Queen Mary, University of London

Singular Chains on Topological Stacks

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Statement of Originality

I, Thomas William Coyne, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications:

Notation from Chapter 2 and content from Chapter 3 appears in [7]. Both Chapter 4 and Chapter 5 are due to appear as a part of a preprint [8].

The paper [7] and the preprint [8] are joint work with my supervisor, Behrang Noohi.

Abstract

The main objective of this thesis is to introduce the concept of 'singular chains on topological stacks'. The idea is to functorially associate to a topological stack, a simplicial set which captures its homotopy type.

This will allow us to compute the singular homology and cohomology of topological stacks. Noohi and Behrend have given several approaches to this problem, however all of these approaches rely on the choice of an atlas for a topological stack. We shall show that our new approach agrees with the existing approaches but has the advantage of being functorial.

Noohi has introduced weak equivalences and fibrations of topological stacks. In analogy to the singular chains functor for topological spaces, we shall show that the functor Sing preserves the weak equivalences and fibrations defined by Noohi under certain 'fibrancy conditions'.

In the second part, we shall push the analogy with the topological singular chains further by considering the adjunction with the geometric realization and the associated counit. We develop a corresponding (but weaker) notion for topological stacks.

We shall give a method for computing the homotopy type of a stack which has a groupoid presentation. Finally, we shall compute the homotopy type of certain mapping stacks and develop the totalization of a cosimplicial topological stack. We shall indicate how this (using the approach of Cohen and Jones) gives a method for computing the string topology of a topological stack.

Contents

Acknowledgments

I owe a huge debt of gratitude to my supervisor, Behrang Noohi, for his patience and the many hours he has spent helping me to become a better mathematician. His help and guidance have been invaluable both for this thesis and for my own development.

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Finally, I would like to thank my close friends and family. They have helped form the person I am today, given encouragement when necessary and supported me through good times and bad.

Chapter 1

Introduction

Topological stacks are a generalization of topological spaces. They can be seen as a categorification of the notion of a presheaf of sets to a presheaf of groupoids. Examples of topological stacks include topological spaces, orbifolds, foliated manifolds and principal G-bundles amongst many others. Topological stacks are better behaved than topological spaces when taking the quotient of a topological space by a group action.

The main goal of this thesis is to define a functor Sing : $\mathsf{topStack} \to \mathsf{sSet}$ from the category of topological stacks to simplicial sets. We aim to define this functor to have similar properties to the usual singular chains functor Sing : $\mathsf{Top} \to \mathsf{sSet}$.

Some of the key properties that we would like our functor Sing to possess are:

- it generalizes the usual singular chains Sing : $\mathsf{Top} \to \mathsf{sSet}$;
- it preserves weak equivalences;
- it preserves Serre fibrations;
- we can define an analogue of the counit map .

After we have defined Sing and shown that it generalizes the classical definition of Sing, we will show that under certain technical assumptions, it does indeed preserve weak equivalences and Serre fibrations (see Theorem 3.5.2 and Theorem 3.4.8).

This means that we can use the functor Sing to compute the homotopy type of a topological stack. This has been considered before ([14, 1, 27]) but our approach has the advantage of being functorial since it does not rely on a choice of groupoid presentation or a choice of atlas.

In another analogy with the adjunction between Sing and the geometric realization, we shall develop the idea of a 'counit map' for topological stacks (for topological spaces this is the natural map $|\operatorname{Sing}(X)| \to X$) and show that it is a weak equivalence. Since the functor Sing is not part of an adjunction, this 'counit map' is not actually a counit. We shall show that there is a map from the geometric realization of the singular chains to the topological stack and that this map is a weak equivalence.

There is a technical aspect that we must consider in order to make the functor Sing well behaved which is related to coherence. Topological stacks naturally form a 2-category whilst we consider the category of simplicial sets as a 1-category. This means that we will have to impose coherence conditions on our topological stacks and morphisms of topological stacks since we require diagrams of topological stacks to commute up to isomorphism whereas in the category of simplicial sets, we require diagrams to commute on the nose.

In order to overcome the issue of coherence, we will firstly define topological stacks in a slightly non-standard way (we shall define them as strict presheaves of groupoids which satisfy descent) and we will use the theory of model categories to ensure the appropriate diagrams strictly commute.

In the case that our topological stack has a groupoid presentation, we shall consider the relationship between the singular chains and the simplicial set generated from the topological groupoid presenting it. We shall show that there is a weak equivalence between them. This will give us a method to compute the homotopy type of a topological stack.

We shall also consider the singular chains of mapping stacks. As a tool for computing the homotopy type of a mapping stack, we shall consider the totalization of a cosimplicial topological stack and develop some basic properties. For topological spaces, totalization commutes with taking the singular chains. Whilst this is not true for topological stacks, we shall give conditions under which totalization commutes with taking the singular chains up to weak equivalence.

We shall show in Section 5.3.2 that all of the work in the second part of the thesis can be used to consider the string topology of topological stacks (using ideas from [11] or [6]). We shall not pursue these ideas in this thesis, but they are due to appear in [8].

Outline

We shall give an outline of this thesis.

In Chapter 2, we shall recall the background material that we will use in this thesis. We will define topological stacks both using our non-standard definition and a more standard definition using categories fibered in groupoids. We shall also introduce the material that we require on model categories.

Chapter 3 will be the technical heart of this thesis. Here we shall define the functor Sing and show that it generalizes the usual singular chains functor. We shall also introduce the notion of a 'Reedy topological stack' which is a fibrancy condition that is required for Sing to preserve weak equivalences and Serre fibrations.

At the end of Chapter 3, we shall show that the functor Sing can also be defined for categories fibered in groupoids.

We recall some results introduced by Shulman [31] in Chapter 4. In this chapter, we shall define the totalization of a cosimplicial topological stack. The main result in this section is Theorem 4.1.30 which says that under certain fibrancy conditions, the functors Sing and the totalization commute up to weak equivalence.

In Chapter 5, we will explore the comparison between the classical Sing functor and our extended definition. In particular, we shall consider the classical adjunction between Sing and the geometric realization $\mathrm{Sing}(-) \vdash |-|$.

We shall show that under certain conditions, we have a functor that is analogous to a 'realization functor'. This does not give us a 2-adjunction involving Sing, but it does share many properties. We shall also consider a map $|\mathrm{Sing}(\mathcal{X})| \to \mathcal{X}$ for a topological stack $\mathcal{X}$, which has similar properties to the classical counit map of the above classical adjunction.

Here, we give a method for computing the homotopy type of a topological stack which has a groupoid presentation. We extend these methods to compute the homotopy type of a mapping stack of the form $\mathrm{Map}_{\mathsf{topStack}}(|K|, \mathcal{X})$ where K is a simplicial set and $\mathcal{X}$ is a topological stack.

Finally we give an indication of how these results allow us to extend the results of [6] and [11] to topological stacks.

Chapter 2

Background

2.1 Basic Notation

For two categories C and D, let $[\mathsf{C},\mathsf{D}]$ denote the category of covariant functors from C to D. The category of groupoids will be denoted by Gpd.

We shall let Top denote the category of compactly generated Hausdorff topological spaces. We will denote the topological unit interval by I or $[0,1]$.

Category theoretic issues

We can overcome any set theoretic issues in this thesis by fixing a Grothendieck universe. The main reason that we consider this concept is in order to be able to treat Top as a small category. We shall point out when we are required to fix a universe, but since this is merely a technical point, we shall not emphasize it within this thesis.

Notation for equivalences

We shall use $\cong$ to denote an isomorphism. The symbol $\simeq$ will be used for weak equivalences whilst $\sim$ will be used for equivalences of categories.

2.1.1 Simplicial sets

The category of finite ordinal numbers with order preserving maps between them is denoted by $\mathbf{\Delta}$. The simplicial n-simplex is denoted by $\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-,[n])$. The topological n-simplex is denoted by

$$|\Delta^n| = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}$$

We denote the cosimplicial object $n \mapsto |\Delta^n|$ in Top by $|\Delta^\bullet|$. The k^{th} horn in Δ^n, namely, the sub-simplicial set of Δ^n generated by the i^{th} faces of the n-cell Δ^n, $i \in \{0, 1, \cdots, \hat{k}, \cdots, n\}$, is denoted by Λ^n_k. When talking about homotopies between maps we often use the notation $[0,1]$ instead of $|\Delta^1|$.

The bisimplex $\Delta^{m,n}$ is the bisimplicial set $\Delta^{m,n} : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ represented by $([m],[n]) \in \mathbf{\Delta} \times \mathbf{\Delta}$. That is, $\Delta^{m,n} := \mathrm{Hom}_{\mathbf{\Delta}\times\mathbf{\Delta}}(-,([m],[n])) = \Delta^m \boxtimes \Delta^n$ (see Section 3.4.3).

For a simplicial set $X \in \mathsf{sSet}$, we use the notation $\tilde{X} \in \mathsf{pshGpd}$ for the left Kan extension of X along $\mathbf{\Delta} \to \mathsf{Top}$ (more details can be found in Section 3.3.1).

2.2 Presheaves

2.2.1 Presheaves of sets

We shall denote the functor category $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ by $\mathsf{pshSet}(\mathsf{C})$ which we refer to as the category of presheaves of sets over C.

Definition 2.2.1. Let C be a category. For every $X \in \mathsf{C}$, there is a *representable presheaf* X which is defined as

$$\begin{aligned} X : \mathsf{C}^{\mathrm{op}} &\to \mathsf{Set} \\ c &\mapsto \mathrm{Hom}_{\mathsf{C}}(c, X). \end{aligned}$$

Note that we refer to both the underlying object and the presheaf that it represents using the same notation. This is for clarity of exposition, since it is clear from the context as to which object we are considering. Further justification for our choice of notation is provided by the following lemma.

Yoneda's Lemma

The following result is elementary, but very important.

Lemma 2.2.2. *Let C be a category. Given $F \in [\mathsf{C}^{op}, \mathsf{Set}]$ and $X \in \mathsf{C}$, there is a bijection of sets*

$$\begin{aligned} \mathrm{Hom}_{\mathsf{pshSet}(\mathsf{C})}(X, F) &\cong F(X) \\ \alpha &\mapsto \alpha(\mathrm{id}_X). \end{aligned}$$

Proof. See page 61 of [22]. □

Yoneda's embedding

We can use Yoneda's Lemma to show that we may 'extend' the category C to the category of presheaves over C. We note in passing that the category $\mathsf{pshSet}(\mathsf{C})$ is both complete and cocomplete.

Lemma 2.2.3. *Let C be a category. There is a fully faithful functor $\mathsf{C} \hookrightarrow \mathsf{pshSet}(\mathsf{C})$ sending $X \mapsto \mathrm{Hom}_{\mathsf{C}}(-, X)$.*

Proof. This follows from Lemma 2.2.2. □

2.2.2 Strict presheaves of groupoids

In addition to considering presheaves of sets, we will also consider presheaves of other target categories. In particular, we shall study presheaves of groupoids. This is a 'categorification' of the notion of a presheaf of sets. Presheaves of groupoids naturally have a 2-categorical structure which is inherited from the 2-categorical structure on Gpd, the 2-category of groupoids. Instead of a set of morphisms between presheaves, we will now have a category (in this case a groupoid) of morphisms between them.

We will consider strict presheaves of groupoids for the majority of this thesis for reasons that can be found in Remark 3.1.1.

Definition 2.2.4. A *strict presheaf of groupoids* $\mathcal{F}$ over a category C is a functor $\mathcal{F} : \mathsf{C}^{\mathrm{op}} \to \mathsf{Gpd}$. Explicitly, a strict presheaf of groupoids is given by the following data:

- for each $c \in C$, a groupoid $\mathcal{F}(c)$;
- for each morphism $f : c \to d$ in C, a functor $\mathcal{F}(f) : \mathcal{F}(d) \to \mathcal{F}(c)$;
- for each composable pair of morphisms $a \xrightarrow{f} b \xrightarrow{g} c$, we have an equality $\mathcal{F}(f) \circ \mathcal{F}(g) = \mathcal{F}(g \circ f)$;
- for each $c \in \mathsf{C}$, we have an equality $\mathcal{F}(\mathrm{id}_c) = \mathrm{id}_{\mathcal{F}(c)}$.

Strict presheaves of groupoids form a strict 2-category. Objects are strict presheaves of groupoids over C, the 1-morphisms are given by natural transformations and the 2-morphisms are given by modifications. Note that both the natural transformations and the modifications are associative up to equality (not just up to a natural 2-isomorphism). We denote this 2-category by $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]$.

Definition 2.2.5. Let f be a morphism of strict presheaves of groupoids over C. The map $f : \mathcal{X} \to \mathcal{Y}$ is an *equivalence* if $f(c) : \mathcal{X}(c) \to \mathcal{Y}(c)$ is an equivalence of groupoids for each $c \in \mathsf{C}$.

There is a version of Lemma 2.2.2 for strict presheaves of groupoids.

Lemma 2.2.6. *Let $\mathcal{X} : \mathsf{C}^{op} \to \mathsf{Gpd}$ be a strict presheaf of groupoids. Given $C \in \mathsf{C}$, there is a natural isomorphism of groupoids (not just an equivalence)*

$$\mathrm{Hom}_{\mathsf{pshGpd}}(C, \mathcal{X}) \xrightarrow{\sim} \mathcal{X}(C).$$

Proof. View $\mathcal{X}$ as a groupoid object in the category of presheaves of sets. Then the lemma follows from the usual Yoneda's lemma (Lemma 2.2.2). □

Remark 2.2.7. We shall often drop the prefix 'strict'. If we say a presheaf of groupoids, then we mean a *strict* presheaf of groupoid. We will only use the word strict for emphasis or clarity.

Remark 2.2.8. We make the observation here that the functor $\mathsf{Set} \hookrightarrow \mathsf{Gpd}$, induces a fully faithful functor

$$\mathsf{pshSet}(\mathsf{C}) \hookrightarrow [C^{\mathrm{op}}, \mathsf{Gpd}].$$

For the majority of this thesis, we will opt to define stacks as strict presheaves of groupoids which satisfy a 'descent condition' (this will be defined precisely in Definition 2.3.15). The reason that we take this approach is explained in Remark 3.1.1.

2.2.3 Categories fibered in groupoids

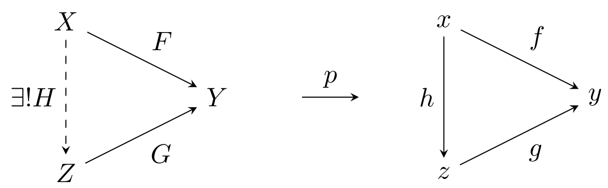
A common way to define stacks is with categories fibered in groupoids. As noted above, we shall not focus on categories fibered in groupoids in this thesis, but we introduce the concept here for completeness. In Section 3.6, we shall show that singular chain can also be defined for categories fibered in groupoids.

In this section we shall introduce categories fibered in groupoids and how they relate to strict presheaves of groupoids.

Definition 2.2.9. A category fibered in groupoids over C is a pair $(\mathcal{X}, p)$ where $\mathcal{X}$ is a category and p is a functor $p : \mathcal{X} \to \mathsf{C}$ with the properties:

1. For each $f : x \to y$ in C and for each $Y \in \mathcal{X}$ such that $p(Y) = y$, there exists a map $F \in \mathcal{X}$ such that $p(F) = f$;

2. Given a commutative triangle $g \circ h = f$ in $\mathrm{Mor}(\mathsf{C})$ and $F, G \in \mathrm{Mor}(\mathcal{X})$ such that $p(F) = f$ and $p(G) = g$, there exists a unique $H \in \mathrm{Mor}(\mathcal{X})$ such that $p(H) = h$ and $G \circ H = F$. We can phrase this in terms of the following diagram:



We note that the definition implies that the subcategory $p^{-1}(x) \subset \mathcal{X}$ which has objects $X \in \mathcal{X}$ such that $p(X) = x$ and morphisms $F : X \to X'$ such that $p(F) = \mathrm{id}_x$ is a groupoid. We can say that the 'fibers' of the functor p are groupoids. We shall denote the fiber over an object $U \in \mathsf{C}$ by $\mathcal{X}(U)$.

Categories fibered in groupoids form a 2-category which we describe below.

Definition 2.2.10. The 2-category of categories fibered in groupoids over C is denoted by CFG_C. It is defined as follows:

- Objects are categories fibered in groupoids $(\mathcal{X}, p)$;

- 1-morphisms between $(\mathcal{X}, p_{\mathcal{X}})$ and $(\mathcal{Y}, p_{\mathcal{Y}})$ are functors $F : \mathcal{X} \to \mathcal{Y}$ such that $p_{\mathcal{X}} = p_{\mathcal{Y}} \circ F$;

- 2-morphisms are natural transformations $\varphi : F \Rightarrow G$ such that $p_{\mathcal{Y}} \cdot \varphi$ is the identity transformation from $p_{\mathcal{X}}$ to itself;

2.2.4 Grothendieck construction

The 2-categories CFG_C and $[C^{\mathrm{op}}, \mathsf{Gpd}]$ are related by the Grothendieck construction.

Definition 2.2.11. The Grothendieck construction is a functor of 2-categories

$$\int_{\mathsf{C}} : [C^{\mathrm{op}}, \mathsf{Gpd}] \to \mathsf{CFG}_C$$

sending F to $(\int_{\mathsf{C}} F, p)$. The objects of $\int_{\mathsf{C}} F$ are pairs (x, X) where $x \in \mathsf{C}$ and $X \in F(x)$. The morphisms of $\int_{\mathsf{C}} F$ between (x, X) and (y, Y) are pairs (f, ϕ) where $f : x \to y$ and $\phi : X \to F(f)(Y)$. The functor $p : \int_{\mathsf{C}} F \to \mathsf{C}$ sends (x, X) to x and (f, ϕ) to f.

Proposition 2.2.12. *The Grothendieck construction is an equivalence of 2-categories.*

Proof. See (Theorem B.4, [16]). □

2.2.5 Yoneda's Lemma for categories fibered in groupoids

We have a version of Yoneda's lemma (Lemma 2.2.2) for categories fibered in groupoids.

Lemma 2.2.13. *Let $\mathcal{F} \in \mathsf{CFG}_C$ be a category fibered in groupoids and let $X \in \mathsf{C}$. Then there is a natural equivalence of groupoids*

$$\mathrm{Hom}_{\mathsf{CFG}_C}(X, \mathcal{F}) \xrightarrow{\simeq} \mathcal{F}(X)$$
$$\alpha \mapsto \alpha(\mathrm{id}_X)$$

Proof. This appears in Section 3.6.2 of [34]. □

Remark 2.2.14. Given a category fibered in groupoids, there is a natural way to associate to it a strict presheaf of groupoids. More precisely, there is a *strictification functor* Strict : $\mathsf{CFG}_C \to \mathsf{pshGpd}(C)$.

The Yoneda Lemma for bicategories can be used to embed a bicategory into the strict 2-category $[\mathsf{C}^{\mathrm{op}}, \mathsf{Cat}]$. The strictification functor is then constructed by sending a category fibered in groupoids to its full image under the Yoneda embedding.

This justifies our use of strict presheaves throughout this thesis.

2.2.6 Fiber products

Consider the following diagram in the 2-category Gpd of groupoids:

$$\begin{array}{ccc} & & K \\ & & \downarrow{\scriptstyle p} \\ H & \xrightarrow[\mathrm{q}]{} & G \end{array}$$

The *2-fiber product* (or 2-categorical fiber product) is denoted by the groupoid $H \tilde{\times}_G K$. The objects of this groupoid are triples (x, y, φ) where $x \in H$, $y \in K$ are objects of H and K respectively and $\varphi : q(x) \to p(y)$ is a morphism in G. A morphism from (x, y, φ) to (x', y', φ') is a pair of morphisms $\alpha : x \to x'$ and $\beta : y \to y'$, in H and K respectively, such that $\varphi' \circ q(\alpha) = p(\beta) \circ \varphi$.

A *strict product* is denoted by the groupoid $H \times_G K$. The objects are pairs (x, y) where $x \in H$, $y \in K$ and $q(x) = p(y)$. Morphisms are pairs (f, g) of morphisms where $f : x \to x' \in H$ and $g : y \to y' \in K$ such that $p(f) = \mathrm{id}_{p(x)} = \mathrm{id}_{q(y)} = q(g)$.

There is a fully faithful functor

$$H \times_G K \to H \tilde{\times}_G K$$

from the strict fiber product to the 2-fiber product, sending a pair $(x, y) \in H \times_G K$ to the triple (x, y, id). The image consists of those triples (x, y, φ) with $\varphi = \mathrm{id}$. This map is sometimes an equivalence (Lemma 2.7.2) but not always.

Fiber products of presheaves of groupoids

The strict and 2-categorical product are defined object-wise for presheaves of groupoids, namely

$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})(c) = \mathcal{X}(c) \times_{\mathcal{Z}(c)} \mathcal{Y}(c), \quad \forall c \in \mathsf{C}$$

and

$$(\mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y})(c) := \mathcal{X}(c) \tilde{\times}_{\mathcal{Z}(c)} \mathcal{Y}(c), \quad \forall c \in \mathsf{C}.$$

2.3 Sheaves

2.3.1 Grothendieck topology

We want to be able to define the categorified version of a sheaf. In order to do this, we shall first define the notion of a Grothendieck topology and a site.

First, we shall introduce the concept of a sieve. Then we shall show how this fits in with the more common definition in terms of coverings.

Definition 2.3.1. Given an object $X \in \mathsf{C}$, a *sieve* S on X is a *subfunctor* of the representable functor $\mathrm{Hom}_{\mathsf{C}}(-, X)$.

Definition 2.3.2. If S is a sieve on $X \in \mathsf{C}$ and $f : Y \to X$, then the *pullback sieve* f^*S is defined as the fiber product

$$f^*S := S \times_{\mathrm{Hom}_{\mathsf{C}}(-,X)} \mathrm{Hom}_{\mathsf{C}}(-, Y).$$

We shall recall the definition of a Grothendieck topology ([23]; Definition 1, page 110).

Definition 2.3.3. A *Grothendieck topology* on a category C is a function $\mathcal{T}$ which for each $X \in \mathsf{C}$ assigns a collection of sieves, $\mathcal{T}(X)$, which satisfy the following axioms:

1. The maximal sieve is in $\mathcal{T}(X)$, i.e. $\mathrm{id}_X : \mathrm{Hom}_{\mathsf{C}}(-, X) \to \mathrm{Hom}_{\mathsf{C}}(-, X) \in \mathcal{T}(X)$;

2. If $S \in \mathcal{T}(X)$ and $f : Y \to X$, then $f^*S \in \mathcal{T}(Y)$;

3. Let $S \in \mathcal{T}(X)$ and let R be any sieve on X. If $h^*(R) \in \mathcal{T}(Y)$ for any $h : Y \to X$, then $R \in \mathcal{T}(X)$.

Definition 2.3.4. A site is a pair $(\mathsf{C}, \mathcal{T})$ where C is a category and $\mathcal{T}$ is a Grothendieck topology on C.

We shall state the definition of a site in terms of coverings in order to give a more intuitive definition which is closer to the ideas commonly used when defining sheaves.

Definition 2.3.5. Let C be a category. A *covering* of an object $X \in \mathsf{C}$ is a collection of morphisms $\{f_i : T_i \to X\}_{i \in I}$ indexed by a set I.

We recall from ([23]; Definition 2, page 111) the notion of a basis.

Definition 2.3.6. A *basis for a Grothendieck topology* on a category C is given by a function $\mathcal{B}$ which assigns for each $X \in \mathsf{C}$, a collection of coverings $\mathcal{B}(X)$ which obey the following axioms:

(i) If $f : T \to X$ is an isomorphism, then $\{f : T \to X\} \in \mathcal{B}(X)$;

(ii) If $\{f_i : T_i \to X\}_{i \in I} \in \mathcal{B}(X)$ is a covering and $g : S \to X$ is any map, then the pullbacks $T_i \times_X S$ exist for each $i \in I$ and $\{f_i \times_X g : T_i \times_X S \to X\}_{i \in I} \in \mathcal{B}(X)$ is a covering;

(iii) If $\{f_i : T_i \to X\}_{i \in I} \in \mathcal{B}(X)$ is a covering and for each T_i, $\{g_j : S_j \to T_i\}_{j \in J} \in \mathcal{B}(T_i)$ is a covering, then $\{f_i \circ g_j : S_j \to X\}_{i \in I, j \in J} \in \mathcal{B}(X)$ is a covering.

Remark 2.3.7. A basis for a Grothendieck topology is not a Grothendieck topology. It does however generate one by defining a sieve S to be a covering sieve $S \in \mathcal{T}(X)$ if and only if

$$\exists R \in \mathcal{B}(X), \quad R \subseteq S.$$

Example 2.3.8 (The global classical topology)**.** *Recall that* Top *is the category of Hausdorff topological spaces. We equip* Top *with the Grothendieck topology where a basis is given by:* $\{f_i : U_i \to X\}_I$ *is a covering if each* f_i *is an open embedding (i.e. an inclusion of an open subspace) and* $\cup_I f(U_i) \supseteq X$*. This is called either the* global classical topology *or the* open-cover topology*.*

2.3.2 Sheaves

Before defining stacks, we shall define sheaves of sets. Stacks will be a categorified version of sheaves.

Let $(\mathsf{C}, \mathcal{T})$ be a site and let $\mathcal{F}$ be a presheaf of sets on C (Section 2.2.1). If $\{U_i \to X\}_I$ is a covering of X, then we may form the diagram

$$\prod_I \mathcal{F}(U_i) \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \prod_{I \times I} \mathcal{F}(U_i \times_X U_j)$$

where $U_i \times_X U_j$ exists by Definition 2.3.6(ii) and the maps pr_1 and pr_2 are the projection maps $U_i \times_X U_j \to U_i$ and $U_i \times_X U_j \to U_j$ respectively.

There is also a natural map

$$\mathcal{F}(X) \to \prod_I \mathcal{F}(U_i)$$
$$s \mapsto (s|_{U_i})_{i \in I}.$$

Definition 2.3.9. Let $(\mathsf{C}, \mathcal{T})$ be a site and $\mathcal{F}$ a presheaf of sets on C. The presheaf $\mathcal{F}$ is a *sheaf* if for each $X \in \mathsf{C}$ and for each cover $\{U_i \to X\}_I$, the natural map

$$\mathcal{F}(X) \longrightarrow \lim \Big(\prod_I \mathcal{F}(U_i) \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \prod_{I \times I} \mathcal{F}(U_i \times_X U_j) \Big)$$

is a bijection.

Notation 2.3.10. The category of sheaves on C is denoted by $\mathsf{Sh}(C)$.

Remark 2.3.11. The sheaves on C are a full subcategory of presheaves on C,

$$\mathsf{Sh}(C) \hookrightarrow \mathsf{pshSet}(\mathsf{C}).$$

Proposition 2.3.12. *The open-cover topology on* Top *(which was described in Example 2.3.8) has the property that every representable presheaf is a sheaf. (If the representable objects are sheaves, we say the topology is subcanonical.)*

Proof. Any function of topological spaces is defined locally. This it is if $f_1 : U_1 \to X$ and $f_2 : U_2 \to X$ are continuous maps such that $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$ then there is a unique function $f : U_1 \cup U_2 \to X$ such that $f|_{U_i} = f_i$.

Also, if two continuous maps agree locally, then they agree globally. □

From Remark 2.3.11, we see that every sheaf is a presheaf but not every presheaf is a sheaf. However there is a functor that 'sheafifies' presheaves.

Proposition 2.3.13. *The functor* $\mathsf{Sh}(C) \hookrightarrow \mathsf{pshSet}(\mathsf{C})$ *described in Remark 2.3.11 admits a left adjoint called the sheafification functor.*

Proof. This appears in ([23]; Corollary 4). □

2.3.3 Stacks and descent

We shall categorify Definition 2.3.9 to define a gluing condition for strict presheaves of groupoids and for categories fibered in groupoids. We shall define this in terms of coverings.

Definition 2.3.14. Let C be a site and $\{U_i \to X\}$ be a cover of $X \in \mathsf{C}$. Let $\mathcal{X}$ be a presheaf of groupoids or a category fibered in groupoids. *Descent data for the object* X is given by the groupoid

$$\mathsf{Desc}(\mathcal{X}, X) = \mathrm{holim}\Big(\prod_I \mathcal{X}(U_i) \rightrightarrows \prod_{I \times I} \mathcal{X}(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{I \times I \times I} \mathcal{X}(U_{ijk}) \Big)$$

where $U_{ij} = U_i \times U_j$ and $U_{ijk} = U_i \times U_j \times U_k$.

Definition 2.3.15. By a *stack* over C with respect to the site $(\mathsf{C}, \mathcal{T})$, we mean a presheaf of groupoids (or a category fibered in groupoids) $\mathcal{X}$, which satisfies *descent*. This means that for each $X \in \mathsf{C}$ and for each covering $\{U_i \to X\}_{i \in I}$

$$\mathcal{X}(X) \xrightarrow{\sim} \mathrm{holim}\Big(\prod_I \mathcal{X}(U_i) \rightrightarrows \prod_{I \times I} \mathcal{X}(U_{ij}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{I \times I \times I} \mathcal{X}(U_{ijk}) \Big) = \mathsf{Desc}(\mathcal{X}, X)$$

is an equivalence of groupoids. Here $U_{ij} := U_i \times_X U_j$ and $U_{ijk} := U_i \times_X U_j \times_X U_k$.

Morphisms and 2-isomorphisms of stacks are the ones of the underlying presheaves of groupoids. Stacks form a full sub-2-category of $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]$. The 2-category of stacks over C will be denoted by $\mathsf{St}(C)$. The sub-2-category of categories fibered in groupoids which satisfy descent will be denoted by $\mathsf{St}_{\mathsf{CFG}}(\mathsf{C})$.

We can unpack this definition to give a more explicit description in terms of local covering data.

Let $\mathcal{X} : \mathsf{C}^{\mathrm{op}} \to \mathsf{Gpd}$ be a presheaf of groupoids (or a category fibered in groupoids) and let $(\mathsf{C}, \mathcal{T})$ be a site. For each $X \in \mathsf{C}$ and for each covering $\{U_i \to X\}$ assume that we have the data $\eta_i \in U_i$ along with isomorphisms $\varphi_{i,j} : \eta_i|_{U_{ij}} \to \eta_j|_{U_{ij}}$ such that $\varphi_{i,j} \circ \varphi_{j,k} = \varphi_{i,k}$. (Here we are using the notation that $U_{ij} = U_i \times_X U_j$ and $U_{ijk} = U_i \times_X U_j \times_X U_k$.)

Then $\mathcal{X}$ is a stack if it satisfies the following axioms (see Section 3, [20]):

1. (Glue objects) given the data $(\eta_i, \varphi_{i,j})$ described above, there exists $\eta \in \mathcal{X}(X)$ and isomorphisms $\lambda_i : \eta|_{U_i} \to \eta_i$ such that $\lambda_j|_{U_{ij}} = \varphi_{i,j} \circ \lambda_i|_{U_{ij}}$ for each i, j;

2. (Glue morphisms) given $\eta, \mu \in \mathcal{X}(X)$ and isomorphisms $\eta|_{U_i} \xrightarrow{\lambda_i} \mu|_{U_i}$ such that $\lambda_i|_{U_{ij}} = \lambda_j|_{U_{ij}}$, then there exists a unique isomorphism $\lambda : \eta \to \mu$ such that $\lambda|_{U_i} = \lambda_i$.

Remark 2.3.16. We note that we are not being precise with our restriction notation. Instead of restriction, we should pullback along the appropriate inclusion. For example, by $\lambda|_{U_i}$ we really mean the pullback of $\lambda \in \mathcal{X}(X)$ along $U_i \to X$ rather than restriction. The reason that we have chosen not to be explicit about this here is to increase the clarity of the exposition.

Proposition 2.3.17. *All sheaves of sets over* C *are stacks over* C.

Proof. See Proposition 4.9 in [34]. □

Remark 2.3.18. Stacks over C form a 2-category which is denoted $\mathsf{St}(C)$. This is a full sub-2-category of $\mathsf{pshGpd}(\mathsf{C})$.

2.3.4 Stackification

Analogously to Proposition 2.3.13, there is a stackification functor. The stackification functor is the 2-categorical left adjoint to the inclusion $\mathsf{St}_{\mathsf{CFG}}(\mathsf{C}) \hookrightarrow \mathsf{CFG}_{\mathsf{C}}$ (see Appendix A in [5]).

We recall from Chapter 3 of [20] that there is an analogue of sheafification for stacks which we shall outline briefly.

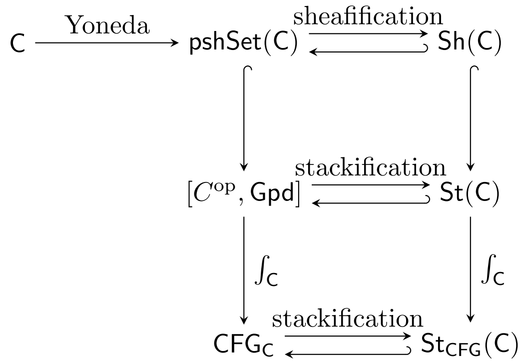
Proposition 2.3.19. *There exists a* stackification functor $a_J : \mathsf{CFG}_{\mathsf{C}} \to \mathsf{St}_{\mathsf{CFG}}(\mathsf{C})$ *which preserves finite limits. This fuctor is defined as follows: given a category fibered in groupoids* $\mathcal{X}$*, the category* $a_J(\mathcal{X})$ *has objects given by tripples* $(\{U_i \to X\}, f_i \in \mathcal{X}(U_i), \alpha_{i,j} : f_i \Rightarrow f_j)$ *where* $\{U_i \to X\}$ *is an open cover. Morphisms* $(\{U_i \to X\}, f_i \in \mathcal{X}(U_i), \alpha_{i,j}) \to (\{V_i \to Y\}, g_i \in \mathcal{X}(U_a), \beta_{a,b})$ *are given by* $\gamma_{i,a} : f_i \Rightarrow g_a$ *making the appropriate diagrams commute.*

Proof. This can be found in Chapter 3 of [20]. The stackification functor is defined by a two step process going from a category fibered in groupoids to an object defined in [20] called a 'pre-stack'. This is a category fibered in groupoids which satisfies the condition that for each cover $\{U_i \to X\}$, $\mathcal{X}(X) \to \mathsf{Desc}(\mathcal{X}, X)$ is fully faithful. The functor from a category fibered in groupoids to a pre-stack can be found in Remark 3.2.1(3) of [20]. Once we have formed a pre-stack, we apply Lemma 3.2 from [20] to get a stack. □

Remark 2.3.20. There is also a strict stackification functor $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}] \to \mathsf{St}(\mathsf{C})$.

2.3.5 Summary

We shall summarize the important relationships between the various categories that we have discussed so far.



Note that the functor $\int_{\mathsf{C}} : \mathsf{St}(\mathsf{C}) \to \mathsf{St}_{\mathsf{CFG}}(\mathsf{C})$ is defined to be the functor which renders the square commutative.

2.4 Topological stacks

From this point on, we shall fix $\mathsf{C} = \mathsf{Top}$, the category of Hausdorff topological spaces. We equip Top with the open-cover topology which was described in Example 2.3.8, where $\{f_i : U_i \to X\}_I$ is a covering if each f_i is an open embedding and $X = \cup_I f_i(U_i)$.

To ease notation, we shall refer to the aforementioned site simply by Top. This will not cause any confusion, as we will not consider other Grothendieck topologies on Top.

Notation 2.4.1.

- The 2-category $\mathsf{St}(\mathsf{Top})$ will be simply denoted as Stack.
- The 2-category of categories fibered in groupoids over Top which satisfy descent will be denoted as $\mathsf{Stack}_{\mathsf{CFG}}$.

Definition 2.4.2. Given a topological groupoid $[R \rightrightarrows X]$, we define the *pre-stack* $\lfloor X/R \rfloor$ by

$$\lfloor X/R \rfloor(T) := [\mathrm{Hom}_{\mathsf{Top}}(T, R) \rightrightarrows \mathrm{Hom}_{\mathsf{Top}}(T, X)]$$

and the topological stack

$$[X/R] := a_J(\lfloor X/R \rfloor)$$

where a_J is the stackification functor.

Definition 2.4.3. Recall from Definition 3.1 of [26], that a morphism of categories fibered in groupoids $f : \mathcal{X} \to \mathcal{Y}$ is a *monomorphism* if for each topological space T, $f(T) : \mathcal{X}(T) \to \mathcal{Y}(T)$ is fully faithful.

Proposition 2.4.4. *The natural map*

$$\mathrm{Hom}_{\mathsf{topGpd}}([Z_1 \rightrightarrows Z_0], [X_1 \rightrightarrows X_0]) \to \mathrm{Hom}_{\mathsf{pshGpd}}([Z_0/Z_1], [X_0/X_1])$$

is fully faithful.

Proof. From Proposition 3.2 (i) of [26], the natural map $\lfloor X_0/X_1 \rfloor \to [X_0/X_1]$ is a monomorphism. This implies that the induced map

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\lfloor Z_0/Z_1 \rfloor, \lfloor X_0/X_1 \rfloor) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\lfloor Z_0/Z_1 \rfloor, [X_0/X_1])$$

is fully faithful. Using the universal property of stackification,

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\lfloor Z_0/Z_1 \rfloor, [X_0/X_1]) \to \mathrm{Hom}_{\mathsf{pshGpd}}([Z_0/Z_1], [X_0/X_1])$$

is an equivalence of groupoids.

Finally, using Yoneda's lemma,

$$\mathrm{Hom}_{\mathsf{topGpd}}([Z_1 \rightrightarrows Z_0], [X_1 \rightrightarrows X_0]) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\lfloor Z_0/Z_1 \rfloor, \lfloor X_0/X_1 \rfloor)$$

is an equivalence. □

2.5 Atlas for a stack

The 2-category of stacks is still too big to work with. In order to gain some control over a stack, we will consider stacks that come equipped with a map into them which is an epimorphism and the source is a topological space. We shall use this map to extend the definition of geometric properties from topological spaces to stacks.

We shall introduce the concept of a topological stack in this section which will be the main object of study in this thesis.

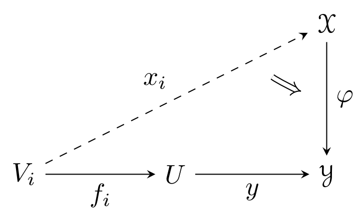
Definition 2.5.1. A stack is representable if there is an equivalence of stacks $\mathcal{X} \simeq T$ where T is in the image of the Yoneda embedding.

Remark 2.5.2. We shall adopt a common abuse of notation and say that if a stack is representable then it is equal to a topological space.

Definition 2.5.3. A morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is *representable* if for any map $T \to \mathcal{Y}$ where T is a representable stack, the base extension $\mathcal{X} \tilde{\times}_{\mathcal{Y}} T$ is a representable stack.

Definition 2.5.4. A morphism of stacks $\varphi : \mathcal{X} \to \mathcal{Y}$ is an *epimorphism* if for each topological space U and for each $y \in \mathcal{Y}(U)$, there exists a cover $\{f_i : V_i \to U\}_{i \in I}$ such that for each $y|_{V_i} \in \mathcal{Y}(V_i)$ there exists a lift (up to isomorphism) $x_i \in \mathcal{X}(V_i)$.

Alternatively, (using Yoneda's Lemma) we can phrase this as saying for each $i \in I$, there exists a 2-commutative diagram



i.e. we have local sections of φ.

Definition 2.5.5. A stack $\mathcal{X}$ is a *topological stack* if there exists a map $\varphi : X \to \mathcal{X}$ which has the properties:

(i) X is a topological space;

(ii) φ is representable;

(iii) φ is an epimorphism.

We say that φ is an *atlas* for $\mathcal{X}$.

2.5.1 Examples

We shall give some of the main examples of topological stacks that we shall consider in this thesis.

Topological spaces

Every topological space is a topological stack. We have already shown that the category Top is a subcategory of Stack in Proposition 2.3.12 and Proposition 2.3.17. For a topological space X, the identity morphism id_X is an atlas.

Moduli of G-torsors

Let G be a topological group. We can form the category fibered in groupoids $[\mathrm{pt}/G] \xrightarrow{\pi} \mathsf{Top}$ as follows. The objects of the category $[\mathrm{pt}/G]$ over a topological space T are given by G-torsors $P \to T$ and the morphisms are given by Cartesian squares

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ \downarrow & & \downarrow \\ T & \xrightarrow[g]{} & T' \end{array}$$

where f is a G-equivariant morphism. The functor $\pi : [\mathrm{pt}/G] \to \mathsf{Top}$ is defined by $\pi(P \to T) = T$ and $\pi\big((P \to T) \xrightarrow{f,g} (P' \to T')\big) = g$. The axioms of a category fibered in groupoids are easily verified.

A group acting on a topological space

We can generalize the above example to any topological groupoid $[X \times G \rightrightarrows X]$ where G is a topological group acting on a topological space X (on the right).

The objects of the category fibered in groupoids $[X/G](T)$ are pairs $(\varphi : P \to T, q : P \to X)$ where φ is a G-torsor and q is a G-equivariant map. A morphism from (φ, q) to (φ', q') is a Cartesian square

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \varphi} \\ T & \xrightarrow[g]{} & T' \end{array}$$

such that

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & X & \end{array}$$

commutes.

We note that $[X/G]$ is a generalization of $[\mathrm{pt}/G]$ since if $X = \mathrm{pt}$, then $[X/G] = [\mathrm{pt}/G]$.

The natural map $X \to [X/G]$ is an atlas which makes $[X/G]$ a topological stack.

Topological stack associated to a topological groupoid

We shall briefly outline two (equivalent) methods to produce a topological stack from a topological groupoid. The first will follow the analogy above and will present the topological stack $\mathrm{Tors}_{[X/R]}$ as groupoid torsors. In the second method, we shall define an equivalent stack $[X/R]$ using the stackification functor (see Section 2.3.4 and Definition 2.4.2). Note here that the first construction gives a category fibered in groupoids whereas the second gives a presheaf of groupoids.

We recall the following definition from [26], Section 12. A map of topological groupoids $[R \rightrightarrows X] \to [R' \rightrightarrows X']$ is *Cartesian* if

$$\begin{array}{ccc} R & \longrightarrow & R' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ X & \longrightarrow & X' \end{array}$$

is Cartesian.

Let $\mathrm{Tors}_{[X/R]}$ denote the category fibered in groupoids of $[R \rightrightarrows X]$-torsors which we shall describe below.

Given a topological groupoid $\mathbb{X} = [R \rightrightarrows X]$, an $\mathbb{X}$-torsor is a map of topological spaces $P \to T$ and a Cartesian morphism of groupoids $[P \times_T P \rightrightarrows T] \to [R \rightrightarrows X]$.

A map $(P \xrightarrow{f} T) \to (P' \xrightarrow{f'} T')$ between $\mathbb{X}$-torsors is given by a Cartesian square

$$\begin{array}{ccc} P & \longrightarrow & P' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & T' \end{array}$$

which induces a commutative diagram of groupoids

$$
\begin{array}{ccc}
 & [R \rightrightarrows X] & \\
\nearrow & & \nwarrow \\
[P \times_T P \rightrightarrows T] & \longrightarrow & [P \times_T P \rightrightarrows T]
\end{array}
$$

The above category of $\mathbb{X}$-torsors is a category fibered in groupoids over Top. In fact, this category fibered in groupoids is a topological stack. The atlas is given by the $\mathbb{X}$-torsor $R \to X$ which corresponds under the Yoneda lemma (Lemma 2.2.13) to a map $X \to \mathrm{Tors}_{[X/R]}$.

Now we shall define $[X/R]$ using the stackification functor. Given a topological groupoid $[R \rightrightarrows X]$, (where X and R are topological spaces and the source and target maps are continuous maps of topological spaces), we can form the quotient stack $[X/R]$. The quotient stack is defined in Definition 2.4.2.

This construction gives rise to a functor $\mathsf{topGpd} \to \mathsf{topStack}$.

By Theorem 12.6 from [26], there is an equivalence of stacks $[X/R] \simeq \mathrm{Tors}^{\mathrm{op}}_{[X/R]}$.

Remark 2.5.6. A stack $\mathcal{X}$ is a topological stack if and only if it is equivalent to a quotient stack of a topological groupoid $[R \rightrightarrows X]$.

Given a topological stack $\mathcal{X}$, with an atlas $\varphi : X \to \mathcal{X}$, we can form the groupoid

$$[X \tilde{\times}_{\mathcal{X}} X \rightrightarrows X]$$

where the source and target maps are the two projection maps. We define $R = X \tilde{\times}_{\mathcal{X}} X$. This is a topological groupoid since φ is an atlas and hence representable (meaning that $R \in \mathsf{Top}$).

Other examples

In this section, we shall briefly outline a few additional examples of topological stacks. We shall not need any of these examples in the remainder of the thesis, but we record them here for the sake of interest.

Every orbifold can be written as a topological stack which is locally of the form $[M/G]$ where M is a manifold and G is a finite group.

Another source of motivation for topological stacks is from algebraic stacks. To every algebraic stack, we may associate a topological stack. This is described in Section 20 of [26]. Thus by computing invariants for the underlying topological stacks, we also compute invariants for the algebraic stacks.

2.6 Model categories

Model categories will be a key tool that we shall use in this thesis. We shall recall some well known results in this section.

2.6.1 Definition of model categories

We shall use the definition of a model category from ([21]; Definition A.2.1.1).

We shall recall some standard terminology below.

Definition 2.6.1. Given two maps $f : X \to Y$ and $g : X' \to Y'$, f is a *retract* of g if there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & X' & \xrightarrow{r} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i'} & Y' & \xrightarrow{r'} & Y \end{array}$$

where $r \circ i = \mathrm{id}_X$ and $r' \circ i' = \mathrm{id}_Y$.

Definition 2.6.2. Given two maps i and p, we say that i has the *left lifting property with respect to p* (LLP) and p has the *right lifting property with respect to i* (RLP) if for any commutative diagram below,

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

there exists a map h such that $h \circ i = f$ and $p \circ h = g$.

Definition 2.6.3 ([21])**.** A *model category* is a category C which is equipped with three classes of morphism: weak equivalence, cofibration and fibration. In addition, the following axioms are satisfied:

(i) The category C admits all small limits and colimits;

(ii) For any pair of composable arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$, if two of the three maps $g \circ f, g, f$ are weak equivalences, then so is the third.

(iii) Let f be a retract of g (see Definition 2.6.1), if g is a cofibration or a fibration or a weak equivalence then so is f;

(iv) (a) If i is a cofibration and p is a weak equivalence and a fibration, then i has the LLP with respect to p.

(b) If i is a cofibration and a weak equivalence and p is a fibration, then i has the LLP with respect to p.

(v) Any map $h : A \to D$ admits factorizations

$$A \xrightarrow{f} B \xrightarrow{g} D$$
$$A \xrightarrow{f'} C \xrightarrow{g'} D$$

where in the first factorization, f is a cofibration and g is a weak equivalence and a fibration. In the second factorization, f' is a cofibration and a weak equivalence and g' is a fibration.

Definition 2.6.4. If a map is a fibration and a weak equivalence, we say it is a *trivial fibration.* Similarly, if a map is a cofibration and a weak equivalence, we say it is a *trivial cofibration.* We note that by Definition 2.6.3.(i), there is a terminal object $*$ and an initial object $\emptyset$. An object X is *fibrant* if the map $X \to *$ is a fibration. Similarly, an object X is *cofibrant* if the map $\emptyset \to X$ is a cofibration.

Definition 2.6.5. A *category with weak equivalences* is a category C which is equipped with a class of morphisms called weak equivalences which satisfy axiom (ii) from Definition 2.6.3.

We shall give examples of model categories in Section 2.8 which will appear later in this thesis. But first, we shall give some elementary examples.

Example 2.6.6. *Let C be any category with all small limits and colimits. Then we can define three model structures on C by setting one of the three classes of weak equivalence, fibrations or cofibrations to be the class of isomorphisms and the other two classes to be all morphisms.*

Example 2.6.7. *Given a model category C, there is a model category structure on C^{op} where the fibrations of C^{op} are the cofibrations of C and the cofibrations of C^{op} are the fibrations of C. The weak equivalences of C^{op} are the weak equivalences of C.*

The above examples show that there is not a unique way to put a model structure onto a category.

Remark 2.6.8. In order to define a model category structure, it suffices to specify the weak equivalences and either the fibrations or the cofibration. The third class of morphism is then defined as those morphisms which satisfy the appropriate lifting conditions.

If we are given the classes of cofibrations and fibrations then by using the appropriate lifting diagrams, we know the classes of trivial fibrations and of trivial cofibrations. We can then define the class of weak equivalences as the class of maps which factorize as a trivial cofibration and a trivial fibration.

The correct notion of an equivalence between model categories is a Quillen equivalence. We shall recall the precise definition below.

Definition 2.6.9. [([18]; Definition 1.3.1)] Let C and D be model categories and let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be adjoint functors. The adjunction is a *Quillen adjunction* if one of the following equivalent conditions hold:

1. The left adjoint F preserves cofibrations and trivial cofibrations;
2. The right adjoint G preserves fibrations and trivial fibrations.

Definition 2.6.10. A Quillen adjunction (F, G, φ) is a *Quillen equivalence* if and only if for every cofibrant $X \in \mathsf{C}$ and every fibrant $Y \in \mathsf{D}$, the map $f : X \to G(Y)$ is a weak equivalence if and only if its adjoint $\varphi(f) : F(X) \to Y$ is a weak equivalence.

We shall give an example of a Quillen equivalence in Section 2.8.2.

For completeness, we shall recall some definitions of properties of model categories. These properties will not be of central importance but will be required in order to state certain theorems.

Definition 2.6.11. A model category is *cofibrantly generated* if there is a set of cofibrations $\mathcal{I}$ and a set of trivial cofibrations $\mathcal{J}$ such that the trivial fibrations are precisely those maps which have the RLP for each $i \in \mathcal{I}$ and the fibrations are precisely the maps which have the RLP for each $j \in \mathcal{J}$.

Definition 2.6.12. A model category is *right proper* if weak equivalences are preserved by pullback along fibrations.
A model category is *left proper* if weak equivalences are preserved by pushout along cofibrations.
A model category is *proper* if it is right proper and left proper.

Definition 2.6.13. A model category is *combinatorial* if it is locally presentable and cofibrantly generated.

Remark 2.6.14. We shall not define a locally presentable model category here. For full details, see Section A.2.6 of [21].

2.6.2 Enriched model categories

We shall make use of the theory of enriched model categories in Section 4.1. We recall some results from [19] in this section.

An enriched model category is a model category M which is enriched over a category V such that these two structures are compatible (this compatibility is encapsulated in an axiom which is often called SM7).

Definition 2.6.15. A *closed monoidal category* is a monoidal category C which has internal hom objects $[a, b]$ which satisfy the adjunction

$$\mathrm{Hom}_{\mathsf{C}}(a \otimes b, c) \cong \mathrm{Hom}_{\mathsf{C}}(a, [b, c]).$$

The closed monoidal category that we will be most interested in is Gpd, the category of groupoids.

Definition 2.6.16. Let V be a bicomplete closed symmetric monoidal category with unit E. A V-*category* C is defined by the following data:

- A collection of objects such that for any pair of objects $A, B \in \mathsf{C}$, there is an object $\mathsf{C}(A, B) \in \mathsf{V}$;
- For each $A \in \mathsf{C}$, a morphism $E \to \mathsf{C}(A, A)$;
- For each triple of objects $A, B, C \in \mathsf{C}$, a morphism $\mathsf{C}(A, B) \otimes \mathsf{C}(B, C) \to \mathsf{C}(A, C)$ which satisfy associativity and unit axioms.

A V-*functor* $f : \mathsf{C} \to \mathsf{C}'$ between V-categories consists of the following data:

- For each $A \in \mathsf{C}$, an object $f(A) \in \mathsf{C}'$;
- For each pair of objects $A, B \in \mathsf{C}$, f induces maps (in V) $\mathsf{C}(A, B) \to \mathsf{C}'(f(A), f(B))$ which satisfy the appropriate axioms.

For more details see [19].

Given a V-category M, the *underlying category of* M *is denoted by* M_0. The category M_0 has the same objects as M and for each pair of objects $A, B \in \mathsf{M}$, $\mathsf{M}_0(A, B) := \mathrm{Hom}_{\mathsf{V}}(E, \mathsf{M}(A, B))$.

Definition 2.6.17. Let V be a closed monoidal category and let C be a V-category. For $v \in \mathsf{V}$ and $c, d \in \mathsf{C}$ the *tensor* of v by c is an object $v \otimes c \in \mathsf{C}$ defined by

$$\mathrm{Hom}_{\mathsf{C}}(v \otimes c, d) \cong \mathrm{Hom}_{\mathsf{V}}(v, \mathsf{C}(c, d))$$

and the *cotensor* of v by d is an object $d^v \in \mathsf{C}$ defined by

$$\mathrm{Hom}_{\mathsf{C}}(c, d^v) \cong \mathrm{Hom}_{\mathsf{V}}(v, \mathsf{C}(c, d)).$$

We note that for a general V-category, the tensor and cotensor might not exist.

Below, we shall recall from Definition A.3.1.1 of [21] the notion of a left Quillen bifunctor. This will enable us to define an enriched model category.

Definition 2.6.18. Let C, D and M be model categories. A functor $F : \mathsf{C} \times \mathsf{D} \to \mathsf{M}$ is a *left Quillen bifunctor* if

1. for any cofibration $i : c \to c'$ in C and any cofibration $j : d \to d'$ in D, the map

$$F(c', d) \coprod_{F(c,d)} F(c, d') \to F(c', d')$$

is a cofibration and is a weak equivalence if either i or j is;

2. F preserves small colimits in each variable.

We shall now consider how to combine the closed monoidal structure with the model structure for a category. This can be found in Definition A.3.1.2 in [21].

Definition 2.6.19. A *monoidal model category* is a model category V, which also has a closed monoidal structure $(\mathsf{V}, \otimes, E)$ such that

- $\otimes : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ is a left Quillen bifunctor;
- The unit of the monoidal structure E is cofibrant.

Definition 2.6.20. Let V be a monoidal model category. A V-*enriched model category* is a V-category C which is tensored and cotensored over V such that the underlying category C_0 has a model category structure and where the tensor functor $\otimes : \mathsf{V} \times \mathsf{C} \to \mathsf{C}$ is a left Quillen bifunctor (Definition 2.6.18).

Example 2.6.21. *We will show in Section 2.8.2 that* sSet *is a monoidal model category. If a model category* M *is* sSet*-enriched as above (setting* $\mathsf{V} = \mathsf{sSet}$*), then we say that* M *is a* simplicial model category*. Often, the axiom that states that the tensor is a left Quillen bifunctor is called SM7.*

2.7 Model category structures on sGpd and $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]$

We shall recall from [21] results that will enable us to put a model category structure on sGpd and $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]$. In order to achieve this, we shall first recall a model category structure on Gpd. Then, we shall consider Reedy and global model category structures on sGpd and $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]$ and show how they relate to each other.

2.7.1 Model structure on Gpd

In this section we discuss the model structure on the underlying 1-category of Gpd.

Definition 2.7.1. Let $p : G \to H$ be a morphism in Gpd. We say that p is a *fibration* if for any $x \in G$ and any isomorphism $\varphi : y \to p(x)$ in H, there exists an isomorphism $\psi : z \to x$ in G such that $p(\psi) = \varphi$. In the literature, this is commonly referred to as an *isofibration*.

There is a model category structure on the category Gpd of groupoids where

- weak equivalences are equivalences of groupoids;
- cofibrations are maps that are injective on the set of objects;
- fibrations are as in Definition 2.7.1.

We refer the reader to ([17], Theorem 2.1) for more detail and further references.

Lemma 2.7.2. *Consider the following diagram in* Gpd*:*

$$\begin{array}{ccc} & & K \\ & & \downarrow{\scriptstyle p} \\ H & \longrightarrow & G \end{array}$$

Suppose that p *is a fibration. Then, the natural map of groupoids*

$$H \times_G K \to H \tilde{\times}_G K$$

is an equivalence.

Proof. This functor is always fully faithful (see Section 2.2.6). It is straightforward that fibrancy of p implies essential surjectivity. □

Lemma 2.7.3. *A morphism* $i : G \to H$ *in* Gpd *is a trivial cofibration if and only if it is essentially surjective and induces an isomorphism of groupoids between* G *and a full subcategory of* H*. When this is the case,* $G \times K \to H \times K$ *is a trivial cofibration for every groupoid* K*.*

Proof. Straightforward. □

Proposition 2.7.4. *The above model structure on* Gpd *is left proper, simplicial, cofibrantly generated, combinatorial and monoidal (with respect to Cartesian product).*

Proof. The properties left proper, simplicial and cofibrantly generated are proved in ([17], Theorem 2.1). Since Gpd is cofibrantly generated and locally presentable, it is, by definition, combinatorial.

To check that the model structure is monoidal we need to verify conditions (i)-(iii) of ([21], Definition A.3.1.2). Conditions (ii) and (iii) are obvious. To check (i) we have to show that the Cartesian product $\times : \mathsf{Gpd} \times \mathsf{Gpd} \to \mathsf{Gpd}$ is a left Quillen bifunctor. Recall from Definition 2.6.18 that this means the following two conditions are satisfied:

(a) Let $i : A \to A'$ and $j : B \to B'$ be cofibrations in Gpd. Then, the induced map

$$i \wedge j : (A' \times B) \coprod_{A \times B} (A \times B') \to A' \times B'$$

is a cofibration in Gpd. Moreover, if either i or j is a trivial cofibration, then $i \wedge j$ is also a trivial cofibration.

(b) The Cartesian product preserves small colimits separately in each variable.

The first part of (a) is easy as it only concerns the object sets of the groupoids in question, and the corresponding statement is true in the category of sets. To prove the second part of (a), assume that $i : A \to A'$ is a trivial cofibration. The claim follows from Lemma 2.7.3 and two-out-of-three applied to

$$A \times B' \to (A' \times B) \coprod_{A \times B} (A \times B') \to A' \times B'.$$

To verify condition (b), let K be an arbitrary groupoid. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Gpd}}(\operatorname*{colim}_{\alpha}(G_\alpha \times H), K) &\cong \lim_{\alpha} \mathrm{Hom}_{\mathsf{Gpd}}(G_\alpha \times H, K) \\
&\cong \lim_{\alpha} \mathrm{Hom}_{\mathsf{Gpd}}(G_\alpha, \mathrm{Hom}_{\mathsf{Gpd}}(H, K)) \\
&\cong \mathrm{Hom}_{\mathsf{Gpd}}(\operatorname*{colim}_{\alpha} G_\alpha, \mathrm{Hom}_{\mathsf{Gpd}}(H, K)) \\
&\cong \mathrm{Hom}_{\mathsf{Gpd}}((\operatorname*{colim}_{\alpha} G_\alpha) \times H, K).
\end{aligned}$$

Thus, $\mathrm{colim}_\alpha(G_\alpha) \times H \cong (\mathrm{colim}_\alpha G_\alpha) \times H$.

□

Below, we shall show that the model category Gpd satisfies a technical assumption which gives a good theory of Gpd-enriched model categories.

Proposition 2.7.5. *The model structure on* Gpd *is excellent in the sense of ([21], Definition A.3.2.16).*

Proof. Axioms (A1)-(A4) of [21] are straightforward to check. Axiom (A5), the Invertibility Hypothesis, follows from ([21], Lemma A.3.2.20) applied to the fundamental groupoid functor $\Pi_1 : \mathsf{sSet} \to \mathsf{Gpd}$. □

2.7.2 Injective model structure on $[\mathsf{C}, \mathsf{Gpd}]$

Proposition 2.7.6. *If* C *is a small category and* M *is a combinatorial model category, then there are global injective and global projective model category structures on the functor category* $[\mathsf{C}, \mathsf{M}]$.

The weak equivalences for both model categories are the level-wise weak equivalences. The fibrations in the projective model category are the level-wise fibrations. The cofibrations in the injective model category are the level-wise cofibrations.

We shall denote these model category structures by $[\mathsf{C}, \mathsf{M}]_{proj}$ *and* $[\mathsf{C}, \mathsf{M}]_{inj}$ *respectively.*

Proof. See Proposition A.2.8.2 of [21] □

Let C be a small category. Since the model structure on Gpd is combinatorial (Proposition 2.7.4), by ([21], Proposition A.2.8.2) there is a model structure on the category $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]$ of presheaves of groupoids, called the *injective model structure*, where

- weak equivalences are the object-wise weak equivalences as in Section 2.7.1;

- cofibrations are the object-wise cofibrations as in Section 2.7.1;

- fibrations have the right lifting property with respect to the trivial cofibrations.

Proposition 2.7.7. *The injective model structure on* $[\mathsf{C}^{op}, \mathsf{Gpd}]$ *is* Gpd*-enriched in the sense of ([21], Definition A.3.1.5).*

Proof. This follows from ([21], Remark A.3.3.4). □

We are particularly interested in the cases $\mathsf{C} = \mathsf{Top}$ and $\mathsf{C} = \Delta$. In the case $\mathsf{C} = \Delta$, we have an explicit description of fibrations thanks to Corollary 2.7.17 below.

Remark 2.7.8. The smallness assumption on C is to allow us to quote results from Appendix A of [21]. As indicated at the beginning of Appendix A of [21], this is not a restrictive assumption as we can always fix a Grothendieck universe. For this reason, our treating $\mathsf{C} = \mathsf{Top}$ as a small category is not problematic.

2.7.3 Reedy model category

There is another model category structure on the functor category if the source category is the category $\mathbf{\Delta}$. This model category structure is called the Reedy model category structure. More generally, we have this intermediate model structure if the source category is a 'Reedy category'. The Reedy model category structure fits in-between the global injective and global projective category. We shall make this statement precise in Proposition 2.7.19.

A general definition of a 'Reedy category' is given in ([18]; Definition 5.2.1) and is designed to generalize the ordinal category $\mathbf{\Delta}$. However, since we shall only require two Reedy categories in this thesis, namely the categories $\mathbf{\Delta}$ and $\mathbf{\Delta}^{\mathrm{op}}$. We describe these categories precisely below.

Example 2.7.9. *The category* $\mathbf{\Delta}$ *has objects finite sets* $[n] := \{0, 1, \cdots, n\}$*. The morphisms are the order preserving maps. There are two subcategories* $\mathbf{\Delta}_+, \mathbf{\Delta}_- \subseteq \mathbf{\Delta}$ *which have the same objects. A morphism* $f : [n] \to [m]$ *to be in* $\mathbf{\Delta}_+$ *if* f *is injective and* $f : [n] \to [m]$ *to be in* $\mathbf{\Delta}_-$ *if* f *is surjective. The degree function* $deg : \mathbf{\Delta} \to \aleph_0$ *sends* $[n] \mapsto n$ *and hence morphisms in* $\mathbf{\Delta}_+$ *clearly raise the degree and morphisms in* $\mathbf{\Delta}_-$ *clearly lower the degree.*

Example 2.7.10. *Note that the category* $\mathbf{\Delta}^{op}$ *has subcategories* $\mathbf{\Delta}_+^{op}$ *and* $\mathbf{\Delta}_-^{op}$ *which have the same objects with morphisms in* $\mathbf{\Delta}_+^{op}$ *precisely the surjective morphisms of* $\mathbf{\Delta}^{op}$ *and the morphisms of* $\mathbf{\Delta}_-^{op}$ *precisely the morphisms of* $\mathbf{\Delta}^{op}$ *which are injective. The degree function is maps* $[n] \mapsto n$ *as before.*

Let M be a model category. We shall describe the Reedy model structures on the categories of functors $[\mathbf{\Delta}, \mathsf{M}]$.

In order to do this, we first define the matching and latching objects.

Definition 2.7.11. Given a functor $X : \mathbf{\Delta} \to \mathsf{M}$, the *latching object* $L_n X$ is defined as

$$L_n X := \operatorname*{colim}_{k \to n \in \mathbf{\Delta}_+} X_k,$$

and the *matching object* $M_n X$ is defined as

$$M_n X := \lim_{n \to k \in \mathbf{\Delta}_-} X_k.$$

The latching and matching objects for a simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{M}$ are defined dually.

Example 2.7.12. *Let* $X : \mathbf{\Delta}^{op} \to \mathsf{Gpd}$ *be a simplicial groupoid. The* latching object $L_n X$ *is defined as*

$$L_n X := \operatorname*{colim}_{[k] \underset{\neq}{\twoheadrightarrow} [n]} X_k,$$

and the matching object $M_n X$ *is defined as*

$$M_n X := \lim_{[n] \underset{\neq}{\hookrightarrow} [k]} X_k.$$

Now we may define the Reedy model structure on a functor category where the source category is a Reedy category (in this thesis this means the source is $\mathbf{\Delta}$ or $\mathbf{\Delta}^{\mathrm{op}}$) and the target category has a model category structure.

Definition 2.7.13. Let D be a Reedy category and let M be a model category. The *Reedy model category structure* on the functor category $[\mathsf{D}, \mathsf{M}]$ is defined as follows:

The cofibrations are the maps $i : A \to B$ such that

$$L_n B \coprod_{L_n A} A_n \to B_n$$

is a cofibration in M.

The fibrations are the maps $f : X \to Y$ such that

$$X_n \to M_n X \times_{M_n Y} Y_n$$

is a fibration in M.

The weak equivalences are the maps $f : X \to Y$ such that $f(d) : X(d) \to Y(d)$ is a weak equivalence in M for each $d \in \mathsf{D}$.

2.7.4 Reedy model structure on sGpd

The Reedy model structure on the category of simplicial groupoids $\mathsf{sGpd} = [\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Gpd}]$ is defined as follows:

- weak equivalences are the object-wise weak equivalences;

- cofibrations are morphisms $X \to Y$ such that for every n the map

$$L_nY \coprod_{L_nX} X_n \to Y_n$$

is a cofibration of groupoids (as in Section 2.7.1);

- fibrations are morphisms $X \to Y$ such that for every n the map

$$X_n \to M_nX \times_{M_nY} Y_n$$

is a fibration of groupoids (as in Section 2.7.1).

Let us unravel the above definitions. First of all, note that the matching object M_nX can be alternatively described by

$$M_nX = \mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, X),$$

where we regard the simplicial set $\partial\Delta^n$ as a simplicial groupoid. The map $X_n \to M_nX$ is the one induced by the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$.

The Reedy fibration condition can now be restated as saying that

$$\mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, X) \to \mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, X) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, Y)} \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, Y)$$

is a fibration of groupoids.

Lemma 2.7.14. *Let X and Y be simplicial sets, regarded as objects in* sGpd*. Then, any morphism $p : X \to Y$ is a Reedy fibration.*

Proof. This follows from the definition of a Reedy fibration and the fact that every map of sets, regarded as objects in Gpd, is a fibration of groupoids. □

Lemma 2.7.15. *Let $N : [\mathsf{C}^{op}, \mathsf{Gpd}] \to [\mathsf{C}^{op}, \mathsf{sSet}]$ be the level-wise nerve functor and Π_1 its left adjoint. A morphism $f : X \to Y$ is an injective fibration in $[\mathsf{C}^{op}, \mathsf{Gpd}]$ if and only if $N(f)$ is an injective fibration in $[\mathsf{C}^{op}, \mathsf{sSet}]$.*

Proof. By ([21], Remark A.2.8.6) N and Π_1 form a Quillen adjunction which proves sufficiency. To prove necessity we use the fact that N preserves trivial cofibrations (for this use Lemma 2.7.3) and that $\Pi_1 \circ N = \mathrm{id}_{[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]}$. More precisely, to solve a lifting problem in $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]$, we can first apply N, solve the lifting problem in $[\mathsf{C}^{\mathrm{op}}, \mathsf{sSet}]$, and then apply Π_1 to obtain a solution to the original lifting problem. □

Proposition 2.7.16. *Let* C *be a category. The Reedy model structure and the injective model structure on $[\boldsymbol{\Delta}^{op}, [\mathsf{C}^{op}, \mathsf{Gpd}]_{inj}]$ coincide. Here, $[\mathsf{C}^{op}, \mathsf{Gpd}]_{inj}$ denotes the functor category $[\mathsf{C}^{op}, \mathsf{Gpd}]$ endowed with the injective model structure.*

Proof. We know that, by definition, the two model structures have the same weak equivalences. It remains to show that they have the same fibrations. Let $N : [\mathbf{\Delta}^{\mathrm{op}} \times \mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}] \to [\mathbf{\Delta}^{\mathrm{op}} \times \mathsf{C}^{\mathrm{op}}, \mathsf{sSet}]$ be the object-wise nerve functor, and let Π_1 be its left adjoint, the object-wise fundamental groupoid functor. We show that the following are equivalent:

(1) $p : X \to Y$ is a Reedy fibration in $[\mathbf{\Delta}^{\mathrm{op}}, [\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]_{\mathrm{inj}}]$.

(2) For all n, $N(X_n) \to M_n(N(X)) \times_{M_n(N(Y))} N(Y_n)$ is an injective fibration in $[\mathsf{C}^{\mathrm{op}}, \mathsf{sSet}]$.

(3) $N(p) : N(X) \to N(Y)$ is a Reedy fibration in $[\mathbf{\Delta}^{\mathrm{op}}, [\mathsf{C}^{\mathrm{op}}, \mathsf{sSet}]_{\mathrm{inj}}]$.

(4) $N(p) : N(X) \to N(Y)$ is an injective fibration in $[\mathbf{\Delta}^{\mathrm{op}}, [\mathsf{C}^{\mathrm{op}}, \mathsf{sSet}]_{\mathrm{inj}}]$.

(5) $p : X \to Y$ is an injective fibration in $[\mathbf{\Delta}^{\mathrm{op}}, [\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]_{\mathrm{inj}}]$.

(1) $\Leftrightarrow$ (2) follows from Lemma 2.7.15 and the fact that the nerve functor preserves fiber products. (2) $\Leftrightarrow$ (3) is true by definition. (3) $\Leftrightarrow$ (4) follows from the fact that the injective model structure on $[\mathbf{\Delta}^{\mathrm{op}}, [\mathsf{C}, \mathsf{sSet}]_{\mathrm{inj}}]$ is the same as the Reedy model structure (by [3], Proposition 3.9 and the fact that $\mathbf{\Delta}$ is an 'elegant Reedy category'). (4) $\Leftrightarrow$ (5) again follows from Lemma 2.7.15 by noting that

$$[\mathbf{\Delta}^{\mathrm{op}}, [\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]_{\mathrm{inj}}]_{\mathrm{inj}} = [\mathbf{\Delta}^{\mathrm{op}} \times \mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]_{\mathrm{inj}}.$$

□

In sGpd, the Reedy cofibrations turn out to coincide with the object-wise cofibrations. That is, a morphism $p : X \to Y$ of simplicial groupoids is a Reedy cofibration if and only if $X_n \to Y_n$ is a cofibration of groupoids (in the sense of Section 2.7.1) for all n.

Corollary 2.7.17. *The Reedy model structure and the injective model structure on* $\mathsf{sGpd} = [\mathbf{\Delta}^{op}, \mathsf{Gpd}]$ *coincide.*

Proof. This follows from Proposition 2.7.16, setting $\mathsf{C} = \mathrm{pt}$. □

Corollary 2.7.18. *The Reedy model structure and the injective model structure on* $[\mathbf{\Delta}^{op}, \mathsf{pshGpd}]$ *coincide.*

Proof. This follows from Proposition 2.7.16, setting $\mathsf{C} = \mathsf{Top}$. □

The above result does not hold in general. We shall state a result that shows how the injective, Reedy and projective model structures are related to each other.

Proposition 2.7.19. *Let* D *be a Reedy category and let* M *be a combinatorial model category. The identity functors* $[\mathsf{D}, \mathsf{M}]_{proj} \xrightarrow{\mathrm{id}} [\mathsf{D}, \mathsf{M}]_{Reedy} \xrightarrow{\mathrm{id}} [\mathsf{D}, \mathsf{M}]_{inj}$ *are left Quillen functors.*

Proof. See Remark A.2.9.23 of [21]. □

2.8 Examples of model categories

In this section, we shall recall model structures on the categories of topological spaces, simplicial sets and bisimplicial sets.

2.8.1 Model category structure on topological spaces

Definition 2.8.1. A morphism of topological spaces p has the *homotopy lifting property* with respect to an object A if $A \to A \times [0,1]$ has the LLP with respect to p.

The only model category structure that we will consider on the category of topological spaces is the Quillen model structure (although there are others).

A morphism of topological spaces p is a Serre fibration if it has the homotopy lifting property with respect to all CW-complexes (or equivalently with respect to all topological spaces of the form $[0,1]^n$ for $n \in \mathbb{N}$.)

The Quillen model structure on the category Top is described as follows: Weak equivalences are morphisms $f : X \to Y$ that induce isomorphisms of homotopy groups $\pi_n(f) : \pi_n(X,x) \to \pi_n(Y, f(x))$ for each $x \in X$. Fibrations are Serre fibrations. Cofibrations have the LLP with respect to these fibrations.

We shall list some properties that this model structure has:

1. The model structure is cofibrantly generated. Cofibrations are generated by morphisms of the form $\{|\partial\Delta^n| \to |\Delta^n|\}$ for each n. Trivial cofibrations are generated by morphisms of the form $\{|\Lambda^n_k| \to |\Delta^n|\}$ for each n and for each $0 \le k \le n$;

2. It is a simplicial model category. This means that it is a sSet-enriched model category in the sense of Section 2.6.2. The enriched homomorphisms are given by $\mathrm{Sing}(\mathrm{Map}_{\mathsf{Top}}(X,Y))$. Given a simplicial set K and a topological space X the tensor and cotensor are given by $|K| \times X$ and $\mathrm{Map}_{\mathsf{Top}}(|K|, X)$ respectively;

3. This model structure is proper;

4. This model structure is not combinatorial. But we will see in Section 2.8.2 that it is Quillen equivalent to a combinatorial model category.

2.8.2 Model category structure on simplicial sets

There are also many different model category structures on the category of simplicial sets. We shall be considering the classical Quillen model structure.

- Weak equivalences are morphisms $f : X \to Y$ such that the geometric realization $|f| : |X| \to |Y|$ is a weak equivalence of topological spaces (see Section 2.8.1);

- Cofibrations are morphisms that are level-wise injections of sets;
- Fibrations are the morphisms that have the RLP with respect to the horn inclusions $\Lambda^n_k \to \Delta^n$ for each n and for each $0 \leq k \leq n$.

Again, we shall list some of the properties of this model structure.

1. The model structure is cofibrantly generated. Cofibrations are generated by morphisms of the form $\{\partial\Delta^n \to \Delta^n\}$ for each n. Trivial cofibrations are generated by morphisms of the form $\{\Lambda^n_k \to \Delta^n\}$ for each n and for each $0 \leq k \leq n$;
2. It is a simplicial model category. This means that it is a sSet-enriched model category in the sense of Section 2.6.2. The enrichment is given by the mapping space. The tensor is the Cartesian product.

The model structures of simplicial sets and of topological spaces given in this section are related by a Quillen equivalence. The singular chains and geometric realization functors form a Quillen equivalence

$$\mathrm{Sing} : \mathsf{Top} \rightleftarrows \mathsf{sSet} : |-|.$$

Loosely speaking, this means that the homotopy theory of sSet and Top are the same.

2.8.3 Model category structure on bisimplicial sets

A bisimplicial set is an object in the functor category $[\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set}]$ or equivalently $[\mathbf{\Delta}^{\mathrm{op}}, \mathsf{sSet}]$. The category of bisimplicial sets is denoted by bsSet. Recall that $\Delta^{n,m}$ denotes the bisimplicial set $\mathrm{Hom}_{\mathbf{\Delta}\times\mathbf{\Delta}}(-, [n] \times [m])$. There is a model category structure on bsSet which is inherited from the Quillen model structure on sSet (described in Section 2.8.2) under the functor $\mathrm{Diag} : \mathsf{bsSet} \to \mathsf{sSet}$, where for $X \in \mathsf{bsSet}$, $\mathrm{Diag}(X)_n := X_{n,n}$.

In bsSet, we define the weak equivalences (resp. fibrations) to be maps which are weak equivalences (resp. fibrations) in sSet after we apply the functor Diag. This model structure is called the Moerdijk structure. See ([13], Chapter IV, Section 3.3) for more details.

We shall recall two adjoint functors involving bisimplicial sets which will be useful later.

Definition 2.8.2. The functor Diag has a left adjoint $d^* : \mathsf{sSet} \to \mathsf{bsSet}$ and a right adjoint $d_* : \mathsf{sSet} \to \mathsf{bsSet}$ which are defined in [13] Chapter IV Section 3.3. The functor d^* is defined by the two properties:

1. $d^*(\Delta^n) = \Delta^{n,n}$;
2. d^* preserves colimits.

whilst d_* is defined by the property that for a simplicial set K,

$$d_*(K)_{m,n} := \mathrm{Hom}_{\mathsf{sSet}}(\Delta^m \times \Delta^n, K).$$

Definition 2.8.3. Given two simplicial sets $X, Y \in \mathsf{sSet}$, we may form the bisimplicial set $X \boxtimes Y$ where $(X \boxtimes Y)_{n,m} := X_n \times Y_m$. We note the following adjunction: for $X \in \mathsf{sSet}$ and $Y \in \mathsf{bsSet}$

$$\mathrm{Hom}_{\mathsf{bsSet}}(X \boxtimes \Delta^n, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, Y_{*,n}).$$

2.9 Homotopy limits

In [31], Shulman considers an approach to homotopy limits which works under quite general assumptions. This approach also takes into account enriched category theory. We shall follow Shulman and show that these homotopy limits and homotopy colimits agree with the classical approaches to homotopy limits and colimits outlined in [12].

For additional details on unenriched homotopy limits and colimits the reader may consult the notes of Dugger [9]. For enriched category theory (using weighted limits) two good references are [29] and [31].

2.9.1 Bar construction

In this section, we shall introduce the background required to define the bar and cobar construction. The definitions in this section are taken from [31]. Once we have this, we may define the homotopy limit and homotopy colimit for diagrams $F : I \to \mathsf{M}$ where M is a suitably enriched model category. We shall also fix a fibrant replacement functor R and a cofibrant replacement functor Q.

Recall from Definition 2.6.15 and Definition 2.6.16 the notion of a V-category where V is a closed monoidal model category.

We recall from (Definition 4.1.12 of [18]) the notion of a two variable adjunction.

Definition 2.9.1. Let M, N and P be V-categories. A *two variable* V*-adjunction* is given by a triple of V-functors $(\circledast, \mathrm{Hom}_r, \mathrm{Hom}_l)$ and a pair of isomorphisms (ϕ, ψ) where

$$\circledast : \mathsf{M} \times \mathsf{N} \to \mathsf{P}$$
$$\mathrm{Hom}_r : \mathsf{N}^{\mathrm{op}} \times \mathsf{P} \to \mathsf{M}$$
$$\mathrm{Hom}_l : \mathsf{M}^{\mathrm{op}} \times \mathsf{P} \to \mathsf{N}$$

and these V-functors satisfy the isomorphisms

$$\mathrm{Hom}_P(a \circledast b, c) \xrightarrow{\phi} \mathrm{Hom}_M(a, \mathrm{Hom}_r(b, c)) \xrightarrow{\psi} \mathrm{Hom}_N(b, \mathrm{Hom}_l(a, c)).$$

We shall often drop the above isomorphisms from the notation.

Definition 2.9.2. A V-category M is V-*tensored* if there is a two variable V-adjunction arising from a V-enriched tensor hom adjunction. Here, $\circledast : \mathsf{M} \times \mathsf{V} \to \mathsf{M}$ is the tensor, and Hom_r and Hom_l become the internal hom and cotensor.

We recall from Definition 19.1 in [31] the tensor product of V-functors.

Definition 2.9.3. Let $(\circledast, \mathrm{Hom}_r, \mathrm{Hom}_l)$ be a two variable V-adjunction with $\circledast : \mathsf{M} \times \mathsf{N} \to \mathsf{P}$ where P is a cocomplete V-tensored category. Let D be a small V-category and consider two V-functors $G : \mathsf{D}^{\mathrm{op}} \to \mathsf{M}$ and $F : \mathsf{D} \to \mathsf{N}$. We define the following object in P

$$G \circledast_{\mathsf{D}} F := \mathrm{coeq}\left(\coprod_{d,d'} \mathsf{D}(d,d') \odot (G(d) \circledast F(d')) \rightrightarrows \coprod_{d} G(d) \circledast F(d) \right) \in \mathsf{P},$$

to be the tensor product of G and F over D. Here $\odot$ denotes the tensor product $\odot : \mathsf{P} \times \mathsf{V} \to \mathsf{P}$.

Definition 2.9.4. Let $(\circledast, \mathrm{Hom}_r, \mathrm{Hom}_l)$ be a two variable V-adjunction with $\circledast : \mathsf{M} \times \mathsf{N} \to \mathsf{P}$ where N is a V-tensored category and M is complete. Given V-functors $G : \mathsf{D} \to \mathsf{N}$ and $F : \mathsf{D} \to \mathsf{P}$, we define the following object in M

$$\mathrm{Hom}_r^{\mathsf{D}}(G,F) = \mathrm{eq}\left(\prod_{d \in \mathsf{D}} \mathrm{Hom}_r(G(d), F(d)) \rightrightarrows \prod_{d,d' \in \mathsf{D}} \mathrm{Hom}_r(G(d) \odot \mathsf{D}(d,d'), F(d')) \right) \in \mathsf{M}.$$

We note that the above definition is dual to Definition 2.9.3.

Remark 2.9.5. Let us consider the specific case when $\mathsf{V} = \mathsf{Gpd}$ and $\mathsf{D} = \mathbf{\Delta}$. Since $\mathbf{\Delta}$ is enriched over groupoids in a trivial way (since $\mathsf{Set} \subset \mathsf{Gpd}$), we note that for two Gpd-functors, $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{M}$ and $F : \mathbf{\Delta} \to \mathsf{N}$ and a two variable Gpd-adjunction $\circledast : \mathsf{M} \times \mathsf{N} \to \mathsf{P}$,

$$\begin{aligned} G \circledast_{\mathsf{D}} F &= \mathrm{coeq}\left(\coprod_{d,d'} \mathsf{D}(d,d') \odot (G(d) \circledast F(d')) \rightrightarrows \coprod_{d} G(d) \circledast F(d) \right) \\ &= \int^{\mathbf{\Delta}} G([n]) \circledast F([n]). \end{aligned}$$

Similarly, for $G : \mathbf{\Delta} \to \mathsf{N}$ and $F : \mathbf{\Delta} \to \mathsf{P}$,

$$\mathrm{Hom}_r^{\mathbf{\Delta}}(G,F) = \int_{\mathbf{\Delta}} \mathrm{Hom}_r(G(d), F(d)).$$

In order to define homotopy limits and colimits, we shall recall the bar and cobar construction from [31] and establish some results in the specific case of cosimplicial presheaves of groupoids. The following definition is from (Definition 19.4, [31]).

Definition 2.9.6. Given the conditions from Definition 2.9.3, the simplicial bar construction is the following simplicial object of P

$$B_n(G, \mathsf{D}, F) := \coprod_{\{\alpha_0, \cdots, \alpha_n : \alpha_i \in \mathsf{D}_0\}} \mathsf{D}(\alpha_0, \alpha_1) \otimes \cdots \otimes \mathsf{D}(\alpha_{n-1}, \alpha_n) \odot (G(\alpha_0) \circledast F(\alpha_n))$$

where D_0 denotes the underlying category of the V-category D. The face and degeneracy maps are defined in the obvious way. If we have a functor $|\Delta| : \mathbf{\Delta} \to \mathsf{V}$, we form the bar construction

$$B(G, \mathsf{D}, F) := B_\bullet(G, \mathsf{D}, F) \odot_{\mathbf{\Delta}} |\Delta| \in \mathsf{P}.$$

Definition 2.9.7. Given the conditions from Definition 2.9.4, the cosimplicial cobar construction is the following cosimplicial object of M

$$C^n(G, D, F) = \prod_{\{\alpha_0, \cdots, \alpha_n : \alpha_i \in \mathsf{D}_0\}} \mathrm{Hom}_r((\mathsf{D}(\alpha_0, \alpha_1) \otimes \cdots \mathsf{D}(\alpha_{n-1}, \alpha_n)) \odot G(\alpha_n), F(\alpha_0)).$$

Again, if we have a functor $|\Delta| : \boldsymbol{\Delta} \to \mathsf{V}$, the cobar construction $C(F, \mathsf{D}, G)$ is the totalization of $C^n(F, \mathsf{D}, G)$.

In what follows, we shall fix a functor $|\Delta| : \boldsymbol{\Delta} \to \mathsf{V}$.

Remark 2.9.8. We note that since the tensor $\circledast$ has two right adjoints, there are two cobar constructions. These are given by using Hom_r or Hom_l.

Definition 2.9.9. Given $F : \mathsf{D} \to \mathsf{M}$, we define the functor

$$B(\mathsf{D}, \mathsf{D}, F) : \mathsf{D} \to \mathsf{M}$$
$$d \mapsto B(\mathsf{D}(-, d), \mathsf{D}, F).$$

We can now define the homotopy limit and the homotopy colimit as follows

Definition 2.9.10. Let M be a V enriched model category and let $F : \mathsf{D} \to \mathsf{M}$ be a V-functor from a small index V-category D. The homotopy limit of F is

$$\mathrm{holim}(F) := C(\mathsf{D}, \mathsf{D}, RF)$$

where R is the level-wise fibrant replacement functor in M. The homotopy colimit of F is

$$\mathrm{hocolim}(F) := B(\mathsf{D}, \mathsf{D}, QF)$$

where Q is the level-wise cofibrant replacement functor in M.

Remark 2.9.11. A common approach to homotopy limits is as the derived versions of the usual limit and colimit functors. We can use a model structure on the diagram category to fibrantly or cofibrantly replace the diagrams before taking the usual limit or colimit.

By Corollary 13.12 and Corollary 13.17 of [31], these two notions agree under certain conditions (which will hold for all examples in this thesis) listed in [31].

2.10 Homotopy between morphisms of presheaves of groupoids

2.10.1 Homotopies of pointed topological stacks

We review the notion of homotopy between morphisms of stacks from [28], and introduce a variant called restricted homotopy which is also defined in [28].

In order to introduce homotopy groups, we shall first introduce the concept of pointed maps of topological stacks and homotopies between these maps. In [26], Noohi introduces the more general notion of a 'pair'. Since we do not require this level of generality, we shall focus on pointed stacks; this can be recovered from [26] by setting $\mathcal{A} = \mathcal{B} = \mathrm{pt}$.

Definition 2.10.1. A *pointed topological stack* is a pair $(\mathcal{X}, x)$, where $\mathcal{X}$ is a topological stack and $x : \mathrm{pt} \to \mathcal{X}$ is a morphism.

A *morphism of pointed topological stacks* $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ is a morphism of topological stacks $f : \mathcal{X} \to \mathcal{Y}$ along with a 2-isomorphism $\varphi : f \circ x \Rightarrow y$.

A *2-morphism of pointed topological stacks* between pointed morphisms of pointed topological stacks $f, g : (\mathcal{X}, x) \to (\mathcal{Y}), y$ is a 2-morphism $\varphi : f \Rightarrow g$.

Definition 2.10.2. Given two morphisms of topological stacks $f, g : \mathcal{X} \to \mathcal{Y}$, a *homotopy from f to g* is given by a triple $(H, \epsilon_0, \epsilon_1)$. Here, $H : \mathcal{X} \tilde{\times} [0, 1] \to \mathcal{Y}$ is a morphism of topological stacks where $[0, 1]$ is the topological unit interval. Both $\epsilon_0 : f \Rightarrow H_0$ and $\epsilon_1 : H_1 \Rightarrow g$ are 2-isomorphisms. Here H_0 and H_1 denotes the restriction of H to $\mathcal{X} \tilde{\times} \{0\}$ and $\mathcal{X} \tilde{\times} \{1\}$ respectively.

Definition 2.10.3. Given two morphisms of pointed topological stacks $f, g : (\mathcal{X}, x) \to (\mathcal{Y}, y)$, a *homotopy from f to g* is given by a triple $(H, \epsilon_0, \epsilon_1)$. Here, $H : (\mathcal{X} \tilde{\times} [0, 1], x \tilde{\times} [0, 1]) \to (\mathcal{Y}, y)$ is a morphism of pointed topological stacks where $[0, 1]$ is the topological unit interval. Both $\epsilon_0 : f \Rightarrow H_0$ and $\epsilon_1 : H_1 \Rightarrow g$ are 2-isomorphisms. Here H_0 and H_1 denotes the restriction of H to $(\mathcal{X} \tilde{\times} \{0\}, x \tilde{\times} \{0\})$ and $(\mathcal{X} \tilde{\times} \{1\}, x \tilde{\times} \{1\})$ respectively.

Remark 2.10.4. Given a 2-isomorphism $\varphi : f \Rightarrow g$ between morphisms of pointed topological stacks $(f, g : (\mathcal{X}, x) \to (\mathcal{Y}, y))$, then we have a *ghost homotopy* $(H, \epsilon_0, \epsilon_1)$ where $H : \mathcal{X} \tilde{\times} [0, 1] \xrightarrow{\mathrm{pr}} \mathcal{X} \xrightarrow{f} \mathcal{Y}$, $\epsilon_0 = \mathrm{id}$ and $\epsilon_1 = \varphi$.

2.10.2 Weak equivalences of topological stacks

In order to define weak equivalences of topological stacks, first we will define homotopy groups for topological stacks. Then we can define weak equivalences to be the morphisms which induce isomorphisms between the homotopy groups. We shall define the homotopy groups in such as way that they generalize homotopy groups of topological spaces (when they are considered as topological stacks under Yoneda's lemma).

Definition 2.10.5. The n^{th} *homotopy group* of a topological stack $\mathcal{X}$ with a basepoint $x : \mathrm{pt} \to \mathcal{X}$ is

$$\pi_n(\mathcal{X}, x) := [(S^n, s), (\mathcal{X}, x)]_{\mathrm{htpy}}$$

where $[-, -]_{\mathrm{htpy}}$ denotes the pointed homotopy classes of morphisms.

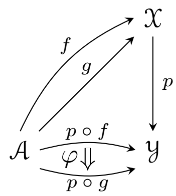
For more details and a description of the group structure, see Section 17 of [26]. Also, from Section 17 of [26], we recall the following definition.

Definition 2.10.6. A map of topological stacks $f : \mathcal{X} \to \mathcal{Y}$ is a *weak equivalence* if for every $x : \mathrm{pt} \to \mathcal{X}$, the induced morphism of sets, $\pi_0(f) : \pi_0(\mathcal{X}, x) \to \pi_0(\mathcal{Y}, f(x))$ is a bijection and for each $n > 0$, the group morphism $\pi_n(f) : \pi_n(\mathcal{X}, x) \to \pi_n(\mathcal{Y}, f(x))$ is an isomorphism.

2.10.3 Fiberwise homotopy

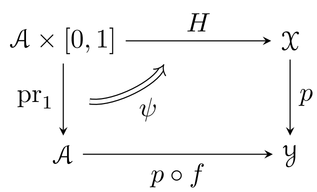
In Section 2.10.1, we defined the notion of homotopies of topological stacks. In this section, we shall develop the notion of a 'fiberwise homotopy' relative to a map $p : \mathcal{X} \to \mathcal{Y}$. The idea is that we shall define a homotopy between two maps $f, g : \mathcal{A} \to \mathcal{X}$ such that when it is restricted to each point $a : \mathrm{pt} \to \mathcal{A}$ the image of the homotopy lies in the fiber of p. We shall make this precise below.

Definition 2.10.7. Let $f, g : \mathcal{A} \to \mathcal{X}$ and $p : \mathcal{X} \to \mathcal{Y}$ be morphisms of presheaves of groupoids, and $\varphi : p \circ f \Rightarrow p \circ g$ a 2-isomorphism:



A *fiberwise homotopy from f to g relative to φ* is a quadruple $(H, \epsilon_0, \epsilon_1, \psi)$ where

- $H : \mathcal{A} \times [0, 1] \to \mathcal{X}$ is a morphism of presheaves of groupoids;
- $\epsilon_0 : f \Rightarrow H_0$ and $\epsilon_1 : H_1 \Rightarrow g$ are 2-isomorphisms;
- $\psi : p \circ f \circ \mathrm{pr}_1 \Rightarrow p \circ H$ is a 2-isomorphism such that $\psi_0 = p \circ \epsilon_0$, $\psi_1 \cdot (p \circ \epsilon_1) = \varphi$. I.e. the following diagram 2-commutes



(Notation: $H_i := H|_{\mathcal{A} \times \{i\}}$, $\psi_i := \psi|_{\mathcal{A} \times \{i\}}$, for $i = 0, 1$.) In the case where φ and ψ are both identity 2-isomorphisms (so $p \circ f = p \circ g$ and $p \circ f \circ \mathrm{pr}_1 = p \circ H$) we say that H is a *homotopy relative to* $\mathcal{Y}$.

A fiberwise homotopy as above is called *strict* if ϵ_0 and ϵ_1 are the identity 2-isomorphisms.

A *ghost fiberwise homotopy from f to g relative to φ* is a 2-isomorphism $\xi : f \Rightarrow g$ such that $\varphi = p \circ \xi$.

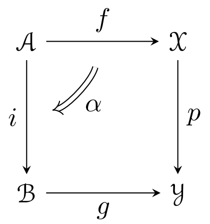
Ghost homotopies typically arise from those quadruples $(H, \epsilon_0, \epsilon_1, \psi)$ for which H and ψ remain constant along $[0,1]$, that is, they factor through pr_1. In this case, $\xi := \epsilon_0 \cdot \epsilon_1$ is a ghost fiberwise homotopy from f to g relative to φ. Conversely, from a ghost homotopy φ we can construct quadruples $(g \circ \mathrm{pr}_1, \xi, \mathrm{id}, \varphi \circ \mathrm{pr}_1)$ and $(f \circ \mathrm{pr}_1, \mathrm{id}, \xi, \mathrm{id} \circ \mathrm{pr}_1)$.

Remark 2.10.8. There is some flexibility in choosing H. More precisely, if $H' : \mathcal{A} \times [0,1] \to \mathcal{X}$ is 2-isomorphic to H via $\alpha : H \Rightarrow H'$, then $(H', \epsilon'_0, \epsilon'_1, \psi')$ is also a fiberwise homotopy from f to g relative to φ, where $\epsilon'_0 = \epsilon_0 \cdot \alpha_0$, $\epsilon'_1 = \alpha_1^{-1} \cdot \epsilon_1$ and $\psi' = \psi \cdot (p \circ \alpha)$.

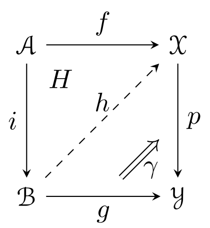
2.10.4 Serre Fibrations and cofibration of topological stacks

We shall review some of the material from [28] and recall the notion of (weak) Serre fibration between stacks. For a full account see Sections 2 and 3 of [28]. Before we start, it is worthwhile to emphasize the difference between the notion of fibration in this section and the standard ones in well known model category structures on the category of presheaves of groupoids: our notion is more geometric, in the sense that it does not distinguish between equivalent presheaves; in particular, any equivalence of presheaves of groupoids is a fibration in our sense.

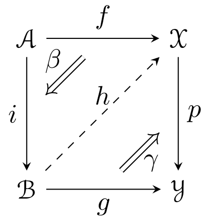
Definition 2.10.9. Let $i : \mathcal{A} \to \mathcal{B}$ and $p : \mathcal{X} \to \mathcal{Y}$ be morphisms of presheaves of groupoids. Then, i has the *weak left lifting property (WLLP)* with respect to p if given



there is a morphism $h : \mathcal{B} \to \mathcal{X}$, a 2-isomorphism $\gamma : g \Rightarrow p \circ h$ and a fiberwise homotopy H from f to $h \circ i$ relative to $\alpha(\gamma \circ i)$:



We say that i has the *left lifting property (LLP)* with respect to p if H can be taken to be a ghost homotopy. In other words, there are 2-isomorphisms $\beta : f \Rightarrow h \circ i$ and $\gamma : g \Rightarrow p \circ h$ such that the following diagram commutes (α is not shown in the diagram):



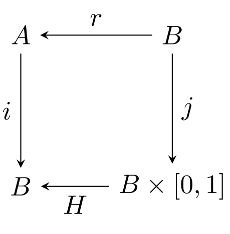
We say that p has the *(weak) covering homotopy property* with respect to $\mathcal{A}$, if the inclusion $\mathcal{A} \to \mathcal{A} \times [0,1]$, $a \mapsto (a,0)$, has (W)LLP with respect to p.

Remark 2.10.10. The usage of the term 'weak' (which means, 'up to homotopy') in the above definition is in conflict with our usual usage of the term weak (which means, 'up to 2-isomorphism', as opposed to 'strict'). But since the above definition is quite standard in the homotopy theory literature, we deemed it inappropriate to change it. We apologize for the confusion this may cause.

Definition 2.10.11 ([28], Definitions 3.6, 3.7)**.** A morphism of presheaves of groupoids $p : \mathcal{X} \to \mathcal{Y}$ is called a *(weak) Serre fibration* if it has the (weak) covering homotopy property with respect to every finite CW complex A. That is, $A \to A \times [0,1]$ has the (W)LLP with respect to p. It is called a *(weak) trivial Serre fibration* if every finite CW inclusion $i : A \hookrightarrow B$ has the (W)LLP with respect to p.

Lemma 2.10.12 (see [28], Proposition 3.21)**.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a (weak) Serre fibration. Then, every cellular inclusion $i : A \hookrightarrow B$ of finite CW complexes that induces isomorphisms on all π_n has the (W)LLP with respect to p.*

Proof. The map $i : A \hookrightarrow B$ being as above, A becomes a deformation retract of B. Therefore, the map i is a retract of the the map $j : B \to B \times [0,1]$, $j(b) = (b,0)$,



Here, r is the retraction and $H : B \times [0,1] \to B$ is a homotopy with $H_0 = r$ and $H_1 = \mathrm{id}_B$. Since j has (W)LLP with respect to p, so does its retract i. $\square$

Remark 2.10.13. As opposed to the notion of Reedy fibration that we will introduce in Definition 3.3.9, the notion of (weak) Serre fibration is "intrinsic" (or "geometric") in the sense that if $p : \mathcal{X} \to \mathcal{Y}$ is a (weak) Serre fibration and $p' : \mathcal{X}' \to \mathcal{Y}'$ is a morphism equivalent to it, then p' is also a (weak) Serre fibration.

Proposition 2.10.14. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks, and assume that $\mathcal{X}$ is Serre. Then, p is a (weak) trivial Serre fibration if and only if it is a (weak) Serre fibration and a weak equivalence.*

Proof. By ([28], Lemma 2.4), every morphism $p : \mathcal{X} \to \mathcal{Y}$ of topological stacks with $\mathcal{X}$ a Serre stack is a Serre morphism (in the sense of [28], Definition 2.2). The result now follows from ([28], Proposition 5.4). Note that ([28], Proposition 5.4) is only stated for trivial Serre fibration, but it is also true for trivial weak Serre fibration; the first paragraph of the given proof (minus the last sentence) is in fact the proof of the statement for trivial weak Serre fibration. □

2.10.5 Classifying atlas

In [27], Noohi gives another method for considering the homotopy type of a topological stack. We shall recall some of the main details below.

Recall from Definition 2.5.3, a morphism of stacks $\mathcal{X} \to \mathcal{Y}$ is *representable* if for any map $Y \to \mathcal{Y}$ where Y is a topological space, the 2-fiber product $\mathcal{X} \tilde{\times}_{\mathcal{Y}} Y$ is (equivalent to) a topological space.

In fact, for a topological stack $\mathcal{X}$ and a topological space X, any map $\varphi : X \to \mathcal{X}$ is representable.

Definition 2.10.15. A map of topological spaces $f : X \to Y$ is *shrinkable* if there exists a section $s : Y \to X$ such that there is a strong deformation retract of X onto $s(Y)$.

Definition 2.10.16. A map of topological spaces $f : X \to Y$ is *locally shrinkable* if there exists a cover $\{U_i \to Y\}_{i \in I}$ such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is shrinkable for each $i \in I$.

We shall now show how we define some property for morphisms of stacks (once we have defined it for topological spaces).

Definition 2.10.17. A representable morphism of topological stacks $\varphi : \mathcal{X} \to \mathcal{Y}$ has property $\mathcal{P}$ if for each topological space T, the base extension $\varphi_T : \mathcal{X} \tilde{\times}_{\mathcal{Y}} T \to T$ has property $\mathcal{P}$.

Here property $\mathcal{P}$ could be shrinkable, locally shrinkable, étale, etc.

Below, we shall define a classifying atlas. This is an atlas which captures the homotopy type of the topological stack. There are many different ways that we could define a classifying atlas. In [27], Noohi defines the classifying atlas as an atlas which is also locally shrinkable, whilst in [25] a classifying atlas is a universal weak equivalence (meaning each base extension from any topological space is a weak equivalence). In this thesis, we opt for a stronger condition than [25] but a weaker condition than [27].

Definition 2.10.18. An atlas $\varphi : X \to \mathcal{X}$ for a topological stack $\mathcal{X}$ is a *classifying atlas* if φ is a weak trivial Serre fibration (see Definition 2.10.11). We say that X is a *classifying space* for $\mathcal{X}$.

The main properties that we want any definition of a classifying atlas to possess are: it is invariant under base change, is a weak equivalence and that it has certain lifting properties that allow us to define induced classifying atlases for mapping stacks (for a precise statement, see Theorem 2.10.25).

We claim that Definition 2.10.18 satisfies all three of these properties. The first is clear, the second follows from Proposition 2.10.14 and the third appears as Theorem 2.10.25.

In order to show that the results about classifying atlases from [27] are still applicable for our definition of classifying atlas in Definition 2.10.18, we recall the following result.

Theorem 2.10.19. *If φ is a locally shrinkable morphism then φ is a weak trivial Serre fibration.*

Proof. This appears as ([28]; Corollary 3.17). □

Theorem 2.10.20. *Every topological stack has a classifying atlas.*

Proof. By Theorem 6.1 of [27], every topological stack $\mathcal{X}$ has an atlas $\varphi : X \to \mathcal{X}$ which is locally shrinkable. By Theorem 2.10.19, φ is a classifying atlas. □

We recall a useful theorem from [27] (Theorem 1.2) and show that it is still true with our stronger definition of a classifying atlas.

Theorem 2.10.21. *For any diagram $P : D \to \mathsf{topStack}$, indexed by a small category D, with a terminal object $pt \in \mathsf{D}$, there exists a diagram $Q : D \to \mathsf{Top}$ and a natural transformation $Q \Rightarrow P$ such that for each $d \in D$, $Q(d) \to P(d)$ is a classifying atlas. The diagram Q is natural up to object-wise weak equivalence of diagrams.*

Proof. By Theorem 2.10.20, there exists a classifying atlas $\varphi : X_\bullet \to \mathcal{X}_\bullet$ where $\mathcal{X}_\bullet = P(\mathrm{pt})$. Pulling back φ along each morphism $P(\alpha)$ for each morphism α in the diagram gives a diagram $Q : \mathsf{D} \to \mathsf{Top}$ of classifying spaces such that $Q(d) \Rightarrow P(d)$ is a pullback of φ.

Any other choice of classifying atlas gives a level-wise weakly equivalent lift to the diagram. □

Examples of classifying atlases

- For a topological space, the identity map is a classifying atlas;
- For the stack $[X/R]$ associated to a topological groupoid $[R \rightrightarrows X]$, there is a classifying atlas which is constructed in Section 6 of [27] which is the natural map $\|N[R \rightrightarrows X]\| \to [X/R]$. Here, $N : \mathsf{Gpd} \to \mathsf{sSet}$ is the nerve functor and $\| - \|$ denotes the fat realization. The fat realization is defined as $\| - \| : \mathsf{sSet} \to \mathsf{Top}$, sending $K \mapsto \int^{\mathbf{\Delta}^+} K_n \times |\Delta^n|$ where $\mathbf{\Delta}^+ \subset \mathbf{\Delta}$ is the subcategory of the ordinal category without degeracy maps.
- In the special case of the stacks $[\mathrm{pt}/G]$ and $[X/G]$, the above construction gives the classifying atlases $BG \to [\mathrm{pt}/G]$ and $EG \times_G X \to [X/G]$ respectively (where BG is the classifying space and $EG \times_G X$ is the Borel construction).

2.10.6 Mapping stack

As an application of our results, we shall consider the *free loop stack* and more general *mapping stacks*. Mapping stacks generalize mapping spaces (with the compact open topology). The mapping stack between two topological stacks $\mathcal{X}$ and $\mathcal{Y}$ is a topological stack which is defined as follows.

Definition 2.10.22. Given two topological stacks $\mathcal{X}$ and $\mathcal{Y}$, the *mapping stack* $\mathrm{Map}_{\mathsf{topStack}}(\mathcal{Y}, \mathcal{X})$ is the topological stack defined by

$$\mathrm{Map}_{\mathsf{topStack}}(\mathcal{Y}, \mathcal{X})(T) := \mathrm{Hom}_{\mathsf{pshGpd}}(\mathcal{Y} \tilde{\times} T, \mathcal{X}).$$

Proposition 2.10.23. *Given topological stacks $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$, we have a natural equivalence of stacks*

$$\mathrm{Map}_{\mathsf{topStack}}(\mathcal{X}, \mathrm{Map}_{\mathsf{topStack}}(\mathcal{Y}, \mathcal{Z})) \sim \mathrm{Map}_{\mathsf{topStack}}(\mathcal{X} \tilde{\times} \mathcal{Y}, \mathcal{Z})).$$

We recall Theorem 1.1 from [25]. This allows us to give conditions to ensure that the mapping stack is a topological stack.

Theorem 2.10.24. *If $\mathcal{X}$ and $\mathcal{Y}$ are topological stacks and $\mathcal{Y}$ admits a groupoid presentation $[Y_1 \rightrightarrows Y_0]$ where Y_0 and Y_1 are compact topological spaces, then $\mathrm{Map}_{\mathsf{topStack}}(\mathcal{Y}, \mathcal{X})$ is a topological stack.*

Proof. See Theorem 1.1 of [25]. □

We would also like to be able to compute the classifying atlas for a mapping stack. The next result gives one case where this is possible.

Theorem 2.10.25. *Given a classifying atlas $\varphi : X \to \mathcal{X}$ and a finite CW-complex Y, the induced map*

$$\varphi_* : \mathrm{Map}_{\mathsf{topStack}}(Y, X) \to \mathrm{Map}_{\mathsf{topStack}}(Y, \mathcal{X})$$

is a classifying atlas.

Proof. If $i : A \to B$ is a cellular inclusion of finite CW-complexes, then $i \tilde{\times} \mathrm{id}_Y : A \tilde{\times} Y \to B \tilde{\times} Y$ is also a cellular inclusion. By the exponential law in Proposition 2.10.23, i has the (W)LLP with respect to φ_* if and only if $i \tilde{\times} \mathrm{id}_Y$ has the (W)LLP with respect to φ.

Note that we require Y to be a finite CW-complex in order to ensure that $\mathrm{Map}_{\mathsf{topStack}}(Y, \mathcal{X})$ is a topological stack by Theorem 2.10.24. □

The main application of Theorem 2.10.25 in this thesis will be in the example below.

Example 2.10.26. *Let K be a finite simplicial set and let $\mathcal{X}$ be a topological stack with a classifying atlas $\varphi : X \to \mathcal{X}$. Since $|K| \in \mathsf{Top}$ is a finite CW-complex,*

$$\varphi_* : \mathrm{Map}_{\mathsf{topStack}}(|K|, X) \to \mathrm{Map}_{\mathsf{topStack}}(|K|, \mathcal{X})$$

is a classifying atlas.

The free loop stack (for a topological stack $\mathcal{X}$) is then defined as $\mathbb{L}\mathcal{X} := \mathrm{Map}_{\mathsf{topStack}}(S^1, \mathcal{X})$ where S^1 is the topological circle. We shall consider the free loop stack in Corollary 5.3.13.

2.10.7 Serre stacks

Throughout this paper, we shall restrict our attention to Serre stacks. This is just a technical assumption, however we shall record it here for completeness.

Definition 2.10.27. A topological stack is *Serre* if it admits a groupoid presentation such that $s : R \to X$ is locally (on source and target) a Serre fibration. That is, for every $y \in R$, s is a Serre fibration from a neighborhood of y to a neighborhood of $s(y)$.

We consider Serre stacks since this is a condition that we require in Lemma 3.3.3. This result is a key technical step in the proof of one our our main results (Theorem 3.5.2).

We invite the reader to look at the statement of Lemma 3.3.3 in conjunction with the following definition.

Definition 2.10.28. A stack $\mathcal{X}$ is a generalized Serre stack if for any closed embeddings of topological spaces $A \hookrightarrow B$ and $A \hookrightarrow C$ which are locally trivial Serre cofibrations, the map

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B \coprod_A C, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(B {\coprod_A}' C, \mathcal{X})$$

induced by the natural map $B \coprod'_A C \to B \coprod_A C$ is an equivalence of groupoids. Here $\coprod'$ denotes the coproduct in pshGpd and $\coprod$ denotes the coproduct in Top (viewed as a stack via the Yoneda embedding).

We shall see in Lemma 3.3.3 that any Serre stack is a generalized Serre stack. The reason that we consider the concept of a generalized Serre stack is due to the following result:

Lemma 2.10.29. *Let $\mathcal{X}$ be a generalized Serre stack and T any compactly generated topological space. Then $\mathrm{Map}_{\mathsf{topStack}}(T, \mathcal{X})$ is a generalized Serre stack.*

Proof. Let $i : A \hookrightarrow B$ and $j : A \hookrightarrow C$ be any closed embeddings of topological spaces which are locally trivial Serre cofibrations. From the exponential property of the mapping stack Proposition 2.10.23,

$$\mathrm{Hom}_{\mathsf{topStack}}(D, \mathrm{Map}_{\mathsf{topStack}}(T, \mathcal{X})) \cong \mathrm{Hom}_{\mathsf{topStack}}(D \times T, \mathcal{X})$$

is an equivalence of groupoids.

In order to show that $\mathrm{Map}_{\mathsf{topStack}}(T, \mathcal{X})$ is a generalized Serre stack it is enough to show that $i \times \mathrm{id}_T$ and $j \times \mathrm{id}_T$ are closed embeddings which are local trivial Serre cofibrations.

For any topological space T, $i \times \mathrm{id}_T : A \times T \to B \times T$ is a closed embedding. For any compactly generated topological space T, $i \times \mathrm{id}_T : A \times T \to B \times T$ is a local trivial cofibration. This follows from the fact that the category of compactly generated topological space with the Cartesian product is a monoidal model category (see [18], Section 4.2). Since i is locally a trivial Serre cofibration and id_T is a trivial Serre cofibration, $i \times \mathrm{id}_T$ is a local trivial Serre cofibration. The same argument shows that $j \times \mathrm{id}_T$ is also a local trivial Serre cofibration. $\square$

Lemma 2.10.30. *The 2-categories of stacks, topological stacks and Serre topological stacks are all closed under 2-fiber products (in fact, all finite limits), and these are computed as presheaves of groupoids.*

Proof. In the case of stacks this is well known (homotopy limit commutes with fiber product). For the other two cases see ([26], page 30) for the construction of a groupoid presentation for $\mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y}$ out of those for $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. □

The following result requires Serre stacks to allow us to form a group structure on $\pi_n(\mathcal{X}, x)$ for $n \geq 1$.

Theorem 2.10.31. *Given a classifying atlas for a Serre topological stack $\varphi : X \to \mathcal{X}$, for every basepoint $x : pt \to \mathcal{X}$ there is an isomorphism $\pi(\mathcal{X}, x) \cong \pi(X, \hat{x})$ where $\hat{x}$ is a lift of the basepoint such that there is a 2-isomorphism $p \circ \hat{x} \Rightarrow x$. Note that $\pi(\mathcal{X}, x)$ is as defined in Definition 2.10.5 and $\pi(X, \hat{x})$ is the usual homotopy group of a pointed topological space.*

Proof. This follows from ([27]; Theorem 10.5). □

2.10.8 Hollander model structure

In this section, we shall briefly describe the work of Hollander [17] and the differences between that approach and ours.

In Section 2.10.2 we have described weak equivalences and in Section 2.10.4 we defined Serre fibrations and cofibrations of topological stacks. We could ask ourselves if this gives rise to a model category structure. Unfortunately, it does not. One obvious problem is that topological stacks are not closed under pushouts. But there is also a problem with the factorization axioms. Namely, it is not possible to factorize every morphism of topological stacks as a trivial cofibration followed by a fibration. We claim, however that Noohi's construction satisfies a weaker set of axioms called a category of fibrant objects. However, we shall not prove that here.

In [17], Hollander puts a model structure on the category of presheaves of groupoids where the fibrant objects are exactly the stacks (note these are stacks and not topological stacks). The choice of cofibrations, fibrations and weak equivalences is different to ours. In fact, the model structure is a localization of the global projective model category structure on pshGpd by the maps $\{\mathsf{Desc}(\mathcal{X}, \mathcal{U}) \to \mathcal{X}\}$.

The model structure considered by Hollander is not as geometric as ours in the sense that fibrations and cofibrations for Hollander are not invariant under equivalences of stacks.

The model structure given by Hollander does not restrict to the Quillen model structure on Top. If $f : X \to Y$ is a weak equivalence in Top, it does not imply that after applying the Yoneda embedding, $f : X \to Y$ is a weak equivalence in Hollander's model category structure. But in the framework developed by Noohi, weak equivalences, Serre fibrations and Serre cofibrations are preserved by the Yoneda embedding.

Chapter 3

Singular chains functor for topological stacks

3.1 Conventions and notation

To be clear, we shall reiterate the conventions that we will use in this chapter.

In this chapter, we shall only consider strict presheaves of groupoids over Top. By a stack we mean a strict presheaf of groupoids which satisfies the descent condition.

At the end of the chapter in Section 3.6, we shall give a summary of how the results can be adapted to categories fibered in groupoids.

Remark 3.1.1. The reason that we choose to use strict presheaves of groupoids is in order to give us a functorial definition for the singular chains.

We want to have the category of simplicial sets as the target category for our singular chains functor. The category sSet is a strict category and the category pshGpd has a simplicial model category structure. We shall use both of these facts to ensure that the singular chains functor that we define gives a 'strict' simplicial set as the output.

We also know by Remark 2.2.14, any category fibered in groupoids can be 'strictified'.

3.2 Singular chains on presheaves of groupoids

We shall define the functor $\mathrm{Sing} : \mathsf{pshGpd} \to \mathsf{sSet}$ of singular chains and establish some of its basic properties. This functor will be the focus of the rest of this thesis.

3.2.1 The functors B and Sing

Definition 3.2.1. Let

$$\begin{aligned} B : \mathsf{pshGpd} &\to \mathsf{bsSet}, \\ \mathcal{X} &\mapsto N(\mathcal{X}_{\boldsymbol{\Delta}}), \end{aligned}$$

$$\begin{aligned} \mathrm{Sing} : \mathsf{pshGpd} &\to \mathsf{sSet}, \\ \mathcal{X} &\mapsto \mathrm{Diag}(N(\mathcal{X}_{\boldsymbol{\Delta}})). \end{aligned}$$

Here, bsSet stands for the category of bisimplicial sets and $\mathrm{Diag} : \mathsf{bsSet} \to \mathsf{sSet}$ refers to taking the diagonal of a bisimplicial set.

In the above definition, $\mathcal{X}_{\boldsymbol{\Delta}} := \mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^{\bullet}|, \mathcal{X})$, and $N : \mathsf{sGpd} \to \mathsf{bsSet}$ is the level-wise nerve functor obtained from $N : \mathsf{Gpd} \to \mathsf{sSet}$.

Remark 3.2.2. When restricted to Top, the functor Sing coincides with the usual singular chains functor $\mathrm{Sing} : \mathsf{Top} \to \mathsf{sSet}$. More precisely, the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Top} & \longrightarrow & \mathsf{pshGpd} \\ & \underset{\mathrm{Sing}}{\searrow} & \downarrow{\scriptstyle \mathrm{Sing}} \\ & & \mathsf{sSet} \end{array}$$

The top arrow in this diagram is the Yoneda embedding.

Lemma 3.2.3. *Let $f : X \to Y$ be a morphism of simplicial groupoids that induces equivalences of groupoids $X_n \to Y_n$ for all n. Then, the induced map $\mathrm{Diag}(NX) \to \mathrm{Diag}(NY)$ is a weak equivalence of simplicial sets.*

Proof. This follows from ([13], Chapter IV, Proposition 1.7). □

Corollary 3.2.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be an equivalence of presheaves of groupoids. Then, $\mathrm{Sing}(f) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ is a weak equivalence of simplicial sets.*

Proposition 3.2.5. *The functors $\mathrm{Diag} : \mathsf{bsSet} \to \mathsf{sSet}$ and $N : \mathsf{sGpd} \to \mathsf{bsSet}$ have left adjoints:*

$$\mathsf{sGpd} \underset{\Pi_1}{\overset{N}{\rightleftarrows}} \mathsf{bsSet} \underset{d^*}{\overset{\mathrm{Diag}}{\rightleftarrows}} \mathsf{sSet}.$$

Here, Π_1 denotes the fundamental groupoid functor, and d^* is as in (Definition 2.8.2). Therefore, $\mathrm{Diag} \circ N$ also has $\Pi_1 \circ d^*$ as left adjoint. In particular, the functors N, Diag and $\mathrm{Sing} = \mathrm{Diag} \circ N \circ ()_{\boldsymbol{\Delta}}$ preserve limits.

Proof. For the first adjunction see ([17], Corollary 2.3). The second adjunction is discussed in ([13], Chapter IV, Section 3.3). □

Lemma 3.2.6. *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be morphisms of presheaves of groupoids.*

(i) If $\alpha : f \Rightarrow g$ is a 2-isomorphism, then we have an induced homotopy $\hat{\alpha}$ from $\mathrm{Sing}(f)$ *to* $\mathrm{Sing}(g)$.

(ii) If h is a strict homotopy from f to g (see Definition 2.10.7), then we have an induced homotopy $\hat{h}$ from $\mathrm{Sing}(f)$ *to* $\mathrm{Sing}(g)$.

Proof. Part (ii) follows from the fact that Sing commutes with products (Proposition 3.2.5). To prove part (i), let $\mathcal{I}$ be the constant presheaf of categories $\mathcal{I} : T \mapsto \{0 \to 1\}$, where $\{0 \to 1\}$ is the ordinal category (also denoted [1]). A 2-isomorphism α as above is the same thing as a morphism

$$\Phi_\alpha : \mathcal{X} \times \mathcal{I} \to \mathcal{Y}$$

whose restrictions to $\{0\}$ and $\{1\}$ are f and g, respectively. It is easy to see that $\mathrm{Sing}(\mathcal{I}) = \Delta^1$. (Note that we have only defined Sing for presheaves of groupoids, but clearly the same definition makes sense for presheaves of categories as well.) By Proposition 3.2.5, we obtain a map of simplicial sets

$$\hat{\alpha} := \mathrm{Sing}(\Phi_\alpha) : \mathrm{Sing}(\mathcal{X}) \times \Delta^1 \to \mathrm{Sing}(\mathcal{Y}).$$

This is the desired homotopy. □

Remark 3.2.7.

(1) The operation $\alpha \mapsto \hat{\alpha}$ respects composition of 2-isomorphisms in the sense that $\widehat{\alpha \cdot \beta}$ is canonically homotopic to the "composition" of $\hat{\alpha}$ and $\hat{\beta}$. More precisely, $\hat{\alpha}$, $\hat{\beta}$ and $\widehat{\alpha \cdot \beta}$ are the three faces of a canonical map

$$\mathrm{Sing}(\mathcal{X}) \times \Delta^2 \to \mathrm{Sing}(\mathcal{Y}).$$

We also have higher coherences. That is, every string of k composable 2-isomorphisms defines a canonical map

$$\mathrm{Sing}(\mathcal{X}) \times \Delta^k \to \mathrm{Sing}(\mathcal{Y})$$

whose restriction to various faces represent different ways of composing (a subset) of homotopies associated to these 2-isomorphisms.

(2) In the statement of Lemma 3.2.6(ii) we could use a general homotopy $h = (H, \epsilon_0, \epsilon_1)$ from f to g (see Definition 2.10.7), but in this case instead of a homotopy from f to g we obtain a sequence of three composable homotopies $\hat{\epsilon_0}$, $\hat{\epsilon_1}$ and $\hat{h}$.

Example 3.2.8. *In Lemma 3.2.9 below we will discuss the effect of* Sing *on 2-fiber products of presheaves of groupoids. To motivate the assumptions made there, we look at the following examples.*

(1) *The functor* Sing *does not respect 2-fiber products. For example, let $\mathcal{Z}$ be the constant presheaf on* Top *with value J (viewed as a stack), where $J = \{0 \longleftrightarrow 1\}$ is the interval groupoid, and let $\mathcal{X} = \mathcal{Y} = *$ be singletons mapping to the points 0 and 1 in $\mathcal{Z}$, respectively. Then,*

$$\mathcal{X}\tilde{\times}_{\mathcal{Z}}\mathcal{Y} = *\tilde{\times}_{\mathcal{Z}}*$$

is equivalent to a point, while

$$\mathrm{Sing}(\mathcal{X}) \times_{\mathrm{Sing}(\mathcal{Z})} \mathrm{Sing}(\mathcal{Y}) = * \times_{\mathrm{Sing}(\mathcal{Z})} *$$

is the empty set.

(2) *It is not reasonable to expect that* Sing *takes 2-fiber products to homotopy fiber products either. For example, let $\mathcal{Z} = [0,1]$ be the unit interval, and let $\mathcal{X} = \mathcal{Y} = *$ be singletons mapping to the points 0 and 1 in $\mathcal{Z}$, respectively. Then,*

$$\mathcal{X}\tilde{\times}_{\mathcal{Z}}\mathcal{Y} = *\tilde{\times}_{\mathcal{Z}}* = * \times_{[0,1]} *$$

is the empty set, while

$$\mathrm{Sing}(\mathcal{X}) \overset{h}{\times}_{\mathrm{Sing}(\mathcal{Z})} \mathrm{Sing}(\mathcal{Y}) = * \overset{h}{\times}_{\mathrm{Sing}(\mathcal{Z})} *$$

is non-empty (in fact, homotopy equivalent to a point).

Lemma 3.2.9. *Consider the following diagram in* pshGpd*:*

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow{\scriptstyle p} \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

Suppose that p is a Reedy fibration (by Lemma 3.3.10 this is automatic if $\mathcal{X}$ and $\mathcal{Z}$ are presheaves of sets). Then, there is a natural weak equivalence of simplicial sets

$$\mathrm{Sing}(\mathcal{X}) \times_{\mathrm{Sing}(\mathcal{Z})} \mathrm{Sing}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{Sing}(\mathcal{X}\tilde{\times}_{\mathcal{Z}}\mathcal{Y}).$$

Proof. Since p is a Reedy fibration (hence object-wise fibration when restricted to $\boldsymbol{\Delta}$), the natural map

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X}\tilde{\times}_{\mathcal{Z}}\mathcal{Y}$$

is an object-wise weak equivalence when restricted to $\boldsymbol{\Delta}$ (see Lemma 2.7.2). It follows from Lemma 3.2.3 that the induced map

$$\mathrm{Sing}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}) \xrightarrow{\sim} \mathrm{Sing}(\mathcal{X}\tilde{\times}_{\mathcal{Z}}\mathcal{Y})$$

is a weak equivalence of simplicial sets. Precomposing with the isomorphism of Proposition 3.2.5, we obtain the desired weak equivalence

$$\mathrm{Sing}(\mathcal{X}) \times_{\mathrm{Sing}(\mathcal{Z})} \mathrm{Sing}(\mathcal{Y}) \xrightarrow{\cong} \mathrm{Sing}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}) \xrightarrow{\sim} \mathrm{Sing}(\mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y}).$$

□

3.2.2 Explicit description of the bisimplicial set $B\mathcal{X}$

For $\mathcal{X} \in \mathsf{pshGpd}$, we give an explicit description of the elements of the bisimplicial set $B(\mathcal{X})$ and the simplicial set $\mathrm{Sing}(\mathcal{X})$. This description will not be used anywhere else in the paper.

An element of the set $B(\mathcal{X})_{m,n}$ is given by a chain

$$\eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n,$$

where $\eta^i : |\Delta^m| \to \mathcal{X}$ are objects and α^i are morphisms in the groupoid $\mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^m|, \mathcal{X})$.

The vertical face and degeneracy maps of $B(\mathcal{X})$ are 'nerve-wise', e.g.,

$$\begin{aligned} d_i^V : B(\mathcal{X})_{m,n} &\to B(\mathcal{X})_{m,n-1}, \\ \eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n &\mapsto \eta^0 \Rightarrow \cdots \Rightarrow \eta^{i-1} \xRightarrow{\alpha^{i+1} \circ \alpha^i} \eta^{i+1} \Rightarrow \cdots \overset{\alpha_n}{\Longrightarrow} \eta^n, \end{aligned}$$

for $i \neq 0, n$. For $i = 0$ we have

$$\begin{aligned} d_0^V : B(\mathcal{X})_{m,n} &\to B(\mathcal{X})_{m,n-1}, \\ \eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n &\mapsto \eta^1 \overset{\alpha^2}{\Longrightarrow} \eta^2 \overset{\alpha^3}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n. \end{aligned}$$

For $i = n$ we have

$$\begin{aligned} d_n^V : B(\mathcal{X})_{m,n} &\to B(\mathcal{X})_{m,n-1}, \\ \eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n &\mapsto \eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \xRightarrow{\alpha^{n-1}} \eta^{n-1}. \end{aligned}$$

The horizontal face and degeneracy maps of $B(\mathcal{X})$ are 'geometric', e.g.,

$$\begin{aligned} d_i^H : B(\mathcal{X})_{m,n} &\to B(\mathcal{X})_{m-1,n}, \\ \eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n &\mapsto \eta_i^0 \overset{\alpha_i^1}{\Longrightarrow} \eta_i^1 \overset{\alpha_i^2}{\Longrightarrow} \cdots \overset{\alpha_i^n}{\Longrightarrow} \eta_i^n \end{aligned}$$

is induced from $d^i : |\Delta^{m-1}| \to |\Delta^m|$. That is, $\eta_i^j = \eta^j \circ d^i : |\Delta^{m-1}| \to \mathcal{X}$ and $\alpha_i^j = \alpha^j \circ d^i$.

An element of $\mathrm{Sing}(\mathcal{X})_n = B(\mathcal{X})_{n,n}$, can be described as a chain

$$\eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n$$

with $\eta^i : |\Delta^n| \to \mathcal{X}$. For $i \neq 0, n$ the effect of the face map $d_i := d_i^H \circ d_i^V : \mathrm{Sing}(\mathcal{X})_n \to \mathrm{Sing}(\mathcal{X})_{n-1}$ is given by

$$\begin{aligned} d_i : \mathrm{Sing}(\mathcal{X})_n = B(\mathcal{X})_{n,n} &\to B(\mathcal{X})_{n-1,n-1} = \mathrm{Sing}(\mathcal{X})_{n-1}, \\ \eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n &\mapsto \eta_i^0 \overset{\alpha_i^1}{\Longrightarrow} \eta_i^1 \overset{\alpha_i^2}{\Longrightarrow} \cdots \xRightarrow{\alpha_i^{i-1}} \eta_i^{i-1} \xRightarrow{\alpha_i^{i+1} \circ \alpha_i^i} \eta_i^{i+1} \Rightarrow \cdots \overset{\alpha_i^n}{\Longrightarrow} \eta_i^n. \end{aligned}$$

For $i = 0$ we have

$$\begin{aligned} d_0 : \mathrm{Sing}(\mathcal{X})_n &\to \mathrm{Sing}(\mathcal{X})_{n-1}, \\ \eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n &\mapsto \eta_0^1 \overset{\alpha_0^2}{\Longrightarrow} \cdots \overset{\alpha_0^{n-1}}{\Longrightarrow} \eta_0^{n-1} \overset{\alpha_0^n}{\Longrightarrow} \eta_0^n. \end{aligned}$$

For $i = n$ we have

$$\begin{aligned} d_n : \mathrm{Sing}(\mathcal{X})_n &\to \mathrm{Sing}(\mathcal{X})_{n-1}, \\ \eta^0 \overset{\alpha^1}{\Longrightarrow} \eta^1 \overset{\alpha^2}{\Longrightarrow} \cdots \overset{\alpha^n}{\Longrightarrow} \eta^n &\mapsto \eta_n^0 \overset{\alpha_n^1}{\Longrightarrow} \eta_n^1 \overset{\alpha_n^2}{\Longrightarrow} \cdots\cdots \overset{\alpha_n^{n-1}}{\Longrightarrow} \eta_n^{n-1}. \end{aligned}$$

3.3 Lifting lemmas

In this section we prove some lifting lemmas which will be used in the subsequent sections in the proofs of our main results. We invite the reader to consult Remark 2.10.10 before reading this section to prevent possible confusion caused by our usage of the term 'weak' in what follows.

3.3.1 Tilde construction

There is a technical obstacle that we consider in this section. The problem that we will face is that the Yoneda embedding (defined in Lemma 2.2.3) does not preserve colimits in general. We will show that when we restrict ourselves to Serre stacks we can overcome this obstacle with the 'tilde construction' that we shall define below.

Consider the inclusion $\mathbf{\Delta} \to \mathsf{Top}$, $[n] \mapsto |\Delta^n|$. Its left Kan extension

$$\begin{aligned} \mathsf{sSet} &\to \mathsf{pshSet} \quad (\hookrightarrow \mathsf{pshGpd}) \\ A &\mapsto \tilde{A} \end{aligned}$$

is uniquely determined by the property that it preserves colimits and sends Δ^n to $|\Delta^n|$ (rather, the presheaf represented by it). It is left adjoint to the restriction functor

$$\begin{aligned} -_{\mathbf{\Delta}} : \mathsf{pshSet} &\to \mathsf{sSet} \quad (\hookrightarrow \mathsf{sGpd}) \\ X &\mapsto X_{\mathbf{\Delta}} = \mathrm{Hom}_{\mathsf{pshSet}}(|\Delta^\bullet|, X). \end{aligned}$$

More explicitly, $\tilde{A}$ is constructed exactly like the colimit construction of the geometric realization of A, except that instead of using the topological simplices $|\Delta^n|$ as building blocks we use the presheaves in pshSet represented by them.

We have a natural map

$$\psi_A : \tilde{A} \to |A|. \tag{3.1}$$

This is adjoint to the map $A \to \mathrm{Sing}(|A|) = |A|_{\mathbf{\Delta}}$, the unit of the adjunction $|-| : \mathsf{Top} \rightleftharpoons \mathsf{sSet} : \mathrm{Sing}$. Note that the Yoneda embedding $\mathsf{Top} \to \mathsf{pshSet}$ (or pshGpd) does not necessarily preserve colimits, so ψ_A is often not an isomorphism (but it is when $A = \Delta^n$).

Example 3.3.1. *Write Λ^n_k as the coequalizer of*

$$\coprod_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \coprod_{i\in\{0,1,\dots,n\}, i\neq k} \Delta^{n-1} \to \Lambda^n_k$$

Then, we can write $\tilde{\Lambda}^n_k$ as the coequalizer

$$\coprod_{0\le i<j\le n} |\Delta^{n-2}| \rightrightarrows \coprod_{i\in\{0,1,\dots,n\}, i\neq k} |\Delta^{n-1}|$$

in pshSet*. The map $\psi_{\Lambda^n_k} : \tilde{\Lambda}^n_k \to |\Lambda^n_k|$ is almost never an isomorphism.*

We can extend the restriction functor $-_{\mathbf{\Delta}}$ defined above to pshGpd:

$$\begin{aligned} -_{\mathbf{\Delta}} : \mathsf{pshGpd} &\to \mathsf{sGpd}, \\ \mathcal{X} &\mapsto \mathcal{X}_{\mathbf{\Delta}} = \mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^\bullet|, \mathcal{X}). \end{aligned}$$

We have the following lemma.

Lemma 3.3.2. *Let A be a simplicial set and $\mathcal{X}$ a presheaf of groupoids. Then, we have an isomorphism (and not just an equivalence) of groupoids*

$$\begin{aligned} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}), \\ f &\mapsto f_{\mathbf{\Delta}} \circ \iota_A. \end{aligned}$$

Here, $\iota_A : A \to \tilde{A}_{\mathbf{\Delta}}$ is the unit of adjunction. In particular, we have the following natural isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^n|, \mathcal{X}) & \xrightarrow{\quad\cong\quad} & \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, \mathcal{X}_{\mathbf{\Delta}}) \\ & \underset{\cong}{\searrow} \quad \underset{\cong}{\swarrow} & \\ & \mathcal{X}(|\Delta^n|) & \end{array}$$

Proof. In the case where $\mathcal{X}$ is a presheaf of sets, i.e., $\mathcal{X} \in \mathsf{pshSet}$, this is just the left adjointness of the left Kan extension. For the general case view $\mathcal{X}$ as a groupoid object in pshSet and apply the above isomorphisms to $\mathrm{Ob}(\mathcal{X})$ and $\mathrm{Mor}(\mathcal{X}) \in \mathsf{pshSet}$. □

3.3.2 Yoneda and colimits

As we pointed out above, unless A is representable, the natural map $\psi_A : \tilde{A} \to |A|$ is not in general an isomorphism of presheaves of sets. This is due to the fact that the Yoneda functor $\mathsf{Top} \to \mathsf{pshSet}$ (or $\mathsf{Top} \to \mathsf{pshGpd}$) does not preserve colimits.

In certain situations, however, the following lemma is useful.

Lemma 3.3.3. *Let $\mathcal{X}$ be a Serre topological stack. Let $A \hookrightarrow B$ and $A \hookrightarrow C$ be closed embeddings of topological spaces. Assume both maps are locally Serre cofibrations. Then, the map*

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B\coprod_A C, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(B{\coprod_A}' C, \mathcal{X})$$

induced by the natural map $B\coprod'_A C \to B\coprod_A C$ is an equivalence of groupoids. Here, $\coprod$ stands for colimit in Top *and $\coprod'$ stands for colimit in* pshSet *(which is the same as colimit in* pshGpd*).*

Proof. This is an easy consequence of ([2], Proposition 1.3). Note that ([2], Proposition 1.3) is proved for Hurewicz stacks; the proof for the case of Serre topological stacks is entirely similar.

To prove the lemma, note that the groupoid

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B{\coprod_A}' C, \mathcal{X}) = \mathrm{Hom}_{\mathsf{pshGpd}}(B, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C, \mathcal{X})$$

can be identified with the full subgroupoid of the groupoid

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B, \mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C, \mathcal{X})$$

consisting of those triples (f, g, φ),

$$f : B \to \mathcal{X},\ g : G \to \mathcal{X},\ \varphi : f|_A \Rightarrow g|_A$$

for which $f|_A = g|_A$ and $\varphi = \mathrm{id}$. The composition

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B\coprod_A C, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(B{\coprod_A}' C, \mathcal{X})$$
$$\hookrightarrow \mathrm{Hom}_{\mathsf{pshGpd}}(B, \mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C, \mathcal{X})$$

is an equivalence of groupoids by (the Serre version) of ([2], Proposition 1.3). Since the second functor is fully faithful, it follows that both functors are equivalences of groupoids. □

Remark 3.3.4. In the course of the proof of the above lemma we have also shown that the natural map

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C, \mathcal{X})$$
$$\hookrightarrow \mathrm{Hom}_{\mathsf{pshGpd}}(B, \mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C, \mathcal{X})$$

is an equivalence of groupoids. In other words, the strict and the 2-fiber product are equivalent.

Definition 3.3.5. We say that a simplicial set A has the *gluing property* with respect to a presheaf of groupoids $\mathcal{X}$ if the map

$$\Sigma_{A,\mathcal{X}} : \mathrm{Hom}_{\mathsf{pshGpd}}(|A|, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X})$$
$$f \mapsto f \circ \psi_A$$

is an equivalence of groupoids.

Lemma 3.3.6. *The simplicial n-simplex Δ^n has the gluing property with respect to any presheaf of groupoids $\mathcal{X}$.*

Proof. This follows from the fact that $\psi_A : |A| \to \tilde{A}$ is an isomorphism when $A = \Delta^n$. In fact, in this case the maps $\Sigma_{\Delta^n,\mathcal{X}}$ are isomorphisms of groupoids. □

Lemma 3.3.7. *Let $A \hookrightarrow B$ and $A \hookrightarrow C$ be monomorphisms of simplicial sets. If A, B and C have the gluing property with respect to a Serre topological stack $\mathcal{X}$, then so does $B \coprod_A C$.*

Proof. We have

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{pshGpd}}(|B\coprod_A C|,\mathcal{X}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathsf{pshGpd}}(|B|\coprod_{|A|}|C|,\mathcal{X}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pshGpd}}(|B|\coprod_{|A|}{}'|C|,\mathcal{X}) && \text{(Lemma 3.3.3)} \\
&\xrightarrow{\cong} \mathrm{Hom}_{\mathsf{pshGpd}}(|B|,\mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(|A|,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(|C|,\mathcal{X}) \\
& && \text{(Definition of colimit)} \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pshGpd}}(|B|,\mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(|A|,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(|C|,\mathcal{X}) && \text{(Remark 3.3.4)} \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B},\mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{C},\mathcal{X}) && \text{(Assumption)}
\end{aligned}
$$

Notice that the above equivalence is equal to the following composition:

$$
\begin{aligned}
&\mathrm{Hom}_{\mathsf{pshGpd}}(|B\coprod_A C|,\mathcal{X}) \xrightarrow{\Sigma_{B\coprod_A C,\mathcal{X}}} \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{B\coprod_A C},\mathcal{X}) \cong \\
&\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}\coprod_{\tilde{A}}\tilde{C},\mathcal{X}) \cong \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B},\mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{C},\mathcal{X}) \\
&\hookrightarrow \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B},\mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{C},\mathcal{X}).
\end{aligned}
$$

Since the last functor is fully faithful and the composition is shown above to be an equivalence, it follows that $\Sigma_{B\coprod_A C,\mathcal{X}}$ is also an equivalence. □

Recall that a simplicial set X is called *non-singular* ([35], Definition 1.2.2) if for every non-degenerate n-simplex x, the corresponding map $\bar{x} : \Delta^n \to X$ is a monomorphism. Examples we will encounter include A=$\partial\Delta^n$, Λ^n_k and $\Lambda^n_k \times \Delta^1$. Non-singular simplicial sets are closed under taking sub-objects and products.

Corollary 3.3.8. *Let D be a finite non-singular simplicial set. Then, D has the gluing property with respect to every Serre topological stack $\mathcal{X}$. That is, for every Serre topological stack $\mathcal{X}$, the map $\psi_D : \tilde{D} \to |D|$ induces an equivalence of groupoids*

$$
\begin{aligned}
\Sigma_{D,\mathcal{X}} : \mathrm{Hom}_{\mathsf{pshGpd}}(|D|,\mathcal{X}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{D},\mathcal{X}) \\
f &\mapsto f \circ \psi_D.
\end{aligned}
$$

Proof. Proof proceeds by induction on the total number of non-degenerate simplices of D. Choose a maximal non-degenerate simplex x, and write $B \subset D$ for the sub simplicial set of D generated by the rest of the non-degenerate simplices. Set $A := B \cap \bar{x}(\Delta^n)$, where $\bar{x} : \Delta^n \to D$ is the map corresponding to x; note that this map is a monomorphism by assumption. Also, note that the sets of non-degenerate simplices of A and B are both properly contained in the set of non-degenerate simplices of D, so they have a smaller size. By the induction hypothesis, the claim is true for A and B, and by Lemma 3.3.6 it is also true for Δ^n. Therefore, by Lemma 3.3.7, the claim is true for $D = B \coprod_A \Delta^n$. □

As we pointed out above, in the case $D = \Delta^n$ the above equivalence is indeed an isomorphism of groupoids.

3.3.3 Reedy fibrations in pshGpd

We begin with our main definition.

Definition 3.3.9. We say that a map of presheaves of groupoids $p : \mathcal{X} \to \mathcal{Y}$ is a *Reedy fibration* if $p_{\mathbf{\Delta}} : \mathcal{X}_{\mathbf{\Delta}} \to \mathcal{Y}_{\mathbf{\Delta}}$ is a Reedy fibration in sGpd (see Section 2.7.4).

Lemma 3.3.10. *Let X and Y be presheaves of simplicial sets, regarded as objects in* pshGpd*. Then, any morphism $p : X \to Y$ is a Reedy fibration.*

Proof. This follows from Lemma 2.7.14. □

Proposition 3.3.11. *If $p : \mathcal{X} \to \mathcal{Y}$ is an injective fibration of presheaves of groupoids, then p is a Reedy fibration.*

Proof. We have to show that, for every n, the map

$$\mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, \mathcal{X}_{\mathbf{\Delta}}) \to \mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, \mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, \mathcal{Y}_{\mathbf{\Delta}})$$

is a fibration of groupoids. Via the tilde construction, the above map is isomorphic to

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{\Delta^n}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{\partial\Delta^n}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{\partial\Delta^n}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{\Delta^n}, \mathcal{Y}).$$

This map is a fibration of groupoids because $p : \mathcal{X} \to \mathcal{Y}$ is a fibration and $\widetilde{\partial\Delta^n} \to \widetilde{\Delta^n} = \Delta^n$ is a cofibration in the injective model structure on pshGpd (to see the latter, write $\partial\Delta^n$ as the colimit of its faces and use the fact that the tilde construction preserves colimits). The claim now follows from Proposition 2.7.7 (also see [21], Remark A.3.1.6(2')). □

Proposition 3.3.12. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a Reedy fibration of presheaves of groupoids, and let $A \to B$ be a monomorphism of simplicial sets. Then, the map*

$$\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}) \to \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}})$$

and, equivalently (see Lemma 3.3.2), the map

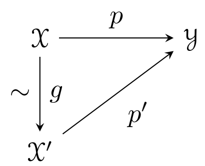
$$\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y})$$

are fibrations of groupoids.

Proof. In fact, the first map is a fibration of groupoids for any Reedy fibration $X \to Y$ in sGpd (in our case $X = \mathcal{X}_{\mathbf{\Delta}}$ and $Y = \mathcal{Y}_{\mathbf{\Delta}}$). In view of Corollary 2.7.17 this follows from Proposition 2.7.7 with $\mathsf{C} = \Delta$ (also see [21], Remark A.3.1.6(2')).

Alternatively, use ([10], Lemma 4.5), with $M = \mathsf{Gpd}$, $K = A$, $L = B$, $X = \mathcal{X}_{\mathbf{\Delta}}$ and $Y = \mathcal{Y}_{\mathbf{\Delta}}$. □

Proposition 3.3.13. *For any morphism of presheaves of groupoids $p : \mathcal{X} \to \mathcal{Y}$, there exists a strictly commutative diagram*



where p' is an injective (hence, also Reedy) fibration and $g : \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ is an equivalence of presheaves of groupoids.

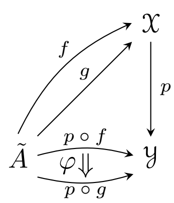
Proof. Take the usual fibrant replacement in the injective model structure on pshGpd and use Proposition 3.3.11. □

3.3.4 Restricted fiberwise homotopy

We want to consider a version of fiberwise homotopy considered in Section 2.10.3 which is compatible with the tilde construction.

The notion of restricted homotopy we introduce below only applies to morphisms of the form $\tilde{A} \to \mathcal{X}$, where A is a simplicial set and $\mathcal{X}$ is a presheaf of groupoids.

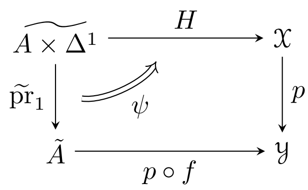
Definition 3.3.14. Let A be a simplicial set. Let $f, g : \tilde{A} \to \mathcal{X}$ and $p : \mathcal{X} \to \mathcal{Y}$ be morphisms of presheaves of groupoids, and $\varphi : p \circ f \Rightarrow p \circ g$ a 2-isomorphism:



A *restricted fiberwise homotopy from f to g relative to φ* is a quadruple $(H, \epsilon_0, \epsilon_1, \psi)$ where

- $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ is a morphism of presheaves of groupoids;
- $\epsilon_0 : f \Rightarrow H_0$ and $\epsilon_1 : H_1 \Rightarrow g$ are 2-isomorphisms;

- $\psi : p \circ f \circ \widetilde{\mathrm{pr}}_1 \Rightarrow p \circ H$ is a 2-isomorphism such that $\psi_0 = p \circ \epsilon_0$, $\psi_1 \cdot (p \circ \epsilon_1) = \varphi$, i.e.

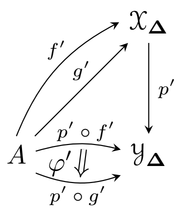


commutes.

(Notation: $H_0 := H \circ \tilde{i}$, where $i : A \to A \times \Delta^1$ is the time 0 map.) In the case where φ and ψ are both identity 2-isomorphisms (so $p \circ f = p \circ g$ and $p \circ f \circ \widetilde{\mathrm{pr}}_1 = p \circ H$) we say that H is a *restricted homotopy relative to* $\mathcal{Y}$.

A restricted fiberwise homotopy as above is called *strict* if ϵ_0 and ϵ_1 are the identity 2-isomorphisms.

Remark 3.3.15. In view of the adjunction of Lemma 3.3.2, we can replace the diagrams above with their corresponding diagram in the category of simplicial groupoids. For example,



Thus, we can regard a restricted homotopy as a homotopy in the category of simplicial groupoids.

Remark 3.3.16. As in Remark 2.10.8, there is some flexibility in choosing H, namely, we are allowed to replace H by any map 2-isomorphic to it (and adjust ϵ_0, ϵ_1 and ψ accordingly).

An ordinary homotopy gives rise to a restricted homotopy.

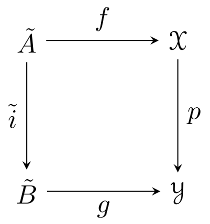
Lemma 3.3.17. *Let A be a simplicial set and let $\mathcal{A} := \tilde{A}$. Notation being as in Definition 2.10.7, suppose that we are given a fiberwise homotopy $(H, \epsilon_0, \epsilon_1, \psi)$ from f to g relative to φ. Then, precomposing with the natural map $\widetilde{A \times \Delta^1} \to \tilde{A} \times [0,1]$ gives rise to a restricted fiberwise homotopy from f to g relative to φ.*

Proof. Straightforward. □

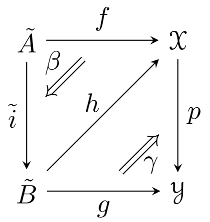
3.3.5 Strictifying lifts

The following lemma is useful when we want to replace a lax solution to a strict lifting problem with a strict solution.

Lemma 3.3.18. *Consider the following strictly commutative diagram, where p is a Reedy fibration of presheaves of groupoids (Definition 3.3.9) and i is a monomorphism of simplicial sets:*



Suppose that there exists a lift h and 2-isomorphisms β and γ making the following diagram 2-commutative:



Then, we can replace h by a 2-isomorphic morphism h' so that β and γ become the identity 2-isomorphisms. More precisely, $h' \circ \tilde{i} = f$, $p \circ h' = g$, and there is $\theta : h' \Rightarrow h$ such that $\theta \circ \tilde{i} = \beta$ and $p \circ \theta = \gamma$.

Proof. By Proposition 3.3.12, the natural map

$$\Psi : \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y})$$

is a fibration of groupoids. The map h can be regarded as an object on the left hand side, with $\Psi(h) = (h \circ \tilde{i}, p \circ h)$. Since Ψ is a fibration, we can lift the 2-isomorphism $(\beta, \gamma) : (f, g) \Rightarrow (h \circ \tilde{i}, p \circ h)$ to a 2-isomorphism $\theta : h' \to h$. This is exactly what we need. $\square$

In most of our applications of the above lemma, we will have $B = \Delta^n$, in which case $\tilde{B} = |\Delta^n|$.

Corollary 3.3.19. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a Reedy fibration of presheaves of groupoids, A a simplicial set, and $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ a restricted homotopy (see Section 3.3.4) relative to $\mathcal{Y}$ starting at $H_0 := H|_{\tilde{A} \times \{0\}} : \tilde{A} \to \mathcal{X}$. Then, for every 2-isomorphism $\beta : f' \Rightarrow H_0$, there exists a restricted homotopy*

$$H' : \widetilde{A \times \Delta^1} \to \mathcal{X} \textit{ relative to } \mathcal{Y}$$

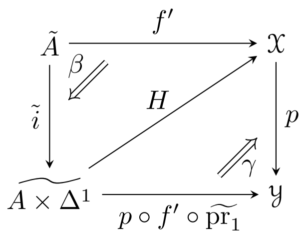
and a 2-isomorphism $\Theta : H' \Rightarrow H$ such that $p \circ \Theta = p \circ \beta \circ \widetilde{\mathrm{pr}_1}$ as 2-isomorphisms

$$p \circ f' \circ \widetilde{\mathrm{pr}_1} \Rightarrow p \circ H_0 \circ \widetilde{\mathrm{pr}_1} (= p \circ H)$$

i.e., Θ is relative to $p \circ \beta \circ \widetilde{\mathrm{pr}_1}$ and that

$$f' = H'_0 := H'|_{\tilde{A} \times \{0\}} \textit{ and } \beta = \Theta_0 := \Theta|_{\tilde{A} \times \{0\}}.$$

Proof. With the notation of Lemma 3.3.18, let $B = A \times \Delta^1$, $i : A \to A \times \Delta^1$ the inclusion at time 0, $f = f'$, $g = p \circ f' \circ \widetilde{\mathrm{pr}_1}$, $h = H$, $\beta = \beta$ and $\gamma = p \circ \beta \circ \widetilde{\mathrm{pr}_1}$, as in the diagram



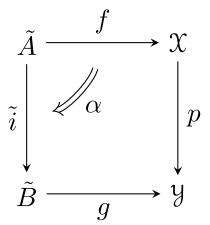
The result now follows from Lemma 3.3.18. □

3.3.6 Strict lifts for Serre+Reedy fibrations

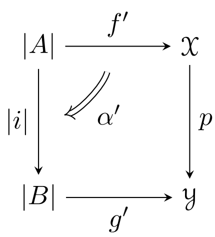
From now on, we will assume that our simplicial sets A and B are finite non-singular simplicial sets (see Corollary 3.3.8 and the preceding paragraph). For example, Δ^n, $\partial\Delta^n$ and Λ^n_k have this property. If A and B have this property, then $A \times B$ also has this property, and so does any colimit $A \coprod_C B$, as long as the maps $C \to A$ and $C \to B$ are monomorphisms. In particular, if $i : A \to B$ is a monomorphism, then the mapping cylinder $\mathrm{Cyl}(i)$ has this property.

Lemma 3.3.20. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks and $i : A \to B$ a monomorphism of finite non-singular simplicial sets. If p is a (weak) Serre fibration and either p or i is a weak equivalence, then $\tilde{i} : \tilde{A} \to \tilde{B}$ has (weak) LLP with respect to p (see Definition 2.10.9).*

Proof. Consider the lifting problem



First note that to solve it we are allowed to replace each of f and g with a 2-isomorphic morphism (and adjust α accordingly). So, we may assume, by Corollary 3.3.8, that there are maps $f' : |A| \to \mathcal{X}$ and $g' : |B| \to \mathcal{Y}$ such that $f = f' \circ \psi_A$ and $g = g' \circ \psi_B$. Here, $\psi_A : \tilde{A} \to |A|$ is as in Eq. (3.1).Thus, our lifting problem translates to



(The existence of the unique α' is guaranteed by Corollary 3.3.8.) This problem can now be solved under the given assumptions. Precomposing with the ψ maps, we obtain a solution to the original lifting problem. (Also see Proposition 2.10.14.) ☐

Lemma 3.3.21. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks and $i : A \to B$ a monomorphism of finite non-singular simplicial sets. If p is a Serre fibration and also a Reedy fibration, and either p or i is a weak equivalence, then $\tilde{i} : \tilde{A} \to \tilde{B}$ has strict LLP with respect to p. That is, if in the diagram*

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \tilde{i}}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \tilde{B} & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

the outer square is strictly commutative, then there exists a lift h making both triangles strictly commutative.

Proof. First use Lemma 3.3.20 to find a solution h which makes the two triangles commutative up to 2-isomorphism. Then use Lemma 3.3.18 to rectify h to make the triangles strictly commutative. ☐

Corollary 3.3.22. *Assumptions being as in Lemma 3.3.21, the map*

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y})$$

and, equivalently (see Lemma 3.3.2), the map

$$\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}) \to \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}})$$

are fibration of groupoids that are surjective on objects (hence, also on morphisms).

Proof. Surjectivity on objects is simply a restatement of Lemma 3.3.21. They are fibrations by Proposition 3.3.12. ☐

3.3.7 Strict lifts for weak Serre+Reedy fibrations

Lemma 3.3.23. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks and $i : A \to B$ a monomorphism of finite non-singular simplicial sets. If p is a weak Serre fibration and also a Reedy fibration, and either p or i is a weak equivalence, then $\tilde{i} : \tilde{A} \to \tilde{B}$ has* strict *WLLP with respect to p in the following sense. If in the diagram*

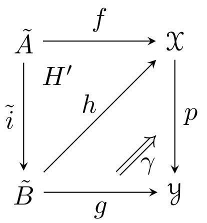
$$\begin{array}{ccc} \tilde{A} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \tilde{i}}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \tilde{B} & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

the outer square is strictly commutative, then there exists a lift h *and a morphism* $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ *such that*

i) the lower triangle is strictly commutative, and

ii) H *is a strict restricted fiberwise homotopy from* f *to* $h \circ \tilde{i}$ *relative to* $\mathcal{Y}$ *(see Section 3.3.4), where strictness means that* $H_0 = f$ *and* $H_1 = h \circ \tilde{i}$.

Proof. First use Lemma 3.3.20 to find a solution h which makes the lower triangle commutative up to a 2-isomorphism and the upper triangle commutative up to fiberwise homotopy $H' : \tilde{A} \times [0,1] \to \mathcal{X}$, as in the diagram



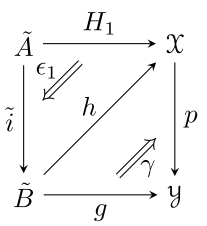
First, we rectify H' using Corollary 3.3.19. (Note that Corollary 3.3.19 only applies to restricted homotopy and not ordinary homotopy. So, we need to replace H' by the corresponding restricted homotopy $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$; see Lemma 3.3.17). Since p is a Reedy fibration and $\widetilde{A \times \Delta^1}$ is cofibrant, Proposition 3.3.12 (with $B = A \times \Delta^1$ and A the empty set) implies that

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{Y})$$

is a fibration of groupoids. So, we can replace H by a 2-isomorphic map so that it becomes relative to $\mathcal{Y}$ (namely, $p \circ f \circ \tilde{\mathrm{pr}}_1 = p \circ H$); see Remark 3.3.16 to see why this is allowed.

We can now use Corollary 3.3.19 to rectify H so that $H_0 = f$.

There are two more things to do now: ensure that the 2-isomorphism $\epsilon_1 : H_1 \Rightarrow h \circ \tilde{i}$ becomes an equality, and that $\gamma = \mathrm{id}$. This is achieved by applying Lemma 3.3.18 to the diagram



to adjust h so that ϵ_1 and γ become the identity 2-isomorphisms. $\square$

Let $p : \mathcal{X} \to \mathcal{Y}$ be a map of presheaves of groupoids and $i : A \to B$ a map of simplicial sets. Let L be the groupoid

$$\begin{aligned} L := \; & \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X})_p \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \\ \cong \; & \mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{X}_{\mathbf{\Delta}})_p \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}) \end{aligned}$$

of pairs (H, h), where $h : \tilde{B} \to \mathcal{X}$ is a morphism and $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ is a restricted fiberwise homotopy relative to $\mathcal{Y}$ such that $H_1 = h \circ \tilde{i}$. Here $H_1 : \tilde{A} \to \mathcal{X}$ stands for the precomposition of H with the time 1 inclusion map $\tilde{A} \to \widetilde{A \times \Delta^1}$, and the subscript p stands for 'fiberwise relative to $\mathcal{Y}$'. More precisely,

$$\begin{aligned}\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X})_p :=\ & \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y}) \\ \cong\ & \mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}}),\end{aligned}$$

where the first map in the fiber product is induced by p, and the second map is induced by the projection $\widetilde{A \times \Delta^1} \to \tilde{A}$.

Thus, we have isomorphisms of groupoids

$$\begin{aligned}L \cong\ & \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y}) \\ & \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \\ \cong\ & \mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}}) \\ & \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}).\end{aligned}$$

Corollary 3.3.24. *Notation being as above and assumptions being as in Lemma 3.3.23, the map*

$$\begin{aligned}L &\to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y}) \\ (H, h) &\mapsto (H_0, p \circ h)\end{aligned}$$

and, equivalently (see Lemma 3.3.2), the map

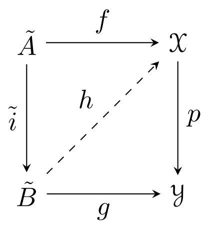
$$L \to \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}})$$

are fibrations of groupoids that are surjective on objects (hence, also on morphisms, as well as tuples of composable morphisms).

Proof. Let us denote the map in question by Ψ. The surjectivity of Ψ on objects is simply a restatement of Lemma 3.3.23. Let us spell this out. Consider an object in

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y}),$$

namely a pair (f, g) making the outer square in the following diagram strictly commutative:



By Lemma 3.3.23, this lifting problem has a weak solution (H, h), namely $h : \tilde{B} \to \mathcal{X}$ and $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ such that

i) the lower triangle is strictly commutative, and

ii) H is a strict restricted fiberwise homotopy from f to $h \circ \tilde{i}$ relative to $\mathcal{Y}$, where strictness means that $H_0 = f$ and $H_1 = h \circ \tilde{i}$.

By definition of L, such a pair determines an object in L mapping to the pair (f, g), $\Psi(H, h) = (f, g)$. This proves surjectivity on objects.

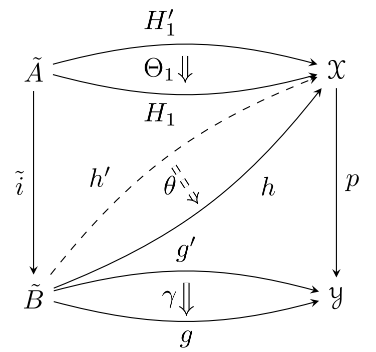
To prove fibrancy, suppose in the above setting that we are also given 2-isomorphisms $\beta : f' \Rightarrow f$ and $\gamma : g' \Rightarrow g$ such that $p \circ \beta = \gamma \circ \tilde{i}$. We need to construct a pair $(\Theta, \theta) \in \mathrm{Mor}(L)$ with the following properties:

i) $\theta : h' \Rightarrow h$ is relative to γ (that is, $p \circ \theta = \gamma$),

ii) $\Theta : H' \Rightarrow H$ is relative to $\gamma \circ \tilde{i} \circ \widetilde{\mathrm{pr}_1} = p \circ \beta \circ \widetilde{\mathrm{pr}_1}$ (that is, $p \circ \Theta = \gamma \circ \tilde{i} \circ \widetilde{\mathrm{pr}_1}$), $\Theta_0 = \beta$ and $\Theta_1 = \theta \circ \tilde{i}$.

By Corollary 3.3.19, we have a restricted fiberwise homotopy $H' : \widetilde{A \times \Delta^1} \to \mathcal{X}$ relative to $\mathcal{Y}$, and a 2-isomorphism $\Theta : H' \Rightarrow H$ relative to $p \circ \beta \circ \widetilde{\mathrm{pr}_1}$ such that $f' = H'_0$ and $\beta = \Theta_0$. This is our desired Θ.

To find θ, note that its restriction to $\tilde{A}$ is already determined, namely Θ_1. So, we need to extend Θ_1 to the whole of $\tilde{B}$ in such a way that $p \circ \theta = \gamma$. We do this by solving the following lifting problem for (h', θ):



Existence of a solution is guaranteed by Proposition 3.3.12. ☐

3.4 Sing preserves fibrations

In this section we study the effect of the functor Sing on fibrations of stacks. We begin with a simple example to show why the Reedy condition is necessary in the statement of our main result (Theorem 3.4.8).

Example 3.4.1. *Let X be a trivial groupoid with more than one point, namely one that is equivalent but not equal to a point. (For example we could take $X = \pi_1(\Delta^1)$) Let $\mathcal{X}$ be the constant presheaf with value X (viewed as a stack). Pick a point in X and consider the map $* \to \mathcal{X}$. This* *map is an equivalence of stacks, hence is a Serre fibration. However, the induced map of simplicial sets*

$$\mathrm{Sing}(*) = * \to N(X) = \mathrm{Sing}(\mathcal{X})$$

is not a Kan fibration.

3.4.1 Weak Kan fibrations

In what follows, the homotopy groups $\pi_n(X, x)$ of a simplicial set X which is not necessarily Kan are taken to be those of its geometric realization.

Definition 3.4.2. We say that a map of simplicial sets $p : X \to Y$ is a *weak Kan fibration* if for any trivial cofibration $i : A \to \Delta^n$, every lifting problem

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[g]{} & Y \end{array}$$

has a weak solution; namely, there exists $h : \Delta^n \to X$ such that the bottom triangle commutes and $f : A \to X$ is fiberwise homotopic to $h \circ i : A \to X$ relative to Y. We say that p is a *weak trivial Kan fibration* if it is a weak Kan fibration and, in addition, it has the weak lifting property with respect to the inclusions $\partial\Delta^n \to \Delta^n$, $n \geq 0$.

In the above definition, a fiberwise homotopy relative to Y means a map of simplicial sets $H : A \times \Delta^1 \to X$ such that $p \circ H$ is the trivial homotopy from $p \circ f$ to itself.

Remark 3.4.3. We do not know if the above definition is the "correct" simplicial counterpart of the notion of a weak Serre fibration, but it serves our purposes in this paper (thanks to Lemma 3.4.6). It is not clear to us whether a weak Kan fibration will have the weak left lifting property with respect to *all* trivial cofibrations.

Lemma 3.4.4. *Let $p : X \to Y$ be a trivial weak Kan fibration. Assume that Y is a Kan simplicial set, and that there exists a Kan simplicial set X' together with a weak equivalence $X' \to X$. Then, p is a weak equivalence.*

Proof. First we prove that $\pi_n(p)$ is injective. Let x be a base point that is in the image of X', and let $y = p(x)$. The fact that X' is Kan guarantees that any class in $\pi_n(X, x)$ is represented by a pointed map $f : \partial\Delta^n \to X$. If the image of this class in $\pi_n(Y, y)$ is trivial, we will have, since Y is

Kan, a filling g for $p \circ f$, as in the diagram

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\ f\ } & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[\ g\ ]{} & Y \end{array}$$

So, a lift h exists which makes the diagram commutative (possibly after replacing f by a fiberwise homotopic map). This implies that the class represented by f in $\pi_n(X, x)$ is trivial.

To prove surjectivity of $\pi_n(p)$, let $g : \partial\Delta^n \to Y$ represent an arbitrary class in $\pi_n(Y, y)$. To lift this to X, we begin by lifting $g|_{\Lambda_0^n} : \Lambda_0^n \to Y$ to X. To do so, first extend $g|_{\Lambda_0^n}$ to the whole Δ^n using the Kan property of Y. Then, apply the weak lifting property to the trivial cofibration $\{0\} \to \Delta^n$. Restricting the outcome to Λ_0^n, we find a lift $\hat{g} : \Lambda_0^n \to X$, sending 0 to a point that is fiberwise homotopic to x. There is no harm in replacing x with $\hat{g}(0)$, so we may assume $\hat{g}(0) = x$. Now consider the following lifting problem

$$\begin{array}{ccc} \partial\Delta^{n-1} & \xrightarrow{\ \hat{g} \circ (d_0|_{\partial\Delta^{n-1}})\ } & X \\ {\scriptstyle j}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^{n-1} & \xrightarrow[\ g \circ d_0\ ]{} & Y \end{array}$$

Here, $d_0 : \Delta^{n-1} \to \partial\Delta^n$ is the 0^{th} face of $\partial\Delta^n$ and $j : \partial\Delta^{n-1} \to \Delta^{n-1}$ is the inclusion map. A weak solution to this problem can be glued to $\hat{g}$ to give a map

$$G : \Lambda_0^n \coprod_{\partial\Delta^{n-1}} (\partial\Delta^{n-1} \times \Delta^1) \coprod_{\partial\Delta^{n-1}} \Delta^{n-1} \longrightarrow X$$

making the following diagram commutative

$$\begin{array}{ccc} \Lambda_0^n \coprod_{\partial\Delta^{n-1}} (\partial\Delta^{n-1} \times \Delta^1) \coprod_{\partial\Delta^{n-1}} \Delta^{n-1} & \xrightarrow{\ G\ } & X \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle p} \\ \partial\Delta^n & \xrightarrow[\ g\ ]{} & Y \end{array}$$

Here,

$$P : \Lambda_0^n \coprod_{\partial\Delta^{n-1}} (\partial\Delta^{n-1} \times \Delta^1) \coprod_{\partial\Delta^{n-1}} \Delta^{n-1} \to \partial\Delta^n$$

is the map that collapses $\partial\Delta^{n-1} \times \Delta^1$ to $\partial\Delta^{n-1}$ via the first projection; note that

$$|\Lambda_0^n \coprod_{\partial\Delta^{n-1}} (\partial\Delta^{n-1} \times \Delta^1) \coprod_{\partial\Delta^{n-1}} \Delta^{n-1}|$$

is homeomorphic to an n-sphere. The (geometric realization of the) map G represents a lift of the class in $\pi_n(Y, y)$ represented by g to a class in class in $\pi_n(X, x)$. This completes the proof of surjectivity. □

Lemma 3.4.5. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks that is a (weak) (trivial) Serre fibration and a Reedy fibration. Let $R_0(\mathcal{X}) = \mathrm{Ob}(\mathcal{X}_{\mathbf{\Delta}})$, $R_1(\mathcal{X}) = \mathrm{Mor}(\mathcal{X}_{\mathbf{\Delta}})$ and*

$$R_m(\mathcal{X}) = R_1(\mathcal{X}) \times_{R_0(\mathcal{X})} \times \cdots \times_{R_0(\mathcal{X})} R_1(\mathcal{X}).$$

Then, for every $m \geq 0$, the induced map

$$R_m(\mathcal{X}) \to R_m(\mathcal{Y})$$

is a (weak) (trivial) Kan fibration of simplicial sets.

Proof. First, we prove the statement in the case of a Serre fibration. Let $A = \Lambda^n_k$ and $B = \Delta^n$, and let $i : A \to B$ be the horn inclusion. By Corollary 3.3.22, we have a fibration of groupoids

$$\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}) \to \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}})$$

which is surjective on objects. Taking nerves on both sides, we find a fibration of simplicial sets

$$N\,\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}) \to N\,\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{N\,\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}})} N\,\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}})$$

which is surjective on m-simplices, for all m. The surjectivity on m-simplices precisely translates to the fact that i has LLP with respect to $R_m(\mathcal{X}) \to R_m(\mathcal{Y})$, as the above map on the level on m-simplices is, term by term, equal to the map

$$\mathrm{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{X})) \to \mathrm{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X})) \times_{\mathrm{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{Y}))} \mathrm{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{Y})).$$

This shows that $R_m(\mathcal{X}) \to R_m(\mathcal{Y})$ is a Kan fibration. The case of a trivial Serre fibration is proved similarly (taking $A = \partial\Delta^n$ instead of Λ^n_k).

Now consider the case where p is a weak Serre fibration. Let $B = \Delta^n$ and $i : A \to B$ be as in Definition 3.4.2. By Corollary 3.3.24, we have a fibration of simplicial sets

$$NL \to N\,\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{N\,\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}})} N\,\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}}) \tag{3.2}$$

which is surjective on m-simplices, for all m. By the discussion just before Lemma 3.3.24, and the fact that taking nerves commutes with fiber products, NL is isomorphic to

$$N\,\mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{X}_{\mathbf{\Delta}}) \times_{N\,\mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{Y}_{\mathbf{\Delta}})} N\,\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}})$$
$$\times_{N\,\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}})} N\,\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}).$$

Its set of m-simplices is then equal to

$$\begin{aligned}(NL)_m =& \operatorname{Hom}_{\mathsf{sSet}}(A \times \Delta^1, R_m(\mathcal{X})) \times_{\operatorname{Hom}_{\mathsf{Set}}(A\times\Delta^1, R_m(\mathcal{Y}))} \operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{Y})) \\ & \times_{\operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X}))} \operatorname{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{X})) \\ \cong & \operatorname{Hom}_{\mathsf{sSet}}(A \times \Delta^1, R_m(\mathcal{X}))_p \times_{\operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X}))} \operatorname{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{X})),\end{aligned}$$

where

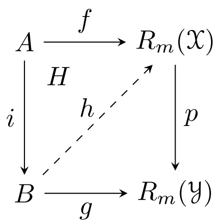
$$\operatorname{Hom}_{\mathsf{sSet}}(A \times \Delta^1, R_m(\mathcal{X}))_p := \operatorname{Hom}_{\mathsf{sSet}}(A \times \Delta^1, R_m(\mathcal{X})) \times_{\operatorname{Hom}_{\mathsf{Set}}(A\times\Delta^1, R_m(\mathcal{Y}))} \operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{Y}))$$

is the set of fiberwise homotopies. Thus, we can think of $(NL)_m$ as the set of pairs (H, h), where $h : B \to R_m(\mathcal{X})$ is a map of simplicial sets and $H : A \times \Delta^1 \to R_m(\mathcal{X})$ is a fiberwise homotopy relative to $R_m(\mathcal{Y})$ such that $H_1 = h \circ i$.

Hence, on the level of m-simplices, the map 3.2 above can be identified with the natural map

$$\begin{aligned}&\operatorname{Hom}_{\mathsf{sSet}}(A \times \Delta^1, R_m(\mathcal{X}))_p \times_{\operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X}))} \operatorname{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{X})) \\ &\to \operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X})) \times_{\operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{Y}))} \operatorname{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{Y})).\end{aligned}$$

which assigns to any weak solution (H, h), viewed as an element in the left hand side, its associated lifting problem (f, g), viewed as an element in the right hand side, as in the diagram



The surjectivity of this map precisely means that any such lifting problem has a weak solution.

The case of a weak trivial Serre fibration is proved similarly. □

3.4.2 A lemma on $d^* : \mathsf{sSet} \to \mathsf{bsSet}$

In this section we prove a lemma which is used in the proof of Lemma 3.4.7, which in turn plays an important role in the proof of our first main result, Theorem 3.4.8.

First, we briefly recall notion of exterior product of simplicial sets. Given simplicial sets X and Y, their *exterior product* is the bisimplicial set $X \boxtimes Y$ defined by

$$(X \boxtimes Y)_{m,n} := X_m \times Y_n.$$

We have $\operatorname{Diag}(X \boxtimes Y) = X \times Y$. The exterior product has the property that the functor

$$\begin{aligned}&\mathsf{sSet} \to \mathsf{bsSet}, \\ &A \mapsto A \boxtimes \Delta^n,\end{aligned}$$

is left adjoint to

$$\mathsf{bsSet} \to \mathsf{sSet},$$
$$X \mapsto X_{*,n}.$$

Let $A \to B$ be a map of simplicial sets. Recall the functor $d^* : \mathsf{sSet} \to \mathsf{bsSet}$ from Definition 2.8.2. We have a natural map

$$d^*(A) \to A \boxtimes B,$$

namely, the adjoint (see Proposition 3.2.5) to the diagonal inclusion

$$A \to \mathrm{Diag}(A \boxtimes B) = A \times B.$$

In the next lemma we show that for any monomorphism $A \to \Delta^n$, the map $d^*(A) \to A \boxtimes \Delta^n$ is a trivial cofibration. The case $A = \Lambda^n_k$ of the following lemma is proved in [13] (see [13], top of the page 221, just before Lemma 3.12).

Lemma 3.4.6. *Let $\gamma : A \to \Delta^n$ be a cofibration (not necessarily trivial) of simplicial sets. Then, for every m, we have*

$$(d^*A)_{m,*} = \coprod_{\alpha \in A_m} C_\alpha,$$

where $C_\alpha \subseteq A$ is the union of all faces of A that contain α. The natural map of bisimplicial sets

$$i : d^*(A) \to A \boxtimes \Delta^n,$$

namely, the left adjoint to the diagonal inclusion

$$(\mathrm{id}, \gamma) : A \to \mathrm{Diag}(A \boxtimes \Delta^n) = A \times \Delta^n,$$

is given on the m^{th} column by the inclusion

$$i_m : \coprod_{\alpha \in A_m} C_\alpha \hookrightarrow \coprod_{\alpha \in A_m} \Delta^n.$$

In particular, i_m is a trivial cofibration of simplicial sets for every m (thus, i is a vertical point-wise trivial cofibration of bisimplicial sets).

In the above lemma, by a *face* of A we mean the sub simplicial set generated by a (non-degenerate) simplex in A. Note that such a face is isomorphic to some simplex Δ^m and that $A \subseteq \Delta^n$ is necessarily a union of a collection of faces in Δ^n.

Proof. First we consider the case where γ is the inclusion of a face (so $A = \Delta^d$ for some d). In this case, $d^*(A) = A \boxtimes A$, which can be identified with a sub bisimplicial set of $A \boxtimes \Delta^n$ via the map whose effect on the m^{th} column is given by

$$i^A_m : \coprod_{\alpha \in A_m} A \overset{\gamma}{\hookrightarrow} \coprod_{\alpha \in A_m} \Delta^n \ (\subseteq \coprod_{\alpha \in \Delta^n_m} \Delta^n).$$

The key observation here is that, for any two faces F_i and F_j of A, the image of $i_m^{F_i \cap F_j}$ in the m^{th} column $\coprod_{\alpha \in \Delta^n_m} \Delta^n$ of $\Delta^n \boxtimes \Delta^n$ is equal to the intersection of the images of $i_m^{F_j}$ and $i_m^{F_k}$.

Now, for general A, write it as a coequalizer of the inclusions of its faces, namely

$$A = \operatorname{coeq}\left(\coprod_{j,k} F_j \cap F_k \rightrightarrows \coprod_j F_j \right).$$

Since d^* commutes with colimits, we have

$$d^*(A) = \operatorname{coeq}\left(\coprod_{j,k} d^*(F_j \cap F_k) \rightrightarrows \coprod_j d^*(F_j) \right).$$

The observation above that i_m respects intersections implies that $i_m^A : d^*(A) \to \Delta^n \boxtimes \Delta^n$ is injective and the image under i_m^A of $d^*(A)$ in the m^{th} column $\coprod_{\alpha \in \Delta^n_m} \Delta^n$ of $\Delta^n \boxtimes \Delta^n$ is the union of images of all $i_m^{F_j}$. This is precisely $\coprod_{\alpha \in A_m} C_\alpha$. □

3.4.3 A criterion for diagonal fibrations

To prove our first main result we need a generalization of Lemma 4.8 of ([13], Chapter IV) which we now prove.

In the next lemma, we are regarding X as the simplicial object $[m] \mapsto X_{m,*}$ in sSet.

Lemma 3.4.7. *Let $f : X \to Y$ be a Reedy fibration of bisimplicial sets. Let $\gamma : A \to \Delta^n$ be a monomorphism. Suppose that γ has (weak) left lifting property with respect to $f_{*,n} : X_{*,n} \to Y_{*,n}$, for all n (Definition 3.4.2). Then, γ has (weak) left lifting property with respect to* $\operatorname{Diag}(f) : \operatorname{Diag}(X) \to \operatorname{Diag}(Y)$. *In particular, if each $f_{*,n} : X_{*,n} \to Y_{*,n}$ is a (weak) (trivial) Kan fibration, then so is* $\operatorname{Diag}(f)$.

Proof. We want to show that $\gamma : A \to \Delta^n$ has (W)LLP with respect to $\operatorname{Diag}(f)$. By adjunction, this is equivalent to showing that the lifting problem

$$\begin{array}{ccc} d^*(A) & \xrightarrow{\;u\;} & X \\ {\scriptstyle d^*(\gamma)}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle f} \\ d^*(\Delta^n) & \xrightarrow[\;v\;]{} & Y \end{array} \qquad (*)$$

in bisimplicial sets has a solution, with the caveat that, in the 'weak' setting, instead of a fiberwise homotopy in the upper triangle of $(*)$ we should be asking for a map $d^*(A \times \Delta^1) \to X$ (with the obvious properties).

We solve $(*)$ in two steps, by writing the left vertical map

$$d^*(A) \to d^*(\Delta^n) = \Delta^{n,n} = \Delta^n \boxtimes \Delta^n$$

as composition of two inclusions

$$d^*(A) \xrightarrow{i} A \boxtimes \Delta^n \xrightarrow{j} \Delta^n \boxtimes \Delta^n.$$

Here, the map i is adjoint to the diagonal inclusion

$$(\mathrm{id}, \gamma) : A \to \mathrm{Diag}(A \boxtimes \Delta^n) = A \times \Delta^n;$$

see the paragraph before Lemma 3.4.6.

Step 1. We first solve the lifting problem

$$\begin{array}{ccc} d^*(A) & \xrightarrow{u} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle f} \\ A \boxtimes \Delta^n & \xrightarrow[v \circ j]{} & Y \end{array} \tag{**}$$

By Lemma 3.4.6, i is a point-wise trivial cofibration, so it has strict LLP with respect to f, as f is a Reedy fibration (see [13], Chapter IV, Lemma 3.3(1)). Therefore, our lifting problem has indeed a strict solution.

Step 2. We now solve the lifting problem

$$\begin{array}{ccc} A \boxtimes \Delta^n & \xrightarrow{h} & X \\ {\scriptstyle j}\downarrow & \overset{l'}{\nearrow} & \downarrow{\scriptstyle f} \\ \Delta^n \boxtimes \Delta^n & \xrightarrow[v]{} & Y \end{array} \tag{***}$$

Consider the adjoint lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X_{*,n} \\ {\scriptstyle \gamma}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle f_{*,n}} \\ \Delta^n & \longrightarrow & Y_{*,n} \end{array}$$

If γ has strict LLP with respect to $f_{*,n} : X_{*,n} \to Y_{*,n}$, this problem has a strict solution. Hence, our original problem $(*)$ also has a strict solution, and we are done.

If γ has weak LLP with respect to $f_{*,n} : X_{*,n} \to Y_{*,n}$, a lift $l : \Delta^n \to X_{*,n}$ exists, but the upper triangle commutes only up to a fiberwise homotopy $H : A \times \Delta^1 \to X_{*,n}$ (relative to $Y_{*,n}$). By adjunction, this gives rise to a lift

$$l' : \Delta^n \boxtimes \Delta^n \to X$$

in $(***)$. The upper triangle in $(***)$, however, is not, strictly speaking, homotopy commutative. Rather, instead of a homotopy we have a map $H' : (A \times \Delta^1) \boxtimes \Delta^n \to X$, the adjoint of H. Let H'' be the composition

$$H'' : d^*(A \times \Delta^1) \to (A \times \Delta^1) \boxtimes \Delta^n \xrightarrow{H'} X.$$

Here, the first map is adjoint to

$$(\mathrm{id}, \gamma) \times \mathrm{id}_{\Delta^1} : A \times \Delta^1 \to \mathrm{Diag}((A \times \Delta^1) \boxtimes \Delta^n) = (A \times \Delta^1) \times \Delta^n = A \times \Delta^n \times \Delta^1,$$

where $(\mathrm{id}, \gamma) : A \to A \times \Delta^n$ is the diagonal inclusion; see the paragraph before Lemma 3.4.6. It follows that the pair

$$l' : \Delta^n \boxtimes \Delta^n \to X, \quad H'' : d^*(A \times \Delta^1) \to X$$

is the desired solution to $(*)$. □

3.4.4 Singular functor preserves fibrations

We are finally ready to prove one of our main results.

Theorem 3.4.8. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks that is a (weak) (trivial) Serre fibration and also a Reedy fibration. Then,* $\mathrm{Sing}(p) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ *is a (weak) (trivial) Kan fibration.*

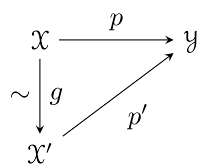
Proof. Let $R_m(\mathcal{X}) := B(\mathcal{X})_{*,m}$ be the m^{th} row of the bisimplicial set $B(\mathcal{X})$ (where $B(\mathcal{X})$ is defined in Definition 3.2.1). Note that we have

$$R_0(\mathcal{X}) = \mathrm{Ob}(\mathcal{X}_{\mathbf{\Delta}}),\ R_1(\mathcal{X}) = \mathrm{Mor}(\mathcal{X}_{\mathbf{\Delta}}),\ R_m(\mathcal{X}) = R_1(\mathcal{X}) \times_{R_0(\mathcal{X})} \times \cdots \times_{R_0(\mathcal{X})} R_1(\mathcal{X}).$$

It follows from Lemma 3.4.5 that, for every m, $B(p)_{*,m} : B(\mathcal{X})_{*,m} \to B(\mathcal{Y})_{*,m}$ is a (weak) (trivial) Kan fibration. Furthermore, $B(p)$ is a Reedy fibration of bisimplicial sets because, by assumption, $p_{\mathbf{\Delta}} : \mathcal{X}_{\mathbf{\Delta}} \to \mathcal{Y}_{\mathbf{\Delta}}$ is a Reedy fibration of simplicial groupoids, and the nerve functor $N : \mathsf{Gpd} \to \mathsf{sSet}$ preserves fibrations and limits (see the proof of Proposition 2.7.17). It follows now from Lemma 3.4.7 that $B(p)$ is a diagonal (weak) (trivial) Kan fibration. In other words, $\mathrm{Sing}(p) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ is a (weak) (trivial) Kan fibration. □

Corollary 3.4.9. *Let $\mathcal{X}$ be a Reedy fibrant Serre topological stack. Then,* $\mathrm{Sing}(\mathcal{X})$ *is a Kan simplicial set.*

Corollary 3.4.10. *For every (weak) (trivial) Serre fibration of Serre stacks $p : \mathcal{X} \to \mathcal{Y}$, there exists a strictly commutative diagram*



where p' is a (weak) (trivial) Serre fibration as well as an injective (hence, also Reedy) fibration, and $g : \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ is an equivalence of Serre stacks. In particular, $\mathrm{Sing}(p') : \mathrm{Sing}(\mathcal{X}') \to \mathrm{Sing}(\mathcal{Y})$ *is a (weak) (trivial) Kan fibration.*

Proof. This follows from Proposition 3.3.13 and Theorem 3.4.8. (Also see Remark 2.10.13.) □

Corollary 3.4.11. *For every Serre stack $\mathcal{X}$ there exists a Serre stack $\mathcal{X}' \sim \mathcal{X}$ equivalent to it that is Reedy fibrant (hence,* $\mathrm{Sing}(\mathcal{X}')$ *is a Kan simplicial set).*

3.5 Singular functor preserves weak equivalences

In this section, we prove that the singular functor has the correct homotopy type by showing that it takes a weak equivalence of topological stacks to a weak equivalence of simplicial sets (Theorem 3.5.2). We begin with a special case.

Proposition 3.5.1. *Let $\mathcal{X}$ be a Serre stack, and let $\varphi : X \to \mathcal{X}$ be a trivial weak Serre fibration with X (equivalent to) a topological space (i.e., X is a classifying space for $\mathcal{X}$ in the sense of Definition 2.10.18). Then,* $\mathrm{Sing}(\varphi) : \mathrm{Sing}(X) \to \mathrm{Sing}(\mathcal{X})$ *is a weak equivalence of simplicial sets.*

Proof. We may assume that $\mathcal{X}$ is Reedy fibrant (Corollary 3.2.4 and Corollary 3.4.11). By Corollary 3.2.4 and Corollary 3.4.10, we may assume that $\varphi : X \to \mathcal{X}$ is a trivial weak Serre fibration as well as a Reedy fibration. Note that we are not insisting on X being *isomorphic* to but only equivalent to a topological space X'.

Observe that we can always find a pair of inverse equivalences between X and X'. On the one hand, we have that $\pi_0(X(T)) = X'(T)$ for every $T \in \mathsf{Top}$, so we have an equivalence $p : X \to X'$. In particular, $X(X') \to X'(X')$ is an equivalence of groupoids (the latter is actually a set). Picking $f \in X(X')$ in the inverse image of $\mathrm{id} \in X'(X')$ and applying Yoneda's lemma, we find the desired inverse $f : X' \to X$ to p.

Now, by Theorem 3.4.8, $\mathrm{Sing}(\varphi) : \mathrm{Sing}(X) \to \mathrm{Sing}(\mathcal{X})$ is a trivial weak Kan fibration, and $\mathrm{Sing}(\mathcal{X})$ is Kan. Furthermore, the conditions of Lemma 3.4.4 are satisfied as the map $\mathrm{Sing}(X') \to \mathrm{Sing}(X)$ is a weak equivalence (Corollary 3.2.4) and $\mathrm{Sing}(X')$ is Kan. So, by Lemma 3.4.4, $\mathrm{Sing}(\varphi)$ is a weak equivalence. □

Theorem 3.5.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a weak equivalence of Serre stacks. Then,* $\mathrm{Sing}(f) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ *is a weak equivalence of simplicial sets.*

Proof. We can choose classifying atlases $\varphi : X \to \mathcal{X}$ and $\psi : Y \to \mathcal{Y}$ (in the sense of Definition 2.10.18) fitting in a 2-commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ {\scriptstyle\varphi}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle\psi} \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y} \end{array}$$

This is done as follows. Choose classifying atlases $\psi : Y \to \mathcal{Y}$ and $h : X \to \mathcal{X}\tilde{\times}_{\mathcal{Y}}Y$. Set $\varphi = \mathrm{pr}_1 \circ h$ and $f' = \mathrm{pr}_2 \circ h$; by ([28], Lemma 3.8), φ is again a trivial weak Serre fibration.

Now, by the two-out-of-three property, f' is a weak equivalence. Applying Sing, we find a homotopy commutative diagram in simplicial sets where $\mathrm{Sing}(f')$, $\mathrm{Sing}(\varphi)$ and $\mathrm{Sing}(\psi)$ are weak equivalences of simplicial sets (by Proposition 3.5.1, also see Remark 3.2.2). Therefore, $\mathrm{Sing}(f)$ is also a weak equivalence by the two-out-of-three property. □

Corollary 3.5.3. *Let $\mathcal{X}$ be a Serre topological stack, and let $\mathbb{X} = [R \rightrightarrows X]$ be a groupoid presentation for it. Then, there is a natural weak equivalence*

$$\mathrm{Sing}(\|N(\mathbb{X})\|) \to \mathrm{Sing}(\mathcal{X}),$$

of simplicial sets, where the left-hand occurrence of Sing *is the classical singular chains functor, and $\| - \|$ denotes the fat geometric realization.*

Proof. This follows from the fact that there is a natural map $\|N(\mathbb{X})\| \to \mathcal{X}$, and this map is a classifying space for $\mathcal{X}$; see [28], Corollary 3.17 and [27], Theorem 6.3. □

3.6 Singular chains of categories fibered in groupoids

We shall give a definition of Sing for categories fibered in groupoids and show that it agrees with the functor defined in Chapter 3.

Categories fibered in groupoids over Top were defined in Definition 2.2.10. Recall that the 2-category of categories fibered in groupoids over Top is denoted by $\mathsf{CFG}_{\mathsf{Top}}$. For a precise definition of $\mathsf{CFG}_{\mathsf{Top}}$ and the descent condition, see Section 2.2.3.

Recall from Proposition 2.2.12 that the Grothendieck construction is a functor

$$\int : \mathsf{pshGpd} \to \mathsf{CFG}_{\mathsf{Top}}$$

which is an equivalence of 2-categories.

In Definition 3.2.1, we defined the functor $\mathrm{Sing} : \mathsf{pshGpd} \to \mathsf{sSet}$ from the category of strict presheaves of groupoids to the category of simplicial sets. We wish to give another functor Sing :

$\mathsf{CFG}_{\mathsf{Top}} \to \mathsf{sSet}$ which has the property that the following diagram commutes up to weak equivalence

$$\begin{array}{ccc} \mathsf{pshGpd} & \xrightarrow{\text{Sing}} & \mathsf{sSet} \\ {\scriptstyle\int}\downarrow & \nearrow_{\text{Sing}} & \\ \mathsf{CFG}_{\mathsf{Top}} & & \end{array}$$

Definition 3.6.1. Let $\mathcal{X} \in \mathsf{CFG}_{\mathsf{Top}}$, we define the functor Sing : $\mathsf{CFG}_{\mathsf{Top}} \to \mathsf{sSet}$ by

$$\text{Sing}(\mathcal{X}) := N\mathcal{X}_{\boldsymbol{\Delta}}$$

where $\mathcal{X}_{\boldsymbol{\Delta}}$ denotes restricting the category fibered in groupoids $\mathcal{X}$ to $\boldsymbol{\Delta} \hookrightarrow \mathsf{Top}$ and $N : \mathsf{Gpd} \to \mathsf{sSet}$ is the nerve functor.

Recall a theorem of Thomason ([33],Theorem 1.2):

Theorem 3.6.2. *Let I be a small category and $F : I^{op} \to \mathsf{Gpd}$ be a functor. Then there is a map*

$$T : \text{hocolim}(N \circ F) \xrightarrow{\simeq} N \int F$$

which is a homotopy equivalence of simplicial sets. For a definition of T, see [33].

Corollary 3.6.3. *For a presheaf of groupoids $\mathcal{X} \in \mathsf{pshGpd}$, there are weak equivalences*

$$\text{Sing}(\mathcal{X}) \xleftarrow{\simeq} \text{hocolim}(N\mathcal{X}_{\boldsymbol{\Delta}}) \xrightarrow{\simeq} N \int \mathcal{X}_{\boldsymbol{\Delta}}$$

Proof. This follows from Theorem 3.6.2 and the following fact. Given a bisimplicial set $X : \boldsymbol{\Delta} \to \mathsf{sSet}$, there is a natural weak equivalence

$$\text{hocolim}\, X \xrightarrow{\simeq} \text{Diag}X.$$

This fact is proven in [21] Corollary A.2.9.30 (where we take $\mathbf{A} = \mathsf{sSet}$ and note that all bisimplicial sets are Reedy cofibrant). □

Note that $\int \mathcal{X}_{\boldsymbol{\Delta}} = \left(\int \mathcal{X}\right)_{\boldsymbol{\Delta}}$ and so for a presheaf of groupoids $\mathcal{X}$,

$$\text{Sing}(\mathcal{X}) \simeq N \int \mathcal{X}_{\boldsymbol{\Delta}} = \text{Sing}(\int \mathcal{X}).$$

Remark 3.6.4. We will not show that the properties of Sing proven for strict presheaves of groupoids also hold for categories fibered in groupoids. We expect that with the appropriate modifications to the Reedy condition, this is possible and will appear in [8].

Chapter 4

Totalization

4.1 Totalization of cosimplicial topological stacks

In Section 2.9.1 we recalled the definition of a two variable V-adjunction, the bar construction and the cobar construction. We shall use these definitions to define the totalization of a cosimplicial topological stack and establish some basic properties.

In Proposition 4.1.4, we shall require the two variable adjunction $(\circledast, \mathrm{Hom}_r, \mathrm{Hom}_l)$ to have the property that $\circledast$ is a left Quillen bifunctor. We shall demonstrate that this is true in the case that we are most interested in which is outlined in Corollary 4.1.3.

Definition 4.1.1. ([21], Definition A.3.1.5) For a symmetric monoidal category V, a V-*enriched model category* M is a V-category with a model structure such that M is tensored and cotensored over V and the tensor product functor is a left Quillen bifunctor.

Proposition 4.1.2. *If* V *is an excellent model category and* M *is a combinatorial* V*-enriched model category, then for any small* V*-enriched category* D*, the injective model structure on* M^{D} *is a* V*-enriched combinatorial model category.*

Proof. This appears as Proposition A.3.3.2 and Remark A.3.3.4 in [21]. □

Corollary 4.1.3. *Both the category of presheaves of groupoids and the category of simplicial groupoids, equipped with the global injective model structure are* Gpd*-enriched combinatorial model categories.*

Proof. This follows from Proposition 2.7.5 and Proposition 4.1.2. □

The case that we are most interested in is when $\mathsf{V} = \mathsf{Gpd}$ and the index category is $\mathbf{\Delta}$. When $\mathsf{V} = \mathsf{Gpd}$, we tacitly fix the functor $|\Delta| : \mathbf{\Delta} \to \mathsf{Gpd}$, sending $[n] \mapsto \pi_1(\Delta^n)$ where π_1 is the fundamental groupoid functor. This allows us to define the bar and cobar constructions as in Definition 2.9.6 and Definition 2.9.7.

Proposition 4.1.4. *Let $F : \mathbf{\Delta} \to \mathsf{pshGpd}$ and $G, G' : \mathbf{\Delta}^{op} \to \mathsf{pshGpd}$ be Gpd-functors and let $\varphi : G' \to G$ be a level-wise cofibration. Then*

$$B(G, \mathbf{\Delta}, F) \to B(G', \mathbf{\Delta}, F)$$

is an injective cofibration in pshGpd.

Proof. Clearly, $B_\bullet(G, \mathbf{\Delta}, F) \to B_\bullet(G', \mathbf{\Delta}, F)$ is a level-wise cofibration. This follows from the definition given in Definition 2.9.6. By Corollary 2.7.18,

$$B_\bullet(G, \mathbf{\Delta}, F) \to B_\bullet(G', \mathbf{\Delta}, F)$$

is a Reedy cofibration. Finally, by Lemma A.8 of [31],

$$B(G, \mathbf{\Delta}, F) \to B(G', \mathbf{\Delta}, F)$$

is an injective cofibration in pshGpd since $|\Delta| : \mathbf{\Delta} \to \mathsf{Gpd}$ is Reedy cofibrant. □

4.1.1 The internal hom of cosimplicial presheaves of groupoids

In Section 4.1.1, we shall define the internal hom between two cosimplicial presheaves of groupoids. This will allow us to define the totalization of a cosimplicial presheaf of groupoids.

We shall also define a 'homotopy fattened' version of the internal hom. This will allow us to show that under certain fibrancy conditions, the totalization of a level-wise equivalence of presheaves of groupoids between cosimplicial presheaves of groupoids is itself an equivalence of presheaves of groupoids.

Throughout Section 4.1.1 we shall restrict our attention to the category $[\mathsf{C}, \mathsf{Gpd}]$ where C is an index category. Although we require C to be small, by fixing a Grothendieck universe, this will not present us with any problems. We shall be using two different two variable Gpd-adjunctions, so for clarity, we shall explicitly state them both here.

Proposition 4.1.5. *The* Gpd*-category* $[\mathsf{C}, \mathsf{Gpd}]$ *carries a two variable* Gpd*-adjunction*

$$(\times, \mathrm{Map}_{[\mathsf{C},\mathsf{Gpd}]}(-,-), \mathrm{Map}_{[\mathsf{C},\mathsf{Gpd}]}(-,-))$$

where $\times$ is the (level-wise) cartesian product and

$$\mathrm{Map}_{[\mathsf{C},\mathsf{Gpd}]}(G, F)(c) := \mathrm{Hom}_{[\mathsf{C},\mathsf{Gpd}]}(G \times c, F)$$

where c denotes the image of c under the Yoneda embedding.

(Note that using the notation of Definition 2.9.1, we are setting $\mathsf{M} = \mathsf{N} = \mathsf{P} = [\mathsf{C}, \mathsf{Gpd}]$.*)*

Proposition 4.1.6. *The* Gpd*-category* $[\mathsf{C}, \mathsf{Gpd}]$ *is enriched, tensored and cotensored over* Gpd. *Equivalently, this can be stated as a two variable* Gpd*-adjunction*

$$(\times, \mathrm{Map}_{[\mathsf{C},\mathsf{Gpd}]}(-,-), \mathrm{Map}_{[\mathsf{C},\mathsf{Gpd}]}(-,-))$$

where $\mathrm{Map}_{[\mathsf{C},\mathsf{Gpd}]}(-,-)$ *is defined as in Proposition 4.1.5 and* Gpd *is viewed as a subcategory of* $[\mathsf{C},\mathsf{Gpd}]$*, sending a groupoid to the constant presheaf.*

(Note that using the notation of Definition 2.9.1, we are setting $\mathsf{M} = \mathsf{P} = [\mathsf{C},\mathsf{Gpd}]$ *and* $\mathsf{N} = \mathsf{Gpd}$*.)*

Let D be an index category. Given $F, G \in [\mathsf{D}, [\mathsf{C},\mathsf{Gpd}]]$, we shall use Proposition 4.1.6 and Proposition 4.1.5 to allow us to define the natural transformations from G to F which are enriched over Gpd and $[\mathsf{C},\mathsf{Gpd}]$ respectively.

Definition 4.1.7. Given two diagrams of presheaves of groupoids $F, G : \mathsf{D} \to [\mathsf{C},\mathsf{Gpd}]$, there is a groupoid of Gpd-natural transformations between them. This is defined by

$$\mathrm{Hom}_{[\mathsf{D},[\mathsf{C},\mathsf{Gpd}]]}(G, F) := \mathrm{Hom}_r^{\mathsf{D}}(G, F) \in \mathsf{Gpd},$$

where $\mathrm{Hom}_r^{\mathsf{D}}(G, F)$ is as in Definition 2.9.4 and the 2-variable adjunction is given by Proposition 4.1.6.

We shall also give a 'homotopy' version of this definition.

Definition 4.1.8. Given two diagrams of presheaves of groupoids $F, G : \mathsf{D} \to [\mathsf{C},\mathsf{Gpd}]$, there is a groupoid of homotopy coherent transformations between them. This is defined by

$$\mathrm{hHom}_{[\mathsf{D},[\mathsf{C},\mathsf{Gpd}]]}(G, F) := \mathrm{Hom}_{[\mathsf{D},[\mathsf{C},\mathsf{Gpd}]]}(B(\mathsf{D}, \mathsf{D}, G), F),$$

where $B(\mathsf{D}, \mathsf{D}, G)$ is defined in Definition 2.9.9.

Remark 4.1.9. The above definition is given by Shulman in ([31], Section 10).

Proposition 4.1.10. *Let* M *be a cocomplete* V*-category which is enriched, tensored and cotensored over* V *(e.g.* $\mathsf{M} = [\mathsf{C},\mathsf{Gpd}]$*). Let* D *be a* V*-category and* $F, G : \mathsf{D} \to \mathsf{M}$ *be two diagrams over* D*. Then*

$$\mathrm{hHom}_{[\mathsf{D},\mathsf{M}]}(G, F) = C(G, \mathsf{D}, F).$$

Proof. Use Lemma 19.7 in [31], where the two variable adjunction is the one arising from the internal hom, tensor and cotensor. □

As well as being enriched over Gpd, we showed in Proposition 4.1.5 that $[\mathsf{D}, [\mathsf{C},\mathsf{Gpd}]]$ is also enriched over $[\mathsf{C},\mathsf{Gpd}]$. This gives rise to the following definition.

Definition 4.1.11. Given two diagrams of presheaves of groupoids $F, G : \mathsf{D} \to [\mathsf{C},\mathsf{Gpd}]$, we define the mapping object between them as the presheaf of groupoids defined as follows.

$$\mathrm{Map}_{[\mathsf{D},[\mathsf{C},\mathsf{Gpd}]]}(G, F)(c) := \mathrm{Hom}_{[\mathsf{D},[\mathsf{C},\mathsf{Gpd}]]}(G \times \hat{c}, F) \in [\mathsf{C},\mathsf{Gpd}]$$

where $\hat{c}$ denotes the diagram of presheaves of groupoids $d \mapsto \mathrm{Hom}_{\mathsf{C}}(-, c)$.

Alternatively, we can phrase this as

$$\mathrm{Map}_{[\mathsf{D},[\mathsf{C},\mathsf{Gpd}]]}(G,F) := \mathrm{Hom}_r^{\mathsf{D}}(G,F) \in [\mathsf{C},\mathsf{Gpd}]$$

where we take the 2-variable adjunction to be as in Proposition 4.1.5.

We shall also include a 'homotopy version' of the mapping object.

Definition 4.1.12. Given two functors $F, G : \mathsf{D} \to [\mathsf{C},\mathsf{Gpd}]$, we define $\mathrm{hMap}_{[\mathsf{D},[\mathsf{C},\mathsf{Gpd}]]}(G,F) \in [\mathsf{C},\mathsf{Gpd}]$ by the property

$$\mathrm{hMap}_{[\mathsf{D},[\mathsf{C},\mathsf{Gpd}]]}(G,F)(c) := \mathrm{hHom}_{[D,[\mathsf{C},\mathsf{Gpd}]]}(G \times \hat{c}, F)$$

where we again use $\hat{c}$ to denote the constant diagram $\hat{c} : d \mapsto \mathrm{Hom}_{\mathsf{C}}(-,c)$.

4.1.2 Totalization

From this point onwards, we shall set $\mathsf{D} = \mathbf{\Delta}$ and $\mathsf{C} = \mathsf{Top}^{\mathrm{op}}$ or $\mathsf{C} = \mathbf{\Delta}^{\mathrm{op}}$, so that $[\mathsf{C},\mathsf{Gpd}]$ is the category pshGpd or sGpd respectively.

Definition 4.1.13. We define the cosimplicial presheaf of groupoids

$$|\Delta^\bullet| : \mathbf{\Delta} \to \mathsf{pshGpd}$$
$$[n] \mapsto |\Delta^n|$$

and we define the cosimplicial simplicial groupoid

$$\Delta^\bullet : \mathbf{\Delta} \to \mathsf{sGpd}$$
$$[n] \mapsto \Delta^n,$$

where $\Delta^n \in \mathsf{sSet} \subset \mathsf{sGpd}$ and $|\Delta^n| \in \mathsf{Top} \subset \mathsf{pshGpd}$ via the Yoneda embedding.

Proposition 4.1.14. *The category $\mathbf{\Delta}$ is good for $\circledast, \mathrm{Hom}_r$ and Hom_l in the sense of Shulman [31].*

Proof. This follows from ([31], Theorem 23.12) since $\mathbf{\Delta}$ is a cofibrant Gpd-category. □

Proposition 4.1.15. *Let M be a Gpd-category. Let $[\mathbf{\Delta},\mathsf{M}]_Q$ denote the cosimplicial objects of M composed with the level-wise cofibrant replacement functor. Similarly, let $[\mathbf{\Delta},\mathsf{M}]_R$ denote the cosimplicial objects of M composed with the level-wise fibrant replacement functor.*

- *$[\mathbf{\Delta},\mathsf{M}]_Q$ is a left deformation retract of $[\mathbf{\Delta},\mathsf{M}]$*
- *$[\mathbf{\Delta},\mathsf{M}]_R$ is a right deformation retract of $[\mathbf{\Delta},\mathsf{M}]$.*

(See Definition 3.1 in [31] for a definition of left and right deformation retracts.)

Proof. This follows from ([31], Proposition 24.1). □

Definition 4.1.16. The *totalization of a cosimplicial presheaf of groupoids* $\mathcal{X}^\bullet : \mathbf{\Delta} \to \mathsf{pshGpd}$, is given by the presheaf of groupoids

$$\mathrm{Tot}(\mathcal{X}^\bullet) := \mathrm{Map}_{[\mathbf{\Delta},\mathsf{pshGpd}]}(|\Delta^\bullet|, \mathcal{X}^\bullet).$$

Definition 4.1.17. The *totalization of a cosimplicial simplicial groupoid* $X^\bullet : \mathbf{\Delta} \to \mathsf{sGpd}$, is given by the simplicial groupoid

$$\mathrm{Tot}(X^\bullet) := \mathrm{Map}_{[\mathbf{\Delta},\mathsf{sGpd}]}(\Delta^\bullet, X^\bullet).$$

We also have the 'homotopy coherent' versions of the above definitions, which we shall record below.

Definition 4.1.18. The *homotopy totalization of a cosimplicial presheaf of groupoids* $\mathcal{X}^\bullet : \mathbf{\Delta} \to \mathsf{pshGpd}$, is given by the presheaf of groupoids

$$\mathrm{hTot}(\mathcal{X}^\bullet) := \mathrm{hMap}_{[\mathbf{\Delta},\mathsf{pshGpd}]}(|\Delta^\bullet|, \mathcal{X}^\bullet).$$

Definition 4.1.19. The *homotopy totalization of a cosimplicial simplicial groupoid* $X^\bullet : \mathbf{\Delta} \to \mathsf{sGpd}$, is given by the simplicial groupoid

$$\mathrm{hTot}(X^\bullet) := \mathrm{hMap}_{[\mathbf{\Delta},\mathsf{sGpd}]}(\Delta^\bullet, X^\bullet).$$

4.1.3 Some properties of the totalization functor

In this section we will will state some results that will be used to prove that Sing and Tot commute up to weak equivalence (which will appear in Section 4.1.4).

Lemma 4.1.20. *Let $\mathcal{X}^\bullet : \mathbf{\Delta} \to \mathsf{pshGpd}$ be a cosimplicial presheaf of groupoids where each $\mathcal{X}^n$ is fibrant in the injective model structure and a Serre stack. Then there is a level-wise equivalence*

$$\mu : \mathrm{hTot}(\mathcal{X}^\bullet)_{\mathbf{\Delta}} \xrightarrow{\simeq} \mathrm{hTot}(\mathcal{X}^\bullet_{\mathbf{\Delta}})$$

between simplicial groupoids.

Proof. Let us first consider a general property of the bar construction and then use this to prove Lemma 4.1.20. Given functors $F, G \in [\mathbf{\Delta}, [\mathsf{C}, \mathsf{Gpd}]]$, by Proposition 4.1.10, $\mathrm{hHom}_{[\mathbf{\Delta},[\mathsf{C},\mathsf{Gpd}]]}(G, F)$ is isomorphic to the cobar construction $C(G, \mathbf{\Delta}, F) \in \mathsf{Gpd}$. By Proposition 4.1.14 and Proposition 4.1.15 we may use ([31], Theorem 20.7) (or the dual of the comments following Theorem 20.7), and if we further assume G is object-wise cofibrant and F is object-wise fibrant (both under the injective model structure on $[\mathsf{C}, \mathsf{Gpd}]$), then the cobar construction is a right derived functor of $\mathrm{Hom}_{[\mathbf{\Delta},[\mathsf{C},\mathsf{Gpd}]]}(G^\bullet, F^\bullet) = \mathrm{Hom}^{\mathbf{\Delta}}_r(G^\bullet, F^\bullet)$.

Since our indexing category is $\mathbf{\Delta}$, by Remark 2.9.5,

$$\mathrm{hHom}_{\mathbf{\Delta}}(G,F) = C(G,\mathbf{\Delta},F) = \mathbb{R}\int_{\mathbf{\Delta}} \mathrm{Hom}_{[\mathsf{C},\mathsf{Gpd}]}(G([n]),F([n]))$$

where $\mathbb{R}\int_{\mathbf{\Delta}} \mathrm{Hom}_{[\mathsf{C},\mathsf{Gpd}]}(-,-)$ is a right derived functor of the usual end functor.

We shall use the above to prove Lemma 4.1.20. It is enough to show that there is a natural equivalence of groupoids between the $m-$cells

$$(\mathrm{hTot}(\mathcal{X}^{\bullet})_{\mathbf{\Delta}})_m = \mathrm{hHom}_{\mathbf{\Delta}}(|\Delta^{\bullet}| \times |\Delta^m|, \mathcal{X}^{\bullet})$$

and

$$(\mathrm{hTot}(\mathcal{X}^{\bullet}_{\mathbf{\Delta}}))_m = \mathrm{hHom}_{\mathbf{\Delta}}(\Delta^{\bullet} \times \Delta^m, \mathcal{X}^{\bullet}_{\mathbf{\Delta}})$$

(the above equalities follow directly from Definition 4.1.18 and Definition 4.1.19).

If in the above discussion, we take $[\mathsf{C},\mathsf{Gpd}] = \mathsf{pshGpd}$, let $G = |\Delta^{\bullet} \times \Delta^m|$ and $F = \mathcal{X}^{\bullet}$ then

$$(\mathrm{hTot}(\mathcal{X}^{\bullet})_{\mathbf{\Delta}})_m = \mathbb{R}\int_{\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^m \times \Delta^n|, \mathcal{X}^n).$$

This follows, since $|\Delta^n \times \Delta^m|$ is cofibrant in pshGpd (all objects are cofibrant in the injective structure) and $\mathcal{X}^n$ is fibrant in pshGpd by assumption.

If we take $[\mathsf{C},\mathsf{Gpd}] = \mathsf{sGpd}$, let $G = \Delta^{\bullet} \times \Delta^m$ and $F = \mathcal{X}^{\bullet}_{\mathbf{\Delta}}$ then we obtain the homotopy limit

$$(\mathrm{hTot}(\mathcal{X}^{\bullet}_{\mathbf{\Delta}}))_m = \mathbb{R}\int_{\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^m \times \Delta^n, \mathcal{X}^n_{\mathbf{\Delta}}).$$

This follows, since $\Delta^m \times \Delta^n$ is cofibrant in the injective model structure on $[\mathbf{\Delta}^{\mathrm{op}},\mathsf{Gpd}]$ and since $\mathcal{X}^n$ is injective fibrant in pshGpd, this implies $\mathcal{X}^n_{\mathbf{\Delta}}$ is injective fibrant in $[\mathbf{\Delta}^{\mathrm{op}},\mathsf{Gpd}]$.

Finally, we note that if each $\mathcal{X}^n$ is a Serre stack, there are natural equivalences of groupoids

$$\mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^m \times \Delta^a|, \mathcal{X}^b) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{\Delta^m \times \Delta^a}, \mathcal{X}^b) \cong \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^m \times \Delta^a, \mathcal{X}^b_{\mathbf{\Delta}})$$

which are described in Corollary 3.3.8 and Lemma 3.3.2 respectively. Since a map which is levelwise a weak equivalence induces a weak equivalence between the associated derived limits, we obtain a natural equivalence of groupoids

$$\mu_m : \mathbb{R}\int_{\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^m \times \Delta^n|, \mathcal{X}^n) \to \mathbb{R}\int_{\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{pshGpd}}(\Delta^m \times \Delta^n, \mathcal{X}^n_{\mathbf{\Delta}})$$

which completes the proof. □

Proposition 4.1.21. *Let* E *be a category, let* V *be a monoidal model category and let* $G : \mathsf{E} \to [\mathbf{\Delta}^{op},\mathsf{V}]$ *be a diagram which indexes functors* $G(e) : \mathbf{\Delta}^{op} \to \mathsf{V}$. *Given any functor* $F : \mathbf{\Delta} \to \mathsf{M}$, *the bar resolution commutes with colimits in the first variable. In other words,*

$$\operatorname*{colim}_{E} B(G(e),\mathbf{\Delta},F) = B(\operatorname*{colim}_{E} G(e),\mathbf{\Delta},F)$$

is an isomorphism.

Proof. This follows from the fact that colimits commute with colimits and that the tensor is a left adjoint and so also commutes with colimits. □

Lemma 4.1.22. *Let* M *be a* Gpd*-tensored,* Gpd*-enriched model category. If* $F : \mathbf{\Delta} \to M$ *is a Reedy cofibrant cosimplicial object, then the cosimplicial object* $B(\mathbf{\Delta}, \mathbf{\Delta}, F)$ *is Reedy cofibrant in* $[\mathbf{\Delta}, \mathsf{M}]$.

Proof. We shall show that the natural latching map $L_n(B(\mathbf{\Delta}, \mathbf{\Delta}, F)) \to B(\mathbf{\Delta}, \mathbf{\Delta}, F)([n]) = B(\Delta^n, \mathbf{\Delta}, F)$ is a cofibration in M.

By Proposition 4.1.21, $L_n(B(\mathbf{\Delta}, \mathbf{\Delta}, F)) = B(\partial\Delta^n, \mathbf{\Delta}, F)$. Since $\partial\Delta^n \to \Delta^n$ is a Reedy cofibration in $[\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Gpd}]$, by Proposition 4.1.4, $B(\partial\Delta^n, \mathbf{\Delta}, F) \to B(\Delta^n, \mathbf{\Delta}, F)$ is a cofibration in M.

□

Proposition 4.1.23. *Let* M *be a* Gpd*-tensored,* Gpd*-enriched model category. If* G *is a Reedy cofibrant functor* $G : \mathbf{\Delta} \to M$ *and* $F : \mathbf{\Delta} \to M$ *is a Reedy fibrant functor, then the map*

$$\psi : B(\mathbf{\Delta}, \mathbf{\Delta}, G) \xrightarrow{\simeq} G$$

defined in Lemma 13.5 of [31] induces a map

$$\psi^* : \mathrm{Hom}_{[\mathbf{\Delta}, M]}(G, F) \xrightarrow{\sim} \mathrm{hHom}_{[\mathbf{\Delta}, M]}(G, F)$$

which is an equivalence of groupoids.

Proof. We recall from [31] Lemma 13.5, that there is a level-wise weak equivalence

$$B(\mathbf{\Delta}, \mathbf{\Delta}, G) \xrightarrow{\simeq} G.$$

By Proposition 4.1.2 $[\mathbf{\Delta}, \mathsf{M}]$ carries a Gpd-enriched Reedy model category structure. Hence, by [24] Lemma 1.22, it is enough to show that F is Reedy fibrant and both G and $B(\mathbf{\Delta}, \mathbf{\Delta}, G)$ are Reedy cofibrant.

By assumption, G and F are Reedy cofibrant and Reedy fibrant respectively. By Lemma 4.1.22, $B(\mathbf{\Delta}, \mathbf{\Delta}, G)$ is Reedy cofibrant. □

Proposition 4.1.24. *If* $\mathcal{X}^\bullet : \mathbf{\Delta} \to \mathsf{pshGpd}$ *is a Reedy fibrant cosimplicial presheaf of groupoids, then the natural map*

$$\theta : \mathrm{Tot}(\mathcal{X}^\bullet) \to \mathrm{hTot}(\mathcal{X}^\bullet)$$

is an equivalence of presheaves of groupoids. Here θ *is defined for each topological space* T *by the natural map* ψ^* *defined in Proposition 4.1.23 (where we set* $M = \mathsf{pshGpd}$*,* $G = |\Delta^\bullet|$ *and* $F = \mathcal{X}^\bullet$*).*

Proof. In order to show that $\theta : \mathrm{Map}_{[\mathbf{\Delta}, \mathsf{pshGpd}]}(|\Delta^\bullet|, \mathcal{X}^\bullet) \to \mathrm{hMap}_{\mathsf{pshGpd}}(|\Delta^\bullet|, \mathcal{X}^\bullet)$ is an equivalence of presheaves of groupoids, it is sufficient to show that for each topological space T, $\theta(T)$ is an equivalence of groupoids. This follows from Proposition 4.1.23, since $\mathcal{X}^\bullet$ is Reedy fibrant by assumption and $|\Delta^\bullet| \times T$ is Reedy cofibrant. The later is Reedy cofibrant since $\widetilde{\partial\Delta^n} \to |\Delta^n|$ is an injective cofibration in pshGpd and hence $|\Delta^\bullet| \times T$ is Reedy cofibrant. □

Proposition 4.1.25. *If $X^\bullet$ is a Reedy fibrant cosimplicial simplicial groupoid, then the natural map*

$$\theta : \mathrm{Tot}(X^\bullet) \to \mathrm{hTot}(X^\bullet)$$

is an equivalence of simplicial groupoids. Here θ is defined for each $[n] \in \mathbf{\Delta}$ by the natural map ψ^* defined in Proposition 4.1.23 (where we set $M = \mathsf{sGpd}$, $G = \Delta^\bullet$ and $F = X^\bullet$).

Proof. The argument is the same as the proof of Proposition 4.1.24. □

Proposition 4.1.26. *If $\mathcal{X}^\bullet : \mathbf{\Delta} \to \mathsf{pshGpd}$ is a Reedy fibrant cosimplicial presheaf of groupoids and each $\mathcal{X}^n$ is injective fibrant, then there is a natural weak equivalence $\lambda : \mathrm{Tot}(\mathcal{X}^\bullet_{\mathbf{\Delta}}) \to (\mathrm{Tot}\, \mathcal{X}^\bullet)_{\mathbf{\Delta}}$ which is induced by the natural maps*

$$\mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^m \times \Delta^a|, \mathcal{X}^b) \to \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^m \times \Delta^a, \mathcal{X}^b_{\mathbf{\Delta}}).$$

defined in Corollary 3.3.8 and Lemma 3.3.2.

Proof. The natural maps

$$\mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^m \times \Delta^a|, \mathcal{X}^b) \to \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^m \times \Delta^a, \mathcal{X}^b_{\mathbf{\Delta}})$$

induce a commutative diagram

$$\begin{array}{ccc}
\mathrm{hTot}(\mathcal{X}^\bullet)_{\mathbf{\Delta}} & \xleftarrow{\ \mu\ } & \mathrm{hTot}(\mathcal{X}^\bullet_{\mathbf{\Delta}}) \\
\uparrow \theta & & \uparrow \theta \\
(\mathrm{Tot}\, \mathcal{X}^\bullet)_{\mathbf{\Delta}} & \xleftarrow[\ \lambda\ ]{} & \mathrm{Tot}(\mathcal{X}^\bullet_{\mathbf{\Delta}})
\end{array}$$

where μ is also induced by the maps

$$\mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^m \times \Delta^a|, \mathcal{X}^b) \to \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^m \times \Delta^a, \mathcal{X}^b_{\mathbf{\Delta}})$$

and is defined in Lemma 4.1.20.

By Proposition 4.1.24 and Proposition 4.1.25 the two vertical morphisms are weak equivalences. By Lemma 4.1.20, $\mu : \mathrm{hTot}(\mathcal{X}^\bullet_{\mathbf{\Delta}}) \to (\mathrm{hTot}(\mathcal{X}^\bullet)_{\mathbf{\Delta}}$ is a weak equivalence. The result then follows from the 2-out-of-3 property. □

Remark 4.1.27. Recall that both the categories of simplicial groupoids and bisimplicial sets are tensored, cotensored and enriched over simplicial sets. For $K \in \mathsf{sSet}$, $X \in \mathsf{sGpd}$ and $Y \in \mathsf{bsSet}$ the tensors are defined as

$$X \otimes K := X \times K$$
$$Y \otimes K := Y \times \bar{K}$$

and the cotensors are defined as

$$X^K := \mathrm{Map}_{[\boldsymbol{\Delta}^{\mathrm{op}},\mathsf{Gpd}]}(K, X)$$
$$Y^K := \mathrm{Map}_{[\boldsymbol{\Delta}^{\mathrm{op}},\mathsf{sSet}]}(\bar{K}, Y)$$

where we view K as a simplicial groupoid via $\mathsf{sSet} \hookrightarrow \mathsf{sGpd}$ and $\bar{K}$ is the functor obtained by precomposing K with the projection functor $\mathrm{pr}_1 : \boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}} \to \boldsymbol{\Delta}^{\mathrm{op}}$ which projects onto the first factor.

Note that the level-wise fundamental groupoid functor $\pi : \mathsf{bsSet} \to \mathsf{sGpd}$ has the property that $\pi(Y \otimes K) = \pi(Y) \otimes K$. This is because

$$\begin{array}{ccc} \mathsf{Set} & \xrightarrow{\text{const}} & \mathsf{sSet} \\ & \searrow & \downarrow \pi \\ & & \mathsf{Gpd} \end{array}$$

commutes.

This definition of tensor and cotensor is compatible with Definition 4.1.17.

Proposition 4.1.28. *Let $X^\bullet \in [\boldsymbol{\Delta}, \mathsf{sGpd}]$ be a cosimplicial simplicial groupoid. Then the map*

$$N\,\mathrm{Tot}(X^\bullet) \cong \mathrm{Tot}(N(X^\bullet))$$

is an isomorphism of bisimplicial sets.

Proof. First, we note that both sGpd and bsSet are tensored and cotensored over sSet and there is an adjunction $N : \mathsf{sGpd} \rightleftarrows \mathsf{bsSet} : \pi$. As remarked in Remark 4.1.27, this adjunction has the property that for $Y \in \mathsf{bsSet}$ and $K \in \mathsf{sSet}$, $\pi(Y \otimes K) = \pi(Y) \otimes K$. Using Proposition 2.9(2) from [13], $N(X^n)^{\Delta^n} \cong N(X^{n\Delta^n})$. Since N commutes with limits,

$$N\,\mathrm{Tot}(X^\bullet) = N \int_{\boldsymbol{\Delta}} X^{n\Delta^n} \cong \int_{\boldsymbol{\Delta}} N(X^{n\Delta^n}) \cong \int_{\boldsymbol{\Delta}} N(X^n)^{\Delta^n} = \mathrm{Tot}(NX^\bullet)$$

as required. □

Let $X^\bullet : \boldsymbol{\Delta} \to \mathsf{bsSet}$ be a cosimplicial bisimplicial set. Note that

$$(\mathrm{Tot}(X^\bullet))_{m,n} = \mathrm{Hom}_{[\boldsymbol{\Delta},\mathsf{bsSet}]}(\Delta^\bullet \times \Delta^{m,n}, X^\bullet)$$

and that

$$\begin{aligned} d_*(\mathrm{Tot}\,\mathrm{Diag}X^\bullet)_{m,n} &= \mathrm{Hom}_{\mathsf{sSet}}(\Delta^m \times \Delta^n, \mathrm{Tot}\,\mathrm{Diag}X^\bullet) \\ &= \mathrm{Hom}_{[\boldsymbol{\Delta},\mathsf{sSet}]}(\Delta^\bullet \times \Delta^m \times \Delta^n, \mathrm{Diag}X^\bullet), \end{aligned}$$

where d_* is defined in Definition 2.8.2.

Define a map $\xi : \mathrm{Tot}(X^\bullet) \to d_*(\mathrm{Tot}\,\mathrm{Diag}X^\bullet)$ on each (m, n) simplex by $\xi_{m,n} = \mathrm{Diag}$. Finally, define $\zeta : \mathrm{Diag}\,\mathrm{Tot}(X^\bullet) \to \mathrm{Tot}\,\mathrm{Diag}X^\bullet$ to be adjoint to ξ.

Proposition 4.1.29. *If $X : \mathbf{\Delta} \to \mathsf{bsSet}$ is a Reedy fibrant cosimplicial bisimplicial set (where bsSet is endowed with the diagonal model structure), then the map*

$$\zeta : \mathit{Diag}\,\mathrm{Tot}\,X \xrightarrow{\simeq} \mathrm{Tot}\,\mathit{Diag}X$$

defined above, is a weak equivalence.

Proof. Since we assume that X is Reedy fibrant, $\mathrm{Diag}(X)$ is also Reedy fibrant. This is because Diag is a right Quillen adjoint and so preserves limits and (trivial) fibrations.

This implies that the Bousfield-Kan maps $\mathrm{BK} : \mathrm{Diag}\,\mathrm{Tot}(X) \to \mathrm{Diag}\,\mathrm{holim}\,X$ and $\mathrm{BK} : \mathrm{Tot}\,\mathrm{Diag}(X) \to \mathrm{holim}\,\mathrm{Diag}X$ are weak equivalences.

Finally, we note that since Diag is a right Quillen functor and preserves level-wise weak equivalences, it commutes with homotopy limits up to canonical weak equivalence. This means there is a natural weak equivalence $\mathrm{Diag}\,\mathrm{holim}\,X \xrightarrow{\simeq} \mathrm{holim}\,\mathrm{Diag}X$ and that the following square commutes

$$\begin{array}{ccc}
\mathrm{Diag}\,\mathrm{Tot}\,X & \xrightarrow{\zeta} & \mathrm{Tot}\,\mathrm{Diag}X \\
\Big\downarrow{\scriptstyle \mathrm{BK}} & & \Big\downarrow{\scriptstyle \mathrm{BK}} \\
\mathrm{Diag}\,\mathrm{holim}\,X & \xrightarrow[\simeq]{} & \mathrm{holim}\,\mathrm{Diag}X
\end{array}$$

where the vertical maps are the Bousfield-Kan maps. By the 2-of-3 property, ζ is a weak equivalence. □

4.1.4 The functors Sing and Tot commute up to weak equivalence

Theorem 4.1.30. *If $\mathcal{X}^\bullet$ is a Reedy fibrant cosimplicial presheaf of groupoids such that each $\mathcal{X}^n$ is an injective fibrant Serre stack, then there exists a natural weak equivalence of simplicial sets $\mathrm{Sing}\,\mathrm{Tot}(\mathcal{X}^\bullet) \to \mathrm{Tot}\,\mathrm{Sing}(\mathcal{X}^\bullet)$.*

Proof. By definition, $\mathrm{Sing}\,\mathrm{Tot}(\mathcal{X}^\bullet) = \mathrm{Diag}N(\mathrm{Tot}\,\mathcal{X}^\bullet)_{\mathbf{\Delta}}$. From Proposition 4.1.26, there is a weak equivalence

$$\mathrm{Diag}N(\mathrm{Tot}\,\mathcal{X}^\bullet)_{\mathbf{\Delta}} \xrightarrow{\simeq} \mathrm{Diag}N\,\mathrm{Tot}(\mathcal{X}^\bullet_{\mathbf{\Delta}}).$$

By Proposition 4.1.28 and Proposition 4.1.29 respectively, there exists an isomorphism and a weak equivalences

$$\mathrm{Diag}N\,\mathrm{Tot}(\mathcal{X}^\bullet_{\mathbf{\Delta}}) \cong \mathrm{Diag}\,\mathrm{Tot}(N\mathcal{X}^\bullet_{\mathbf{\Delta}}) \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{Diag}N\mathcal{X}^\bullet_{\mathbf{\Delta}}) = \mathrm{Tot}\,\mathrm{Sing}(\mathcal{X}^\bullet).$$ □

Theorem 4.1.31. *Let $f : \mathcal{X}^\bullet \to \mathcal{Y}^\bullet$ be a level-wise weak equivalence of cosimplicial Serre stacks. If $\mathrm{Sing}(\mathcal{X}^\bullet)$ and $\mathrm{Sing}(\mathcal{Y}^\bullet)$ are Reedy fibrant (in the category $[\boldsymbol{\Delta}, \mathsf{sSet}]$) then the map of simplicial sets $\mathrm{Sing}\,\mathrm{Tot}(f)$ (equivalently $\mathrm{Tot}(f)$) is a weak equivalence.*

Proof. Since f is a level-wise weak equivalence of cosimplicial stacks, $\mathrm{Sing}(f)$ is a level-wise weak equivalence of cosimplicial simplicial sets. Since $\mathrm{Sing}(\mathcal{X}^\bullet)$ and $\mathrm{Sing}(\mathcal{Y}^\bullet)$ are Reedy fibrant, $\mathrm{Tot}\,\mathrm{Sing}(f)$ is a weak equivalence of simplicial sets.

By Theorem 4.1.30 and using the 2-of-3 axiom, we see that $\mathrm{Sing}\,\mathrm{Tot}(f)$ is a weak equivalence as required. □

Chapter 5

Computations

5.1 Adjunction and unit

In this section we restrict our attention to Serre stacks. We shall again consider strict presheaves of groupoids.

Classically, there is a well-known adjunction for a simplicial set X and a topological space Y:

$$\mathrm{Hom}_{\mathsf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, \mathrm{Sing}(Y))$$

(which is described in Section 2.8.2). This adjunction gives rise to the counit $|\,\mathrm{Sing}(Y)| \to Y$, which is a weak equivalence of topological spaces (since the above adjunction is a Quillen equivalence).

We aim to define a loose analogue to this adjunction for topological stacks involving the functor $\mathrm{Sing} : \mathsf{pshGpd} \to \mathsf{sSet}$. We shall also define a version of the unit map. This would give a model for computing the homotopy type of a topological stack which is potentially easier to compute than Sing directly.

5.1.1 Adjunction for Sing

Definition 5.1.1. Consider the two functors

$$\mathsf{sGpd} \underset{\mathrm{Hom}_{\mathsf{topStack}}(\Delta^\bullet, -)}{\overset{Q}{\rightleftarrows}} \mathsf{topStack}$$

where the quotient functor Q is defined by $Q : [X_1 \rightrightarrows X_0] \mapsto [|X_0|/|X_1|]$ where $|-|$ denotes the usual geometric realization.

We shall show that when we restrict the categories sGpd and $\mathsf{topStack}$ to appropriate subcategories, these functors form something similar to a 2-adjunction (see Proposition 5.1.12 for the precise statement).

Lemma 5.1.2. *Let $[Z_1 \rightrightarrows Z_0]$ be a topological groupoid and any let $\alpha : Y \to \mathcal{Y}$ be an atlas for a topological stack $\mathcal{Y}$. Let $[R \rightrightarrows Y]$ be the groupoid associated to α where $R = Y \times_{\mathcal{Y}} Y$. Then the natural functor*

$$\psi : \mathrm{Hom}_{\mathsf{topGpd}}\left([Z_1 \rightrightarrows Z_0], [R \rightrightarrows Y]\right) \to \mathrm{Hom}_{\mathsf{topStack}}\left([Z_0/Z_1], \mathcal{Y}\right)$$

is fully faithful. If Z_0 is paracompact and the atlas $\alpha : Y \to \mathcal{Y}$ is a classifying atlas, then ψ is an equivalence of groupoids.

Proof. Given a map $f : [Z_1 \rightrightarrows Z_0] \to [R \rightrightarrows Y]$, this descends to the quotient $\bar{f} : [Z_0/Z_1] \to [Y/R]$. By Proposition 2.4.4, ψ is fully faithful.

If we assume $\alpha : Y \to \mathcal{Y}$ is a classifying atlas and Z_0 is paracompact, we claim that ψ is essentially surjective. Given a map $g : [Z_0/Z_1] \to \mathcal{Y}$ we may consider

$$\begin{array}{ccc} Z_0 & \dashrightarrow & Y \\ \downarrow & \swArrow & \downarrow{\scriptstyle \alpha} \\ [Z_0/Z_1] & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

where the dashed arrow exists due to Lemma 5.5 in [27]. This induces a map

$$[Z_1 \rightrightarrows Z_0] \to [R \rightrightarrows Y]$$

which shows that ψ is essentially surjective. $\square$

Corollary 5.1.3. *Let $\mathcal{Y}$ be a topological stack and let $[R \rightrightarrows Y]$ be the groupoid associated to the atlas $\alpha : Y \to \mathcal{Y}$ where $R = Y \times_{\mathcal{Y}} Y$. For any topological space Z, the natural functor*

$$\psi : [\mathrm{Hom}_{\mathsf{Top}}(Z, R) \rightrightarrows \mathrm{Hom}_{\mathsf{Top}}(Z, Y)] \to \mathrm{Hom}_{\mathsf{topStack}}\left(Z, \mathcal{Y}\right)$$

is fully faithful. If Z is paracompact and the atlas $\alpha : Y \to \mathcal{Y}$ is a classifying atlas, then ψ is an equivalence of groupoids.

Proof. Since

$$[\mathrm{Hom}_{\mathsf{Top}}(Z, R) \rightrightarrows \mathrm{Hom}_{\mathsf{Top}}(Z, Y)] = \mathrm{Hom}_{\mathsf{topGpd}}\left([Z \rightrightarrows Z], [R \rightrightarrows Y]\right),$$

this follows from Lemma 5.1.2. $\square$

Suppose $[R \rightrightarrows Y]$ is the groupoid associated to an atlas $Y \to \mathcal{Y}$. There is a natural map of simplicial groupoids

$$\theta : [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)] \to \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^{\bullet}|, \mathcal{Y}) \tag{5.1}$$

which is defined level-wise by

$$\theta_n : [\mathrm{Sing}(R)_n \rightrightarrows \mathrm{Sing}(Y)_n] = \mathrm{Hom}_{\mathsf{Gpd}}([\Delta^n \rightrightarrows \Delta^n], [R \rightrightarrows Y]) \to \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^n|, \mathcal{Y}),$$

sending an n-cell $x : |\Delta^n| \to Y$ to the composition $x : |\Delta^n| \to Y \to \mathcal{Y}$.

Proposition 5.1.4. *Let $\mathcal{Y}$ be a stack and $[R \rightrightarrows Y]$ be the topological groupoid associated to the atlas $\alpha : Y \to \mathcal{Y}$. If α is a Serre classifying atlas then the simplicial groupoid $[\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]$ is Reedy fibrant.*

Proof. This amounts to showing that the natural map

$$[\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]_n \to M_n([\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)])$$

is a fibration of groupoids (see Definition 2.7.1).

We note that

$$M_n([\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]) = [\mathrm{Hom}_{\mathsf{Top}}(|\partial\Delta^n|, R) \rightrightarrows \mathrm{Hom}_{\mathsf{Top}}(|\partial\Delta^n|, Y)]$$

this means that showing the above map is a fibration is equivalent to finding a lift for the following diagram

$$\begin{array}{ccc} |\partial\Delta^n| & \longrightarrow & R \\ \big\downarrow & \nearrow & \big\downarrow s \\ |\Delta^n| & \longrightarrow & Y \end{array}$$

if $\alpha : Y \to \mathcal{Y}$ is a Serre classifying atlas, then $s : R \to Y$ is a trivial Serre fibration, meaning such a lift exists. ☐

Proposition 5.1.5. *Given a topological stack $\mathcal{Y}$ and a classifying atlas $\alpha : Y \to \mathcal{Y}$ with corresponding groupoid $[R \rightrightarrows Y]$, we shall list some properties of θ which is defined in Eq.* (5.1)*.*

1. *The morphism θ is a level-wise weak equivalence of simplicial groupoids;*

2. *The morphism θ induces a morphism*

$$\mathit{DiagN}(\theta) : \mathit{DiagN}\,\mathrm{Sing}[R \rightrightarrows Y] \to \mathrm{Sing}(\mathcal{Y})$$

which is a weak equivalence;

3. *If α is also a Serre fibration and $\mathcal{Y}$ is Reedy fibrant, then θ is a homotopy equivalence of simplicial groupoids.*

Proof. (1) follows from Corollary 5.1.3.

Now we shall prove (2). By (1) Corollary 5.1.3, θ is a level-wise equivalence of simplicial groupoids $\theta : [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)] \to \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})$. The result then follows from Corollary 3.2.4.

Finally, we shall prove (3). By (2), θ is a level-wise equivalence of groupoids meaning it is a weak equivalence in the Reedy model structure.

By Proposition 5.1.4, $[\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]$ is Reedy fibrant. If $\mathcal{Y}$ is Reedy fibrant, then by definition $\mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})$ is Reedy fibrant. (This is because $\mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y}) \cong \mathcal{Y}_{\boldsymbol{\Delta}}$ are isomorphic by Lemma 3.3.2.)

All objects are Reedy cofibrant (since the monomorphisms in Gpd are the cofibrations), hence $[\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(X)]$ and $\mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})$ are Reedy cofibrant.

We have now verified the conditions to use Whitehead's theorem, which says that

$$\theta : [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)] \to \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})$$

is a homotopy equivalence. □

Remark 5.1.6. We shall recall the notion of a homotopy in the category of simplicial groupoids. Let us define the constant simplicial groupoid $\mathbb{I}$ sending $[n]$ to the groupoid with two objects and one non identity isomorphism.

Recall from ([15], Definition 7.3.2) the notion of a cylinder object and note that for the Reedy model structure on simplicial groupoids, we may choose a cylinder object for a simplicial groupoid X to be $\mathrm{cyl}(X) = X \times \mathbb{I}$.

This means that if two maps of simplicial groupoids $f, g : X \to Y$ are homotopic, then there exists a map $H : X \times \mathbb{I} \to Y$ which induces level-wise 2-isomorphisms $f_n \Rightarrow g_n$.

Lemma 5.1.7. *Let $\mathcal{Y}$ be Reedy fibrant. If $\alpha : Y \to \mathcal{Y}$ is a classifying atlas which is also a Serre fibration then the map induced by θ (see Eq.* (5.1)*),*

$$\theta_* : \mathrm{Hom}_{\mathsf{sGpd}}([X_1 \rightrightarrows X_0], [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]) \to \mathrm{Hom}_{\mathsf{sGpd}}([X_1 \rightrightarrows X_0], \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y}))$$

is an equivalence of groupoids.

Proof. Under these assumptions, by Proposition 5.1.5 the map θ is a homotopy equivalence. By Remark 5.1.6 and functoriality, the induced map θ_* is an equivalence of groupoids. □

Definition 5.1.8. Define the a natural map

$$\varphi : \mathrm{Hom}_{\mathsf{sGpd}}\left([X_1 \rightrightarrows X_0], [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]\right) \to \mathrm{Hom}_{\mathsf{topStack}}\left([|X_0|/|X_1|], \mathcal{Y}\right)$$

as the composition of the natural isomorphism of groupoids

$$\mathrm{Hom}_{\mathsf{topGpd}}([X_1 \rightrightarrows X_0], [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]) \cong \mathrm{Hom}_{\mathsf{topGpd}}([|X_1| \rightrightarrows |X_0|], [R \rightrightarrows Y])$$

arising from the usual adjunction between $|\cdot|$ and Sing$(\cdot)$, with the quotient map defined in Lemma 5.1.2

$$\mathrm{Hom}_{\mathsf{topGpd}}([|X_1| \rightrightarrows |X_0|], [R \rightrightarrows Y]) \to \mathrm{Hom}_{\mathsf{topStack}}([|X_0|/|X_1|], \mathcal{Y}).$$

Lemma 5.1.9. *If $\alpha : Y \to \mathcal{Y}$ is a classifying atlas for a topological stack $\mathcal{Y}$ and $[X_1 \rightrightarrows X_0]$ is a simplicial groupoid then the natural map defined in Definition 5.1.8,*

$$\varphi : \mathrm{Hom}_{\mathsf{sGpd}}\left([X_1 \rightrightarrows X_0], [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]\right) \to \mathrm{Hom}_{\mathsf{topStack}}\left([|X_0|/|X_1|], \mathcal{Y}\right)$$

is an equivalence of groupoids.

Proof. Let $[R \rightrightarrows Y]$ be the groupoid associated to the classifying atlas $\alpha : Y \to \mathcal{Y}$. By Lemma 5.1.2,

$$\mathrm{Hom}_{\mathsf{topGpd}}([|X_1| \rightrightarrows |X_0|], [R \rightrightarrows Y]) \to \mathrm{Hom}_{\mathsf{topStack}}([|X_0|/|X_1|], \mathcal{Y})$$

is an equivalence of groupoids. Using the usual adjunction between $|\cdot|$ and Sing$(\cdot)$, we have the following equivalence of groupoids

$$\mathrm{Hom}_{\mathsf{sGpd}}([X_1 \rightrightarrows X_0], [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]) \to \mathrm{Hom}_{\mathsf{topGpd}}([|X_1| \rightrightarrows |X_0|], [R \rightrightarrows Y]).$$

Since φ is the composition of these two maps, φ is an equivalence of groupoids. ☐

Remark 5.1.10. If a map of groupoids $f : \mathbb{X} \to \mathbb{Y}$, has two inverses σ and σ' so that $f \circ \sigma \overset{\alpha_\sigma}{\Longrightarrow} \mathrm{id}_{\mathbb{X}}$ and $f \circ \sigma' \overset{\alpha_{\sigma'}}{\Longrightarrow} \mathrm{id}_{\mathbb{X}}$ then there exists a unique isomorphism $\theta : \sigma \Rightarrow \sigma'$ such that $f \circ \theta \cdot \alpha_{\sigma'} = \alpha_\sigma$.

Remark 5.1.11. Remark 5.1.10 is also true for presheaves of groupoids. If a map of presheaves of groupoids $f : \mathcal{X} \to \mathcal{Y}$, has two inverses σ and σ' so that $f \circ \sigma \overset{\alpha_\sigma}{\Longrightarrow} \mathrm{id}_{\mathcal{X}}$ and $f \circ \sigma' \overset{\alpha_{\sigma'}}{\Longrightarrow} \mathrm{id}_{\mathcal{X}}$ then there exists a unique isomorphism $\theta : \sigma \Rightarrow \sigma'$ such that $f \circ \theta \cdot \alpha_{\sigma'} = \alpha_\sigma$.

Proposition 5.1.12. *Consider the sub-2-category of simplicial groupoids with objects $[X_1 \rightrightarrows X_0]$ where X_0 is paracompact and the sub-2-category of topological stacks which are Reedy fibrant and have a Serre classifying atlas.*

Between these sub-categories, there is a natural equivalence of groupoids

$$\mathrm{Hom}_{\mathsf{sGpd}}([X_1 \rightrightarrows X_0], \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})) \to \mathrm{Hom}_{\mathsf{topStack}}\left([|X_0|/|X_1|], \mathcal{Y}\right).$$

which is unique up to unique isomorphism.

Proof. We define this equivalence of groupoids to be the composition of the two equivalences of groupoids

$$\mathrm{Hom}_{\mathsf{sGpd}}([X_1 \rightrightarrows X_0], \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})) \xrightarrow{\rho_\alpha} \mathrm{Hom}_{\mathsf{sGpd}}([X_1 \rightrightarrows X_0], [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)])$$
$$\mathrm{Hom}_{\mathsf{sGpd}}([X_1 \rightrightarrows X_0], [\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)]) \xrightarrow{\varphi} \mathrm{Hom}_{\mathsf{topStack}}\left([|X_0|/|X_1|], \mathcal{Y}\right)$$

where ρ_α is the inverse of the map θ_*, defined in Lemma 5.1.7, for a Serre classifying atlas α and φ is defined in Lemma 5.1.9.

In order to show that this gives a 2-adjunction, we need to show that the map

$$\varphi \circ \rho_\alpha : \mathrm{Hom}_{\mathsf{sGpd}}([X_1 \rightrightarrows X_0], \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})) \to \mathrm{Hom}_{\mathsf{topStack}}([|X_0|/|X_1|], \mathcal{Y})$$

is natural and does not depend on the choice of classifying atlas.

We shall show that the map is well defined up to unique isomorphism. Firstly, by Remark 5.1.11, if $Y \to \mathcal{Y}$ is a Serre classifying atlas, the map

$$[\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(Y)] \to \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})$$

has an inverse which is unique up to unique isomorphism.

Secondly, if we chose two Serre classifying atlases $Y \to \mathcal{Y}$ and $Y' \to \mathcal{Y}$ then there exists an equivalence of simplicial groupoids and a 2-isomorphism (f, ψ) which make the following diagram commutative

$$\begin{array}{ccc}
\mathrm{Sing}[R \rightrightarrows Y] & \xrightarrow{\ f\ } & \mathrm{Sing}[R' \rightrightarrows Y'] \\
& {\scriptstyle\theta} \searrow \quad \Downarrow \psi \quad \swarrow {\scriptstyle\theta'} & \\
& \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y}) &
\end{array}$$

If (f, ψ) and (f', ψ') are two such equivalences, then there exists a unique $\lambda : f \Rightarrow f'$ which is relative to $\mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})$ (making the appropriate tetrahedron commute).By Remark 5.1.11, this implies that there is a unique isomorphism $\rho \Rightarrow f \circ \rho'$.

The upshot of the previous discussion is that up to unique isomorphism, there is a well defined map

$$\mathrm{Hom}_{\mathsf{sGpd}}([X_1 \rightrightarrows X_0], \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y})) \to \mathrm{Hom}_{\mathsf{topStack}}([|X_0|/|X_1|], \mathcal{Y})$$

for each simplicial groupoid $[X_1 \rightrightarrows X_0]$ and each Reedy topological stack $\mathcal{Y}$ with a Serre atlas.

To show this gives rise to a 2-adjunction, it remains to show that it is natural in each argument. Clearly, it is natural in the first argument.

To see that it is natural in the second argument, consider a morphism of topological stacks $f : \mathcal{X} \to \mathcal{Y}$ and a Serre atlas $Y \to \mathcal{Y}$. By taking the fiber product, $\mathcal{X} \tilde{\times}_{\mathcal{Y}} Y$ is a Serre atlas for $\mathcal{X}$ which induces a natural map between the corresponding groupoid presentations $\bar{f} : [R \rightrightarrows Y] \to [R' \rightrightarrows \mathcal{X} \tilde{\times}_{\mathcal{Y}} Y]$. This induces a diagram of simplicial groupoids

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{X}) & \xrightarrow{\ f_*\ } & \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{Y}) \\
{\scriptstyle\rho} \downarrow & \swarrow\!\!\Downarrow & \downarrow {\scriptstyle\rho'} \\
\mathrm{Sing}[R \rightrightarrows Y] & \xrightarrow{\ \mathrm{Sing}(\bar{f})\ } & \mathrm{Sing}[R' \rightrightarrows Y']
\end{array}$$

which by Remark 5.1.11, is level-wise 2-commutative up to a unique isomorphism. □

Remark 5.1.13. Note that the above result does not give a 2-adjunction. This is because the image of the functors Q and $\mathrm{Hom}_{\mathsf{topStack}}(\Delta^\bullet, -)$ do not land in the correct target category.

5.1.2 Counit map for Sing

We shall recall the definition of the 'fat realization'. This realization is in some sense better behaved than the usual geometric realization (at least from a homotopy theoretic point of view). For more details on the fat realization, see [30].

Definition 5.1.14. Given a simplicial space $X : \mathbf{\Delta} \to \mathsf{Top}$, the *fat realization* of X is the coend

$$\|X\| := \int^{\mathbf{\Delta}^+} X_n \times |\Delta^n|$$

where $\mathbf{\Delta}^+ \subset \mathbf{\Delta}$ is the subcategory with the same objects but generated by only the face maps (no degeneracy maps).

Definition 5.1.15. Let X be a simplicial space. We say X is *good* if the degeneracy maps of X are closed Hurewicz cofibrations. For more details, see Appendix A of [30].

Remark 5.1.16. A simplicial space X is good if and only if it is Reedy cofibrant in

$$[\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Top}_{\mathrm{Strøm}}]$$

where $\mathsf{Top}_{\mathrm{Strøm}}$ denotes the Strøm model category structure on Top. For more details, see [32].

For a topological groupoid $\mathbb{X}$, the simplicial space $N(\mathbb{X})$ is good if the identity map $\mathbb{X}_0 \to \mathbb{X}_1$ is a neighborhood deformation retract. If $\mathbb{X}_0 \to \mathbb{X}_1$ is a neighborhood deformation retract, we shall say the groupoid $\mathbb{X}$ is *good.*

Proposition 5.1.17. *Let $\mathbb{X}$ be a topological groupoid, then the bisimplicial sets*

$$N\,\mathrm{Sing}\,\mathbb{X} = \mathrm{Sing}\,N\mathbb{X}$$

are isomorphic.

Proof. This follows from the fact that Sing and N are both right adjoints. □

The classical adjunction between $\mathrm{Sing}(-)$ and $|-|$ has a counit $|\,\mathrm{Sing}(X)| \to X$ which is a weak equivalence of topological spaces. We shall prove an analogue of this result for topological stacks, however we are not able to show that there is a canonical map.

Proposition 5.1.18. *Let $\mathcal{X}$ be a Reedy fibrant topological stack with a classifying atlas $\alpha : X \to \mathcal{X}$ which is a Serre fibration. Then there exists a map $\lambda : |\,\mathrm{Sing}(\mathcal{X})| \to \mathcal{X}$ which is a weak equivalence. We define λ in the proof below.*

Proof. We shall define the simplicial space $\mathcal{L}_\bullet := N([R \rightrightarrows X])$ where $[R \rightrightarrows X]$ is the groupoid associated to α. From Proposition 5.1.5, there exists a map $\mathrm{Diag}(N(\rho)) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Diag}(\mathrm{Sing}\,\mathcal{L}_\bullet)$ which is a weak equivalence. After we apply the geometric realization functor, $|\mathrm{Diag}(N(\rho))|$ is a weak equivalence.

In [27] Noohi constructs the classifying space of a topological stack as the fat realization of the simplicial space $\mathcal{L}_\bullet$ and constructs a trivial Serre fibration (in fact a locally shrinkable map) $\mu : \|\mathcal{L}_\bullet\| \to \mathcal{X}$. Thus, in order to complete the proof, we will construct a map

$$\gamma : |\mathrm{Diag}(\mathrm{Sing}\,\mathcal{L}_\bullet)| \xrightarrow{\text{w.e.}} \|\mathcal{L}_\bullet\|,$$

which is a weak equivalence. We shall then define $\lambda = \mu \circ \gamma \circ |\mathrm{Diag}(N(\rho))|$.

Consider the following morphisms

$$|\mathrm{Diag}\,\mathrm{Sing}\,\mathcal{L}_\bullet| \xrightarrow{f} ||\,\mathrm{Sing}\,\mathcal{L}_\bullet|| \xleftarrow{\Omega} \|\,|\,\mathrm{Sing}\,\mathcal{L}_\bullet|\,\| \xrightarrow{g} \|\mathcal{L}_\bullet\|$$

where $f : |\mathrm{Diag}\,\mathrm{Sing}\,\mathcal{L}_\bullet| \cong ||\,\mathrm{Sing}\,\mathcal{L}_\bullet||$ is an isomorphism and $\Omega : \|\,|\,\mathrm{Sing}\,\mathcal{L}_\bullet|\,\| \to ||\,\mathrm{Sing}\,\mathcal{L}_\bullet||$ is a homotopy equivalence since $|\,\mathrm{Sing}\,\mathcal{L}_\bullet|$ is good for any simplicial space $\mathcal{L}_\bullet$. Also, we note that by Proposition A.1(ii) of [30], since $|\,\mathrm{Sing}\,\mathcal{L}_\bullet| \to \mathcal{L}_\bullet$ is a level-wise weak equivalence,

$$g : \|\,|\,\mathrm{Sing}\,\mathcal{L}_\bullet|\,\| \to \|\mathcal{L}_\bullet\|$$

is a weak equivalence.

Since $\Omega : \|\,|\,\mathrm{Sing}\,\mathcal{L}_\bullet|\,\| \to ||\,\mathrm{Sing}\,\mathcal{L}_\bullet||$ is a homotopy equivalence, choose a homotopy inverse ω (which will also be a weak equivalence).

Finally, define $\gamma = g \circ \omega \circ f$. This gives the weak equivalence required. □

Remark 5.1.19. The map $\lambda : |\,\mathrm{Sing}(\mathcal{X})| \to \mathcal{X}$ is non-canonical as it requires a choice of an inverse to the map $[\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(X)] \to \mathrm{Hom}_{\mathsf{topStack}}(|\Delta^\bullet|, \mathcal{X})$ and to $\Omega : \|\,|\,\mathrm{Sing}\,\mathcal{L}_\bullet|\,\| \to ||\,\mathrm{Sing}\,\mathcal{L}_\bullet||$. If we pass to the homotopy category, there is a canonical map $|\,\mathrm{Sing}(\mathcal{X})| \to \mathcal{X}$.

5.2 Stacks with a given groupoid presentation

5.2.1 Computing the homotopy type of a stack using a groupoid presentation

Using the functor Sing, we may take a topological stack and extract its homotopy type. If we have a groupoid presentation $[R \rightrightarrows X]$ for a stack, we may also form the simplicial set $\mathrm{Diag}(N[\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(X)])$. For specific examples, this can be easier to compute than $\mathrm{Sing}([X/R])$. In fact, these two simplicial sets are weakly equivalent if X is paracompact (we shall prove this in Proposition 5.2.1).

We reiterate that throughout this section, by 'stack' we mean 'Serre stack' (see Section 2.10.7 for more details). This is just a technical assumption which ensures that $\mathrm{Sing}(\mathcal{X})$ computes the correct homotopy type of the stack $\mathcal{X}$.

Proposition 5.2.1. *Consider a topological stack* $\mathcal{X}$ *with a groupoid presentation* $\mathbb{X} = [R \rightrightarrows X]$ *where* X *is paracompact, then the natural map (defined in Eq.* (5.1)*)*

$$\theta_{\mathbb{X}} : \mathit{Diag}(N[\mathrm{Sing}(R) \rightrightarrows \mathrm{Sing}(X)]) \to \mathrm{Sing}(\mathcal{X})$$

is a weak equivalence.

Note that in Proposition 5.2.1 we take any groupoid presentation $[R \rightrightarrows X]$ (with X paracompact) for the stack $\mathcal{X}$, whilst in Proposition 5.1.5 the groupoid presentation must have the additional property that $X \to \mathcal{X}$ is a classifying atlas. This additional property can be restrictive in computations, which is why we consider Proposition 5.2.1 as well.

Proof. Let $\mathbb{X}' = [R' \rightrightarrows X']$ be a presentation for $\mathcal{X}$ where $X' \to \mathcal{X}$ is a classifying atlas. For notational brevity, we denote $X_\bullet = N\mathbb{X}$ and $X'_\bullet = N\mathbb{X}'$.

Since we assume that X is paracompact, by [27] (Lemma 5.5) there exists a map $X \to X'$. This induces a map of groupoids $\phi : \mathbb{X} \to \mathbb{X}'$, which is unique up to homotopy.

Note that

$$\begin{array}{ccc} \mathrm{Sing}[R \rightrightarrows X] & \xrightarrow{\mathrm{Sing}(\phi)} & \mathrm{Sing}[R' \rightrightarrows X'] \\ & \Swarrow & \\ {\scriptstyle \theta_{\mathbb{X}}} \searrow & & \swarrow {\scriptstyle \theta_{\mathbb{X}'}} \\ & \mathrm{Hom}_{\mathsf{topStack}}(\Delta^\bullet, \mathcal{X}) & \end{array}$$

is 2-commutative. This implies that after applying $\mathrm{Diag} \circ N$,

$$\begin{array}{ccc} \mathrm{Diag}N\,\mathrm{Sing}[R \rightrightarrows X] & \xrightarrow{\mathrm{Diag}N\,\mathrm{Sing}(\phi)} & \mathrm{Diag}N\,\mathrm{Sing}[R' \rightrightarrows X'] \\ {\scriptstyle \mathrm{Diag}N(\theta_{\mathbb{X}})} \searrow & & \swarrow {\scriptstyle \mathrm{Diag}N(\theta_{\mathbb{X}'})} \\ & \mathrm{Sing}(\mathcal{X}) & \end{array}$$

is homotopy commutative.

By the above remark and the 2-of-3 property, it suffices to show that $\mathrm{Diag}N\,\mathrm{Sing}(\phi) : \mathrm{Diag}N\,\mathrm{Sing}\,\mathbb{X} \to \mathrm{Diag}N\,\mathrm{Sing}\,\mathbb{X}'$ is a weak equivalence. Consider the following commutative diagram:

$$\begin{array}{ccc} \|X_\bullet\| & \xrightarrow{\|N(\phi)\|} & \|X'_\bullet\| \\ a \uparrow & & \uparrow b \\ \|\,|\mathrm{Sing}(X_\bullet)|\,\| & \xrightarrow[\|\,|\mathrm{Sing}\,N(\phi)|\,\|]{} & \|\,|\mathrm{Sing}(X'_\bullet)|\,\| \\ c \downarrow & & \downarrow d \\ |\,|\mathrm{Sing}(X_\bullet)|\,| & \xrightarrow[|\,|\mathrm{Sing}\,N(\phi)|\,|]{} & |\,|\mathrm{Sing}(X'_\bullet)|\,| \end{array}$$

where the vertical maps a and b are defined by the fat realization of the maps of simplicial spaces which are defined level-wise by the natural maps $|\operatorname{Sing}(X_\bullet)| \to X_\bullet$ and $|\operatorname{Sing}(X'_\bullet)| \to X'_\bullet$ respectively. Both a and b are weak equivalences since the map of simplicial spaces defined above is a level-wise weak equivalence.

Since $|\operatorname{Sing}(X_\bullet)|$ and $|\operatorname{Sing}(X'_\bullet)|$ are 'good' simplicial spaces (in the sense of Segal [30]), there are natural weak equivalences $c : \| \, |\operatorname{Sing}(X_\bullet)| \, \| \to | \, |\operatorname{Sing}(X_\bullet)| \, |$ and $d : \| \, |\operatorname{Sing}(X'_\bullet)| \, \| \to | \, |\operatorname{Sing}(X'_\bullet)| \, |$.

The map $\|N(\phi)\|$ is also a weak equivalence since $\|X_\bullet\|$ and $\|X'_\bullet\|$ are both classifying atlases for $\mathcal{X}$. After applying the '2 out of 3 property' twice,

$$| \, |\operatorname{Sing} N\phi| \, | : | \, |\operatorname{Sing}(X_\bullet)| \, | \to | \, |\operatorname{Sing}(X'_\bullet)| \, |$$

is a weak equivalence.

Finally, note that $| \, |\operatorname{Sing}(X_\bullet)| \, |$ is homeomorphic to $|\operatorname{Diag}\operatorname{Sing}(X_\bullet)|$. This implies that

$$\operatorname{Diag} N \operatorname{Sing}(\phi) = \operatorname{Diag}\operatorname{Sing} N(\phi) : \operatorname{Diag}\operatorname{Sing}(X_\bullet) \to \operatorname{Diag}\operatorname{Sing}(X'_\bullet)$$

is a weak equivalence as required (here we use Proposition 5.1.17 to say Sing and N commute). □

This gives a 'convenient model' for calculating $\operatorname{Sing}(\mathcal{X})$. Namely, we can choose a groupoid presentation under conditions that are easy to satisfy and then form a simplicial set in the obvious way. We consider a special case of the above result.

Example 5.2.2. *Let us consider the case when a topological group G acts on a paracompact manifold X. From this group action we can form the action groupoid $[G \times X \rightrightarrows X]$ and from this the quotient stack $[X/G]$. By Proposition 5.2.1, we see that*

$$\mathit{Diag}(N[\operatorname{Sing}(G) \times \operatorname{Sing}(X) \rightrightarrows \operatorname{Sing}(X)] \to \operatorname{Sing}([X/G])$$

is a weak equivalence.

We shall expand on Example 5.2.2 in Section 5.3.2.

5.2.2 Fibrancy condition on groupoids

We shall show in this section that $\operatorname{Diag} N \operatorname{Sing} \mathbb{X}$ is a fibrant simplicial set if the source and target maps of the groupoid $\mathbb{X}$ are Serre fibrations. In conjunction with Proposition 5.3.5, this will enable us to describe the homotopy type of certain mapping stacks.

Is is well known that a category C is a groupoid if and only if upon taking the nerve $N(\mathsf{C})$ has the unique lifting property with respect to all maps of the form $\Lambda^n_k \to \Delta$. We consider a slight variant of this property below.

Definition 5.2.3. Let $\alpha \in (\Delta^n)_m$. We define the simplicial set C_α to be the subsimplicial set of Δ^n generated by all faces of Δ^n which contain α.

For example, if the image of $\alpha : \Delta^0 \to \Delta^n$ is the k^{th} vertex, then $C_\alpha = \Lambda^n_k$.

Proposition 5.2.4. *Let $\mathbb{X}$ be a groupoid. If α is a k-cell of Δ^n with $k < n$ which is contained in at least two $(n-1)$-cells, then $i : C_\alpha \to \Delta^n$ has the unique lifting property with respect to the terminal morphism $N(\mathbb{X}) \to \Delta^0$.*

Proof. We know that $N(\mathbb{X})$ is 1-coskeletal. We also know that since α is contained in at least two $(n-1)$-cells, $\mathrm{Sk}_0(\Delta^n) = \mathrm{Sk}_0(C_\alpha)$ and C_α is connected. We can use the property of 1-coskeletal simplicial sets now to inductively produce a unique lift $\Delta^n \to N(\mathbb{X})$. $\square$

Remark 5.2.5. We note that a sufficient condition for $\mathrm{Diag} N\,\mathrm{Sing}\,\mathbb{X}$ to be fibrant is that $N\,\mathrm{Sing}\,\mathbb{X}$ is Reedy fibrant (where we consider the bisimplicial set to be a functor $[\Delta^{\mathrm{op}}, \mathsf{sSet}]$ sending $[n] \mapsto (N\,\mathrm{Sing}_\bullet\,\mathbb{X})_n$). This is true if and only if the source map $s : R \to X$ in the groupoid presentation is a trivial Serre fibration. Although this can often be arranged, since we want to compute examples we opt for the less strict requirements of Proposition 5.2.10.

5.2.3 Bisimplicial set notation

In order to prove Proposition 5.2.10, we shall work in the category of bisimplicial sets. We will define the various morphisms that we shall use below.

We define $I^n_k \subset \Lambda^n_k \subset \Delta^n$ to be the simplicial set generated by the $n-2$ cells which are represented by subsets of $\{0, 1, \cdots, n\}$ containing k.

Consider the map $a : d^*(\Lambda^n_k) \to \Lambda^n_k \boxtimes \Delta^n$, which is the adjoint to the inclusion

$$\Lambda^n_k \to \mathrm{Diag}(\Lambda^n_k \boxtimes \Delta^n) = \Lambda^n_k \times \Delta^n.$$

Recall that

$$d^*(\Lambda^n_k)_{m,*} = \coprod_{\alpha \in (\Lambda^n_k)_m} C_\alpha$$

where C_α is the subsimplicial set of Λ^n_k generated by all faces of Λ^n_k which contain α (this can be found on page 221 of [13]). Using this description, we may write the map a as

$$a : \coprod_{\alpha \in (\Lambda^n_k)_m} C_\alpha \to \coprod_{\alpha \in (\Lambda^n_k)_m} \Delta^n = \Lambda^n_k \boxtimes \Delta^n$$

defined by the inclusions $C_\alpha \hookrightarrow \Delta^n$ for each α (note that the above equality is obvious from the definition).

We shall define the bisimplicial set S by

$$S_{m,*} = \coprod_{\alpha \in (I^n_k)_m} C_\alpha.$$

We define the inclusion of bisimplicial sets $b : S \to d^*(\Lambda^n_k)$ level-wise via the inclusions

$$S_{m,*} = \coprod_{\alpha \in (I^n_k)_m} C_\alpha \hookrightarrow \coprod_{\alpha \in (\Lambda^n_k)_m} C_\alpha = d^*(\Lambda^n_k)_{m,*}$$

where $C_\alpha \subseteq \Lambda^n_k$ has the same definition as above.

We shall also consider the inclusion of bisimplicial sets $c : S \to I^n_k \boxtimes \Delta^n$ which is defined level-wise by

$$c_{m,*} : S_{m,*} = \coprod_{\alpha \in (I^n_k)_m} C_\alpha \to \coprod_{\alpha \in (I^n_k)_m} \Delta^n = (I^n_k \boxtimes \Delta^n)_{m,*}$$

where the map $C_\alpha \hookrightarrow \Delta^n$ is the obvious inclusion.

Finally, we shall consider the maps

$$i : I^n_k \boxtimes \Delta^n \to \Lambda^n_k \boxtimes \Delta^n$$
$$j : \Lambda^n_k \boxtimes \Delta^n \to \Delta^n \boxtimes \Delta^n$$

which are adjoint to the inclusions $I^n_k \to \Lambda^n_k$ and $\Lambda^n_k \hookrightarrow \Delta^n$ respectively (where the adjunction is described in Definition 2.8.3).

5.2.4 Fibrancy of Diag$N \operatorname{Sing} \mathbb{X}$

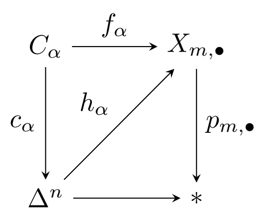
Most of the work in this section will be to show that $a : d^*(\Lambda^n_k) \to \Lambda^n_k \boxtimes \Delta^n$ has the LLP with respect to $p : \operatorname{Sing} N\mathbb{X} \to *$. We briefly outline the strategy here.

Firstly, in Lemma 5.2.6 we shall show that Proposition 5.2.4, allows us to find a unique lift for the pair $(c : d^*(I^n_k) \to I^n_k \boxtimes \Delta^n, p)$. (Informally, this means that there is a unique lift in the nerve-wise direction over I^n_k.)

After doing this, we shall extend these isomorphisms over the 'geometric' $(n-1)$ faces of $\Lambda^n_k \supset I^n_k$. This will appear in Corollary 5.2.8.

Lemma 5.2.6. *Consider the map $c : d^*(I^n_k) = S \to I^n_k \boxtimes \Delta^n$, defined in Section 5.2.3. Let $\mathbb{X}$ be a topological groupoid. Then the terminal morphism of bisimplicial sets $p : N \operatorname{Sing} \mathbb{X} \to *$ has the unique right lifting property with respect to c.*

Proof. Write $X := N \operatorname{Sing} \mathbb{X}$. We shall use the convention that $X_{m,\bullet} = N(\operatorname{Sing}(\mathbb{X})_m)$. We first notice that each lifting problem



always has a unique lift when $\alpha \in (I^n_k)_m$ for some m. This follows from Proposition 5.2.4 since $X_{m,*} = N(\operatorname{Sing}(\mathbb{X})_m)$.

The uniqueness of the above lifts means that the following diagram

$$\begin{array}{ccc} \coprod\limits_{\alpha\in(I^n_k)_m} C_\alpha & \xrightarrow{f_m} & X_{m,\bullet} \\ {\scriptstyle c_m}\Big\downarrow & \overset{h_m}{\nearrow} & \Big\downarrow{\scriptstyle p_{m,\bullet}} \\ \coprod\limits_{\alpha\in(I^n_k)_m} \Delta^n & \longrightarrow & * \end{array} \tag{5.2}$$

also has a unique lift h_m.

To show that the level-wise lift constructed in 5.2 gives a map $I^n_k \boxtimes \Delta^n \to X$ which is a lift for the pair (c, p), it is enough to show that for every map $q : [a] \to [b]$ in $\mathbf{\Delta}$ the induced diagram

$$\begin{array}{ccc} X_{a,*} & \xleftarrow{q^*} & X_{b,*} \\ {\scriptstyle h_\alpha}\Big\uparrow & \nearrow{\scriptstyle h_\beta} & \\ \Delta^n & & \end{array}$$

commutes for each $\alpha \in (I^n_k)_a$ and for each $\beta \in (I^n_k)_b$ such that $q^*(\beta) = \alpha$. This follows from the uniqueness of the lifts. □

Lemma 5.2.7. *Let* $X = \operatorname{Sing} N\mathbb{X}$ *for a topological groupoid* $\mathbb{X}$*. If the source and hence target maps are Serre fibrations and the following diagram commutes*

$$\begin{array}{ccccc} & & I^n_k \boxtimes \Delta^n & & \\ & \overset{c}{\nearrow} & {\scriptstyle i}\Big\downarrow & \overset{f}{\searrow} & \\ S & \hookrightarrow & \Lambda^n_k \boxtimes \Delta^n & \overset{h}{\dashrightarrow} & X \\ & \underset{b}{\searrow} & {\scriptstyle a}\Big\uparrow & \underset{g}{\nearrow} & \\ & & d^*(\Lambda^n_k) & & \end{array}$$

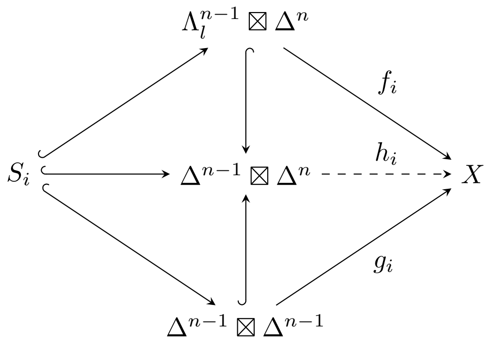
then the dashed morphism exists and makes the diagram commutative.

Proof. We shall prove this by considering $\Lambda^n_k \boxtimes \Delta^n$ as a colimit of $\Delta^{n-1} \boxtimes \Delta^n$ indexed by the face maps $d^i : \Delta^{n-1} \to \Lambda^n_k$. We are using the fact that $(\operatorname{colim}_I T_i) \boxtimes \Delta^n = \operatorname{colim}_I (T_i \boxtimes \Delta^n)$ since $- \boxtimes \Delta^n$ is a left adjoint and we are writing Λ^n_k as a colimit of its faces. We shall show that there is a lift for each 'face' and that these lifts agree on their intersection.

We shall define the map $h : \Lambda^n_k \boxtimes \Delta^n \to X$ on each 'face'. Define Λ^{n-1}_l to be the pullback of $I^n_k \hookrightarrow \Delta^n$ along $d_i : \Delta^{n-1} \to \Delta^n$ (so l depends on k and i). We also define S_i to be the bisimplicial set given by

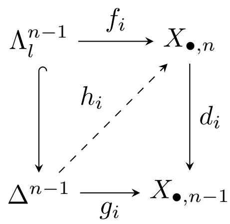
$$(S_i)_{m,*} = \coprod_{\alpha \in (\Lambda_l^{n-1})_m} C_\alpha.$$

For each $i = 0, \cdots, \hat{k}, \cdots, n$, consider the diagram below which is induced by the 'ith face' of the original diagram



(5.3)

By Definition 2.8.3, diagram 5.3 is equivalent to the commutative square

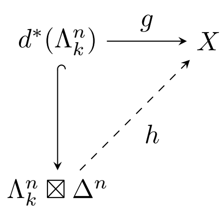


(5.4)

where the commutativity is guaranteed by the map $S_i \to \Lambda_k^n \boxtimes \Delta^n$. Since $d_i : X_{\bullet,n} \to X_{\bullet,n-1}$ is a Serre fibration, diagram 5.4 has a lift h_i.

It remains to show that the maps h_i agree on their intersection. A map from the intersection of two faces is determined by f, thus the commutativity of diagram 5.4 for each i ensures that the maps h_i agree on their intersection (since the maps f_i are compatible). □

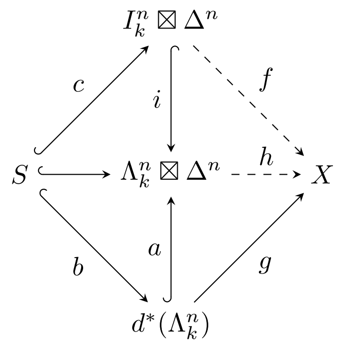
Corollary 5.2.8. *Let $X = \operatorname{Sing} N\mathbb{X}$ for a topological groupoid $\mathbb{X}$. If the source and hence target maps are Serre fibrations, then given*



there exists a lift h.

Proof. Recall that there are natural maps $b : S \to d^*(\Lambda_k^n)$ and $c : S \to I_k^n \boxtimes \Delta^n$ (defined in

Section 5.2.3), which gives rise to the commutative diagram



In the above diagram, f exists by Lemma 5.2.6. Then we can use Lemma 5.2.7 to construct the map h. This gives the lift that we require. □

Lemma 5.2.9. *Consider the map $j : \Lambda_k^n \boxtimes \Delta^n \to \Delta^n \boxtimes \Delta^n = \Delta^{n,n}$, which is adjoint to the inclusion $\alpha : \Lambda_k^n \hookrightarrow \Delta^n$, and a map of bisimplicial sets $p : X \to Y$. If $p_{*,n} : X_{*,n} \to Y_{*,n}$ is a Kan fibration, then j has the LLP with respect to p.*

Proof. Recall the adjunction

$$\mathrm{Hom}_{\mathsf{bsSet}}(A \boxtimes \Delta^n, X) \cong \mathrm{Hom}_{\mathsf{sSet}}(A, X_{*,n})$$

where A is a simplicial set and X is a bisimplicial set. This means that j has the LLP with respect to p if and only if α has the LLP with respect to $p_{*,n}$. This is true since α is a trivial cofibration and $p_{*,n}$ is a Kan fibration by assumption. □

Finally, we can prove the main result of this section.

Proposition 5.2.10. *If $\mathbb{X}$ is a topological groupoid where the source (and hence target) map is a Serre fibration, then Diag$N\,\mathrm{Sing}\,\mathbb{X}$ is fibrant.*

Proof. Under the adjunction between d^* and Diag (see Definition 2.8.2), we shall show that $d^*(\Lambda_k^n) \to \Delta^{n,n}$ has the LLP with respect to $p : N\,\mathrm{Sing}\,\mathbb{X} \to *$. Note that $d^*(\Lambda_k^n) \to \Delta^{n,n}$ factorizes as

$$d^*(\Lambda_k^n) \xrightarrow{a} \Lambda_k^n \boxtimes \Delta^n \xrightarrow{j} \Delta^n \boxtimes \Delta^n = \Delta^{n,n}$$

where a and j are defined in Section 5.2.3. In Corollary 5.2.8, we show that a has the LLP with respect to p. In Lemma 5.2.9, by setting $X = N\,\mathrm{Sing}\,\mathbb{X}$ and $p : X \to *$ as the terminal morphism, we show that j has the LLP with respect to p. This completes the proof. □

5.3 Homotopy type of mapping stacks

We shall give a concrete way to calculate the homotopy type of a mapping stack of the form $\mathrm{Map}_{\mathsf{topStack}}(|K|, [X/R])$ where K is a finite simplicial set and $[R \rightrightarrows X]$ is a topological groupoid with the source (and target) morphisms Serre fibrations which present the Serre stack $[X/R]$.

For completeness, we collect here some elementary results that we will use. Recall from Section 2.8.2 that the mapping simplicial set for two simplicial sets K and X is defined as

$$\mathrm{Map}_{\mathsf{sSet}}(K, X)_n := \mathrm{Hom}_{\mathsf{sSet}}(K \times \Delta^n, X).$$

Proposition 5.3.1. *Given a diagram $K : I \to \mathsf{sSet}$ and a simplicial set X, we have the following natural isomorphism*

$$\lim_I \mathrm{Map}_{\mathsf{sSet}}(K_i, X) \cong \mathrm{Map}_{\mathsf{sSet}}(\operatorname{colim}_I K_i, X).$$

Proof. See [15] Proposition 9.2.2. □

The totalization of a cosimplicial stack, $\mathcal{X}^\bullet$ is defined in Definition 4.1.16 to be

$$\mathrm{Tot}(\mathcal{X}^\bullet) = \mathrm{eq}\left(\prod_{[m]\to[n]} \mathrm{Map}_{\mathsf{topStack}}(|\Delta^m|, \mathcal{X}^n) \rightrightarrows \prod_{[n]} \mathrm{Map}_{\mathsf{topStack}}(|\Delta^n|, \mathcal{X}^n)\right).$$

Recall from Definition 4.1.17, that we have also a totalization functor for cosimplicial simplicial groupoids,

$$\mathrm{Tot}(X^\bullet) = \mathrm{eq}\left(\prod_{[m]\to[n]} \mathrm{Map}_{\mathsf{sSet}}(\Delta^m, X^n) \rightrightarrows \prod_{[n]} \mathrm{Map}_{\mathsf{sSet}}(\Delta^n, X^n)\right).$$

We may alternatively describe the totalization of a cosimplicial simplicial set $X^\bullet$ as

$$\mathrm{Tot}(X) = \mathrm{Map}_{[\boldsymbol{\Delta}, \mathsf{sSet}]}(\Delta, X),$$

for more details see Chapter VIII in [13].

Remark 5.3.2. We recall from [13], Chapter VIII, that if $p : X^\bullet \to Y^\bullet$ is a level-wise weak equivalence of cosimplicial simplicial sets and both $X^\bullet$ and $Y^\bullet$ are Reedy fibrant, then since the cosimplicial simplicial set $\Delta^\bullet$ is cofibrant, by SM7 and Ken Brown's Lemma (page 4 of [4]), $\mathrm{Tot}(p)$ is a weak equivalence.

Remark 5.3.3. Note that

$$\mathrm{Tot}(\mathcal{X}^\bullet) \subset \prod_{i=0}^{\infty} \mathrm{Map}_{\mathsf{topStack}}(|\Delta^i|, \mathcal{X}^i).$$

For a stack $\mathcal{X}^n$, since the topological space $|\Delta^n|$ is compact, from ([25]; Theorem 1.1) we know that $\mathrm{Map}_{\mathsf{topStack}}(|\Delta^n|, \mathcal{X}^n)$ is also a topological stack. The infinite product of topological stacks will not always be a topological stack, however it will be a paratopological stack. This means that for a cosimplicial stack $\mathcal{X}^\bullet$, $\mathrm{Tot}(\mathcal{X}^\bullet)$ is a paratopological stack. For a definition of a paratopological stack, see Definition 9.1 of [27].

Remark 5.3.4. We note that if we apply the functor Sing to a paratopological stack, we still get the same properties as for topological stacks. This is because by Section 9 of [27], there exists a classifying atlas for paratopological stacks.

Proposition 5.3.5. *For any simplicial set A and any two fibrant simplicial sets X and Y, if there is a weak equivalence $f : X \to Y$, then*

$$\mathrm{Map}_{\mathsf{sSet}}(A, X) \xrightarrow{f_*} \mathrm{Map}_{\mathsf{sSet}}(A, Y)$$

is a weak equivalence of simplicial sets.

Proof. This appears in [24] as Lemma 1.22. □

We aim to use Proposition 5.3.5 to show that $\mathrm{Map}_{\mathsf{sSet}}(S^1_\bullet, \mathrm{Sing}(\mathcal{X}))$ and $\mathrm{Map}_{\mathsf{sSet}}(S^1_\bullet, \mathrm{Diag}N\,\mathrm{Sing}\,\mathbb{X})$ are weakly equivalent. In order to do this, we will give conditions which ensure that both $\mathrm{Sing}(\mathcal{X})$ and $\mathrm{Diag}N\,\mathrm{Sing}\,\mathbb{X}$ are fibrant.

We have already considered under which conditions $\mathrm{Sing}(\mathcal{X})$ is fibrant in Chapter 3. In Section 5.2.2, we considered conditions which ensure that $\mathrm{Diag}N\,\mathrm{Sing}\,\mathbb{X}$ is fibrant.

5.3.1 Homotopy type of $\mathrm{Map}_{\mathsf{topStack}}(|K|, \mathcal{X})$

In this section, we shall compute the homotopy type of the topological stack $\mathrm{Map}_{\mathsf{topStack}}(|K|, \mathcal{X})$ where $\mathcal{X}$ is a Serre stack and K is a finite simplicial set. Recall from Lemma 2.10.29 that if $\mathcal{X}$ is a Serre stack, then $\mathrm{Map}_{\mathsf{topStack}}(|K|, \mathcal{X})$ is a generalized Serre stack.

As discussed in Section 16 of [26] and also in Section 3.3.2, we have to be careful when we take colimits in $\mathsf{topStack}$. The Yoneda embedding does not preserve colimits.

Remark 5.3.6. For any diagram $F : I \to \mathsf{Top}$ there is a natural morphism

$$\mathrm{Map}_{\mathsf{topStack}}(\operatorname*{colim}_I^{\mathsf{Top}} F, \mathcal{X}) \to \lim_I^{\mathsf{topStack}} \mathrm{Map}_{\mathsf{topStack}}(F(i), \mathcal{X})$$

however, this will not in general be an equivalence of stacks.

For an example of when this fails to be an equivalence, see (Section 3.3.1). The problem is that the colimit of topological spaces is not preserved under the Yoneda embedding.

Definition 5.3.7. There exists a natural map

$$\nu_{\mathcal{X}} : \mathrm{Map}_{\mathsf{topStack}}(|K_\bullet|, \mathcal{X}) \xrightarrow{w.e.} \mathrm{Tot}(\mathrm{Map}_{\mathsf{topStack}}(K_\bullet, \mathcal{X})).$$

For each $n \in \mathbb{N}$, there are maps

$$\mathrm{Map}_{\mathsf{topStack}}(|K_\bullet|, \mathcal{X}) \to \mathrm{Map}_{\mathsf{topStack}}(K_n \times |\Delta^n|, \mathcal{X}) \to \mathrm{Map}_{\mathsf{topStack}}(|\Delta^n|, \mathrm{Map}_{\mathsf{topStack}}(K_n, \mathcal{X})$$

which define $\nu_{\mathcal{X}}$ by the properties of limits (see Remark 5.3.6).

Remark 5.3.8. Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. Then ϕ induces 2-commutative squares

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathsf{topStack}}(|K_\bullet|, \mathcal{X}) & \longrightarrow & \mathrm{Map}_{\mathsf{topStack}}(K_n \times |\Delta^n|, \mathcal{X}) & \xrightarrow{\cong} & \mathrm{Map}_{\mathsf{topStack}}(|\Delta^n|, \mathrm{Map}_{\mathsf{topStack}}(K_n, \mathcal{X})) \\
{\scriptstyle \phi_*}\downarrow & \swarrow\!\!\!\!\swarrow & & & \downarrow{\scriptstyle \psi_*} \\
\mathrm{Map}_{\mathsf{topStack}}(|K_\bullet|, \mathcal{Y}) & \longrightarrow & \mathrm{Map}_{\mathsf{topStack}}(K_n \times |\Delta^n|, \mathcal{Y}) & \xrightarrow{\sim} & \mathrm{Map}_{\mathsf{topStack}}(|\Delta^n|, \mathrm{Map}_{\mathsf{topStack}}(K_n, \mathcal{Y}))
\end{array}$$

where $\psi : \mathrm{Map}_{\mathsf{topStack}}(K_\bullet, \mathcal{X}) \to \mathrm{Map}_{\mathsf{topStack}}(K_\bullet, \mathcal{Y})$ is the map induced by ϕ. This shows that $\nu_{\mathcal{Y}} \circ \phi_* \Rightarrow \mathrm{Tot}(\psi) \circ \nu_{\mathcal{X}}$.

For topological spaces, we have the following well known result.

Theorem 5.3.9. *For a simplicial set $K_\bullet$ and a topological space X,*

$$\nu_X : \mathrm{Map}_{\mathsf{Top}}(|K_\bullet|, X) \xrightarrow{\cong} \mathrm{Tot}\,\mathrm{Map}_{\mathsf{Top}}(K_\bullet, X)$$

is a homeomorphism.

Proof. This follows from the fact that Map(,) preserves limits. Note that,

$$\begin{aligned}
\mathrm{Map}_{\mathsf{Top}}(|K_\bullet|, X) &= \mathrm{Map}_{\mathsf{Top}}\Big(\int^{\boldsymbol{\Delta}} K_\bullet, X\Big) \\
&\cong \int_{\boldsymbol{\Delta}} \mathrm{Map}_{\mathsf{Top}}(K_\bullet, X) \\
&= \mathrm{Tot}\,\mathrm{Map}_{\mathsf{Top}}(K_\bullet, X).
\end{aligned}$$

□

Lemma 5.3.10. *If $\mathcal{X}$ is Reedy and K is any simplicial set, then* $\mathrm{Sing}\,\mathrm{Map}_{\mathsf{topStack}}(K, \mathcal{X})$ *is a Reedy fibrant cosimplicial simplicial set.*

Proof. We are required to show that

$$\mathrm{Sing}\,\mathrm{Map}_{\mathsf{topStack}}(K_n, \mathcal{X}) \to M_n(\mathrm{Sing}\,\mathrm{Map}_{\mathsf{topStack}}(K, \mathcal{X}))$$

is a Kan fibration.

Since each K_n is a discrete set and since Sing commutes with limits,

$$\begin{aligned}
M_n(\mathrm{Sing}\,\mathrm{Map}_{\mathsf{topStack}}(K, \mathcal{X})) &= \mathrm{Sing}\, M_n\, \mathrm{Map}_{\mathsf{topStack}}(K, \mathcal{X}) \\
&= \mathrm{Sing} \lim_{[m] \twoheadrightarrow [n]} \mathrm{Map}_{\mathsf{topStack}}(K_m, \mathcal{X}) \\
&= \lim_{[m] \twoheadrightarrow [n]} \mathrm{Map}_{\mathsf{sSet}}(K_m, \mathrm{Sing}(\mathcal{X})) \\
&= \mathrm{Map}_{\mathsf{sSet}}(L_n(K), \mathrm{Sing}(\mathcal{X})).
\end{aligned}$$

It remains to show that the map $\mathrm{Map}_{\mathsf{sSet}}(K_n, \mathrm{Sing}(\mathcal{X})) \to \mathrm{Map}_{\mathsf{sSet}}(L_n(K), \mathrm{Sing}(\mathcal{X}))$ induced by $L_n(K) \to K_n$ is a Kan fibration. Since $L_n(K) \to K_n$ is a cofibration of simplicial sets (where K_n is considered as a constant simplicial set) and $\mathrm{Sing}(\mathcal{X})$ is Kan, this follows from SM7. □

In the case of topological stacks, due to the failure of the Yoneda embedding to preserve arbitrary colimits, we have the following weakening of Theorem 5.3.9.

Theorem 5.3.11. *If $\mathcal{X}$ is a Reedy Serre stack and $K_\bullet$ is a simplicial set with the property that each K_n is finite, then $\nu_{\mathcal{X}}$ is a weak equivalence.*

Proof. Let $\phi : X \to \mathcal{X}$ be a classifying atlas. By Theorem 2.10.25, $\phi_* : \mathrm{Map}(|K_\bullet|, X) \to \mathrm{Map}(|K_\bullet|, \mathcal{X})$ is a classifying atlas.

By Remark 5.3.8, ϕ induces a 2-commutative diagram:

$$\begin{array}{ccc}
\mathrm{Map}_{\mathsf{topStack}}(|K_\bullet|, X) & \xrightarrow[\cong]{\nu_X} & \mathrm{Tot}\,\mathrm{Map}_{\mathsf{topStack}}(K_\bullet, X) \\
{\scriptstyle \phi_*}\downarrow & \swArrow & \downarrow{\scriptstyle \mathrm{Tot}(\psi)} \\
\mathrm{Map}_{\mathsf{topStack}}(|K_\bullet|, \mathcal{X}) & \xrightarrow{\nu_{\mathcal{X}}} & \mathrm{Tot}\,\mathrm{Map}_{\mathsf{topStack}}(K_\bullet, \mathcal{X})
\end{array}$$

where $\psi : \mathrm{Map}(K_\bullet, X) \to \mathrm{Map}(K_\bullet, \mathcal{X})$ is induced by ϕ.

By Lemma 5.3.10, if each K_n is finite then ψ is a level-wise weak equivalence of Reedy fibrant cosimplicial stacks. Consequently, $\mathrm{Tot}(\psi)$ is a weak equivalence of simplicial sets by Theorem 4.1.31. Finally, by the 2-of-3 axiom, $\nu_{\mathcal{X}}$ is a weak equivalence. $\square$

Proposition 5.3.12. *For a simplicial set $K_\bullet$ and a Reedy Serre stack $\mathcal{X}$, if $\mathbb{X} = [R \rightrightarrows X]$ is a groupoid presentation for $\mathcal{X}$ where the source and target morphisms are Serre fibrations and X is paracompact, then the simplicial sets $\mathrm{Sing}\,\mathrm{Map}_{\mathsf{topStack}}(|K_\bullet|, \mathcal{X})$ and $\mathrm{Map}_{\mathsf{sSet}}(K_\bullet, \mathit{DiagN}\,\mathrm{Sing}\,\mathbb{X})$ have a natural zigzag of weak equivalences between them.*

Proof. Using the map ν_X as defined in Theorem 5.3.9, by Theorem 3.5.2, we see that a classifying atlas, $\phi : X \to \mathcal{X}$ induces a weak equivalence

$$\mathrm{Sing}(\phi_*) \circ \mathrm{Sing}(\nu_X^{-1}) : \mathrm{Sing}\,\mathrm{Tot}\,\mathrm{Map}_{\mathsf{Top}}(K_\bullet, X) \to \mathrm{Sing}\,\mathrm{Map}_{\mathsf{topStack}}(|K_\bullet|, \mathcal{X}).$$

Since we assume that $\mathcal{X}$ is Reedy fibrant, by Proposition 5.3.5 and Proposition 5.2.10, we have weak equivalences

$$\mathrm{Map}_{\mathsf{sSet}}(K_\bullet, \mathrm{Sing}(X)) \xrightarrow{\mathrm{Sing}(\phi)_*} \mathrm{Map}_{\mathsf{sSet}}(K_\bullet, \mathrm{Sing}(\mathcal{X})) \xleftarrow{f_*} \mathrm{Map}_{\mathsf{sSet}}(K_\bullet, \mathrm{Diag}N\,\mathrm{Sing}\,\mathbb{X})$$

where $\mathrm{Sing}(\phi)_*$ is induced by ϕ and f_* is induced by the map defined in Proposition 5.2.1.

Finally, the result follows from the fact that $\mathrm{Sing}\,\mathrm{Tot}\,\mathrm{Map}_{\mathsf{Top}}(K_\bullet, X) \cong \mathrm{Map}_{\mathsf{sSet}}(K_\bullet, \mathrm{Sing}(X))$ are isomorphic simplicial sets. This is proved by firstly noting that

$$\mathrm{Tot}\,\mathrm{Map}_{\mathsf{Top}}(K_\bullet, X) \cong \mathrm{Map}_{\mathsf{Top}}(\int^{\mathbf{\Delta}} K_n \times |\Delta^n|, X) \cong \mathrm{Map}_{\mathsf{Top}}(|K_\bullet|, X).$$

Secondly, there is a bijection between the n-simplicies of $\operatorname{Sing}\operatorname{Map}_{\mathsf{Top}}(|K_\bullet|, X)$ and $\operatorname{Map}_{\mathsf{sSet}}(K_\bullet, \operatorname{Sing}(X))$

$$\begin{aligned}\operatorname{Hom}_{\mathsf{Top}}(|\Delta^n|, \operatorname{Map}_{\mathsf{Top}}(|K_\bullet|, X)) &\cong \operatorname{Hom}_{\mathsf{Top}}(|\Delta^n \times K_\bullet|, X)\\ &\cong \operatorname{Hom}_{\mathsf{sSet}}(\Delta^n \times K_\bullet, \operatorname{Sing}(X))\\ &\cong \operatorname{Hom}_{\mathsf{sSet}}(\Delta^n, \operatorname{Map}_{\mathsf{sSet}}(K_\bullet, \operatorname{Sing}(X))).\end{aligned}$$

□

Corollary 5.3.13. *For a Reedy fibrant Serre stack $\mathfrak{X}$, if $\mathbb{X} = [R \rightrightarrows X]$ is a groupoid presentation for $\mathfrak{X}$ where the source (and target) morphism are Serre fibrations and X is paracompact, then* $\operatorname{Sing}\operatorname{Map}_{\mathsf{topStack}}(S^1, \mathfrak{X})$ *and* $\operatorname{Map}_{\mathsf{sSet}}(S^1_\bullet, \mathit{Diag}N \operatorname{Sing}\mathbb{X})$ *have a natural zigzag of weak equivalences between them.*

Proof. This is immediate from Proposition 5.3.12. □

5.3.2 Example: Computing the homotopy type of $\operatorname{Map}_{\mathsf{topStack}}(S^1, [X/G])$

We shall show that we can use the above results to explicitly compute the cochain complex of the free loop stack $\operatorname{Map}_{\mathsf{topStack}}(S^1, [X/G])$.

Let G be a topological group acting on a paracompact topological space X (on the right) and define the action groupoid $[R \rightrightarrows X] := [G \times X \rightrightarrows X]$. Recall from Proposition 5.2.1 that $\operatorname{Diag}N \operatorname{Sing}[R \rightrightarrows X] \xrightarrow{w.e.} \operatorname{Sing}([X/R])$ is a weak equivalence.

Note that $[\operatorname{Sing}(X \times G)_n \rightrightarrows \operatorname{Sing}(X)_n]$ is the groupoid arising from the action of the discrete group $\operatorname{Sing}(G)_n$ on the discrete space $\operatorname{Sing}(X)_n$. This means we can describe the nerve as follows:

$$N([\operatorname{Sing}(X \times G)_n \rightrightarrows \operatorname{Sing}(X)_n])_m = \operatorname{Sing}(X)_n \times \operatorname{Sing}(G)_n^{\times m}$$

and the face maps are

$$d_i(x, g_1, \cdots, g_m) = \begin{cases} (x \cdot g_1, \cdots, g_m) & \text{if } i = 0\\ (x, g_1, \cdots, g_i \cdot g_{i+1}, \cdots, g_m) & \text{if } 0 < i < n\\ (x, g_1, \cdots, g_{m-1}) & \text{if } i = n \end{cases}$$

(the degeneracy maps add a repeated entry in the i^{th} position.) Upon taking the diagonal of the bisimplicial set $\operatorname{Sing}(X)_n \times \operatorname{Sing}(G)_n^{\times m}$ we obtain the following

$$\operatorname{Diag}(\operatorname{Sing}(X)_n \times \operatorname{Sing}(G)_n^{\times m})_k = \operatorname{Sing}(X)_k \times \operatorname{Sing}(G)_k^{\times k}$$

and the face maps are

$$d_i(x, g_1, \cdots, g_k) = \begin{cases} (x \cdot d_0(g_1), \cdots, d_0(g_k)) & \text{if } i = 0\\ (x, d_i(g_1), \cdots, d_i(g_i \cdot g_{i+1}), \cdots, d_i(g_k)) & \text{if } 0 < i < k\\ (x, d_k(g_1), \cdots, d_k(g_{m-1})) & \text{if } i = k \end{cases}$$

In Theorem 5.3.11, we showed that there is a natural weak equivalence

$$\nu_{[X/G]} : \mathrm{Map}_{\mathsf{topStack}}(S^1, [X/G]) \to \mathrm{Tot}\,\mathrm{Map}_{\mathsf{topStack}}(S^1_\bullet, [X/G]).$$

Using this in combination with results from [6] (the details of which will appear in [8]), this allows us to explicitly compute the cochains of the free loop stack $C^\bullet(\mathrm{Map}_{\mathsf{topStack}}(S^1, [X/G]))$ with coefficients in a field $\mathbf{k}$. In [8], we show that this chain complex is quasi-isomorphic to the chain complex

$$\mathrm{Tot}\,C^\bullet\,\mathrm{Hom}_{\mathsf{topStack}}(S^1_\bullet, [X/G]).$$

We may write the n^{th} component of this cochain complex (which computes the cohomology of the free loop stack) as

$$\begin{aligned}\mathrm{Tot}(C^\bullet(\mathrm{Hom}(S^1_\bullet, \mathfrak{X})))^n &= \prod_{p-q=n} \left(\mathbf{k}\cdot \mathrm{Sing}(X)_p \otimes \mathbf{k}\cdot \mathrm{Sing}(G)_p^{\otimes p}\right)^{\otimes(q+1)} \\ &= \prod_{p-q=n} (\mathbf{k}\cdot \mathrm{Sing}(X)_p)^{\otimes(q+1)} \otimes (\mathbf{k}\cdot \mathrm{Sing}(G)_p)^{\otimes p(q+1)}\end{aligned}$$

The differential may be written as $d = d^s - d^c$ where d^s is the differential in the simplicial direction and d^c is the differential in the cosimplicial direction

$$\begin{aligned}d^c &: \left(\mathbf{k}\cdot \mathrm{Sing}(X)_p \otimes \mathbf{k}\cdot \mathrm{Sing}(G)_p^{\otimes p}\right)^{\otimes(q+1)} \to \left(\mathbf{k}\cdot \mathrm{Sing}(X)_{p+1} \otimes \mathbf{k}\cdot \mathrm{Sing}(G)_{p+1}^{\otimes p+1}\right)^{\otimes(q+1)} \\ d^s &: \left(\mathbf{k}\cdot \mathrm{Sing}(X)_p \otimes \mathbf{k}\cdot \mathrm{Sing}(G)_p^{\otimes p}\right)^{\otimes(q+1)} \to \left(\mathbf{k}\cdot \mathrm{Sing}(X)_p \otimes \mathbf{k}\cdot \mathrm{Sing}(G)_p^{\otimes p}\right)^{\otimes q}\end{aligned}$$

which are both obtained by taking the alternating sum of the dual face maps (or 'coface maps') of the simplicial (cosimplicial) set which gives rise to the horizontal (vertical) part of the bicomplex.

Glossary

For the convenience of the reader, we shall list some of the notation for the categories that we will commonly use throughout this thesis:

- We shall use $\cong$ to denote an isomorphism, $\simeq$ to denote a weak equivalence and $\sim$ to denote an equivalence of stacks;
- Top is the category of compactly generated Hausdorff spaces;
- Gpd is the category of groupoids;
- $\mathbf{\Delta}$ is the simplex category;
- $[\mathsf{C}, \mathsf{D}]$ is the functor category of functors between the categories C and D;
- $\mathsf{pshGpd} := [\mathsf{Top}^{\mathrm{op}}, \mathsf{Gpd}]$ is the 2-category of strict presheaves of groupoids;
- sSet is the category of simplicial sets;
- bsSet is the category of bisimplicial sets;
- $\mathsf{sGpd} := [\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Gpd}]$ the category of simplicial groupoids.

We shall also collate all the notation that we shall use for the stacks and topological stacks:

- $\mathsf{pshGpd} = [\mathsf{Top}^{\mathrm{op}}, \mathsf{Gpd}]$ is the 2-category of strict presheaves of groupoids;
- $\mathsf{St}(\mathsf{C})$ is the 2-category of strict presheaves of groupoids over C which satisfy descent;
- CFG_C is the 2-category of categories fibered in groupoids over C;
- $\mathsf{St}_{\mathsf{CFG}}(\mathsf{C})$ is the 2-cateogry of categories fibered in groupoids over C which satisfy descent;
- pshSet is the category of presheaves of sets over Top;
- $\mathsf{topStack}$ is the 2-category of topological stacks;

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