. Furthermore the matrices ξ_p will reduce in our case to vectors $\xi_p = [\alpha_p, \alpha_p^*]^T$. This furthermore implies that Y_p only comprises two independent Grassmann variables, and hence Y_p^3 and Y_p^4 featured in the formula for v_p vanish.

Summarising the results from this section, a parametrisation of $\mathcal{Q}$ is given by

$$\mathcal{Q} = \hat{U}^{-1}\begin{bmatrix} -iD_1 & D_{12} \\ D_{21} & iD_1 \end{bmatrix}\hat{U}, \quad \hat{U} = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}, \tag{A.42a}$$

$$D_1 = \mathrm{diag}(\lambda_1,\lambda_0,\lambda_0), \tag{A.42b}$$

$$D_{12} = \mathrm{diag}\left(\sqrt{\lambda_1^2-1}, i\sqrt{1-\lambda_0^2}\,U\right), \tag{A.42c}$$

$$D_{21} = \mathrm{diag}\left(\sqrt{\lambda_1^2-1}, i\sqrt{1-\lambda_0^2}\,U^\dagger\right), \tag{A.42d}$$

$$U = \frac{1}{\sqrt{1+m^2+r^2+s^2}}\begin{bmatrix} 1+im & -(r+is) \\ r-is & 1-im \end{bmatrix}, \tag{A.42e}$$

[38] Notice that we use different variables as Verbaarschot et al. where $\lambda_{p,\mathrm{VWZ}} = \mu_p^2$, $p=0,1,2$.

$$u_p^{\pm 1} = \mathbb{1}_3 \pm i^{p-1} Y_p + \frac{1}{2} i^{2(p-1)} Y_p^2, \quad Y_p = \begin{bmatrix} 0 & -\alpha_p^* & \alpha_p \\ \alpha_p & 0 & 0 \\ \alpha_p^* & 0 & 0 \end{bmatrix}, \quad p = 1, 2, \tag{A.42f}$$

where $\lambda_1 \in (1, \infty)$, $\lambda_0 \in (-1, 1)$, $m, r, s \in \mathbb{R}$ are commuting variables and α_1, α_1^*, α_2 and α_2^* are Grassmann variables.

A.4.2. Jacobian

To calculate the Jacobian (or Berezinian) of the parametrisation (A.42), we follow closely the route described in Appendix K of [8]. The authors have shown that the parametrisation of T_0 in terms of t_{12} and t_{21} in (A.33) induces the measure

$$\mathrm{d}\mu = |\mathrm{sdet}\,[((\mathrm{d}T_0)T_0^{-1})_{12}/\mathrm{d}t_{12}]|\mathrm{d}t_{12}. \tag{A.43}$$

However, for computational convenience they introduce new supermatrices τ_{12} and τ_{21} via $t_{12} = 2(1 - \tau_{12}\tau_{21})^{-1}\tau_{12}$ and $t_{21} = 2(1 - \tau_{21}\tau_{12})^{-1}\tau_{21}$ which implies

$$\mathrm{d}\mu = |\mathrm{sdet}\,[((\mathrm{d}T_0)T_0^{-1})_{12}/\mathrm{d}\tau_{12}]||\mathrm{sdet}\,[\mathrm{d}\tau_{12}/\mathrm{d}t_{12}]|\mathrm{d}t_{12}. \tag{A.44}$$

Furthermore the authors show that the first term can be expressed by the common "eigenvalues" (modulo the special unitary matrix U) of τ_{12} and τ_{21}, which are combined in the matrix $\theta_{\mathrm{VWZ}} = \mathrm{diag}(\theta_1, \theta_2, i\theta_0, i\theta_0)$ as $|\mathrm{sdet}\,[\mathrm{d}X/\mathrm{d}\tilde{t}_{12}]|$, where

$$X = 2i(1 - \theta^2)^{-1}\tilde{t}_{12}(1 - \theta^2)^{-1} - 2i(1 - \theta^2)^{-1}\theta\tilde{t}_{21}(1 - \theta^2)^{-1}\theta, \tag{A.45}$$

with $\tilde{t}_{12}$ and $\tilde{t}_{21}$ being two supermatrices having same structure as t_{12} and t_{21}, respectively. We may compute this term in our case in the exact same fashion, but with $\theta = \mathrm{diag}(\theta_1, i\theta_0, i\theta_0)$ instead. $\tilde{t}_{12}$ has the same structure as t_{12} given in Eq. (A.34), and $\tilde{t}_{21}$ follows from the relation $\tilde{t}_{12}^{\dagger} = -k\tilde{t}_{21}$ with $k = \mathrm{diag}(-1, +1, +1)$ (see previous

section). X is then explicitly given as

$$X = 2\begin{bmatrix} \frac{ia}{1-\theta_1^2} & -\frac{\eta_1-\eta_2\theta_0\theta_1}{(1+\theta_0^2)(1-\theta_1^2)} & -\frac{\eta_1^*-\eta_2^*\theta_0\theta_1}{(1+\theta_0^2)(1-\theta_1^2)} \\ -i\frac{\eta_2^*+\eta_1^*\theta_0\theta_1}{(1+\theta_0^2)(1-\theta_1^2)} & -\frac{z+z^*\theta_0^2}{(1+\theta_0^2)^2} & -\frac{w^*(1-\theta_0^2)}{(1+\theta_0^2)^2} \\ i\frac{\eta_2+\eta_1\theta_0\theta_1}{(1+\theta_0^2)(1-\theta_1^2)} & \frac{w(1-\theta_0^2)}{(1+\theta_0^2)^2} & -\frac{z^*+z\theta_0^2}{(1+\theta_0^2)^2} \end{bmatrix}, \tag{A.46}$$

where a, z, z^*, w, w^* η_1, η_1^*, η_2 and η_2^* are the entries of $\tilde{t}_{12}$ as in (A.34). We now need to construct the Jacobian supermatrix of X with respect to all these variables, $\mathrm{d}X/\mathrm{d}\tilde{t}_{12}$, and then take its superdeterminant. However, note that both the boson-boson block and the fermion-fermion block of X are composed of only ordinary variables (i.e. it does not contain any commuting terms composed of Grassmann variables like $\eta_1^*\eta_1$ etc.), whereas the boson-fermion and fermion boson block do not depend on any ordinary variable of $\tilde{t}_{12}$. This suggests that the off-diagonal blocks in $\mathrm{d}X/\mathrm{d}\tilde{t}_{12}$ are zero and hence $\operatorname{sdet}\mathrm{d}X/\mathrm{d}\tilde{t}_{12} = \frac{\det(\mathrm{d}X/\mathrm{d}\tilde{t}_{12})_{BB}}{\det(\mathrm{d}X/\mathrm{d}\tilde{t}_{12})_{FF}}$. Due to the structure of X calculating the determinants of the boson-boson and fermion-fermion blocks of $\mathrm{d}X/\mathrm{d}\tilde{t}_{12}$ is particularly easy as they are given by

$$|\det(\mathrm{d}X/\mathrm{d}\tilde{t}_{12})_{BB}| = \left|\frac{\partial X_{11}}{\partial a}\frac{\partial X_{23}}{\partial w^*}\frac{\partial X_{32}}{\partial w}\det\begin{bmatrix}\frac{\partial X_{22}}{\partial z} & \frac{\partial X_{22}}{\partial z^*} \\ \frac{\partial X_{33}}{\partial z} & \frac{\partial X_{33}}{\partial z^*}\end{bmatrix}\right| = \frac{32|1-\theta_0^2|^3}{(1+\theta_0^2)^7|1-\theta_1^2|} \tag{A.47}$$

and

$$|\det(\mathrm{d}X/\mathrm{d}\tilde{t}_{12})_{FF}| = \left|\det\begin{bmatrix}\frac{\partial X_{12}}{\partial \eta_1} & \frac{\partial X_{12}}{\partial \eta_2} \\ \frac{\partial X_{31}}{\partial \eta_1} & \frac{\partial X_{31}}{\partial \eta_2}\end{bmatrix}\det\begin{bmatrix}\frac{\partial X_{13}}{\partial \eta_1^*} & \frac{\partial X_{13}}{\partial \eta_2^*} \\ \frac{\partial X_{21}}{\partial \eta_1^*} & \frac{\partial X_{21}}{\partial \eta_2^*}\end{bmatrix}\right| = \frac{16(1+\theta_0^2\theta_1^2)^2}{(1+\theta_0^2)^4(1-\theta_1^2)^4}. \tag{A.48}$$

Hence the first term of Eq. (A.44) is finally given by

$$\left|\operatorname{sdet}\left[\frac{((\mathrm{d}T_0)T_0^{-1})_{12}}{\mathrm{d}\tau_{12}}\right]\right| = \left|\frac{\det(\mathrm{d}X/\mathrm{d}\tilde{t}_{12})_{BB}}{\det(\mathrm{d}X/\mathrm{d}\tilde{t}_{12})_{FF}}\right| = \frac{2}{(1+\theta_0^2\theta_1^2)^2}\left|\frac{(1-\theta_0^2)(1-\theta_1^2)}{1+\theta_0^2}\right|^3. \tag{A.49}$$

Notice that this result looks quite different from the one obtained by Verbaarschot et al. for the 4×4 case.

Next we need to calculate $\mathrm{d}t_{12}$. To that end we recall that t_{12} was quasi-diagonalised by $t_{12} = u_1^{-1}M_{12}u_2$, where $M_{12} = \operatorname{diag}(\mu_1, i\mu_0 U)$. For notational convenience we introduce the matrices $\tilde{U} = \operatorname{diag}(1, U)$ and $\tilde{M}_{12} = \operatorname{diag}(\mu_1, i\mu_0\mathbb{1}_2)$ such that $M_{12} =$

$\tilde{U}\tilde{M}_{12}$. Then we can write the differential as

$$\begin{aligned} \mathrm{d}t_{12} &= u_1^{-1}\tilde{U}[\tilde{U}^{-1}u_1\mathrm{d}(u_1^{-1})\tilde{U}\tilde{M}_{12} + \tilde{U}^{-1}\mathrm{d}\tilde{U}\tilde{M}_{12} + \mathrm{d}\tilde{M}_{12} + \tilde{M}_{12}\mathrm{d}u_2u_2^{-1}]u_2 \\ &= u_1^{-1}\tilde{U}[-\delta u_1\tilde{M}_{12} + \delta\tilde{U}\tilde{M}_{12} + \mathrm{d}\tilde{M}_{12} + \tilde{M}_{12}\delta u_2]u_2 \end{aligned} \tag{A.50}$$

with $\delta u_1 = \tilde{U}^{-1}\mathrm{d}u_1u_1^{-1}\tilde{U}$ (we used $u_1\mathrm{d}(u_1^{-1}) = -\mathrm{d}u_1u_1^{-1}$), $\delta u_2 = \mathrm{d}u_2u_2^{-1}$ and $\delta\tilde{U} = \tilde{U}^{-1}\mathrm{d}\tilde{U}$. Since $|\mathrm{sdet}\,\tilde{U}| = |\mathrm{sdet}\,u_1^{-1}| = |\mathrm{sdet}\,u_2| = 1$[39] it suffices to concentrate on the bracket in (A.50).

$\mathrm{d}\tilde{M}_{12}$ is given by $\mathrm{d}\tilde{M}_{12} = \mathrm{diag}(\mathrm{d}\mu_1, i\mathrm{d}\mu_0\mathbb{1}_2)$, and with the explicit formulas for U (recall that $\tilde{U} = \mathrm{diag}(1, U)$, hence $\delta\tilde{U} = \mathrm{diag}(0, \delta U)$) and $u_p^{\pm1}$ given in (A.42), it is straightforward to calculate δU and δu_p, $p = 1, 2$ and one gets

$$\delta U = \begin{bmatrix} i\frac{\mathrm{d}m+s\mathrm{d}r-r\mathrm{d}s}{1+m^2+r^2+s^2} & -\frac{i(r+is)\mathrm{d}m+(1-im)\mathrm{d}r+i(1-im)\mathrm{d}s}{1+m^2+r^2+s^2} \\ \frac{-i(r-is)\mathrm{d}m+(1+im)\mathrm{d}r-i(1+im)\mathrm{d}s}{1+m^2+r^2+s^2} & -i\frac{\mathrm{d}m+s\mathrm{d}r-r\mathrm{d}s}{1+m^2+r^2+s^2} \end{bmatrix}, \tag{A.51}$$

$$\delta\tilde{u}_p = \begin{bmatrix} 0 & f(\alpha_p\alpha_p^*)\mathrm{d}\alpha_p^* & f(\alpha_p\alpha_p^*)\mathrm{d}\alpha_p \\ f(\alpha_p\alpha_p^*)\mathrm{d}\alpha_p & \frac{1}{2}(-)^p(\alpha_p^*\mathrm{d}\alpha_p + \alpha_p\mathrm{d}\alpha_p^*) & (-)^{p-1}\alpha_p\mathrm{d}\alpha_p \\ f(\alpha_p\alpha_p^*)\mathrm{d}\alpha_p^* & (-)^p\alpha_p^*\mathrm{d}\alpha_p^* & \frac{1}{2}(-)^{p-1}(\alpha_p^*\mathrm{d}\alpha_p + \alpha_p\mathrm{d}\alpha_p^*) \end{bmatrix}, \tag{A.52}$$

where $f(\alpha_p\alpha_p^*) = i^{p-1}(1 + \frac{1}{2}(-)^{p-1}\alpha_p\alpha_p^*)$, $\delta u_1 \equiv \tilde{U}^{-1}\delta\tilde{u}_1\tilde{U}$ and $\delta u_2 \equiv \delta\tilde{u}_2$. In order to evaluate the Jacobian, we first use the elements of $\delta\tilde{U}$, δu_1 and δu_2 as independent variables and then compute the Jacobians $\mathcal{J}(\delta U/\mathrm{d}U)$, $\mathcal{J}(\delta u_1/\mathrm{d}u_1)$ and $\mathcal{J}(\delta u_2/\mathrm{d}u_2)$. The form of the matrices in the above equations suggests that one can parametrise

$$\delta U' = \begin{bmatrix} i\mathrm{d}m' & -\mathrm{d}m_1' \\ \mathrm{d}m_1^{*\prime} & -i\mathrm{d}m' \end{bmatrix}, \quad \delta u_p' = i^{p-1}\begin{bmatrix} 0 & -\mathrm{d}\alpha_p^{*\prime} & \mathrm{d}\alpha_p' \\ \mathrm{d}\alpha_p' & \sim & \sim \\ \mathrm{d}\alpha_p^{*\prime} & \sim & \sim \end{bmatrix}. \tag{A.53}$$

Notice that the fermion-fermion block of $\delta u_p'$ is not vanishing, however it will not play a role in the computation of the Jacobian as we will see. Using further $\mathrm{d}\tilde{M}_{12} =$

[39] For U and u_1 this follows because they are unitary. For u_2, recall that it obeys $u_2^{-1} = ku_2^\dagger k$ and hence $1 = \mathrm{sdet}\,(u_2u_2^{-1}) = \mathrm{sdet}\,(u_2ku_2^\dagger k) = \mathrm{sdet}\,(u_2u_2^\dagger k^2) = (\mathrm{sdet}\,u_2)^2$.

$\mathrm{diag}(\mathrm{d}\mu_1, i\mathrm{d}\mu_0, i\mathrm{d}\mu_0)$, the term in brackets in (A.50) is hence given by

$$\begin{bmatrix} \mathrm{d}\mu_1 & i\mu_0\mathrm{d}\alpha_1^{*\prime} - i\mu_1\mathrm{d}\alpha_2^{*\prime} & -i\mu_0\mathrm{d}\alpha_1' + i\mu_1\mathrm{d}\alpha_2' \\ -\mu_1\mathrm{d}\alpha_1' - \mu_0\mathrm{d}\alpha_2' & i\mathrm{d}\mu_0 - \mu_0\mathrm{d}m' + \dots & -i\mu_0\mathrm{d}m_1' + \dots \\ -\mu_1\mathrm{d}\alpha_1^{*\prime} - \mu_0\mathrm{d}\alpha_2^{*\prime} & i\mu_0\mathrm{d}m_1^{*\prime} + \dots & i\mathrm{d}\mu_0 + \mu_0\mathrm{d}m' + \dots \end{bmatrix}, \tag{A.54}$$

where the dots indicate additional terms coming from the fermion-fermion blocks of $\delta u_1'$ and $\delta u_2'$. Only the boson-boson block depends on $\mathrm{d}\mu_1$, whereas only the fermion-fermion block depends on $\mathrm{d}\mu_0$ and the variables of $\delta U'$. This implies like in the previous calculation that the superdeterminant of the Jacobian supermatrix reduces to the ratio of boson-boson and fermion-fermion block, and moreover the various dotted terms are not needed. We get for the Jacobian the ratio

$$\left| \frac{(i\mu_0)(-i\mu_0)\det \begin{bmatrix} i & -\mu_0 \\ i & \mu_0 \end{bmatrix}}{\det \begin{bmatrix} -i\mu_0 & i\mu_1 \\ -\mu_1 & -\mu_0 \end{bmatrix} \det \begin{bmatrix} i\mu_0 & -i\mu_1 \\ -\mu_1 & -\mu_0 \end{bmatrix}} \right| = \frac{2\mu_0^3}{(\mu_0^2+\mu_1^2)^2}, \tag{A.55}$$

where the numerator is the part coming from the boson-boson block of the Jacobian supermatrix and the denominator coming from its fermion-fermion part.

In order to get the full Jacobian of $\mathrm{d}t_{12}$ we need to multiply this result with the Jacobians $\mathcal{J}(\delta U/\mathrm{d}U)$, $\mathcal{J}(\delta u_1/\mathrm{d}u_1)$ and $\mathcal{J}(\delta u_2/\mathrm{d}u_2)$. To compute $\mathcal{J}(\delta u_1/\mathrm{d}u_1)$ we note that the matrix $\tilde{U}$ in the definition of δu_1 yields a factor of unity, $\mathrm{sdet}\,\tilde{U} = 1$ and hence $\mathcal{J}(\delta u_1/\mathrm{d}u_1) = \mathcal{J}(\delta\tilde{u}_1/\mathrm{d}u_1)$. Calculating the remaining three Jacobians can be done easily now by simply comparing the expressions for δU, $\delta\tilde{u}_p$, Eqs. (A.51), (A.52), with their dashed counterparts $\delta U'$ and $\delta u_p'$ given in Eq. (A.53). For $\mathcal{J}(\delta U/\mathrm{d}U)$ one gets

$$\begin{aligned} \mathcal{J}(\delta U/\mathrm{d}U) &= \left| \frac{1}{(1+m^2+r^2+s^2)^3} \det \begin{bmatrix} 1 & s & -r \\ ir-s & 1-im & i+m \\ -ir-s & 1+im & -i+m \end{bmatrix} \right| \\ &= \frac{2}{(1+m^2+r^2+s^2)^2}. \end{aligned} \tag{A.56}$$

This is the same result as in [8]. For $\mathcal{J}(\delta\tilde{u}_p/\mathrm{d}u_p)$ it suffices to compare the entries

in the first column of (A.52) and (A.53) with each other. This yields immediately $\mathrm{d}\alpha'_p = (1 + \frac{1}{2}(-)^{p-1}\alpha_p\alpha_p^*)\mathrm{d}\alpha_p$ and $\mathrm{d}\alpha_p^{*\prime} = (1 + \frac{1}{2}(-)^{p-1}\alpha_p\alpha_p^*)\mathrm{d}\alpha_p^*$. The Jacobian is hence given by

$$\mathcal{J}(\delta\tilde{u}_p/\mathrm{d}u_p) = \left(1 + \tfrac{1}{2}(-)^{p-1}\alpha_p\alpha_p^*\right)^2 = 1 + (-)^{p-1}\alpha_p\alpha_p^*, \quad p = 1, 2. \tag{A.57}$$

Collecting the results from (A.55), (A.56) and (A.57) we hence finally get

$$\mathrm{d}t_{12} = \frac{4(1-\alpha_1\alpha_1^*)(1+\alpha_2\alpha_2^*)}{(1+m^2+r^2+s^2)^2}\frac{\mu_0^3}{(\mu_0^2+\mu_1^2)^2}\mathrm{d}m\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\mu_0\,\mathrm{d}\mu_1\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_1^*\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2^*. \tag{A.58}$$

Notice that the Jacobian depends on the Grassmann variables. This is a new feature which was not present in the work of Verbaarschot et al. where the analogous expression for $\mathcal{J}(\delta\tilde{u}_p/\mathrm{d}u_p)$ evaluated to unity.

Since we evaluated the first term in (A.43) in terms of the eigenvalues θ_0 and θ_1 of the supermatrix τ_{12}, it is advantageous to express the result in terms of $\mathrm{d}\tau_{12}$ instead of $\mathrm{d}t_{12}$. It is given by the same equation (A.58) with μ_0 replaced with θ_0 and μ_1 replaced with θ_1 and accordingly for the differentials. The full measure is then given by

$$\begin{aligned}|\mathrm{sdet}\,[((\mathrm{d}T_0)T_0^{-1})_{12}/\mathrm{d}\tau_{12}]|\mathrm{d}\tau_{12} &= \frac{8\theta_0^3}{(1+\theta_0^2\theta_1^2)^2(\theta_0^2+\theta_1^2)^2}\left|\frac{(1-\theta_0^2)(1-\theta_1^2)}{1+\theta_0^2}\right|^3 \\ &\times \frac{(1-\alpha_1\alpha_1^*)(1+\alpha_2\alpha_2^*)}{(1+m^2+r^2+s^2)^2}\mathrm{d}m\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\theta_0\,\mathrm{d}\theta_1\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_1^*\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2^*.\end{aligned} \tag{A.59}$$

The last step is to rewrite this measure in terms of the variables λ_0 and λ_1 from the parametrisation (A.42) instead of θ_0 and θ_1. The definitions of of τ_{12} and τ_{21} (see the paragraph below Eq. (A.43)) imply $\mu_1 = 2(1-\theta_1^2)^{-1}\theta_1$ and $\mu_0 = 2(1+\theta_0^2)^{-1}\theta_0$, where μ_0 and μ_1 are the "eigenvalues" (modulo the special unitary matrix U) of t_{12} and t_{21}. They are furthermore related to the λ's via $\lambda_1 = 1 + 2\mu_1^2$ and $\lambda_0 = 1 - 2\mu_0^2$ (see the paragraph below Eq. (A.41d)). This suggests the relations

$$\lambda_1 = 1 + \frac{8\theta_1^2}{(1-\theta_1^2)^2}, \quad \mathrm{d}\lambda_1 = \frac{16\,\theta_1(1+\theta_1^2)}{|1-\theta_1^2|^3}\,\mathrm{d}\theta_1, \tag{A.60a}$$

$$\lambda_0 = 1 - \frac{8\theta_0^2}{(1+\theta_0^2)^2}, \quad \mathrm{d}\lambda_0 = \frac{16\,\theta_0(1-\theta_0^2)}{|1+\theta_0^2|^3}\,\mathrm{d}\theta_0. \tag{A.60b}$$

With these relations at hand we can express the measure (A.59) in terms of λ_1 and λ_0 and finally get

$$\begin{aligned} \mathrm{d}\mu(\mathcal{Q}) =& \frac{1-\lambda_0^2}{2^4\sqrt{\lambda_1^2-1}(\lambda_1-\lambda_0)^2}\frac{(1-\alpha_1\alpha_1^*)(1+\alpha_2\alpha_2^*)}{(1+m^2+r^2+s^2)^2} \\ &\times \mathrm{d}m\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\lambda_0\,\mathrm{d}\lambda_1\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_1^*\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2^*. \end{aligned} \tag{A.61}$$

Notice that this result differs significantly from the standard expressions one gets for models where the dimensions of the boson-boson block and fermion-fermion block are equal (e.g. in the case of the the two-point correlation function) which do not show a dependence on the Grassmann variables.

A.5. Proof of Eq. (4.158)

We show the validity of the claim

$$\int_1^\infty \mathrm{d}\lambda_1 \,\frac{\operatorname{arcoth}\lambda_1}{\sqrt{\lambda_1^2-1}}\exp\left(-a\lambda_1\right)\left(a+\lambda_1-a\lambda_1^2\right) = K_0(a). \tag{A.62}$$

The first step is to take the Laplace transform on both sides. For the right-hand side one gets (see e.g. 6.611.9 in [59])

$$\int_0^\infty \mathrm{d}a\, K_0(a)\exp(-a\alpha) = \frac{\operatorname{arcosh}\alpha}{\sqrt{\alpha^2-1}}. \tag{A.63}$$

Taking the Laplace transform on the left-hand side is trivial since it amounts to perform integrals of the form $\int_0^\infty a^n \exp[-a(\alpha+\lambda_1)] = \frac{n!}{(\alpha+\lambda_1)^{n+1}}$ and the result is

$$\mathcal{L}(\alpha) = \int_1^\infty \mathrm{d}\lambda_1 \,\frac{\operatorname{arcoth}\lambda_1}{\sqrt{\lambda_1^2-1}}\frac{1+\alpha\lambda_1}{(\alpha+\lambda)^2}. \tag{A.64}$$

By definition, the hyperbolic area cotangent can be expressed as a logarithm via $\operatorname{arcoth}\lambda_1 = \frac{1}{2}\ln\frac{\lambda_1+1}{\lambda_1-1}$ for $\lambda_1 > 1$. This suggests the substitution $\frac{\lambda_1+1}{\lambda_1-1} = t^2$, which implies $t \in (1,\infty)$, $\lambda_1 = \frac{t^2+1}{t^2-1}$ and $\mathrm{d}\lambda_1 = \frac{4t}{(t^2-1)^2}\mathrm{d}t$. The choice of t^2 instead of just t in the substitution is such that $\sqrt{\lambda_1^2-1} = \frac{2t}{1-t^2}$ becomes rational. The integral is then

given by

$$\mathcal{L}(\alpha) = 2\int_1^\infty \mathrm{d}t\ \ln t \frac{\alpha - 1 + (\alpha+1)t^2}{(\alpha - 1 - (\alpha+1)t^2)^2}. \tag{A.65}$$

To make further progress we use that the rational part of the integrand can be represented as a derivative,

$$\frac{\alpha - 1 + (\alpha+1)t^2}{(\alpha - 1 - (\alpha+1)t^2)^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{\alpha - 1 - (\alpha+1)t^2}\right). \tag{A.66}$$

This enables us to simplify the integral performing integration by parts. Note that the boundary term vanishes since $\lim_{t\to\infty} \frac{t\ln t}{\alpha-1-(\alpha+1)t^2} = 0$ and $\ln(t=1) = 0$. Hence we are left with

$$\mathcal{L}(\alpha) = -2\int_1^\infty \frac{\mathrm{d}t}{\alpha - 1 - (\alpha+1)t^2} = -\frac{2}{\sqrt{\alpha^2-1}}\int_{\sqrt{\frac{\alpha+1}{\alpha-1}}}^\infty \frac{\mathrm{d}t}{1-t^2}, \tag{A.67}$$

where we rescaled $t \to \sqrt{\frac{\alpha-1}{\alpha+1}}t$. The above integral can be performed easily and the Laplace transform is given by

$$\mathcal{L}(\alpha) = \frac{2}{\sqrt{\alpha^2-1}}\,\mathrm{arcoth}\left(\sqrt{\frac{\alpha+1}{\alpha-1}}\right). \tag{A.68}$$

The last step is to realise that one can rewrite the hyperbolic area cotangent, using its definition in terms of a logarithm, in the following way,

$$\begin{aligned}\mathrm{arcoth}\left(\sqrt{\frac{\alpha+1}{\alpha-1}}\right) &= \frac{1}{2}\ln\left(\frac{\sqrt{\frac{\alpha+1}{\alpha-1}}+1}{\sqrt{\frac{\alpha+1}{\alpha-1}}-1}\right) = \frac{1}{2}\ln\left(\frac{\sqrt{\alpha+1}+\sqrt{\alpha-1}}{\sqrt{\alpha+1}-\sqrt{\alpha-1}}\right)\\ &= \frac{1}{2}\ln\left(\frac{(\sqrt{\alpha+1}+\sqrt{\alpha-1})^2}{(\alpha+1)-(\alpha-1)}\right) = \frac{1}{2}\ln(\alpha+\sqrt{\alpha^2-1}).\end{aligned} \tag{A.69}$$

The logarithm in the last expression is nothing else than the definition of the hyperbolic area cosine, and hence we have shown

$$\mathcal{L}(\alpha) = \frac{\mathrm{arcosh}\,\alpha}{\sqrt{\alpha^2-1}}. \tag{A.70}$$

This is precisely the Laplace transform of the Bessel function $K_0(a)$, Eq. (A.63). We have hence shown that the left-hand side and right-hand side of our claim (4.158) share the same Laplace transform. Taking the inverse Laplace transform on both sides completes the proof.

A.6. Integration Over the Vectors h_1 and h_2 in Eq. (4.170)

In this appendix we perform integration over the $N-2$ component vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ in (4.170), thus deriving the result (4.171). Note that y_{11}, y_{22} and y_{12} in (4.170) is a short-hand notation with $y_{jk} = \boldsymbol{h}_j^T H_{N-2}^{-1} \boldsymbol{h}_k$. This suggests we need to perform the following integrals

$$I_{1,m} = \int \mathrm{d}\boldsymbol{h}\, e^{-\frac{c}{2}\boldsymbol{h}^T\boldsymbol{h}} (\boldsymbol{h}^T A\, \boldsymbol{h})^m, \quad m = 0, 1, 2, \tag{A.71}$$

$$I_{2,m} = \int \mathrm{d}\boldsymbol{h}_1 \int \mathrm{d}\boldsymbol{h}_2\, e^{-\frac{c_1}{2}\boldsymbol{h}_1^T\boldsymbol{h}_1 - \frac{c_2}{2}\boldsymbol{h}_2^T\boldsymbol{h}_2} (\boldsymbol{h}_1^T A\, \boldsymbol{h}_2)^m, \quad m = 2, 4, \tag{A.72}$$

$$I_3 = \int \mathrm{d}\boldsymbol{h}_1 \mathrm{d}\boldsymbol{h}_2\, e^{-\frac{c_1}{2}\boldsymbol{h}_1^T\boldsymbol{h}_1 - \frac{c_2}{2}\boldsymbol{h}_2^T\boldsymbol{h}_2} (\boldsymbol{h}_1^T A\, \boldsymbol{h}_1)(\boldsymbol{h}_2^T A\, \boldsymbol{h}_2)(\boldsymbol{h}_1^T A\, \boldsymbol{h}_2)^2, \tag{A.73}$$

where we abbreviated $A = H_{N-2}^{-1}$. Note that this matrix is real symmetric. The simplest integral is $I_{1,0}$ which is Gaussian with the solution

$$I_{1,0} = \left(\frac{2\pi}{c}\right)^{(N-2)/2}. \tag{A.74}$$

$I_{1,1}$ can be calculated by diagonalising $A = O^T \operatorname{diag}(a_1 \dots a_n) O$ and absorbing O into $\boldsymbol{h}$ by changing $O\boldsymbol{h} \to \boldsymbol{h}$. The Jacobian of this transformation is unity and $\boldsymbol{h}^T\boldsymbol{h}$ remains invariant. The resulting integral reads

$$I_{1,1} = \sum_{j=1}^{n} a_j \int \mathrm{d}\boldsymbol{h}\, e^{-\frac{c}{2}\boldsymbol{h}^T\boldsymbol{h}} h_j^2 = \sum_{j=1}^{n} a_j \left(\frac{2\pi}{c}\right)^{(N-2)/2} \frac{1}{c} = \left(\frac{2\pi}{c}\right)^{(N-2)/2} \frac{\operatorname{Tr} A}{c}. \tag{A.75}$$

In the same fashion we calculate $I_{1,2}$ and after diagonalising we get

$$\begin{aligned} I_{1,2} &= \sum_j a_j^2 \int \mathrm{d}\boldsymbol{h}\, e^{-\frac{c}{2}\boldsymbol{h}^T\boldsymbol{h}} h_j^4 + \sum_{j\neq k} a_j a_k \int \mathrm{d}\boldsymbol{h}\, e^{-\frac{c}{2}\boldsymbol{h}^T\boldsymbol{h}} h_j^2 h_k^2 \\ &= \left(\frac{2\pi}{c}\right)^{(N-2)/2} \frac{1}{c^2}\left(3\sum_i a_i^2 + \sum_{i\neq j} a_i a_j\right) \\ &= \left(\frac{2\pi}{c}\right)^{(N-2)/2} \frac{1}{c^2}\left[2\sum_i a_i^2 + \left(\sum_i a_i\right)^2\right] \\ &= \left(\frac{2\pi}{c}\right)^{(N-2)/2} \frac{1}{c^2}\left[2\,\mathrm{Tr}\,A^2 + (\mathrm{Tr}\,A)^2\right], \end{aligned} \tag{A.76}$$

where we used that $(\sum_j x_j)^2 = \sum_j x_j^2 + \sum_{j\neq k} x_j x_k$. Next we also need the integrals which feature two vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$. We start with $I_{2,2}$, and after diagonalising and using the same identities as above we get

$$\begin{aligned} I_{2,2} =& \sum_j a_j^2 \int \mathrm{d}\boldsymbol{h}_1\, e^{-\frac{c_1}{2}\boldsymbol{h}_1^T\boldsymbol{h}_1} (h_1^j)^2 \int \mathrm{d}\boldsymbol{h}_2\, e^{-\frac{c_2}{2}\boldsymbol{h}_2^T\boldsymbol{h}_2} (h_2^j)^2 \\ &+ \sum_{j\neq k} a_j a_k \int \mathrm{d}\boldsymbol{h}_1\, e^{-\frac{c_1}{2}\boldsymbol{h}_1^T\boldsymbol{h}_1} h_1^j h_1^k \int \mathrm{d}\boldsymbol{h}_2\, e^{-\frac{c_2}{2}\boldsymbol{h}_2^T\boldsymbol{h}_2} h_2^j h_2^k \\ =& \sum_j a_j^2 \left(\frac{4\pi^2}{c_1 c_2}\right)^{(N-2)/2} \frac{1}{c_1 c_2} = \left(\frac{4\pi^2}{c_1 c_2}\right)^{(N-2)/2} \frac{\mathrm{Tr}\,A^2}{c_1 c_2}. \end{aligned} \tag{A.77}$$

Note that the integrands in the second line feature odd powers of h_1^j and h_2^j, such that those integrals vanish completely. This fact also helps in calculating the next integral, $I_{2,4}$, which is a bit more complicated because of the 4th power, which gives $(\boldsymbol{h}_1^T A \boldsymbol{h}_2)^4 = \sum_{i,j,k,l} a_i a_j a_k a_l h_1^i h_1^j h_1^k h_1^l h_2^i h_2^j h_2^k h_2^l$. However, we only need to keep the terms with even powers, i.e. for $i = j = k = l$ and if two indices coincide pairwise, viz, $i = j \neq k = l$, $i = k \neq j = l$ and $i = l \neq j = k$, such that the integration reduces

to

$$
\begin{aligned}
I_{2,4} &= \sum_j a_j^4 \int \mathrm{d}\boldsymbol{h}_1\, e^{-\frac{c_1}{2}\boldsymbol{h}_1^T\boldsymbol{h}_1}(h_1^j)^4 \int \mathrm{d}\boldsymbol{h}_2\, e^{-\frac{c_2}{2}\boldsymbol{h}_2^T\boldsymbol{h}_2}(h_2^j)^4 \\
&\quad + 3\sum_{j\neq k} a_j^2 a_k^2 \int \mathrm{d}\boldsymbol{h}_1\, e^{-\frac{c_1}{2}\boldsymbol{h}_1^T\boldsymbol{h}_1}(h_1^j)^2(h_1^k)^2 \int \mathrm{d}\boldsymbol{h}_2\, e^{-\frac{c_2}{2}\boldsymbol{h}_2^T\boldsymbol{h}_2}(h_2^j)^2(h_2^k)^2 \\
&= \left(\frac{4\pi^2}{c_1c_2}\right)^{(N-2)/2} \frac{3}{c_1^2c_2^2}\left(3\sum_j a_j^4 + \sum_{j\neq k} a_j^2a_k^2\right) \\
&= \left(\frac{4\pi^2}{c_1c_2}\right)^{(N-2)/2} \frac{3}{c_1^2c_2^2}\left[2\,\mathrm{Tr}\,A^4 + (\mathrm{Tr}\,A^2)^2\right]. \qquad \text{(A.78)}
\end{aligned}
$$

Last we need to calculate I_3, and the term in the integrand written in index-notation reads $(\boldsymbol{h}_1^T A\,\boldsymbol{h}_1)(\boldsymbol{h}_2^T A\,\boldsymbol{h}_2)(\boldsymbol{h}_1^T A\,\boldsymbol{h}_2)^2 = \sum_{i,j,k,l}(h_1^i)^2h_1^jh_1^l(h_2^k)^2h_2^jh_2^l$. Again, only terms with even powers contribute, and hence we can restrict to terms with $j = l$, such that the summation reduces to

$$
\begin{aligned}
&\sum_{i,j,k} a_ia_j^2a_k(h_1^i)^2(h_1^j)^2(h_2^k)^2(h_2^j)^2 \\
&= \sum_i a_i^4(h_1^i)^4(h_2^i)^4 + \sum_{i\neq j} a_i^3a_j(h_1^i)^4(h_2^i)^2(h_2^j)^2 \\
&\quad + \sum_{i\neq j} a_ia_j^3(h_1^i)^2(h_1^j)^2(h_2^j)^4 + \sum_{i\neq j,k\neq j} a_ia_j^2a_k(h_1^i)^2(h_1^j)^2(h_2^i)^2(h_2^j)^2.
\end{aligned} \qquad \text{(A.79)}
$$

Integration of this then yields

$$
\begin{aligned}
I_3 &= \left(\frac{4\pi^2}{c_1c_2}\right)^{(N-2)/2} \frac{1}{c_1^2c_2^2}\left(9\sum_i a_i^4 + 6\sum_{i\neq j} a_i^3a_j + \sum_{i\neq j,k\neq j} a_ia_j^2a_k\right) \\
&= \left(\frac{4\pi^2}{c_1c_2}\right)^{(N-2)/2} \frac{1}{c_1^2c_2^2}\left[\mathrm{Tr}\,A^2(\mathrm{Tr}\,A)^2 + 4\,\mathrm{Tr}\,A^3\,\mathrm{Tr}\,A + 4\,\mathrm{Tr}\,A^4\right]. \qquad \text{(A.80)}
\end{aligned}
$$

With these six identities for $I_{1,0}$, $I_{1,1}$, $I_{1,2}$, $I_{2,2}$, $I_{2,4}$ and I_3 we have now all ingredients at hand to perform the $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ integration in (4.170), which leads to the solution (4.171).

A.7. Simplification of Eq. (4.186)

The correlation function

$$\mathcal{R}(x) = \left\langle \frac{|\det H|}{\det(H^2 + \frac{x^2}{N^2})^{1/2}} \right\rangle_{\mathrm{GOE},\, N\to\infty}, \tag{A.81}$$

which can be interpreted as the characteristic function of the probability distribution of a single off-diagonal element K_{ab} of the K-matrix (see Section 5.2.2), is a special case of the correlation function $\mathcal{C}^{(\mathrm{bulk})}_{2,4}(0;0,\omega_{B1},-\omega_{B1},\omega_{B2},-\omega_{B2})$ calculated in Section 4.5.2 with $\omega_{B1} \equiv x$ and $\omega_{B2} \equiv 0$. In this appendix we show that its representation as two-fold integral, Eq. (4.186), can be simplified to

$$\mathcal{R}(x) = \frac{2}{\pi}\left(\frac{|x|}{J}K_0(|x|/J) + \int_{|x|/J}^{\infty} dy\, K_0(y)\right). \tag{A.82}$$

First note that the integrand is a function of x^2/J^2. Hence for simplicity we choose $J = 1$ and restrict to the case of $x > 0$. The result for arbitrary J and x can be recovered by replacing x with $|x|/J$.

While $\mathcal{R}(x)$ in (A.82) cannot be expressed by elementary functions, its first derivative takes the particularly easy form

$$\mathcal{R}'(x) = \frac{2}{\pi}xK_1(x). \tag{A.83}$$

Taking the derivative of the two-fold integral representation on the other hand gives

$$\begin{aligned}\mathcal{R}'(x) = \frac{Cx}{2}\int_0^\infty \mathrm{d}q_1 \int_0^\infty \mathrm{d}q_2\, \frac{|q_1-q_2|}{q_1q_2\sqrt{q_1+q_2}} \exp\left[-\frac{1}{2}\left(\frac{1}{q_1}+\frac{1}{q_2}+\frac{x^2}{2}(q_1+q_2)\right)\right] \\ \times\left\{(q_1-q_2)I_1\left[\frac{x^2}{4}(q_1-q_2)\right] - (q_1+q_2)I_0\left[\frac{x^2}{4}(q_1-q_2)\right]\right\} \\ \times\left\{\frac{(1+q_1)(1+q_2)}{q_1^2q_2^2} + \frac{3}{(q_1+q_2)^2} + \frac{2}{q_1q_2(q_1+q_2)}\right\}.\end{aligned} \tag{A.84}$$

To prove the above statement we hence need to show

$$\begin{aligned} K_1(x) =& \frac{\pi C}{4} \int_0^\infty \mathrm{d}q_1 \int_0^\infty \mathrm{d}q_2 \, \frac{|q_1 - q_2|}{q_1 q_2 \sqrt{q_1 + q_2}} \exp\left[-\frac{1}{2} \left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{x^2}{2}(q_1 + q_2) \right) \right] \\ & \times \left\{ (q_1 - q_2) I_1 \left[\frac{x^2}{4}(q_1 - q_2) \right] - (q_1 + q_2) I_0 \left[\frac{x^2}{4}(q_1 - q_2) \right] \right\} \\ & \times \left\{ \frac{(1+q_1)(1+q_2)}{q_1^2 q_2^2} + \frac{3}{(q_1+q_2)^2} + \frac{2}{q_1 q_2 (q_1 + q_2)} \right\}, \end{aligned} \tag{A.85}$$

with some constant C independent of x (the same C as in (4.186)). The first step is to replace x by $\sqrt{x}$ (this is valid because $x > 0$) and to take the Laplace transform on both sides of Eq. (A.85). For the left-hand side we use formula 6.614.5 from [59],

$$\int_0^\infty \mathrm{d}x \, K_1(\sqrt{x}) \exp(-\alpha x) = \frac{1}{8} \sqrt{\frac{\pi}{\alpha^3}} \, \exp\left(\frac{1}{8\alpha} \right) \left[K_1\left(\frac{1}{8\alpha} \right) - K_0\left(\frac{1}{8\alpha} \right) \right], \tag{A.86}$$

where α is the Laplace variable. On the right-hand side we use 6.611.4 from [59] (valid for $\mathrm{Re}(\beta) > |\,\mathrm{Re}(\gamma)|$ and $\mathrm{Re}(\nu) > -1$)

$$\int_0^\infty \mathrm{d}x \, I_\nu(\gamma x) \exp(-\beta x) = \gamma^{-\nu} \frac{(\beta - \sqrt{\beta^2 - \gamma^2})^\nu}{\sqrt{\beta^2 - \gamma^2}}. \tag{A.87}$$

Here $\beta = \alpha + \frac{1}{4}(q_1 + q_2)$, $\nu = 0$ or $\nu = 1$ and $\gamma = \frac{1}{4}(q_1 - q_2)$. After some cancellation, this allows us to write the right-hand side of the Laplace transform as

$$\begin{aligned} \mathcal{L}(\alpha) =& \pi C \int_0^\infty \mathrm{d}q_1 \int_0^\infty \mathrm{d}q_2 \, \frac{|q_1 - q_2|}{q_1 q_2 \sqrt{q_1 + q_2}} \exp\left[-\frac{1}{2} \left(\frac{1}{q_1} + \frac{1}{q_2} \right) \right] \\ & \times \left(\frac{2\alpha}{\sqrt{2\alpha + q_1}\sqrt{2\alpha + q_2}} - 1 \right) \Psi(q_1, q_2), \end{aligned} \tag{A.88}$$

where we abbreviated the last line in Eq. (A.85) with $\Psi(q_1, q_2)$. The above equation

consists of two terms, where one is independent of α, viz.,

$$\mathcal{L}(\alpha) = \mathcal{L}(0) + 2\pi C\alpha \int_0^\infty \mathrm{d}q_1 \int_0^\infty \mathrm{d}q_2 \, \frac{|q_1 - q_2|}{q_1 q_2 \sqrt{q_1 + q_2}} \exp\left[-\frac{1}{2}\left(\frac{1}{q_1} + \frac{1}{q_2}\right)\right] \times \frac{\Psi(q_1, q_2)}{\sqrt{2\alpha + q_1}\sqrt{2\alpha + q_2}}, \tag{A.89}$$

where

$$\mathcal{L}(0) = -\pi C \int_0^\infty \mathrm{d}q_1 \int_0^\infty \mathrm{d}q_2 \, \frac{|q_1 - q_2|}{q_1 q_2 \sqrt{q_1 + q_2}} \exp\left[-\frac{1}{2}\left(\frac{1}{q_1} + \frac{1}{q_2}\right)\right] \Psi(q_1, q_2). \tag{A.90}$$

If our assumption (A.85) is correct, one should expect $\mathcal{L}(0) = \int_0^\infty \mathrm{d}x \, K_1(\sqrt{x}) = \pi$.

Next we substitute $q_1 \to \alpha/q_1$ and $q_2 \to \alpha/q_2$, which implies $\mathrm{d}q_1\mathrm{d}q_2 \to \frac{\alpha^2}{q_1^2 q_2^2}\mathrm{d}q_1\mathrm{d}q_2$. This gives

$$\mathcal{L}(\alpha) = \mathcal{L}(0) + \frac{2\pi C}{\alpha^{7/2}} \int_0^\infty \mathrm{d}q_1 \int_0^\infty \mathrm{d}q_2 \, \frac{|q_1 - q_2| \exp\left[-\frac{1}{2\alpha}(q_1 + q_2)\right] \tilde{\Psi}(q_1, q_2, \alpha)}{\sqrt{q_1 + q_2}\sqrt{1 + 2q_1}\sqrt{1 + 2q_2}}, \tag{A.91}$$

where

$$\tilde{\Psi}(q_1, q_2, \alpha) = (\alpha + q_1)(\alpha + q_2) + \frac{2\alpha q_1 q_2}{q_1 + q_2} + \frac{3\alpha^2 q_1 q_2}{(q_1 + q_2)^2}. \tag{A.92}$$

Now we change variables again with the transformation $u = \alpha(q_1 + q_2)$, $v = q_1 - q_2$. Since q_1 and q_2 are positive, and the integrand is invariant under exchanging q_1 and q_2, the integration range is given by $v \in (0, u)$ and $u \in (0, \infty)$, and

$$\mathcal{L}(\alpha) = \mathcal{L}(0) + \frac{\pi C}{2\alpha^3} \int_0^\infty \mathrm{d}u \int_0^{\alpha u} \mathrm{d}v \, \frac{v \exp\left[-\frac{1}{2}u\right] \hat{\Psi}(u, v, \alpha)}{\sqrt{u}\sqrt{(1 + \alpha u)^2 - v^2}}, \tag{A.93}$$

where

$$\hat{\Psi}(u, v, \alpha) = \alpha^2(u^2 + 6u + 7) - v^2\left(1 + \frac{2}{u} + \frac{3}{u^2}\right). \tag{A.94}$$

The v-integration thus takes the particularly simple form

$$\int_0^{\alpha u} \mathrm{d}v \, \frac{\alpha^2 A(u) v - B(u) v^3}{\sqrt{(1 + \alpha u)^2 - v^2}}, \tag{A.95}$$

which can be done in closed form using the identities

$$\int \frac{\mathrm{d}v\, v}{\sqrt{a^2-v^2}} = -\sqrt{a^2-v^2}, \quad \int \frac{\mathrm{d}v\, v^3}{\sqrt{a^2-v^2}} = -\frac{\sqrt{a^2-v^2}}{3}(2a^2+v^2). \tag{A.96}$$

Then, with $A(u) = u^2+6u+7$ and $B(u) = 1+\frac{2}{u}+\frac{3}{u^2}$ the Laplace transform of the right-hand side is given by

$$\begin{aligned}\mathcal{L}(\alpha) = \mathcal{L}(0) + \frac{\pi C}{6\alpha^3}\int_0^\infty \mathrm{d}u\, \frac{\exp\left[-\frac{1}{2}u\right]}{\sqrt{u}}\Big\{&3\alpha^2 A(u)(1+\alpha u) - 2B(u)(1+\alpha u)^3 \\ &- \sqrt{1+2\alpha u}[3\alpha^2 A(u) - B(u)(2+4\alpha u+3\alpha^2u^2)]\Big\}.\end{aligned} \tag{A.97}$$

Note that $A(u)$ and $B(u)$ are related, $A(u) = 4(u+1)+u^2B(u)$. Hence we express the Laplace transform in terms of just $B(u)$ as

$$\begin{aligned}\mathcal{L}(\alpha) = \mathcal{L}(0) + \frac{\pi C}{6\alpha^3}\int_0^\infty \mathrm{d}u\, \frac{\exp\left[-\frac{1}{2}u\right]}{\sqrt{u}}\Big\{&(1+\alpha u)[12\alpha^2(u+1) + B(u)(\alpha^2u^2-4\alpha u-2)] \\ &- \sqrt{1+2\alpha u}[12\alpha^2(u+1) - 2B(u)(1+2\alpha u)]\Big\}.\end{aligned} \tag{A.98}$$

We can calculate $\mathcal{L}(0)$ following the exact steps as before, which gives the result

$$\mathcal{L}(0) = -\frac{\pi C}{6}\int_0^\infty \mathrm{d}u\, \frac{\exp\left[-\frac{1}{2}u\right]}{\sqrt{u}}(12u(u+1)+u^3B(u)) = -12\pi\sqrt{2\pi}C, \tag{A.99}$$

where the integral was calculated using (see e.g. formula 3.381.4 in [59])

$$\int_0^\infty \mathrm{d}x\, x^{\nu-1}e^{-\mu x} = \mu^{-\nu}\Gamma(\nu), \quad \mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0. \tag{A.100}$$

As mentioned above one should actually expect $\mathcal{L}(0) = \pi$ which suggest that one has to choose $C = -\frac{1}{12\sqrt{2\pi}}$. Note that the integral representation of $\mathcal{L}(0)$ appears also in (A.98), but with a different sign, such that those terms cancel and

$$\begin{aligned}\mathcal{L}(\alpha) = \frac{\pi C}{6\alpha^3}\int_0^\infty \mathrm{d}u\, \frac{\exp\left[-\frac{1}{2}u\right]}{\sqrt{u}}\Big\{&[12\alpha^2(u+1) - B(u)(3\alpha^2u^2+6\alpha u+2)] \\ &- \sqrt{1+2\alpha u}[12\alpha^2(u+1) - 2B(u)(1+2\alpha u)]\Big\}.\end{aligned} \tag{A.101}$$

Next we pick the polynomial term of the integrand, given by $12\alpha^2(u+1)-3\alpha^2u^2B(u) = 3\alpha^2(-u^2+2u+1)$. With (A.100), integration over these terms vanishes since one gets a term proportional to $\Gamma(1/2) + 4\Gamma(3/2) - 4\Gamma(5/2) = 0$, and the Laplace transform simplifies to

$$\begin{aligned}\mathcal{L}(\alpha) = -\frac{\pi C}{3\alpha^3}\int_0^\infty \mathrm{d}u\,\frac{\exp\left[-\frac{1}{2}u\right]}{\sqrt{u}}\Big\{&6\alpha^2(u+1)\sqrt{1+2\alpha u}\\ &+\left(1+\frac{2}{u}+\frac{3}{u^2}\right)\left[3\alpha u+1-(1+2\alpha u)^{3/2}\right]\Big\}.\end{aligned} \tag{A.102}$$

The integrand is now in a form which makes it amenable to series expansion. To that end we use the Taylor expansion of the square root,

$$\sqrt{1+2\alpha u} = \sum_{j=0}^{\infty} b_j(2\alpha u)^j, \quad b_j = \frac{(-1)^j(2j)!}{(1-2j)(j!)^2 4^j}. \tag{A.103}$$

The square-bracket in the second line of Eq. (A.102) can then be written as

$$\begin{aligned}&3\alpha u+1-(1+2\alpha u)^{3/2} = 3\alpha u+1-\sqrt{1+2\alpha u}-2\alpha u\sqrt{1+\alpha u}\\ &= 3\alpha u+1-b_0-\sum_{j=1}^{\infty}(b_j+b_{j-1})(2\alpha u)^j = -\sum_{j=0}^{\infty}(b_{j+2}+b_{j+1})(2\alpha u)^{j+2},\end{aligned} \tag{A.104}$$

where in the second line we used $b_0 = 1$ and $b_1 = 1/2$, such that the zeroth and first order terms cancel. Eq. (A.102) can hence be written as

$$\begin{aligned}\mathcal{L}(\alpha) = -\frac{2\pi C}{3\alpha}\int_0^\infty \mathrm{d}u\,\frac{\exp\left(-\frac{u}{2}\right)}{\sqrt{u}}\sum_{j=0}^{\infty}(2\alpha u)^j\,\{&3(u+1)b_j\\ &-2(u^2+2u+3)(b_{j+2}+b_{j+1})\big\}.\end{aligned} \tag{A.105}$$

Next we interchange summation and integration and use

$$\int_0^\infty \frac{\exp\left(-\frac{u}{2}\right)}{\sqrt{u}}u^j = 2^{j+1/2}\Gamma(j+1/2) = \frac{\sqrt{2\pi}(2j)!}{j!\,2^j}, \tag{A.106}$$

where we used that the Gamma function for integer j can be expressed via the given

factorial ratio. This allows us to obtain the solution in terms of a series,

$$\mathcal{L}(\alpha) = -\frac{2\pi\sqrt{2\pi}C}{3\alpha}\sum_{j=0}^{\infty}\alpha^j\left\{3\left(\frac{(2j+2)!}{2(j+1)!}+\frac{(2j)!}{j!}\right)b_j \right. \\ \left. -2\left(\frac{(2j+4)!}{4(j+2)!}+\frac{(2j+2)!}{(j+1)!}+3\frac{(2j)!}{j!}\right)(b_{j+2}+b_{j+1})\right\}. \tag{A.107}$$

Using the definition of the b_j, Eq. (A.103), one notices that all terms in the curly bracket have the common factor $(-4)^{-j}[(2j)!]^2/(j!)^3$. Writing this factor in front of the bracket it is then easy to verify that the remaining terms in the bracket simplify to the term $18j/(1-2j)$. This implies that the term for $j=0$ in the sum vanishes, and combining the results one finally gets

$$\mathcal{L}(\alpha) = -\frac{12\pi\sqrt{2\pi}C}{\alpha}\sum_{j=1}^{\infty}\left[\frac{(2j)!}{j!}\right]^2\frac{(-\alpha)^j}{(j-1)!(1-2j)4^j}. \tag{A.108}$$

We compare this result to the series expansion of the Laplace transform of $K_1(\sqrt{x})$, Eq. (A.86). To that end we use the following formula for the asymptotic expansion of $K_\nu(1/z)$ (see e.g. 8.451.6 in [59])

$$K_\nu\left(\frac{1}{z}\right) = \sqrt{\frac{\pi z}{2\pi}}e^{-1/z}\sum_{j=0}^{\infty}\frac{\Gamma(1/2+\nu+j)}{j!\Gamma(1/2+\nu-j)}\left(\frac{z}{2}\right)^j. \tag{A.109}$$

For $\nu=1$ we use the recurrence relation $x\Gamma(x)=\Gamma(x+1)$ to express the ratio of the two Gamma-functions as $\frac{1+2j}{1-2j}\frac{\Gamma(1/2+j)}{\Gamma(1/2-j)}$. Furthermore one can express this ratio via factorials since j is an integer, $\frac{\Gamma(1/2+j)}{\Gamma(1/2-j)} = \left[\frac{(2j)!}{j!}\right]^2\frac{(-1)^j}{4^{2j}}$. Hence Eq. (A.86) can be represented by the series

$$\frac{1}{8}\sqrt{\frac{\pi}{\alpha^3}}\exp\left(\frac{1}{8\alpha}\right)\left[K_1\left(\frac{1}{8\alpha}\right)-K_0\left(\frac{1}{8\alpha}\right)\right] \\ = \frac{\pi}{4\alpha}\sum_{j=0}^{\infty}\left[\frac{(2j)!}{j!}\right]^2\frac{(-1)^j}{j!4^{2j}}(4\alpha)^j\left[\frac{1+2j}{1-2j}-1\right] = \frac{\pi}{\alpha}\sum_{j=1}^{\infty}\left[\frac{(2j)!}{j!}\right]^2\frac{(-\alpha)^j}{(j-1)!4^j}. \tag{A.110}$$

This is precisely the series expansion of $\mathcal{L}(\alpha)$ in Eq. (A.108) for the choice $C = -\frac{1}{12\sqrt{2\pi}}$. This shows that the Laplace transform of the right-hand side of Eq. (A.85)

(with $x \to \sqrt{x}$) is equal to the Laplace transform of $K_1(\sqrt{x})$. Taking the inverse Laplace transform of both sides finally proves that the statement (A.85) is true which furthermore implies that the correlation function $\mathcal{R}(x)$, Eq. (A.81), for $J = 1$ and $x > 0$ is proportional to $xK_0(x) + \int_x^\infty \mathrm{d}y\, K_0(y)$. The result for arbitrary x and J can be obtained by replacing $x \to |x|/J$ as discussed in the paragraph after Eq. (A.82), and the proportionality constant can be obtained by considering the limit $x \to 0$. In this limit Eq. (A.81) becomes unity and

$$\lim_{x\to 0} \frac{|x|}{J} K_0\left(\frac{|x|}{J}\right) + \int_{|x|/J}^\infty \mathrm{d}y\, K_0(y) = \int_0^\infty \mathrm{d}y\, K_0(y) = \frac{\pi}{2}, \tag{A.111}$$

such that the proportionality constant is $2/\pi$ and the final solution of the correlation function is given by Eq. (A.82).

A.8. Consistency Between (4.195) and Brouwer's Conjecture

This appendix is taken from [14]. We show that the characteristic function of the probability density $\mathcal{P}(K)$ in the case $M = 2$ given in Eq. (4.195) is fully consistent with the claim that $\mathcal{P}(K) \propto \det[1 + K^2]^{-3/2}$. For the particular choice $\gamma_1 x_1 = -\gamma_2 x_2 \equiv \gamma x$ the expression Eq. (4.195) is equivalent to Eq. (4.188) (for brevity we choose $\gamma = 1$). Our task then amounts to demonstrating that

$$\int \mathrm{d}K\, \exp\left(\frac{i}{2}\,\mathrm{Tr}\, KX\right) \det[1 + K^2]^{-3/2} \propto xK_0(x) + \int_x^\infty \mathrm{d}y\, K_0(y), \tag{A.112}$$

where X can be chosen diagonal, $X = \mathrm{diag}(x, -x)$. Since K is symmetric we can diagonalise it by an orthogonal transformation, $K = O\,\mathrm{diag}(k_1, k_2)O^T$. Choosing for O the standard parametrization of a 2×2 orthogonal matrix, the left-hand side of Eq. (A.112) then simplifies to

$$\int_{-\infty}^\infty \mathrm{d}k_1 \int_{-\infty}^\infty \mathrm{d}k_2\, \frac{|k_1 - k_2|}{(1 + k_1^2)^{3/2}(1 + k_2^2)^{3/2}} \int_0^\pi \mathrm{d}\phi\, e^{\frac{i}{2}x(k_1 - k_2)\cos(2\phi)}. \tag{A.113}$$

The integral over the angle yields the Bessel function $J_0(\frac{(k_1-k_2)x}{2})$, which can also be written in the form $\int_0^\pi d\phi\, e^{\frac{i}{2}x(k_1-k_2)\sin(2\phi)}$. Now note that $\frac{1}{2}(k_1-k_2)\sin(2\phi) \equiv -K_{12}$, which allows to present Eq. (A.113) in the form

$$\int \mathrm{d}K\, \exp(-ixK_{12}) \det[1+K^2]^{-3/2}. \tag{A.114}$$

This is precisely the Fourier transform of $\mathcal{P}(K_{12})$, and in Section 5.2.1 we show that it is proportional to $xK_0(x) + \int_x^\infty dy K_0(y)$. This shows the validity of the claim (A.112).

A.9. Derivation of the Level Density at the Edge for the GUE

In this appendix we derive the level density at the spectral edge for the GUE. At the same time this can be seen as a verification of the characteristic function of the GUE edge distribution for a diagonal K-matrix element.

As explained in Section 2.2.4 the level density $\rho(E)$ can be obtained from a generating function, see Eq. (2.67), which we write in the form

$$\rho(E) = \frac{1}{\pi N^{1/3}} \mathrm{Re}\left[\frac{\mathrm{d}}{\mathrm{d}x} Z(E^-,x)\Big|_{x=0}\right], \quad Z(E^-,x) = \left\langle \frac{\det(E^- - H)}{\det(E^- + ixN^{-2/3} - H)} \right\rangle_H, \tag{A.115}$$

where $E^- = E - i\epsilon$ has a small negative imaginary part, the limit $\epsilon \to 0$ is implied. Compared to (2.67) we changed $s = -ixN^{-2/3}$. Since the whole imaginary part of the denominator should be negative (to ensure the correct sign of ρ) we require $x \leq 0$. Now notice that the form of the generating function is the same as the characteristic function of a diagonal entry of the K-matrix, compare e.g. with (3.7) or (5.71). This suggests one can compute the level density at the edge from the result we obtained for the characteristic function at the edge, given in Eqs. (5.72a,5.72b), i.e.

$$Z(2+\xi N^{-2/3}, x \leq 0) \approx ie^{-iN^{1/3}x}[\mathrm{Ai}(\xi)\,\alpha'(\xi,x) - \mathrm{Ai}'(\xi)\,\alpha(\xi,x)], \tag{A.116a}$$

$$\alpha(\xi,x) = \int_0^\infty \mathrm{d}\tau\, e^{-i(\tau\xi+\tau^3/3)+x\tau} - i\int_0^\infty \mathrm{d}\tau\, e^{\tau\xi-\tau^3/3+ix\tau}, \tag{A.116b}$$

where the limit $\epsilon \to 0$ has already been performed. The dashes denote the derivative

with respect to ξ. To obtain the level density we need to take the derivative of Z with respect to x at $x = 0$ and take the real part of the result. This yields

$$\begin{aligned}\rho(2+\xi N^{-2/3}) = &-\frac{1}{\pi}[\mathrm{Ai}(\xi)\ \mathrm{Re}\,\alpha'(\xi,0) - \mathrm{Ai}'(\xi)\ \mathrm{Re}\,\alpha(\xi,0)] \\ &-\frac{1}{\pi N^{1/3}}[\mathrm{Ai}(\xi)\ \mathrm{Im}\,\tfrac{\mathrm{d}}{\mathrm{d}x}\alpha'(\xi,x)|_{x=0} - \mathrm{Ai}'(\xi)\ \mathrm{Im}\,\tfrac{\mathrm{d}}{\mathrm{d}x}\alpha(\xi,x)|_{x=0}].\end{aligned} \tag{A.117}$$

Next we need to compute the α-related terms in the above equation. We start with

$$\begin{aligned}\mathrm{Re}[\alpha(\xi,0)] &= \mathrm{Re}\left[\int_0^\infty \mathrm{d}\tau\, e^{i(\tau\xi+\tau^3/3)} + i\int_0^\infty \mathrm{d}\tau\, e^{\tau\xi-\tau^3/3}\right] \\ &= \int_0^\infty \mathrm{d}\tau\ \cos(\tau\xi+\tau^3/3) = \pi\,\mathrm{Ai}(\xi),\end{aligned} \tag{A.118}$$

where we used that the integral in the second line is an integral representation of the Airy function $\mathrm{Ai}(\xi)$. This result further implies $\mathrm{Re}[\alpha'(\xi,0)] = \pi\,\mathrm{Ai}'(\xi)$. On the other hand one has

$$\begin{aligned}\mathrm{Im}[\tfrac{\mathrm{d}}{\mathrm{d}x}\alpha(\xi,x)|_{x=0}] &= \mathrm{Im}\left[-\int_0^\infty \mathrm{d}\tau\,\tau e^{i(\tau\xi+\tau^3/3)} - \int_0^\infty \mathrm{d}\tau\,\tau e^{\tau\xi-\tau^3/3}\right] \\ &= -\int_0^\infty \mathrm{d}\tau\,\tau\sin(\tau\xi+\tau^3/3) = \pi\,\mathrm{Ai}'(\xi),\end{aligned} \tag{A.119}$$

which further implies $\mathrm{Im}[\frac{d}{dx}\alpha'(\xi,x)|_{x=0}] = \pi\,\mathrm{Ai}''(\xi)$. Substituting these expressions back into Eq. (A.117) we observe that its first line vanishes, and the second line yields the desired result

$$\rho(2+\xi N^{-2/3}) = \frac{1}{N^{1/3}}[\mathrm{Ai}'(\xi)^2 - \mathrm{Ai}(\xi)\ \mathrm{Ai}''(\xi)]. \tag{A.120}$$

This is precisely the level density at the edge of the GUE-spectrum as given in Eq. (1.11).

A.10. Limit of the Edge Distribution

In Section 5.5 the probability distribution of a diagonal K-matrix element at the edge $E = 2+\xi N^{-2/3}$ of the GUE-spectrum was calculated, see Eq. (5.78). In this appendix it is shown that in the bulk limit $\xi \to -\infty$ the Cauchy distribution is recovered.

The distribution (5.78) comprises two terms, $\mathcal{P}_\xi(\kappa) = \mathcal{P}_{\xi,1}(\kappa) + \mathcal{P}_{\xi,2}(\kappa)$, we start with

$$\mathcal{P}_{\xi,1}(\kappa) = \frac{1}{\pi} \operatorname{Re}\left\{ \int_0^\infty \mathrm{d}\tau \, \exp\left[i\left(\xi\tau + \frac{\tau^3}{3}\right)\right] \frac{\operatorname{Ai}(\xi)\tau + i\operatorname{Ai}'(\xi)}{\tau + i\kappa} \right\}. \tag{A.121}$$

We rewrite this term in the following way: First introduce the integral

$$I(\xi) = \int_0^\infty \mathrm{d}\tau \, \frac{\exp\left[i(\tau\xi + \tau^3/3)\right]}{\tau + i\kappa}. \tag{A.122}$$

Then the term (A.121) can be written in terms of I as

$$\mathcal{P}_{\xi,1}(\kappa) = \frac{1}{\pi} \operatorname{Im}\left[\operatorname{Ai}(\xi) I'(\xi) - \operatorname{Ai}'(\xi) I(\xi)\right]. \tag{A.123}$$

This means we need to determine the limit of large negative ξ for $\operatorname{Ai}(\xi)$ and $\operatorname{Im} I(\xi)$ and their derivatives with respect to ξ.

We start the analysis with the Airy function. We use its integral representation

$$\operatorname{Ai}(-\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t \, e^{-\frac{i}{3}t^3 + i\xi t} = \frac{\sqrt{\xi}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t \, e^{-i\xi^{3/2}(\frac{1}{3}t^3 - t)}, \tag{A.124}$$

where we substituted $t \to \sqrt{\xi}t$. The exponent is purely imaginary and its derivative vanishes at ± 1. This makes the large-ξ analysis particularly easy since there is no need to deform any contours and one can follow the standard procedure expanding the exponent to second order around $+1$ and -1, respectively, performing the Gaussian integrals and adding both contributions together. As they are complex conjugates of each other we may combine them to a trigonometric function and the final result is

$$\operatorname{Ai}(-\xi) \approx \frac{1}{\sqrt{\pi}\xi^{1/4}} \cos\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right), \quad (\xi \to \infty). \tag{A.125}$$

The derivative of $\mathrm{Ai}(-\xi)$ can now be obtained in the same manner, or simply by taking the derivative of (A.125) w.r.t. $-\xi$ (only keeping the highest order term). Either way yields

$$\mathrm{Ai}'(-\xi) \approx \frac{\xi^{1/4}}{\sqrt{\pi}} \sin\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right), \quad (\xi \to \infty). \tag{A.126}$$

Next we need to perform the integral $I(-\xi)$ given in Eq. (A.122), in the limit of large ξ. Substituting $\tau \to \sqrt{\xi}\tau$ we can compute its large-N limit in the same fashion as for the Airy function. The saddle points are again given by ± 1, but only the point at $+1$ contributes due to the integration range. The imaginary part of the solution then takes the form

$$\begin{aligned}\mathrm{Im}[I(-\xi)] &\approx \sqrt{\pi}\,\xi^{-1/4}\,\mathrm{Im}\left\{\frac{\exp\left(-\frac{2i}{3}\xi^{3/2} + \frac{i\pi}{4}\right)}{\sqrt{\xi} + i\kappa}\right\} \\ &= -\frac{\sqrt{\pi}}{\xi + \kappa^2}\left[\kappa\,\xi^{-1/4}\cos\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right) + \xi^{1/4}\sin\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right)\right].\end{aligned} \tag{A.127}$$

The derivative of the above equation with respect to $-\xi$, keeping only the highest order term, is consequently given by

$$\begin{aligned}\mathrm{Im}[I'(-\xi)] &\approx -\sqrt{\pi}\,\xi^{1/4}\,\mathrm{Im}\left\{\frac{\exp\left(-\frac{2i}{3}\xi^{3/2} + \frac{i\pi}{4}\right)}{\kappa - i\sqrt{\xi}}\right\} \\ &= -\frac{\sqrt{\pi}}{\xi + \kappa^2}\left[\kappa\,\xi^{1/4}\sin\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right) - \xi^{3/4}\cos\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right)\right].\end{aligned} \tag{A.128}$$

After substituting our findings into Eq. (A.123) and simplifying the expression, all the sines and cosines vanish and we are left with

$$\mathcal{P}_{\xi,1}(\kappa) \approx \frac{\sqrt{\xi}/\pi}{\xi + \kappa^2}. \tag{A.129}$$

The second term in (5.78) is given by

$$\mathcal{P}_{\xi,2}(\kappa) = \Theta(\kappa)\left[\mathrm{Ai}(\xi)\kappa - \mathrm{Ai}'(\xi)\right]\exp\left(\kappa\xi - \frac{1}{3}\kappa^3\right). \tag{A.130}$$

We now replace the Airy function and its derivative by their appropriate asymptotics. This yields

$$\mathcal{P}_{-\xi,2}(\kappa) \approx \frac{\Theta(\kappa)}{\sqrt{\pi}} \exp\left(-\kappa\xi - \frac{1}{3}\kappa^3\right) \times \left[\frac{1}{\xi^{1/4}} \cos\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right)\kappa - \xi^{1/4} \sin\left(\frac{2}{3}\xi^{3/2} - \frac{\pi}{4}\right)\right], \quad (\xi \to \infty). \tag{A.131}$$

For $\kappa \leq 0$ this term vanishes due to the Heaviside-Theta function. For $\kappa > 0$, on the other hand, the term is exponentially small in ξ. Hence $\mathcal{P}_{\xi,2}(\kappa)$ is always negligible compared to $\mathcal{P}_{\xi,1}(\kappa)$ for large negative ξ, and the final result is already given by (A.129) which is a Cauchy distribution as expected.

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