# Aspects of Supersymmetric Field Theories in Four and Six Dimensions

by

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# Abstract

Supersymmetry is an important concept in modern high energy physics. It has found many applications in theoretical considerations of supersymmetric gauge theories as well as in phenomenological approaches to physics beyond the Standard Model. In this report we discuss some recent progress in supersymmetric field theories in four and six dimensions.

After introducing basic ideas and properties of supersymmetry we review the concept of scattering amplitudes in maximally supersymmetric theories in four dimensions before constructing a related framework in six dimensions. Here, the spinor helicity formalism and on-shell superspace were recently developed for six-dimensional gauge theories with (1,1) supersymmetry. We combine these two techniques with (generalised) unitarity, which is a powerful technique to calculate scattering amplitudes in any massless theory. As an application we calculate one-loop superamplitudes with four and five external particles in the (1,1) theory and perform several consistency checks on our results.

Within the area of phenomenological applications of supersymmetric gauge theories, we briefly review basic properties of supersymmetry breaking and gauge mediation in four dimensions. An important recent development has been the concept of theories with broken supersymmetry and metastable vacua. By using the advances of Seiberg duality, we examine a metastable $\mathcal{N} = 1$ Macroscopic $\mathsf{SO}(N)$ SQCD model of Intriligator, Seiberg and Shih (ISS). We introduce various baryon and meson deformations, including multitrace operators. In this setup, direct fundamental messengers and the symmetric pseudomodulus messenger mediate supersymmetry breaking to a minimal supersymmetric Standard Model. We compute gaugino and sfermion masses and compare them for each deformation type. We also explore reducing the rank of the magnetic quark matrix of the ISS model and find an additional fundamental messenger in the theory.

# Declaration

I hereby declare that the material presented in this thesis is a representation of my own personal work, unless otherwise stated, and is a result of collaborations with Andreas Brandhuber, Dimitrios Korres, Moritz McGarrie, Steven Thomas and Gabriele Travaglini.

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# Contents

# 1 | An Invitation

Over the centuries, our understanding of nature has developed constantly. From the days of Newton to current research experiments at the Large Hadron Collider (LHC), physicists have always tried to push the research frontier further. Historically, this has always happened as a fruitful interplay between theory and experiment. Nowadays, probably the most successful, experimentally validated theory is the Standard Model (SM) of high energy physics. However, we know that it cannot be the full story: The SM is only a low energy effective theory.

Nevertheless, one of its fundamental building blocks, the concept of symmetries, turned out to be extremely useful. Symmetries can be used to classify the particle content of a theory and play an important role in describing interactions between the different particles by the means of gauge theories. Furthermore, they constrain a physical system and quite often, this makes it possible to extract information about the underlying theory which governs the system's behaviour. Over the last forty years a rather peculiar symmetry has edged ever closer to the spotlight. True to the motto 'nomen est omen', this special type of symmetry is know as supersymmetry. It is the only symmetry under which a fundamental property of a particle changes: Supersymmetry transforms bosonic degrees of freedom into fermionic ones and vice versa. This concept has proven to be extremely rich of consequences. Supersymmetry is applied over a wide range of topics in high energy physics. It ranges from highly theoretical considerations in string and M-theory, over more formal studies in supersymmetric gauge theories to quite phenomenological ideas for physics beyond the SM. The vast range of areas that are influenced by supersymmetry indicates its importance in physics, but makes it also difficult to cover all of these ideas thoroughly.

However, in this thesis we will try to bridge between different arenas of applications and discuss both technical and phenomenological reasons for why supersymmetry plays a central role in modern theoretical physics. To make it somewhat tangible we will focus on maximally (rigid) supersymmetric theories in four (the $\mathcal{N} = 4$ super Yang-Mills theory) and six (the $\mathcal{N} = (1, 1)$ super Yang-Mills theory) dimensions as well as $\mathcal{N} = 1$ supersymmetric gauge theories where we study the effects of supersymmetry breaking. The first part of this thesis concentrates on more formal aspects of supersymmetric field theories. Here, we will focus on the applications of supersymmetry in perturbative quantum field theories. By using the framework of superamplitudes in four and six dimensions we will demonstrate recent advances in efficient calculations of scattering amplitudes. Also theories with non-extended supersymmetry are of high importance. They provide the leading candidates for physics beyond the SM. However, experimental bounds dictates that supersymmetry, if realised in nature, is only an approximate symmetry. In the second half of this thesis we will discuss ways to incorporate the SM into a supersymmetric theory.

In Chapter 2 we begin by briefly reminding ourselves of the concept of spacetime and internal symmetries. We then have a look at possible extension to the Poincaré algebra and find that supersymmetry is a natural extension of it. Furthermore, we will discuss some details of maximally supersymmetric theories in four and also six dimensions. After this digression we focus on non-extended supersymmetry in four dimensions. We discuss the off-shell superspace construction for $\mathcal{N} = 1$ supersymmetry and review supersymmetric chiral and gauge theories.

Chapter 3 is devoted to the concept of scattering amplitudes in four dimensions. After a short revision of amplitudes we introduce the four-dimensional spinor helicity formalism. In the following sections we will utilise this construction to discuss several techniques for efficient calculations of scattering amplitudes at tree-level. We then combine these ideas with $\mathcal{N} = 4$ maximal supersymmetry in four dimensions. The solely supermultiplet of this theory can be conveniently described by an on-shell super-wavefunction. A superamplitude is then a scattering amplitude of super-wavefunctions. We investigate this approach by discussing several supersymmetrised techniques of amplitude calculation. Finally, we move on to loop-level and introduce the unitarity method as a convenient approach for calculating loop amplitudes.

These ideas are taken to the next level in Chapter 4 where we discuss superamplitudes for the maximally supersymmetric gauge theory in six dimensions. We begin by introducing the recently developed six-dimensional spinor helicity formalism before constructing a super-wavefunction and on-shell superspace for this $\mathcal{N} = (1, 1)$ super Yang-Mills theory. After a brief review of three-, four- and five-point tree-level superamplitudes in this theory we will move on to the one-loop level. Here, we discuss in detail two- and four-particle cuts for four- and five-point amplitudes. We also perform several consistency checks of our results using dimensional reduction to four dimensions in order to compare with the corresponding amplitudes in $\mathcal{N} = 4$ SYM. This concludes the first part of the thesis.

In the second half of the thesis, beginning with Chapter 5, we focus on the applications of supersymmetry for physics beyond the SM where we will limit ourselves to $\mathcal{N} = 1$ supersymmetry in four dimensions. We begin the discussion with a short introduction into the relation between supersymmetry and the SM and explore the necessity of supersymmetry breaking. Then, we follow up on these ideas and discuss how supersymmetric and non-supersymmetric vacua are realised in quantum field theory. After a short digression on the relation between supersymmetry breaking and global symmetries in field theory we review some aspects of supersymmetry QCD for different flavours. This will lead us to the introduction of the ISS model of metastable supersymmetry breaking. We conclude the chapter by some general remarks on gauge-mediated models of supersymmetry breaking.

In the last two Chapters we will explore a specific examples for the ISS construction in the context of $\mathsf{SO}(N)$ symmetry groups. We begin in Chapter 6 with a discussion of the dual pictures in SQCD with $\mathsf{SO}(N)$ groups. After that we discuss the effect of the supersymmetry breaking sector on the masses of the SM gauginos and sfermions where we focus on the contributions to the gaugino masses. Before introducing a deformation to the ISS superpotential we consider R-symmetry breaking in $\mathsf{SO}(N)$ theories. After deforming the superpotential we will discuss the effect on the gaugino and sfermion masses of the SM. We conclude this chapter by considering the supersymmetric vacua introduced by the non-perturbative superpotential which make the SUSY-breaking states metastable and estimate the lifetime of the metastable states.

Finally, in Chapter 7, we consider deformations of the basic ISS model which had been investigated in the context of $\mathsf{SU}(N)$ theories before. We deform the superpotential by multitrace operators utilising the magnetic quarks and the meson field of the magnetic theory. This will offer the opportunity to highlight differences of these models compared to the vanilla ISS construction. We also consider the possibility of reducing the rank of the magnetic quark matrix which results in the appearance of an additional fundamental messenger field. In all these cases we explore the effect on the gaugino and sfermion masses of the SM. The thesis concludes with a series of appendices.

# 2 | Spacetime and Supersymmetries

What is a symmetry? The answer to this questions seems to be well known[1]. Almost everybody has some kind of understanding of the word 'symmetry'. However, its meaning can range from the subjects of arts to science. Different people might have a different notion of what a symmetry is. Therefore, it is important to provide a common framework. Within mathematics and physics, the concepts of groups (more specifically Lie groups) and algebras provide such a framework. Combined with the concept of quantum field theory, the construct which comprises quantum mechanics and special relativity, symmetries enable us to study the fundamental processes in physics. In the following we will briefly highlight some important implications of group theoretical studies of symmetries that ultimately lead to the concept of 'supersymmetry'.

## 2.1 A brief introduction

When we talk about a symmetry we are normally referring to some kind of invariance of an object under a certain transformation. Hence, a symmetry is always defined with respect to an operation that an object is undergoing. If we consider symmetries in physics we think of a symmetry as the group of transformations that leaves the Lagrangian $\mathcal{L}$ (actually the action $S$) of a particular theory invariant.

In general, symmetries are a basic, yet powerful concept in theoretical physics, making it somewhat hard to overestimate their importance for understanding the underlying physical theory. Using symmetry arguments one can even get information about a physical system without understanding and/or knowing the exact physical laws which govern it. All this is mainly due to two reasons. Firstly, we have the fact that Noether's theorem applies to our formulation of field theories. It relates the symmetries of a system to conservation laws. For each continuous symmetry we have an associate conserved quantity, a so called symmetry current. Secondly and even more important, symmetries have a strong interrelation with experiments since nature seems to respect many of them. The conserved quantities (the physical observables) coming from the symmetry currents (spatial volume integrals of components of the symmetry currents leading to conserved charges) are measurable. Some of the most important ones are the conservation of energy, spatial momentum and angular momentum. If these quantities are conserved in a physical system they are linked to the invariance under time translations, spatial translations and spatial rotations of the system.

[1] For a beautiful discussion see the text of the late Julius Wess [1]. Parts of our introduction are based on this nice review.

So far, we have mentioned only symmetries of spacetime. Over the last decades it turned out that another type, so called *internal symmetries*, are just as important as spacetime symmetries. In general, these correspond to transformations of the different fields in a field theory. Starting with the ideas of Heisenberg who generalised $\mathsf{SU}(2)$ spatial rotations to rotations in an internal space which led to the concept of isospin, this idea was later generalised to 'internal' $\mathsf{SU}(3)$ rotations. This was the first attempt to understand the structure of hadrons, leading to the so called 'eightfold way'. All these ideas turned out to be the right concept for phase transformations of particle wave functions. Ultimately, this approach yielded a generalisation of global symmetries towards local symmetries and the introduction of gauge theories. In this context, locality means that the group transformations acting in the internal space depend on spacetime parameters. Since gauge symmetries describe the interactions between matter particles they are of fundamental importance for modern particle physics.

A concrete example of this is the *Standard Model* (SM), a quantum field theory based on the non-abelian gauge symmetries $\mathsf{SU}(3)_C, \mathsf{SU}(2)_L$ and $\mathsf{U}(1)$. These correspond to the strong, weak and electromagnetic interactions between the matter particles. Compared with experimental measurements, the SM is a highly successful theory. This alone shows the importance of the symmetry concept. Despite the huge success in predicting experimental data, the SM cannot be a fundamental theory. First of all, the SM does not include gravitational interactions since a viable quantum theory of gravity is not known yet. Usually it is expected that a fundamental theory should describe all interactions of nature. Although the SM offers an excellent description of the strong and electro-weak interactions at low energies, however, at high energies problems arise. Quantum corrections of scalar masses are quadratically divergent within the SM. These are just a few of the issues which indicate that the SM cannot offer a full description of nature to arbitrarily high energies. Therefore, one needs to look for new ideas to go beyond the SM. There are numerous ways to extend the SM or incorporate it within a new, more fundamental theory. Most important for this thesis is the approach of generalising the symmetry concept of ordinary quantum field theories. A starting point is to add more internal symmetries, leading to the 'grand unified theory' (GUT) approach: The gauge symmetries of the SM could be only part of a bigger, unified symmetry group. Unfortunately, the SM itself is not a GUT since the gauge couplings do not unify at a high energy scale. This contrasts with the *Minimal Supersymmetric Standard Model* (MSSM) where unification happens at the order of $10^{16}$ GeV. Another approach is to generalise the concept of spacetime symmetries, also in combination with adding more spacetime dimensions. As we will see, this leads directly to the introduction of supersymmetry. Before discussing this idea, it is instructive to briefly review the symmetries of spacetime.

## 2.2 Lorentz and Poincaré symmetries

Since the days of Einstein we are familiar with the concept of *spacetime*. The unification of space and time in the context of his special theory of relativity, based on independent contributions of Hendrik Lorentz and Henri Poincaré, marked one of the important steps towards our current understanding of the modern quantum field theories. Special relativity incorporates two basic principles:

- The principle of relativity (based on Galileo's principle of relativity and formulated by Poincaré) introduces the concept of inertial frames states that the laws of physics are independent from the choice of an inertial frame.
- The second principle states the fact that the speed of light $c$ is a constant in nature and hence the same in all inertial frames.

Einstein's theory makes use of the Lorentz transformations which became apparent in the formulation of Maxwell's theory of electrodynamics. It was shown by Henri Poincaré that Lorentz transformations form a subset of a larger symmetry group, named the Poincaré group. This group incorporates the Lorentz transformations and translations of spacetime and describe the basic symmetries in special relativity. The isometries of spacetime act as

$$x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu}\, x^\nu + a^\mu \ . \tag{2.1}$$

The first term corresponds to the Lorentz transformations whereas the second one gives a translation by a constant spacetime vector $a^\mu$. The group of these transformations is also often denoted as inhomogeneous Lorentz group, the semi-direct product of the $D$-dimensional Lorentz group $\mathsf{O}(D-1,1)$ and the translations in $D$ dimensions. Restricting to those Lorentz transformations with $\det(\Lambda) = +1$ and $\Lambda_{00} \geq 1$ yields the subgroup $\mathsf{ISO}(D-1,1)$ of the inhomogeneous Lorentz group which is usually denoted as the *Poincaré symmetry group*. The elements Poincaré group are generated by the momentum and rotation generators $P^\mu$ and $M^{\mu\nu}$. The generators fulfill the commutation relations

$$\begin{aligned}
[P^\mu, P^\nu] &= 0\,, \\
[M^{\mu\nu}, P^\rho] &= i\left(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu\right)\,, \\
[M^{\mu\nu}, M^{\rho\sigma}] &= i\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho}\right)\ .
\end{aligned} \tag{2.2}$$

The generators are all bosonic. Depending on the type of field under consideration, one has to look for the correct representation of the generators which generate the corresponding elements of the algebra. For further information we refer the reader to the literature where a lot of excellent reviews can be found, see for instance [2–4]. We also provide some discussions in Appendix A.3.

At this point, an important question arises: Is it possible to extend the symmetry group of quantum field theory? Since nature respects the Poincaré symmetries, we should be looking for extensions of the Poincaré algebra. We already know that this is possible since we describe the fundamental interactions of nature by gauge theories. Hence, we could easily combine the Poincaré and gauge algebras by adding the generators $T^a$ with $a = 1, \ldots, N^2-1$ for a $\mathsf{SU(N)}$ gauge group, obeying

$$[T^a, T^b] = i f^{abc} T^c\ , \qquad [T^a, P^\mu] = 0\ , \qquad [T^a, M^{\mu\nu}] = 0\ . \tag{2.3}$$

We immediately see that this extended symmetry group is a direct product of the Poincaré and gauge groups since the generators $T^a, P^\mu$ and $M^{\mu\nu}$ commute. What we are really after is a non-trivial extension of the symmetry group of nature. This requirement[2] leads us directly to the notion of supersymmetry (SUSY) and the introduction of the super-Poincaré algebra.

## 2.3 The super-Poincaré algebra

As we have seen, trivial extensions of the Poincaré algebra can be easily constructed. In order to introduce non-trivial extensions one has to look for new symmetry generators that mix with the ones generating the Poincaré symmetries. An important step in this direction was taken by Coleman and Mandula in 1967 [5]. They studied restrictions for possible extensions of the Poincaré group which led to the famous *Coleman-Mandula No-Go Theorem*. It states that the only possible symmetries compatible with an interacting quantum field theory are direct products of the Poincaré symmetries with an internal symmetry group $G$ (global or local). An important assumption in proving this theorem is that only bosonic generators of symmetries are allowed. By bosonic generators we mean scalar, vector or tensor generator that do not change the spin of a state. Hence, the generators do not transform as spinors under the Lorentz group. The main step towards a realisation of an extension of the Poincaré algebra is the weakening of this assumption, namely to allow commuting as well as anti-commuting generators for the symmetry algebra which bypasses the Coleman-Mandula theorem. This leads to the idea of considering generators with half-integer spin. Historically, this is how supersymmetry was discovered.

Hence, one introduces supersymmetry generators $Q$ that change the spin of a state by units of $\frac{1}{2}$. Being of fermionic nature, they transform as spinors under the Lorentz group. Therefore the new symmetry is not an internal one, rather it is a non-trivial extension of the Poincaré symmetries. In 1971, a first step was taken into this direction in the former Soviet Union when the Poincaré algebra was extended to include spinor generators [6]. In the same year, supersymmetry appeared in string theory in the context of a two-dimensional field theory [7–9]. It took another 3 years until Wess and Zumino published their famous work on supersymmetry in four-dimensional quantum field theory [10, 11] which made supersymmetry a widely known subject. Finally, in 1975 it was shown that supersymmetry is actually the only possible extension of the Poincaré algebra if one requires a non-trivial S-matrix in an interacting quantum field theory [12]. This statement is known as the *Haag-Lapusyanksi-Sohnius theorem*.

Therefore, we can non-trivially extend the Poincaré algebra only by including the fermionic supersymmetry generators (often also called supercharges). The new algebra comprise the supercharges which transform as spinors under the Lorentz group and the rotation and translation generators. The new algebra can also contain other additional generators which commute with the supercharges. These generators are usually denoted as 'central charges' of the algebra, although this is not very precise: Often, the central charges do not commute with the Lorentz or Poincaré generators. In any case, in this thesis we will not consider any extensions of the algebra, we set the central charges to zero. For further information we refer the reader to the literature.

[2] Here, we assume that the requirements of the Coleman-Mandula theorem discussed further below need to be fulfilled for a non-trivial extension.

For a classification of the supersymmetry algebras in more than two dimensions see the work of Nahm [13].

The defining anti-commutation relation between the supercharges holds for arbitrary spacetime dimensions. Hence, we consider algebras of the form (a useful review can be found in [14])

$$\{Q^I_a, \bar{Q}^b_J\} = 2P^\mu \, (\Gamma_\mu)_a^{\ b} \, \delta^I_J \ , \tag{2.4}$$

where $a, b$ are spinor indices, $\Gamma^\mu$ are the Dirac matrices in $D$ dimensions with $\mu = 0, \ldots, D-1$ whereas $I, J$ label different sets of supersymmetry generators in the case of extended supersymmetry. In addition, we have introduced the Dirac conjugate supercharge $\bar{Q} = Q^\dagger \Gamma^0$. The relation (2.4) together with the algebra of the Poincaré transformations gives the *super-Poincaré algebra* (plus anti-commutation relations among the supercharges $Q^I_a$). Although we will focus on supersymmetry in four and six dimensions, we find it instructive to briefly discuss some basic properties of spinors in arbitrary dimensions.

The dimensionality of a Dirac spinor as a solution to the Dirac equation in $D$ spacetime dimensions is given by the dimension of the Dirac matrices. These objects obey a Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I} \ , \tag{2.5}$$

where $\mathbb{I}$ is an unit matrix. The dimensionality of the Dirac matrices is also defined by these anti-commutation relations. Following general theorems of representation theory the matrix dimension of $\Gamma^\mu$ is given by $(D_\Gamma \times D_\Gamma)$ where

$$D_\Gamma = \begin{cases} 2^{\frac{D}{2}} & D \text{ even} \\ 2^{\frac{D-1}{2}} & D \text{ odd} \ . \end{cases} \tag{2.6}$$

This states the fact that Dirac spinors in $D$ dimensions have $2^{D/2}$ complex components. In the familiar example of four dimensions, Dirac spinors belong to reducible representations of the Lorentz group. This is most easily seen by choosing a chiral basis for the Dirac matrices. The upshot of this is that that additional constraints can be applied to a Dirac spinor which reduce its degrees of freedom. For arbitrary spacetime dimensions one might wonder what dimensionality an irreducible spinor has got. The answer can be conveniently presented as follows which is taken from the nice discussion in [15].

In general, one has five different sequences of spinors of different dimensionalty for a metric with Minkowski signature. The results are summarised in Table 2.1. We start with odd spacetime dimensions. For some of them it is possible to apply the Majorana reality condition (MRC) on the spinors, leading to the first sequence of Table 2.1. The second line corresponds to the case of an odd dimension where a Majorana condition cannot be applied. Coming to even dimensions one finds that charge conjugation relates (irreducible) spinors of positive and negative chirality. This is the case in $4k, k \in \mathbb{N}$ dimensions and is comprised in the third line. Now, we are left with even dimensions and chirality-conserving charge conjugations. For a six-dimensional spacetime the MRC cannot be applied and we have complex Weyl spinors as the only irreducible representation[3].

[3]We will make use of this fact in Chapter 4 when discussing the spinor helicity formalism in six dimensions.

| Dimension $D$ | $\Psi_{\text{irr}}$ | spinor type | Automorphism group |
|---|---|---|---|
| 1, 3, 9, 11 | $2^{(D-1)/2}$ | Majorana | $\mathsf{SO(N)}$ |
| 5, 7 | $2^{(D+1)/2}$ | Dirac | $\mathsf{USp(2N)}$ |
| 4, 8 | $2^{D/2}$ | Majorana | $\mathsf{U(N)}$ |
| 6 | $2^{D/2}$ | Weyl | $\mathsf{USp(2N_+)} \times \mathsf{USp(2N_-)}$ |
| 2, 10 | $2^{D/2-1}$ | Majorana-Weyl | $\mathsf{SO(N_+)} \times \mathsf{SO(N_-)}$ |

Table 2.1: *Overview of various spacetime dimensions, the corresponding number of real dimensions of an irreducible spinor, the spinor type and the automorphism group of the supersymmetry algebra.*

If an application of the MRC is possible we have Majorana-Weyl spinors. This is case in two and ten spacetime dimensions. Furthermore, for any dimension, there exists a group under which the supercharges transform into each other. This is the automorphism group of the super-Poincaré algebra. The last column of Table 2.1 states the corresponding symmetry group for the spinor representations for the various dimensions.

Some additional comments about the super-Poincaré algebra are in order here. We know that an irreducible representation of the Poincaré algebra corresponds to a particle state. In contrast, an irreducible representation of the super-Poincaré algebra corresponds to several particle states which differ in their spin by units of $\frac{1}{2}$. We understand this since the (on-shell) states in an irreducible representation are related by the fermionic generators $Q_a^I$ or $\bar{Q}_{aJ}$ which change the spin of the states. A collection of (in general off-shell) fields that transform irreducibly under the super-Poincaré transformations is conventionally called a *supermultiplet*. In our discussions we also denote an irreducible representation of the supersymmetry algebra as a supermultiplet. Due to the structure of the super-Poincaré algebra we immediately have two important properties of a supermultiplet:

- All particle states within one supermultiplet have the same mass since the operator $P^2$ commutes with all other generators of the super-Poincaré algebra.
- A supermultiplet contains an equal number of bosonic and fermionic (physical) degrees of freedom.

From now on we will specialise our discussion to four or six spacetime dimensions. In the next two chapters we will be mainly interested in the maximal amount of supersymmetry in these spacetime dimensions. For the second half of this thesis, starting with Chapter 5, we will limit ourselves to four spacetime dimensions and non-extended supersymmetry which offers many phenomenologically viable inputs.

### 2.3.1 Maximal supersymmetry in four dimensions

In four spacetime dimensions, the Lorentz group is $\mathsf{SO}(3,1)$. An irreducible spinor has four real degrees of freedom. One can choose between a Majorana spinor (four real components) or a complex Weyl spinor (two complex components) to represent the irreducible supercharges. Here and throughout the thesis, we use the two-component notation, i.e. we introduce fermionic generators $q^I_\alpha$ and $\bar{q}_{\dot{\beta}J}$ which transform in the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the Lorentz group respectively. Here, $\alpha,\dot{\beta}=1,2$ are spinor indices and $I,J=1,\ldots,\mathcal{N}$ with $\mathcal{N}\geq 1$ label the different set of supercharges. The anti-commutation relation between the Weyl supercharges is then given by

$$\{q^I_\alpha,\bar{q}_{\dot{\beta}J}\}=2p_\mu\sigma^\mu_{\alpha\dot{\beta}}\delta^I_J\ ,\qquad \{\bar{q}^{\dot{\alpha}}_J,q^{\beta I}\}=2p_\mu\bar{\sigma}^{\mu,\dot{\alpha}\beta}\delta^I_J\ . \tag{2.7}$$

The super-Poincaré algebra is then given by the above anti-commutators combined with the commutators of the usual Poincaré generators $P^\mu$ and $M^{\mu\nu}$. The product of a left-handed and a right-handed spinor is proportional to a vector and hence, the $\sigma^\mu_{\alpha\dot{\beta}}$ and $\bar{\sigma}^{\mu,\dot{\alpha}\beta}$ are the Clebsch-Gordon coefficients that provide a dictionary between the spinor and vector representations. This will be useful in Chapter 3 when we express four-dimensional momenta as bispinors in the (four-dimensional) spinor helicity formalism.

At this point one might asked if there is any limit on the number $\mathcal{N}$ of possible sets of supersymmetry generators. The answer to this question is closely connected to the possible representations of the super-Poincaré algebra. From an algebraic point of view there is no limit on $\mathcal{N}$. However, with increasing $\mathcal{N}$ the corresponding supersymmetric quantum field theory contains particles of increasing spin. As a physical requirement we impose the condition that no particle should have a spin higher than $s=1$ (we do not consider gravity states where the graviton has spin $s=2$). This leads to the bound of $\mathcal{N}\leq 4$ for non-gravitational theories.

The representation of the super-Poincaré algebra are obtained by acting with the supercharges on a vacuum state. Since $q^I_\alpha$ and $\bar{q}_{\dot{\beta}J}$ are fermionic generators, they change the spin of the states they act an. This can be easily seen from the super-Poincaré algebra. In particular when we identify $M_{12}=J_3$ we have $[M_{12},q^I_\alpha]=(\sigma_{12})_\alpha^{\ \beta}q^I_\beta$ and $[M_{12},\bar{q}^{\dot{\alpha}}_J]=(\bar{\sigma}_{12})^{\dot{\alpha}}_{\ \dot{\beta}}\bar{q}^{\dot{\beta}}_J$. From this we deduce that

$$\begin{aligned}[][J_3,q^I_1]&=\frac{1}{2}q^I_1\ , &\qquad [J_3,q^I_2]&=-\frac{1}{2}q^I_2\ ,\\ [J_3,\bar{q}_{\dot{1}I}]&=-\frac{1}{2}\bar{q}_{\dot{1}I}\ , &\qquad [J_3,\bar{q}_{\dot{2}I}]&=\frac{1}{2}\bar{q}_{\dot{2}I}\ .\end{aligned} \tag{2.8}$$

Hence, the action of the supercharges is such that $q^I_1$ and $\tilde{q}_{2I}$ increase the helicity of a state by $\frac{1}{2}$ whereas $q^I_2$ and $\tilde{q}_{1I}$ decreases it by $\frac{1}{2}$. Furthermore, since the momentum operator $P^\mu$ commutes with the supercharges, we can consider the states at arbitrary but fixed momentum. This is helpful when we consider different representations. For a massive particle, one can boost to the rest frame. Here, we purely focus on massless states. Further information can be found in many reviews of supersymmetry, see for instance [2–4] and also the books by Wess and Bagger [16] or by Terning [17].

In the massless case one cannot go to a restframe. However, the condition $P^2=0$ is fulfilled for instance in the frame $P_\mu=(E,0,0,E)$. From the algebra we find that half of the supercharges (namely $q^I_2$ and $\bar{q}_{2J}$) can be set to zero. The other half can be interpreted as creation and annihilation operators which raise or lower the spin of the state they act on. Actually, in the massless case, the states are labeled by their helicity which is the eigenvalue of the Lorentz generator $M_{12}$. Therefore, by repeatedly acting with $\bar{q}_{1I}$ on a vacuum of helicity $\lambda$ we can decrease its helicity by multiples of $\frac{1}{2}$.


Figure 2.1: *A pictorial representation of the states in the massless $\mathcal{N} = 4$ vector supermultiplet. The creation operators are understood to act from the left to the right in this diagram.*

A comment about invariance under CPT (the combined transformation of the discrete symmetries of charge conjugation C, partiy conjugation P and time reversal T) is in order here. Since CPT flips the sign of the helicity label, a supermultiplet is only CPT invariant if the helicities are symmetrically distributed about $\lambda = 0$. This is only the case if the *highest helicity* is $\lambda_{\max} = \frac{\mathcal{N}}{4}$. For all other vacua one has to add the CPT conjugate multiplet to obtain a CPT invariant supermultiplet.

Let us discuss this on two examples. We begin with unextended supersymmetry. For $\mathcal{N} = 1$ we have two possible states, $|\lambda_{\max}\rangle$ and $\left|\lambda_{\max} - \frac{1}{2}\right\rangle$. Adding their CPT conjugates we arrive at the following $\mathcal{N} = 1$ massless supermultiplets where we limit ourselves to non-gravitational theories:

**Chiral supermulitplet** - It consists of the helicity configuration $(0, \frac{1}{2})$ and its CPT conjugate $(-\frac{1}{2}, 0)$. The degrees of freedom belong to a Weyl fermion and a complex scalar.

**Vector supermulitplet** - It consists of the configuration $(\frac{1}{2}, 1)$ and $(-\frac{1}{2}, -1)$. We have a massless gauge boson and a real fermion field, represented by a complex Weyl spinor. The gauge boson is in the adjoint representation of the gauge group and so is the corresponding fermion.

If we do not consider theories with particles of helicities $|\lambda| > 1$ then we have constructed all possible massless supermultiplets. Allowing $|\lambda| \leq 2$ we also have the gravitino supermultiplet (containing a gravitino and a gauge boson) and the graviton supermulitplet (containing the graviton and a gravitino) for $\mathcal{N} = 1$ unextended supersymmetric theories.

Finally, we turn to the maximally supersymmetric theories in four dimensions. Again we do not consider gravitational theories. Hence, we have $\mathcal{N} = 4$ for maximal supersymmetry. We construct the massless supermultiplet as before, starting with the highest possible helicity state. By applying the four creation operators $a_I^\dagger = 1/\sqrt{2}\,\bar{q}_{1I}$ for $I = 1, \ldots, 4$ we get:

| State | Helicity | Multiplicity |
|---|---|---|
| $\lvert\Omega, \lambda_{\max}\rangle$ | $\lambda_{\max}$ | 1 |
| $a_I^\dagger\lvert\Omega, \lambda_{\max}\rangle$ | $\lambda_{\max} - \frac{1}{2}$ | 4 |
| $a_I^\dagger a_J^\dagger\lvert\Omega, \lambda_{\max}\rangle$ | $\lambda_{\max} - 1$ | 6 |
| $\epsilon^{IJKL} a_I^\dagger a_J^\dagger a_K^\dagger\lvert\Omega, \lambda_{\max}\rangle$ | $\lambda_{\max} - \frac{3}{2}$ | 4 |
| $\epsilon^{IJKL} a_I^\dagger a_J^\dagger a_K^\dagger a_L^\dagger\lvert\Omega, \lambda_{\max}\rangle$ | $\lambda_{\max} - 2$ | 1 |

Here, the multiplicity of each state is given by the antisymmetry of the labels $I, J, \ldots$. For non-gravitational theories we have $\lambda_{\max} = 1$ and we see that the supermultiplet is automatically CPT invariant, having states helicities in the range of $+1, \ldots, -1$. Conventionally, the supermultiplet is called the massless $\mathcal{N} = 4$ vector supermultiplet. It contains states of massless gauge bosons/gluons (helicities $\pm 1$), gauginos/gluinos (helicities $\pm\frac{1}{2}$) and scalars (helicity 0).

Here we have constructed the supermultiplets as states. The corresponding realisations in terms of quantum fields can be deduced from these representations. However, this will result in 'on-shell' fields. We will construct such a 'superfied' explicitly in Chapter 3 for the case of $\mathcal{N} = 4$ massless representations. A construction of off-shell superfields, i.e. fields that do not obey their equations of motions, is possible for non-extended supersymmetry. We discuss the off-shell superspace approach for $\mathcal{N} = 1$ supersymmetry in Section 2.4.

### 2.3.2 Maximal supersymmetry in six dimensions

The Lorentz group in six dimensions is $\mathsf{SO}(5,1)$ which is isomorphic to the non-compact group $\mathsf{SU}^*(4)$. Spinors transforming under the six-dimensional Lorentz group carry therefore a fundamental or anti-fundamental Lorentz index $A = 1, \ldots, 4$ which corresponds to the chiral or anti-chiral spinor representation, respectively. Here, the fundamental and anti-fundamental representations of $\mathsf{SU}(4)$ are inequivalent, there is no tensor which could raise or lower the Lorentz indices.

As we have mentioned before, a general spinor in six dimensions is of Weyl-type[4]. Since the six-dimensional $\gamma^5$-equivalent is the identity, a charge conjugated spinor has the same chirality. We have independent rotations for the chiral and anti-chiral supercharges, namely $\mathsf{USp}(2N_+) \times \mathsf{USp}(2N_-)$. Just as for the well known Weyl spinors in four dimensions, it is useful to understand what Lorentz invariant *reality conditions* we can impose on the six-dimensional spinors. The usual Majorana reality condition in four dimensions relates a spinor to its charge conjugate, i.e. $\psi = \psi^c = B\psi^*$ which is only possible for $BB^* = \mathbb{I}$. Here, $B$ plays the role of a charge conjugation matrix which has to be an unitary matrix, $BB^\dagger = \mathbb{I}$. However, in six dimensions we have $BB^* = -\mathbb{I}$ and hence, Majorana spinors do not exist in six-dimensional Minkowski spacetime. For details on this fact see for instance the review [18]. Then, it is a question of how to introduce a reality condition on the spinors. In six dimensions, we can impose the so called *symplectic Majorana condition*. In order to do so, one introduces an additional $\mathsf{SU}(2)$ label $a = 1, 2$ on the six-dimensional spinors, such that the reality condition reads

$$\left(\psi_a^A\right)^c = \epsilon_{ab} \tilde{B}^A_B \left(\psi^{Bb}\right)^* \ , \tag{2.9}$$

where $A, B$ are $\mathsf{SU}(4)$ indices and $a, b$ are the newly introduced $\mathsf{SU}(2)$ indices. This relation yields

$$\psi_a^A = \left(\left(\psi_a^A\right)^c\right)^c = \epsilon_{ab} \tilde{B}^A_B \left(\left(\psi^{Bb}\right)^c\right)^* = \epsilon_{ab} \epsilon^{bd} \tilde{B}^A_B \tilde{B}^{*B}_C \psi^C_d = \delta^d_a \delta^A_C \psi^C_d = \psi^A_a \ , \tag{2.10}$$

with $\tilde{B}^A_B \tilde{B}^{*B}_C = -\delta^A_C$ such that the complex conjugate of the spinor is equal to itself. By introducing the additional $\mathsf{SU}(2)$ indices we have doubled the components of the six-dimensional spinors. However, by imposing the reality condition we have halved the number of components again. In the end, we are left with four complex degrees of freedom of the symplectic spinor. One might wonder of how to interpret the additional $\mathsf{SU}(2)$ indices which are needed to impose a reality condition in six dimensions. It turns out that we can identify the additional $\mathsf{SU}(2)$ index on the spinors with

[4]Here, we follow the discussion and notation of [15].

the six-dimensional little group, i.e. the part of the Lorentz group which leaves the momentum invariant. Whereas in four dimensions the little group is $\mathsf{SO}(2) \cong \mathsf{U}(1)$ and labels the helicity of a particle state, in the six-dimensional context it is $\mathsf{SO}(4) \cong \mathsf{SU}(2) \times \mathsf{SU}(2)$. The two copies of $\mathsf{SU}(2)$ represent the fact that so far, we have considered spinors with fundamental $\mathsf{SU}(4)$ indices only. However, we also have to include those spinors with anti-fundamental $\mathsf{SU}(4)$ indices. These spinors come with their own little group index, usually denoted by a dotted index $\dot{a} = 1, 2$. This leads to spinors $\tilde{\psi}_A^{\dot{a}}$ for which a reality condition similar to (2.9) can be imposed. The upshot of this construction is the fact that the invariance under the $\mathsf{SU}(2)$ little group transformations reduces the components of a six-dimensional spinor even further. Due to the reality condition, we have 4 complex or 8 real components and the $\mathsf{SU}(2)$ invariance reduces these components by a factor of 3, yielding 5 real components. Since we have both a chiral and a reality condition, six-dimensional real spinors are usually denoted as *symplectic Majorana-Weyl spinors*. This construction makes real momenta and real representations of the supersymmetry algebra in six dimensions possible. For further discussions on this topic we refer the reader to the literature, see for instance the recent work [19] and also the original works [20] and [?]. Especially the latter one provides also a nice overview on spinors in various dimensions.

The minimal amount of supersymmetry in six dimensions is given by $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (0, 1)$. If we limit ourselves again to the massless representations we have in the $\mathcal{N} = (1, 0)$ case the *hypermultiplet* (one complex fermion and two complex scalars), the *tensor multiplet* (one 2nd rank tensor, one complex fermion and one real scalar) and the *vector multiplet* (one massless vector field and one complex fermion) when focusing only on non-gravitational theories. All multiplets have in total eight real degrees of freedom. Increasing the number of chiral or anti-chiral supercharges by one we get the $\mathcal{N} = (2, 0)$ or $\mathcal{N} = (0, 2)$ theories with maximal supersymmetry. In the case of $\mathcal{N} = (2, 0)$ we encounter the *tensor multiplet* (one 2nd rank tensor, five real scalars and four complex fermions) with in total 16 degrees of freedom (again we limit ourselves to non-gravitational theories).

However, what we are really after is the non-chiral maximally supersymmetric theory with 16 supercharges, namely the $\mathcal{N} = (1, 1)$ theory. Although the $\mathcal{N} = (2, 0)$ is also maximal, it does not contain a vector gauge field which we need for a description of six-dimensional gauge interactions. The $\mathcal{N} = (1, 0)$ theory contains a massless vector field, however, it is not maximal and therefore we cannot combine all on-shell states into a single (on-shell) superfield[5]. Therefore, from now on, we solely consider the $\mathcal{N} = (1, 1)$ super Yang-Mills theory when we discuss supersymmetric theories in six dimensions. As usual we consider particles with spins up to $s = 1$ and hence, this supersymmetric Yang-Mills theory contains a single supermultiplet only, namely the *vector multiplet*. It consists of a six-dimensional vector field and four real scalars (yielding eight bosonic degrees of freedom) and also two fermion fields, one chiral and one anti-chiral one (yielding eight fermionic degrees of freedom).

The $\mathcal{N} = (1, 1)$ theory contains 16 supercharges, eight chiral and eight anti-chiral ones. They carry the usual Lorentz indices of the $\mathsf{SU}(4)$ and transform also under the $\mathsf{USp}(2) \times \mathsf{USp}(2) \cong$

[5] See the discussions in Sections 3.3.1 and 4.2 for further details.

$\mathsf{SU}(2) \times \mathsf{SU}(2)$ R-symmetry group. Hence, we introduce spinorial generators $q^{AI}$ as chiral and $\tilde{q}_{AI'}$ as anti-chiral supercharges where $A = 1, \ldots, 4$ and $I$ and $I'$ are $\mathsf{SU}(2)$ R-symmetry indices. They obey the anti-commutation relations

$$\begin{aligned} \{q^{AI}, q^{BJ}\} &= p^{AB}\epsilon^{IJ} \ , \\ \{\tilde{q}_{AI'}, \tilde{q}_{BJ'}\} &= p_{AB}\epsilon_{I'J'} \ , \end{aligned} \tag{2.11}$$

where $p^{AB}$ and $p_{AB}$ is a six-dimensional vector which is in the anti-symmetric representation of $\mathsf{SU}(4)$. The total anti-symmetric tensor $\epsilon^{ABCD}$ can be used to raise the indices of a vector $p_{AB}$. Similarly, one can lower indices with $\epsilon_{ABCD}$. The anti-commutators (2.11) together with the six-dimensional Poincaré algebra and additional vanishing anti-commutators between supercharges of opposite chirality comprise the super-Poincaré algebra of the $\mathcal{N} = (1,1)$ super Yang-Mills theory in six dimensions. In addition to our brief overview in this section we will discuss six-dimensional spinors, on-shell supercharges and an on-shell $\mathcal{N} = (1,1)$ superspace construction in Sections 4.1 and 4.2.

## 2.4 The $\mathcal{N} = 1$ Off-shell Superspace

In the second half of this thesis we focus on non-extended $\mathcal{N} = 1$ supersymmetry in four dimensions and on the important question of how supersymmetry can be broken. We find it therefore instructive to introduce a convenient method to discuss the supermultiplet structure of the $\mathcal{N} = 1$ algebra and the corresponding field content, namely the $\mathcal{N} = 1$ off-shell superspace. This section also sets the notations and conventions for our discussions in the Chapters 5, 6 and 7.

So far, we have constructed irreducible representations of the super-Poincaré algebra and found that they are realised as supermultiplets. This is an on-shell construction since it is based on on-shell asymptotic states which the generators act on. For quantum fields, irreducible representations can be constructed under the requirement that the fields are complex and on-shell, i.e. obeying their equation of motion. However, there is a problem if one tries to construct a field content corresponding to a supermultiplet in an off-shell fashion, at least in the case of extended supersymmetry. With the exception of theories with $\mathcal{N} = 2$ supersymmetry, a closure of the supersymmetry algebra where fields are not obeying their equations of motions requires an infinite set of auxiliary fields. For more information about extended superspace we refer the interested reader to the rather detailed review in [21]. Here, we will focus on the relevant case for our discussions, namely the off-shell superspace for unextended $\mathcal{N} = 1$ supersymmetry and its field content.

To construct such an off-shell superspace we remind ourselves that any quantum field is parameterised by spacetime coordinates $x^\mu$ which themselves parameterise the coset $K = P/L$. Here, $P$ is the Poincaré group and $L$ is the Lorentz group. In general, for a coset $K = G/H$ we can identify the elements $g$ of $G$ through an exponential map

$$g = e^{i\epsilon^a L_a} e^{i\xi^b H_b} \ , \tag{2.12}$$

where $\epsilon_a$ with $a = 1, \ldots, (\dim G - \dim H)$ are a set of coordinates parameterising the coset. The generators of $G$ separate into the disjunct sets of generators $H_b$ of $H$ and the remaining generators $L_a$. The elements of the coset $K$ can be then obtained by letting $\xi_b = 0$.

In order to use this prescription to construct the off-shell $\mathcal{N} = 1$ superspace we write the supersymmetry algebra as a Lie algebra, i.e. we rewrite the anti-commutators as commutators

$$[\theta^\alpha q_\alpha, \bar{\theta}^{\dot{\alpha}} \bar{q}_{\dot{\alpha}}] = 2p_\mu \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \ . \tag{2.13}$$

Here, we introduced two-component fermionic Grassmann variables $\theta^\alpha$ and $\bar{\theta}^{\dot{\alpha}}$ to obtain the commutator between the supercharges. An element of the corresponding group is then written as

$$g(x, \theta, \bar{\theta}, w) = e^{i(-a^\mu p_\mu + \theta q + \bar{\theta}\bar{q})} e^{i w^{\mu\nu} M_{\mu\nu}} \ , \tag{2.14}$$

where the sign of the parameter $a$ is a convention. Now, the $\mathcal{N} = 1$ superspace is defined as the coset

$$K(\mathcal{N} = 1) = \text{super-Poincaré / Lorentz} = \{w^{\mu\nu}, a^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}\}/\{w^{\mu\nu}\} \ , \tag{2.15}$$

and an element in this superspace is given by elements $k$ of the super-Poincaré group with $w^{\mu\nu} = 0$,

$$k = e^{i(-x^\mu p_\mu + \theta q + \bar{\theta}\bar{q})} \equiv e^{z^a K_a} \ . \tag{2.16}$$

Here, the elements are parameterised by the coordinates $z_a = (x_\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$ with generators $K_a = (P_\mu, q_\alpha, \bar{q}^{\dot{\alpha}})$. The elements are unitariy since $(\theta q)^\dagger = \bar{\theta}\bar{q}$.

Having identified the coordinates of the $\mathcal{N} = 1$ superspace we can follow a construction analog to the well known case of scalar fields in Minkowski space in order to introduce scalar *superfields*. The usual scalar fields $\phi$ are functions of the spacetime coordinates $x^\mu$ and transform under the Poincaré symmetries, for instance for translations we have

$$\phi(x) \longrightarrow e^{ia_\mu P^\mu} \phi(x)\, e^{-ia_\mu P^\mu} = \phi(x + a) \ . \tag{2.17}$$

Here, the field is an operator in a Fockspace $\mathcal{F}$ such that it can be written in terms of creation and annihilation operators. However, the field $\phi(x)$ can also be expressed as a vector in a Hilbert space where the functions are act on by differential operators. In our example of translations we have

$$\phi(x) \longrightarrow e^{-ia^\mu \hat{P}_\mu} \phi(x) = \phi(x + a) \ , \tag{2.18}$$

where $\hat{P}_\mu$ is a representation of the abstract operator $P_\mu$ as an differential operator, $\hat{P}_\mu = i\partial_\mu$.

A scalar superfield $\Omega(x, \theta, \bar{\theta})$ can then be introduced by requiring similar transformational properties. Introducing constant Grassmann spinors $\xi_\alpha$ and $\bar{\xi}_{\dot{\alpha}}$, one defines a supertranslation operator

$$S(a, \xi, \bar{\xi}) = e^{i(-a^\mu P_\mu + \xi^\alpha q_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{q}^{\dot{\alpha}})} \ , \tag{2.19}$$

where $P_\mu, q_\alpha$ and $\bar{q}_{\dot{\alpha}}$ are abstract operators in Fock space. We notice that by letting $\xi_\alpha = \bar{\xi}_{\dot{\alpha}} = 0$ and acting on a scalar field we immediately obtain the result of (2.17). We can combine two supertranslations by usage of the Baker-Campbell-Hausdorff formula for matrix exponentials, which yields the result

$$S(a, \xi, \bar{\xi}) S(x, \theta, \bar{\theta}) = S(x^\mu + a^\mu - i\xi\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}) \ . \tag{2.20}$$

Here, we used the non-zero commutators $[\xi q, \bar\theta\bar q] = 2\xi\sigma^\mu\bar\theta P_\mu$ and $[\bar\xi\bar q, \theta q] = -2\theta\sigma^\mu\bar\xi P_\mu$. Induced by this transformation we have a translation of the coordinates $z^a$, a generalisation of the corresponding relation (2.1) in Minkowski space. The transformation of a scalar superfield as a field operator in Fock space is then

$$\begin{aligned}\Omega(x,\theta,\bar\theta) &\longrightarrow e^{-i(-x^\mu P_\mu+\xi^\alpha q_\alpha+\bar\xi_{\dot\alpha}\bar q^{\dot\alpha})}\Omega(x,\theta,\bar\theta)e^{i(-x^\mu P_\mu+\xi^\alpha q_\alpha+\bar\xi_{\dot\alpha}\bar q^{\dot\alpha})} \\ &= \Omega(x^\mu+a^\mu-i\xi\sigma^\mu\bar\theta+i\theta\sigma^\mu\bar\xi,\theta+\xi,\bar\theta+\bar\xi)\ .\end{aligned} \tag{2.21}$$

Just as before, the abstract operators $P^\mu, q_\alpha, \bar q_{\dot\alpha}$ have a representation as superspace differential operators when acting on Hilbert vectors. Hence, we can consider the left multiplication of a generator $S(a,\xi,\bar\xi)$ on a general coset element in the Hilbert space of states, yielding

$$\begin{aligned}S(a,\xi,\bar\xi)\Omega(x,\theta,\bar\theta) &= e^{i(-a^\mu\hat P_\mu+\xi^\alpha\hat q_\alpha+\bar\xi_{\dot\alpha}\hat{\bar q}^{\dot\alpha})}e^{i(-x^\mu\hat P_\mu+\theta\hat q_\alpha+\bar\theta\hat{\bar q})} \\ &= \Omega(x^\mu+a^\mu-i\xi\sigma^\mu\bar\theta+i\theta\sigma^\mu\bar\xi,\theta+\xi,\bar\theta+\bar\xi)\ .\end{aligned} \tag{2.22}$$

For infinitesimal parameters $a,\xi$ and $\bar\xi$ we can expand both sides of this relation and obtain, by comparing the coefficients of the parameters (from now on we neglect the ˆ on the operators),

$$\begin{aligned}P_\mu &= i\partial_\mu\ , \\ q_\alpha &= \partial_\alpha - i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu\ , \\ \bar q_{\dot\alpha} &= \overline{\partial}_{\dot\alpha} - i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu\ .\end{aligned} \tag{2.23}$$

One can explicitly check that the differential operators obey the supersymmetry algebra for the abstract supercharges. From this we have for the change of a general coset element $\Omega(x,\theta,\bar\theta)$ under a superspace translation

$$\Omega \longrightarrow \Omega + \delta_\xi\Omega = \Omega + i(-a^\mu P_\mu + \xi^\alpha q_\alpha + \bar\xi_{\dot\alpha}\bar q^{\dot\alpha})\Omega\ . \tag{2.24}$$

Ignoring normal spacetime translations for a moment, we obtain

$$\delta_\xi\Omega = i(\xi^\alpha q_\alpha + \bar\xi_{\dot\alpha}\bar q^{\dot\alpha})\Omega\ . \tag{2.25}$$

We can utilise these results to briefly note some properties of superfields. Firstly, if $\Omega_1$ and $\Omega_2$ are superfields then the product $\Omega_1\Omega_2$ is also a superfield:

$$\begin{aligned}\delta_\xi(\Omega_1\Omega_2) &= i\big[\Omega_1\Omega_2,\xi^\alpha q_\alpha+\bar\xi_{\dot\alpha}\bar q^{\dot\alpha}\big] = i\Omega_1\big[\Omega_2,\xi^\alpha q_\alpha+\bar\xi_{\dot\alpha}\bar q^{\dot\alpha}\big] + i\big[\Omega_1,\xi^\alpha q_\alpha+\bar\xi_{\dot\alpha}\bar q^{\dot\alpha}\big]\Omega_2 \\ &= \Omega_1\big(i(\xi^\alpha q_\alpha+\bar\xi_{\dot\alpha}\bar q^{\dot\alpha})\Omega_2\big) + \big(i(\xi^\alpha q_\alpha+\bar\xi_{\dot\alpha}\bar q^{\dot\alpha})\Omega_1\big)\Omega_2 \\ &= i(\xi^\alpha q_\alpha+\bar\xi_{\dot\alpha}\bar q^{\dot\alpha})(\Omega_1\Omega_2)\ .\end{aligned} \tag{2.26}$$

Note that in the first line the supercharges are abstract operators whereas after that they are given by their represenation as differential operators. This is important in the last step where the product rule is applied to combine both terms. Secondly, we note that although $\partial_\mu\Omega$ is a superfield, the object $\partial_\alpha\Omega$ is not because the operator $\partial_\alpha$ mixes with the representation of the supercharges as differential operators, i.e. $[\partial_\alpha, \xi q + \bar\xi\bar q] \neq 0$. At this point it is useful to introduce objects that anti-commute with the differential represenation of the supercharges. They are usually denoted as *covariant derivatives* and are defined by

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \quad \overline{D}_{\dot\alpha} = -\overline{\partial}_{\dot\alpha} - i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \ . \tag{2.27}$$

They satisfy the relations

$$\{D_\alpha, q_\beta\} = \{D_\alpha, \bar q_{\dot\beta}\} = \{\overline{D}_{\dot\alpha}, q_\beta\} = \{\overline{D}_{\dot\alpha}, \bar q_{\dot\beta}\} = 0 \ , \tag{2.28}$$

and we have

$$[D_\alpha, \xi q + \bar\xi\bar q] = 0 \ . \tag{2.29}$$

Therefore,

$$\delta_\xi(D_\alpha S) = iD_\alpha(\xi^\alpha q_\alpha + \bar\xi_{\dot\alpha}\bar q^{\dot\alpha})\Omega = i(\xi^\alpha q_\alpha + \bar\xi_{\dot\alpha}\bar q^{\dot\alpha})D_\alpha\Omega \tag{2.30}$$

and $D_\alpha\Omega$ is a superfield. We conclude that if $\Omega$ is a general scalar superfield, then $\partial_\mu\Omega, D_\alpha\Omega$ and $\overline{D}_{\dot\alpha}\Omega$ are also superfields.

The covariant derivatives inherit their name from the well-know covariant derivates of gauge theory where the objects $\phi$ and $D_\mu\phi$ transform in the same way under gauge transformations. The same holds for their superspace cousins which we will use when we discuss supersymmetric gauge theories.

The fact the the scalar superfield $\Omega(x,\theta,\bar\theta)$ is a function in the Grassmann parameters $\theta_\alpha$ and $\bar\theta_{\dot\alpha}$ makes it possible to Taylor-expand it in $\theta_\alpha$ and $\bar\theta^{\dot a}$. The expansion terminates since higher powers of these fermionic parameters vanish. This gives the component expansion of a scalar superfield:

$$\begin{aligned}\Omega(x,\theta,\bar\theta) = c(x) + \theta\psi(x) + \bar\theta\,\bar\psi'(x) + (\theta\theta)f(x) + (\bar\theta\bar\theta)f'(x) + \theta\sigma^\mu\bar\theta\; V_\mu(x)\\ + (\theta\theta)\bar\theta\bar\lambda'(x) + (\bar\theta\bar\theta)\theta\lambda(x) + (\theta\theta)(\bar\theta\bar\theta)D(x).\end{aligned} \tag{2.31}$$

Here, the primed fields are not related to the unprimed fields. Furthermore, all component fields are complex and additionally, $\psi(x), \bar\psi'(x), \lambda(x), \bar\lambda'(x)$ are Grassmann odd, hence, they anti-commute with all other fermionic objects in this expansion. Notice that in principle, there are four terms in the $\theta_\alpha, \bar\theta_{\dot\alpha}$. Using standard Fierz identities they can be combined into one single term which is conveniently written in terms of the vector field $V_\mu$. Hence, we have for the component fields four complex Weyl spinors $\psi, \bar\psi', \lambda$ and $\bar\lambda'$, four complex scalar fields $c, f, f'$ and $D$ and the vector field $V_\mu$. This gives in total eight complex fermionic and eight complex bosonic degrees of freedom which nicely states the fact of equal number of fermionic and bosonic degrees of freedom in a supermultiplet. Using this explicit expansion and the represenation of the supercharges as differential operators one can deduce from the (2.25) the transformation of the component fields under a supertranslation (again with $a^\mu = 0$ for simplicity) by defining

$$\begin{aligned}\delta_\xi\Omega(x,\theta,\bar\theta) =&\delta_\xi c(x) + \theta^\alpha\delta_\xi\psi_\alpha(x) + \bar\theta_{\dot\alpha}\delta_\xi\bar\psi'^{\dot\alpha}(x) + \theta\theta\delta_\xi f(x) + \bar\theta\bar\theta\delta_\xi f'(x) + \theta\sigma^\mu\bar\theta\delta_\xi V_\mu(x)\\ &+ \theta\theta\bar\theta_{\dot\alpha}\delta_\xi\bar\lambda'^{\dot\alpha}(x) + \bar\theta\bar\theta\theta^\alpha\delta_\xi\lambda_\alpha(x) + \theta\theta\bar\theta\bar\theta\delta_\xi D(x)\end{aligned} \tag{2.32}$$

and matching the appropriate powers in the Grassmann spinors on both sides of the relation (2.25).

One might ask if the component field expansion yields an irreducible representation for off-shell $\mathcal{N} = 1$ supersymmetry. To see this we might impose constraints on the component fields of $\Omega$. One can then check that the supersymmetry transformations of the component fields respect the imposed constraints. This shows that the general scalar superfield gives a reducible representation of the off-shell $\mathcal{N} = 1$ algebra. Hence, in general we can impose consistent (with the superspace transformations) constraints on $\Omega$, leading to smaller superfields which can give irreducible representations of the algebra. In the following, we discuss some very important irreducible representations for four-dimensional supersymmetric field theories, namely the chiral superfield and the vector superfield.

### 2.4.1 The chiral superfield

We have seen that the covariant derivatives are important objects in superspace, especially since they impose certain constraints on the superfields. Let $\Phi(x, \theta, \bar{\theta})$ be such a general scalar superfield. Then, $\overline{D}_{\dot{\alpha}}\Phi$ is also a superfield. Therefore, we can impose the constraint

$$\overline{D}_{\dot{\alpha}}\Phi = 0. \tag{2.33}$$

A superfield that fulfills this condition is called *chiral superfield*. We now want to find the most general solution to the covariant constraint (2.33). To do this we define a new set of coordinates $y^\mu$ in superspace,

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \ . \tag{2.34}$$

Using $\overline{D}_{\dot{\alpha}}y^\mu = \overline{D}_{\dot{\alpha}}\theta^\alpha = 0$ it is easy to see that any function $\Phi(y^\mu, \theta)$ which is not a function of $\bar{\theta}_{\dot{\alpha}}$ satisfies

$$\overline{D}_{\dot{\alpha}}\Phi(y^\mu, \theta) = 0 \tag{2.35}$$

which is the most general solution to the chiral constraint. We can therefore write

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y) \ , \tag{2.36}$$

where the factor of $\sqrt{2}$ is conventional. Notice that terms with higher powers in the Grassmann spinor $\theta_\alpha$ are not possible. Counting the degrees of freedom we see that the two complex scalars $\phi$ and $F$ give four real bosonic degrees of freedom whereas the left-handed Weyl spinor gives four real fermionic degrees of freedom. As before, the compontent fields are off-shell in this construction.

Using the explicit form of $y^\mu$ we can expand $\Phi(y, \theta)$ in powers of $\theta$ and $\bar{\theta}$. The result, after using some Fierz identities, is given by

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) = {} & \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} \\ & - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^\mu\partial_\mu\phi(x) + (\theta\theta)F(x).\end{aligned} \tag{2.37}$$

This is the full expansion of a left-handed chiral superfield. We also have anti-chiral superfields which are also denoted as right-handed chiral superfields. The relation between chiral and anti-chiral superfields is obvious. If $\Phi$ is a chiral superfield, then $\Phi^\dagger$ is an anti-chiral one, obeying the relation

$$D_\alpha \Phi^\dagger = 0 \ , \qquad \text{with } \Phi^\dagger = \Phi^\dagger(y^\dagger, \bar{\theta}) \ , \quad y^{\dagger\mu} = x^\mu - i\theta\sigma^\mu\bar{\theta} \ . \tag{2.38}$$

Expanding in $\theta$ and $\bar{\theta}$ we have[6]

$$\begin{aligned}\Phi^\dagger(x,\theta,\bar{\theta}) = \phi^\dagger(x) + \sqrt{2}\,\bar{\theta}\bar{\psi}(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^\dagger(x) + \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})\partial_\mu\bar{\psi}(x)\sigma^\mu\bar{\theta} \\ - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^\mu\partial_\mu\phi^\dagger(x) + (\bar{\theta}\bar{\theta})F^\dagger(x).\end{aligned} \tag{2.39}$$

Using the relations (2.25) and (2.32) we can deduce the transformations of the components of the chiral superfield under the infinitesimal transformation $\delta_\xi$, yielding

$$\begin{aligned}\delta_\xi\phi &= \sqrt{2}\xi\psi \ , \\ \delta_\xi\psi &= \sqrt{2}\xi F + i\sqrt{2}\sigma^\mu\bar{\xi}\delta_\mu\phi \ , \\ \delta_\xi F &= -i\sqrt{2}\partial_\mu\psi\sigma^\mu\bar{\xi} \ .\end{aligned} \tag{2.40}$$

Here, we note that the variantion of $F(x)$, the so called $F$-term of the chiral superfield, transforms as a total derivative.

We finish our discussion by quickly stating some comments about chiral superfields. As we have seen, the product of two general superfields is a superfield itself. And since the covariant derivatives obey the usual chain rule as differential operators, any product of chiral superfields $\Phi_i, \Phi_j$ is also a chiral superfield. The same holds for products of anti-chiral superfields. However, whereas $\Phi^\dagger\Phi$ and $\Phi + \Phi^\dagger$ are real superfields, they are neither chiral nor anti-chiral. Furthermore, since $D^3 = \overline{D}^3 = 0$, we have the simple fact

$$\begin{aligned}\overline{D}^2\Omega &= \Phi \qquad \text{with } \Phi \text{ as chiral superfield} \ , \\ D^2\Omega &= \Phi^\dagger \qquad \text{with } \Phi^\dagger \text{ as anti-chiral superfield} \ .\end{aligned}$$

### 2.4.2 The vector superfield

From the chiral superfield we move on to an off-shell representation of the supermultiplet of next higher spin in the $\mathcal{N} = 1$ theory. Here, one can introduce the *vector superfield* $V(x,\theta,\bar{\theta})$. Its definition is deduced from the general scalar superfield by imposing the constraint

$$V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta}) \ , \tag{2.41}$$

such that it is a real superfield. This constraint yields a component expansion

$$\begin{aligned}V(x,\theta,\bar{\theta}) = C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{\sqrt{2}}\theta\theta S(x) - \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}S^\dagger(x) - \theta\sigma^\mu\bar{\theta}V_\mu(x) \\ + i\theta\theta\bar{\theta}\big[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\big] - i\bar{\theta}\bar{\theta}\theta\big[\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\big] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\big[D(x) + \frac{1}{2}\Box C(x)\big] \ .\end{aligned} \tag{2.42}$$

Hence, the representation of the vector superfield has two real scalars $C, D$, the complex scalar $S$, four Weyl spinors $\xi, \bar{\xi}, \lambda, \bar{\lambda}$ and a real vector $V_\mu$ as component fields. The appearance of a

[6] We follow the usual convention to denote the complex conjugate of the complex scalars $\phi$ and $F$ by $\phi^\dagger$ and $F^\dagger$.

real four-dimensional vector field makes it natural to use vector superfields as building blocks for supersymmetric gauge interactions.

The first step in this directions is to extend the usual gauge transformations in the $\mathcal{N}=1$ framework by noting that one can build a vector superfield of the form $\Phi+\Phi^\dagger$ from a chiral superfield $\Phi$. This yields for the component fields in the expansion (2.42) the identifications

$$\begin{aligned} C &= (\phi+\phi^\dagger)\ , & \chi &= -i\sqrt{2}\psi\ , \\ S &= -i\sqrt{2}F\ , & V_\mu &= -i\partial_\mu\big(\phi-\phi^\dagger\big)\ , \\ D &= 0\ , & \lambda &= 0\ . \end{aligned} \tag{2.43}$$

If we then define the supersymmetric generalisation of an infinitesimal abelian gauge transformation of the vector superfield [11] as

$$V \longrightarrow V+\Phi+\Phi^\dagger\ , \tag{2.44}$$

one can immediately deduce how the component transforms and finds for the vector $V_\mu$, the scalar $D$ and the Weyl spinor $\lambda$

$$\begin{aligned} V_\mu &\longrightarrow V_\mu - i\partial_\mu\big(\phi-\phi^\dagger\big) \\ D &\longrightarrow D \\ \lambda &\longrightarrow \lambda\ . \end{aligned} \tag{2.45}$$

Hence, the supersymmetric abelian construction (2.44) gives the correct transformation of a gauge field $V_\mu \to V_\mu - \partial_\mu\Lambda$ with $\Lambda = i\big(\phi-\phi^\dagger\big)$. In addition, we can choose the component fields of $\Phi$ in (2.44) in such a way that

$$C=\chi=S=0\ . \tag{2.46}$$

The freedom in setting these fields to zero is similar to a gauge choice. Hence, this particular choice is called the *Wess Zumino (WZ) gauge*. In this gauge the vector superfield takes the simple form

$$V_{\mathrm{WZ}} = -\theta\sigma^\mu\bar\theta V_\mu(x) + i\theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x)\ . \tag{2.47}$$

The component fields of the superfield in the WZ gauge are the vector field $V^\mu$, corresponding to gauge bosons, the gauginos $\lambda$ and $\bar\lambda$ and the real scalar $D$ which is an auxiliary field. We note that fixing the supersymmetric gauge freedom does not fix the remaining abelian gauge freedom for the gauge field $V^\mu$. The simple structure of the vector superfield in WZ gauge yields for products of $V_{\mathrm{WZ}}$

$$V_{\mathrm{WZ}}^2 = \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x)\ , \qquad V_{\mathrm{WZ}}^{n+2} = 0\ \ \forall n\in\mathbb{N}\ . \tag{2.48}$$

Unfortunately, the vector superfield in the WZ gauge is not invariant under supersymmetry transformations since the superfield $V_{\mathrm{WZ}}$ does not transform into a vector superfield of the same WZ gauge under a supersymmetry transformation. Hence, one has to deal with the higher amount of component fields when using the full vector superfield in a supersymmetric formulation. However, one can define another superfield that contains only the fields of the WZ gauge. In addition, it provides the field strength for the vector field and products with itself can provide the gauge kinetic terms for the gauge field $V^\mu$.

We can use the covariant derivatives to define the (abelian) supersymmetric field strength

$$\begin{aligned}\mathbb{W}_\alpha(x,\theta,\bar{\theta}) &= -\frac{1}{4}\left(\overline{DD}\right) D_\alpha V(x,\theta,\bar{\theta}) \\ \overline{\mathbb{W}}_{\dot{\alpha}}(x,\theta,\bar{\theta}) &= -\frac{1}{4}\left(DD\right) \overline{D}_{\dot{\alpha}} V(x,\theta,\bar{\theta}) \ .\end{aligned} \tag{2.49}$$

We see that they are chiral or anti-chiral superfields, $\overline{D}_{\dot{\alpha}}\mathbb{W}_\beta = 0$ because higher powers in the covariant derivatives $\overline{D}^3$ vanish. However, the chiral and anti-chiral $\mathbb{W}_\alpha$ and $\overline{\mathbb{W}}_{\dot{\alpha}}$ obey the additional constraint

$$\overline{D}_{\dot{\alpha}}\overline{\mathbb{W}}^{\dot{\alpha}} = D^\alpha \mathbb{W}_\alpha \ . \tag{2.50}$$

The anti-commutation relation of the covariant derivatives yield the invariance of the supersymmetric field strength under the generalised abelian gauge transformation (2.44), for instance

$$\mathbb{W}_\alpha \longrightarrow \mathbb{W}'_\alpha = -\frac{1}{4}\left(\overline{DD}\right) D_\alpha\big(V + \Phi + \Phi^\dagger\big) \ . \tag{2.51}$$

This allows us to compute the component expansion of the field strength superfield in the WZ gauge where it is convenient to switch to the superspace coordinate $y^\mu$. Using the relation $x^\mu = y^\mu - i\theta\sigma^\mu\bar{\theta}$ we can expand the component fields as[7]

$$\begin{aligned}\mathbb{W}_\alpha &= -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\right)_\alpha^{\ \beta}\theta_\beta\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right)(y) + (\theta\theta)\,\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\bar{\lambda}^{\dot{\beta}}(y) \\ &= -i\lambda_\alpha(y) + \left[\delta_\alpha^\beta D(y) - \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\right)_\alpha^{\ \beta} F_{\mu\nu}(y)\right]\theta_\beta + (\theta\theta)\,\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\bar{\lambda}^{\dot{\beta}}(y)\end{aligned} \tag{2.52}$$

where we have used the abelian field strength $F_{\mu\nu}(y) = \partial_\mu V_\nu - \partial_\nu V_\mu$. An analogue expansion holds for the anti-chiral supersymmetric field strength $\overline{\mathbb{W}}_{\dot{\alpha}}$. Note that we have only discussed the abelian case so far, i.e. a supersymmetric gauge transformation of the form (2.44) where $V, \Phi$ and $\Phi^\dagger$ commute. We will generalise this construction to the general non-abelian case in the following.

### 2.4.3 The non-abelian field strength superfield

In the general non-abelian case, the superfields in the infinitesimal transformation (2.44) are matrix valued fields. We can exponentiate the expression (2.44) and obtain for a finite gauge transformation

$$e^V \longrightarrow e^{-i\Lambda^\dagger} e^V e^{i\Lambda} \tag{2.53}$$

with the conventional definitions of $\Phi = i\Lambda$ and $\Phi^\dagger = -i\Lambda$ [22]. In the abelian case, the infinitesimal transformation follows directly from (2.53) since all fields commute. For a non-abelian gauge group, we have

$$\begin{aligned}V &\equiv (T^A)^a_{\ b} V^A \ , \\ \Lambda &\equiv (T^A)^a_{\ b} \Lambda^A\end{aligned} \tag{2.54}$$

where the $T^A$ are the generators of the gauge group and a sum over the gauge index $A = 1, \ldots, N^2 - 1$ for $\mathsf{SU}(N)$ gauge groups is understood. In this case, the superfields $\Lambda$ and $V$ do not commute. Therefore, a priori it is not clear that the WZ gauge can be applied for non-abelian gauge symmetries

[7]The reader should not confuse the covariant derivative $D_\alpha$ with the auxiliary component field $D(x)$.

because the relation between the vector superfield $V$ and $\Lambda$ and $\Lambda^\dagger$ are more complicated as can be seen from the infinitesimal non-abelian gauge transformation. Following (2.53) one can use the Baker-Campbell-Hausdorff relation for matrix exponentials which yields

$$V \longrightarrow V + i(\Lambda - \Lambda^\dagger) - \frac{i}{2}[(\Lambda + \Lambda^\dagger), V] \ . \tag{2.55}$$

This transformation holds to first order in $\Lambda$ where we neglect an infinite series of higher commutators $[V, [V \dots [V, (\Lambda - \Lambda^\dagger)]$. Although the relation between the vector and chiral superfields in the non-abelian transformation is more involved, the transformation (2.55) suggests that one can still arrange $\Lambda$ and $\Lambda^\dagger$ such that only the component fields $V^\mu, \lambda, \bar\lambda$ and $D$ are non-vanishing. Hence, the WZ gauge holds also for non-abelian gauge transformations. Since $V^\mu$ is now a matrix valued gauge field, $\lambda$ and $D$ are now (matter) fields in the adjoint representation.

For the non-abelian case we also have to modify the definition of the supersymmetric field strength because the expressions (2.49) are not gauge invariant. One defines for non-abelian gauge symmetries

$$\begin{aligned} \mathbb{W}_\alpha(x,\theta,\bar\theta) &= -\frac{1}{4}\left(\overline{DD}\right) e^{-V} D_\alpha\, e^{V} \\ \overline{\mathbb{W}}_{\dot\alpha}(x,\theta,\bar\theta) &= -\frac{1}{4}\left(DD\right) e^{V} \overline{D}_{\dot\alpha} e^{-V} \end{aligned} \tag{2.56}$$

where the superfields are also matrix valued, $\mathbb{W}_\alpha = \mathbb{W}^A_\alpha T^A$. This definition is compatible with the abelian case as can be seen from expanding the exponentials. Without loss of generality we choose $V$ to be in the $WZ$ gauge. Expanding with $V^3_{\mathrm{WZ}} = V^2_{\mathrm{WZ}}\,(D_\alpha V_{\mathrm{WZ}}) = 0$ and $D_\alpha V^2_{\mathrm{WZ}} = (D_\alpha V_{\mathrm{WZ}})\, V_{\mathrm{WZ}} + V_{\mathrm{WZ}}\,(D_\alpha V_{\mathrm{WZ}})$ yields

$$\mathbb{W}_\alpha = -\frac{1}{4}\left(\overline{DD}\right)\left(D_\alpha V + [D_\alpha V, V]\right) \ , \qquad \overline{\mathbb{W}}_{\dot\alpha} = -\frac{1}{4}\left(DD\right)\left(\overline{D}_{\dot\alpha} V - [\overline{D}_{\dot\alpha} V, V]\right) \ . \tag{2.57}$$

For an abelian theory the commutators vanish and we obtain (2.49). We also note that the non-abelian field strengh superfields are not invariant under gauge transformations but rather covariant,

$$\mathbb{W}_\alpha \longrightarrow e^{-i\Lambda}\mathbb{W}_\alpha\, e^{i\Lambda} \ . \tag{2.58}$$

Tracing over the gauge index $A$, the object $\mathrm{Tr}[\mathbb{W}^\alpha \mathbb{W}_\alpha]$ is gauge invariant. This is the same as in the non-supersymmetric case where the field strength $F_{\mu\nu}$ is not invariant under non-abelian gauge transformation but the quantity $\mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}] = \frac{1}{2}F^{A,\mu\nu}F^A_{\mu\nu}$ is[8].

We obtain the component expansion of the non-abelian version of $\mathbb{W}_\alpha$ by working in the WZ gauge and expressing the superfield in the coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$,

$$\mathbb{W}_\alpha = -i\lambda_\alpha(y) + \left[\delta^\beta_\alpha D(y) - \frac{i}{2}\left(\sigma^\mu\bar\sigma^\nu\right)^\beta_\alpha F_{\mu\nu}(y)\right]\theta_\beta + (\theta\theta)\,\sigma^\mu_{\alpha\dot\beta} D_\mu \bar\lambda^{\dot\beta}(y) \ , \tag{2.59}$$

where the field strength and gauge covariant derivate are given by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu] \ , \\ D_\mu \bar\lambda^{\dot\beta} &= \partial_\mu \bar\lambda^{\dot\beta} + i[V_\mu, \bar\lambda^{\dot\beta}] \ . \end{aligned} \tag{2.60}$$

This shows that $\mathbb{W}_\alpha = \mathbb{W}^A_\alpha T^A$. In the abelian limit, the commutators in the above expression vanish and one obtains (2.52).

[8]For generators in the adjoint representation of an $\mathsf{SU}(N)$ gauge group one has $\mathrm{Tr}[T^A T^B] = \frac{1}{2}\delta^{AB}$.

## 2.5 Supersymmetric field theories

Having introduced the machinery of the $\mathcal{N}=1$ superspace and the corresponding superfield notation, we are now ready to describe a supersymmetric version of familiar non-SUSY quantum field theories in a superfield approach. The formulation is based on the observation that the F-term of a chiral superfield and the D-term of a vector superfield transform into themselves plus a term which is a total derivative under the infinitesimal SUSY transformations (2.25). Hence, the action does not change. We can implement this feature by integrations of the Grassmann spinors, leading to the terms

$$S_F = \int d^4x \left\{ \int d^2\theta \Phi + \int d^2\bar{\theta}\Phi^\dagger \right\} , \qquad S_D = \int d^4x \int d^4\theta V$$

which are automatically invariant under supersymmetry transformations. In the following we quickly discuss possible theories that incorporate these constructions, firstly theories that are constructed out of chiral superfields only and then theories including gauge superfields.

### 2.5.1 Supersymmetric chiral models

Any holomorphic function of a chiral superfield is also a chiral superfield. Denoting such a holomorphic function by $W(\Phi)$, we know that its F-term transforms as a total derivative. If we generalise this to a theory with several chiral superfields $\Phi_i$, the term

$$\mathcal{L}_W = \int d^2\theta W(\Phi_i) + \text{h.c.} \ , \tag{2.61}$$

together with its hermitian conjugate function $W(\Phi^\dagger)$, give rise to supersymmetric interactions among the component fields. The function $W$ is called the *superpotential*. Note that no spacetime derivatives can occur in this construction and hence, this Lagrangian does not generate kinetic terms for the fields.

However, we already know that the object $\Phi\Phi^\dagger$ is a real superfield. If we again generalise to several superfields in the theory, the term

$$\mathcal{L}_K = \int d^4\theta K \ , \qquad \text{with } K = \sum_i \Phi_i^\dagger \Phi_i \tag{2.62}$$

is invariant and give rise to canonical kinetic terms. Here, $K$ is the so called canonical *Kähler potential*. In general, it is possible to have a general Kähler potential which leades to kinetic terms

$$\mathcal{L}_K = \int d^4\theta K \supset g^{ij} \left( -(\partial_\mu \phi_i)^\dagger (\partial^\mu \phi_j) - i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_j + F_i^\dagger F_j \right) \tag{2.63}$$

where $g^{ij} = \partial^2 K/(\partial\Phi_i^\dagger \partial\Phi_j)\big|_{\Phi=\phi}$ is a Kähler metric. For the rest of our discussion we will assume a canonical Kähler potential $K_{\text{can}}$.

The simplest supersymmetric Lagrangian that contains only chiral superfields is the Wess-Zumino model [11]. Its action is given by

$$S_{WZ} = \int d^4x \left[ \int d^4\theta K_{\text{can}} + \int d^2\theta \left\{ \lambda_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} y_{ijk} \Phi_i \Phi_j \Phi_k + \text{h.c.} \right\} \right] \tag{2.64}$$

for $i$ flavours of chiral superfields $\Phi_i$. Sometimes one can find definitions that include the numerical factors in the couplings $m_{ij}$ and $y_{ijk}$. By construction, the Lagrangian is supersymmetric and the most general one leading to a renormalisable theory. The component expansion of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{WZ} = &-(\partial_\mu\phi_i)^\dagger(\partial^\mu\phi_i) - i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i + F_i^\dagger F_i + \left[\lambda_i F_i + m_{ij}\left(\phi_i F_j - \frac{1}{2}\psi_i\psi_j\right)\right.\\ &\left.+ y_{ijk}\left(\phi_i\phi_j F_k - \psi_i\psi_j\phi_k\right) + \text{h.c.}\right]\end{aligned} \tag{2.65}$$

We note that there are no kinetic terms for the $F$ field in this expansion. This justifies to call it an auxiliary field where its equations of motions are purely algebraic and we obtain (for just one chiral superfield)

$$0 = -\frac{\partial\mathcal{L}}{\partial F} = F^\dagger + \lambda + m\phi + \frac{1}{2}y\phi\phi\ . \tag{2.66}$$

If we denote the interaction terms in (2.64) more generally by $W(\Phi)$, we can write the WZ Lagrangian in a compact form. Solving the auxiliary equations for $F$ and $F^\dagger$ and plugging the result back into the component expansions, the Lagrangian contains the terms

$$\mathcal{L}_{WZ} \supset F^\dagger F - (\lambda F + m\phi F + \frac{1}{2}y\phi\phi F + \text{h.c.}) = \left|\frac{\partial W(\Phi)}{\partial\Phi}\right|^2_{\Phi=\phi}\ . \tag{2.67}$$

Therefore, we have the most general expression

$$\begin{aligned}\mathcal{L}_{WZ} = &-(\partial_\mu\phi_i)^\dagger(\partial^\mu\phi_i) - i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i - \sum_i\left|\frac{\partial W(\Phi_i)}{\partial\Phi_i}\right|^2_{\Phi=\phi}\\ &-\frac{1}{2}\left(\frac{\partial^2 W(\Phi_i)}{\partial\Phi_i\partial\Phi_j}\right)\bigg|_{\Phi=\phi}\psi_i\psi_j - \frac{1}{2}\left(\frac{\partial^2 W^\dagger(\Phi_i^\dagger)}{\partial\Phi_i^\dagger\partial\Phi_j^\dagger}\right)\bigg|_{\Phi=\phi}\bar{\psi}_i\bar{\psi}_j\ ,\end{aligned} \tag{2.68}$$

where we identify

$$V_F = \sum_i\left|\frac{\partial W(\Phi_i)}{\partial\Phi_i}\right|^2_{\Phi=\phi}\ , \tag{2.69}$$

as the scalar potential of the Wess Zumino model. In the next step, we will add gauge interactions to the supersymmetric formalism.

### 2.5.2 Supersymmetric gauge theories

Before introducing vector superfields into the theory we remind ourselves about the definition of the field strength superfield $\mathbb{W}_\alpha$ for the abelian and non-abelian case we discussed before. As we have seen, it can be used to construct gauge kinetic terms. When discussing supersymmetric gauge theories, we want invariance of the action under supersymmetry *and* gauge transformations.

We begin our discussion with an abelian gauge theory. Under a $\mathsf{U}(1)$ gauge transformations, a chiral superfield $\Phi$ transforms as

$$\Phi' = e^{-i\Lambda}\Phi\ , \qquad \Phi'^\dagger = e^{i\Lambda^\dagger}\Phi^\dagger\ , \tag{2.70}$$

with $\Lambda$ and $\Lambda^\dagger$ as chiral superfields. This immediately shows that the Kähler potential term from the Wess-Zumino model is not invariant under local gauge transformations. However, if we remember the form of the infinitesimal abelian gauge transformation for a vector superfield V,

$$V' = V - i\Lambda^\dagger + i\Lambda \tag{2.71}$$

we find that the term $\Phi^\dagger e^V \Phi$ is gauge and supersymmetry invariant. Combing gauge and matter fields, the invariant Lagrangian takes the form

$$\mathcal{L} = \int d^4\theta \sum_i \Phi_i^\dagger e^V \Phi_i + \frac{1}{4g^2}\left[\int d^2\theta\, \mathbb{W}^\alpha \mathbb{W}_\alpha + \int d^2\bar\theta \overline{\mathbb{W}}_{\dot\alpha}\overline{\mathbb{W}}^{\dot\alpha}\right] + \left[\int d^2\theta\, W(\Phi_i) + \text{h.c.}\right] , \tag{2.72}$$

where $W$ is again the superpotential. The Lagrangian of the chiral and vector component fields can be extracted as usual from the superfield Lagrangian where one usually replaces $V \to 2gV$ to obtain the standard normalisation with respect to the coupling $g$ of the component terms.

The non-abelian generalisation was originally discussed in [22, 23]. As mentioned before, the superfields are now matrix valued

$$(V)^a_b = (T^A)^a_b V^A \ , \quad (\Lambda)^a_b = (T^A)^a_b \Lambda^A \ , \quad \left(\Lambda^\dagger\right)^a_b = (T^A)^a_b \Lambda^{\dagger A} \ , \tag{2.73}$$

where the $T^A$ are the hermitian generators of the corresponding Lie algebra. The form of the gauge transformation of the chiral superfields is

$$\Phi' = e^{-i\Lambda}\Phi \ , \qquad \Phi'^\dagger = e^{i\Lambda^\dagger}\Phi^\dagger \tag{2.74}$$

and the finite supergauge transformation for the vector superfield is given by (2.53). Using these results, the non-abelian generalisation is straight forward. The general non-abelian Lagrangian of interacting chiral and vector superfields is given by

$$\mathcal{L} = \int d^4\theta \sum_i \Phi_i^\dagger e^V \Phi_i + \frac{1}{4g^2}\left[\int d^2\theta\, \mathbb{W}^{A\alpha} \mathbb{W}^A_\alpha + \int d^2\bar\theta \overline{\mathbb{W}}^a_{\dot\alpha}\overline{\mathbb{W}}^{a\dot\alpha}\right] + \left[\int d^2\theta\, W(\Phi_i) + \text{h.c.}\right] . \tag{2.75}$$

A comment about the terms of the full Lagrangian that contain the component field $D$ of the vector superfield is in order here. We find in the general non-abelian case terms of the form (after integration over the Grassmann variables)

$$\mathcal{L} \supset \frac{1}{2}D^2 + D^A \phi_i^\dagger T^A \phi_i \tag{2.76}$$

where the first part is coming from the supersymmetric field strength and the second part form the expansion of the gauge kinetic piece to first order in $V^A$. We have also suppressed all gauge group indices. Just as in the case of the auxiliary field $F$, we see that its equation of motion is purely algebraic and we can therefore eliminate the auxiliary field $D$:

$$0 = -\frac{\partial \mathcal{L}}{\partial D} = D^A + \phi_i^\dagger T^A \phi_i \ , \tag{2.77}$$

such that the terms containing the field $D$ can be written as

$$\frac{1}{2}D^2 + D^A \phi_i^\dagger T^A \phi_i = -\frac{1}{2}\left(\phi_i^\dagger T^A \phi_i\right)^2 \equiv -\frac{1}{2}\left(((\phi^\dagger)^{ia}(T^A)^b_a \phi_{bi}\right)^2 \ . \tag{2.78}$$

In the last step we have explicitly written out all indices of the fields. This so called D-term is another contribution to the scalar potential of the supersymmetric theory. We conclude our discussion by giving the component expansion of this general super-gauge invariant theory. In order to obtain the standard normalisation we let $V \to 2gV$ in (2.73). Suppressing all colour indices we get

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - i\bar{\lambda}^A\bar{\sigma}D_\mu\lambda^A - (D_\mu\phi)^\dagger_i(D^\mu\phi)_i - i\bar{\psi}_i\bar{\sigma}^\mu D_\mu\psi_i \qquad (2.79)\\ & -i\sqrt{2}g\bar{\psi}_i\lambda^A T^A\phi_i + i\sqrt{2}g\phi^\dagger_i T^A\phi_i\bar{\lambda}^A\\ & -\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2 W^\dagger}{\partial\phi^\dagger_i\partial^\dagger_j}\bar{\psi}_i\bar{\psi}_j - V(\phi_i,\phi^\dagger_j)\ . \qquad (2.80)\end{aligned}$$

Here, the component fields $V^A_\mu$ and $\lambda^A$ belong to the vector superfield $V^A$ and $\phi_i$ and $\psi_i$ to the chiral superfields $\Phi_i$. The gauge covariant derivatives are given by

$$\begin{aligned}D_\mu\phi &= \partial_\mu\phi + igV^A_\mu T^A\phi \qquad (2.81)\\ D_\mu\psi &= \partial_\mu\psi + igV^A_\mu T^A\psi\\ D_\mu\lambda^A &= \partial_\mu\lambda^A + igf^{ABC}V^B_\mu\lambda^C\ .\end{aligned}$$

The general scalar potential is now a some of the F-terms and D-terms,

$$V(\phi_i,\phi^\dagger_j) = F^\dagger_i F_i + \frac{1}{2}(D^A)^2 = \sum_i\left|\frac{\partial W}{\partial\phi_i}\right|^2 + \frac{1}{2}\sum_A\left(g\phi^\dagger_i T^A\phi_i\right)^2 \qquad (2.82)$$

where as usual the superpotential derivatives are evaluated with respect to the scalar component of the chiral superfields. This concludes our discussion of the $\mathcal{N}=1$ superfields approach. We will use some of the discussed machinery to describe a certain class of models of supersymmetry breaking in the Chapters 6 and 7.

At this point we would like to move on and discuss another highly interesting application of supersymmetry in four dimensions, namely the idea to construct superamplitudes in the maximally supersymmetric Yang-Mills theory. We will return to four-dimensional, non-extended supersymmetry in Chapter 5.

# 3 Amplitudes in Four-Dimensional $\mathcal{N} = 4$ SYM Theory

Within the framework of perturbation theory, the concept of perturbative scattering amplitudes provides a unique link between experimental data and a mathematical description of the underlying theory. Since the beginning of 20th century, starting with the ideas of quantum mechanics, physicists have been trying to formulate a theory that can describe the fundamental interactions in nature. The concept of quantum field theories has been an important step towards this goal. Starting with quantum mechanics and groundbreaking works on quantum electrodynamics in the thirties and forties of the previous century, Feynman's ideas about path integrals in quantum mechanics and especially his works on a diagrammatical interpretation of scattering processes of subatomic particles have led to an understanding which provides a highly impressive match between theoretical predictions and their experimental checks. Over the last 60 years, a lot of progress has been made on these subjects. Nowadays, the perturbative analysis of a gauge theory in terms of Feynman diagrams is in principle well understood. However, with increasing complexity of the scattering process, the number of Feynman diagrams one has to calculate grows rapidly. Therefore, it is a rather striking fact that the final result of a Feynman calculation can be elegant and simple.

In this chapter we introduce basic concepts and properties of scattering amplitudes and combine these ideas with maximal supersymmetry in four dimensions. We will use techniques for efficient calculation of superamplitudes both at tree- as well as loop-level. Furthermore, this chapter provides some intuition on amplitudes in supersymmetric theories which will be useful for our discussion of perturbative calculations in a six-dimensional context in Chapter 4.

## 3.1 Some preliminaries on scattering amplitudes

We begin by reminding ourselves what we actually mean by a scattering amplitude. In general, a theory's field content and corresponding interactions are defined by a Lagrangian density $\mathcal{L}$. A scattering event between particles of the theory is then described by elements of the so called S-matrix. In order to define these matrix elements, we need to introduce the concept of asymptotic states in the interacting quantum field theory. We define initial and final states for any number of particles by states in a Fock space by

$$|i\rangle = |a_1(p_1,)\ldots,a_k(p_k),\ldots t=-\infty\rangle, \quad |f\rangle = |\ldots,b_m(p_m),\ldots,t=+\infty\rangle \; . \tag{3.1}$$

Then, an interaction between particles in these asymptotic states is described by the time evolution operator $U$. Asymptotically, one defines

$$\lim_{t\to\infty} U(t,-\infty)\,|i\rangle = S|i\rangle, \qquad \text{with} \quad S^\dagger S = \mathbb{I} \tag{3.2}$$

where $S$ is the S-operator or S-matrix which is an unitary operator. The individual matrix elements are then given by considering the operator between initial and final (asymptotic) scattering states

$$\langle f|S|i\rangle = S_{fi}\ . \tag{3.3}$$

For the actual purpose of calculating scattering events it is often practical to define all particles involved in the process as incoming states. One can use the crossing-symmetry[1] of the S-matrix and rewrite the matrix-elements as

$${}_{\text{final}}\langle \text{vac}|S|a_1(p_1),\ldots,a_1(p_n)\rangle_{\text{initial}} \tag{3.4}$$

for the $n$ incoming particles. The assumption of momentum conservation (sum of initial momenta $P_i$ is equal to the sum of the outgoing momenta $P_f$) in the scattering process leads to the identity

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(P_i - P_f)T_{fi}\ . \tag{3.5}$$

Here, $T_{fi}$ is a T-matrix element, usually denoted as the *scattering amplitude* $\mathcal{A}$. In general, the amplitude is a function of kinematical and structure variables like momenta, couplings, colour factors and polarisation tensors. It is this part of the full S-matrix that is of high value in order to make connections to experiments. Measurable observables are cross sections and decay widths and these quantities are directly proportional to the squared amplitude $|\mathcal{A}|^2$ (integrated over the corresponding phase space).

A comment is in place here: For most parts of this chapter we consider amplitudes in the $\mathcal{N}=4$ Super Yang-Mills theory which is a conformal field theory. Strictly speaking, asymptotic states are not well defined in a conformal theory because of its scale invariance. A separation of interaction and asymptotic regions is problematic because of the theory's long-range interactions. In practice this problem can be avoided by using regulators in the conformal theory. A convenient method is dimensional regularisation in which one regularises the theory in the infrared by continuing the four-dimensional spacetime to $D = 4-2\epsilon$ dimensions for $\epsilon < 0$. In maximally supersymmetric theories this can be done such that all supersymmetries are preserved [24]. This procedure breaks the conformal symmetry and hence, asymptotic states can be defined. In that sense one has a well-defined S-matrix, however, at loop-level it is divergent when we remove the regulator.

In the following section we present techniques that are helpful for efficient computations of amplitudes in gauge theories. As mentioned before, one of the main difficulties of a perturbative Feynman diagram approach is the increasing number of diagrams that have to be dealt with in processes with more than 4 external particles or higher order corrections in the coupling constant. It turns out that it is efficient to explicitly use the quantum numbers of the external particles (their colour) to group similar diagrams together, leading to the so-called colour decomposition. Our discussion of tree-level gluon amplitudes will loosely follow the work [25] and the reviews [26, 27].

[1] In quantum field theory, crossing-symmetry is the fact that one can exchange initial particle states with final anti-particle states going backwards in time. This can be understood from the usual Feynman diagram approach.

### 3.1.1 Colour structure of the amplitude

Even at lowest order in a perturbative expansion (tree-level), the form of the scattering amplitudes can become quite complicated, especially when particles carrying colour are involved in the scattering process. The complication arises from the non-albelian nature of the gauge symmetry. The most prominent example is QCD with gauge group $\mathsf{SU}(3)$. It is therefore important to understand the colour structure of the amplitude. In the following we study scattering of $\mathsf{SU}(N)$ Yang-Mills gauge bosons for arbitrary $N$. Here, quarks and antiquarks carry fundamental or anti-fundamental indices $\mathsf{SU}(N)$ $i, j = 1, \ldots, N$ whereas the gauge bosons (the gluons) are in the adjoint representation and therefore carry a colour index $a = 1, \ldots, N^2 - 1$. If we are considering an amplitude of a scattering process of $n$ of these gauge bosons, we are dealing with $n$ incoming states. These states are labelled by the particles' momenta $p_i$, helicities $h_i$ and colours $c_i$ and we write for a generic amplitude

$$A_{(n)} = A(p_1, h_1, c_1, \ldots p_n, h_n, c_n) \ . \tag{3.6}$$

If one constructs such an amplitude from $\mathsf{SU}(N)$ Feynman rules one finds that the general structure is a product between a kinematical part (containing momenta and coupling constants) and a colour part (containing the colour factors). The latter one can be represented by the structure constants of the corresponding gauge group, defined by the relation

$$[T^a, T^b] = i f_{abc} T^c \ . \tag{3.7}$$

Note that we have slightly changed our notation compared to the previous section in order to follow the usual conventions of the amplitudes' literature. The generators $T^a$ are in the fundamental representation of $\mathsf{SU}(N)$, i.e. they are $N \times N$ matrices and there are $N^2 - 1$ of them. In the context of amplitude calculations it is convenient to normalise them as[2]

$$\mathrm{Tr}[T^a T^b] = \delta^{ab} \ . \tag{3.8}$$

The Feynman rules are such that each quark-gluon vertex contributes a factor of $T^a$, each three gluon vertex a factor $f^{abc}$ and each four gluon interaction give a structure containing products of the structure constants like $f^{abe} f^{cde}$. One can use the defining relation of the structure constants to replace all factors of $f^{abc}$ in the Feynman rules by linear combinations of strings of generators $T^a$. This is done by using the definition of the generators Lie-algebra to write the structure constants in terms of products of generators,

$$f^{abc} = -i \mathrm{Tr}[T^a, [T^b, T^c]] \ . \tag{3.9}$$

Furthermore, one can reduce the number of trace factors by using the identify[3]

$$\sum_a (T^a)_i^{\ j} (T^a)_k^{\ l} = \delta_i^l \delta_k^j - \frac{1}{N} \delta_i^j \delta_k^l \ . \tag{3.10}$$

[2]The usual normalisation of the generators of the fundamental representation is $\mathrm{Tr}[T^a T^b] = \delta^{ab}/2$. In order to avoid any proliferation of factors of 2 in the amplitudes it is convenient in the context of scattering amplitudes to normalise the generators without the factor of 1/2. Although Feynman rules are normally based on the normalisation $\mathrm{Tr}[T^a T^b] = \delta^{ab}/2$, one can simply rescale the generators and structure constants as $T^a \to T^a/\sqrt{2}$ and $f^{abc} \to f^{abc}/\sqrt{2}$ in order to obtain a description which is compatible with our normalisation condition Tr $[T^a T^b] = \delta^{ab}$.

[3]Again, there is no factor of $\frac{1}{2}$ on the RHS of this relation due to our normalisation condition.

We see that the $\mathsf{SU}(N)$ generators are traceless due to the subtraction of the term proportional to $1/N$ which comes from the $\mathsf{U}(N)$ group into which $\mathsf{SU}(N)$ is embedded. By repeated application of these relations one can show that any tree level amplitude of a scattering of $n$ gauge bosons can be written as a sum of single trace terms. This fact enables us to separate the colour part of the tree level amplitude from the bit describing the kinematics. This is achieved by the decomposition of the gluon amplitude [28], [29]

$$\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}})\; A_{n;0}(\sigma(p_1^{h_1}), \dots, \sigma(p_n^{h_n}))\;, \tag{3.11}$$

where $g$ is the $SU(N)$ gauge coupling and $A_{n;0}$ is the *colour-ordered partial amplitude* at tree-level. In this expression we are summing over all permutations $S_n$ of $n$ objects but we have to account for the invariance of the trace under cyclic permutations $Z_n$. Therefore, the sum is only performed for permutations over the set $S_n/Z_n$, giving a total of $(n-1)!$ permutations.

A similar structure holds at loop-level [30], however, expressions become even more complicated. For an $n$-gluon amplitude, in general multi traces appear. Furthermore, we have to sum over the different particles that circulate in the loop, i.e. one sums over their spins. If only particles in the adjoint representation are propagating, the leading contribution for large $N$ is a single trace term (times a factor of $N$) and gives rise to planar partial amplitudes. The subleading terms which are proportional to multi-trace terms are giving non-planar contributions. At one-loop one has an expansion

$$\begin{aligned}\mathcal{A}_n^{\text{1-loop}} = g^n \Bigg[ &\sum_{\sigma \in S_n/Z_n} N\text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}})\; A_{n;c=1}^{\text{1-loop}}(\sigma(p_1^{h_1}), \dots, \sigma(p_n^{h_n})) \\ &+ \sum_{c=2}^{\lfloor n/2 \rfloor +1} \sum_{\sigma \in S_n/S_{n;c}} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(c-1)}})\text{Tr}(T^{a_{\sigma(c)}} \dots T^{a_{\sigma(n)}}) \\ &\qquad \times A_{n;c=1}^{\text{1-loop}}(\sigma(p_1^{h_1}), \dots, \sigma(p_n^{h_n}))\Bigg]\;.\end{aligned} \tag{3.12}$$

Here, $\lfloor n \rfloor$ is the largest integer less than or equal to $n$ and $S_{n;c}$ is the set of permutations that leaves the double-trace structure invariant. In [31] it is discussed that for generic $\mathsf{SU}(N)$ theories, including supersymmetric ones, the non-planar contributions can be obtained as a sum over permutations of the planar terms. This holds as long as the contribution particles (external and internal) are in the adjoint of the gauge group.

In these decompositions, $A_n$ does not contain any information about the colour structure of the full amplitude but provides full information about the kinematics of the process. A specific partial amplitude receives only contributions for a specific ordering of the external gauge boson states (again up to cyclic permutations). Hence, these objects have a simpler analytic structure. Obviously, they are invariant under gauge transformations (due to gauge invariance of the theory) and cyclic permutations. As indicated above, any partial amplitude depends on the gauge bosons momenta and helicites. However, we notice that the momenta are direct parameters of the amplitude whereas the helicites label different physical amplitudes (i.e. changing the helicities $h_i$ leads to a different scattering process). In the following we use the short hand notation

$$A_n(p_1^{h_1}, p_2^{h_2}, \dots, p_n^{h_n}) \equiv A_n(1, 2, \dots, n) \tag{3.13}$$

or even $A_n(h_1, h_2, \cdots, h_n)$ with $h_i = +/-$ whenever the momentum structure of the amplitude is clear from its context. We can reverse the helicity configuration of the partial amplitude by applying parity (recalling that we define all external gauge boson momenta to be incoming). In addition, partial amplitudes with reversed order of the gauge fields are related to each other by the identity

$$A_n(n, n-1, \ldots, 1) = (-1)^n A_n(1, 2, \ldots, n). \tag{3.14}$$

There exist more relations and an overview of partial amplitude identities can be found in [32]. In general, these relations help to reduce the number of partial amplitudes one needs to calculate in order to describe the full scattering process, either because the result for a partial amplitude can be obtained from another calculation or the specific amplitude just vanishes (due to symmetry relations). From now on we will refer to the partial amplitudes just as the amplitude and will not consider any colour structure. If necessary, the full amplitude can easily be reconstructed by using the discussed colour decomposition.

### 3.1.2 Spinor helicity formalism

As we have seen in the last section the complexity of calculating a scattering amplitude for an arbitrary process can be reduced by using the colour structure of $\mathsf{SU}(N)$ gauge groups and considering the partial amplitudes. Although this is a nice feature, the kinematical structure of the remaining amplitude can still be very complicated. In general, the particles' wave functions $\psi$ need to be taken into account, yielding a structure

$$A_n = A_n(p_i, \psi_i)\, \delta^{(4)}(p_1 + \cdots + p_n) \tag{3.15}$$

where the delta function ensures 4-momentum conservation. The wave function used in these expressions depends on the particles under consideration. Normally, one uses Dirac spinors $u(p)$ and $v(p)$ whenever spin 1/2 particles are among the external states for a description of the wave function dependence. In the case of spin 1 particles (e.g. gauge bosons), the particles' polarisation vectors $\epsilon^\mu$ capture the needed behaviour. In any case, the amplitude is a scalar quantity (although it might be complex) and hence it must be constructed out of Lorentz scalars. This condition is fulfilled by the usual Minkowski four-vector products like $p_i \cdot p_j$ or $\epsilon_k \cdot p_l$. A description that captures the behaviour of spin 1 and spin 1/2 particles neatly would lead to further simplifications on the structure of the partial amplitudes. Indeed, such a unified scheme exist in the case of massless states, the so called *spinor helicity formalism*[4] [33], [34]. In this prescription one uses spinor inner products instead of Minkowski vector products. They are scalar quantities as well, capture collinear behaviour of the momenta in a nice way and are in some sense more 'fundamental' objects, considering the fact that Minkowski products can be derived from spinor products quite easily. This framework works for two-component (Weyl) as well as four-component (Dirac/Majorana) spinors where in four-dimensions the general structure of the expressions is the same. Whereas Dirac spinors are conventionally used in the normal Feynman diagram approach one finds the Weyl spinor

[4]This powerful method is not limited to four-dimensions. As one of the main results of this thesis we will apply a six-dimensional spinor helicity formalism to scattering amplitudes at one-loop in Chapter 4.

expressions in the context of the supersymmetric two-component description. To some extent, the two-component objects are therefore more fundamental, not least due to the fact that one may construct Dirac spinors by combining two Weyl spinors in four dimensions. In the following, we will present the spinor helicity formalism in the context of two-dimensional Weyl spinors.

We start with the fact that the complexified Lorentz group in four dimensions is locally isomorphic to (we omit a $\mathbb{Z}_2$ periodicity)

$$\mathsf{SO}(3,1,\mathbb{C}) \cong \mathsf{Sl}(2,\mathbb{C}) \times \mathsf{Sl}(2,\mathbb{C}) \; . \tag{3.16}$$

The finite dimensional representations are then classified by $(p,q)$ where $p$ and $q$ are integers or half-integers. In this context, the simplest non-trivial objects which transform under the complexified Lorentz group are the previously introduced two-component Weyl spinors. Conventionally, one assigns $\lambda_\alpha$ to a negative chirality Weyl spinor (for a massless theory equal to the helicity) which transforms in the $(\frac{1}{2},0)$ representation of the Lorentz group and $\tilde{\lambda}_{\dot{\alpha}}$ to a positive chirality Weyl spinor transforming in the $(0,\frac{1}{2})$ representation, where $\alpha = 1,2$ and $\dot{\alpha} = 1,2$. In a shortened matrix notation, one may represents these Weyl spinors in a bra-ket notation,

$$\begin{aligned} \langle\lambda| = \lambda^\alpha, \qquad & |\lambda\rangle = \lambda_\alpha \, , \\ [\lambda| = \tilde{\lambda}^{\dot{\alpha}} \qquad & |\lambda] = \tilde{\lambda}_{\dot{\alpha}} \, . \end{aligned} \tag{3.17}$$

In this context of the spinor helicity formalism, these objects are defined to be commuting spinors, in contrast to the usual two-component spinors describing spinor fields. This bra-ket notation can be used to conveniently define products of the spinors. Before doing so, we note that we may raise and lower the index of the $(\frac{1}{2},0)$ objects by using the antisymmetric tensor $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ as $\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta$ and $\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta$. Similarly relations hold for the $(0,\frac{1}{2})$ objects $\tilde{\lambda}_{\dot{\alpha}}$ with dotted indices. The epsilon tensor obeys $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_\gamma$ with

$$\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.18}$$

and acts as a metric in a two-dimensional spinor space. Using this metric we can define, given two negative chirality spinors $\lambda$ and $\mu$, the Lorentz invariant spinor product as

$$\langle\lambda,\mu\rangle \equiv \langle\lambda|\mu\rangle = \lambda^\alpha\mu_\alpha = \lambda^\alpha\epsilon_{\alpha\beta}\mu^\beta \; . \tag{3.19}$$

Due to the antisymmetry of the spinor indices we have $\langle\mu,\lambda\rangle = -\langle\lambda,\mu\rangle$. Obviously, a spinor product for the positive chirality objects is defined in a similar fashion. To distinguish it from the spinor product of the $(\frac{1}{2},0)$ representation, we uses the notation

$$[\tilde{\lambda},\tilde{\mu}] \equiv [\tilde{\lambda}|\tilde{\mu}] = \tilde{\lambda}_{\dot{\alpha}}\tilde{\mu}^{\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\mu}_{\dot{\beta}} \tag{3.20}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are two Weyl spinors of positive chirality and we have again $[\tilde{\mu},\tilde{\lambda}] = -[\tilde{\lambda},\tilde{\mu}]$. In addition, in the case of $\langle\lambda,\mu\rangle = 0$ (or $[\tilde{\lambda},\tilde{\mu}] = 0$) one finds that the two spinors are equal up to a complex scaling, i.e. $\mu_a = c\lambda_a$ with $c \in \mathbb{C}$, and similar for the positive chirality spinor product. In order to shorten the notation of the spinor product even further, we will use abbreviations like

$$\langle\lambda_i,\lambda_j\rangle \equiv \langle i\; j\rangle \;\text{ and }\; [\tilde{\lambda}_i,\tilde{\lambda}_j] \equiv [i\; j] \; . \tag{3.21}$$

Since both types of spinors are two-dimensional objects one might decompose them along two independent directions in spinor space. Denoting the basis spinors by $\lambda_i$ and $\lambda_j$ we can decompose any spinor $\lambda_k$ as

$$\lambda_{k\alpha} = a\lambda_{i\alpha} + b\lambda_{j\alpha} \tag{3.22}$$

with $a, b \in \mathbb{C}$ and $\langle ij\rangle \neq 0$. Therefore, the complex coefficients are given by

$$a = \frac{\langle jk\rangle}{\langle ji\rangle}\ , \qquad b = \frac{\langle ik\rangle}{\langle ij\rangle} \tag{3.23}$$

such that we can write for an arbitrary spinor $\lambda_k$

$$\lambda_{k\alpha} = \frac{1}{\langle ij\rangle}\left(\langle kj\rangle\lambda_{i\alpha} + \langle ik\rangle\lambda_{j\alpha}\right)\ . \tag{3.24}$$

Contracting again with a spinor $\lambda_l$ which is not a multiple of $\lambda_i, \lambda_j$ or $\lambda_k$ we arrive at the useful Schouten identities

$$\langle i\ j\rangle\,\langle k\ l\rangle + \langle j\ k\rangle\,\langle i\ l\rangle + \langle k\ i\rangle\,\langle j\ l\rangle = 0\ , \tag{3.25}$$

$$[i\ j][k\ l] + [j\ k][i\ l] + [k\ i][j\ l] = 0\ . \tag{3.26}$$

These relations are often used for spinor product manipulations.

In order to write Minkowski products of four-vectors in terms of spinor products we need to express objects like $p_\mu$ in terms of Weyl spinors. By considering the fact that the vector representation of the complex $\mathsf{SO}(1,3,\mathbb{C})$ is the $(\frac{1}{2},\frac{1}{2})$ representation we can express any four-vector as a product of spinors, a so called bi-spinor $p_{\alpha\dot\alpha}$. We use the chiral representation of the 4 dimensional $\gamma$ matrices and follow the usual convention of $+---$ as the metric's signature and write the $\gamma$ matrices as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \tag{3.27}$$

where $\sigma^\mu = (1, \vec\sigma)$ and $\bar\sigma^\mu = (1, -\vec\sigma)$. The $\sigma^i$ are the well known $\mathsf{SU}(2)$ generators, the Pauli matrices. Using the $\sigma$'s we can then write any four-vector $p^\mu$ as

$$\begin{aligned} p_{\alpha\dot\alpha} &= p_\mu \sigma^\mu_{\alpha\dot\alpha} \\ &= \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_3 & p_0 - p_3 \end{pmatrix}. \end{aligned} \tag{3.28}$$

Hence, one can express a vector $p^\mu$ as a $2\times 2$ matrix $p_{\alpha\dot\alpha}$. We might invert this relation by noting that $\sigma^\mu_{\alpha\dot\alpha}\sigma^{\alpha\dot\alpha}_\nu = 2\delta^\mu_\nu$ and find

$$p_\mu = \frac{1}{2}\sigma^{\alpha\dot\alpha}_\mu p_{\alpha\dot\alpha}. \tag{3.29}$$

This gives us an expression for the Minkowski product of 4-vectors. Using the relation $\sigma^\mu_{\alpha\dot\alpha}\sigma^{\beta\dot\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}$ one has

$$p^\mu p'_\mu = \frac{1}{2}p^{\alpha\dot\alpha}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}p'^{\beta\dot\beta} \tag{3.30}$$

where we write the last expression as $\det[p_{\alpha\dot\alpha}]$ after contraction of all indices. This leads to the important observation that the determinant of $p_{\alpha\dot\alpha}$ vanishes for lightlike 4-vectors. Since we are dealing with $2\times 2$ matrices this means rank $[p_{\alpha\dot{\alpha}}] \leq 1$ in the case of massless particles. This enables us to write every lightlike 4-momentum as a product of two Weyl spinors,

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \ . \tag{3.31}$$

The spinors $\lambda_\alpha$ and $\tilde{\lambda}_{\dot{\alpha}}$ are unique up to scaling by a complex number,

$$(\lambda, \tilde{\lambda}) \to (c\lambda, c^{-1}\tilde{\lambda}) \ \forall c \in \mathbb{C}, c \neq 0 \ . \tag{3.32}$$

In general, these two spinors are independent complex variables. However, this results in complex momenta $p_\mu$. Under the condition of working with real momenta (for Lorentz signature $+ - --$), we have to impose a conjugation condition on the spinors, $\tilde{\lambda} = \pm\bar{\lambda}$, i.e. the spinors are complex conjugates of each other. The sign in the above relation determines whether the corresponding null vector points into the positive lightcone ('future') or the negative one ('past'). It is a standard terminology to call the negative chirality spinor $\lambda$ the holomorphic and the positive chirality spinor $\tilde{\lambda}$ anti-holomorphic. As mentioned above, the representation of two Weyl spinors and the relation to each other depend on the chosen signature of the metric. For instance, it is also possible to choose them to be real and independent for the case of a metric with $(+ + --)$ signature.

In a final step we might generalise the relation (3.30) for Minkowski vector products. Given two 4-momenta $p^{(i)}_{\alpha\dot{\alpha}} = \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}$ and $p^{(j)}_{\alpha\dot{\alpha}} = \lambda_{j\alpha}\tilde{\lambda}_{j\dot{\alpha}}$ we can write the vector product as

$$p_i \cdot p_j = \frac{1}{2}\langle \lambda_i, \lambda_j\rangle [\tilde{\lambda}_j, \tilde{\lambda}_i] = \frac{1}{2}\langle i\ j\rangle [j\ i]. \tag{3.33}$$

We would like to stress that there are different conventions in the literature of how to define this vector product, differing by a sign. This choice is related to the convention of how to contract indices by the epsilon tensor. It is useful to introduce another convention when dealing with momentum vectors in a matrix representation. For an arbitrary null vector $p$ we define

$$\langle \lambda_i |\, p | \tilde{\lambda}'_l] \equiv \lambda_i^\alpha p_{\alpha\dot{\alpha}} \tilde{\lambda}'^{\dot{\alpha}}_l = \lambda_i^\alpha \lambda_{p\alpha} \tilde{\lambda}_{p\dot{\alpha}} \tilde{\lambda}'^{\dot{\alpha}}_l = \langle ip\rangle [pl] \ . \tag{3.34}$$

In general, $p$ can also be a sum over lightlike momenta, $\sum_i p_i$ and one writes in this case

$$\langle i| \sum_j p_j |l] = \sum_j \langle i\ j\rangle [j\ l]. \tag{3.35}$$

By rewriting the momenta in terms of strings of spinor brackets, the usual spinor manipulations can be applied to these objects.

Let us now turn back to the wavefunction $\psi_i$ that we mentioned at the beginning of this section. As described over there, one normally uses the polarisation vector $\epsilon^\mu$ to describe a spin 1 particle's wavefunction. Since we are interested in gauge boson descriptions, this is an appropriate choice. Obviously, one can choose different polarisation vectors for the same physical situation. The idea is that this choice is equivalent to specifying which Weyl spinor $\lambda$ we use for the description of a given 4-momentum $p_\mu$ (normally we want the momentum to be real and therefore only one spinor needs to be specified). One starts with a description of the wavefunction for a massless spin 1/2 particle by considering the Dirac equation for a spinor $\psi_a$ of negative chirality,

$$i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \psi^\alpha = 0. \tag{3.36}$$

Solutions for $\psi^\alpha$ are the plane waves $\xi^\alpha \exp(ix\cdot p)$ with momentum $p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$ if and only if $p_{\alpha\dot\alpha}\xi^\alpha = 0$. This implies that $\lambda$ and $\xi$ are related to each other by $\xi^\alpha = c\lambda^\alpha$ for $c \in \mathbb{C}, c \neq 0$. Therefore, the wavefunction of a negative helicity fermion is

$$\psi^\alpha = c\lambda^\alpha e^{ix_{\alpha\dot\alpha}\lambda^\alpha\tilde\lambda^{\dot\alpha}} \tag{3.37}$$

where we write for 4-vector $x_\mu = \sigma^{\alpha\dot\alpha}_\mu x_{\alpha\dot\alpha}$. This relation states the already mentioned fact that the additional information about the spinor $\lambda$ is carried by the corresponding wavefunction. Obviously, this also holds in the case of a massless particle with positive chirality, suggesting the form

$$\psi^{\dot\alpha} = c\tilde\lambda^{\dot\alpha} e^{(ix_{\alpha\dot\alpha}\lambda^\alpha\tilde\lambda^{\dot\alpha})} \ . \tag{3.38}$$

Similarly, one can construct a relation between the spinors describing the momentum of a spin 1 gauge boson and its polarisation vector [32], see also [25]. Choosing the momentum of the negative helicity boson to be $p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$, we write for the polarisation vector

$$\epsilon^-_{\alpha\dot\alpha} = \sqrt{2}\frac{\lambda_\alpha\tilde\mu_{\dot\alpha}}{[\tilde\lambda,\tilde\mu]} \ , \tag{3.39}$$

where $\tilde\mu$ is an arbitrary (but not a multiple of $\lambda$) spinor of positive chirality. Similarly, we take for a gauge boson of positive chirality an arbitrary spinor $\mu$ of negative chirality and construct

$$\epsilon^+_{\alpha\dot\alpha} = \sqrt{2}\frac{\mu_\alpha\tilde\lambda_{\dot\alpha}}{\langle\mu,\lambda\rangle} \ . \tag{3.40}$$

In general, polarisation vectors must fulfill the constraint $p_\mu\epsilon^\mu = 0$, i.e. stating that momentum and polarisation are always orthogonal to each other. Hence, the particle has no longitudinal polarisation states. We can explicitly check that this is fulfilled for our choice due to the fact that the spinor products $\lambda^\alpha\lambda_\alpha$ and $\tilde\lambda_{\dot\alpha}\tilde\lambda^{\dot\alpha}$ vanish (the polarisation is related to the spinors describing the momentum). Since the choice of the spinor $\mu$ or $\tilde\mu$ is almost arbitrary, this corresponds to the gauge freedom of the $SU(N)$ gauge theory as is discussed in [25]. In general, the polarisation vector of a gauge field transforms under a gauge transformation as

$$\epsilon'^\mu = \epsilon^\mu + \omega\, p^\mu \tag{3.41}$$

where $p^\mu$ is the associated momentum of the polarisation vector and $\omega$ is the transformation parameter. Since we are dealing with light-like particles, this transformation fulfills the transverse condition. If we now consider the arbitrariness of the choice of the reference spinor $\mu$, we observe that any change of this spinor is of the form

$$\mu'_\alpha = \mu_\alpha + \delta\mu_\alpha = \mu_\alpha + A\mu_\alpha + B\lambda_\alpha. \tag{3.42}$$

One can understand this relation by considering the fact that spinors are 2-dimensional objects. Hence, two spinors $\lambda$ and $\mu$ are a basis of this space if they are not multiples of each other and one might build linear combinations of these two spinors. The first part which scales with $A$ is just rescaling of the spinor $\mu_\alpha$ and hence - due to the definition of the polarisation vectors - leaves the vector $\epsilon^+$ unchanged. The other term, proportional to $B$, generates a change of the polarisation vector $\epsilon^\mu + \delta\epsilon^\mu$ of the form

$$\delta\epsilon^+_{\alpha\dot\alpha} = B\frac{\lambda_\alpha\tilde\lambda_{\dot\alpha}}{\langle\mu\ \lambda\rangle} \ . \tag{3.43}$$

Because of the fact that the gauge boson's momentum takes the form $p_{\alpha\dot{\alpha}}$ we see that this shift of the polarisation vector is proportional to the momentum, hence fulfilling the gauge transformation property of $\epsilon^{\mu}$. Obviously, this also applies to the polarisation vector $\epsilon^{-}_{\mu}$. We just have to apply a change to the spinor variable $\tilde{\mu}$ of the same form. This yields a change $\delta\epsilon^{-}_{\alpha\dot{\alpha}}$, being proportional to the momentum of the negative helicity gauge boson.

We conclude this section by introducing some useful relations between the polarisation vectors. Since products of spinors with themselves vanish we easily see (using 4-vector notation)

$$\epsilon^{+}(p,q)\cdot\epsilon^{+}(p,q) = 0 = \epsilon^{-}(p,q)\cdot\epsilon^{-}(p,q) \tag{3.44}$$

In addition we find relations between the polarisation vectors for different gauge boson momenta

$$\epsilon^{+}(p,q)\cdot\epsilon^{+}(p',q) = \quad 0 \quad = \epsilon^{-}(p,q)\cdot\epsilon^{-}(p',q), \tag{3.45}$$

$$\epsilon^{+}(p,q)\cdot\epsilon^{+}(p,q') = \quad 0 \quad = \epsilon^{-}(p,q)\cdot\epsilon^{-}(p,q'). \tag{3.46}$$

Furthermore, one can shown that

$$\epsilon^{+}(p,q=p')\cdot\epsilon^{-}(p',q) = 0 = \epsilon^{-}(p,q=p')\cdot\epsilon^{+}(p',q). \tag{3.47}$$

These relations between the polarisation vectors are useful if one wants to calculate scattering amplitude 'by hand', using Feynman rules. However, these vector products can also be related to spinor products since they are just 'ordinary' 4-vectors. In addition, we notice that by choosing the reference momenta $q_i^{\mu}$ of all gauge bosons to be the same, this can simplify the structure of products of polarisations vectors and, hence, the final form of the corresponding amplitude. Especially the last relation suggests to choose the $q_i^{\mu}$ not only equal to each other but also equal to one of the external gauge bosons of an opposite helicity.

### 3.1.3 MHV tree-level amplitudes

The machinery developed in the last section allows us to focus on a reduced set of problems, namely to calculate the partial amplitudes for a given process within the spinor helicity formalism. In the following we would like to give some simple examples of scattering amplitudes of $\mathsf{SU}(N)$ gauge bosons. In general, we are interested in processes involving $n$ of these particles as external states. In our discussions we will use the convention of labeling the external particle (here gauge bosons) as *incoming states*. Hence, all given quantum numbers such as momentum and chirality (which sometimes we will also refer to as the particle's helicity due to its zero mass) are given according to this convention. All the presented information can be found at various places in the literature, for instance a detailed review is given in [27].

Let us start with the simplest case and consider the amplitude $A_n(1^+,2^+,\ldots,n^+)$ with $n>3$, i.e. the process where all gluons are incoming and have the same helicity. The case of $n=3$ needs special treatment as we will see further down in our discussions. According to our convention, this is the case for a scattering process $++\rightarrow -\ldots-$ with two incoming and n-2 outgoing bosons (whichever they may be). For this class of amplitudes one has

$$A_n(1^+,2^+,\ldots,n^+) = 0 \tag{3.48}$$

for arbitrary $n$. To see this, let us choose the reference momentum for all boson polarisation vectors to be the same null vector $q_{\alpha\dot{\alpha}} = \mu_\alpha \tilde{\mu}_{\dot{\alpha}}$ [27]. Hence, all products of polarisation vectors $\epsilon_i^+ \cdot \epsilon_j^+$ are equal to zero. The choice of the arbitrary momentum $q$ is almost 'free' since we cannot choose $q$ to be equal to one of the external boson's momentum because then the spinor product in the polarisation vector's denominator would vanish. At the level of the partial amplitude, every interaction vertex of gauge bosons comes with maximal one momentum vector $k_i$ (which might be a combination of external momenta). For an scattering of $n$ gauge bosons there is a maximum of $n-2$ interaction vertices (coming from the 3-point interaction of the gauge bosons - an insertion of a 4-point interaction increases the number of the gauge bosons states). There are in total $n$ polarisation vectors $\epsilon_i$ of the $n$ external states. Since the total amplitude is a scalar quantity all vectors must be contracted with each other. We have $n-2$ momentum vectors to be contracted with $n$ polarisation vectors, which means that two polarisation vectors must be contracted with each other, yielding a factor $\epsilon_i \cdot \epsilon_j$. By our choice of the reference vector $q$ this product vanishes and so does the whole amplitude.

Similarly, we can deal with the class of amplitudes like $A_n(1^-, 2^+, \ldots, n^+)$. We can arrange that one contraction of the polarisation vectors vanishes by choosing $q_2 = q_3 = \cdots = q_n = p_1$ and $q_1 = p_n$ which leads to a vanishing amplitude. Obviously, we can rearrange the amplitude to obtain $A_n(1^+, 2^+, \ldots, r^- \ldots, n^+)$ from the previous case. Here, it does not matter which boson carries the negative helicity because we can rearrange its position by cyclic symmetry of the amplitude. Hence, we have discussed that certain classes of scattering amplitudes vanishes, namely

$$A_n(1^+, 2^+, \ldots, r^\pm, \ldots, n^+) = 0 \tag{3.49}$$

for arbitrary $n$ and some $r$ in the range $1 \leq r \leq n$.

We now turn to the first non-vanishing partial amplitude, namely the case where the helicity of two of the incoming gauge bosons are different from the other helicities, i.e. we consider the amplitude $A_n(1^+, 2^+, \ldots, r^-, \ldots, s^-, \ldots, n^+)$ with $1 \leq r \leq s \leq n$. If we denote the total helicity of an amplitude by $h_{tot}$ then this type of amplitudes has $h_{tot} = n - 4$. And since amplitudes with $h_{tot} = n, n-2$ vanish, the helicity changes in such a scattering process by the maximal possible amount. Hence, this type of amplitudes is called *maximally helicity violating* (MHV). It turns out that these objects are on the one hand fundamental (first non-vanishing gauge boson amplitudes) but on the other hand also very powerful in describing more complicated interactions. Obviously, we can also consider the 'reversed' MHV amplitude where most of the gauge bosons are negative helicity states, $A_n(1^-, 2^-, \ldots, r^+, \ldots, s^+, \ldots, n^-)$. Conventionally, these are called $\overline{\text{MHV}}$ or anti-MHV amplitudes.

In both cases, these amplitudes take a surprisingly simple form, considering the fact that they are valid for an arbitrary number of external gauge boson states. Their form in terms of spinor products was first conjectured by Parke and Taylor [35] and was subsequently proven by Berends and Giele [36]. The Parke-Taylor formula takes the form[5]

$$A_n(1^+,2^+,\ldots,r^-,\ldots,s^-,\ldots,n^+) = ig^{n-2}\frac{\langle r\ s\rangle^4}{\prod_{j=1}^n \langle j\ j+1\rangle},\ \ n+1\equiv 1 \tag{3.50}$$

in the case of a mostly plus MHV amplitude (we have omitted a general delta function stating the momentum conservation within the scattering process). Here, $g$ is the YM coupling constant (the power $n-2$ states the fact that we have $n-2$ interaction vertices at tree level). For the $\overline{\text{MHV}}$ amplitudes we have

$$A_n(1^-,2^-,\ldots,r^+,\ldots,s^+,\ldots,n^-) = ig^{n-2}(-1)^n\frac{[r\ s]^4}{\prod_{j=1}^n [j\ j+1]},\ \ n+1\equiv 1. \tag{3.51}$$

It is remarkable that the amplitudes take such a simple form. They are just functions of holomorphic (MHV) or anti-holomorphic (anti-MHV) spinor products with a sequence of spinor products in the denominator.

So far, our discussion is valid for $n$ particle states with $n>3$. Let us now turn to the special case of $n=3$. The tree-level amplitude with just three external gauge boson states is the simplest possible one, based on just a three-point interaction between the particles as shown in Figure 3.1. It is interesting to note that this simplest amplitude is somewhat special, as we will see[6]. If one denotes the momenta of the three particles by $p_1, p_2$ and $p_3$ we have the following kinematical constraints

$$\begin{aligned} 0 =& p_1^2 = 2p_2\cdot p_3 = \langle 23\rangle[32]\ , \\ 0 =& p_2^2 = 2p_1\cdot p_3 = \langle 13\rangle[31]\ , \\ 0 =& p_1^2 = 2p_1\cdot p_2 = \langle 12\rangle[21]\ . \end{aligned} \tag{3.52}$$

For real momenta, the positive and negative helicity spinors are related by the constraints $\tilde\lambda = \pm\bar\lambda$. Hence, all spinor products vanish for real momenta and so does the three-point amplitude,

$$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = [12] = [23] = [31] = 0\ . \tag{3.53}$$

Relaxing the constraint of real momenta we find that this is not the case anymore. Since the holomorphic and anti-holomorphic spinors are independent for complex momenta, only the $\lambda_i$ or the $\tilde\lambda_i$ are proportional to each other for the three-point case. Hence, one can choose one of the conditions

$$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0\ , \tag{3.54}$$

$$[12] = [23] = [31] = 0\ , \tag{3.55}$$

which follow individually from momentum conservation. Note that they do not mix holomorphic with anti-holomorphic spinors. We can choose the anti-holomorphic spinors to vanish and see that this results in an non-vanishing amplitude with two negative and one positive helicity gluon is non zero. This is of MHV type. Alternatively, choosing the first relation in (3.54) leads to a non-vanishing anti-MHV amplitude.

[5]The gauge coupling is explicitly shown here. In our discussions, we will mostly omit this factor.

[6]For gauge bosons, the speciality is just the fact that one needs complex momenta to define the three-point amplitude. This is different in the case of superamplitudes in four and six dimensions where the three-point amplitude takes a rather 'special form' as we will see later.

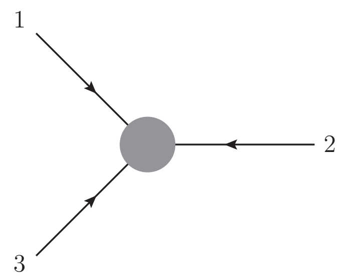



Figure 3.1: *A generic three-point amplitude with all momenta defined to be incoming.*

We can see this also from applying the usual colour-ordered Feynman rules for a three gauge boson vertex. For example the amplitude $A_3(1^-, 2^-, 3^+)$ is given by [37]

$$A_3 = \frac{i}{\sqrt{2}}\big[\epsilon_1^- \cdot \epsilon_2^- \epsilon_3^+ \cdot (p_1 - p_2) + \epsilon_2^- \cdot \epsilon_3^+ \epsilon_1^- \cdot (p_2 - p_3) + \epsilon_3^+ \cdot \epsilon_1^- \epsilon_2^+ \cdot (p_3 - p_1)\big] \ . \tag{3.56}$$

One might use the freedom of choosing the reference momenta of the polarisation vectors such that $q_1 = q_2$ and $q_3 = p_1$. Then, only one of the terms of amplitude is non-zero and rewriting the four-vector quantities into spinor objects yields

$$\begin{aligned} A_3(1^-, 2^-, 3^+) &= i\sqrt{2}\epsilon_2^- \cdot \epsilon_3^+ \epsilon_1^- \cdot p_2 \\ &= i\frac{[q_1 3]}{[q_1 2]}\frac{\langle 12\rangle}{\langle 13\rangle}\frac{[q_1 2]\langle 21\rangle}{[q_1 1]} = i\frac{[q_1 3]}{[q_1 1]}\frac{\langle 12\rangle^2}{\langle 31\rangle}\frac{\langle 32\rangle}{\langle 32\rangle} \\ &= i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \ . \end{aligned} \tag{3.57}$$

We arrived at this result by using normal spinor manipulations and momentum conservation. All intermediate factors cancel out. This short calculation confirms the general structure of the MHV amplitudes even for three gluons. A similar calculation, based on the assumption that the holomorphic spinors are proportional to each other, yields the corresponding anti-MHV amplitude. The approach outlined above can be used to derive all tree-level amplitudes. The calculation becomes more involved for increasing number of external particles since it is based on the Feynman rules. In the next section we will see how one can does better in terms of efficient calculations.

We conclude this section by collecting the obtained results for tree-level $n$-point gluon scattering amplitudes[7], see also [27]:

$$\begin{aligned} A_n(1^+, \ldots, r^\pm, \ldots, n^+) &= 0 \\ A_n(1^-, \ldots, r^\mp, \ldots, n^-) &= 0 \\ A_n(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) &= i\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} \\ A_n(1^-, \ldots, i^+, \ldots, j^+, \ldots, n^-) &= i(-1)^n\frac{[ij]^4}{[12][23]\ldots[n1]} \ . \end{aligned} \tag{3.58}$$

[7] We omit the coupling constant and the momentum conservation delta function.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| # of diagrams | 4 | 25 | 220 | 2485 | 34300 | 559405 |

Table 3.1: *The table shows the number of Feynman diagrams that have to be taken into account in a scattering process of $gg \to n \times g$ as presented in [32]. Here, $n$ stands for number of outgoing particles, so in total $n+2$ gauge bosons are involved in the scattering.*

## 3.2 Novel techniques for perturbative calculations

In principle, interactions of an arbitrary number of $\mathsf{SU}(N)$ gauge bosons can be described by considering all Feynman diagrams, calculating their individual contribution and summing over all diagrams. The techniques and tools that we have described in the last section simplify this task (and can reduce the number of diagrams that need to be calculated). However, the more particles contribute to the scattering process the more diagrams have to be considered and hence, the calculations get more complicated, see for instance Table 3.1 for an example of this fact. It would be highly appreciated if one could determine the form of an amplitude with a certain number of external states from an amplitude with less external particles, i.e. a recursive structure of the scattering amplitudes would be of great advantage. It turns out that it is indeed possible to construct such recursion relations. In this section we introduce some of the concepts that have been developed within the last few years for scattering amplitudes in $\mathsf{SU}(N)$ gauge theories, with or without supersymmetry.

A first step in this direction dates back to the 1980s. In [36], Berends and Giele (BG) introduced their recursion relation for scattering processes at tree-level involving an arbitrary number of gluons. The main idea which is also continued in other recursion relations is the possibility to use an *off-shell description* for gauge bosons. The result is the generation of tree-level amplitudes recursively in the number of legs.

For these BG recursion relations one introduces an off-shell gauge boson current $J^\mu(1,2,\ldots,n)$ which we define to be colour ordered. The current itself has $n+1$ external legs where the legs with $p_1, p_2, \ldots, p_n$ are external on-shell gauge boson states and leg $n+1$ (denoted by leg $\mu$) is taken off-shell. One can think of the current as the partial amplitude $A_{n+1}(1,\ldots,n,n+1)$ where the polarisation vector for leg $\mu$ is replaced by an off-shell propagator, i.e. $p_{n+1} \neq 0$. However, one still requires total momentum conservation. The off-shell leg is defined to be included in the current $J^\mu$.

The idea is now to construct a recursion for $J^\mu$. We start at the off-shell leg, follow it into the diagram and arrive at a three- or four-point gluon interaction vertex. In both cases, we have off-shell (two or three) internal propagators branching out from this vertex, approaching a subdiagram with less external on-shell gluons. We may than apply the same steps to each subdiagram, i.e. following the off-shell leg into a three or four-point vertex. This procedure yields the following form of the

Berends-Giele current [36]

$$J^{\mu}(1,\dots,n) = -\frac{i}{P_{1,n}^2}\Bigg[\sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i},P_{i+1,n})\; J_\nu(1,\dots,i)\; J_\rho(i+1,\dots,n) \tag{3.59}$$
$$+\sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2} V_4^{\mu\nu\rho\sigma}\; J_\nu(1,\dots,i)\; J_\rho(i+1,\dots,j)\; J_\sigma(j+1,\dots,n)\Bigg]$$

Here, we have used the colour ordered three- and four-point self-interactions of the gauge bosons, $V_3$ and $V_4$ (with appropriate tensor structure, for their concrete form see e.g. [27]) and the $P_{i,j}$ are defined as the sum of consecutive momenta $p_i + \cdots + p_j$.

One can see the recursive approach in the structure of the current $J^\mu$. Starting with the off-shell propagator we have one sum for a possible three-point interaction and another sum for the four-point vertex. Once we have constructed $J^\mu$, depending on $n$ on-shell and one off-shell leg, we can obtain the $(n+1)$-point amplitude $A_{n+1}$ in the following way: We amputate the off-shell leg by multiplying with the inverse propagator $iP_{1,n}^2$ and take care of the Minkowski index $\mu$ by contracting the current with an appropriate polarisation vector $\epsilon_{n+1}^\mu$. We take the off-shell leg to be on-shell by letting $P_{1,n} = -p_{n+1}$ and $P_{1,n}^2 \to 0$. Letting $n+1 \to n$ yields

$$A_n(1,2,\dots,n) = \left( iP_{1,n-1}^2\; J_\mu(1,2,\dots,n-1)\; \epsilon_n^\mu(p_n) \right)\Bigg|_{p_n^2\to 0}. \tag{3.60}$$

In some cases, these recursions can be solved in a closed form and lead to a answers for the partial amplitudes. It is interesting to note that the Berends-Giele recursion relations can be used to prove the Parke-Tayler forumla for MHV/anti-MHV scattering amplitudes. Although these relations fulfill a striking recursive structure, they still suffer from rather long calculations, leading to rather unhandy expressions. In the following we present methods that are generic and lead to more compact expressions.

### 3.2.1 CSW construction

A novel diagrammatic approach to calculate tree-level amplitudes, initially constructed for scattering of gauge bosons, was introduced by Cachazo, Svrcek and Witten in [38]. The authors introduced a prescription of how to construct amplitudes in a recursive fashion by using MHV tree-level amplitudes as 'building blocks' of the total amplitude. Their construction was subsequently proven in [39] by using a generalisation of the BCFW recursion relations which we will introduce in the next section.

The CSW construction is based on the duality between the weakly gauged $\mathcal{N}=4$ YM theory and a string theory in super twistor space $\mathbb{CP}^{3|4}$ as proposed by Witten in [25]. Based on the twistor space ideas originally introduced by Penrose in the late 1960s [40], it was discussed that MHV amplitudes localise on degree one, genus zero curves in twistor space, i.e. on lines. This statement is equivalent to the fact that MHV amplitudes are only functions of the holomorphic and not of the anti-holomorphic spinors. It is than an interesting fact that lines in twistor space map to points in Minkowski spacetime [40]. This makes it somewhat natural to think of a MHV amplitude as a local interaction in four-dimensional spacetime which leads to the central idea of the CSW construction: Any tree-level amplitude can be build out of MHV diagrams interpreted as interaction vertices and then connected by internal propagators. This setup corresponds in twistor space to the geometrical picture of two intersecting lines where the intersection point corresponds to the internal gauge boson.

The MHV amplitudes are evaluated by the means of the Parke-Taylor formula where the internal gauge boson propagators carry a positive helicity label on one side and the opposite label on the other side, according to the fact that all subamplitudes are MHV. However, we face the problem that the MHV amplitude in (3.50) is defined for on-shell spinors only and a general internal propagator is given by an off-shell momentum with $P^2 \neq 0$. For the momenta not being light-like, an application of the spinor helicity formalism seems to fail since it is not clear what we mean by a spinor $\lambda_\alpha$ if the corresponding momentum $p_{\alpha\dot\alpha}$ is not light-like. This can be solved by observing that any off-shell vector can be decomposed as [41]

$$P = l + zp \tag{3.61}$$

where $l$ and $p$ are null vectors and $z$ is a real number. It follows that $z$ is given as a function of $l$ since

$$z = \frac{P^2}{2l \cdot p} \ . \tag{3.62}$$

In spinor notation we may write for $l_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$ and $p_{\alpha\dot\alpha} = \eta_\alpha \tilde\eta_{\dot\alpha}$. By considering $\tilde\eta^{\dot\alpha} P_{\alpha\dot\alpha}$ we obtain

$$\lambda_\alpha = \frac{P_{\alpha\dot\alpha}\, \tilde\eta^{\dot\alpha}}{[\tilde\lambda, \tilde\eta]} \ . \tag{3.63}$$

A similar relation holds for the anti-holomorphic spinor $\tilde\lambda_{\dot\alpha}$. Since the spinor products are simply scalar quantities we might rescale the anti-holomorphic spinors and define the negative helicity spinor $\lambda_\alpha$ of an arbitrary off-shell vector $p^\mu$ as

$$\lambda_\alpha = p_{\alpha\dot\alpha}\, \tilde\eta^{\dot\alpha} \ . \tag{3.64}$$

Thus, one can define MHV amplitudes according to the Parke-Taylor expression with external off-shell legs. Since the choice of $\tilde\eta$ is arbitrary, one has to use the same spinors $\eta$ and $\tilde\eta$ for all internal off-shell lines. One then has to calculate all contributing MHV diagrams and sum in a final step over all of them. The dependence on the fixed reference spinor $\tilde\eta$ drops out in the final expression, i.e. the sum over all possible diagrams contributing to the total amplitude will not depend on the choice of the arbitrary spinor. In addition it was shown that the total amplitude is indeed Lorentz covariant (the interested reader finds more information in section 5 of [38]).

**The Yang-Mills amplitude $A_4(1^+, 2^-, 3^-, 4^-)$ as an example**

The CSW prescription is as easy as it is powerful. To see some applications of the construction it is useful to consider a concrete example. In the following we calculate the four-point amplitude $A_4(1^+, 2^-, 3^-, 4^-)$ by the means of the CSW rules. The amplitude has three negative-helicity gluons and vanishes due to our discussions in section 3.1.3.

The number of vertices is $v = q - 1$ where $q$ is the number of external gauge bosons with negative helicity. Hence, in our example $v = 2$ and we have to consider two distinct MHV diagrams. They

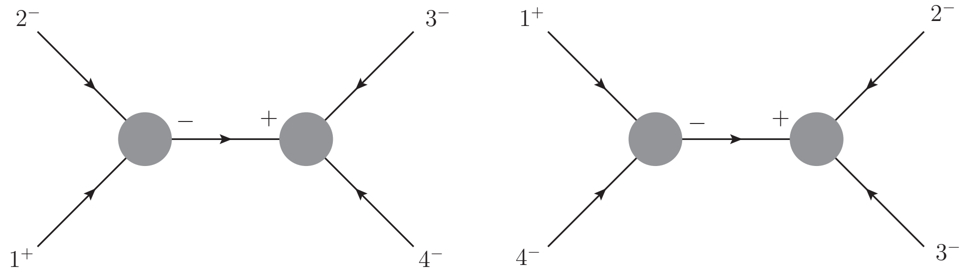



Figure 3.2: *The two contributing diagrams in the CSW construction of the amplitude $A_4(1^+,2^-,3^-,4^-)$. All vertices in the diagrams are of the MHV type.*

are shown in Figure 3.2. Denoting the momentum of the internal propagator by $q$ we start by considering the s-channel diagram, shown on the left in the figure. We have $q = (p_1 + p_2) = -(p_3 + p_4)$ and choosing an arbitrary spinor $\tilde{\eta}_{\dot{\alpha}}$ we define the holomorphic spinor of q as

$$\begin{aligned}\lambda_{q\alpha} = q_{\alpha\dot{\alpha}}\tilde{\eta}^{\dot{\alpha}} &= (p_{1\alpha\dot{\alpha}} + p_{2\alpha\dot{\alpha}})\,\tilde{\eta}^{\dot{\alpha}} \\ &= (\lambda_{1\alpha}\tilde{\lambda}_{1\dot{\alpha}} + \lambda_{2\alpha}\tilde{\lambda}_{2\dot{\alpha}})\,\tilde{\eta}^{\dot{\alpha}} = \lambda_{1\alpha}[1,\tilde{\eta}] + \lambda_{2\alpha}[2,\tilde{\eta}]\ . \end{aligned} \tag{3.65}$$

Similarly, we get

$$\lambda_{q\alpha} = -\lambda_{3\alpha}[3,\tilde{\eta}] - \lambda_{4\alpha}[4,\tilde{\eta}]\ . \tag{3.66}$$

Following the CSW prescription, the s-channel diagram gives

$$A_s = \frac{\langle 2q\rangle^3}{\langle 12\rangle\langle q1\rangle}\,\frac{1}{q^2}\,\frac{\langle 34\rangle^3}{\langle 4q\rangle\langle q3\rangle}$$

where we evaluate

$$\begin{aligned}\langle 2q\rangle &= \lambda_{2\alpha}\lambda_{q\beta}\epsilon^{\alpha\beta} = \langle 21\rangle\,[1\tilde{\eta}]\ , \\ \langle q1\rangle &= \langle 21\rangle\,[2\tilde{\eta}]\ . \end{aligned} \tag{3.67}$$

The spinor products involving $\lambda_3$ and $\lambda_4$ can be rewritten in a similar way and we obtain

$$A_s = -\frac{[1\tilde{\eta}]^3}{[2\tilde{\eta}][3\tilde{\eta}][4\tilde{\eta}]}\,\frac{\langle 21\rangle^3}{\langle 21\rangle\langle 12\rangle}\,\frac{1}{q^2}\,\frac{\langle 34\rangle^3}{\langle 43\rangle\langle 43\rangle}. \tag{3.68}$$

Finally, rewriting the propagator in terms of spinors as $q^2 = 2p_1\cdot p_2 = \langle 12\rangle\,[21]$ gives

$$A_s = \frac{[1\tilde{\eta}]^3}{[2\tilde{\eta}][3\tilde{\eta}][4\tilde{\eta}]}\,\frac{\langle 34\rangle}{[12]}. \tag{3.69}$$

A similar construction holds for the t-channel diagram where $q = (p_1 + p_4) = -p_2 + p_3$. Following the CSW prescription (or just using crossing symmetry with $2 \longleftrightarrow 4$) yields

$$A_t = \frac{[1\tilde{\eta}]^3}{[4\tilde{\eta}][3\tilde{\eta}][2\tilde{\eta}]}\,\frac{\langle 32\rangle}{[14]}. \tag{3.70}$$

We now sum over both contributions and find

$$\begin{aligned}A_4(1^+,2^-,3^-,4^-) &= \frac{[1\tilde{\eta}]^3}{[2\tilde{\eta}][3\tilde{\eta}][4\tilde{\eta}]}\left(\frac{\langle 34\rangle}{[12]} + \frac{\langle 32\rangle}{[14]}\right) \\ &= \frac{[1\tilde{\eta}]^3}{[2\tilde{\eta}][3\tilde{\eta}][4\tilde{\eta}]}\left(\frac{[12]\,\langle 32\rangle + [14]\,\langle 34\rangle}{[14][12]}\right). \end{aligned} \tag{3.71}$$

Now, using momentum conservation we see that $[12]\langle 23\rangle + [14]\langle 43\rangle = [1|\sum_j p_j|3\rangle = 0$ and hence, the amplitude $A_4(1^+, 2^-, 3^-, 4^-)$ vanishes.

In a similar fashion, one can construct arbitrary tree-level amplitudes, not only limited to the MHV or anti-MHV case. Obviously, the number of diagrams one has to calculate increases with the number of external legs. However, for $n$ external states the number of diagrams grows at most as $n^2$. For instance, in the case of the five-point anti-MHV amplitude $A_5(1^-, 2^-, 3^-, 4^+, 5^+)$ four different diagrams have to be considered and six diagrams contribute to the next-to-MHV amplitude $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. This is still a remarkable improvement compared with the more than 200 diagrams if one follows the Feynman rules approach [28]. Successively, the CSW ideas of an off-shell continuation have been applied to many cases of tree-level scattering amplitudes, not only limited to pure gluon case, see for instance the applications in [42–48]. For instance, the inclusion of massless scalar or fermions is straight forward. If one includes quarks, some of the internal propagators may become fermionic. As long as we stick to the CSW prescription (all vertices of MHV type, final summation over all including diagrams), the prescription works also in these cases.

We conclude this section by a comment about the validity of the CSW construction at the quantum level. So far, the MHV diagram method was discussed for tree-level amplitudes. Initial considerations for the duality of the $\mathcal{N} = 4$ SYM theory and the twistor string theory showed that conformal supergravity enters the game from the loop-level on which can not be decoupled [49]. In the remarkable paper [50], Brandhuber, Spence and Travaglini (BST) showed that an application of the CSW idea to one-loop amplitudes in $\mathcal{N} = 4$ SYM gives the correct $n$-point MHV amplitude of [31]. Following the BST approach, applications of the CSW idea were successfully applied to amplitudes in theories with less supersymmetry ($\mathcal{N} = 1$ SYM) in [51] and the case of pure YM theory in [52], highlighting the remarkable properties of this off-shell recursive prescription.

### 3.2.2 BCFW recursion relations

The previously discussed CSW prescription can be used to calculate arbitrary tree-level amplitudes of gauge bosons. However, it uses an off-shell continuation for the internal boson legs. One might ask if we can do better by only using on-shell information. Indeed, there is an on-shell prescription based on the work of Britto, Cachazo and Feng (BCF) presented in [53] and proven by the previous authors in collaboration with Witten in [54]. Therefore, this construction is normally referred to as the BCFW recursion relations. Their construction is based on two fundamental properties of scattering amplitudes at tree-level. Firstly, analyticity of the amplitude [55] and secondly the factorisation properties of a general tree-level amplitude on multi-particle poles. Hence, the BCFW relations can find applications in many different contexts within perturbative field theories.

A key observation is the fact that one can consider a general scattering amplitude as an analytic function of complex variables by introducing complex momenta. Complex analysis provides a powerful set of theorems that one can apply to analytic functions in order to obtain information about their properties. To make use of these theorems it is necessary to express the amplitude as a function of one complex variable $z$ only. It was the main observation of [54] that this can be done in terms of complex momenta by introducing momentum deformations for two of the external states. If one labels these momenta by $k$ and $l$, the deformation is written in terms of shifts in the spinor variables as

$$\tilde{\lambda}_k \longrightarrow \tilde{\lambda}_k(z) = \tilde{\lambda}_k + z\tilde{\lambda}_l \ , \qquad \lambda_l \to \lambda_l(z) = \lambda_l - z\lambda_k \tag{3.72}$$

where $z$ is the only complex parameter of this $[kl\rangle$-shift. The spinors $\lambda_k$ and $\tilde{\lambda}_l$ as well as all spinors belonging to the remaining external momenta are unchanged. This means to shift the momenta $p_l$ and $p_k$ according to

$$p_k \longrightarrow p_k(z) = \lambda_k\tilde{\lambda}_k + z\lambda_k\tilde{\lambda}_l \ , \quad p_l \longrightarrow p_l(z) = \lambda_l\tilde{\lambda}_l - z\lambda_k\tilde{\lambda}_l \ . \tag{3.73}$$

We immediately see that this complex deformation is not effecting the total momentum $P = \sum_i p_i$ since $p_k(z) + p_l(z) = p_k + p_l$ and so $P$ is still conserved. Denoting the shift of the momenta $p_l$ and $p_k$ by a bi-spinor $\rho = \lambda_k\tilde{\lambda}_l$, we see that $\rho^2 = 0$ and $p_k(z)\cdot\rho = p_l(z)\cdot\rho = 0$ and hence, the shifted momenta are on-shell. Furthermore, since the unshifted amplitude is a rational function in the spinor products and the $z$-dependence of the amplitude enters only through the shifts in two spinors as rational functions, the amplitude $A_n(z)$ is a rational function in $z$ with

$$A(z) = A_n(p_1, p_2, \ldots, p_k(z), \ldots, p_l(z), \ldots, p_n) \ . \tag{3.74}$$

Notice that a priori $p_k$ and $p_l$ do not need to be adjacent legs.

An important property of $A(z)$ is the fact that it has only simple poles as a rational function of $z$. This follows from the factorisation properties of the amplitude on multi-particle or collinear singularities, for a discussion see for instance [32]. Multi-particle singularities of diagrams arise when an internal propagator goes on-shell, i.e. $P_{i,j}^2 \to 0$ where $P_{i,j}$ is the momentum flowing in this channel and is the usual sum of adjacent particle momenta $p_i + \cdots + p_j$. Under the shifts the spinors of the internal propagator's momentum may become $z$-dependent, $P_{i,j}(z)$. This depends on the shifted legs. If none or both of the $z$-dependent legs fall into the range $i, j$, the dependence is not existent or cancels out explicitly. Only in the case where a single leg belongs to the range of momenta we have a $z$-dependent propagator $1/P_{i,j}^2(z)$. For instance by letting $p_l$ be in the range of $p_i, \ldots p_j$ we have $P_{i,j}(z) = P_{i,j} - z\lambda_k\tilde{\lambda}_l$. One might square this expression and solve for $P_{i,j}^2(z) \to 0$. Since $P_{i,j}^2(z)$ is linear in $z$ as $P_{i,j}^2(z) = P_{i,j}^2 - z(P_{i,j})_{\alpha\dot{\alpha}}\,\lambda_k^\alpha\tilde{\lambda}_l^{\dot{\alpha}}$ this yields in the on-shell limit

$$z \to z_P = \frac{P_{i,j}^2}{(P_{i,j})_{\alpha\dot{\alpha}}\,\lambda_k^\alpha\tilde{\lambda}_l^{\dot{\alpha}}} = \frac{P_{i,j}^2}{\langle k|P_{i,j}|l]} \ , \tag{3.75}$$

where $z_P$ is the value of $z$ at the propagators pole. For a general momentum configuration of the external states the different poles in $z_P$ are distinct for different pairs $i, j$. This proves the statement that the general amplitude $A_n(z)$ has simple poles in the complex variable. We are now ready to apply the standard theorems of complex analysis.

The BCFW recursion relations emerge from the application of Cauchy's theorem. To that extent we consider the integral of the amplitude over a closed contour $\mathcal{C}$ at infinity

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} dz\ \frac{A(z)}{z} \tag{3.76}$$

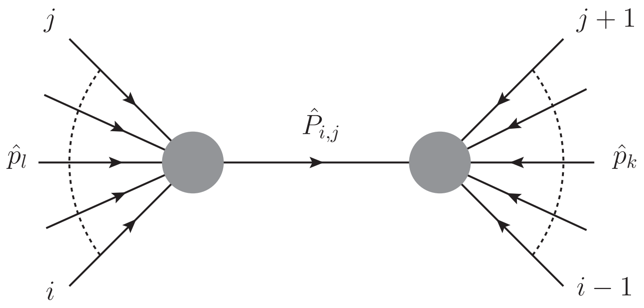



Figure 3.3: *A recursive diagram in the limit of $P^2_{i,j} \to 0$ which the BCFW relations are build upon. We denote the shifted momenta by $\hat{p}_l$ and $\hat{p}_k$ which lay in the range of $i \leq l \leq j$ or $j+1 \leq k \leq i-q$, respectively. In order to extract the full $n$-point amplitude one has to sum over all possible distributions of $n$ momenta such that $\hat{p}_l$ and $\hat{p}_k$ are always on opposite sites of the internal on-shell propagator.*

and $A(z)$ contains only poles and no branch cuts in the complex plane. Hence, the integrand in (3.76) contains all physical poles plus the pole at $z = 0$. Under the requirement that $A(z)$ vanishes for $z \to \infty$ we find that the whole integral vanishes. Hence, Cauchy's theorem implies

$$0 = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \, \frac{A(z)}{z} = A(0) + \sum_{\text{poles } z_p} \text{Res} \left\{ \frac{A(z)}{z} \right\} . \tag{3.77}$$

The sum on the RHS runs over all poles $z_p$ of the function $A(z)/z$ and the residue of pole at $z_p = 0$ is the amplitude $A(0)$, i.e. exactly the object one would like to calculate recursively,

$$A_n(p_1, \ldots, p_n) = A(z = 0) = - \sum_{\text{poles } z_p} \text{Res} \left\{ \frac{A(z)}{z} \right\} . \tag{3.78}$$

One can then evaluate the sum over the poles in the following way: As we approach a pole in the complex plane, $z \to z_P$, we have $P^2_{i,j} \to 0$ and the internal propagator becomes on-shell, hence dividing the total amplitude into two physical subamplitudes,

$$A(z) \to \sum_{h=\pm 1} A^h_L(z_P) \frac{1}{P^2_{i,j}(z)} A^{-h}_R(z_P) \qquad \text{as } z \to z_P. \tag{3.79}$$

Here, we have to sum over the possible helicity assignments of the internal gluon propagator. With the help of (3.75) we can substitute $P^2_{i,j}(z)$ and arrive at

$$A(z) \to - \sum_{h=\pm 1} \frac{A^h_L(z_P) A^{-h}_R(z_P)}{(z - z_p) \, \langle \lambda_k | \, P_{i,j} | \tilde{\lambda}_l]} \qquad \text{as } z \to z_P \; . \tag{3.80}$$

Then, the residue of the amplitude at a pole $z_p$ is just given by

$$\text{Res} \left\{ \frac{A(z)}{z} \right\} \bigg|_{z=z_p} = - \sum_h \frac{A^h_L(z_P) A^{-h}_R(z_P)}{\langle \lambda_k | \, P_{i,j} | \tilde{\lambda}_l]} \frac{1}{z_p} \; . \tag{3.81}$$

In a final step we replace the value of $z_p$ due to (3.75) and arrive at BCFW recursion relation

$$A_n(p_1, \ldots, p_n) = A(0) = \sum_{i,j} \sum_{h=\pm 1} A^h_L(z_P) \frac{1}{P^2_{i,j}} A^{-h}_R(z_P) \tag{3.82}$$

where we simply replaced the sum over the poles by a sum over all momenta $p_i, \ldots, p_j$ such that one of the shifted legs lays within that range. A pictorial representation of the recursion relations is given in Figure 3.3. The subamplitudes are evaluated at the pole of the corresponding sum of external momenta and have the momentum dependence

$$A_L^h(-P_{i,j}^h, i, \ldots, j) \; , \qquad A_R^{-h}(j+1, \ldots, i-1, P_{i,j}^{-h}) \; . \tag{3.83}$$

Notice that the internal propagator $1/P_{i,j}^2$ is evaluated with unshifted kinematics. In order to reduce the terms in the BCFW expansion it is convenient to shift adjacent legs, i.e. considering shifts of the form $[l\, l+1\rangle$.

The power of these relations lay in their recursive structure. In principle, they allow us to derive any $n$-point amplitude starting from the basic amplitude in a gauge theory, the three-point amplitude. Once this object is defined, we can calculate higher-point amplitudes recursively. The only requirement for the application of the BCFW construction is the vanishing of the amplitude for infinite complex momenta, $A(z) \to 0$ for $z \to \infty$. Different theories might behave differently for large momenta. We will return to the question of vanishing amplitudes for large $z$ in the context of supersymmetric theories in Section 3.4.

## 3.3 Amplitudes for maximal supersymmetric theories

So far, we discussed amplitudes in four-dimensional Yang-Mills theory, focusing on pure gauge boson interactions. In a realistic theory, like QCD, one has to deal with additional particles, both at tree-level as external states as well as internal particles at loop-level. Similarly, the number of different particle species in a supersymmetric theory is larger. However, the additional symmetries can put constraints on the amplitudes, leading in many cases to simpler forms or can be useful as calculation tools. Especially theories with the maximal amount of supersymmetry in four dimensions are important examples.

For theories of massless particles with spin equal or less than 1 this is the $\mathcal{N} = 4$ super Yang-Mills theory (SYM) which we already discussed in Section 2.3.1. Its field content is given by a vector supermultiplet in the adjoint representation, containing one gauge boson $A_\mu$ with 2 real degrees of freedom (d.o.f.), 4 complex fermions $\chi_\alpha$ with 8 real d.o.f. finally 6 real scalars with 6 d.o.f. As we have seen in Section 2.3.1, for this maximally supersymmetric theory it is possible to combine all states into a single supermultiplet by merely acting with the supersymmetry generator $\bar{q}_{\dot\alpha}$ on the vacuum state with the highest helicity possible. In the following we discuss the structure of scattering amplitudes in this theory with maximal supersymmetry. It is important to note that here, we are considering an on-shell description of the $\mathcal{N} = 4$ superspace. Hence, no action can be written down for this superspace. Merely, supersymmetry is used to relate scattering amplitudes of different particles of the $\mathcal{N} = 4$ theory. Via the power of supersymmetry, related amplitudes[8] are combined into a single object, the so called *superamplitude*. Our discussion is mainly based on the publications [56–58].

[8]Supersymmetric Ward Identities are used to relate amplitudes with a fixed number of external states.

### 3.3.1 The $\mathcal{N} = 4$ on-shell superspace and superamplitudes

Although the $\mathcal{N} = 4$ SYM theory has the largest particle content of the four-dimensional supersymmetric theories, it offers the feature of an unique combination of all particles states into a single object, the $\mathcal{N} = 4$ supermultiplet. We can use supersymmetry as a nice bookkeeping tool and combine all states of the theory into a single superwavefunction. This idea goes back to Nair [59] who proposed to use an on-shell superspace for the $\mathcal{N} = 4$ theory with massless particles. The generators of $\mathcal{N} = 4$ algebra fulfill the following anti-commutation relations

$$\{q^I_\alpha, \bar{q}_{J\dot{\alpha}}\} = \delta^I_J \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \ , \tag{3.84}$$

for a massless particle with momentum $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ where $I, J$ are $\mathsf{SU}(4)$ $R$-symmetry indices and $\alpha, \dot{\alpha}$ are the usual $\mathsf{SU}(2)$ spinor indices in four dimensions. Then, one can decompose the supercharge $q^I_\alpha$ along two independent directions $\lambda$ and $\mu$ in spinor space,

$$q^I_\alpha = \lambda_\alpha q^I_{(1)} + \mu_\alpha q^I_{(2)} \ , \tag{3.85}$$

where $\langle \lambda\mu \rangle \neq 0$. A similar decomposition is performed for $\bar{q}_{I\dot{\alpha}}$. Substituting this decomposition into (3.84) and multiplying with $\lambda^\alpha$ one can easily see that the charges $q_{(2)}$ and $\tilde{q}_{(2)}$ anti-commute among themselves and the other generators, and can therefore be set to zero. The supersymmetry algebra becomes

$$\{q^I_{(1)}, \bar{q}_{(1)J}\} = \delta^I_J \ . \tag{3.86}$$

Denoting $q^I_{(1)} = q^I$ and $\bar{q}_{(1)J} = \bar{q}_J$, the Clifford algebra can be naturally realised in terms of Grassmann variables $\eta^I$ with $\{\eta^I, \eta^J\} = 0$ as

$$q^I = \eta^I \ , \qquad \bar{q}_J = \frac{\partial}{\partial \eta^J} \ . \tag{3.87}$$

Hence, we write for the supercharge $q^I_\alpha = \lambda_\alpha \eta^I$ and $\bar{q}_{I\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \eta^I}$. Note that this representation of the algebra is chiral. One could have chosen an anti-chiral representation, where the roles of $q$ and $\bar{q}$ in (4.12) are interchanged. If we are dealing with a scattering process where $n$ particles are scattered and the total momentum is $P_{\alpha\dot{\alpha}}$, we write for supersymmetry generators $\{Q^I_\alpha, \bar{Q}_{J\dot{\alpha}}\} = \delta^I_J P_{\alpha\dot{\alpha}}$ with

$$Q^I_\alpha = \sum_{i=1}^n \lambda_{i\alpha} \eta^I_i \ , \qquad \bar{Q}_{I\dot{\alpha}} = \sum_{i=1}^n \tilde{\lambda}_{i\dot{\alpha}} \frac{\partial}{\partial \eta^I_i} \ , \qquad P_{\alpha\dot{\alpha}} = \sum_{i=1}^n \lambda_{i\alpha} \tilde{\lambda}_{i\dot{\alpha}} \ . \tag{3.88}$$

Due to our conventions for the super-Poincaré algebra, the Grassmann variables $\eta^I_i$ carry a helicity of $\frac{1}{2}$ whereas the $\bar{\eta}_I$ have helicity $-\frac{1}{2}$.

We can use these fermionic variables to describe arbitrary external states of a scattering process. To that extent one can reproduce the full content of the $\mathcal{N} = 4$ supermultiplet in the following compact super-wavefunction [57, 59]

$$\begin{aligned} \Phi(p, \eta) = {} & G^+(p) + \eta^I \Gamma_I(p) + \frac{1}{2} \eta^I \eta^J S_{IJ}(p) + \frac{1}{3!} \eta^I \eta^J \eta^K \epsilon_{IJKL} \bar{\Gamma}^L(p) \\ & + \frac{1}{4!} \eta^I \eta^J \eta^K \eta^L \epsilon_{IJKL} G^-(p) \ . \end{aligned} \tag{3.89}$$

Here, $G^\pm$ are the gauge boson states with helicities $\pm 1$, $\Gamma_I$ and $\bar{\Gamma}^I$ are the eight fermion states in the theory with helicities $1/2$ and $-1/2$, respectively, and the helicity zero states are the six real scalar states $S_{IJ}$ with the reality condition $S_{IJ} = \frac{1}{2}\epsilon_{IJKL}\bar{S}^{KL}$. The Grassmann variables are used for tracking the corresponding states in the super-wavefunction. For instance, the gluon of positive helicity is just the power-zero component of $\Phi(p,\eta)$, so $G^+(p) = \Phi(p,0)$. The next state can be obtained by projecting out the power-one component. Since $\bar{q}_J$ acts as a derivate in $\eta^J$, the projection is just a multiplication with $\bar{q}_J$ and we have $\Gamma_I(p) = \bar{q}_I\Phi(p,\eta)\big|_{\eta=0}$ and so on. The powers of $\eta^I$ matches also the helicity counting: Since $G^+$ has helicity $h = +1$ the whole super-wavefunction $\Phi(p,\eta)$ has $h = +1$. Now each power of $\eta^I$ increases the helicity by $+1/2$ and hence, all terms in the expansion have $h = +1$. As we have seen in (3.86), the representation in terms of Grassmann spinors $\eta^I$ is chiral and leads to a chiral description of the super-wavefunction in (3.89) and ultimately to a chiral on-shell superspace. However, we could have chosen an antichiral description where the $\bar{q}_I$ act multiplicative and the $q^I$ are represented by differentiation with respect to $\bar{\eta}_I$. In the case of real momenta this leads to a antichiral super-wavefunction $\bar{\Phi} = (\Phi(p,\eta))^*$ where the antichiral Grassmann spinors are given by $\bar{\eta}_I = (\eta^I)^*$ [57]. For our discussion we will choose the chiral super-wavefunction as it is commonly done in the literature.

Since the super-wavefunction $\Phi(p,\eta)$ combines all kind of particles of the $\mathcal{N} = 4$ SYM theory into a single object it is possible to use it to construct amplitudes for all kind of scattering processes in that maximally supersymmetric theory. We can combine all scattering amplitudes of $n$ external particles into a single superamplitude $\mathcal{A}_n$ which is defined as

$$\mathcal{A}_n(\lambda,\tilde{\lambda},\eta) = \mathcal{A}_n(\Phi_1 \dots \Phi_n) \; . \tag{3.90}$$

Here, the $\Phi_i$ stand for the super-wavefunctions $\Phi_i = \Phi(p_i,\eta_i)$. In that sense, a superamplitude is a scattering amplitude of super-wavefunctions which depends on the spinors $\lambda_i$ and $\tilde{\lambda}_i$ and the Grassmann variables $\eta_i$ of all external states with $i = 1,\dots,n$. The usual scattering amplitudes of all sort of particle types are then component amplitudes of the superamplitude. Since every component state in the super-wavefunction (3.89) carries a different power in the $\eta_i$ one can obtain the subamplitudes $A_n$ by simpliy expanding the superamplitude $\mathcal{A}_n(\lambda,\tilde{\lambda},\eta)$ in powers of the Grassmann spinors $\eta_i$. The expansion of $\mathcal{A}_n$ contains for instance terms like

$$\begin{aligned}\mathcal{A}_n =& (\eta_1)^4 A_n(G^-G^+\dots G^+) + (\eta_1)^4 (\eta_2)^4 A_n(G^-G^-\dots G^+) \\ &+ \frac{1}{3!}(\eta_1)^4 \eta_2^I\eta_2^J\eta_2^K\eta_3^M\epsilon_{IJKL}A_n(G^-\bar{\Gamma}_2^L\Gamma_{3M}G^+\dots G^+) + \dots\end{aligned} \tag{3.91}$$

where we defined $(\eta_i)^4 = \frac{1}{4!}\eta_i^I\eta_i^J\eta_i^K\eta_i^L\epsilon_{IJKL}$. Here, the first term is the gluon component amplitude with one negative helicity gauge boson and $n-1$ positive helicity gauge bosons. This amplitude is zero, however, it is still a subamplitude in the expansion in the Grassmann variables. In a similar fashion all possible scattering amplitudes can be extracted from one n-point superamplitude.

Having defined a general superamplitude in the $\mathcal{N} = 4$ theory we might wonder if the presence of supersymmetry puts any restrictions on their specific form. Indeed, this is the case. In general, the superamplitude should be an inhomogenious polynomial of degree $4n$ in the $\eta_i^I$ due to invariance under the $\mathsf{SU}(4)$ R-symmetry group. However, a superamplitude should also be invariant under the supersymmetry transformations and this puts further restrictions on its form and its degree in $\eta_i^I$. The supersymmetry on-shell generators take the form of (3.88) and act multiplicatively on the superamplitude. Therefore, invariance under $q^I_\alpha$ constrains the superamplitude to live on a surface in superspace defined by

$$Q^I_\alpha \mathcal{A}_n = \sum_{i=1}^{n} \lambda_{i\alpha}\eta^I_i \mathcal{A}_n = 0 \ . \tag{3.92}$$

Similar to the constraint of momentum conservation, one can implement this constraint as a conservation of total supermomentum $Q^I_\alpha$ which restricts the generic form of the superamplitude to be

$$\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)}(P_{\alpha\dot{\alpha}})\, \delta^{(8)}(Q^I_\alpha)\, \mathcal{P}_n(\lambda, \tilde{\lambda}, \eta) \ . \tag{3.93}$$

Here, the fermionic $\delta$-function is defined as a product of the individual supercharge components,

$$\delta^{(8)}(Q^I_\alpha) = \prod_{I=1}^{4}\prod_{\alpha=1}^{2}\left(\sum_{i=1}^{n} \lambda_{i\alpha}\eta^I_i\right) \tag{3.94}$$

and $\mathcal{P}_n$ is a polynomial in the Grassmann parameters $\eta_i$. This form of the superamplitude is valid for $n \geq 4$ in the case of real momenta. As we will see, a three-point superamplitude can be defined only for complex momenta, just as in the case of the usual scattering amplitudes which we discussed in Section 3.1.3.

The generators $\bar{Q}_{I\dot{\alpha}}$ are given by derivatives with respect to the Grassmann variables $\eta^I_i$. Requiring invariance under supersymmetry we have $\bar{Q}_{I\dot{\alpha}}\mathcal{A}(\lambda, \tilde{\lambda}, \eta) = 0$. Acting on a superamplitude of the form (3.93) one obtains the expression

$$\left(\sum_{i=1}^{n} \tilde{\lambda}_{i\dot{\alpha}} \frac{\partial}{\partial \eta^I_i}\, \delta^{(8)}(\sum_j \eta^I_j \lambda_{j\alpha})\right) \mathcal{P}_n + \delta^{(8)}(Q^I_\alpha) \left(\sum_{i=1}^{n} \tilde{\lambda}_{i\dot{\alpha}} \frac{\partial}{\partial \eta^I_i} \mathcal{P}_n\right) \tag{3.95}$$

which should vanish for invariance. The first term just simplifies to the total momentum of the superamplitude and vanishes due to momentum conservation and the second term results in the constraint

$$\bar{Q}_{I\dot{\alpha}}\mathcal{P}_n(\lambda, \tilde{\lambda}, \eta)) = 0 \ . \tag{3.96}$$

The function $\mathcal{P}$ is a polynomial in the Grassmann variables $\eta^I_i$ and must be a singlet under the $\mathsf{SU}(4)$ R-symmetry. Just as we can expand the superamplitude in powers of $\eta^I_i$ we can expand the polynomial $\mathcal{P}_n$. Since it should be invariant under the $\mathsf{SU}(4)$ R-symmetry, $\mathcal{P}_n$ can be expanded into a sum of $\mathsf{SU}(4)$ singlet homogeneous polynomial functions, all having a degree of multiples of 4 in $\eta^I_i$. The expansion is given by [57]

$$\mathcal{P}_n = \mathcal{P}^{(0)}_n + \mathcal{P}^{(4)}_n + \mathcal{P}^{(8)}_n + \cdots + \mathcal{P}^{(4n-16)}_n \ . \tag{3.97}$$

It turns out that each of these terms represent a certain class of subamplitudes with a specific helicity configuration. To see this we have to consider the total degree in the Grassmann variables of the full superamplitude. Firstly, supersymmetry invariance requires the appearance of the fermionic $\delta$-function $\delta^{(8)}(Q^I_\alpha)$ in the superamplitude which has degree 8 in $\eta$. This is actually the minimal degree in the fermionic variables a superamplitude can have since the first term in the expansion of $\mathcal{P}_n$ has degree 0 in $\eta$. It follows that $\mathcal{P}^{(0)}_n$ represents subamplitudes with a total helicity of $h_{tot} = n - 4$ since the degree 8 in the Grassmann spinors includes maximal two gauge bosons of negative helicity. Hence, this term represents subamplitudes of MHV type where all sorts of particle configurations are allowed as long as $h_{tot} = n - 4$, for instance $A_n(G^-\Gamma_I\bar{\Gamma}^J G^+ \dots G^+)$ is such a subamplitude. The remaining terms include all other helicity configurations, starting with the next-to-MHV (NMHV) subamplitudes contained in the term $\mathcal{P}_n^{(4)}$ up to the anti-MHV subamplitudes with total helicity $h_{tot} = -(n-4)$ , being represented by $\mathcal{P}_n^{(4n-16)}$. In all cases one has to include the $\delta^{(8)}(Q^I_\alpha)$ when identifying the total helicity configuration of the subamplitudes which raises the specific degree in $\eta$ to $8 + 4k = 4(k+2)$ for $k \geq 1$. Note that the maximal degree of a superamplitude is therefore not $4n$ but rather $4n - 8$. The reason for this is the absence of a term $\mathcal{P}_n^{(4n-8)}$ in the expansion of $\mathcal{P}_n$ which would include subamplitudes with only one gauge boson of positive helicity [56, 57].

In our discussions of four-dimensional superamplitudes and their applications for unitarity cuts we will focus on the MHV case, namely the first term in the expansion (3.97). The form of the polynomial $\mathcal{P}_n^{(0)}$ can be easily deduced by comparing the result of an integration over the corresponding Grassmann variables with the know MHV tree-level all gluon amplitude. Choosing the $i$-th and $j$-th gauge bosons to be of negative helicity, we have to extract factors of $(\eta_i)^4$ and $(\eta_j)^4$ from the fermionic $\delta$-function in (3.93) since $\mathcal{P}_n^{(0)}$ cannot contribute any powers of $\eta$. Integrating over the Grassmann variables for the two negative helicity gluons one obtains the bosonic factor $\langle ij\rangle^4$ from the $\delta^{(8)}(Q^I_\alpha)$. By comparison with the Parke-Taylor expression (3.50) one finds

$$\mathcal{P}_n^{(0)} = (\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle)^{-1} \tag{3.98}$$

Hence, in the MHV case, the $n$-point superamplitude ($n \geq 4$) is given by

$$\mathcal{A}_n^{\mathrm{MHV}}(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)}(\sum_{i=1}^{n} \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}})\ \frac{\delta^{(8)}(\sum_{i=1}^{n} \lambda_{i\alpha}\eta_i^I)}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}\ , \tag{3.99}$$

which was first presented by Nair in [59]. For further discussions of the terms in the polynomial $\mathcal{P}_n$ we refer the interested reader to the original article [56]. In the following we focus on some examples for tree-level superamplitudes, namely the case of external states with three, four and five particles. Especially interesting is the three-point superamplitude since it is the only one required when constructing higher-point superamplitudes in a recursive fashion. This can be done by 'supersymmetrising' the BCFW construction as we will discuss in Section 3.4.

### 3.3.2 Tree-level superamplitudes

To get some intuition on the superamplitude structure of the $\mathcal{N} = 4$ theory we start with the four- and five-point case as examples. For $n = 4$, only the first term in the expansion (3.97) is non-zero, stating the fact that all scattering amplitudes with four external states are of MHV-type and we have

$$\mathcal{P}_4^{(0)} = (\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle)^{-1}\ . \tag{3.100}$$

In the case of $n = 5$ two terms contribute to the polynomial $\mathcal{P}_5$, namely $\mathcal{P}_5^{(0)}$ and $\mathcal{P}_5^{(4)}$. So all amplitudes are either of MHV or anti-MHV since $4n - 16 = 4$ for $n = 5$. The five-particle MHV polynomial has the same structure as before whereas anti-MHV superamplitudes are described by [57]

$$\mathcal{P}_5^{(4)}(\lambda,\tilde{\lambda},\eta) = \left(\langle 12\rangle^4[12][23]\ldots[51]\right)^{-1}\delta^{(4)}(\eta_3[45]+\eta_4[53]+\eta_5[34])\ . \tag{3.101}$$

The $\delta^{(4)}$ ensures that the polynomial is of degree four in $\eta$. By acting with $\bar{Q}_{I\dot{\alpha}}$ on $\mathcal{P}_5^{(4)}$ one can show that the term is invariant under $\bar{q}$-supersymmetry as stated in (3.96). As we will see in the following, the form of the fermionic $\delta$-function is important for the special case of $n=3$ anti-MHV superamplitudes.

Just as discussed in Section 3.1.3 for the usual tree-level amplitudes, kinematical constraints lead to vanishing three-point superamplitudes for real momenta since $\tilde{\lambda}=\pm\bar{\lambda}$. Relaxing this condition and going to complex momenta, we can choose between the constraints in (3.54) which leads to a MHV or anti-MHV configuration. Choosing the $[ij]=0$ we have the MHV tree-level superamplitude with $n=3$,

$$\mathcal{A}_3^{\text{MHV}}(\lambda,\tilde{\lambda},\eta) = \delta^{(4)}(\sum_{i=1}^{3}\lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}})\frac{\delta^{(8)}(\sum_{i=1}^{3}\lambda_{i\alpha}\eta_i^I)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\ , \tag{3.102}$$

which is holomorphic in the spinor variables. Choosing the holomorphic spinor brackets to vanish, $\langle ij\rangle=0$, we have the $n=3$ anti-MHV superamplitude. In principle, we should assume that it is described by the last term in the expansion (3.97). However, for $n=3$ this would lead to a polynomial $\mathcal{P}_3^{(-4)}$ of negative degree in $\eta$. This is due to our assumption that invariance under $q$-supersymmetry is imposed by the $\delta^{(8)}(Q_\alpha^I)$ which is not the case for the three-point anti-MHV superamplitude. For $n=3$ the anti-MHV superamplitude is given by [58,60]

$$\mathcal{A}_3^{\overline{\text{MHV}}} = \delta^{(4)}(\sum_{i=1}^{3}\lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}})\frac{\delta^{(4)}(\eta_1[23]+\eta_2[31]+\eta_3[12])}{[12][23][31]}\ . \tag{3.103}$$

We notice that it is a anti-holomorphic function of degree four in $\eta$, corresponding to the fact that the total degree in $\eta$ of an anti-MHV superamplitude is $4n-8$. Although it is not proportional to a $\delta^{(8)}(Q_\alpha^I)$, it is invariant under $q$-supersymmetry. To see this we use the fermionic $\delta$-function in (3.103) to solve for instance for $\eta_1$,

$$\eta_1^I\lambda_{1\alpha} = \frac{-\eta_2^I[31]-\eta_3^I[12]}{[23]}\lambda_{1\alpha}\ . \tag{3.104}$$

Hence, the supersymmetry generator $Q_\alpha^I$ is given

$$Q_\alpha^I = \sum_{i=1}^{3}\lambda_{i\alpha}\eta_i^I = \eta_2^I\frac{\lambda_{1\alpha}[13]+\lambda_{2\alpha}[23]}{[23]} + \eta_3^I\frac{\lambda_{1\alpha}[21]+\lambda_{3\alpha}[23]}{[23]} \tag{3.105}$$

which vanishes due to momentum conservation, $\sum_{i=1}^{3}\lambda_i\tilde{\lambda}_i=0$, and the three-point anti-MHV superamplitude is automatically invariant under $q$-supersymmetry. Likewise, $\mathcal{A}_3^{\overline{\text{MHV}}}$ is invariant under $\bar{q}$-supersymmetry: Acting with the generator $\bar{Q}_{I\dot{\alpha}}$ on the superamplitude yields

$$\bar{Q}_{I\dot{\alpha}}\mathcal{A}_3^{\overline{\text{MHV}}} \to \sum_{i=1}^{3}\tilde{\lambda}_i\frac{\partial}{\partial\eta_i}(\eta_1[23]+\eta_2[31]+\eta_3[12]) = \tilde{\lambda}_1[23]+\tilde{\lambda}_2[31]+\tilde{\lambda}_3[12] \tag{3.106}$$

which vanishes as well after an application of an anti-holomorphic version of the Schouten-identity (3.25). Hence, the anti-MHV superamplitude for $n=3$ is invariant under the full $\mathcal{N}=4$ supersymmetries.

Both the MHV and anti-MHV three-point superamplitudes are important when constructing higher point superamplitudes in a recursive fashion. The explicit form of both configurations is needed: Since they are defined for different kinematical configurations, they cannot be combined into a single three-point superamplitude.

## 3.4 Super-BCFW recursion relations

The motivation for a recursive construction of a $n$-point superamplitude at tree-level in the $\mathcal{N} = 4$ theory is the same as for the more familiar tree-level amplitudes we discussed in Section 3.2.2. The supersymmetric version of the BCFW recursion relations was first discussed in [58] and [60]. In the following we will briefly motivated the construction for a supersymmetric theory. We will then apply this technique to derive the five-point anti-MHV superamplitude as an example.

Just as in the non-supersymmetric case, one chooses to shift two external momenta by a complex variable $z$ according two

$$\tilde{\lambda}_k \longrightarrow \tilde{\lambda}_k(z) = \tilde{\lambda}_k + z\tilde{\lambda}_l \ , \qquad \lambda_l \rightarrow \lambda_l(z) = \lambda_l - z\lambda_k \tag{3.107}$$

which we denote as a $[kl\rangle$ shift. As we have seen, the total momentum is conserved by this shift. In the supersymmetric framework, one has to deal with supermomentum as well. Since we are shifting the holomorphic spinor with label $l$, the supermomentum changes as

$$q_l \rightarrow q_l(z) = q_l - z\eta_l\lambda_k \ . \tag{3.108}$$

Shifting the Grassmann spinor with label $k$ according to

$$\eta_k \rightarrow \eta_k(z) = \eta_k + z\eta_l \ , \tag{3.109}$$

leads to supermomentum conservation since $q_k(z) + q_l(z) = q_k + q_l$. The remaining construction of the recursion relations follows suit. The shifted tree-level superamplitude is a rational function of the spinor variables and a polynomial in the Grassmann parameters $\eta_i^I$. Since the dependence on $z$ enters only through the spinor-shifts in $\lambda$ and $\eta$, the superamplitude $\mathcal{A}(z)$ is an analytic function in $z$ and just as before it contains only poles and no cuts over the complex plane. Therefore, the application of Cauchy's theorem follows the case of the ordinary tree-level scattering amplitudes $A_n(z)$. The result are the supersymmetric recursion relations

$$\mathcal{A}_n = \sum_{\text{poles } z_P} \int d^4\eta_{\hat{P}} \ \mathcal{A}_L(z_P) \frac{i}{P^2} \mathcal{A}_R(z_p) \tag{3.110}$$

where we have to sum over the simple poles and both subamplitudes $\mathcal{A}_L$ and $\mathcal{A}_R$ are evaluated at these poles. The main difference compared to the usual BCFW recursion relations is the assignment of helicities. Whereas in Section 3.2.2 we had to sum over all helicity configurations of the subamplitudes, the sum is replaced by an integration over the Grassmann variable $\eta_{\hat{P}}$ assigned to the internal propagator. This is a manifestation of the fact that a superamplitude contains all possible particle and helicity configurations for a fixed number of external states.

The subamplitudes $\mathcal{A}_L$ and $\mathcal{A}_R$ are superamplitudes themselves, i.e. they can be expanded in terms of the fermionic variabels $\eta_i$ just as in (3.93) and (3.97). Since a specific superamplitude is defined by the number of external states and its total helicity, this puts a constraint on the helicity assignments. The Grassmann integration reduces the power of $\eta$ on the LHS of the relation 3.110 by four. Hence, the sum of the total helicities of the two subamplitudes reduced by four should be equal to the total helicity of the recursive superamplitude $\mathcal{A}_n$.

One of the main requirements in the derivation of the BCFW recursion relation (supersymmetric and non-supersymmetric ones) is the vanishing of the (super)amplitude for large $z$, i.e. $\mathcal{A}(z) \to 0$ for $z \to \infty$. In the original papers [53, 54] this was checked for specific helicity configurations using the MHV diagram approach we discussed in Section 3.2.1. However, an amplitude with a general helicity assignment does not need to vanish at infinite complex momentum. Indeed, in [61] it was shown that amplitudes in pure Yang-Mills vanish for large $z$ only under certain shifts. If one chooses the leg of the shifted holomorphic spinor $\lambda_l$ and the anti-holomorphic spinor $\tilde{\lambda}_k$ both to be of positive helicity then the whole (bosonic) amplitude vanishes for $z \to \infty$ (for a shift with helicities $(+,+)$ the amplitudes scales as $\propto 1/z$ [54]). Although not focus of our discussion, the recursion relations can be applied to other theories as well, for instance gravity. Amplitudes in this theory do behave similarly and vanish at infinite complex momentum for certain shifts. For further discussions see the articles [61–63].

In a supersymmetric theory one does not need to distinguish between different helicity assignments for the superamplitude. Hence, a large class of different subamplitudes can be considered in terms of their large-$z$ behaviour when considering a theory with maximal supersymmetry. The behaviour of superamplitudes for large complex momenta was discussed in [58] and [60] and it was found that the maximally supersymmetric Yang-Mills ($\mathcal{N} = 4$ SYM) as well as maximally supersymmetric gravity ($\mathcal{N} = 8$ supergravity) vanish for $z \to \infty$ since

$$\mathcal{A}(z) \propto \frac{1}{z}\ , \qquad \text{and} \qquad \mathcal{M}(z) \propto \frac{1}{z^2} \tag{3.111}$$

where $\mathcal{M}(z)$ is the shifted super-gravity amplitude. In [60] this is proven by using the action of the supersymmetry generators. Considering a shift of legs $l$ and $l+1$ we know that the whole amplitudes vanishes for $z \to \infty$ in the case of $\eta_l = \eta_{l+1} = 0$ since this resembles the case of leg $l$ and $l+1$ having positive helicity. A supersymmetry transformation with the $q$-supercharges results in a translation of the Grassmann parameters. The parameter $\xi^I_\alpha$ of such a supersymmetry transformation can be decomposed along two independent directions in spinor space. By choosing these two spinor-directions to coincide with the two shifted legs one can make $\eta_l$ and $\eta_{l+1}$ vanish and hence, this results in an amplitude with states of positive helicity at positions $l$ and $l+1$. This amplitude is known to vanish for large $z$. The argument of vanishing amplitudes for large complex momenta was extended to theories with spin-1 and spin-2 gauge bosonos in [64] where the behaviour of (3.111) was confirmed.

### 3.4.1 The five-point anti-MHV superamplitude

In this section we want to present an example of a recursive calculation of the five-point anti-MHV superamplitude in the $\mathcal{N} = 4$ super Yang-Mills theory in order to apply the machinery we have developed so far. We follow the discussion presented in [58] and start by choosing to shift two of

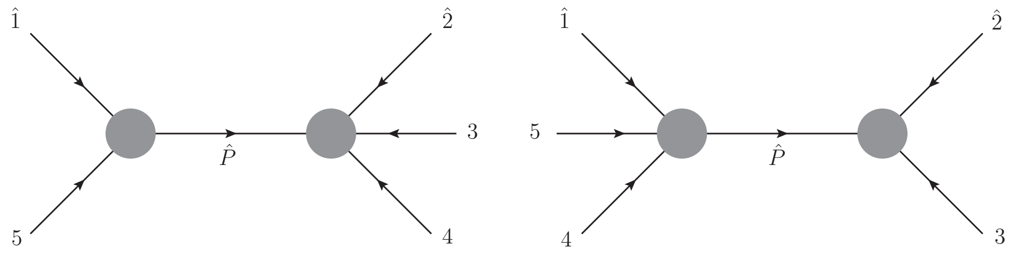



Figure 3.4: *The recursive diagrams which contribute to the BCFW calculation of the five-point anti-MHV superamplitude. Notice that both subamplitudes in all recursive diagrams are of MHV-type. Here, a hatted momentum $\hat{p}_i$ represents a shifted (z-dependent) quantity.*

the external legs. For convenience we shift adjacent legs and choose $l = 1$ and $k = 2$, leading to a $[12\rangle$-shift

$$\tilde{\lambda}_1(z) = \tilde{\lambda}_1 + z\tilde{\lambda}_2 \ , \qquad \lambda_2(z) = \lambda_2 - z\lambda_1 \ . \tag{3.112}$$

In the following we denote $z$-dependent quantities with a hat, i.e. by writing $\hat{\tilde{\lambda}}_1$ or $\hat{\lambda}_2$. According to the super-BCFW construction, a $[12\rangle$-shift leads also to a $z$-dependent Grassmann spinor

$$\hat{\eta}_1 = \eta_1 + z\eta_2 \ . \tag{3.113}$$

Having chosen the shift we can start considering the contributing recursive diagrams. In Figure 3.4 we give the two possible momentum configurations on both sides of the internal propagator. We are interested in the $n = 5$ anti-MHV superamplitude which receives contributions from the term $\mathcal{P}_n^{(4n-16)}$ of the expansion (3.97). Including the $\delta$-function of total supermomentum conservation, the total degree of the superamplitude in the Grassmann spinors is then $4n - 8 = 12$ for the final five-point anti-MHV superamplitude. Since the fermionic integration over the parameter $\eta_{\hat{P}}$ removes fours degrees in $\eta$'s, we have to have a total power of 16 in the Grassmann variables before integrating over the subamplitudes as indicated in (3.110). This means that $\mathcal{A}_L$ and $\mathcal{A}_R$ must have a combined degree of 16 in the $\eta$'s. Since an anti-MHV three-point superamplitude has only a degree of 4, both subamplitudes must be of MHV type in both the recursive diagrams, contributing 8 powers in the Grassmann spinors each.

Inspecting the two recursive diagrams we find that the one on the right in Figure 3.4 vanishes. This is due to the $[12\rangle$-shift which makes the three-point superamplitude on the left-hand side in the right diagram vanish since the shift results in $\langle\hat{2}3\rangle = \langle 3\hat{P}\rangle = \langle\hat{P}\hat{2}\rangle = 0$. Hence, we can focus our discussion on the diagram on the left-hand side in Figure 3.4.

The two subamplitudes are then given by

$$\mathcal{A}_L = \frac{\delta^{(4)}(\hat{p}_1 - \hat{P} + \hat{p}_5)\ \delta^{(8)}(\hat{\eta}_1\lambda_1 - \eta_{\hat{P}}\lambda_{\hat{P}} + \eta_5\lambda_5)}{\langle 1\hat{P}\rangle\langle\hat{P}5\rangle\langle 51\rangle} \ , \tag{3.114}$$

$$\mathcal{A}_R = \frac{\delta^{(4)}(\hat{p}_2 + p_3 + p_4 + \hat{P})\ \delta^{(8)}(\eta_2\hat{\lambda}_2 + \eta_3\lambda_3 + \eta_4\lambda_4 + \eta_{\hat{P}}\lambda_{\hat{P}})}{\langle\hat{2}3\rangle\langle 34\rangle\langle 4\hat{P}\rangle\langle\hat{P}\hat{2}\rangle} \ , \tag{3.115}$$

where the signs are appropriately chosen such that all momenta (and supermomenta) are defined to be incoming. For normal four-momenta this means just a change of sign, however, in terms of spinors, one can choose to change the sign of $\lambda$ or $\tilde{\lambda}$ when reversing the direction. Here, we follow the usual convention to introduce factors of $i$ in both spinors: We shift $\lambda \to i\lambda$ and $\tilde{\lambda} \to i\tilde{\lambda}$ when $p \to -p$. To be consistent with our choice of the supermomenta, we also shift $\eta \to i\eta$ for $q \to -q$. This mimics the choice we will make for six-dimensional spinors in the next chapter.

One can now use the properties of the two fermionic and the two bosonic $\delta$-functions to rewrite them into a single bosonic $\delta$-function imposing total momentum conservation and a single fermionic $\delta$-function imposing total supermomentum conservation, yielding a $\delta^{(4)}(\sum_{i=1}^{5} p_i)$ and a $\delta^{(8)}(\sum_{i=1}^{5} \eta_i \lambda_i)$. In order to combine the denominators of the two subamplitudes we apply momentum conservation on each subamplitude which results in the following spinor identities:

$$\langle \hat{2}3\rangle[34] = -\langle \hat{2}|\hat{P}|4] = -\langle \hat{2}\hat{P}\rangle[\hat{P}4] \; , \tag{3.116}$$

$$\langle 34\rangle[43] = (p_3+p_4)^2 = (\hat{p}_2+\hat{P})^2 = \langle \hat{2}\hat{P}\rangle[\hat{P}2] \; , \tag{3.117}$$

$$\langle 4\hat{P}\rangle[34] = [3|4|\hat{P}\rangle = -[32]\langle \hat{2}\hat{P}\rangle \; . \tag{3.118}$$

Since all spinor brackets in these relations are non-zero, we can solve for the denominator of $\mathcal{A}_R$ and find[9]

$$\langle \hat{2}3\rangle\langle 34\rangle\langle 4\hat{P}\rangle\langle \hat{P}\hat{2}\rangle = \frac{[\hat{P}4][\hat{P}2][23][34]}{[34]^4}\langle \hat{2}\hat{P}\rangle^4 \; . \tag{3.119}$$

Finally, we can combine this relation with the spinor products in the denominator of $\mathcal{A}_L$. Using momentum conservation of the left subamplitude in Figure 3.4 we have

$$\langle 1\hat{P}\rangle[\hat{P}4] = \langle 1|5|4] = \langle 15\rangle[54] \; , \tag{3.120}$$

$$\langle \hat{P}5\rangle[\hat{P}2] = -\langle 5|\hat{P}|2] = -\langle 51\rangle[\hat{1}2] = -\langle 51\rangle[12] \; . \tag{3.121}$$

Combining these results we arrive at

$$\frac{1}{(p_1+p_5)^2}\frac{1}{\langle 1\hat{P}\rangle\langle \hat{P}5\rangle\langle 51\rangle\langle \hat{2}3\rangle\langle 34\rangle\langle 4\hat{P}\rangle\langle \hat{P}\hat{2}\rangle} = \frac{[34]^4}{\langle 15\rangle^4\langle \hat{2}\hat{P}\rangle^4}\frac{1}{\prod_{i=1}^{5}[i\; i+1]} \tag{3.122}$$

where the internal propagator is evaluated with unshifted quantities. Therefore, the whole $n=5$ anti-MHV superamplitude is given by

$$\mathcal{A}_5^{\overline{\mathrm{MHV}}} = \int d^4\eta_{\hat{P}} \Big[\delta^{(8)}(\hat{\eta}_1\lambda_1 - \eta_{\hat{P}}\lambda_{\hat{P}} + \eta_5\lambda_5)\, \delta^{(8)}(\eta_2\hat{\lambda}_2 + \eta_3\lambda_3 + \eta_4\lambda_4 + \eta_{\hat{P}}\lambda_{\hat{P}}) \\ \times \delta^{(4)}(\sum_{i=1}^{5}\lambda_i\tilde{\lambda}_i)\frac{[34]^4}{\langle 15\rangle^4\langle \hat{2}\hat{P}\rangle^4}\frac{1}{\prod_{i=1}^{5}[i\; i+1]}\Big] \; . \tag{3.123}$$

As mentioned in the previous section, this superamplitude combines all possible helicity and particle configurations of five external states into a single object. In the following we discuss some explicit particle and helicity configurations and show how to extract the corresponding amplitude from the superamplitude, as presented in [58].

[9] Although our sign conventions are opposite to those of [58], all signs cancel out and the overall result is the same.

In general, the integration over $\eta_{\hat{P}}$ ensures that when expanding the fermionic $\delta$-functions and integrating over the $\eta_{\hat{P}}$ only terms which are proportional to $(\eta_{\hat{P}})^4$ survive. It is then a computational question of how to extract the corresponding powers of the $\eta_i$ from the fermionic $\delta$-functions that describe the particle and helicity content one is looking for. A convenient way for extracting the sought after powers in the Grassmann spinors is provided by the identity

$$\delta^{(8)}\big(\sum_i \lambda_{i\alpha}\eta_i^I\big) = \frac{1}{16}\prod_{I=1}^{4}\sum_{k,l}\eta_k^I\eta_l^I\langle kl\rangle \ . \tag{3.124}$$

We start with a straight forward example and would like to extract the anti-MHV gluon amplitude $A_5(1_g^-, 2_g^-, 3_g^+, 4_g^+, 5_g^-)$ from (3.123). Hence, one needs to extract the coefficient of $(\eta_1)^4(\eta_2)^4(\eta_5)^4(\eta_{\hat{P}})^4$. By inspection of the two fermionic $\delta$-functions of the superamplitude we see that a contribution for $(\eta_1)^4(\eta_5)^4$ can only come from the first $\delta^{(8)}$ and hence, $(\eta_2)^4(\eta_{\hat{P}})^4$ must come from the second one. Using the expansion (3.124) we generate prefactors $\langle 15\rangle^4$ and $\langle\hat{2}\hat{P}\rangle^4$. Plugging this back into (3.123) and integrating over $\eta_{\hat{P}}$ leads to the result[10]

$$A_5^{\overline{\text{MHV}}}(1_g^-, 2_g^-, 3_g^+, 4_g^+, 5_g^-) = \frac{[34]^4}{[12][23][34][45][51]} \ . \tag{3.125}$$

For more complicated gluonic configurations a few more steps are required. For example, when considering the non-adjacent gluon helicity configuration $(1^-, 2^+, 3^-, 4^+, 5^-)$ we need to extract the coefficient of $(\eta_1)^4(\eta_3)^4(\eta_5)^4(\eta_{\hat{P}})^4$ and integrate over $\eta_{\hat{P}}$. Expanding the $\delta$-functions generates a coefficient of $\langle 15\rangle^4\langle 3\hat{P}\rangle^4$. After integration we can use momentum conservation $p_3 = -\hat{p}_2 - \hat{P} - p_4$ to write

$$[34]^4\langle 3\hat{P}\rangle^4 = ([42]\langle 2\hat{P}\rangle)^4 \ . \tag{3.126}$$

Combining this with the result of the Grassmann integration leads to the gluonic amplitude

$$A_5^{\overline{\text{MHV}}}(1_g^-, 2_g^+, 3_g^-, 4_g^+, 5_g^-) = \frac{[24]^4}{[12][23][34][45][51]} \ . \tag{3.127}$$

In a final step we would like to consider amplitudes with fermionic states. Let us focus on the subamplitude $A_5(1_g^-, 2_g^-, 3_f^+, 4_g^+, 5_f^-)$. For this configuration one has to extract the coefficient of the Grassmann variables $(\eta_1)^4(\eta_2)^4\eta_3(\eta_5)^3(\eta_{\hat{P}})^4$. Just as in the previous cases, by inspecting the two fermionic $\delta$-functions in (3.123) one observes that the four powers in $\eta_1$ and $\eta_2$ are coming from different $\delta$-functions. The first one gives us the coefficient of $(\eta_1)^4$ whereas the second one gives $(\eta_2)^4$. The sought powers of $\eta_3$ and $\eta_5$ have to come from the second and the first $\delta^{(8)}$, respectively. Hence, we have one power of the first and three powers of the second $\delta$-function in a Grassmann variable left which can be assigned to $\eta_{\hat{P}}$. Again, we use the expansion (3.124) and obtain a factor $\langle 15\rangle^3\langle 1\hat{P}\rangle\langle\hat{2}\hat{P}\rangle^3\langle\hat{2}3\rangle$. In a second step, we want to remove the dependence on $\hat{P}$ in these expressions. Here, momentum conservations proves to be useful again. Using the fact that the left-hand subamplitude is a three-point MHV superamplitude we find by momentum conservation

$$\langle\hat{2}3\rangle = -\frac{[54]\langle 43\rangle}{[52]} \ , \qquad \frac{\langle 1\hat{P}\rangle}{\langle\hat{2}\hat{P}\rangle} = \frac{\langle 15\rangle[52]}{\langle 34\rangle[43]} \tag{3.128}$$

[10]We omit the $\delta$-function of momentum conservation.

where the second relation follows after multiplying numerator and denominator by $[\hat{P}2]$ and using the second relation in (3.116). Now, multiplying all factors together yields the final result

$$A_5^{\overline{\text{MHV}}}(1_g^-, 2_g^-, 3_f^+, 4_g^+, 5_f^-) = \frac{[34]^3[45]}{[12][23][34][45][51]} \ . \tag{3.129}$$

This result agrees with the form of the five-point anti-MHV amplitude found in [42].

This concludes our discussion of superamplitudes in the $\mathcal{N} = 4$ super Yang-Mills theory at tree-level. The techniques we discussed, especially how to deal with fermionic $\delta$-functions and how to expand them, will be useful when we discuss superamplitudes at one-loop level. We will do this for the $\mathcal{N} = 4$ theory in four dimensions in the next sections in order to set up some loop-level terminology which will be important when we discuss six-dimensional superamplitudes at one-loop in Chapter 4.

## 3.5 Unitarity for amplitudes in four dimensions

So far, we have discussed the structure of scattering amplitudes in the $\mathcal{N} = 4$ SYM theory at tree-level. It is also interesting to consider their structure at higher orders in perturbation theory. As we will see, similar on-shell constructions to those at tree-level provide powerful techniques for calculating one-loop amplitudes, not only in the maximally supersymmetric theory. In principle, it is possible to reconstruct a scattering amplitude from its properties as a function over the complex plane. This is the basic idea of the *unitarity approach* which we will discuss in the following. It is intriguing to note that historically, an on-shell approach was first realised at one-loop level, see for instance the textbook [55], and was only applied to tree-level amplitudes after the paper [25] where some of the Twistor string theory inspired constructions of Section 3.2 were applied.

### 3.5.1 Unitarity and the optical theorem

We start by reminding ourselves that unitarity is an important property of any interacting quantum field theory, closely tied to conservation of probability, a fundamental requirement on any physical theory. Translating this into the context of scattering processes leads to the requirement of a unitary S-matrix. Then, by interpreting the scattering amplitudes as transition matrix elements, we can apply unitarity at the amplitude level directly. This leads to the so called *optical theorem* which relates the imaginary part of the amplitude to a sum over contributions from all possible intermediate particle states. Discussions of these analytic properties of scattering amplitudes can be found in many textbooks on quantum field theory, see for instance [65] which we will follow loosely.

As discussed in Section 3.1, the scattering amplitudes are the elements of the T-matrix, related to the S-matrix by $S = 1 + iT$. Unitarity of the S-matrix then implies

$$-\,i(T - T^\dagger) = T^\dagger T \ . \tag{3.130}$$

At the level of the transition matrix elements $A$, i.e. we are considering the T-matrix between initial and final particle states, the LHS of (3.130) is then proportional to the imaginary part of the amplitude and one has in a short-hand notation the optical theorem[11]

$$-\,i\big[A(a\to b)-A^*(b\to a)\big]=\sum_f\int d\text{LIPS}\;A^*(b\to f)A(a\to f) \tag{3.131}$$

where $d$LIPS is the $n$-particle Lorentz invariant phase pace measure,

$$d\text{LIPS}=\prod_{i=1}^{n}\frac{d^4q_i}{(2\pi)^4}\delta^{(+)}(q_i^2-m_i^2)\;(2\pi)^4\delta^{(4)}\big(p_a+p_b-\sum_i q_i\big) \tag{3.132}$$

and $a$ and $b$ are initial and final particle states, respectively, and the sum on the RHS runs over all possible intermediate states $f$. The optical theorem of (3.131) states that the imaginary part of a scattering amplitude can be obtained from the sum of phase-space integrals of intermediate multiparticle states. Here, one integrates products of scattering amplitudes which individually transform the initial or final particle states into the intermediate states. Expanding both sides in perturbation theory and matching the powers of the expansion parameter can lead to simplified calculations of scattering amplitudes. For instance, in the case of the imaginary part of a full one-loop amplitude, the two amplitudes on the RHS of (3.131) are tree-level objects.

One can understand the presence of the imaginary part of an amplitude from a concrete consideration of Feynman diagrams. From the Feynman rules of the considered theory one can check that each diagram contributing to the S-matrix element $A$ is purely real unless an internal propagator goes on-shell. In this case, the $i\epsilon$-prescription of the virtual particle's propagator becomes relevant and generates the imaginary part. To see possible implications we might consider the amplitude $A(s)$ as an analytic function of the complex variable $s=E_{\text{cm}}^2$, although physically $s$ is a real variable. If $s_0$ is the threshold energy for creation of the lightest multiparticle state (such that this particle can be created to form a virtual state in a loop-diagram) then $A(s)$ is real in the region $s<s_0$ for $s\in\mathbb{R}$,

$$A(s)=\big[A(s^*)\big]^*\;. \tag{3.133}$$

Since $A(s)$ is an analytic function of (real) $s$ we can continue the function analytically to the entire complex $s$ plane. To that extent we split $A$ into its real and imaginary part, $A(s)=\text{Re}A(s)+i\;\text{Im}A(s)$. Then, for a given $s$ near the real axis with $s>s_0$ we have

$$\begin{aligned}\text{Re}\;A(s+i\epsilon)&=\text{Re}\;A(s-i\epsilon)\;,\\ \text{Im}\;A(s+i\epsilon)&=-\text{Im}\;A(s-i\epsilon)\;.\end{aligned} \tag{3.134}$$

We see that the imaginary part of the amplitude is different above and below the real axis, it has a branch cut, starting at the threshold energy $s_0$. The discontinuity of the analytic function $A(s)$ is given by

$$\text{Disc}\;A(s)=2i\text{Im}\;A(s+i\epsilon) \tag{3.135}$$

and hence, the imaginary part of an amplitude evaluated above the real axis at $s+i\epsilon$ is given by the amplitude's discontinuity along a branch but in a diagrammatic interpretation. By the optical

[11] In many textbooks, the optical theorem is stated in terms of the forward scattering amplitude only where the initial momenta $p_i$ and the final momenta $k_i$ are the same, $p_i=k_i$.

theorem we find that the discontinuity of an amplitude is then related to the sum of phase space integrals as in (3.131).

However, the discontinuity of an amplitude derived from Feynman graphs can be calculated directly by applying a set of cutting rules which are usually denoted as the *Cutkosky Rules*, a development of a series of papers [66–68]. Ultimately, in [69] Cutkosky generalised the analysis of discontinuities of one-loop to multi-loop diagrams[12]. These rules can be used to compute the discontinuity of any amplitude by cutting the corresponding Feynman diagram in a specific way. At one-loop, the discontinuity is given by the following algorithm [65]:

1. Cut for a given kinematic invariant, the diagram such that the two cut propagators can be simultaneously be put on-shell.

2. For each cut, replace the (massive) cut-propagator by a $\delta$-function,

$$\frac{1}{p^2 - m^2 + i\epsilon} \longrightarrow -2\pi i \delta(p^2 - m^2)$$

   Thus, the $\delta$-functions that appear in (3.132) are generated and the propagators are 'put on-shell'. For massless propagators simply set $m^2 = 0$.

3. Perform the phase-space integration over the two-particle $d$LIPS which gives the discontinuity of the diagram in the branch but for the specific kinematical configuration.

This cutting-procedure leads to phase-space integrals which can be, at least in principle, evaluated to extract the discontinuity of the amplitude.

### 3.5.2 Two-particle unitarity cuts and integral basis

Unfortunately, phase-space integrations can become quite cumbersome. The approach of Bern, Dixon, Dunbar and Kosower (BDDK) [31, 70] showed an alternative way of calculating the amplitude's discontinuity by avoiding phase-space integrals. Their idea was to apply unitarity directly at the level of amplitudes and hence, bypassing the use of Feynman diagrams. Rather than integrating over the phase-space, in this *unitarity approach* one replaces the two $\delta$-functions associated with the cuts by propagators and thus generating Feynman integrals instead of phase-space integrals. This procedure is usually denoted as the 'reconstruction of the Feynman integral'. By application of these *two-particle cuts*, BDDK were able to construct many one-loop amplitudes in supersymmetric theories like the $n$-point MHV amplitudes for the $\mathcal{N} = 4$ and $\mathcal{N} = 1$ super Yang-Mills theories [31, 70]. The following discussion is mainly taken from [71].

From this analysis, BDDK could identify which integral functions can appear in the amplitude. At one-loop, all amplitudes in massless gauge theories can be written in a basis of certain integral functions

$$A_{n;1} = \sum_{\mathcal{I}_i} c_i \mathcal{I}_i + \text{rational terms} \tag{3.136}$$

[12] For a detailed discussion, including singularities of the amplitude away from the physical parameter region, see the textbook [55].

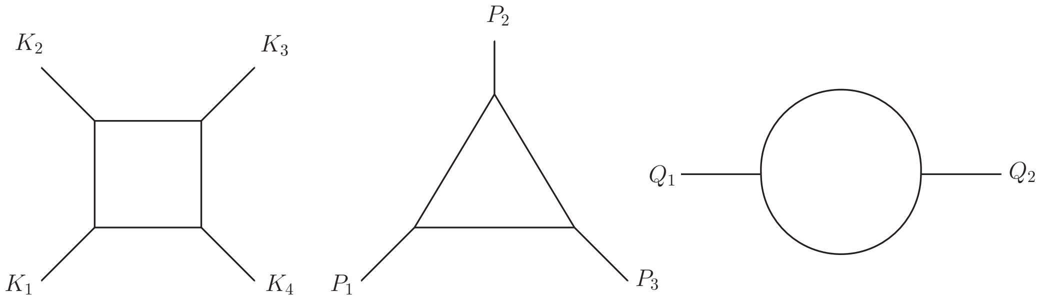



Figure 3.5: *A pictorial representation of the possible integral functions that can appear at one-loop in four-dimensional massless gauge theories. Shown are from left to right the box-, triangle and bubble-integrals. The $K_i$, $P_i$ and $Q_i$ are generic momenta, in the case of $K_i^2, P_i^2, Q_i^2 \neq 0$ the corresponding vertex $i$ is denoted as a massive corner.*

where the expansion is over scalar integral-functions $\mathcal{I}_i$ which are usually denoted as box, triangle and bubble integrals. A pictorial representation of these integral functions is given in Figure 3.5. In addition, rational terms are present. These terms are contributions to the one-loop amplitude which do not contain any branch cuts. The coefficients $c_i$ are rational functions of the external momenta and polarisation vectors. In general, the integral functions can contain products of loop-momenta in their numerator, leading to so-called tensor integrals. We can characterise these general functions by the number of vertices. Conventionally, an integral function with four vertices is called a box integral. Then, a triangle integral function has three and a bubble integral function only two vertices. Furthermore, all integrals can be distinguished in terms of the clusters of their external momenta, $K_i, P_i$ or $Q_i$, compare with Figure 3.5. In the case of $K_i^2, P_i^2$ or $Q_i^2 = 0$ the corresponding vertex is called massless, otherwise it is denoted as a massive vertex. Thus we have for the box integrals four different types, the four-mass box ($K_i^2 \neq 0$ for all $i$), the three-mass box ($K_i^2 = 0$ for one $i$), the two-mass easy (two of the $K_i$ vanish, $K_i^2 = 0$ and $K_{i+2}^2 = 0$ for $i = 1,2$) and the two-mass hard box (two of the $K_i$ vanish, $K_i^2 = 0$ and $K_{i+1}^2 = 0$ for $i = 1,2$) and finally the one-mass box (only one $K_i^2 \neq 0$). Similarly, we can distinguish three-mass, two-mass and one-mass triangles and of course two-mass and one-mass bubbles. In our applications of the unitarity method in four and six dimensions we are mainly interested in the box integral functions since in the maximally supersymmetric theories are UV finite. This excludes the appearance of bubble and triangle integrals. For further discussion and explicit forms of the integral functions see the Appendix I of [70].

In order to construct the full amplitude from the unitarity-cut approach one has to consider all the kinematical channels and perform the double-cut procedure for each of them. However, the two-particle cuts give only the so-called *cut-constructible*[13] part of the amplitude. This is the part of the full one-loop amplitude that contains discontinuities, like logarithms or polylogarithms.

[13]See [31] for a criterion of cut-constructibility which is full-filed by the $\mathcal{N} = 1$ and $\mathcal{N} = 4$ SYM theories. If the degree of the loop-momentum in the numerator-polynomial of a $n$-point one-loop integrand is less then $n$ the supersymmetric theory is called cut-constructible.

Cut-free terms are lost. It is for this reason that the BDDK two-particle cuts are successful in constructing the full amplitude for supersymmetric theories where all rational terms of the amplitude are uniquely linked to terms with discontinuities [31, 70]. However, the two-particle cuts cannot be easily used for non-supersymmetric theories where rational terms in the expansion (3.136) are present. To order $\mathcal{O}(\epsilon^0)$ these rational terms are not linked to terms with discontinuities. If one continues the dimension of the loop-momenta slightly away from four dimensions as $D = 4-2\epsilon$ with $\epsilon < 0$, rational terms develop discontinuities of the form $R(-s)^{-\epsilon} = R - \epsilon\log(-s)R + \mathcal{O}(\epsilon^2)$. Keeping terms at least up to order $\mathcal{O}(\epsilon)$, the amplitudes in the non-supersymmetric theory become cut-constructible. The first applications of cuts in $D = 4-2\epsilon$ dimensions appeared in [72–74].

A further simplification for non-supersymmetric theories comes from the well-known supersymmetric decomposition of one-loop gluon amplitudes in pure Yang-Mills theory. A one-loop amplitude $A_g$ with gluons of the pure YM theory running in the loop can be written as

$$A_g = (A_g + 4A_f + 3A_s) - 4(A_f + A_s) + A_s \tag{3.137}$$

where the first term on the RHS comes from a full $\mathcal{N} = 4$ multiplet running in the loop, the second term likewise from (four times) a $\mathcal{N} = 1$ chiral multiplet whereas the last term comes from just a scalar running in the loop. Since the one-loop amplitudes in the $\mathcal{N} = 4$ and $\mathcal{N} = 1$ theory can be constructed by the double-cut unitarity method, this decomposition allows one to recast the pure gluon amplitude at one-loop by an amplitude with the same external gluon states but with a scalar running in the loop. Again, an efficient way to compute this scalar amplitude $A_s$ is to continue the loop momentum to $4-2\epsilon$ dimensions since a massless scalar in $4-2\epsilon$ dimensions can be described as a massive scalar in four dimensions [73, 74]. This can be easily seen by decomposing the $(4-2\epsilon)$-dimensional loop momentum. If we express the massless scalar as a four-dimensional massive scalar, one has to deal with tree-level amplitudes involving massless gluon states and two massive scalars. This approach is promising since the corresponding amplitudes have been calculated and have a rather simple form [73, 74]. Additionally, amplitudes with massive scalars have been constructed in a BCFW recursive fashion, for more information see [75].

In principle, using the various technical manipulations discussed above, the unitarity-cut technique can be used to calculate amplitudes in various theories. However, the procedure still involves tedious steps and calculations. Firstly, one has to carefully consider the cuts in all the different kinematical channels to obtain the full amplitude. The resulting integral function will have, besides the correct discontinuities of the corresponding channel, additional discontinuities of other channels. This precisely is the reason why one cannot just sum up all contributions from the different kinematical channels to obtain a final expression for the amplitude since this might lead to an over-counting of some discontinuities. In addition, after simplification of the integrand and reconstruction of the Feynman integral, one often ends up with tensor integrals, i.e. with integrands that have products of loop momenta in their numerator. These integrals require reduction techniques such as the Passarino-Veltman (PV) reduction [76] in order to reduce the tensor integrals to scalar ones[14]. Such a procedure results in lengthy expressions for the rational coefficients of the

[14] We will discuss such a reduction in a six-dimensional context in Section 4.5.3 where we find that the one-loop

scalar integrals in the basis (3.136). However, quite often the final results are of a simple form which usually suggests that a more straightforward way for the computation exists. Indeed, a more efficient method for extracting the rational coefficients of the integral basis expansion exists and, intriguingly, it is based on the idea of simply cutting more than two propagators and replacing them with $\delta$-functions. This approach goes under the name of *generalised unitarity* and was known since Cutkosky [69], see also [55] and [37]. In the following sections we present advantages of this procedure and apply it to superamplitudes in the maximally supersymmetric theory in four dimensions.

### 3.5.3 Triple- and quadruple-cuts in generalised unitarity

Cutting more than two propagators leads to triple- (three cut-propagators) and quadruple-cuts (four cut-propagators). Especially quadruple-cuts are of great convenience since they freeze an integration over four-dimensional loop-momenta completely, hence making the calculation of the amplitude a purely algebraic procedure. This is of great importance in theories with maximal amount of supersymmetry as we will see below. In any case, the procedure of putting more than two propagators on-shell (cutting them) offers several advantages over the more restrictive two-particle cut approach. Firstly, it reduces the overlap between different cuts, making it easier to disentangle the individual rational integral-coefficients. Secondly, more on-shell conditions reduce the complexity of the PV reduction of tensor integrals since fewer terms survive the uplift to the Feynman integral. Although more tree-level amplitudes need to be sewn together, the individual tree-level amplitudes are less complex. This all comes at the cost of working in complex momenta: Cutting more than two propagators involves using on-shell three-point tree-level amplitudes that one needs to sew together. As we have seen in Section 3.1.3, these objects are non-vanishing only when working with complex momenta or using a metric of different signature. However, even under the condition of using three-point amplitudes and complex momenta, the more streamlined generalised unitarity approach leads to various applications in supersymmetric and also non-supersymmetric theories.

As we have discussed in the last section, amplitudes in pure Yang-Mills theory are only cut-constructible when considering cuts in $D = 4 - 2\epsilon$ dimensions. The idea of cutting and uplifting more than two propagators can be readily generalised to $(4 - 2\epsilon)$-dimensions in massless, non-supersymmetric theories [71]. The downside of this approach is the fact that higher-dimensional integral functions, in addition to the four-dimensional bubble-, triangle- and box-functions, appear in the integral basis. Nevertheless, to disentangle the various rational coefficients one can use the advanced quadruple and triple cuts. This approach was discussed in [71] where triple and quadruple cuts in $(4 - 2\epsilon)$-dimensions were used to calculate one-loop amplitudes in pure Yang-Mills theory. For instance, to all orders in $\epsilon$, the four- and five-point amplitudes with helicities $(+ + ++)$, $(- + ++)$ and $(+ + + + +)$ (these amplitudes vanish in supersymmetric Yang-Mills theory) were computed and agreement with the original results of [73] and [77] were found.

Moving on to the simplest supersymmetric gauge theory, namely $\mathcal{N} = 1$ SYM theory, the structure becomes simpler. Rational terms do not appear in the expansion of one-loop amplitudes

five-point superamplitude in six dimensions is given by a linear pentagon integral.

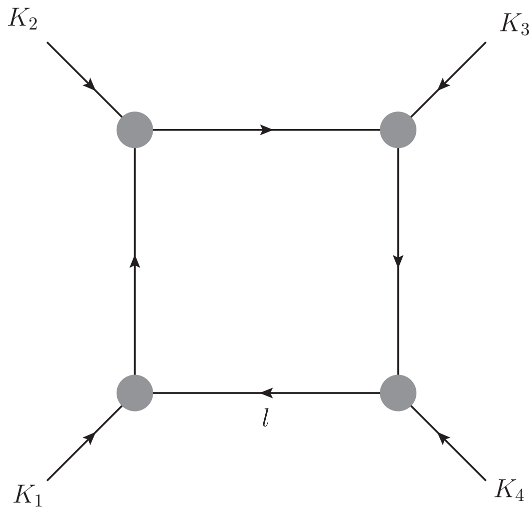



Figure 3.6: *A graphical representation of the scalar box integrals $I_i$ of the expansion* (3.138)*. The cluster $K_i$ of momenta contain all the external momenta. In the case of $K_i^2 \neq 0$ the corresponding corner is called a massive one.*

and one has to deal with linear combinations of scalar box, triangle and bubble functions only. Applying generalised unitarity simplifies the calculation of one-loop amplitudes compared to the original derivation based on two-particle cuts significantly. One may first apply quadruple-cuts to fix the box coefficients. Since the four $\delta$-functions freeze the loop-integration completely, the coefficient of the specific box-function is just a product of four tree-level scattering amplitudes. Then, by a series of algebraic manipulations one rewrites the spinor part of the product of tree-level amplitude such that it is not depending on the loop-momenta. Uplifting the product of the four $\delta$-functions to a full Feynman integral gives the rational coefficient times the corresponding box-function. In a next step one can use triple-cuts to isolate the triangle and bubble coefficients. Cutting three propagators does not freeze the loop-integration completely and hence, after a series of spinor manipulations due to the on-shell conditions one uplifts the cut-integral to a full Feynman integral. Usually, one ends up with a tensor integral which can be reduced to a sum of scalar box-, triangle- and bubble-functions, yielding the final rational coefficients for the specific triple-cut. This procedure was applied to one-loop MHV amplitudes in the $\mathcal{N} = 1$ SYM theory where the two negative helicity gluons are adjacent, confirming the results of the original papers [31, 70]. Also, generalised unitarity for $\mathcal{N} = 1$ SYM theory was used to study next-to-MHV amplitudes at one-loop, again with adjacent negative-helicity gluons. These amplitudes can be expressed by triangle-functions only and where computed by using triple-cuts, see the original paper [78].

The most intriguing example of generalised unitarity is its application to one-loop amplitudes in the four-dimensional $\mathcal{N} = 4$ SYM theory. Although the generlised unitarity approach was in principle known since the sixties [55, 69] it was applied to the maximally supersymmetric case much later [79]. The important difference to theories with less supersymmetry is the fact that one-loop amplitudes in this theory can be written as a linear combination of scalar box functions only[15] [31, 70] and triangle or bubble integral-functions are absent,

$$A_{n;1} = \sum_{\mathcal{P}(\{K_i\})} c_i(K_1, K_2, K_3, K_4)\; I_i(K_1, K_2, K_3, K_4)\;. \tag{3.138}$$

where the $K_i$ are sums of external momenta as shown in Figure 3.6. The scalar box functions are given by the integral[16]

$$I(K_1, K_2, K_3, K_4) = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2(l+K_1)^2(l+K_1+K_2)^2(l-K_4)^2}\;. \tag{3.139}$$

The integral is dimensionally regularised due to the infrared divergences of the one-loop amplitudes. Here, we consider colour-ordered clusters of momenta $K_i$ and each cluster contains consecutive momenta only. The sum in the expansion (3.138) is then over permutations of the consecutive momenta. As discussed in Section 3.5.2, one has four different types of scalar box integrals depending on the number of massive corners. Therefore, it is convenient to expand a one-loop amplitude in the $\mathcal{N}=4$ SYM theory in a basis which distinguishes between the different box integrals as

$$A_{n;1} = \sum \left( c^{1\mathrm{m}} I^{1\mathrm{m}} + c^{2\mathrm{me}} I^{2\mathrm{me}} + c^{2\mathrm{mh}} I^{2\mathrm{mh}} + c^{3\mathrm{m}} I^{3\mathrm{m}} + c^{4\mathrm{m}} I^{4\mathrm{m}} \right) \tag{3.140}$$

where the sum runs over all possible distributions of the $n$ external momenta. For the case of $n=4$ there is just a single box function, the zero-mass integral $I^{0\mathrm{m}}$.

Since the one-loop amplitudes do not contain bubble and triangle integrals, each individual quadruple cut, which is just a specific choice of how to distribute the external states among the $K_i$, singles out a unique box function. The four $\delta$-functions completely localise the integrand on the solutions of the on-shell conditions and the loop integration is completely frozen. The product of the four tree-level amplitudes can be manipulated using the on-shell conditions such that any dependence on the loop momenta is removed. One is then left with an integrand which is just the product of the four $\delta$-functions. Therefore, uplifting all $\delta$-functions to the corresponding Feynman propagators yields directly the appropriate integral representation of the box function associated to the specific cut. The corresponding rational coefficient is then just the (simplified) product of the tree-level amplitudes. Hence, the calculation of the one-loop amplitude is reduced to the algebraic problem of calculating the coefficients $c_i$ [79]. No integrations need to be carried out and in that sense, the generalised unitarity approach is a truly diagrammatic approach for the $\mathcal{N}=4$ SYM theory.

Let us discuss the form of the coefficients $c_i$ in more detail. The scalar box functions are purely kinematical objects and therefore, any dependence on the helicities of the external states are carried by the coefficients $c_i$ which for our purposes are functions of the spinor variables $\lambda_i$ and $\tilde{\lambda}_i$. The specific form of the $c_i$'s are determined by two requirements. Firstly, the four on-shell conditions due to the $\delta$-functions and secondly momentum conservation at the four corners. These conditions

[15] This fact is also true if one allows theories of higher spin. One-loop amplitude in the the maximally supersymmetric gravity theory, $\mathcal{N}=8$ supergravity, can also be written as a sum of box-functions with rational coefficients, see [80] and also [60] for a proof for both maximally supersymmetric theories.

[16] We follow the conventions of [57].

are sufficient to reduce the loop integration to a discrete sum over two solutions $\mathcal{S}_\pm$ with

$$\mathcal{S}_\pm : \qquad l_i^2 = 0 \, , \qquad l_{i+1}^\mu = l_i^\mu + K_i^\mu \, . \tag{3.141}$$

For more details on the explicit solutions see the original work [79] and also the discussions in [81]. For our applications of quadruple cuts of one-loop superamplitudes we do not need the specific form of the solutions. As we will see, it is often sufficient to use the constraints given by momentum conservation to simplify the dependence on the $l_i$ within the product of the four tree-level amplitudes.

This diagrammatic approach simplifies the calculation of the rational coefficients significantly. However, in gauge theories and especially in the maximally supersymmetric SYM theory it is necessary to deal with different distributions of external particle states and their helicity. Furthermore, if we consider one-loop amplitudes, in principle the full $\mathcal{N}=4$ multiplet can run in the loop. That is why one has to carefully consider the external states and sum over all helicity assignments of the particles that propagate inside the loop. One would wish for a more convenient method that captures different particle and helicity configurations within one quadruple cut. In the maximally supersymmetric theory both considerations can be neatly combined by using the superspace/ superamplitude approach which was introduced in Section 3.3.1 and a unitarity method for one-loop superamplitudes.

### 3.5.4 Generalised unitarity for $\mathcal{N}=4$ superamplitudes

The combination of superspace notations for scattering amplitudes and the generalised unitarity method was first discussed in [57]. In this paper a supersymmetric extension of the quadruple cuts was used to calculate the $n$-point MHV and NMHV superamplitudes at one-loop, confirming results for amplitudes in the $\mathcal{N}=4$ theory previously obtained in [82, 83].

Since all one-loop amplitude $A_{n;1}$ in the expansion (3.139) are just subamplitudes of a general superamplitude as discussed in Section 3.3.1, it is easy to extend this approach to the expansion of an one-loop superamplitude $\mathcal{A}_{n;1}$ in the $\mathcal{N}=4$ SYM theory. In the basis of the five different scalar box integral-functions one can expand as

$$\mathcal{A}_{n;1} = \delta^{(4)}\left(P_{\alpha\dot{\alpha}}\right) \sum \left( \mathcal{C}^{1\mathrm{m}} I^{1\mathrm{m}} + \mathcal{C}^{2\mathrm{me}} I^{2\mathrm{me}} + \mathcal{C}^{2\mathrm{mh}} I^{2\mathrm{mh}} + \mathcal{C}^{3\mathrm{m}} I^{3\mathrm{m}} + \mathcal{C}^{4\mathrm{m}} I^{4\mathrm{m}} \right) \, . \tag{3.142}$$

Now, the supercoefficients $C$ are rational functions of the spinor variables but also polynomials in the Grassmann parameters $\eta$ defining the different subamplitudes [57].

In the expansion the same integral functions are present, so the quadrupole cut method discussed in the previous section can be applied to the superamplitude $\mathcal{A}_{n;1}$ straight away. Each specific quadruple cut will single out a specific box function and will result in the corresponding supercoefficient, like the four-mass, three-mass, two-mass easy and two-mass hard or one-mass coefficient. Again, the special case $n = 4$ can be obtained by a single quadruple cut, leading to the zero-mass supercoefficient $\mathcal{C}^{0\mathrm{m}}$ of the four-point one-loop superamplitude. Since the quadruple cut method can be readily extended to the superamplitude formalism, it follows that the supercoefficients $\mathcal{C}$ are given by a product of four tree-level superamplitudes. Performing a quadruple cut as

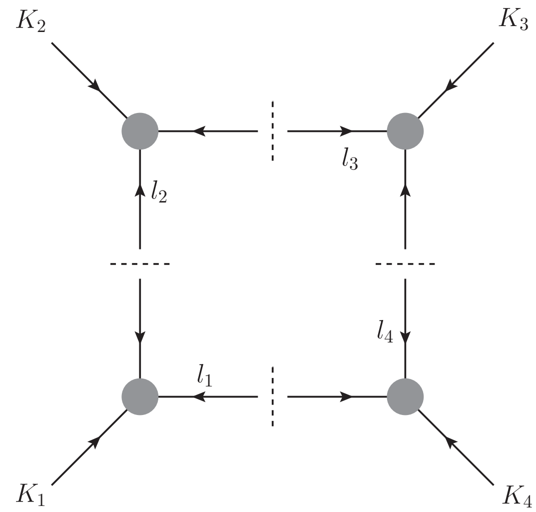



Figure 3.7: *A quadruple cut for a one-loop superamplitude with clusters of external momenta $K_1, K_2, K_3$ and $K_4$. The four cut propagators freeze the loop integration completely.*

shown in Figure 3.7 yields the expression

$$\mathcal{C}(K_i) = \frac{1}{2}\sum_{\mathcal{S}_\pm}\int\left(\prod_{i=1}^{4} d^4\eta_{l_i}\right)\mathcal{A}_{n_1+2;0}(l_1,K_1,-l_2)\,\mathcal{A}_{n_2+2;0}(l_2,K_2,-l_3) \times \mathcal{A}_{n_3+2;0}(l_3,K_3,-l_4)\,\mathcal{A}_{n_4+2;0}(l_4,K_4,-l_1)\ . \tag{3.143}$$

Here one averages over the two solutions $\mathcal{S}_\pm$ due to (3.141). In the superspace formalism one does not have to sum over the different particles of the $\mathcal{N}=4$ multiplet running in the loop, one rather integrates over the Grassmann parameters $\eta_{l_i}$ associated with the internal cut-legs of the four tree-level superamplitudes (see Figure 3.7). The $n_i$ represent the number of external legs for each cluster of momenta $K_i$. This makes it easy to classify the supercoefficients in (3.143) according to the four different types of scalar integral functions. For instance the four-mass supercoefficient is given by $\mathcal{C}^{4\mathrm{m}} = \mathcal{C}(n_{1,2,3,4} \geq 2)$, whereas the two-mass easy supercoefficient is for instance $\mathcal{C}^{2\mathrm{me}} = \mathcal{C}(n_{1,3}=1, n_{2,4}\geq 2)$.

By just extending the amplitudes into superspace, the supercoefficients became polynomials in the Grassmann parameters $\eta$. Just as for tree-level superamplitudes, one can expand them into polynomials of different degree in $\eta$, yielding their most general expression [57] (compare with the expansion (3.97))

$$\mathcal{C}^{\mathrm{m}} = \delta^{(8)}(\sum_{i=1}^{n}\lambda_i\eta_i)\big[\mathcal{P}^{(0),m}_{n;1} + \mathcal{P}^{(4),m}_{n;1} + \cdots + \mathcal{P}^{(4n-16),m}_{n;1}\big]\ . \tag{3.144}$$

where $m$ labels the four different types of integral functions, $m = 4\mathrm{m}, 3\mathrm{m}, 2\mathrm{me}, 2\mathrm{mh}, 1\mathrm{m}$. The above expression gives the most general form. When considering certain helicity configurations of the one-loop superamplitude (for instance a MHV or NMHV configuration), one has to match the total power in the $\eta$'s on both sides of the expression (3.143), taking into account that the integration over the Grassmann variabels $\eta_{l_i}$ reduces the total power in $\eta$ of the left-hand side by 16. This is similar to the case of the supersymmetric recursion relations discussed in Section 3.4. In addition, one has to consider the fact that for specific configurations of the invariant masses $K_i$, terms in the expansion (3.144) of the supercoefficient in polynomials of increasing degree in $\eta$ might be absent. For instance, in the four-mass case, the lowest degree of the one-loop polynomials $\mathcal{P}_{n;1}$ in (3.144) is 8 and the highest degree is $4n - 24$. Therefore, the four-mass supercoefficient $\mathcal{C}^{4\text{m}}$ does not contribute to the MHV and NMHV as well as the anti-MHV and anti-NMHV superamplitudes. To illustrate this behaviour we will discuss the example of a $n$-point one-loop MHV superamplitude in the next section. For further discussions and explicit forms of the supercoefficient we refer the reader to the original paper [57].

As in the case of the individual subamplitudes, the actual calculation of the one-loop superamplitude is then reduced to the derivation of the supercoefficients. One has to simplify the product of the four tree-level superamplitudes in the expression (3.143) as much as possible and perform the integrations over the Grassmann parameters $\eta_{l_i}$. This can be conveniently done by using the fermionic $\delta$-functions of the tree-level superamplitudes. They localise the $\eta_{l_i}$'s on spinor combinations of loop- and external-states. In the case of non-vanishing invariant masses $K_i$ ($n_i > 1$ for all $i$), all tree-level superamplitudes are of the usual form (3.93). One might use the four $\delta^{(8)}(Q^I_\alpha)$ of all corners to isolate a single $\delta^{(8)}(\sum_{\text{ext}} q^I_{i,\alpha})$ for total supermomentum conservation. Upon the identity

$$\delta^{(8)}(\sum_k \lambda_k \eta_k) = \langle ij\rangle^4 \delta^{(4)}\big(\eta_i + \sum_{k\neq j} \frac{\langle jk\rangle}{\langle ij\rangle}\eta_k\big)\ \delta^{(4)}\big(\eta_j - \sum_{k\neq i} \frac{\langle ki\rangle}{\langle ij\rangle}\eta_k\big) \tag{3.145}$$

the individual $\eta_{l_i}$ can be isolated and integrated over if one chooses $i$ and $j$ as the label of the spinors of the two cut propagators of any tree-level vertex.

As soon as one of the invariant masses vanishes (e.g. $K_1^2 = 0$) we have to deal with a three-point vertex and hence a three-point tree-level superamplitude. As discussed in Section 3.3.2, the three-point superamplitudes only exist for the MHV and anti-MHV configuration with complex momenta and their form is given in (3.102) and (3.103), respectively. In the MHV case, the tree-level superamplitude carries the usual $\delta^{(8)}(Q^I_\alpha)$. Since the degree in $\eta$ is the same as before, the separation and integration of the Grassmann spinors $\eta_{l_i}$ follows the same logic. The anti-MHV three-point superamplitude has only degree four in $\eta$ and is proportional to a

$$\delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12]) \tag{3.146}$$

for supercharges $q_1, q_2$ and $q_3$ with external states $p_i$, $i = 1, 2, 3$. Isolating a single Grassmann parameter $\eta_{l_i}$ for an anti-MHV corner is then straight forward. We will see an example of this procedure in the next section when we discuss the one-loop MHV superamplitude.

A complication arises if more than one kinematical invariant $K_i$ vanishes. In this case one has to deal with more than one three-point tree-level superamplitude. We recall that MHV and anti-MHV three-point superamplitudes are defined for different kinematical configurations, see the relations in (3.54). In the case of two adjacent three-point superamplitudes, one has to require that the superamplitudes are not of the same type. If they are both MHV or both anti-MHV as shown

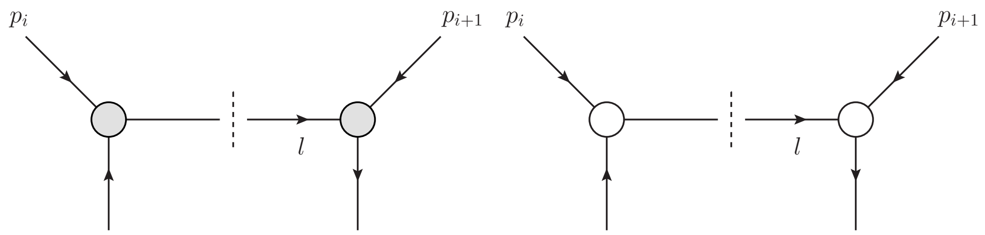



Figure 3.8: *The diagrams for two adjacent MHV (left) or anti-MHV (right) three-point superamplitude. This kinematical configuration does not exist since it would imply the on-shell momentum constraint $(p_i + p_{i+1})^2 = 0$ which is not true for general kinematics.*

in Figure 3.8 the kinematical configurations require that $\tilde{\lambda}_i \propto \tilde{\lambda}_l \propto \tilde{\lambda}_{i+1}$ (MHV) or $\lambda_i \propto \lambda_l \propto \lambda_{i+1}$ (anti-MHV). This would lead to spinor products $[i, i+1] = 0$ or $\langle i, i+1 \rangle = 0$, yielding a vanishing invariant $s_{ii+1} = (p_i + p_{i+1})^2 = \langle i, i+1 \rangle [i+1, i] = 0$. However, for general kinematics this is not fulfilled. Hence, this configuration does not exist and one can only have adjacent MHV and anti-MHV three-point superamplitudes. This is of great importance when calculating superamplitudes at one-loop with certain helicity configurations since it simplifies the possible contributions of the three-point superamplitudes one has to consider. We encounter an example of this structure in the next section where we discuss the derivation of the $n$-point MHV superamplitude at one-loop.

### 3.5.5 The n-point one-loop MHV superamplitude

In order to develop some intuition on the generalised unitarity approach, we want to apply the techniques developed in the last sections to derive the MHV contribution to the one-loop superamplitude with $n$ external states. This result was first presented in [57]. Here, we provide a slightly different derivation and discuss some calculations details in order to illustrate the general procedure.

Since we are considering a MHV configuration we are looking for a Grassmann degree of 8 for the one-loop superamplitude. The generalised unitarity approach reduces the calculation of the superamplitude to the calculation of the supercoefficients in (3.143) of the individual quadruple cuts. The integration over the variables $\eta_{l_i}$ reduces the Grassmann degree of the left-hand side by 16, yielding an allowed Grassmann degree of 24 for the product of the four tree-level superamplitudes. From this observation it is clear that the MHV superamplitude at one-loop only receives contributions from the one-mass and two-mass easy supercoefficients. In the case of three or more non-vanishing kinematical invariants $K_i$ the corresponding tree-level superamplitudes are at least of Grassmann degree 8. This would already exhaust the maximal possible degree in the $\eta$'s of the four tree-level superamplitudes. Hence, we are left with maximal two massive corners which already contribute at least 16 powers in the Grassmann spinors if they are both of MHV type. This fixes the two massless corners to be of anti-MHV type since they both have Grassmann degree 4, resulting in a total degree of $8 + 8 + 4 + 4 - 16 = 8$ for the one-loop MHV superamplitude. This also holds for the exceptional case of $n = 4$ where both MHV superamplitudes are three-point am-

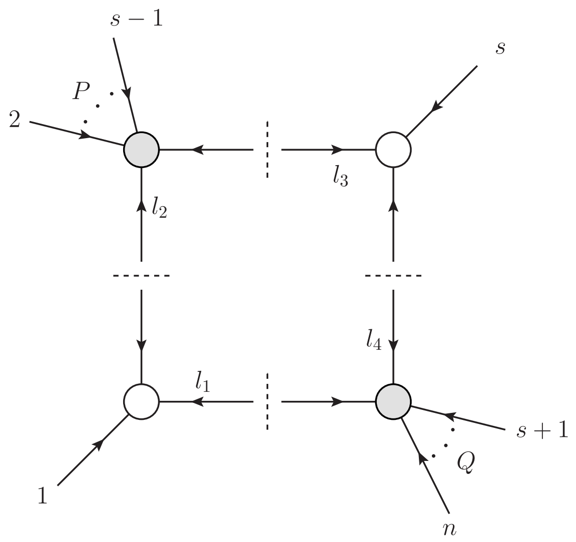



Figure 3.9: *The quadruple cut for an one-loop MHV superamplitude with n external particle states. The two white vertices in the lower left and upper right corner represent three-point anti-MHV superamplitudes whereas the grey vertices in the upper left and lower right higher point represent MHV superamplitudes.*

plitudes, contribution both a Grassmann degree of 8. This results in a zero-mass supercoefficient. Finally, we deduce from this that the two-mass hard coefficient cannot contribute. As explained in the previous section, two three-point MHV or anti-MHV superamplitudes cannot be adjacent. In the two-mass hard case this would result in at least a three-point MHV and a three-point anti-MHV superamplitude. Together with the two massive corners the total Grassmann degree would be $8+8+8+4-16=12$ which does not correspond to the MHV configuration.

We start by computing the supercoefficient $\mathcal{C}^{\text{MHV}}(1,P,s,Q)$ of the quadruple cut shown in Figure 3.9. Cyclic permutation of the external states leads then to the other supercoefficients. For this cut, the momentum $P$ runs in the range of $2,\ldots,s-1$ whereas $Q$ runs between $s+1,\ldots,n$. If one of the corners $P$ or $Q$ become massless then the diagram reduces to the one-mass case and only if both $P$ and $Q$ become massless we have the exceptional case of the four-point MHV superamplitude. Following the generalised unitarity construction, the supercoefficient takes the form

$$\begin{aligned}\mathcal{C}^{\text{MHV}}(1,P,s,Q) =& \frac{1}{2}\sum_{\mathcal{S}_\pm}\int\prod_{i=1}^{4} d\eta_{l_i}\Big[\mathcal{A}_3^{\overline{\text{MHV}}}(l_1,1,-l_2)\,\mathcal{A}^{\text{MHV}}(l_2,2,\ldots,s-1,-l_3)\\ &\times\mathcal{A}_3^{\overline{\text{MHV}}}(l_3,2,-l_4)\,\mathcal{A}^{\text{MHV}}(l_4,s+1,\ldots,n,-l_1)\Big]\,. \qquad (3.147)\end{aligned}$$

Here, the individual tree-level MHV superamplitudes are given by the expressions in (3.99) and (3.103). The general strategy is to integrate over the Grassmann spinors $\eta_{l_i}$ and then simplify the product of the tree-level superamplitudes such that we can remove any dependence on the loop momenta $l_i$. First, we use the fermionic $\delta$-function to simplify the dependence of the integrand on the Grassmann variables $\eta_{l_i}$. The two anti-MHV superamplitudes provides us with a $\delta^{(4)}(\eta_{l_1}[1l_2]+$

$\eta_1[l_2l_1]+\eta_{l_2}[l_11])$ and a $\delta^{(4)}(\eta_{l_3}[sl_4]+\eta_s[l_4l_3]+\eta_{l_4}[l_3s])$. Let us integrate over $\eta_{l_2}$ and $\eta_{l_3}$ first. The two fermionic $\delta$-functions localise the spinors on the two solutions

$$\eta_{l_2} = -\frac{\eta_{l_1}[1l_2]+\eta_1[l_2l_1]}{[l_11]} \qquad \text{and} \qquad \eta_{l_3} = -\frac{\eta_s[l_4l_3]+\eta_{l_4}[l_3s]}{[sl_4]} \ , \tag{3.148}$$

generating the prefactors $[l_11]^4$ and $[sl_4]^4$. We can plug back these solutions in the two remaining $\delta$-functions of the MHV superamplitudes, yielding for instance

$$\delta^{(8)}\big(\frac{\eta_{l_1}[1l_2]+\eta_1[l_2l_1]}{[l_11]}\lambda_{l_2}+\sum_{i=2}^{s-1}\eta_i\lambda_i-\frac{\eta_s[l_4l_3]+\eta_{l_4}[l_3s]}{[sl_4]}\lambda_{l_3}\big) \ . \tag{3.149}$$

In order to combine the arguments of the $\delta^{(8)}$s one might use momentum conservation. The spinor products can be simplified by noting

$$[1l_2]\lambda_{l_2} = [1|l_2 = [1|(-1+l_1) = [1|l_1 = [1l_1]\lambda_{l_1} \tag{3.150}$$

$$[l_2l_1]\lambda_{l_2} = -[1l_1]\lambda_1 \ , \quad [l_4l_3]\lambda_{l_3} = [l_4s]\lambda_s \ , \quad [l_3s]\lambda_{l_3} = [l_4s]\lambda_{l_4} \tag{3.151}$$

leading to an argument of

$$\delta^{(8)}\big(-\eta_{l_1}\lambda_{l_1}+\eta_1\lambda_1+\eta_s\lambda_s+\eta_{l_4}\lambda_{l_4}+\sum_{i=2}^{s-1}\eta_i\lambda_i\big) \ . \tag{3.152}$$

Hence, we have the following product of two $\delta^{(8)}$ to integrate over,

$$\delta^{(8)}\big(-\eta_{l_1}\lambda_{l_1}+\eta_1\lambda_1+\eta_s\lambda_s+\eta_{l_4}\lambda_{l_4}+\sum_{i=2}^{s-1}\eta_i\lambda_i\big)\delta^{(8)}\big(-\eta_{l_4}\lambda_{l_4}+\eta_{l_1}\lambda_{l_1}+\sum_{i=s+1}^{n}\eta_i\lambda_i\big) \ . \tag{3.153}$$

The integration over the remaining Grassmann spinors $\eta_{l_1}$ and $\eta_{l_4}$ can be done by means of the identity (3.145). We use one of the two $\delta^{(8)}$-functions to rewrite it as a product of two $\delta^{(4)}$-functions. and integrate over $\eta_{l_1}$ and $\eta_{l_4}$. This leads to

$$\delta^{(8)}\big(\sum_{i=1}^{n}\eta_i\lambda_i\big)\langle l_1l_4\rangle^4 \tag{3.154}$$

as the result of the integration over the product of the two $\delta^{(8)}$-functions. Collecting the previous results and remaining prefactors we end up with the following expression for the MHV supercoefficient,

$$\begin{aligned}\mathcal{C}^{\text{MHV}}(1,P,s,Q) =& \frac{1}{2}\sum_{\mathcal{S}_\pm}\frac{\delta^{(8)}\big(\sum_{i=1}^n\eta_i\lambda_i\big)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\frac{\langle 12\rangle\langle n1\rangle\langle s-1s\rangle\langle ss+1\rangle}{\langle l_22\rangle\langle s-1l_3\rangle\langle l_4s+1\rangle\langle nl_1\rangle\langle l_2l_3\rangle\langle l_4l_1\rangle}\\ &\times[l_11]^4[sl_4]^4\langle l_4l_1\rangle^4\frac{1}{[l_11][1l_2][l_2l_1]}\frac{1}{[l_3s][sl_4][l_4l_3]} \ .\end{aligned} \tag{3.155}$$

One can simplify the spinor structure further and after a few lines of spinor algebra we arrive at

$$\mathcal{C}^{\text{MHV}}(1,P,s,Q) = \frac{1}{2}\sum_{\mathcal{S}_\pm}\mathcal{A}^{\text{MHV}}_{n;0}\langle n1\rangle\langle 12\rangle\langle s-1s\rangle\langle ss+1\rangle\frac{[1|l_1l_4|s]^2}{\langle s-1|l_3l_4|s+1\rangle\langle 2|l_2l_1|n\rangle} \tag{3.156}$$

where we have used the fact that $[1|l_2l_3|2] = [1|l_1l_4|2]$. However, the supercoefficient still depends on the loop-momenta $l_i$. In the following we focus on the part depending on the $l_i$ and manipulate it further. To that extent we consider the object

$$\Delta_{1,2,s,s+1} \equiv \frac{1}{2}\sum_{\mathcal{S}_\pm}\langle n1\rangle\langle 12\rangle\langle s-1s\rangle\langle ss+1\rangle\frac{[1|l_1l_4|s]^2}{\langle s-1|l_3l_4|s+1\rangle\langle 2|l_2l_1|n\rangle} \tag{3.157}$$

In order to simplify this object we mainly use momentum conservation. This leads to identities like

$$\langle 2|l_2 l_1|n\rangle = \langle 2|(l_1 - 1)l_1|n\rangle = -\langle 21\rangle[1l_1]\langle l_1 n\rangle \tag{3.158}$$

due to the fact that $l_1^2 = 0$ and similarly to

$$\langle s-1|l_3 l_4|s+1\rangle = \langle s-1 s\rangle[s l_4]\langle l_4 s+1\rangle \tag{3.159}$$

since $l_4^2 = 0$. Using these relations, a few lines of spinor algebra yield

$$\Delta_{1,2,s,s+1} = -\sum_{\mathcal{S}_\pm} \frac{1}{2}\frac{\langle s\ s+1\rangle\langle n1\rangle}{\langle l_4\ s+1\rangle\langle l_1 n\rangle}[1l_1]\langle l_1 l_4\rangle^2[l_4 s] \ . \tag{3.160}$$

Furthermore, one might use the proportionality of the holomorphic spinors belonging to the three-point anti-MHV vertex, for instance $\lambda_1 \sim \lambda_{l_1} \sim \lambda_{l_2}$. Combined with momentum conservation, this leads to the expression

$$\frac{\langle s\ s+1\rangle}{\langle l_4\ s+1\rangle}[1l_2]\langle l_2 l_4\rangle\langle 1l_3\rangle[l_3 s] = [1l_2]\langle l_2 s\rangle\langle 1l_3\rangle[l_3 s] = [1|l_2|s\rangle[s|l_3|1\rangle \tag{3.161}$$

where we used the proportionality of $\lambda_{l_4}$ and $\lambda_s$ in the first step. Furthermore, we have from momentum conservation $l_2 = l_1 + p_1 = l_4 + Q + p_1$ which gives $[1|l_2|s\rangle = [1|l_4 - Q|s\rangle = [1|Q|s\rangle$. Similarly, we have $l_3 = l_2 + P$ which yields $[s|l_3|1\rangle = [s|P|1\rangle$ and, hence, we end up with

$$\Delta_{1,2,s,s+1} = \frac{1}{2}\sum_{\mathcal{S}_\pm}[1|Q|s\rangle[s|P|1\rangle = \frac{1}{2}\sum_{\mathcal{S}_\pm}[1|Qp_s P|1\rangle \ . \tag{3.162}$$

By rewriting the spinor expression as Dirac traces we can simplify these spinor products further. Converting the trace structure into products of usual four-dimensional momenta leads to

$$\langle 1|P|s]\langle s|Q|1] = 2\big[(p_1\cdot P)(p_s\cdot Q) - (p_1\cdot p_s)(P\cdot Q) + (p_1\cdot Q)(P\cdot p_s)\big] \ . \tag{3.163}$$

Let us introduce the momentum invariants[17] $s = (p_1 + P)^2$ and $t = (p_s + P)^2$. $s = (p_1 + P)^2$ and $t = (p_s + P)^2$ and apply momentum conservation $p_1 + P + p_s + Q = 0$ we can rewrite the products of four-momenta in the expression (3.163). this expression into a simple form as

$$\Delta_{1,2,s,s+1} = \frac{1}{2}\sum_{\mathcal{S}_\pm}\left(P^2Q^2 - st\right) \ . \tag{3.164}$$

At this stage we observe that due to the presence of the three-point anti-MHV superamplitude, only one of the solutions $\mathcal{S}_\pm$ contributes since $\lambda_1 \propto \lambda_{l_1} \propto \lambda_{l_2}$. Furthermore, the explicit form of the solution is not needed since $P, Q$ and $s$ and $t$ are fixed by the external kinematics. Hence, we can write for the MHV supercoefficient

$$\mathcal{C}^{\text{MHV}}(1,P,s,Q) = \frac{1}{2}\left(P^2Q^2 - st\right)\frac{\delta^{(8)}\left(\sum_{i=1}^n \eta_i\lambda_i\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} = \frac{1}{2}\left(P^2Q^2 - st\right)\mathcal{A}_{n;0}^{\text{MHV}} \ . \tag{3.165}$$

This is its generic form. In the two-mass easy case, we have the condition that $P^2, Q^2 \neq 0$. This leads to the condition $4 \leq s \leq n-2$. If we consider the one-mass case, one of the MHV corners becomes massless and we have an additional three-point MHV superamplitude in the case of either

[17]The reader should note that the momentum $p_s$ should not to be confused with the kinematic invariant $s$.

$P^2 = 0$ or $Q^2 = 0$. This results in $s = 3$ or $s = n - 1$ (compare with Figure 3.9). Hence, the supercoefficient combines both the two- and one-mass case if $s$ ranges between 3 and $n - 1$. This allows us to write the result of the one-loop MHV superamplitude in a compact form [57]

$$\mathcal{A}_{n;1}^{\mathrm{MHV}} = \delta^{(4)}(P_{\alpha\dot{\alpha}})\frac{\delta^{(8)}\left(\sum_{i=1}^{n}\eta_i\lambda_i\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\left[\sum_{s=3}^{n-1} I_{1,2,s,s+1}\Delta_{1,2,s,s+1} + \text{cyclic}\right]. \tag{3.166}$$

We have discussed the specific cut that leads to the first term in the expression for the one-loop superamplitude. The other terms are obtained from cuts where the external states are cyclicly permuted and have the same structure, i.e. they can be obtained from the result of our cut calculation by just shifted in labels of the external states appropriately.

This concludes our discussion of four-dimensional superamplitudes in the $\mathcal{N} = 4$ SYM theory and the generalised unitarity approach. Other supercoefficients of one-loop amplitudes can be found in [57] for the case of $\mathcal{N} = 4$ SYM and in [84] for $\mathcal{N} = 8$ supergravity. In the next chapter we consider applications of the discussed four-dimensional concepts in the maximally supersymmetric gauge theory in six dimensions.

# 4 Superamplitudes and Generalised Unitarity in Six Dimensions

As we have seen in the previous chapter, considering amplitudes in the maximally supersymmetric theory in four dimensions provides a rich field of investigations. The supersymmetric formulation provides not only a nice tool to simplify calculations, it provides us also with the opportunity to combine amplitudes with different particle content and helicity configurations in a compact form. One might wonder if this approach extends to higher dimensions.

Furthermore, there are several reasons why it is interesting to consider scattering amplitudes in six-dimensional theories. Firstly, there is a powerful spinor helicity formalism, introduced in [85] and further discussed in [86] for arbitrary dimensions, which allows one to express scattering amplitudes in a rather compact form. An important difference with respect to the four-dimensional world is that physical states are no longer labeled by their helicity, but carry indices of the little group $\mathsf{SU}(2) \times \mathsf{SU}(2)$ of a massless particle. As a consequence, states in a particular little group representation can be rotated into each other, and hence, at a fixed number of external legs, all scattering amplitudes for different external states are collected into a single object, transforming covariantly under the little group. In [85], an expression for the three-point gluon amplitude in Yang-Mills theory was obtained, and used to derive tree-level four- and five-point amplitudes using on-shell recursion relations [53, 54].

Particularly interesting are the maximally supersymmetric theories in six dimensions, with (1,1) and (2,0) supersymmetry, which arise as the low-energy effective field theories on fivebranes in string/M-theory and upon compactification on a two-torus reduce to $\mathcal{N} = 4$ super Yang-Mills (SYM) in four dimensions. The scattering superamplitudes in the (1,1) theory have been studied in [87] (see also [88]), using supersymmetric on-shell recursion relations [58, 60]. In particular, the three-, four- and five-point superamplitudes at tree-level have been derived, as well as the the one-loop four-point superamplitude, using the unitarity-based approach of [31,70]. Some generalisations to (2,0) theories in six dimensions have been considered in [88].

Six-dimensional tree-level amplitudes take a rather compact form, which can be fed into unitarity [31, 70] and generalised unitarity cuts [79, 89] to generate loop amplitudes. Originally the unitarity methods and their generalisations were formulated in four dimensions but they apply in principle in any number of dimensions, which is also often exploited in calculations of QCD amplitudes in dimensional regularisation (see e.g. [71–73,90]). First applications of unitarity to one-loop four-point amplitudes in six-dimensional (1,1) theories appeared in [87] and more recently in six-dimensional Yang-Mills in [91], where also higher-loop four-point amplitudes in the (1,1) theory were computed.

Gauge theories in more than four dimensions are usually non-renormalisable, but at least for the maximally supersymmetric examples their known embedding into string theory as low-energy theories living on D-branes or M-branes guarantees the existence of a UV completion. In particular, it is known that the (1,1) supersymmetric gauge theory in six dimensions is finite up to two loops [92]. Furthermore, infrared divergences are absent in more than four dimensions, and hence all amplitudes in the (1,1) theory are expected to be finite up to two-loop order and can be calculated without regularisation.

An additional motivation to study higher-dimensional theories stems from the fact that QCD amplitudes in dimensional regularisation naturally give rise to integral functions in higher dimensions, in particular $D = 6$ and $D = 8$ [72, 73]. These integrals are related to finite, rational terms or terms that vanish in the four-dimensional limit. Furthermore, there exists a mysterious dimension shift relation between MHV one-loop amplitudes in the maximally supersymmetric gauge theory in eight dimension (with four-dimensional external momenta) and the finite same-helicity one-loop gluon amplitude in pure Yang-Mills in four dimensions [74].

In this chapter, which is based on the author's original work [93], we focus on the calculation of four- and five-point superamplitudes in the maximally supersymmetric (1,1) theory using one-loop two-particle as well as quadruple cuts. In particular, we show that the five-point superamplitude can be expressed in terms of just a linear pentagon integral in six dimensions, which can be further reduced in terms of scalar pentagon and box functions. Because of the non-chiral nature of the (1,1) on-shell superspace, this superamplitude contains all possible component amplitudes with five particles, in contradistinction with the four-dimensional case where one has to distinguish MHV and anti-MHV helicity configurations.

We begin our discussion in Section 4.1 by briefly reviewing the six-dimensional spinor helicity formalism developed in [85], which is required to present Yang-Mills scattering amplitudes in a compact form. Then, in Section 4.2 we discuss the on-shell (1,1) superspace description of amplitudes in maximally supersymmetric Yang-Mills which was introduced in [87]. We then move on and discuss briefly the three-, four- and five-point amplitudes at tree-level in the six-dimensional theory.

## 4.1 Spinor helicity formalism in six dimensions

The key observation for a compact formulation of amplitudes in six-dimensional gauge and gravity theories is that, similarly to four dimensions, null momenta in six dimensions can be conveniently presented in a spinor helicity formalism, introduced in [85]. Firstly, one rewrites vectors of the Lorentz group $\mathsf{SO}(1,5)$ as antisymmetric $\mathsf{SU}(4)$ matrices

$$p^{AB} \ := \ p^{\mu}\tilde{\sigma}_{\mu}^{AB} \ , \tag{4.1}$$

using the appropriate Clebsch-Gordan symbols $\tilde{\sigma}_\mu^{AB}$, where $A, B = 1, \ldots, 4$ are fundamental indices of $\mathsf{SU}(4)$. One can similarly introduce[1]

$$p_{AB} \;:=\; \frac{1}{2}\epsilon_{ABCD}p^{CD} := p^\mu \sigma_{\mu,AB} \;, \tag{4.2}$$

with $\sigma_{\mu,AB} := (1/2)\epsilon_{ABCD}\tilde{\sigma}_\mu^{CD}$. Here, we generally work with complex momenta. For the case of real momenta, see the discussion on reality conditions of six dimensional momenta in the appendix of [87]. When $p^2 = 0$, it is natural to recast $p^{AB}$ and $p_{AB}$ as the product of two spinors as [85]

$$\begin{aligned} p^{AB} &= \lambda^{Aa}\lambda_a^B \;, \\ p_{AB} &= \tilde{\lambda}_A^{\dot{a}}\tilde{\lambda}_{B\dot{a}} \;. \end{aligned} \tag{4.3}$$

Here $a = 1, 2$ and $\dot{a} = 1, 2$ are indices of the little group[2] $\mathsf{SO}(4) \simeq \mathsf{SU}(2) \times \mathsf{SU}(2)$, which are contracted with the usual invariant tensors $\epsilon_{ab}$ and $\epsilon_{\dot{a}\dot{b}}$. The expression for $p$ given in (4.3) automatically ensures that $p$ is a null vector, since

$$p^2 = -\frac{1}{8}\epsilon_{ABCD}\lambda_a^A\lambda_b^B\lambda_c^C\lambda_d^D\epsilon^{ab}\epsilon^{cd} \;=\; 0 \;. \tag{4.4}$$

The dot product of two null vectors $p_i$ and $p_j$ can also be conveniently written using spinors as

$$p_i \cdot p_j \;=\; -\frac{1}{4}p_i^{AB}p_{j;AB} \;. \tag{4.5}$$

Lorentz invariant contractions of two spinors are expressed as

$$\langle i_a | j_{\dot{a}}] := \lambda_{i,a}^A \tilde{\lambda}_{j,A\dot{a}} = \tilde{\lambda}_{j,A\dot{a}}\lambda_{i,a}^A =: [j_{\dot{a}} | i_a\rangle \;. \tag{4.6}$$

Further Lorentz-invariant combinations can be constructed from four spinors using the $\mathsf{SU}(4)$ invariant $\epsilon$ tensor, as

$$\begin{aligned} \langle 1_a\, 2_b\, 3_c\, 4_d\rangle &:= \epsilon_{ABCD}\lambda_{1,a}^A\lambda_{2,b}^B\lambda_{3,c}^C\lambda_{4,d}^D \;, \\ [1_{\dot{a}}\, 2_{\dot{b}}\, 3_{\dot{c}}\, 4_{\dot{d}}] &:= \epsilon^{ABCD}\tilde{\lambda}_{1,A\dot{a}}\tilde{\lambda}_{2,B\dot{b}}\tilde{\lambda}_{3,C\dot{c}}\tilde{\lambda}_{4,D\dot{d}} \;. \end{aligned} \tag{4.7}$$

This notation may be used to express compactly strings of six-dimensional momenta contracted with Dirac matrices, such as

$$\begin{aligned} \langle i_a | \hat{p}_1\hat{p}_2 \ldots \hat{p}_{2n+1} | j_b\rangle &:= \lambda_{i,a}^{A_1}\, p_{1,A_1A_2}\, p_2^{A_2A_3} \ldots p_{2n+1,A_{2n+1}A_{2n+2}}\, \lambda_{j,b}^{A_{2n+2}} \;, \\ \langle i_a | \hat{p}_1\hat{p}_2 \ldots \hat{p}_{2n} | j_{\dot{a}}] &:= \lambda_{i,a}^{A_1}\, p_{1,A_1A_2}\, p_2^{A_2A_3} \ldots p_{2n}^{A_{2n}A_{2n+1}}\, \tilde{\lambda}_{j,A_{2n+1}\dot{b}} \;. \end{aligned} \tag{4.8}$$

Having discussed momenta, we now consider polarisation states of particles. In four dimensions, these are associated to the notion of helicity. In six dimensions, physical states, and hence their wavefunctions, transform according to representations of the little group, and therefore carry $\mathsf{SU}(2) \times \mathsf{SU}(2)$ indices [85]. In particular, for gluons of momentum $p$ defined as in (4.3) one has

$$\epsilon_{a\dot{a}}^{AB} := \lambda_a^{[A}\mu_b^{B]}\langle\mu_b|\tilde{\lambda}^{\dot{a}}]^{-1} \;, \tag{4.9}$$

[1] Our notation and conventions are outlined in Appendix B.1.

[2] Or $\mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C})$, if we complexify spacetime.

or alternatively

$$\epsilon_{a\dot{a};AB} := \langle \lambda^a | \tilde{\mu}_{\dot{b}}]^{-1} \tilde{\mu}_{\dot{b}[A} \tilde{\lambda}_{\dot{a}B]} \ . \tag{4.10}$$

Here, $\mu$ and $\tilde{\mu}$ are spinors of a reference momentum $q$, and the denominator is defined to be the inverse of the matrices $\langle q^b | p_{\dot{a}}]$ and $\langle p_a | q^{\dot{b}}]$, respectively.[3]

It is amusing to make contact between six-dimensional spinors and momentum twistors [94], employed recently to describe amplitudes in four-dimensional conformal theories. There, one describes a point in (conformally compactified) Minkowski space as a six-dimensional null vector $X$, i.e. one satisfying $\eta_{ij} X^i X^j = 0$, with $\eta = \text{diag}(+ - - -; + -)$. The conformal group $\mathsf{SO}(2,4)$ acts linearly on the $X$ variables, and plays the role of the Lorentz group $\mathsf{SO}(1,5)$ acting on our six-dimensional momenta $p$. Furthermore, in contradistinction with the null six-dimensional momenta, the coordinate $X$ are defined only up to nonvanishing rescalings. For (cyclically ordered) four-dimensional region momenta $x_i$, one defines the corresponding six-dimensional null $X_i$ as $X_i = \lambda_i \wedge \lambda_{i+1}$ , $X_j = \lambda_j \wedge \lambda_{j+1}$, and $X_i \cdot X_j = \langle i\, i+1\, j\, j+1 \rangle$.

## 4.2 The $\mathcal{N} = (1,1)$ on-shell superspace

We will now review the on-shell superspace description of $(1,1)$ theories introduced in [87]. This construction is inspired by the covariant on-shell superspace formalism for four-dimensional $\mathcal{N} = 4$ SYM introduced by Nair in [59]. As we have seen in in Section 3.3.1, by decomposing the supercharges along two independent directions, the four-dimensional on-shell $\mathcal{N} = 4$ algebra can be represented as

$$\{q^I, \bar{q}_J\} = \delta^I_J \tag{4.11}$$

where $I, J$ are $\mathsf{SU}(4)$ $R$-symmetry indices in four dimensions. This yields a chiral representation of the algebra in terms of Grassmann variables $\eta^I$, as

$$q^I = \eta^I \ , \qquad \bar{q}_J = \frac{\partial}{\partial \eta^J} \ . \tag{4.12}$$

For an anti-chiral representation the roles of $q$ and $\bar{q}$ in (4.12) are interchanged.

One can apply similar ideas to the case of the $\mathcal{N} = (1,1)$ superspace of the six-dimensional SYM theory. However, for this on-shell space the chiral and anti-chiral components do not decouple. To see this we start with the algebra

$$\begin{aligned} \{q^{AI}, q^{BJ}\} &= p^{AB} \epsilon^{IJ} \ , \\ \{\tilde{q}_{AI'}, \tilde{q}_{BJ'}\} &= p_{AB} \epsilon_{I'J'} \ , \end{aligned} \tag{4.13}$$

where $A, B$ are the $\mathsf{SU}(4)$ Lorentz index and $I, J$ and $I', J'$ are indices of the $R$-symmetry group $\mathsf{SU}(2) \times \mathsf{SU}(2)$. As before, we decompose the supercharges as

$$\begin{aligned} q^{AI} &= \lambda^{Aa} q^I_{(1)a} + \mu^{Aa} q^I_{(2)a} \ , \\ \tilde{q}_{BI'} &= \tilde{\lambda}^{\dot{b}}_B \tilde{q}_{(1)\dot{b}I'} + \tilde{\mu}^{\dot{b}}_B \tilde{q}_{(2)\dot{b}I'} \ , \end{aligned} \tag{4.14}$$

[3]The reference spinors are chosen such that the matrices $\langle q^b | p_{\dot{a}}]$ and $\langle p_a | q^{\dot{b}}]$ are nonsingular.

with $\det(\lambda^{Aa}\tilde{\mu}^{\dot{a}}_A) \neq 0$ and $\det(\mu^{Aa}\tilde{\lambda}^{\dot{a}}_A) \neq 0$. Multiplying the supercharges in (4.14) by $\tilde{\lambda}_{A\dot{a}}$ and $\lambda^B_b$, respectively, and summing over the $\mathsf{SU}(4)$ indices, one finds that

$$\begin{aligned} \{q^I_{(2)a}, q^J_{(2)b}\} &= 0 \ , \\ \{\tilde{q}_{(2)\dot{a}I'}, \tilde{q}_{(2)\dot{b}J'}\} &= 0 \ . \end{aligned} \tag{4.15}$$

One can thus set all the $q_{(2)}$ and $\tilde{q}_{(2)}$ charges equal to zero, so that $q^{AI} = \lambda^{Aa} q^I_{(1)a}$. The supersymmetry algebra then yields,

$$\begin{aligned} \{q^I_{(1)a}, q^J_{(1)b}\} &= \epsilon_{ab}\epsilon^{IJ} \ , \\ \{\tilde{q}_{(1)I'\dot{a}}, \tilde{q}_{(1)J'\dot{b}}\} &= \epsilon_{\dot{a}\dot{b}}\epsilon_{I'J'} \ . \end{aligned} \tag{4.16}$$

The realisation of (4.16) in terms of anticommuting Grassmann variables is

$$q^{AI} = \lambda^{Aa}\eta^I_a \, , \qquad \tilde{q}_{AI'} = \tilde{\lambda}^{\dot{a}}_A \tilde{\eta}_{I'\dot{a}} \ . \tag{4.17}$$

In contrast to the four-dimensional $\mathcal{N}=4$ SYM theory, the $\mathcal{N}=(1,1)$ on-shell superspace in six dimensions carries chiral and anti-chiral components. The field strength of the six-dimensional SYM theory transforms under the little group $\mathsf{SU}(2)\times\mathsf{SU}(2)$ and therefore carries both indices $a$ and $\dot{a}$. Hence, one needs both $\eta_a$ and $\tilde{\eta}_{\dot{a}}$ to describe all helicity states in this theory.

In order to describe only the physical components of the full six-dimensional SYM theory, one needs to truncate half of the superspace charges in (4.17) [87]. This is performed by contracting the $R$-symmetry indices with fixed two-component (harmonic) vectors, which effectively reduce the number of supercharges by a factor of two. The resulting truncated supersymmetry generators are then [87]

$$q^A = \lambda^{Aa}\eta_a \, , \qquad \tilde{q}_A = \tilde{\lambda}^{\dot{a}}_A \tilde{\eta}_{\dot{a}} \ . \tag{4.18}$$

Using this on-shell superspace, one can neatly package all states of the theory into a six-dimensional analogue of Nair's superfield [59],

$$\begin{aligned} \Phi(p;\eta,\tilde{\eta}) &= \phi^{(1)} + \psi^{(1)}_a\eta^a + \tilde{\psi}^{(1)}_{\dot{a}}\tilde{\eta}^{\dot{a}} + \phi^{(2)}\eta^a\eta_a + A_{a\dot{a}}\eta^a\tilde{\eta}^{\dot{a}} + \phi^{(3)}\tilde{\eta}^{\dot{a}}\tilde{\eta}_{\dot{a}} \\ &+ \psi^{(2)}_a\eta^a\tilde{\eta}^{\dot{a}}\tilde{\eta}_{\dot{a}} + \tilde{\psi}^{(2)}_{\dot{a}}\tilde{\eta}^{\dot{a}}\eta^a\eta_a + \phi^{(4)}\eta^a\eta_a\tilde{\eta}^{\dot{a}}\tilde{\eta}_{\dot{a}} \ . \end{aligned} \tag{4.19}$$

Here $\phi^{(i)}(p)$, $i=1,\ldots,4$ are four scalar fields, $\psi^{(l)}(p)$ and $\tilde{\psi}^{(l)}(p)$, $l=1,2$ are fermion fields and finally $A_{a\dot{a}}(p)$ contains the gluons. Upon reduction to four dimensions, $A_{a\dot{a}}$ provides, in addition to gluons of positive and negative (four-dimensional) helicity, the two remaining scalar fields needed to obtain the matter content of $\mathcal{N}=4$ super Yang-Mills.[4] A pictorial representation of the states in the (1,1) supermultiplet is given in Figure 4.1.

## 4.3 Tree-level amplitudes and their properties

In the following we briefly review the form of the three-, four- and five-point amplitudes in six-dimensional Yang-Mills theory and discuss their supersymmetrisation. For further information we refer the reader to the original papers [85, 87].

[4] More details on the reduction to four dimensions are provided in Section 4.5.5.

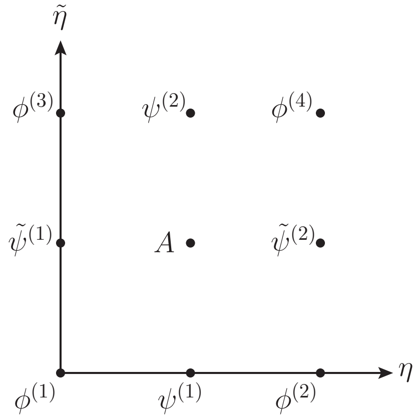



Figure 4.1: *The component fields of the (1,1) superfield given in* (4.19).

### 4.3.1 Three-point amplitude

The smallest amplitude one encounters is the three-point amplitude. In four dimensions, and for real kinematics, three-point amplitudes vanish because $p_i \cdot p_j = 0$ for any of the three particles' momenta, but are non-vanishing upon spacetime complexification [53, 54]. In the six-dimensional spinor helicity formalism, the special three-point kinematics induces the constraint $\mathrm{d}et\langle i_a|j_{\dot a}] = 0$, $i, j = 1, 2, 3$. This allows one to write (see Appendix B.1)

$$\langle i_a|j_{\dot b}] = (-)^{\mathcal{P}_{ij}} u_{ia}\tilde{u}_{j\dot b} \ , \tag{4.20}$$

where we choose $(-)^{\mathcal{P}_{ij}} = +1$ for $(i, j) = (1, 2), (2, 3), (3, 1)$ and $-1$ for $(i, j) = (2, 1), (3, 2), (1, 3)$. One can also introduce the spinors $w_a$ and $\tilde{w}_{\dot a}$ [85], defined as the inverse of $u_a$ and $\tilde{u}_{\dot a}$,

$$u_a w_b - u_b w_a := \epsilon_{ab} \quad \Leftrightarrow \quad u^a w_a := -u_a w^a := 1\,. \tag{4.21}$$

As stressed in [85] the $w_i$ spinors are not uniquely specified. Momentum conservation suggests a further constraint that may imposed in order to reduce this redundancy. This is used in various calculations throughout the present work. Specifically, for a generic three-point amplitude it is assumed that

$$|w_1 \cdot 1\rangle + |w_2 \cdot 2\rangle + |w_3 \cdot 3\rangle = 0 \ , \tag{4.22}$$

where we have used the abbreviation $|w_i \cdot i\rangle = w_i^a \lambda^A_{i,a}$. One may then express the three-point tree-level amplitude for six-dimensional Yang-Mills theory as [85]

$$A_{3;0}(1_{a\dot a}, 2_{b\dot b}, 3_{c\dot c}) = i\Gamma_{abc}\tilde{\Gamma}_{\dot a\dot b\dot c} \ , \tag{4.23}$$

where the tensors $\Gamma$ and $\tilde{\Gamma}$ are given by

$$\begin{aligned}\Gamma_{abc} &= u_{1a}u_{2b}w_{3c} + u_{1a}w_{2b}u_{3c} + w_{1a}u_{2b}u_{3c},\\ \tilde{\Gamma}_{\dot a\dot b\dot c} &= \tilde{u}_{1\dot a}\tilde{u}_{2\dot b}\tilde{w}_{3\dot c} + \tilde{u}_{1\dot a}\tilde{w}_{2\dot b}\tilde{u}_{3\dot c} + \tilde{w}_{1\dot a}\tilde{u}_{2\dot b}\tilde{u}_{3\dot c}\ .\end{aligned} \tag{4.24}$$

As recently shown in [87], this result can be combined with the $\mathcal{N} = (1,1)$ on-shell superspace in six dimensions. The corresponding three-point tree-level superamplitude takes the simple form [87]

$$\mathcal{A}_{3;0}(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{c\dot{c}}) \;=\; i\,\delta(Q^A)\delta(\tilde{Q}_A)\delta(Q^B)\delta(\tilde{Q}_B)\delta(W)\delta(\tilde{W}) \;. \tag{4.25}$$

Here we have introduced the $\mathcal{N} = (1,1)$ supercharges for the external states,

$$Q^A \;:=\; \sum_{i=1}^{n} q_i^A = \sum_{i=1}^{n} \lambda_i^{Aa}\eta_{ia} \;, \qquad\qquad \tilde{Q}_A \;:=\; \sum_{i=1}^{n} \tilde{q}_{iA} = \sum_{i=1}^{n} \tilde{\lambda}_{iA}^{\dot{a}}\tilde{\eta}_{i\dot{a}} \tag{4.26}$$

(with $n = 3$ in the three-point amplitude we are considering in this section). The quantities $W, \tilde{W}$ appear only in the special three-point kinematics case, and are given by

$$W \;:=\; \sum_{i=1}^{3} w_i^a \eta_{ia} \;, \qquad \tilde{W} \;:=\; \sum_{i=1}^{3} \tilde{w}_i^{\dot{a}} \tilde{\eta}_{i\dot{a}} \;. \tag{4.27}$$

In Appendix B.2 we give an explicit proof of the (non-manifest) invariance of the three-point superamplitude under supersymmetry transformations, and hence of the fact that the total supermomentum $Q^A = \sum_i q_i^A$ is conserved.

### 4.3.2 Four-point amplitude

The four-point tree-level amplitude in six dimensions is given by

$$A_{4;0}(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{c\dot{c}}, 4_{d\dot{d}}) = -\frac{i}{st}\langle 1_a 2_b 3_c 4_d\rangle [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}] \;, \tag{4.28}$$

and was derived by using a six-dimensional version [85] of the BCFW recursion relations [53, 54]. The corresponding $\mathcal{N} = (1,1)$ superamplitude is [87]

$$\mathcal{A}_{4;0}(1,\ldots,4) = -\frac{i}{st}\delta^{(4)}(Q)\delta^{(4)}(\tilde{Q}) \;, \tag{4.29}$$

where the $(1,1)$ supercharges are defined in (4.26). In (4.29) we follow [87] and introduce the fermionic $\delta$-functions which enforce supermomentum conservation as

$$\begin{aligned} \delta^{(4)}(Q)\delta^{(4)}(\tilde{Q}) \;&=\; \frac{1}{4!}\epsilon_{ABCD}\delta(Q^A)\delta(Q^B)\delta(Q^C)\delta(Q^D) \\ &\times\; \frac{1}{4!}\epsilon^{A'B'C'D'}\delta(\tilde{Q}_{A'})\delta(\tilde{Q}_{B'})\delta(\tilde{Q}_{C'})\delta(\tilde{Q}_{D'}) \\ &:=\; \delta^{(8)}(Q) \;. \end{aligned} \tag{4.30}$$

Hence, a $\delta^{(4)}(Q)$ sets $Q^A = 0$ whereas the $\delta^{(4)}(\tilde{Q})$ sets $\tilde{Q}_A = 0$.

### 4.3.3 Five-point amplitude

The five-point tree-level amplitude was derived in [85] using recursion relations, and is equal to[5]

$$A_{5;0}(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{c\dot{c}}, 4_{d\dot{d}}, 5_{e\dot{e}}) \;=\; \frac{i}{s_{12}s_{23}s_{34}s_{45}s_{51}}\left(\mathcal{A}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} + \mathcal{D}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}}\right) \tag{4.31}$$

[5] In Appendix B.5 the five-point amplitude (4.31) is reduced to four dimensions and found to be in agreement with the expected Parke-Taylor expression.

where the two tensors $\mathcal{A}$ and $\mathcal{D}$ are given by

$$\mathcal{A}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} = \langle 1_a|\hat{p}_2\hat{p}_3\hat{p}_4\hat{p}_5|1_{\dot{a}}]\langle 2_b 3_c 4_d 5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + \text{cyclic permutations}\ , \tag{4.32}$$

and

$$\begin{aligned} 2\mathcal{D}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} = &\ \langle 1_a|(2\cdot\tilde{\Delta}_2)_{\dot{b}}]\langle 2_b 3_c 4_d 5_e\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + \langle 3_c|(4\cdot\tilde{\Delta}_4)_{\dot{d}}]\langle 1_a 2_b 4_d 5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}5_{\dot{e}}] \\ &+ \langle 4_d|(5\cdot\tilde{\Delta}_5)_{\dot{e}}]\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] - \langle 3_c|(5\cdot\tilde{\Delta}_5)_{\dot{e}}]\langle 1_a 2_b 4_d 5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] \\ &- [1_{\dot{a}}|(2\cdot\Delta_2)_b\rangle\langle 1_a 3_c 4_d 5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] - [3_{\dot{c}}|(4\cdot\Delta_4)_d\rangle\langle 1_a 2_b 3_c 5_e\rangle[1_{\dot{a}}2_{\dot{b}}4_{\dot{d}}5_{\dot{e}}] \\ &- [4_{\dot{d}}|(5\cdot\Delta_5)_e\rangle\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}5_{\dot{e}}] + [3_{\dot{c}}|(5\cdot\Delta_5)_e\rangle\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}}2_{\dot{b}}4_{\dot{d}}5_{\dot{e}}]\ . \end{aligned} \tag{4.33}$$

Here, the spinor matrices $\Delta$ and $\tilde{\Delta}$ are defined by

$$\Delta_1 = \langle 1|\hat{p}_2\hat{p}_3\hat{p}_4 - \hat{p}_4\hat{p}_3\hat{p}_2|1\rangle, \quad \tilde{\Delta}_1 = [1|\hat{p}_2\hat{p}_3\hat{p}_4 - \hat{p}_4\hat{p}_3\hat{p}_2|1]\ , \tag{4.34}$$

where the other quantities $\Delta_i, \tilde{\Delta}_i$ are generated by taking cyclic permutation on (4.34). The contraction between the object $\Delta_i$ and the corresponding spinor $\lambda_i^{Aa}$ is given by $\langle 1_a|(2\cdot\tilde{\Delta}_2)_{\dot{b}}] = \lambda_{1a}^{A}\tilde{\lambda}_{2A}^{\dot{a}'}[2_{\dot{a}'}|\hat{p}_3\hat{p}_4\hat{p}_5 - \hat{p}_5\hat{p}_4\hat{p}_3|2_{\dot{b}}]$.

The five-point superamplitude in the $\mathcal{N}=(1,1)$ on-shell superspace can also be calculated in a recursive fashion. It takes the form [87]

$$\begin{aligned} \mathcal{A}_{5;0} \ = &\ i\,\frac{\delta^{(4)}(Q)\delta^{(4)}(\tilde{Q})}{s_{12}s_{23}s_{34}s_{45}s_{51}}\Bigg[ \\ &+ \frac{3}{10}q_1^A\big[(\hat{p}_2\hat{p}_3\hat{p}_4\hat{p}_5) - (\hat{p}_2\hat{p}_5\hat{p}_4\hat{p}_3)\big]_A^{\ B}\tilde{q}_{2B} + \frac{3}{10}\tilde{q}_{1A}\big[(\hat{p}_2\hat{p}_3\hat{p}_4\hat{p}_5) - (\hat{p}_2\hat{p}_5\hat{p}_4\hat{p}_3)\big]^A_{\ B}q_2^B \\ &+ \frac{1}{10}q_3^A\big[(\hat{p}_5\hat{p}_1\hat{p}_2\hat{p}_3) - (\hat{p}_5\hat{p}_3\hat{p}_2\hat{p}_1)\big]_A^{\ B}\tilde{q}_{5B} + \frac{1}{10}\tilde{q}_{3A}\big[(\hat{p}_5\hat{p}_1\hat{p}_2\hat{p}_3) - (\hat{p}_5\hat{p}_3\hat{p}_2\hat{p}_1)\big]^A_{\ B}q_5^B \\ &+ q_1^A(\hat{p}_2\hat{p}_3\hat{p}_4\hat{p}_5)_A^{\ B}\tilde{q}_{1B} + \text{cyclic permutations}\Bigg]\ , \end{aligned} \tag{4.35}$$

where the supercharges $Q$ and $\tilde{Q}$ are defined in (4.26).

## 4.4 The One-Loop Four-Point Superamplitude

In this section we calculate the four-point one-loop amplitude using two-particle and four-particle cuts. As expected, we find that the one-loop amplitude is proportional to the four-point tree-level superamplitude times the corresponding integral function.

### 4.4.1 The superamplitude from two-particle cuts

As a warm-up exercise, we start by rederiving the one-loop four-point superamplitude in six dimensions using two-particle cuts. This calculation was first sketched in [87]. Here, we will perform it in some detail while setting up our notation. We will then show how to reproduce this result using quadruple cuts.

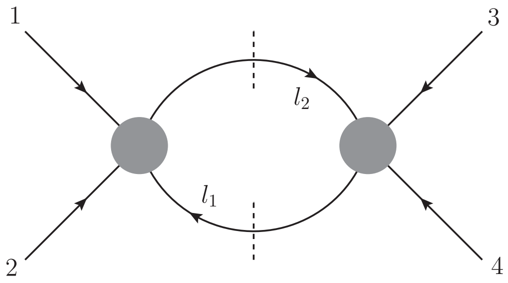



Figure 4.2: *Double cut in the s-channel. The two internal cut-propagators, carrying momenta $l_1$ and $l_2$ set the two four-point subamplitudes on-shell. We identify $l_1 = l$ and $l_2 = l + p_1 + p_2$.*

We begin by considering the one-loop amplitude with external momenta $p_1, \ldots, p_4$, and perform a unitarity cut in the $s$-channel, see Figure 4.2. The $s$-cut of the one-loop amplitude is given by[6]

$$\mathcal{A}_{4;1}\big|_{s\text{-cut}} = \int \frac{d^6 l}{(2\pi)^6} \delta^+(l_1^2)\delta^+(l_2^2) \left[ \prod_{i=1}^{2} \int d^2\eta_{l_i} d^2\tilde{\eta}_{l_i}\, A_{4;0}^{(L)}(l_1, 1, 2, -l_2)\, A_{4;0}^{(R)}(l_2, 3, 4, -l_1) \right] . \tag{4.36}$$

Plugging the expression (4.29) of the four-point superamplitude into (4.36), we get the following fermionic integral,

$$\prod_{i=1}^{2} \int d^2\eta_{l_i} d^2\tilde{\eta}_{l_i} \left( \frac{-i}{s_L t_L} \delta^{(4)}(\sum_L q_i) \delta^{(4)}(\sum_L \tilde{q}_i) \right) \left( \frac{-i}{s_R t_R} \delta^{(4)}(\sum_R q_i) \delta^{(4)}(\sum_R \tilde{q}_i) \right) , \tag{4.37}$$

where the sums are over the external states of the left and right subamplitude in the cut diagram and the kinematical invariants are given by

$$t_L = (l_1 + p_1)^2 \,, \qquad t_R = (l_2 + p_3)^2 \,, \tag{4.38}$$

and

$$s_L = (p_1 + p_2)^2 = (p_3 + p_4)^2 = s_R = s \,. \tag{4.39}$$

Using supermomentum conservation we can remove the dependence of the loop-supermomenta on one side of the cut. For instance a $\delta^{(4)}(Q_R)$ sets $q^A_{l_1} = q^A_{l_2} + q^A_3 + q^A_4$, which can be used in the remaining $\delta^{(4)}(Q_L)$ to write

$$\begin{aligned} \delta^{(4)}(\sum_L q_i) &\to \delta^{(4)}(\sum_{\text{ext}} q_i) \equiv \delta^{(4)}(Q_{\text{ext}}) \,, \\ \delta^{(4)}(\sum_L \tilde{q}_i) &\to \delta^{(4)}(\sum_{\text{ext}} \tilde{q}_i) \equiv \delta^{(4)}(\tilde{Q}_{\text{ext}}) \,. \end{aligned} \tag{4.40}$$

Hence, (4.37) becomes

$$\delta^{(4)}(Q_{\text{ext}})\delta^{(4)}(\tilde{Q}_{\text{ext}}) \prod_{i=1}^{2} \int d^2\eta_{l_i} d^2\tilde{\eta}_{l_i} \delta^{(4)}(\sum_R q_i) \delta^{(4)}(\sum_R \tilde{q}_i) \,. \tag{4.41}$$

[6]See Appendix A for our definitions of fermionic integrals.

To perform the integration, we need to pick two powers of $\eta_{l_i}$ and two powers of $\tilde{\eta}_{l_i}^{\dot{a}}$. Expanding the fermionic $\delta$-functions, we find one possible term with the right powers of Grassmann variables to be

$$\eta_{l_1 a}\eta_{l_1 b}\eta_{l_2 c}\eta_{l_2 d}\tilde{\eta}_{l_1 \dot{a}}\tilde{\eta}_{l_1 \dot{b}}\tilde{\eta}_{l_2 \dot{c}}\tilde{\eta}_{l_2 \dot{d}} \left[\epsilon_{ABCD}\lambda_{l_1}^{Aa}\lambda_{l_1}^{Bb}\lambda_{l_2}^{Cc}\lambda_{l_2}^{Dd}\ \epsilon^{EFGH}\tilde{\lambda}_{l_1 E}^{\dot{a}}\tilde{\lambda}_{l_1 F}^{\dot{b}}\tilde{\lambda}_{l_2 G}^{\dot{c}}\tilde{\lambda}_{l_2 H}^{\dot{d}}\right] . \tag{4.42}$$

Other combinations can be brought into that form by rearranging and relabeling indices. Integrating out the Grassmann variables gives

$$\left(\epsilon_{ABCD}\lambda_{l_1}^{Aa}\lambda_{l_1 a}^{B}\ \lambda_{l_2}^{Cb}\lambda_{l_2 b}^{D}\right)\ \left(\epsilon^{EFGH}\tilde{\lambda}_{l_1 E}^{\dot{a}}\tilde{\lambda}_{l_1 F\dot{a}}\ \tilde{\lambda}_{l_2 G}^{\dot{b}}\tilde{\lambda}_{l_2 H\dot{b}}\right) . \tag{4.43}$$

Hence, the two-particle cut reduces to

$$\begin{aligned}\mathcal{A}_{4;1}\big|_{s\text{-cut}} \propto(-1)\int\frac{d^6 l}{(2\pi)^6}\delta^+(l_1^2)\delta^+(l_2^2)\Big[&\delta^{(4)}(Q_{\text{ext}})\delta^{(4)}(\tilde{Q}_{\text{ext}})\\ &\times\frac{\epsilon_{ABCD}l_1^{AB}l_2^{CD}\ \epsilon^{EFGH}l_{1EF}l_{2GH}}{s^2(l_1+p_1)^2(l_2+p_3)^2}\Big] .\end{aligned} \tag{4.44}$$

Next, we use (B.4) to rewrite

$$\epsilon_{ABCD}\ p_{l_1}^{AB}p_{l_2}^{CD}\ \epsilon^{EFGH}p_{l_1 EF}p_{l_2 GH} = 64\,(l_1\cdot l_2)^2\ . \tag{4.45}$$

Thus we obtain, for the one-loop superamplitude,

$$\begin{aligned}\mathcal{A}_{4;1}\big|_{s\text{-cut}} \propto &-\delta^{(4)}(Q_{\text{ext}})\delta^{(4)}(\tilde{Q}_{\text{ext}})\int\frac{d^6 l}{(2\pi)^6}\delta^+(l_1^2)\delta^+(l_2^2)\left[\frac{64\,(l_1\cdot l_2)^2}{s^2(l_1+p_1)^2(l_2+p_3)^2}\right]\\ =&-\ 16\,\delta^{(4)}(Q_{\text{ext}})\delta^{(4)}(\tilde{Q}_{\text{ext}})\int\frac{d^6 l}{(2\pi)^6}\delta^+(l_1^2)\delta^+(l_2^2)\left[\frac{1}{(l_1+p_1)^2(l_2+p_3)^2}\right]\\ =&-16\,ist\mathcal{A}_{4;0}(1,\ldots,4)\ I_4(s,t)\big|_{s\text{-cut}}\ ,\end{aligned} \tag{4.46}$$

where $\mathcal{A}_{4;0}(1,\ldots,4)$ is the tree-level four-point superamplitude in (4.29), and

$$I_4(s,t) = \int\frac{d^6 l}{(2\pi)^6}\left[\frac{1}{l_1^2 l_2^2(l+p_1)^2(l-p_4)^2}\right] . \tag{4.47}$$

The $t$-channel cut is performed in the same fashion and after inspecting it we conclude that

$$\mathcal{A}_{4;1}(1,\ldots,4)\ =\ st\,\mathcal{A}_{4;0}(1,\ldots,4)\,I_4(s,t)\ , \tag{4.48}$$

in agreement with the result of [87].

### 4.4.2 The superamplitude from quadruple cuts

We now move on to studying the quadruple cut of the one-loop four-point superamplitude, depicted in Figure 4.3. The loop momenta are defined as

$$l_1 = l,\qquad l_2 = l+p_1,\qquad l_3 = l+p_1+p_2,\qquad l_4 = l-p_4\ , \tag{4.49}$$

and all primed momenta $l_i'$ in Figure 4.3 are understood to flow in opposite direction to the $l_i$'s.

Four three-point tree-level superamplitudes enter the quadruple cut expression. Uplifting the cut by replacing cut with uncut propagators, we obtain, for the one-loop superamplitude,

$$\begin{aligned}\mathcal{A}_{4;1} = \int\frac{d^6 l}{(2\pi)^6}\Big[\prod_{i=1}^{4}\int d^2\eta_{l_i}d^2\tilde{\eta}_{l_i}\frac{1}{l_1^2}\mathcal{A}_{3;0}(l_1,1,l_2')\frac{1}{l_2^2}\mathcal{A}_{3;0}(l_2,2,l_3')&\\ \times\frac{1}{l_3^2}\mathcal{A}_{3;0}(l_3,3,l_4')\frac{1}{l_4^2}\mathcal{A}_{3;0}(l_4,4,l_1')\Big] .&\end{aligned} \tag{4.50}$$

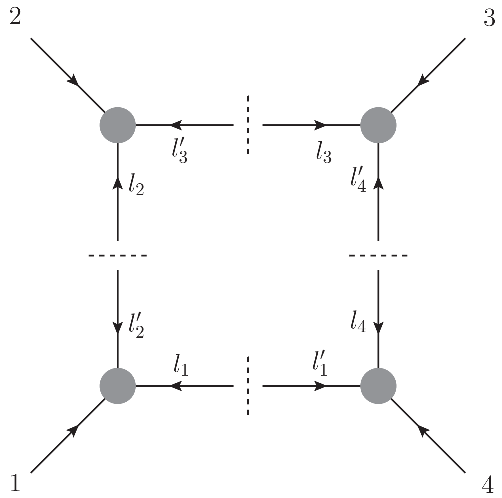



Figure 4.3: *The quadruple cut of a four-point superamplitude. The primed momenta $l'_i$ are defined as $l'_i := -l_i$.*

In the following we will discuss two different but equivalent approaches to evaluate the Grassmann integrals in (4.50).

**Quadruple cut as reduced two-particle cuts**

To begin with, we proceed in way similar to the case of a double cut. The idea is to integrate over two of the internal momenta, say $l_1$ and $l_3$ first, and treat $l_2$ and $l_4$ as fixed, i.e. external lines. In doing so the quadruple cut splits into two four-point tree-level superamplitudes, having the same structure as in case of the BCFW construction for the four-point trevel-level superamplitude [87].

Let us start by focusing on the 'lower' part of the diagram first. Here we have two three-point superamplitudes connected by an internal (cut) propagator carrying momentum $l_1$. Treating $l'_2$ and $l_4$ as external momenta (they are on-shell due to the cut) we can follow the procedure of a four-point BCFW construction. This involves rewriting fermionic $\delta$-functions of both three-point amplitudes and integrating over $d^2\eta_{l_1} d^2\tilde{\eta}_{l_1}$, leading to the result

$$\delta^{(4)}(q_1 + q_{l'_2} + q_{l_4} + q_4)\delta^{(4)}(\tilde{q}_1 + \tilde{q}_{l'_2} + \tilde{q}_{l_4} + \tilde{q}_4)\; w^a_{l_1} w_{l'_1 a}\; \tilde{w}^{\dot{a}}_{l_1} \tilde{w}_{l'_1 \dot{a}}\,. \tag{4.51}$$

Note that the $\delta$-functions are ensuring supermomentum conservation of the 'external momenta' and that we do not have an internal propagator with momentum $l_1$ as in the recursive construction. Here, we get this propagator from uplifting the cut-expression for the one-loop amplitude. Furthermore, note that we do not have to shift any legs in order to use the BCFW prescription since the internal propagator is already on-shell due to the cut.

We may now perform the Grassmann integration over $\eta_{l_1}$ and $\tilde{\eta}_{l_1}$ in (4.51). Since the $w$-spinors are contracted we can simply use the spinor identity

$$w^a_{l_1} w_{l'_1 a} \tilde{w}^{\dot{a}}_{l_1} \tilde{w}_{l'_1 \dot{a}} = -s^{-1}_{l_4 l'_2} = -s^{-1}_{14}\;, \tag{4.52}$$

which is a direct generalisation of the corresponding result from the BCFW construction (see also Appendix B.3).

We can now turn to the 'upper' half of the cut-diagram. Following the description we derived above we get in a similar fashion after integrating over $\eta_{l_3}$ and $\tilde{\eta}_{l_3}$

$$\delta^{(4)}(q_{l_2} + q_2 + q_3 + q_{l'_4})\delta^{(4)}(\tilde{q}_{l_2} + \tilde{q}_2 + \tilde{q}_3 + \tilde{q}_{l'_4}) w^a_{l'_3} w_{l_3 a} \tilde{w}^{\dot{a}}_{l'_3} \tilde{w}_{l_3 \dot{a}} \,. \tag{4.53}$$

We also have

$$w^a_{l_1} w_{l'_1 a} \tilde{w}^{\dot{a}}_{l_1} \tilde{w}_{l'_1 \dot{a}} = -s^{-1}_{l_2 l'_4} = -s^{-1}_{23} \,. \tag{4.54}$$

Uplifting the quadruple cut, we get

$$\begin{aligned} \mathcal{A}_{4;1} = \int \frac{d^6 l}{(2\pi)^6} \int d^2\eta_{l_2} d^2\tilde{\eta}_{l_2} d^2\eta_{l_4} d^2\tilde{\eta}_{l_4} \Bigg[ & \frac{1}{l_1^2 l_2^2 l_3^2 l_4^2} \frac{1}{s_{14} s_{23}} \\ & \times \delta^{(4)}(q_1 + q_{l'_2} + q_{l_4} + q_4)\delta^{(4)}(\tilde{q}_1 + \tilde{q}_{l'_2} + \tilde{q}_{l_4} + \tilde{q}_4) \\ & \times \delta^{(4)}(q_{l_2} + q_2 + q_3 + q_{l'_4})\delta^{(4)}(\tilde{q}_{l_2} + \tilde{q}_2 + \tilde{q}_3 + \tilde{q}_{l'_4}) \Bigg] . \end{aligned} \tag{4.55}$$

Since $l'_i = -l_i$ we can use the constraints given by the $\delta^{(4)}(q_i)$ to eliminate the dependence of the remaining loop momenta in one of the sets of fermionic $\delta$-functions and write it as a sum over external momenta only. The same argument holds for the Grassmann functions $\delta^{(4)}(\tilde{q}_i)$, and we find

$$\begin{aligned} \mathcal{A}_{4;1} = \int \frac{d^6 l}{(2\pi)^6} \int d^2\eta_{l_2} d^2\tilde{\eta}_{l_2} d^2\eta_{l_4} d^2\tilde{\eta}_{l_4} \Bigg[ & \delta^{(4)}(Q_{\text{ext}})\delta^{(4)}(\tilde{Q}_{\text{ext}}) \frac{1}{l_1^2 l_2^2 l_3^2 l_4^2} \frac{1}{s_{14} s_{24}} \\ & \times \delta^{(4)}(q_{l_2} + q_2 + q_3 - q_{l_4})\delta^{(4)}(\tilde{q}_{l_2} + \tilde{q}_2 + \tilde{q}_3 - \tilde{q}_{l_4}) \Bigg], \end{aligned} \tag{4.56}$$

where as before the $Q^A_{\text{ext}}$ and $\tilde{Q}_{A\,\text{ext}}$ are the sums of all external supermomenta in $\eta$ and $\tilde{\eta}$ respectively. The remaining integrations over $\eta_{l_2}$ and $\eta_{l_4}$ and their $\tilde{\eta}$-counterparts yield just as in the case of the two-particle cut

$$\epsilon_{ABCD}\, {l_2}^{AB} {l_4}^{CD}\, \epsilon^{EFGH} p_{l_2 EF} p_{l_4 GH} = 64\, (l_2 \cdot l_4)^2 \,. \tag{4.57}$$

The product of the two loop momenta cancels with the factor

$$s_{14} s_{23} = 2(p_1 \cdot p_4) 2(p_2 \cdot p_3) = 4(l'_2 \cdot l_4)(l_2 \cdot l'_4) = (-1)^2 4 (l_2 \cdot l_4)^2. \tag{4.58}$$

so that our final result for the quadruple cut of the four-point superamplitude is

$$\mathcal{A}_{4;1} \propto\ ist \mathcal{A}_{4;0}(1, \ldots, 4) \int \frac{d^6 l}{(2\pi)^6} \left[ \frac{1}{l^2 (l+p_1)^2 (l+p_1+p_2)^2 (l-p_4)^2} \right]. \tag{4.59}$$

Hence we have shown that the quadruple cut gives the same structure as the two-particle cut discussed in Section 4.4.1.

### Quadruple cut by Grassmann decomposition

In this section we will calculate the quadruple cut of the one-loop four-point superamplitude in an alternative fashion. Whereas in the last section we used the structure of the cut-expression to simplify the fermionic integrations, here we will explicitly perform the integrals by using constraints given by the $\delta$-functions.

To perform the Grassmann integrations we work directly at the level of the three-point superamplitudes. The quadruple cut results in the following four on-shell tree-level amplitudes (see Figure 4.3)

$$\mathcal{A}_3(l_1, 1, l_2'), \qquad \mathcal{A}_3(l_2, 2, l_3'), \qquad \mathcal{A}_3(l_3, 3, l_4'), \qquad \mathcal{A}_3(l_4, 4, l_1') \ . \tag{4.60}$$

Each of the three-point superamplitudes has the usual form [87]

$$\mathcal{A}_{3,i} \ = \ i \left[\delta(Q_i^A)\delta(\tilde{Q}_{iA})\right]^2 \delta(W_i)\delta(\tilde{W}_i) \ , \tag{4.61}$$

where $i = 1, \ldots, 4$ labels the corners. The arguments of the $\delta$-functions are

$$Q_i^A = q_{l_i}^A + q_i^A + q_{l'_{i+1}}^A \ , \qquad W_i = w_{l_i}^a \eta_{l_i a} + w_i^a \eta_{ia} + w_{l'_{i+1}}^a \eta_{l'_{i+1} a} \ , \tag{4.62}$$

with the identification $l_5 \equiv l_1$. Similar expressions hold for for $\tilde{Q}_{iA}$ and $\tilde{W}_i$. Note that since $l_i' = -l_i$ we find it convenient to define spinors with primed momenta $l_i'$ as

$$\lambda_{l_i'}^A = i\lambda_{l_i}^A \ , \qquad \tilde{\lambda}_{l_i' A} = \tilde{\lambda}_{l_i A} \ , \qquad \eta_{l_i'} = i\eta_{l_i} \ , \qquad \tilde{\eta}_{l_i'} = i\tilde{\eta}_{l_i} \ , \tag{4.63}$$

which we will frequently use in the following manipulations.

We can use supermomentum conservation at each corner to reduce the number of $\delta$-functions depending on the loop variables $\eta_{l_i}$ and $\tilde{\eta}_{l_i}$. There is a choice involved and we choose to remove the dependence of $\eta_{\ell_i}$ ($\tilde{\eta}_{\ell_i}$) from one copy of each $[\delta(Q_i^A)\delta(\tilde{Q}_{iA})]^2$. This yields for the Grassmann integrations

$$\delta^{(4)}(Q_{\text{ext}})\delta^{(4)}(\tilde{Q}_{\text{ext}}) \int \prod_{i=1}^{4} d^2\eta_{l_i} d^2\tilde{\eta}_{l_i} \left[\delta(Q_i^A)\delta(\tilde{Q}_{iA})\delta(W_i)\delta(\tilde{W}_i)\right] \ . \tag{4.64}$$

We can simplify the calculation by noticing that we have to integrate over 16 powers of Grassmann variables (8 powers of $\eta$ and $\tilde{\eta}$ each) while at the same time we have 16 $\delta$-functions in total. Therefore, when expanding the fermionic functions, each of them must contribute a power of Grassmann variables we are going to integrate over. Unless this is so, the result is zero. In other words, we can only pick the terms in the $\delta$-functions that contribute an $\eta_{l_i}$ or $\tilde{\eta}_{l_i}$. This simplifies the structure considerably as we can drop all terms depending on external variables.

Equation (4.64) now becomes

$$\begin{aligned} \delta^{(4)}(Q_{\text{ext}})\delta^{(4)}(\tilde{Q}_{\text{ext}}) \int \prod_{i=1}^{4} d^2\eta_{l_i} d^2\tilde{\eta}_{l_i} \Big[\delta(q_{l_i}^A - q_{l_{i+1}}^A)\, \delta(\tilde{q}_{l_i A} - \tilde{q}_{l_{i+1} A}) \\ \times\, \delta(w_{l_i}^a \eta_{l_i a} + i w_{l'_{i+1}}^a \eta_{l_{i+1} a})\delta(\tilde{w}_{l_i}^{\dot{a}} \tilde{\eta}_{l_i \dot{a}} + i\tilde{w}_{l'_{i+1}}^{\dot{a}} \tilde{\eta}_{l_{i+1} \dot{a}})\Big] \ . \end{aligned} \tag{4.65}$$

Notice that the $w$-spinors $w_{l'_{i+1}}^a$ are not identical to $w_{l_{i+1}}^a$.

Since the $\delta$-functions only depend on the $\eta_{l_i}$ and $\tilde{\eta}_{l_i}$, we find convenient to decompose the integration variables as

$$\eta_{l_i}^a = u_{l_i}^a \eta_{l_i}^{\parallel} + w_{l_i}^a \eta_{l_i}^{\perp} \ , \qquad \tilde{\eta}_{l_i}^{\dot{a}} = \tilde{u}_{l_i}^{\dot{a}} \tilde{\eta}_{l_i}^{\parallel} + \tilde{w}_{l_i}^{\dot{a}} \tilde{\eta}_{l_i}^{\perp} \ , \tag{4.66}$$

which implies

$$w_{l_i a}\eta^a_{l_i} = \eta^{\|}_{l_i}\,, \qquad u^a_{l_i}\eta_{l_i a} = \eta^{\perp}_{l_i}\ . \tag{4.67}$$

Hence, we can rewrite the arguments of the $\delta$-functions in the $w$-spinors as

$$\begin{aligned} i w^a_{l'_{i+1}}\eta_{l_{i+1}a} &= i w^a_{l'_{i+1}}\left(u_{l_{i+1}a}\eta^{\|}_{l_{i+1}} + w_{l_{i+1}a}\eta^{\perp}_{l_{i+1}}\right) = \frac{i}{\sqrt{-s_{i,i+1}}} u^a_{l_{i+1}} w_{l_{i+1}a}\eta^{\perp}_{l_{i+1}} \\ &= \frac{i}{\sqrt{-s_{i,i+1}}}\eta^{\perp}_{l_{i+1}}\ , \end{aligned} \tag{4.68}$$

and similarly we have $i\tilde{w}^{\dot a}_{l'_{i+1}}\tilde{\eta}_{l_{i+1}\dot a} = \frac{i}{\sqrt{-s_{i,i+1}}}\tilde{\eta}^{\perp}_{l_{i+1}}$. Notice that we have used the fact that the $w'_{l_{i+1}}$ can be normalised such that they are proportional to the $u_{l_{i+1}}$ if the momenta fulfill the condition $l'_{i+1} = -l_{i+1}$. We give some more detail on such relations in Appendix B.3.2.

Using this, the $\delta$-functions in the $w$-spinors become

$$\delta\big(-\eta^{\|}_{l_i} + \frac{i}{\sqrt{-s_{i,i+1}}}\eta^{\perp}_{l_{i+1}}\big)\delta\big(-\tilde{\eta}^{\|}_{l_i} + \frac{i}{\sqrt{-s_{i,i+1}}}\tilde{\eta}^{\perp}_{l_{i+1}}\big)\ . \tag{4.69}$$

Next we proceed by integrating first over the $\eta^{\|}_{l_i}$ variables. This sets

$$\eta^{\|}_{l_i} = \frac{-i}{\sqrt{-s_{i,i+1}}}\eta^{\perp}_{l_{i+1}}\ , \tag{4.70}$$

with similar expressions for the $\tilde{\eta}^{\|}_{l_i}$. We then plug this into the remaining $\delta$-functions of (4.65). First we notice that

$$\begin{aligned} &\delta\big(\lambda^{Aa}_{l_i}\eta_{l_i a} + \lambda^{Aa}_{l'_{i+1}}\eta_{l'_{i+1}a}\big)\delta\big(\tilde{\lambda}^{A\dot a}_{l_i}\tilde{\eta}_{l_i\dot a} + \tilde{\lambda}^{A\dot a}_{l'_{i+1}}\tilde{\eta}_{l'_{i+1}\dot a}\big) \\ =&\langle l^a_i|l'^{\dot a}_{i+1}]\eta_{l_i a}\tilde{\eta}_{l'_{i+1}\dot a} + \langle l'^a_{i+1}|l^{\dot a}_i]\eta_{l'_{i+1}a}\tilde{\eta}_{l_i\dot a} = -u^a_{l_i}\tilde{u}^{\dot a}_{l'_{i+1}}\eta_{l_i a}\tilde{\eta}_{l'_{i+1}\dot a} + u^a_{l'_{i+1}}\tilde{u}^{\dot a}_{l_i}\eta_{l'_{i+1}a}\tilde{\eta}_{l_i\dot a}\ . \end{aligned} \tag{4.71}$$

The decomposition of the Grassmann spinors then yields

$$u^a_{l'_{i+1}}\eta_{l'_{i+1}a} = i\sqrt{-s_{i,i+1}}\,w^a_{l_{i+1}}u_{l_{i+1}a}\eta^{\|}_{l_{i+1}} = -i\sqrt{-s_{i,i+1}}\ \eta^{\|}_{l_{i+1}} \tag{4.72}$$

and

$$\tilde{u}^{\dot a}_{l'_{i+1}}\tilde{\eta}_{l'_{i+1}\dot a} = -i\sqrt{-s_{i,i+1}}\ \tilde{\eta}^{\|}_{l_{i+1}}\ . \tag{4.73}$$

The remaining Grassmann integrations give

$$\begin{aligned} &\int\prod_{i=1}^{4} d\eta^{\perp}_{l_i}d\tilde{\eta}^{\perp}_{l_i}\left[\delta\big(\lambda^{Aa}_{l_i}\eta_{l_i a} + \lambda^{Aa}_{l'_{i+1}}\eta_{l'_{i+1}a}\big)\delta\big(\tilde{\lambda}^{A\dot a}_{l_i}\tilde{\eta}_{l_i\dot a} + \tilde{\lambda}^{A\dot a}_{l'_{i+1}}\tilde{\eta}_{l'_{i+1}\dot a}\big)\right] \\ =&\int\prod_{i=1}^{4} d\eta^{\perp}_{l_i}d\tilde{\eta}^{\perp}_{l_i}\left[i\sqrt{-s_{i,i+1}}\ \eta^{\perp}_{l_i}\tilde{\eta}^{\|}_{l_{i+1}} - i\sqrt{-s_{i,i+1}}\ \eta^{\|}_{l_{i+1}}\tilde{\eta}^{\perp}_{l_i}\right] \\ =&\int\prod_{i=1}^{4} d\eta^{\perp}_{l_i}d\tilde{\eta}^{\perp}_{l_i}\left[\eta^{\perp}_{l_i}\tilde{\eta}^{\perp}_{l_{i+2}} - \eta^{\perp}_{l_{i+2}}\tilde{\eta}^{\perp}_{l_i}\right]\ , \end{aligned} \tag{4.74}$$

where we have used the solutions for $\eta^{\|}_{l_i}$ and $\tilde{\eta}^{\|}_{l_i}$ following (4.70). The integration is now straightforward, since the integrand is simply given by

$$\begin{aligned} &\big(\eta^{\perp}_{l_1}\tilde{\eta}^{\perp}_{l_3} - \eta^{\perp}_{l_3}\tilde{\eta}^{\perp}_{l_1}\big)\big(\eta^{\perp}_{l_2}\tilde{\eta}^{\perp}_{l_4} - \eta^{\perp}_{l_4}\tilde{\eta}^{\perp}_{l_2}\big)\big(\eta^{\perp}_{l_3}\tilde{\eta}^{\perp}_{l_1} - \eta^{\perp}_{l_1}\tilde{\eta}^{\perp}_{l_3}\big)\big(\eta^{\perp}_{l_4}\tilde{\eta}^{\perp}_{l_2} - \eta^{\perp}_{l_2}\tilde{\eta}^{\perp}_{l_4}\big) \\ =& 4\eta^{\perp}_{l_1}\tilde{\eta}^{\perp}_{l_3}\eta^{\perp}_{l_2}\tilde{\eta}^{\perp}_{l_4}\eta^{\perp}_{l_3}\tilde{\eta}^{\perp}_{l_1}\eta^{\perp}_{l_4}\tilde{\eta}^{\perp}_{l_2}\ . \end{aligned} \tag{4.75}$$

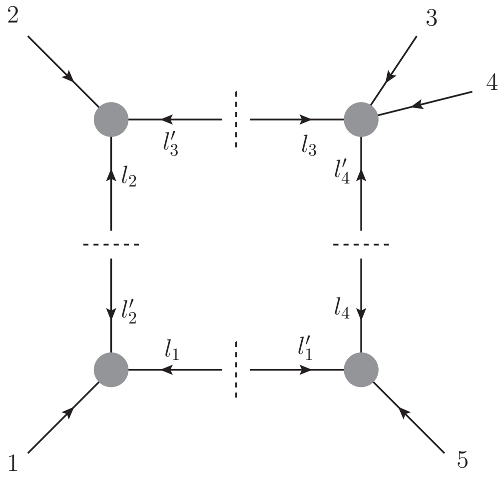



Figure 4.4: *A specific quadruple cut of a five-point superamplitude. We choose to cut the legs such that we have the massive corner for momenta* $p_3, p_4$.

This yields

$$\mathcal{A}_{4;1} \propto -istA_{4;0}(1,\ldots,4)\int \frac{d^6l}{(2\pi)^6}\left[\frac{1}{l^2(l+p_1)^2(l+p_1+p_2)^2(l-p_4)^2}\right], \qquad (4.76)$$

recovering the expected result of [87] from two-particle cuts.

## 4.5 The One-Loop Five-Point Superamplitude

We now move on to the one-loop five-point superamplitude and calculate its quadruple cuts. These cuts will reveal the presence of a linear pentagon integral, which we will reduce using standard Passarino-Veltman (PV) techniques to a scalar pentagon plus scalar box integrals. Note that we are considering here one-loop amplitudes in the maximally supersymmetric theory in six dimensions which are free of IR and UV divergences. Therefore, bubbles and triangles which would be UV divergent in six dimensions must be absent. It is for this reason that it will be enough to consider quadruple cuts, without having to inspect also triple and double cuts, which would be required if triangle and bubble functions were present.

### 4.5.1 Quadruple cuts

The quadruple cut we consider has the structure

$$\begin{aligned}\mathcal{A}_{5;1}\big|_{(3,4)\text{-cut}} = &\int \frac{d^6l}{(2\pi)^6}\delta^+(l_1^2)\,\delta^+(l_2^2)\,\delta^+(l_3^2)\,\delta^+(l_4^2) \\ &\times \mathcal{A}_{3;0}(l_1,p_1,-l_2)\,\mathcal{A}_{3;0}(l_2,p_2,-l_3)\,\mathcal{A}_{4;0}(l_3,p_3,p_4,-l_4)\,\mathcal{A}_{3;0}(l_4,p_5,-l_1)\,,\end{aligned} \qquad (4.77)$$

where the subscript $(3,4)$ indicates where the massive corner is located, see Figure 4.4. In the following we will discuss this specific cut and all other cuts can be treated in an identical way.

From the three three-point superamplitudes and the four-point superamplitude, we have the following fermionic $\delta$-functions,

$$\Big[\delta(Q_1^A)\delta(\tilde{Q}_{1A})\Big]^2\delta(W_1)\delta(\tilde{W}_1)\Big[\delta(Q_2^B)\delta(\tilde{Q}_{2B})\Big]^2\delta(W_2)\delta(\tilde{W}_2) \times\delta^{(4)}(Q_3^C)\delta^{(4)}(\tilde{Q}_{3C})\Big[\delta(Q_4^D)\delta(\tilde{Q}_{4D})\Big]^2\delta(W_4)\delta(\tilde{W}_4)\ , \tag{4.78}$$

where the $Q_i^A$ and the $W_i$ are defined as sums over the supermomenta and products of $w$- and $\eta$-spinors respectively at a given corner (including internal legs). We may now use the supermomentum constraints $Q_i^A = 0$ at all four corners and rewrite the $\delta^{(4)}(Q_3)\delta^{(4)}(\tilde{Q}_3)$ as a total $\delta^{(8)}$ in the external momenta only,

$$\delta^{(4)}(Q_3) = \delta^{(4)}(Q_3 + Q_1 + Q_2 + Q_4) = \delta^{(4)}(Q_{\text{ext}}). \tag{4.79}$$

One is then left with the Grassmann integrations

$$\begin{aligned}\int\prod_{i=1}^{4} d^2\eta_{l_i} d^2\tilde{\eta}_{l_i}\Big\{ &\left[\delta(q_{l_1}^A + q_1^A - q_{l_2}^A)\delta(\tilde{q}_{l_1A} + \tilde{q}_{1A} - \tilde{q}_{l_2A})\right]^2\\ &\delta(w_{l_1}^a\eta_{l_1a} + w_1^a\eta_{1a} + iw_{l_2'}^a\eta_{l_2a})\delta(\tilde{w}_{l_1}^{\dot a}\tilde{\eta}_{l_1\dot a} + \tilde{w}_1^{\dot a}\tilde{\eta}_{1\dot a} + i\tilde{w}_{l_2'}^{\dot a}\tilde{\eta}_{l_2\dot a})\\ &\left[\delta(q_{l_2}^B + q_2^B - q_{l_3}^B)\delta(\tilde{q}_{l_2B} + \tilde{q}_{2B} - \tilde{q}_{l_3B})\right]^2\\ &\delta(w_{l_2}^b\eta_{l_2b} + w_2^b\eta_{2b} + iw_{l_3'}^b\eta_{l_3b})\delta(\tilde{w}_{l_2}^{\dot b}\tilde{\eta}_{l_2\dot b} + \tilde{w}_2^{\dot b}\tilde{\eta}_{2\dot b} + i\tilde{w}_{l_3'}^{\dot b}\tilde{\eta}_{l_3\dot b})\\ &\left[\delta(q_{l_4}^D + q_5^D - q_{l_1}^D)\delta(\tilde{q}_{l_4D} + \tilde{q}_{5D} - \tilde{q}_{l_1D})\right]^2\\ &\delta(w_{l_4}^c\eta_{l_4c} + w_5^c\eta_{5c} + iw_{l_1'}^c\eta_{l_1c})\delta(\tilde{w}_{l_4}^{\dot c}\tilde{\eta}_{l_4\dot c} + \tilde{w}_5^{\dot c}\tilde{\eta}_{5\dot c} + i\tilde{w}_{l_1'}^{\dot c}\tilde{\eta}_{l_1\dot c})\Big\}\ .\end{aligned} \tag{4.80}$$

Unfortunately, a decomposition as used for the quadruple cut of the four-point one-loop superamplitude is not immediately useful here. However, we notice that, due to the particular dependence of the $\delta$-functions on the loop momenta $l_i$, by removing a total $\delta^{(8)}$ from the integrand one can restrict the dependence on the Grassmann variables $\eta_{l_3}$ and $\eta_{l_4}$ to six $\delta$-functions each for this specific cut. This allows us to narrow the possible combinations of coefficients for, say, two powers of $\eta_{l_4a}$ and two powers of $\tilde{\eta}_{l_4\dot a}$. For example, two powers of $\eta_{l_4a}$ can either come both from $\delta(Q_4^A)\delta(Q_4^B)$ or one from $\delta(Q_4^A)$ and one from[7] $\delta(W_4)$, and both possibilities needs to be appropriately contracted with the possible combinations from $\delta(\tilde{Q}_{4A})\delta(\tilde{Q}_{4B})\delta(\tilde{W}_4)$. If we choose both powers of $\eta_{l_4a}$ from $\delta(Q_4^A)\delta(Q_4^B)$ we have a coefficient

$$\lambda_{l_4}^{Aa}\eta_{l_4a}\lambda_{l_4}^{Bb}\eta_{l_4b}\ , \tag{4.81}$$

which will be contracted at least by a $\tilde{\lambda}_{l_4A}^{\dot a}$ or $\tilde{\lambda}_{l_4B}^{\dot a}$ coming from the possible combinations for $\tilde{\eta}_{l_4\dot a}$. Since $\lambda_{ia}^A\tilde{\lambda}_{iA\dot a} = 0$ these terms vanish.

In conclusion, the only non-vanishing combination is

$$\lambda_{l_4}^{Aa}\eta_{l_4a}\delta(\tilde{q}_{5A} - \tilde{q}_{l_1A})\delta(q_5^B - q_{l_1}^B)\tilde{\lambda}_{l_4B}^{\dot a}\tilde{\eta}_{l_4\dot a}w_{l_4}^b\eta_{l_4b}\tilde{w}_{l_4}^{\dot b}\tilde{\eta}_{l_4\dot b}\ . \tag{4.82}$$

The same argument holds for the expansion of the $\delta$-functions depending on $\eta_{l_3a}$ and $\tilde{\eta}_{l_3\dot a}$. Here, we only have to deal with additional signs and factors of $i$. We get for the expansion

$$(-1)\lambda_{l_3}^{Aa}\eta_{l_3a}\delta(\tilde{q}_{l_2A} + \tilde{q}_{2A})\delta(q_{l_2}^B + q_2^B)(-1)\tilde{\lambda}_{l_3B}^{\dot a}\tilde{\eta}_{l_3\dot a}iw_{l_3'}^b\eta_{l_3b}i\tilde{w}_{l_3'}^{\dot b}\tilde{\eta}_{l_3\dot b}\ . \tag{4.83}$$

[7] This is similar to the recursive calculation of the five-point tree-level superamplitude in six dimensions, see also [87].

This leads us to the structure

$$\int \prod_{i=1}^{4} d^2\eta_{l_i} d^2\tilde{\eta}_{l_i} \Big\{ \big[\delta(q^A_{l_1} + q^A_1 - q^A_{l_2})\delta(\tilde{q}_{l_1 A} + \tilde{q}_{1A} - \tilde{q}_{l_2 A})\big]^2$$
$$\delta(w^a_{l_1}\eta_{l_1 a} + w^a_1\eta_{1a} + iw^a_{l'_2}\eta_{l_2 a})\delta(\tilde{w}^{\dot{a}}_{l_1}\tilde{\eta}_{l_1\dot{a}} + \tilde{w}^{\dot{a}}_1\tilde{\eta}_{1\dot{a}} + i\tilde{w}^{\dot{a}}_{l'_2}\tilde{\eta}_{l_2\dot{a}})$$
$$\lambda^{Cc}_{l_3}\eta_{l_3 c}\delta(\tilde{q}_{l_2 C} + \tilde{q}_{2C})\delta(q^D_{l_2} + q^D_2)\tilde{\lambda}^{\dot{c}}_{l_3 D}\tilde{\eta}_{l_3\dot{c}}(i)^2 w^c_{l'_3}\eta_{l_3 c}\tilde{w}^{\dot{c}}_{l'_3}\tilde{\eta}_{l_3\dot{c}}$$
$$\lambda^{Ed}_{l_4}\eta_{l_4 d}\delta(\tilde{q}_{5E} - \tilde{q}_{l_1 E})\delta(q^F_5 - q^F_{l_1})\tilde{\lambda}^{\dot{d}}_{l_4 F}\tilde{\eta}_{l_4\dot{d}} w^d_{l_4}\eta_{l_4 d}\tilde{w}^{\dot{d}}_{l_4}\tilde{\eta}_{l_4\dot{d}}\Big\}. \tag{4.84}$$

Notice that we have not expanded the six $\delta$-functions of the first corner yet, therefore we still have supermomentum conservation $Q^A_1 = 0$, $\tilde{Q}^A_1 = 0$. We can use this constraint to remove the dependence on $\eta_{l_2 a}$ in the third line of the above integrand, using $q^A_{l_2} = q^A_{l_1} + q^A_1$. Our fermionic integral then becomes

$$\int \prod_{i=1}^{4} d^2\eta_{l_i} d^2\tilde{\eta}_{l_i} \Big\{ \big[\delta(q^A_{l_1} + q^A_1 - q^A_{l_2})\delta(\tilde{q}_{l_1 A} + \tilde{q}_{1A} - \tilde{q}_{l_2 A})\big]^2$$
$$\delta(w^a_{l_1}\eta_{l_1 a} + w^a_1\eta_{1a} + iw^a_{l'_2}\eta_{l_2 a})\delta(\tilde{w}^{\dot{a}}_{l_1}\tilde{\eta}_{l_1\dot{a}} + \tilde{w}^{\dot{a}}_1\tilde{\eta}_{1\dot{a}} + i\tilde{w}^{\dot{a}}_{l'_2}\tilde{\eta}_{l_2\dot{a}})$$
$$(i)^2\eta_{l_3 c}\tilde{\eta}_{l_3\dot{c}}\eta_{l_3 c'}\tilde{\eta}_{l_3\dot{c}'}\lambda^{Cc}_{l_3}\tilde{\lambda}^{\dot{c}}_{l_3 D} w^{c'}_{l'_3}\tilde{w}^{\dot{c}'}_{l'_3}\delta(\tilde{q}_{l_1 C} + \tilde{q}_{1C} + \tilde{q}_{2C})\delta(q^D_{l_1} + q^D_1 + q^D_2)$$
$$\eta_{l_4 d}\tilde{\eta}_{l_4\dot{d}}\eta_{l_4 d'}\tilde{\eta}_{l_4\dot{d}'}\lambda^{Ed}_{l_4}\tilde{\lambda}^{\dot{d}}_{l_4 F} w^{d'}_{l_4}\tilde{w}^{\dot{d}'}_{l_4}\delta(\tilde{q}_{5E} - \tilde{q}_{l_1 E})\delta(q^F_5 - q^F_{l_1})\Big\}. \tag{4.85}$$

We immediately see that, just as before, only the first six $\delta$-functions depend on $\eta_{l_2 a}$ and $\tilde{\eta}_{l_2\dot{a}}$ so we can expand straight away (noticing that we get another factor of $(i)^2$ from this expansion)

$$\int \prod_{i=1}^{4} d^2\eta_{l_i} d^2\tilde{\eta}_{l_i} \Big\{(i)^2\eta_{l_2 b}\tilde{\eta}_{l_2\dot{b}}\eta_{l_2 b'}\tilde{\eta}_{l_2\dot{b}'}\lambda^{Ab}_{l_2}\tilde{\lambda}^{\dot{b}}_{l_2 B} w^{b'}_{l'_2}\tilde{w}^{\dot{b}'}_{l'_2}\delta(\tilde{q}_{l_1 A} + \tilde{q}_{1A})\delta(q^B_{l_1} + q^B_1)$$
$$(i)^2\eta_{l_3 c}\tilde{\eta}_{l_3\dot{c}}\eta_{l_3 c'}\tilde{\eta}_{l_3\dot{c}'}\lambda^{Cc}_{l_3}\tilde{\lambda}^{\dot{c}}_{l_3 D} w^{c'}_{l'_3}\tilde{w}^{\dot{c}'}_{l'_3}\delta(\tilde{q}_{l_1 C} + \tilde{q}_{1C} + \tilde{q}_{2C})\delta(q^D_{l_1} + q^D_1 + q^D_2)$$
$$\eta_{l_4 d}\tilde{\eta}_{l_4\dot{d}}\eta_{l_4 d'}\tilde{\eta}_{l_4\dot{d}'}\lambda^{Ed}_{l_4}\tilde{\lambda}^{\dot{d}}_{l_4 F} w^{d'}_{l_4}\tilde{w}^{\dot{d}'}_{l_4}\delta(\tilde{q}_{5E} - \tilde{q}_{l_1 E})\delta(q^F_5 - q^F_{l_1})\Big\}. \tag{4.86}$$

One notes that, by expanding the fermionic $\delta$-functions, the dependence on the Grassmann parameters $\eta_{l_1 a}$ and $\tilde{\eta}_{l_1\dot{a}}$ has reduced to

$$\delta(\tilde{q}_{l_1 A} + \tilde{q}_{1A})\delta(q^B_{l_1} + q^B_1)\delta(\tilde{q}_{l_1 C} + \tilde{q}_{1C} + \tilde{q}_{2C})\delta(q^D_{l_1} + q^D_1 + q^D_2)\delta(\tilde{q}_{5E} - \tilde{q}_{l_1 E})\delta(q^F_5 - q^F_{l_1}) \tag{4.87}$$

only. Expanding this further gives the sought-after coefficient of $\eta_{l_1 a}\eta_{l_1 b}\tilde{\eta}_{l_1\dot{a}}\tilde{\eta}_{l_1\dot{b}}$. The result (in an appropriate order of the Grassmann spinors) of the expansion of the six $\delta$-functions in (4.87) is then given by

$$\tilde{\eta}_{l_1\dot{a}}\tilde{\eta}_{l_1\dot{b}}\Big( - \tilde{\eta}_{1\dot{c}}\tilde{\lambda}^{\dot{a}}_{l_1 A}\tilde{\lambda}^{\dot{c}}_{1C}\tilde{\lambda}^{\dot{b}}_{l_1 E} - \tilde{\eta}_{2\dot{c}}\tilde{\lambda}^{\dot{a}}_{l_1 A}\tilde{\lambda}^{\dot{c}}_{2C}\tilde{\lambda}^{\dot{b}}_{l_1 E} - \tilde{\eta}_{5\dot{c}}\tilde{\lambda}^{\dot{a}}_{l_1 A}\tilde{\lambda}^{\dot{b}}_{l_1 C}\tilde{\lambda}^{\dot{c}}_{5E} + \tilde{\eta}_{1\dot{c}}\tilde{\lambda}^{\dot{c}}_{1A}\tilde{\lambda}^{\dot{a}}_{l_1 C}\tilde{\lambda}^{\dot{b}}_{l_1 E}\Big)$$
$$\times\, \eta_{l_1 a}\eta_{l_1 b}\Big(\eta_{5c}\lambda^{Ba}_{l_1}\lambda^{Db}_{l_1}\lambda^{Fc}_5 + \eta_{1c}\lambda^{Ba}_{l_1}\lambda^{Dc}_1\lambda^{Fb}_{l_1} + \eta_{2c}\lambda^{Ba}_{l_1}\lambda^{Dc}_2\lambda^{Fb}_{l_1} - \eta_{1c}\lambda^{Bc}_1\lambda^{Da}_{l_1}\lambda^{Fb}_{l_1}\Big). \tag{4.88}$$

Having extracted the correct powers of the Grassmann variables from all fermionic $\delta$-functions, we can now integrate over the $\eta_{l_i}$ and $\tilde{\eta}_{l_i}$. The integration is straightforward and yields,

$$\begin{aligned}&\Big(\tilde{\eta}_{1\dot{c}}[1^{\dot{c}}|l_3\rangle\cdot w_{l_3'}\; w_{l_2'}\cdot\langle l_2|\hat{l}_1|l_4\rangle\cdot w_{l_4}-\tilde{\eta}_{1\dot{c}}[1^{\dot{c}}|l_2\rangle\cdot w_{l_2'}\; w_{l_3'}\cdot\langle l_3|\hat{l}_1|l_4\rangle\cdot w_{l_4}\\&+\;\tilde{\eta}_{2\dot{c}}[2^{\dot{c}}|l_3\rangle\cdot w_{l_3'}\; w_{l_2'}\cdot\langle l_2|\hat{l}_1|l_4\rangle\cdot w_{l_4}+\tilde{\eta}_{5\dot{c}}[5^{\dot{c}}|l_4\rangle\cdot w_{l_4}\; w_{l_2'}\cdot\langle l_2|\hat{l}_1|l_3\rangle\cdot w_{l_3'}\Big)\\\times&\Big(\eta_{1c}\langle 1^c|l_3]\cdot\tilde{w}_{l_3'}\;\tilde{w}_{l_2'}\cdot[l_2|\hat{l}_1|l_4]\cdot\tilde{w}_{l_4}-\eta_{1c}\langle 1^c|l_2]\cdot\tilde{w}_{l_2'}\;\tilde{w}_{l_3'}\cdot[l_3|\hat{l}_1|l_4]\cdot\tilde{w}_{l_4}\\&+\;\eta_{2c}\langle 2^c|l_3]\cdot\tilde{w}_{l_3'}\;\tilde{w}_{l_2'}\cdot[l_2|\hat{l}_1|l_4]\cdot\tilde{w}_{l_4}+\eta_{5c}\langle 5^c|l_4]\cdot\tilde{w}_{l_4}\;\tilde{w}_{l_2'}\cdot[l_2|\hat{l}_1|l_3]\cdot\tilde{w}_{l_3'}\Big)\,.\end{aligned}\tag{4.89}$$

Here we introduced the notation that $w_{l_i}\cdot\langle l_i| := w^a_{l_i}\langle l_{i,a}|$, and the $\hat{l}_i$ are slashed momenta, with e.g.

$$w_{l_2'}\cdot\langle l_2|\hat{l}_1|l_3\rangle\cdot w_{l_3'} = w^a_{l_2'}\lambda^A_{l_2,a}l_{1,AB}\lambda^B_{l_3,b}w^b_{l_3'}\,.\tag{4.90}$$

Next, one rewrites the spinor expressions in (4.89) in terms of six-dimensional momenta, thereby removing any dependence on $u$- and $w$-spinors. An important observation to do so is the fact that the expressions depending on $\tilde{\eta}_1$ and/or $\eta_1$ antisymmetrise among themselves[8]. The result of these manipulations is

$$\begin{aligned}&\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{l}_1\hat{p}_5\hat{p}_2|1^c\rangle-\tilde{\eta}_{1\dot{c}}\eta_{2c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{l}_1\hat{p}_1|2^c\rangle+\tilde{\eta}_{2\dot{c}}\eta_{1c}\frac{1}{s_{12}}[2^{\dot{c}}|\hat{p}_1\hat{l}_1\hat{p}_5\hat{p}_2|1^c\rangle\\&+\tilde{\eta}_{1\dot{c}}\eta_{5c}[1^{\dot{c}}|\hat{p}_2\hat{l}_1|5^c\rangle-\tilde{\eta}_{5\dot{c}}\eta_{1c}[5^{\dot{c}}|\hat{l}_1\hat{p}_2|1^c\rangle+\tilde{\eta}_{2\dot{c}}\eta_{2c}\frac{1}{s_{12}}[2^{\dot{c}}|\hat{p}_1\hat{l}_1\hat{p}_5\hat{p}_1|2^c\rangle\\&+\tilde{\eta}_{5\dot{c}}\eta_{5c}\frac{1}{s_{15}}[5^{\dot{c}}|\hat{p}_1\hat{l}_1\hat{p}_2\hat{p}_1|5^c\rangle+\tilde{\eta}_{5\dot{c}}\eta_{2c}[5^{\dot{c}}|\hat{l}_1\hat{p}_1|2^c\rangle-\tilde{\eta}_{2\dot{c}}\eta_{5c}[2^{\dot{c}}|\hat{p}_1\hat{l}_1|5^c\rangle\,.\end{aligned}\tag{4.91}$$

### 4.5.2 Final result (before PV reduction)

Including all appropriate prefactors, our result for the five-point one-loop superamplitude is expressed in terms of a single integral function, namely a linear pentagon integral. Explicitly,

$$A_{5;1} \;=\; \mathcal{C}_\mu\, I^\mu_{5,l_1}\;,\tag{4.92}$$

where

$$I^\mu_{5,l_1}(1,\ldots,5) := \int\frac{d^Dl}{(2\pi)^D}\frac{l_1^\mu}{l_1^2l_2^2l_3^2(p_3+l_3)^2l_5^2}\,,\tag{4.93}$$

is the linear pentagon, and the coefficient $\mathcal{C}_\mu$ is given by

$$\begin{aligned}\mathcal{C}_\mu=&\frac{1}{s_{34}}\Big\{\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{\sigma}_\mu\hat{p}_5\hat{p}_2|1^c\rangle-\tilde{\eta}_{1\dot{c}}\eta_{2c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{\sigma}_\mu\hat{p}_1|2^c\rangle+\tilde{\eta}_{2\dot{c}}\eta_{1c}\frac{1}{s_{12}}[2^{\dot{c}}|\hat{p}_1\hat{\sigma}_\mu\hat{p}_5\hat{p}_2|1^c\rangle\\&+\tilde{\eta}_{1\dot{c}}\eta_{5c}[1^{\dot{c}}|\hat{p}_2\hat{\sigma}_\mu|5^c\rangle-\tilde{\eta}_{5\dot{c}}\eta_{1c}[5^{\dot{c}}|\hat{\sigma}_\mu\hat{p}_2|1^c\rangle+\tilde{\eta}_{2\dot{c}}\eta_{2c}\frac{1}{s_{12}}[2^{\dot{c}}|\hat{p}_1\hat{\sigma}_\mu\hat{p}_5\hat{p}_1|2^c\rangle\\&+\tilde{\eta}_{5\dot{c}}\eta_{5c}\frac{1}{s_{15}}[5^{\dot{c}}|\hat{p}_1\hat{\sigma}_\mu\hat{p}_2\hat{p}_1|5^c\rangle+\tilde{\eta}_{5\dot{c}}\eta_{2c}[5^{\dot{c}}|\hat{\sigma}_\mu\hat{p}_1|2^c\rangle-\tilde{\eta}_{2\dot{c}}\eta_{5c}[2^{\dot{c}}|\hat{\sigma}_\mu\hat{p}_1|5^c\rangle\Big\}\,.\end{aligned}\tag{4.94}$$

The factor of $1/s_{34}$ and the additional propagator in the pentagon appearing in (4.92), are due to the prefactor of the four-point tree-level superamplitude entering the cut. A pictorial representation of a generic pentagon integral is given in Figure 4.5. We now proceed and summarise the result of the PV reduction of (4.93) in the next section.

[8] We give more details on these manipulations in Appendix B.3.2.

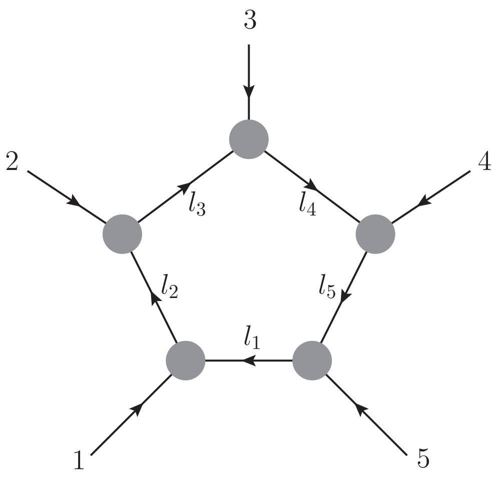



Figure 4.5: *A generic pentagon loop integral.*

### 4.5.3 Final result (after PV reduction)

The PV reduction of (4.93) allows us to re-express a linear pentagon in terms of a scalar pentagon and scalar box functions. Using this, we re-express the one-loop five-point superamplitude as

$$\mathcal{A}_{5;1} = \mathcal{C}^{(5)} I_5(1,\ldots,5) + \sum_{i=1}^{5} \mathcal{C}^{(4,i)} I_{4,i}(1,\ldots,5)\ , \tag{4.95}$$

where we introduced the scalar integral functions $I_5$ for the pentagon and $I_{4,i}$ for the boxes. Here, the index $i$ in $I_{4,i}$ labels the first leg of the massive corner for a clockwise ordering of the external states.

Explicitly, the coefficients for the specific cut we discussed in the previous section are given by

$$\begin{aligned}
\mathcal{C}^{(5)/(4,3)} = {} & \tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}\big(s_{12}[1^{\dot{c}}|\hat{p}_5\hat{p}_2|1^c\rangle A^{(5)/(4,3)} + [1^{\dot{c}}|\hat{p}_2\hat{p}_3\hat{p}_5\hat{p}_2|1^c\rangle C^{(5)/(4,3)}\big) \\
& + \tilde{\eta}_{2\dot{c}}\eta_{2c}\frac{1}{s_{12}}\big(s_{12}[2^{\dot{c}}|\hat{p}_5\hat{p}_1|2^c\rangle B^{(5)/(4,3)} + [2^{\dot{c}}|\hat{p}_1\hat{p}_3\hat{p}_5\hat{p}_1|2^c\rangle C^{(5)/(4,3)}\big) \\
& + \tilde{\eta}_{5\dot{c}}\eta_{5c}\frac{1}{s_{15}}\big([5^{\dot{c}}|\hat{p}_1\hat{p}_3\hat{p}_2\hat{p}_1|5^c\rangle C^{(5)/(4,3)} + s_{15}[5^{\dot{c}}|\hat{p}_2\hat{p}_1|5^c\rangle D^{(5)/(4,3)}\big) \\
& - \tilde{\eta}_{1\dot{c}}\eta_{2c}\frac{1}{s_{12}}\big(s_{12}[1^{\dot{c}}|\hat{p}_2\hat{p}_5|2^c\rangle B^{(5)/(4,3)} + [1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{p}_3\hat{p}_1|2^c\rangle C^{(5)/(4,3)}\big) \\
& + \tilde{\eta}_{2\dot{c}}\eta_{1c}\frac{1}{s_{12}}\big(s_{12}[2^{\dot{c}}|\hat{p}_5\hat{p}_2|1^c\rangle B^{(5)/(4,3)} + [2^{\dot{c}}|\hat{p}_1\hat{p}_3\hat{p}_5\hat{p}_2|1^c\rangle C^{(5)/(4,3)}\big) \\
& + \tilde{\eta}_{1\dot{c}}\eta_{5c}\big([1^{\dot{c}}|\hat{p}_2\hat{p}_1|5^c\rangle A^{(5)/(4,3)} + [1^{\dot{c}}|\hat{p}_2\hat{p}_3|5^c\rangle C^{(5)/(4,3)}\big) \\
& - \tilde{\eta}_{5\dot{c}}\eta_{1c}\big([5^{\dot{c}}|\hat{p}_1\hat{p}_2|1^c\rangle A^{(5)/(4,3)} + [5^{\dot{c}}|\hat{p}_3\hat{p}_2|1^c\rangle C^{(5)/(4,3)}\big) \\
& + \tilde{\eta}_{5\dot{c}}\eta_{2c}\big([5^{\dot{c}}|\hat{p}_2\hat{p}_1|2^c\rangle B^{(5)/(4,3)} + [5^{\dot{c}}|\hat{p}_3\hat{p}_1|2^c\rangle C^{(5)/(4,3)}\big) \\
& - \tilde{\eta}_{2\dot{c}}\eta_{5c}\big([2^{\dot{c}}|\hat{p}_1\hat{p}_2|5^c\rangle B^{(5)/(4,3)} + [2^{\dot{c}}|\hat{p}_1\hat{p}_3|5^c\rangle C^{(5)/(4,3)}\big)\ .
\end{aligned} \tag{4.96}$$

Here, the variables $A^{(5)/(4,3)}, B^{(5)/(4,3)}, C^{(5)/(4,3)}$ and $D^{(5)/(4,3)}$ are the coefficients from the PV reduction of the scalar pentagon $I_5$ or box function $I_{4,3}$ respectively. For the scalar pentagon, we have

$$\begin{aligned} A^{(5)} &= \Delta^{-1}(s_{15}s_{13}s_{23}s_{25} + s_{15}s_{25}s_{23}^2 - s_{13}s_{23}s_{25}^2 - s_{13}^2 s_{25}^2 + 2s_{12}s_{13}s_{25}s_{35} \\ &\quad + s_{12}s_{23}s_{35}s_{15} + s_{12}s_{23}s_{25}s_{35} - s_{12}^2 s_{35}^2) \\ B^{(5)} &= \Delta^{-1}s_{15}(s_{12}s_{23}s_{35} + s_{13}s_{23}s_{25} - s_{12}s_{13}s_{35} - s_{13}s_{23}s_{15} + s_{13}^2 s_{25} - s_{15}s_{23}^2) \\ C^{(5)} &= \Delta^{-1}s_{12}s_{15}(s_{12}s_{35} - s_{15}s_{23} - s_{13}s_{25} - 2s_{23}s_{25}) \\ D^{(5)} &= \Delta^{-1}s_{12}s_{23}(s_{15}s_{23} + s_{13}s_{25} - s_{12}s_{35} + 2s_{15}s_{13}) \end{aligned} \tag{4.97}$$

whereas for the coefficients of the box integral $I_{4,3}$ we find

$$\begin{aligned} A^{(4,3)} &= \Delta^{-1}s_{25}(s_{13}s_{25} - s_{15}s_{23} - s_{12}s_{35}) \\ B^{(4,3)} &= \Delta^{-1}s_{15}(s_{15}s_{23} - s_{13}s_{25} - s_{12}s_{35}) \\ C^{(4,3)} &= \Delta^{-1}2s_{12}s_{15}s_{25} \\ D^{(4,3)} &= \Delta^{-1}s_{12}(s_{12}s_{35} - s_{15}s_{23} - s_{13}s_{25}) \ . \end{aligned} \tag{4.98}$$

Furthermore, we have defined $\Delta$ as the Gram determinant. Explicitly, it is given by

$$\Delta = s_{15}^2 s_{23}^2 + (s_{13}s_{25} - s_{12}s_{35})^2 - 2s_{15}s_{23}(s_{13}s_{25} + s_{12}s_{35}) \, . \tag{4.99}$$

Notice that for the final expression for the amplitude we have to collect the five box integrals $I_{4,i}$ with their respective coefficients which can be obtained by cyclic permutation of the states $(1, \ldots, 5)$. Furthermore, we have to include one copy of the pentagon integral with its coefficient. The pentagon coefficient does not possess manifest cyclic symmetry, and each of the five quadruple cuts produces a different looking expression. However, our tests provided below confirm that the pentagon coefficients have the expected cyclic symmetry.

### 4.5.4 Gluon component amplitude

In this section we extract from the one-loop five-point superamplitude its component where all external particles are six-dimensional gluons. This is useful since, dimensionally reducing this component amplitude to four dimensions, one can access the gluon MHV and anti-MHV amplitudes of $\mathcal{N} = 4$ SYM.

In order to extract this component we have to integrate one power of $\eta_i$ and $\tilde{\eta}_i$ for each external state, here denoted by $1_{a\dot{a}}$, $2_{b\dot{b}}$, $3_{c\dot{c}}$, $4_{d\dot{d}}$, and $5_{e\dot{e}}$. Doing this, one arrives at

$$\begin{aligned} A_{5;1}\big|_{(3,4)\text{-cut}} &\propto \int \frac{d^6 l}{(2\pi)^6} \frac{1}{s_{34}} \delta^+(l_1^2)\delta^+(l_2^2)\delta^+(l_3^2)\delta^+(l_5^2) \frac{1}{(p_3 + l_3)^2} \\ \times\Big\{ & \frac{1}{s_{12}}[1_{\dot{a}}|\hat{p}_2\hat{l}_1\hat{p}_5\hat{p}_2|1_a\rangle\langle 2_b 3_c 4_d 5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + \frac{1}{s_{12}}[2_{\dot{b}}|\hat{p}_1\hat{l}_1\hat{p}_5\hat{p}_1|2_b\rangle\langle 1_a 3_c 4_d 5_e\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] \\ & + \frac{1}{s_{15}}[5_{\dot{e}}|\hat{p}_1\hat{l}_1\hat{p}_2\hat{p}_1|5_e\rangle\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] \\ & - \frac{1}{s_{12}}[2_{\dot{b}}|\hat{p}_1\hat{l}_1\hat{p}_5\hat{p}_2|1_a\rangle\langle 2_b 3_c 4_d 5_e\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + \frac{1}{s_{12}}[1_{\dot{a}}|\hat{p}_2\hat{p}_5\hat{l}_1\hat{p}_1|2_b\rangle\langle 1_a 3_c 4_d 5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] \\ & - [1_{\dot{a}}|\hat{p}_2\hat{l}_1|5_e\rangle\langle 1_a 2_b 3_c 4_d\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + [5_{\dot{e}}|\hat{l}_1\hat{p}_2|1_a\rangle\langle 2_b 3_c 4_d 5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] \\ & - [5_{\dot{e}}|\hat{l}_1\hat{p}_1|2_b\rangle\langle 1_a 3_c 4_d 5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] + [2_{\dot{b}}|\hat{p}_1\hat{l}_1|5_e\rangle\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] \Big\} \ , \end{aligned} \tag{4.100}$$

where $l_i$, $i = 1, \ldots, 5$ are the five propagators in Figure 4.5. In the next section we perform the reduction to four dimension of (4.100), which will give us important checks on our result.

### 4.5.5 4D limit of the one-loop five-point amplitude

An important series of nontrivial consistency checks on our six-dimensional five-point amplitude at one loop can be obtained by performing its reduction to four dimensions, and comparing it to the expected form of the one-loop (MHV or anti-MHV) amplitude(s) directly calculated in four-dimensional $\mathcal{N} = 4$ SYM theory. In performing this reduction, we restrict any six-dimensional spinorial expression to four dimensions which we discuss below. As for the integral functions, we formally evaluate them in $6-2\epsilon$ dimensions. The four-dimensional limit of these higher dimensional integrals is then obtained by simply replacing $\epsilon \to 1+\epsilon$. Then, in order to perform the reduction to four dimensions of various six-dimensional quantities, one can employ the results of [86] (see also [87]). There, it was found that the solutions to the Dirac equation with the external momenta living in a four-dimensional subspace, i.e. $p = (p^0, p^1, p^2, p^3, 0, 0)$, can be written as

$$\lambda_a^A = \begin{pmatrix} 0 & \lambda_\alpha \\ \tilde{\lambda}^{\dot{\alpha}} & 0 \end{pmatrix}, \qquad \tilde{\lambda}_{A\dot{a}} = \begin{pmatrix} 0 & \lambda^\alpha \\ -\tilde{\lambda}_{\dot{\alpha}} & 0 \end{pmatrix}, \tag{4.101}$$

where $\lambda_\alpha$ and $\tilde{\lambda}_{\dot{\alpha}}$ are the usual four-dimensional spinor variables. Hence, the Lorentz invariant, little group covariant quantities $\langle i_a | j_{\dot{a}}]$, $[i_{\dot{a}} | j_a\rangle$ become

$$\langle i_a | j_{\dot{a}}] = \begin{pmatrix} [ij] & 0 \\ 0 & -\langle ij\rangle \end{pmatrix}, \qquad [i_{\dot{a}} | j_a\rangle = \begin{pmatrix} -[ij] & 0 \\ 0 & \langle ij\rangle \end{pmatrix}. \tag{4.102}$$

Here, we follow the standard convention of writing the four-dimensional spinor contractions as $\lambda_i^\alpha \lambda_{j\alpha} = \langle ij\rangle$ and $\tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}} = [ij]$.

The four-dimensional helicity group is a $\mathsf{U}(1)$ subgroup of the six-dimensional little group which preserves the structure of (4.101) and (4.102). In order to determine the (four-dimensional) helicity of a certain state in (4.19), a practical way to proceed is as follows. Each appearance of a dotted or undotted index equal to 1 (2) contributes an amount of $+1/2$ ($-1/2$) to the total four-dimensional helicity. As an example, consider the term $A_{a\dot{a}}$ in (4.19). States with $(a, \dot{a}) = (1, 1)$ correspond, upon reduction, to gluons with positive helicity and states with $(a, \dot{a}) = (2, 2)$ to gluons of negative helicity.

In the four-dimensional limit, the six-dimensional spinor brackets become[9] [87]

$$\begin{aligned} -\langle i_+ | j_+] = [ij] = [i_+ | j_+\rangle\,, \qquad & \langle i_- | j_-] = \langle ij\rangle = -[i_- | j_-\rangle\,, \\ \langle i_- j_- k_+ l_+\rangle = -\langle ij\rangle[kl]\,, \qquad & [i_- j_- k_+ l_+] = -\langle ij\rangle[kl]\,, \\ \langle i_- j_+ k_- l_+\rangle = +\langle ik\rangle[jl]\,, \qquad & [i_- j_+ k_- l_+] = +\langle ik\rangle[jl]\,. \end{aligned} \tag{4.103}$$

In the following we will use these identifications to check the four-dimensional limits of (4.100) for all MHV helicity assignments of the external gluons. As expected, we will always obtain the

[9] Note that our definition of spinors of positive and negative helicities in four dimensions is opposite to that in [87], i.e. the spinor bracket $\langle\,\cdot\,,\,\cdot\,\rangle$ represents a product between spinors of negative helicitiy.

expected $\mathcal{N} = 4$ SYM result, i.e. the appropriate Parke-Taylor MHV prefactor multiplied by a four-dimensional one-loop box function.

To begin with, we recall that upon four-dimensional reduction, a six-dimensional scalar pentagon reduces to five different box functions (plus terms vanishing in four dimensions) [95–97], and hence contributes to the coefficients of the relevant box functions. Schematically,

$$\mathcal{C}^{(5)} I_5 + \mathcal{C}^{(4,3)} I_{4,3} \xrightarrow{4D} \left[ \mathcal{C}^{(5)} \frac{P^{(4,3)}}{2 s_{12} s_{23} s_{34} s_{45} s_{51}} + \mathcal{C}^{(4,3)} \right] I_{4,3} \tag{4.104}$$

where

$$P^{(4,3)} = s_{12} s_{51} (s_{12} s_{23} - s_{12} s_{51} - s_{23} s_{34} - s_{34} s_{45} + s_{45} s_{51}) \,, \tag{4.105}$$

when going to four dimensions. Hence, upon dimensional reduction the coefficients of the PV reduction become

$$\begin{aligned} A &\to -\frac{s_{12}s_{15} - s_{15}s_{45} + s_{34}s_{45}}{2 s_{23} s_{34} s_{45}}, \\ B &\to -\frac{s_{15}s_{12} - s_{15}s_{45}}{2 s_{23} s_{34} s_{45}}, \\ C &\to -\frac{s_{12}s_{15}}{2 s_{23} s_{34} s_{45}}, \\ D &\to -\frac{s_{12}}{2 s_{34} s_{45}} \,. \end{aligned} \tag{4.106}$$

Let us now discuss specific helicity assignments. We start by considering the amplitude with a helicity configuration of $(1^-, 2^-, 3^+, 4^+, 5^+)$. In this case, after PV reduction only the third term in (4.100) is non-vanishing. Hence, we have to consider the four-dimensional limit of

$$\frac{1}{s_{34} s_{15}} \left( [5_{\dot e}|\hat p_1 \hat p_3 \hat p_2 \hat p_1|5_e\rangle C + [5_{\dot e}|\hat p_1 \hat p_5 \hat p_2 \hat p_1|5_e\rangle D \right) \langle 1_a 2_b 3_c 4_d\rangle [1_{\dot a} 2_{\dot b} 3_{\dot c} 4_{\dot d}] \,. \tag{4.107}$$

Upon dimensional reduction, the resulting contribution is

$$\frac{s_{15} s_{12}}{2} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \,. \tag{4.108}$$

Given the relation between the scalar box functions $F_4$ and the corresponding box integrals, $I_4 = 2\,F/(s_{12}s_{15})$, it is immediate to see that the kinematic factors in (4.108) cancel and the final result is the anticipated one:

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \,. \tag{4.109}$$

The fact that the form of the one-loop five-point amplitude upon reduction to four dimensions is precisely the well-known result is an expected, though highly non-trivial, outcome.

As mentioned above, we have performed checks for all external helicity configurations, finding in all cases agreement with the expected four-dimensional result. We would like to highlight a particularly stringent test, namely that corresponding to the helicity configuration $(1^+, 2^+, 3^-, 4^-, 5^+)$, where all terms in (4.100) contribute to the four-dimensional reduction.

A final comment is in order here. It is known that collinear and soft limits put important constraints on tree-level and loop amplitudes in any gauge theory and in gravity. In six dimensions, the lack of infrared divergences makes loop level factorisation trivial, similarly to what happens to four-dimensional gravity because of its improved infrared behaviour compared to four-dimensional Yang-Mills theory amplitudes. Therefore, the factorisation properties we derive below from tree-level amplitudes will apply unmodified to one-loop amplitudes.

We now consider again the five-point amplitude (4.31) derived in [85], and take the soft limit where $p_1 \to 0$. A short calculation shows that

$$A^{(0)}_{5;a\dot{a}\ldots} \to S_{a\dot{a}}(5,1,2) A^{(0)}_{4;\ldots} \, , \tag{4.110}$$

where we find, for the six-dimensional soft function,

$$S_{a\dot{a}}(i,s,j) \;=\; \frac{\langle s_a|\hat{p}_j \hat{p}_i|s_{\dot{a}}]}{s_{is}s_{sj}} \, . \tag{4.111}$$

In (4.110) the dots stand for the little group indices of the remaining particles in the amplitude. Using the results in this section, it is also immediate to check that (4.111) reduces, in the four-dimensional limit, to the expected soft functions of [32]. As a final test on our five-point amplitude we have checked that the soft limits where legs 1, 2 or 5 become soft are all correct.

This provides an exhaustive set of checks of our result for the six-dimensional five-point superamplitude at one-loop. In summary, our checks confirm the appearance of a linear pentagon integral function (see the result (4.92)) in the the six-dimensional, maximally supersymmetric theory at one-loop with five external states.

# 5 Gauge Mediation, SQCD and $\mathcal{N} = 1$ Seiberg Duality

In this chapter we review some basic facts about supersymmetric theories and how supersymmetry can be broken. We will be interested mainly in phenomenological applications and the possibility of *metastable non-supersymmetric vacua* in $\mathcal{N} = 1$ super-QCD (SQCD) theories. The ideas and concepts behind this construction will be reviewed in the following sections.

## 5.1 Some introductory comments

Quite generally, the concept of supersymmetry is a corner stone in modern theoretical physics. Ideas surrounding supersymmetry were involved in many important developments in the last decades. The reason for the wide range of SUSY applications is the observation that quite often, supersymmetric models are easier to understand. Furthermore, they are even easier to solve since the high amount of symmetry constrains a system. We have seen applications of this in the last two chapters where the maximal amount of supersymmetry in a quantum field theory makes the notation of highly constrained superamplitudes natural. Although (maximally) supersymmetric Yang-Mills theories are not part of the framework that ultimately describes nature, they serve as toy models in which analytic results can be derived. These results can then provide deeper insights or simplified computational approaches to more realistic theories.[1]

Especially interesting are supersymmetric versions of non-abelian gauge theories. The studies of non-perturbative effects in supersymmetric QCD has led to insights in the strong-coupling limit of ordinary, non-supersymmetric QCD. An example of this is a better understanding of the dynamics which lead to quark confinement. Two basic principles have heavily influenced the studies of supersymmetric gauge theories, namely the concepts of *duality* and *holomorphy.* Behind the first one lies the idea of mapping strongly and weakly coupled regimes of different supersymmetric theories. This seems to be a rather general pattern. Strong-weak dualities are rather easily realised in supersymmetric theories compared to non-supersymmetric ones. To that extent, several examples of at least (by the means of dualities) partly calculable strongly coupled supersymmetric theories have been found in the past. Examples are dualities in the maximally supersymmetric $\mathcal{N} = 4$ SYM theory [98–102], the Seiberg-Witten duality [103], the highly celebrated gauge/gravity

[1] We have seen an example of this in Chapter 3 where we briefly mentioned the supersymmetric decomposition of one-loop QCD amplitudes.

correspondence in AdS space [104] and also the recently discovered dualities of Wilson-loops and scattering amplitudes in the four-dimensional $\mathcal{N} = 4$ SYM theory [105–107]. Another important example is key to our discussions in the following chapters namely a class of certain dualities in QCD-like $\mathcal{N} = 1$ supersymmetric theories which go under the name of Seiberg dualities [108, 109]. These dualities generalise to some extent the well known electric-magnetic duality of Maxwell's electrodynamics. Seiberg argued that in certain cases the infrared limit of a strongly coupled supersymmetric gauge theory (normally denoted as the 'electric theory') is 'dual' to the infrared limit of another weakly coupled supersymmetric gauge theory (the 'magnetic theory'). If the theories both flow to a (non-trivial) infrared fixed point, they essentially describe the same physics. Since we want to perform perturbation theory, we can then choose the weakly coupled description. Although strictly speaking the Seiberg dualities have not been proven[2], they passed a lot of stringent tests [109–111]. In our discussions of supersymmetric QCD theories we will point out some of the ideas behind the SQCD Seiberg duality.

The other important ingredient is the fact that the superpotential of a supersymmetric theory is a holomorphic function of the chiral superfields only. Otherwise the Lagrangian would not be invariant under supersymmetry. This can be used to prove [112] that the superpotential in a supersymmetric theory is not renormalised to any order in perturbation theory, see also [113, 114]. This can be motivated as follows: Essential is the idea of promoting the couplings (including the gauge coupling) of the specific theory to non-dynamical chiral superfields such that the physical couplings are their vacuum expectation values. In order to be consistent with supersymmetry these spurious[3] superfields must appear holomorphically in the superpotential. This puts constraints on its form. In the literature these are often called 'selection rules', leading to the observation that the effective superpotential of a supersymmetric theory is actually equal to the tree level superpotential [112]. Hence, the superpotential receives no loop corrections and is not renormalised. However, Seiberg's argument does not take account of non-perturbative effects which *can* provide corrections to the superpotential.

### 5.1.1 Supersymmetry and the Standard Model

Supersymmetry is not only of pure theoretical interest, it has also heavily influenced elementary particle physics. Furthermore, it has attracted theorists to more phenomenological ideas and topics. There are two main motivations for why supersymmetry is so attractive to particle physicists. Firstly, the coupling constants of the Standard Model (SM) unify within the minimal supersymmetric extension of the SM at a large energy $E_{\text{GUT}} \sim 10^{16}$ GeV [115–117]. Secondly, there is the important fact that supersymmetry protects certain quantities from receiving quadratically divergent quantum corrections. The most prominent example of this is the solution to the *hierarchy problem* of the Standard Model which supersymmetry can offer. This 'problem' is the famous

[2] From a more 'physical' point of view it is not necessarily a question of 'proving' the duality. As long as the global symmetries in both theories match, we can treat one theory as an effective one which provides a weakly coupled description in a certain energy regime.

[3] Here, the expression 'spurion' just comes from the fact that setting the coupling superfields to the actual value of the coupling breaks certain symmetries which the spurios superfields were chosen to be charged under.

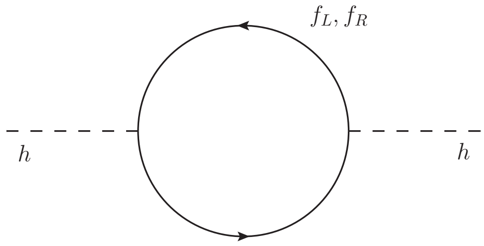



Figure 5.1: *One-loop correction to a scalar field where a left- and right-handed fermion is running in the loop.*

puzzle of why the electroweak scale $M_{\rm ew}$ is so much smaller than the Planck scale $M_{\rm Pl}$ and if this hierarchy between the scales is stable under quantum corrections.

In the SM one finds that scalar masses are indeed not stable under quantum corrections. This is rather important for the mass of the Higgs particle which is the only scalar particle of the SM. In general, a scalar mass receives one-loop corrections from quantum diagrams where a fermion is running in the loop as shown in Figure 5.1. This leads to a correction of the scalar mass squared of the order

$$\delta m_S^2 \simeq -|y_f|^2 \left[ \Lambda^2 + c \log\left(\frac{\Lambda^2}{m_f^2}\right)\right] , \tag{5.1}$$

where $y_f$ is a Yukawa-type coupling of the scalar to the fermions, $c$ is a constant with dimension $[c] = 2$ and $\Lambda$ is a cut-off in the theory. Hence, the scalar mass gets corrected to leading order by $\Lambda^2$ and is therefore quadratically divergent. In contrast, fermion masses - if introduced in the theory at tree-level - grow only logarithmically with the cut-off scale and hence, any corrections are of the same order as the bare fermion mass itself. This observation yields the unpleasant fact that the Higgs mass (and therefore the Higgs vev) would be as heavy as the cut-off scale $\Lambda$. This is somewhat unnatural[4] since the cut-off can be much larger than the electroweak scale where the Higgs mass is supposed to lie. Hence, even a tree-level Higgs mass of an appropriate order would end up at the cut-off scale due to quantum corrections. The only way to obtain a physical Higgs mass of the order of $10^{-2}$ GeV is to highly fine-tune the bare mass of the scalar field to be as well of the order of the cut-off. And this has to be done order by order in perturbation theory. However, such a fine-tuning is considered to be unnatural in the sense that it is not providing an explanation of *why* the electroweak scale can be naturally hierarchically smaller than the cut-off scale $\Lambda$.

The introduction of supersymmetry into the theory can solve the issue of quadratic divergencies. If new scalars are present in the theory which couple to the existing scalar field new corrections to the scalar mass are introduced. The corresponding one-loop diagrams are shown in Figure 5.2.

[4] Here we mean that a Higgs of the order of the cut-off is not inconsistent, it is just not what one would expect from the Higgs-mechanism in the SM.

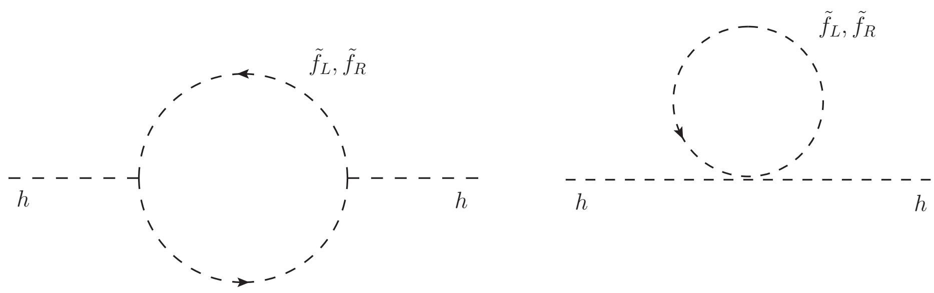



Figure 5.2: *One-loop correction to a scalar field with scalars $\tilde{f}_L, \tilde{f}_R$ running in the loop.*

From the three-point and four-point interactions one finds schematically a correction of

$$\delta m_S^2 \simeq \lambda \left[ \Lambda^2 + \tilde{c}_L \log\left(\frac{\Lambda^2}{\tilde{m}_L^2}\right) + \tilde{c}_R \log\left(\frac{\Lambda^2}{\tilde{m}_R^2}\right)\right] \ . \tag{5.2}$$

Here, we used the same cut-off scale $\Lambda$, $\tilde{m}_{L/R}$ are the scalar masses and again $\tilde{c}_{L/R}$ are coefficients with dimension $[\tilde{c}_{L/R}] = 2$. Note that these terms come from the diagram on the right-hand side of Figure 5.2. The diagram on the left only generates terms with logarithmic divergences times the three-point coupling between the newly introduced scalar fields and the Higgs.

From these oberservations one finds that the quadratic divergences cancel out if $\lambda = |y_f|^2$. This is exactly what supersymmetry provides. Of course, the additional scalar fields are the scalar partners of the fermion yielding the correction to the Higgs mass. Furthermore, the interactions between scalar-fermion (three-point) and scalar-scalar (three- and four-point) are coming from the same term in the superpotential of the supersymmetric theory and hence, the couplings match. However, by supersymmetry, the new scalar particles in the theory are bound to have the same mass as their fermionic partners. And since we have not observed any light scalars at all, we know that at low energies supersymmetry can only be an approximate symmetry of nature. Although this might look disappointing at first sight, it offers a neat solution to the hierarchy problem. If supersymmetry is broken at a scale of the order of the weak scale it protects the Higgs mass from becoming too large since a new supersymmetric theory above the breaking scale provides a natural cut-off for the quantum corrections of the scalar mass in the SM. In that sense, the presence of supersymmetry stabilises the electroweak scale which solves the 'naturalness part' of the hierarchy problem.

In addition to that, supersymmetry might offer a promising way to explain the hierarchy between the supersymmetry breaking and Planck scale itself. The possible solution that a supersymmetric theory provides is strongly connected to the mechanism of how this symmetry is broken. In order to keep the nice features of supersymmetry like the cancellation of quadratic divergences we don't want to break supersymmetry by brute force. We rather prefer to break it spontaneously such that the Lagrangian is still supersymmetry but the vacuum state of the theory does not share this symmetry. If the spontaneous supersymmetry breaking is a dynamical effect as was first suggested by Witten [118] one can naturally implement a large hierarchy between the Planck and the supersymmetry breaking scale (which sets the electroweak scale). By *dynamical supersymmetry breaking* one means that small non-perturbative effects break supersymmetry such that the breaking scale $M_s$ is much smaller (namely exponentially suppressed) than the cut-off scale in the theory

$$M_s = e^{-1/g^2} M_{\text{cut-off}} \ll M_{\text{cut-off}} \ . \tag{5.3}$$

Here, $g$ is a small coupling constant at the cut-off scale $M_{\text{cut-off}}$. The need for non-perturbative effects to break supersymmetry is connected to the non-renormalisation of the superpotential. In general, the scalar potential of supersymmetric gauge theories has a rather complicated vacuum structure. In many cases, the potential of the classical theory also has flat directions which are vacua with vanishing energy. As we will shortly see, these are all supersymmetric vacua of the theory. By the non-renormalisation theorem [112] we know that flat directions of the potential are not lifted by quantum corrections. So if a theory is supersymmetric at the classical level (it has vacua with vanishing energy) then supersymmetry is unbroken to all order in perturbation theory. However, non-perturbative effects can break supersymmetry. Based on this important insight, a vast amount of different models of dynamical supersymmetry breaking where constructed in the past, starting with the early papers [118] and [119, 120] in 1981. It is this non-trivial relation to supersymmetry breaking that made it important to study and understand non-perturbative effects in supersymmetric theories in order to construct phenomenological models with broken supersymmetry.

### 5.1.2 The need for a hidden sector

To that extent, supersymmetry is an attractive candidate for physics beyond the SM. It stabilises the hierarchy between the electroweak and ultraviolet scales and offers an explanation for the large difference in the scales. Furthermore, due to the presence of new degrees of freedom, it provides a vast arena for model-building physics at a rather accessible energy scale. The 'only' missing piece in this nice picture is the mechanism that dynamically breaks supersymmetry. Of course, one could just argue that one is not interested in the specific model and only wants to do low energy phenomenology. This is certainly possible and leads to soft-breaking terms in the minimal supersymmetric extension of the SM. Here, soft-breaking means terms that break supersymmetry but do not disturb the important cancellation of quadratic divergences. This approach is discussed in many places, for instance a nice introduction can be found in [121]. Here, we follow the approach of trying to model the dynamics that break supersymmetry. To do so we need another important ingredient in the supersymmetry breaking approach which is strongly tied to the mass spectrum of supersymmetric theories. In general, the masses of the particles obey a supertrace sum rule [122] where the supertrace is defined as

$$\mathrm{STr}[M^2] \equiv \sum_j (-1)^j (2j+1) \mathrm{Tr}[M_j^2] \ . \tag{5.4}$$

Here, the $M_j^2$ are the squared mass matrices for scalars ($j = 0$), fermions ($j = 1/2$) and gauge bosons ($j = 1$). Calculating the mass matrices for all the different particle species one finds the sum rule [122]

$$\mathrm{STr}[M^2] = \mathrm{Tr}[M_0^2] - 2\mathrm{Tr}[M_{\frac{1}{2}}^\dagger M_{\frac{1}{2}}] + 3\mathrm{Tr}[M_1^2] = -2g^2\langle D^A\rangle \mathrm{Tr}[T^A] \ , \tag{5.5}$$

where $D^A$ is the D-term of (2.76). From this trace relation we can see that the supertrace vanishes if $\langle D^A\rangle = 0$ or $\mathrm{Tr}[T^A] = 0$, i.e. if no $\mathsf{U}(1)$ gauge factors are present in the theory. The physical consequence is that the sum of all squared masses of the bosonic degrees of freedom is equal to the fermionic ones. This is an automatic consequence of supersymmetry since all masses in a supermultiplet are the same. However, even for broken supersymmetry the relation (5.5) holds. In any case, this mass relation is valid at tree-level and receives loop-level corrections, however, these corrections are usually relatively small for weakly coupled theories.

The supertrace relation has important consequences for realistic models of extensions to the SM with broken supersymmetry. From a phenomenological point of view we would like to break supersymmetry such that the superpartners of the SM particles are all heavier. However, the upshot of the supertrace mass formula (5.5) is such that in supersymmetric extensions of the SM with broken supersymmetry one finds a mass splitting between the scalar degrees of freedom of a supermultiplet[5], leading to scalar particles which are lighter than their fermionic partner [115,123]. This is ruled out by experiments which have pushed the mass-bound for the scalar SM partners to be larger than the mass of the SM fermions [124]. The standard lore to circumvent these strong constraints is the introduction of a *hidden sector*. Non-perturbative dynamics are then assumed to break supersymmetry in the hidden sector. In order to transmit the breaking of supersymmetry to the visible sector one needs certain 'mediator' interactions between the hidden and visible sector.

### 5.1.3 Mediation of supersymmetry breaking effects

This leads to the question of what type the mediating interactions are. Just as there exist many different models of how to break supersymmetry in the hidden sector, there are different ways to mediate the supersymmetry breaking effects to the supersymmetric SM. Two constructions have been of major interest in the literature, namely *gravity mediation* and *gauge mediation*. In the first approach it is assumed that the breaking is transmitted through interactions of gravitational strength by Planck suppressed couplings. More precisely, these models are supergravity mediated where supergravity is the combination of gravity and local supersymmetry. In general, it is rather complicated to construct such theories and detailed discussions of this are far beyond the scope of this thesis. Here, we only mention some basics aspects of gravity mediated constructions. In these models, the mass of the gravitino, the superpartner of the spin-two graviton, represents the supersymmetry breaking scale. The mass is of the order

$$m_{3/2} \sim \frac{F_X}{M_{\mathrm{Pl}}} \tag{5.6}$$

where $F_X$ is the F-component of a chiral superfield $X$ which breaks supersymmetry, i.e. $F_X = D_X W \neq 0$ with $D_i$ as the covariant supergravity derivative

$$D_i W = \frac{\partial W}{\partial \phi_i} + \frac{\partial K}{\partial \phi_i} W \ , \tag{5.7}$$

[5] We assume no flavour mixing between different fermion generations.

with $K$ as the Kähler potential. This leads to an order parameter of $F_X \approx \sqrt{m_{3/2} M_{\rm Pl}} \approx (10^{11}\ \text{GeV})^2$ if we require an effective supersymmetry breaking scale (and hence the scale of the soft breaking terms of the supersymmetric SM, including the gravitino mass $m_{3/2}$) of the order of 1 TeV.

Although gravity mediated models have been heavily studied in the development of realistic supersymmetry breaking models, they have drawbacks. Most prominent is the problem of non-degenerate sfermion masses which lead to amplitudes of flavour-changing-neutral-current processes (FCNC, for instance $K\overline{K}$ mixing) which are not consistent with experimental bounds [125]. One can construct gravity mediated models that lead to diagonal mass matrices in flavour space which fulfill experimental bounds. However, these models are quite exotic: In supergravity, nothing forbids a Kähler potential of the general form

$$K = \frac{1}{M_{\rm Pl}^2} f(X, X^\dagger)_i^j \Phi^{\dagger i} \Phi_j \tag{5.8}$$

where $X$ is again the field of the hidden sector triggering supersymmetry breaking and $\Phi_i$ are chiral superfields of the supersymmetric SM. For instance, a $f$ of the form $f_i^j = XX^\dagger [m\delta_i^j + \Delta_i^j]$ gives contributions to the Kähler potential which lead directly to off-diagonal elements in the mass matrix $M_i^j$ which is found in the term

$$\mathcal{L} \supset \int d^4\theta \frac{XX^\dagger}{M_{\rm Pl}^2} M_i^j \Phi^{\dagger i} \Phi_j \ . \tag{5.9}$$

To avoid these contributions one needs additional constraints in the supergravity model. Per se, gravity is flavour-blind. So in order to have a mechanism similar to the GIM-mechanism [126] in the supersymmetric SM, one needs a model which highly restricts terms leading to off-diagonal contributions to the sfermion mass matrices. An approach to this has been studied in [127] where the possibility of a strongly coupled hidden sector was discussed. Another possible scenario is to separate the visible and hidden sectors by an extra dimension, e.g. by working in a five-dimensional spacetime. For early works see for instance [128]. This concludes our brief comments about supergravity and gravity mediation. For the rest of this thesis we will focus on the other key approach in mediating supersymmetry breaking effects which circumvents problems with flavour symmetry, namely gauge mediation.

An economic way to solve flavour issues is to use the gauge interactions of the SM for mediating any supersymmetry breaking from the hidden sector. This is due to the fact that the standard gauge interactions are flavour-blind and therefore, do not give rise to any off-diagonal terms in the sfermion mass matrices of the supersymmetric SM. All gauge-mediated models are based on the simple idea of having a messenger sector of chiral superfields. The messenger fields couple to the hidden sector and hence, supersymmetry breaking is transmitted to the messenger sector such that these have a susy-broken spectrum. In addition to that, the messenger fields are also charged under the SM gauge groups. Therefore, they couple to the visible sector by the usual gauge interactions and communicated the supersymmetry breaking effects to all of the supersymmetric SM. The soft breaking terms[6] (soft masses) for the superpartners of the SM fields are produced

[6]Again, 'soft' refers to the fact that these terms, although breaking supersymmetry, do not spoil the nice UV cancellations.

through loop-effects. A detailed analysis of the contributing processes leads to soft masses of the order

$$m_{\text{soft}} \sim \frac{\alpha_i}{4\pi} \frac{F_X}{M_{\text{mess}}} \tag{5.10}$$

where $\alpha_i$ is the coupling constant of the corresponding SM gauge group, $F_X$ is the F-component of a chiral superfield which triggers supersymmetry breaking and $M_{\text{mess}}$ is the mass-scale of the messenger fields. Under the condition that $\sqrt{F_X}$ and the messenger scale $M_{\text{mess}}$ are of the same order, the supersymmetry breaking scale can be of the order of $10^4$ GeV. We see that the scale of the order parameter can be much lower in gauge mediated models compared to the gravity mediated construction. This nice setup provides a rich area of model-building opportunities. Especially in the last 5 years, the field of gauge mediation was invigorated by Intriligator, Seiberg and Shih in their work [129]. The authors explored the possibility of dynamical supersymmetry breaking and gauge meditation combined with *metastable vacua*, i.e. vacua with non-vanishing energy which are not the *global* minimum of the scalar potential. This ISS construction led to a vast amount of research on different phenomenological models and will be the main ingredient for our discussion of metastable supersymmetry breaking for $\mathsf{SO}(N)$ models in Chapters 6 and 7. To build up some intuition and understanding of these ideas we will discuss some aspects of gauge mediated models in Section 5.6

In general, the field of supersymmetry breaking and its mediation is a broad and fast developing one and therefore, giving a complete overview of all important aspects is far beyond the scope of this thesis. We will rather focus on selected topics which are needed for our discussions on $\mathsf{SO}(N)$ metastable models. To that extent it is sensible to develop some background knowledge of supersymmetric field theories which we will provide in the following sections. Throughout this part of the thesis we will restrict ourselves to theories with $\mathcal{N} = 1$ rigid supersymmetry. We focus on non-extended supersymmetric theories since these provide chiral superfields which enables us to keep the right- and left-handed fermions of the SM in separate superfields. Furthermore, a restriction to global supersymmetry arises from the approach of having a supersymmetry breaking scale of low energies accessible by current and future collider experiments. In the next section we begin by considering vacua in supersymmetric theories and find a criterion for broken supersymmetry. Then we move on and discuss some aspects of supersymmetric QCD, the structure of its vacua for different flavours and some non-perturbative aspects. This leads us to a brief discussion of Seiberg duality in $\mathcal{N} = 1$ SQCD. Before finally discussing gauge mediation and some recent developments, we have a more detailed look at the ISS model which utilises the $\mathcal{N} = 1$ Seiberg duality.

Although we cannot discuss all aspects of supersymmetry breaking and its applications in gauge mediation, fortunately, a lot of excellent reviews can be found in the literature. Most of them focus also on recent developments which we could not discuss here. General discussions and information can be found in the text book by Terning [17] and of course in the evergreen of Wess and Bagger [16]. Furthermore, the reviews [110,130–133] provide detailed discussions of supersymmetry breaking and Seiberg duality whereas further information on supersymmetry breaking mediation and other recent developments can be found in [134–137]. Our introductory review in this section is mainly based on discussions provided in [17] and [131, 136, 137].

## 5.2 Preliminaries on vacua in supersymmetric theories

In the last section we understood that supersymmetry - if realised in nature - must be a broken symmetry. Furthermore, in order to keep its nice features we want to break supersymmetry spontaneously. The question is then how one can characterise such a non-supersymmetric theory. To this extent, much can be learned from simple considerations of vacua in supersymmetric theories. Here, we follow the discussions of [3].

When we work in the regime of perturbation theory we normally expand in powers of the coupling constant around a stable configuration which corresponds to a minimum of the action. In quantum field theory, such a stable configuration which is also Lorentz invariant is called a *vacuum*. The requirement of Lorentz invariance basically implies that only scalar fields are non-vanishing in the vacuum. All other fields and all spacetime derivatives of fields must vanish. Hence, a general vacuum configuration is given by

$$\langle A_\mu^A \rangle = \langle \lambda^A \rangle = \langle \psi_\alpha \rangle = \partial_\mu \langle \phi \rangle = 0 \qquad \text{for } V(\langle \phi \rangle, \langle \phi^\dagger \rangle) \text{ at its minimum,} \tag{5.11}$$

where we have denoted the vacuum expectation value (vev) of a field by $\langle \cdot \rangle$. Here, the minimum of the potential might be a local or a global one. If it is only a local minimum one has the case of a 'false vacuum' which will eventually decay into the true global minimum of the theory via tunneling processes. In a supersymmetric theory the generator algebra puts important constraints on the minima of the potential. From the anti-commutation relation of the $\mathcal{N} = 1$ algebra

$$\{q_\alpha, \bar{q}_{\dot{\beta}}\} = 2 p_\mu \sigma^\mu_{\alpha\dot{\beta}} \tag{5.12}$$

we find for any state $|\phi\rangle$

$$2\sigma^\mu_{\alpha\dot{\beta}} \langle \phi | p_\mu | \phi \rangle = \langle \phi | q_\alpha \bar{q}_{\dot{\beta}} + \bar{q}_{\dot{\beta}} q_\alpha | \phi \rangle \tag{5.13}$$

If we multiply now by $(\bar{\sigma}^0)^{\dot{\beta}\alpha}$ and use $\bar{q}_{\dot{\alpha}} = (q_\alpha)^\dagger$ we get

$$q_1 q_1^\dagger + q_1^\dagger q_1 + q_2 q_2^\dagger + q_2^\dagger q_2 = 4\eta^{0\mu} p_\mu = 4p^0 = 4H \geq 0 \ , \tag{5.14}$$

due to positivity of the Hilbert space. Here, $H$ is the Hamiltonian and we immediately see that it is bounded from below, i.e. $\langle \phi | H | \phi \rangle \geq 0$ for any state $|p\rangle$. In a supersymmetric theory, a vacuum state $|\Omega\rangle$ obeys $S(0, \xi, \bar{\xi})|\Omega\rangle = 0$ and hence, $q_\alpha |\Omega\rangle = (q_\alpha)^\dagger |\Omega\rangle = 0$. This implies that the scalar potential vanishes in the vacuum. For a theory which is supersymmetric at the Lagrangian level but has $q_\alpha |\Omega\rangle \neq 0$, the vacuum is not invariant under supersymmetry. Hence, supersymmetry is spontaneously broken. However, due to the relation (5.13) we have the important restriction that the energy in a supersymmetric theory is always positive, $\langle \Omega | H | \Omega \rangle \geq 0$.

Let us now turn specifically to the vacuum of such a supersymmetric theory. In general, we obtain a vacuum state (true or false) by considering

$$\frac{V}{\partial \phi_i}(\langle \phi_j \rangle, \langle \phi_j^\dagger \rangle) = \frac{V}{\partial \phi_i^\dagger}(\langle \phi_j \rangle, \langle \phi_j^\dagger \rangle) = 0 \ . \tag{5.15}$$

As we have seen in Section 2.5.2, in a supersymmetric theory the scalar potential is given by[7]

$$V(\phi_i, \phi_j^\dagger) = F_i^\dagger F_i + \frac{1}{2}(D^A)^2 \equiv \sum_i \left\{ \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{1}{2} \sum_a \left( g(\phi_i^\dagger T^A \phi_i) \right)^2 \right\} \tag{5.16}$$

Therefore, for a global supersymmetric minimum $V = 0$ of the scalar potential we have to find solutions for the equations[8]

$$F_i(\langle \phi_i^\dagger \rangle) = 0 \ , \qquad\qquad D^A(\langle \phi_i \rangle, \langle \phi_j^\dagger \rangle) = 0 \tag{5.17}$$

which are usually denoted as F-term and D-term equations[9]. One has to solve this system of equations simultaneously in order to obtain a vanishing scalar potential. If the system has a solution, it automatically defines the global minimum of the theory since the potential is always positive. This leads to a stable vacuum state. In principle, there can be many solutions to the F- and D-flatness conditions such that we end up with many (degenerate) supersymmetric vacua.

However, not always do solutions to the relations (5.17) need to exist. Generally, we have as many F-term equations $F_i = 0$ as we have unknown vevs $\langle \phi_i \rangle$ and the same holds for the complex conjugate equation $F_i^\dagger = 0$. In addition, we need to satisfy as many D-term equations $D^A = 0$ as the dimension of the gauge group. If there is no solutions to the F- and D-terms but the condition (5.15) is fulfilled we have a ground state with a strictly positive energy, $V_0 > 0$. In such a case, the vacuum is not invariant under supersymmetry transformations and hence, supersymmetry is broken in any perturbative field theory around this ground state.

We conclude that broken supersymmetry requires some $F_i(\langle \phi^\dagger \rangle) \neq 0$ or some $D^A(\langle \phi \rangle, \langle \phi^\dagger \rangle) \neq 0$. This can also be seen at the level of the supersymmetry transformations of the component fields of a chiral superfield $\Phi$. Recalling the transformations (2.40) we find

$$\delta \langle \phi_i \rangle \stackrel{!}{=} 0 \ , \qquad \delta \langle \psi_i \rangle = -\sqrt{2} \langle F_i \rangle \xi \stackrel{!}{=} 0 \ , \qquad \delta \langle F_i \rangle \stackrel{!}{=} 0 \tag{5.18}$$

since only scalar fields can acquire a vev. This is only consistent for $\langle F_i \rangle = 0$. Similarly, from the component transformations of a vector superfield we deduce that for $\delta \langle \lambda^A \rangle \stackrel{!}{=} 0$ one needs $\langle D^A \rangle = 0$.

In a supersymmetric theory, one has different possibilities of how a ground state with vanishing energy can be realised. The theory might have a single point in field space where the energy is zero. It is also possible that the scalar potential has several isolated minima where the theory is supersymmetric. However, supersymmetric theories often have a whole range of vacua, i.e. the theory has directions in field space which are both F- and D-flat. These flat directions are usually denoted as the *moduli space* of the (classical) theory. The field fluctuations along the flat directions are massless fields and are called moduli. For unbroken supersymmetry, the moduli stay massless to all orders in perturbation theory. However, non-perturbative effects can generate contributions to the scalar potential with non-zero energy and hence, the moduli space *can* be lifted.

[7] Here, we write out the derivatives with respect to the scalar components of the chiral superfields explicitly. In general, we will write $\partial W / \partial \Phi_i$ for an F-term. It is then understood to take only the scalar component of the result.

[8] Here and in the following we always assume that the Kähler potential is a regular function of the superfields and does not have any singularities.

[9] In the literature, the notion of F-flatness and D-flatness are used equivalently.

Finally, a theory for which supersymmetry is broken at tree-level can also have a (classical) moduli-space of degenerate, non-supersymmetric vacua. Since supersymmetry is broken, the potential is not protected in perturbation theory and hence, quantum corrections typically lift the classical degeneracy of the non-supersymmetric vacua. In these cases, we speak of a *pseudomoduli space of vacua*. In the following chapters we will see several examples of this behaviour.

In principle, we have to solve the D- and F-terms in order to find the moduli space of a theory. If we imagine to set all superpotential couplings to zero classically, then the F-terms vanish automatically and the moduli space is given by the D-flat directions. Usually, for small tree-level couplings, the vacua resulting from solving both D- and F-terms will be close to the D-flat directions. Therefore, it is convenient to solve for the D-terms first and then analyse and solve the F-terms along the D-flat directions, for details see the discussions in [138]. The F-flat conditions can lift the D-flat directions, and typically this happens for large vevs of the fields. This is because the F-terms usually grow with the vevs and vanish at the origin of field space since the tree-level superpotential is a polynomial in the chiral superfields. For further discussion on theories with broken or unbroken supersymmetry and their moduli space we refer the reader to the nice review of [133].

Finally, we briefly mention a theorem that will be useful in our discussions of the moduli spaces. It was pointed out by Luty and Taylor that the moduli space of a classical gauge theory (with vanishing superpotential) can always be parameterised by independent, gauge-invariant composite operators [139] (for further details see also the discussions in [110]). By the means of this theorem, it is not necessary to solve the D-flat conditions in order to find the moduli space. We can rather build all possible composite gauge-invariant operators but have to consider all classical relations between them in order to achieve a matching of the degrees of freedom when comparing to the case of solving the D-flat conditions.

## 5.3 Supersymmetry breaking and global symmetries

The concept of global symmetries restricts the behaviour of physical quantities in a theory under the corresponding symmetry transformations. Especially in the study of non-perturbative effects in supersymmetric gauge theories, global symmetries place important constraints on the form of the non-perturbative contributions. Besides the usual global $\mathsf{SU}(N)$ and $\mathsf{U}(1)$ symmetries, supersymmetry possesses an additional class of symmetries: one has symmetry generators $R$ which do not commute with the supersymmetry generators $q_\alpha$ and $\bar{q}_{\dot{\alpha}}$ as

$$[R, q_\alpha] = q_\alpha \ , \qquad [R, \bar{q}_{\dot{\alpha}}] = -\bar{q}_{\dot{\alpha}} \ . \tag{5.19}$$

The corresponding symmetry group is called $\mathsf{U}(1)_R$. Following [16] we assign a R-symmetry transformation to the fermionic superspace coordinate $\theta_\alpha$ and $\bar{\theta}_{\dot{\alpha}}$ as

$$\theta \to e^{i\alpha}\theta \ , \qquad \bar{\theta} \to e^{-i\alpha}\bar{\theta} \ . \tag{5.20}$$

Hence, a general chiral superfield with R-charge $r$ transforms under the $\mathsf{U}(1)_R$ as

$$\Phi(x, \theta, \bar{\theta}) \to e^{-ir\alpha}\Phi(x, e^{i\alpha}\theta, e^{-i\alpha}\bar{\theta}) \tag{5.21}$$

which yields for the R-charges of the component fields $R[\phi] = r, R[\psi] = r - 1$ and $R[F] = r - 2$. Invariance of the theory under the R-symmetry requires the superpotential to have R-charge $R[W] = 2$. Similarly, for the field-strength superfield $\mathbb{W}_\alpha$ we have a supercharge $R[\mathbb{W}_\alpha] = 2$ which fixes the gaugino supercharge to be $R[\lambda] = 1$. With these requirements, the $\mathsf{U}(1)_R$ is a classical symmetry of the action. Quantum mechanically though, the symmetry is be broken. However, there is an anomaly-free R-symmetry, compare with our discussion in Section 5.4.1.

The question of broken or unbroken supersymmetry is strongly tied to the existence of a R-symmetry in the theory. The connection between supersymmetry breaking and R-symmetries was studied by Seiberg and Nelson [140]. They considered supersymmetric theories with generic superpotentials which are described by a low-energy theory with just chiral superfields and no gauge fields. By generic we usually mean that the superpotential contains all by the symmetries of the theory allowed terms. The important assumption here is that one has a theory of chiral superfields only such that the superpotential does not receive any non-perturbative corrections as in supersymmetric gauge theories. In the following we review the nice discussion in [141].

We consider a theory with a generic superpotential with $n$ chiral superfields $\Phi_i$ and a canonical Kähler potential. Hence, the condition for unbroken supersymmetry is[10]

$$\frac{\partial W}{\partial \Phi_i}(\Phi_1, \ldots, \Phi_n) = 0 \qquad \forall i \ . \tag{5.22}$$

Without any global symmetries putting restrictions on the superpotential, this is a system of $n$ complex equations for $n$ complex variables. Hence, a solution exists for a generic superpotential and supersymmetry is not broken.

We move on and consider global symmetries of the theory. The superpotential should be invariant under the global symmetry where we assign the superfields a charge $Q[\Phi_i] = q_i$. For simplicity, we assume a global $\mathsf{U}(1)$ symmetry. If the vacuum state does not break the global symmetry spontaneously, all charged fields must vanish in that vacuum,

$$\langle \Phi_i \rangle = 0 \qquad \text{for } q_i \neq 0 \ . \tag{5.23}$$

Suppose $k$ of the $n$ fields carry a non-zero charge under the $\mathsf{U}(1)$ symmetry. This gives $k$ constraints on the fields. Then, the conditions (5.22), restricted to the subspace of possible solutions under the vanishing vevs for $k$ fields, give $n - k$ constraints for the remaining $n - k$ unknowns. Thus, for a generic superpotential, supersymmetry is still unbroken.

What happens if the global $\mathsf{U}(1)$ symmetry is broken spontaneously? At least one of the charged fields will have a non-zero vev. Suppose that this field is $\Phi_1$ with $q_1 \neq 0$. This theory should still be invariant under the symmetry and the superpotential can be expressed by

$$W(\Phi_1, \ldots, \Phi_n) = f\left(\Phi_2 \Phi_1^{-q_2/q_1}, \ldots, \Phi_n \Phi_1^{-q_n/q_1}\right) \ . \tag{5.24}$$

Here, we have written the superpotential as a function of variables that are not charged under the global symmetry. Under the condition (5.22) this gives a system of $n - 1$ equations for $n - 1$

[10] As usual, we consider the scalar component of the chiral superfields only and write for the vev of the fields the corresponding field itself, i.e. $\langle \phi_i \rangle = \phi_i$.

unknowns (the vevs of the other $\Phi_i$). Again, a solution generically exists and supersymmetry is unbroken.

Things are different in the case of a global $\mathsf{U}(1)_R$ symmetry. The important difference is that the superpotential is *not* invariant under this symmetry but rather carries an R-charge of 2. If the R-symmetry is not broken spontaneously (i.e. all charged fields vanish in the vacuum), our previous analysis goes through accordingly and supersymmetry is not broken. However, this changes if the $\mathsf{U}(1)_R$ symmetry is broken spontaneously. Denoting the R-charges of the fields by $R[\Phi_i] = r_i$ we have to modify the relation (5.24) as

$$W(\Phi_1, \dots, \Phi_n) = \Phi_1^{2/r_1} f\left(\Phi_2 \Phi_1^{-q_2/q_1}, \dots, \Phi_n \Phi_1^{-q_n/q_1}\right) \tag{5.25}$$

in order to maintain $R[W] = 2$. This form of the superpotential yields for the condition of unbroken supersymmetry $\partial_i W = 0$

$$\partial_{i\neq 1}\, f\left(\Phi_2 \Phi_1^{-q_2/q_1}, \dots, \Phi_n \Phi_1^{-q_n/q_1}\right) = 0 \ , \tag{5.26}$$

$$f\left(\Phi_2 \Phi_1^{-q_2/q_1}, \dots, \Phi_n \Phi_1^{-q_n/q_1}\right) = 0 \ , \tag{5.27}$$

where the second relation follows from $\partial_1 W = 0$ and we have used the abbreviation $\partial_i \equiv \partial/\partial\Phi_i$. The first relation leads again to a system of $n-1$ equations for $n-1$ unknowns. However, the relation (5.27) gives an additional constraint, yielding an over-constrained system and hence, generically no solution exists. Since the condition (5.22) is not fulfilled we have a theory with broken supersymmetry.

This yields the main-result of the Nelson-Seiberg theorem: For a supersymmetric theory described by a generic effective Lagrangian, the existence of a R-symmetry is a necessary condition for a supersymmetry breaking vacuum. Furthermore, a spontaneously broken R-symmetry is a sufficient condition for a supersymmetry breaking minimum. In addition, it was shown quite recently that spontaneous R-symmetry breaking in so-called O'Raifeartaigh-type models[11] require fields of a R-charge different from 0 or 2 [143][12]. In all these considerations, the important constraint is a generic superpotential of the theory. Again, generic usually means that the superpotential contains all terms which are allowed by the symmetries of the theory. More specifically, one requires that for a tree-level potential as a polynomial of some degree $n$, no term of degree equal or less than $n$ that is compatible with the global symmetries of the theory is omitted from the superpotential. In general, this is only true for the classical superpotential in perturbation theory since non-perturbative corrections are often not generic, see for instance discussions in [138] and [133].

Unfortunately, for a generic theory one runs into problems when we want to construct realistic models of broken supersymmetry. Since the gaugino carries R-charge $R[\lambda] = 1$, a R-symmetry in the theory forbids a gaugino mass term. And if we spontaneously break the R-symmetry, we have a

[11] The basic O'Raifeartaigh model [142] and its generalisation are generic examples of F-term supersymmetry breaking, i.e. for models where the F-component of a chiral superfield acquires a VEV which breaks supersymmetry. The model is defined by a set of chiral superfields $\Phi$ a superpotential which is a polynomial in the superfields up to degree 3 and a canonical Kähler potential.

[12] This does not hold for models with gauge interactions.

massless Goldstone boson in the spectrum. Hence, we either have massless gauginos with preserved R-symmetry or a massless R-Goldstone boson and spontaneously broken R-symmetry. In both cases we have an additional massless particle in the theory which has not been observed. So in order to have massive gauginos, we should break the R-symmetry. However, we don't want to break it spontaneously in order to avoid the massless R-Goldstone boson and hence, we need to break the R-symmetry explicitly. Following the discussion of the Nelson-Seiberg theorem, the explicit R-symmetry breaking introduces supersymmetry vacua in the theory. Following this approach one has to *accept the inevitable*: If we want to construct generic models of broken supersymmetry, then metastable supersymmetry breaking states cannot be avoided, any 'resistance is futile'[13]. In that sense, metastable supersymmetry breaking is inevitable[14] [145].

Strictly speaking, this is only true when we exclude gravitational effects. In the case of a spontaneously broken R-symmetry it was shown that supergravity contributions give the R-Goldstone boson a mass due to a constant term in the superpotential which is needed to set the cosmological constant to (nearly) zero [149]. It then depends on the various scales in the supergravity theory if the R-Goldstone mass is compatible with experimental bounds. For the purposes of this thesis we do not consider gravity effects for breaking the R-symmetry and focus purely on the discussed field theory reasoning. However, even including supergravity contributions, there are good reasons for taking the concept of metastable supersymmetry breaking seriously. This is directly linked to the observation that in many direct mediation models one has to deal with anomalously small gaugino masses. Generating large enough gaugino masses is directly linked to the vacuum structure of the theory. Gaugino masses vanish in a *stable* supersymmetry breaking vacuum at leading order in an expansion in the supersymmetry breaking order parameter if there is no unstable point anywhere in the pseudo-moduli space. Therefore, one needs to construct a theory of metastable supersymmetry breaking vacua such that an unstable point in the pseudo-moduli space is allowed. For further details see Section 5.6 and also the nice summary in [150]. We will also discuss possible techniques for R-symmetry breaking in the case of $\mathsf{SO}(N)$ metastable ISS-like theories in Chapter 6.

## 5.4 Some results in supersymmetric QCD

Before we discuss some specific gauge mediation models and the ISS construction of metastable supersymmetry breaking, we need to review some basic results in supersymmetric QCD theories and non-perturbative gauge dynamics. Our discussions loosely follow the reviews [133,137] and the corresponding chapters in [17].

[13] See [144] - the author could not resist.

[14] Metastable supersymmetry breaking vacua are also possible for spontaneously broken R-symmetry. As mentioned ealier, one can assign generic R-charges to the chiral superfields [143]. Furthermore, one can couple the effective theory to some broken gauge symmetry [145,146]. Or one can have the case in which the pseudo-moduli are only lifted by two-loop effects, leading to an R-symmetry breaking [147,148]. We do not consider these possibilities here.

### 5.4.1 Supersymmetric gauge theories

By super-QCD or SQCD we normally mean a supersymmetric gauge theory with gauge group $\mathsf{SU}(N)$[15] and $F$ flavours. In the case of $F = 0$ we have a supersymmetric Yang-Mills (SYM) theory. The particle content is given by 'quark' chiral superfields $Q_{ai}$ in the fundamental representation and 'anti-quark' chiral superfields $\tilde{Q}^{ai}$ in the anti-fundamental representation of the gauge group with a flavour index $i = 1, \ldots, F$ and a gauge index $m = 1, \ldots, N$ in the fundamental or anti-fundamental representation. The theory has a $\mathsf{SU}(F) \times \mathsf{SU}(F) \times \mathsf{U}(1)_B \times \mathsf{U}(1)_R$ global symmetry group. We summarise the quantum numbers of the chiral superfields in the following table:

| Field | $\mathsf{SU}(N)$ | $\mathsf{SU}(F)$ | $\mathsf{SU}(F)$ | $\mathsf{U}(1)_B$ | $\mathsf{U}(1)_R$ |
|---|---|---|---|---|---|
| $Q$ | $\square$ | $\square$ | $\mathbf{1}$ | 1 | $\frac{F-N}{F}$ |
| $\tilde{Q}$ | $\overline{\square}$ | $\mathbf{1}$ | $\overline{\square}$ | -1 | $\frac{F-N}{F}$ |

Here, $\square$ and $\overline{\square}$ denote the fundamental and anti-fundamental representation, respectively. The charges of the R-symmetry have been chosen such that the symmetry is non-anomalous. As usual, the R-charge of the chiral superfield $Q$ is given by the charge of its scalar component $\phi$ and the R-charge of the fermion $\psi$ is $R[\psi] = R[\phi] - 1$ with $Q = \phi + \theta\psi + \theta^2 F$. The charges under the other symmetries are the same for all component fields of $Q$ and $\tilde{Q}$.

For a general gauge theory with matter content in some representations, the $\beta$-function of the gauge coupling at one-loop is given by [151–153]

$$\beta_g = \mu\frac{dg}{d\mu} = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C(\mathbf{Ad}) - \frac{2}{3}\sum_f C(\mathbf{r}) - \frac{1}{3}\sum_s C(\mathbf{r})\right) \equiv -\frac{g^3 b}{16\pi^2} \tag{5.28}$$

where the sums run over all fermions $f$ and all scalars $s$ of the corresponding theory. The fermions are in Weyl multiplets and the scalars are complex. For the $\mathcal{N} = 1$ SQCD theory with $2F$ fermions and scalars in the fundamental or anti-fundamental and a gluino we have

$$b = 3N - F \ . \tag{5.29}$$

Solving for the running gauge coupling yields for the coupling at an energy scale $\mu$

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2}\log\left(\frac{|\Lambda|}{\mu}\right) \tag{5.30}$$

where $|\Lambda|$ is the intrinsic scale of the SQCD theory. Within a supersymmetric theory, the gauge coupling can be conveniently combined with the $\Theta_{\mathrm{YM}}$-angle of the theory which represents non-perturbative contributions to the action. Recalling the Lagrangian of pure SYM in Section 2.5.2,

$$\mathcal{L} = \frac{1}{4g^2}\int d^2\theta\ \mathbb{W}^{A\alpha}\mathbb{W}^A_\alpha + \text{h.c.} \tag{5.31}$$

we can incorporate non-perturbative effects by defining the *holomorphic gauge coupling*[16]

$$\tau = \frac{4\pi i}{g^2} + \frac{\Theta_{\mathrm{YM}}}{2\pi} \ . \tag{5.32}$$

[15] Other gauge groups are possible and conventionally these theories are also denoted as SQCD ones.

[16] Unfortunately, different conventions are used in the literature, mainly differing by a factor of $2\pi$. We follow the conventions of [17].

Then, the SYM Lagrangian takes the form

$$\mathcal{L} = \frac{1}{16\pi i}\int d^2\theta\ \tau \mathbb{W}^{A\alpha}\mathbb{W}^A_\alpha + \text{h.c.} \tag{5.33}$$

As usual, one obtains the canonically normalised component expansion by rescaling the component fields (compare with expression (2.79)). Then, the one-loop running expression of the holomorphic coupling $\tau$ is given by

$$\begin{aligned}\tau_{\text{1-loop}} &= \frac{4\pi i}{g^2(\mu)} + \frac{\Theta_{\text{YM}}}{2\pi} \\ &= \frac{1}{2\pi i}\left[b\log\left(\frac{|\Lambda|}{\mu}\right) + i\Theta_{\text{YM}}\right] = \frac{1}{2\pi i}\log\left[\left(\frac{|\Lambda|}{\mu}\right)^b e^{i\Theta_{\text{YM}}}\right] .\end{aligned} \tag{5.34}$$

From this expression we can define the *holomorphic dynamical scale* $\Lambda$ of the theory,

$$\Lambda = |\Lambda| e^{i\Theta_{\text{YM}}/b} = \mu e^{2\pi i\tau/b} \qquad \longrightarrow \qquad \tau_{\text{1-loop}} = \frac{b}{2\pi i}\log\left(\frac{\Lambda}{\mu}\right) . \tag{5.35}$$

Beyond the one-loop order, the holomorphic gauge coupling receives only contributions from non-perturbative instanton effects. In perturbation theory, there is no additional running beyond the one-loop level. For details see for instance [17]. By construction, we can treat $\Lambda$ as a complex parameter with the $\Theta$-angle as its phase. Furthermore, from (5.35) we see that

$$e^{-8\pi^2/g^2 + i\Theta_{\text{YM}}} = \left(\frac{\Lambda}{\mu}\right)^b \sim \Lambda^b \tag{5.36}$$

and hence, one-instanton effects are weighted by $\Lambda^b$ since $S_{\text{inst}} = -8\pi^2/g^2$. The fact that $\Lambda$ is a holomorphic quantity is important since we can apply the usual 'Seiberg holomorphy' construction like assigning charges to holomorphic quantities.

### 5.4.2 Classical moduli space

After introducing SQCD, let us move on and explore the classical moduli space of the theory for different ranges of flavours. As mentioned earlier, it is a useful strategy to solve for the D-terms first. Therefore, we will ignore any superpotential for the moment. We remind ourselves that in the case of $W = 0$ the moduli space arises from solving the D-term constraints only or from constructing independent, gauge-invariant, holomorphic polynomials [139].

**The case $F < N$:** We start with a $\mathsf{SU}(N)$ SQCD theory with $F < N$ flavours. Revelant for the scalar potential are the $F$ 'squarks' $\phi_{ai}$ in the fundamental and $F$ 'anti-squarks' $\tilde{\phi}^{ai}$ in the anti-fundamental representation. Let us first consider the possible gauge-invariant object we can construct with these fields. Since $F < N$, the only possibility is to build 'mesons' $M_i^j$ where we contract on the gauge indices,

$$M_i^j = \tilde{\phi}^{ja}\phi_{ai} . \tag{5.37}$$

Following the Luty-Taylor theorem, the $F \times F$ meson field $M_i^j$ has (massless) $F^2$ degrees of freedom which parameterise the $F^2$-dimensional moduli space.

This result can also be obtained by solving the D-flat conditions explicitly. We briefly want to highlight the main steps. The D-term is given by

$$D^A = \sum_i \phi_i^\dagger T^A \phi_i + \tilde{\phi}^\dagger \tilde{T}^A \tilde{\phi}_i = \sum_i \left[ (\phi^\dagger)^{ia} \phi_{bi} - \tilde{\phi}^{ia} (\tilde{\phi}^\dagger)_{bi} \right] (T^A)_a^b \stackrel{!}{=} 0 \tag{5.38}$$

where $a, b = 1, \ldots, N$ and $i = 1, \ldots, F$ and the second identity follows from rewriting the anti-fundamental generators $\tilde{T}^A$ in terms of fundamental ones, $T^A_{\bar{\Box}} = -(T^A_{\Box})^*$. Defining a matrix $N^a_b = \sum_i \left[ (\phi^\dagger)^{ia} \phi_{bi} - \tilde{\phi}^{ia} (\tilde{\phi}^\dagger)_{bi} \right]$, the D-flat conditions yield

$$N^a_b = \alpha \delta^a_b + \sum_A \beta_A (T^A)^a_b \stackrel{!}{=} 0 \tag{5.39}$$

The $\mathsf{SU}(N)$ generators are traceless, hence $\beta_A = 0$ and we have a solution to the D-flat condition, $N^a_b = c_0 \delta^a_b$. Using appropriate $\mathsf{SU}(N)$ and $\mathsf{SU}(F)$ rotations we can bring the $N \times F$ matrices $\phi_{ai}$ into a form where the first $F \times F$ block is diagonal with elements $v_1, \ldots, v_F$ and the remaining $(N - F) \times F$ block has only zero entries. It follows that $(\phi^\dagger)^{ia} \phi_{bi}$ must be $N \times N$ diagonal matrix with $(\phi^\dagger)^{ia} \phi_{bi} = \mathrm{diag}(|v_1|^2, \ldots, |v_F|^2, 0, \ldots, 0)$. Imposing the D-flat condition with $c_A = 0$ yields that $\tilde{\phi}^{ia} (\tilde{\phi}^\dagger)_{bi} = \mathrm{diag}(|\tilde{v}_1|^2, \ldots, |\tilde{v}_F|^2, 0, \ldots, 0)$, again with $N - F$ zero entries. Since $N^a_b = c_0 \delta^a_b$, this can only be true for $c_0 = 0$ and we have $|v_i|^2 = |\tilde{v}_i|^2$.

Having $F$ vevs for the squarks, the gauge symmetry is spontaneously broken from $\mathsf{SU}(N) \to \mathsf{SU}(N - F)$. This is the supersymmetric version of the well-known Higgs mechanism. Normally, a massless vector boson 'eats' a massless Goldstone boson and becomes massive. In a supersymmetric theory, and under the condition that the scalar vevs do not break supersymmetry, one has to have a similar mechanism to ensure mass degeneracy of the bosonic and fermionic degrees of freedom. In the super Higgs mechanism a massless vector supermultiplet becomes massive by eating the components of a massless chiral supermultiplet [154].

Due to the spontaneous breaking through $F$ vevs we end up with some broken generators,

$$[N^2 - 1] - [(N - F)^2 - 1] = 2NF - F^2 \ . \tag{5.40}$$

By the super Higgs mechanism, $2NF - F^2$ of the total $2NF$ chiral superfields are eaten to form massive vectormultiplets. Hence, some chiral superfields are left over, we have

$$[2NF] - [2NF - F^2] = F^2 \tag{5.41}$$

massless chiral superfields in the theory. Their scalar components parameterise the moduli space of the SQCD theory. These $F^2$ light degrees of freedom are exactly the ones described earlier by the $F \times F$ meson field. Hence, both description lead to the same description of the classical moduli space.

**The case $F \geq N$:** As before we can solve the D-flat conditions or construct all independent, gauge invariant, holomorphic polynomials. The D-terms

$$D^A = \sum_i \left[ (\phi^\dagger)^{ia} \phi_{bi} - \tilde{\phi}^{ia} (\tilde{\phi}^\dagger)_{bi} \right] (T^A)_a^b \stackrel{!}{=} 0 \tag{5.42}$$

can be solved similarly to the previous case. Since $F \geq N$, the matrices $\phi_{ai}$ have more columns than rows. By appropriate gauge and flavour rotations the first $N \times N$ block can be brought to a diagonal form with entries $v_1, \dots v_N$ and the remaining $N \times (F-N)$ block has only zero entries. Hence, the matrix $(\phi^\dagger)^{ia}\phi_{bi}$ is diagonal and of full rank with $(\phi^\dagger)^{ia}\phi_{bi} = \mathrm{diag}(|v_1|^2, \dots, |v_N|^2)$. Now, the D-flat conditions

$$(\phi^\dagger)^{ia}\phi_{bi} - \tilde{\phi}^{ia}(\tilde{\phi}^\dagger)_{bi} = \rho\delta^a_b \tag{5.43}$$

for some constant $\rho$, yield that $\tilde{\phi}^{ia}(\tilde{\phi}^\dagger)_{bi}$ is also a diagonal matrix of full rank with $\tilde{\phi}^{ia}(\tilde{\phi}^\dagger)_{bi} = \mathrm{diag}(|\tilde{v}_1|^2, \dots, |\tilde{v}_F|^2)$. Hence, we have

$$|v_i|^2 = |\tilde{v}_i|^2 + \rho \tag{5.44}$$

and we can maximally 'higgs' $N$ different scalars. At a generic point of the moduli space, the $\mathsf{SU}(N)$ gauge symmetry is completely broken. As before, the concept of the super Higgs mechanism yields the number of light degrees of freedom which are left-over after the spontaneous symmetry breaking. We have $N^2 - 1$ broken generators and a total of $2NF$ chiral superfields in the theory. Therefore, a total of $2NF - [N^2 - 1]$ massless chiral superfields are left over which parameterise the moduli space for $F \geq N$.

This can also be seen from constructing all possible independent, gauge invariant operators. As before, we can build scalar 'meson' fields but since $F \geq N$ we can also construct 'baryons',

$$M^j_i = \tilde{\phi}^{ja}\phi_{ai} \tag{5.45}$$

$$B^{i_{N+1},\dots,i_F} = \phi_{a_1 i_1} \dots \phi_{a_N i_N}\ \epsilon^{a_1 \dots a_N}\epsilon^{i_1 \dots i_N i_{N+1} \dots i_F} \tag{5.46}$$

$$\tilde{B}_{i_{N+1},\dots,i_F} = \tilde{\phi}^{a_1 i_1} \dots \tilde{\phi}^{a_N i_N}\ \epsilon_{a_1 \dots a_N}\epsilon_{i_1 \dots i_N i_{N+1} i_F}\ . \tag{5.47}$$

By the virtue of [139], the description of the moduli space in terms of these gauge-invariant polynomials should match the result of the previous discussion. We have $F^2$ degrees of freedom from $M$ and each $B$ and $\tilde{B}$ has $\binom{F}{N}$ components. This gives $F^2 + 2\binom{F}{N} > 2NF - [N^2 - 1]$. Hence, we have over counted the light degrees of freedom by not considering classical constraints between the composite operators. For instance, multiplying two Baryons yields the relation

$$B^{j_{N+1} \dots j_F}\tilde{B}_{i_{N+1} \dots i_F} = \epsilon_{i_1,\dots,i_N,\dots,i_F}\ \epsilon^{j_1,\dots,j_N,\dots,j_F}\left(M^{j_1}_{i_1} \dots M^{j_N}_{i_N}\right)\ . \tag{5.48}$$

Furthermore, the product of a Baryon and the Meson field gives zero: Anything antisymmetrised with more than $N$ colour indices $a_i$ vanishes.

We end this section by briefly looking at a specific example. Let us consider the case of $F = N$. We have $N + 2\binom{N}{N} = N^2 + 2$ possible gauge-invariant polynomials. Breaking the $\mathsf{SU}(N)$ gauge symmetry completely leaves $N^2+1$ massless chiral superfields. Since all baryons are flavour singlets, the classical constraint (5.48) yields

$$B\tilde{B} = \mathrm{det}M \tag{5.49}$$

which is the needed relation between the gauge-invariant polynomials.

### 5.4.3 Pure super-Yang-Mills, $F = 0$

Although not the focus of this thesis, we briefly summarise the dynamics of a pure Yang-Mills theory with $\mathcal{N} = 1$ supersymmetry. Let us start by considering the global symmetries of the theory. Here, the $\mathsf{U}(1)_R$ symmetry, under which $\lambda \to e^{i\alpha}\lambda$, is anomalous since we do not have any flavours in the theory to build a non-anomalous R-symmetry. An explicit calculation of the triangle diagram with the gaugino (the only fermion of the theory) running in the loop shows that the anomaly coefficient is indeed non-zero. However, a $\mathbb{Z}_{2N}$ subgroup of the $\mathsf{U}(1)_R$ is left unbroken in the quantum theory.

The pure gauge theory is characterised by a dynamical scale $\Lambda$ at which the theory is believed to confine. The initial massless particles form condensates and the theory develops a mass gap. This is similar to the case of usual QCD: the strongly interacting fermions of the theory (quarks) undergo pair condensation and combine into massive colour-singlet bound states. For pure SYM, the gaugino condensate is given by

$$\langle \lambda^A \lambda^A \rangle = a\Lambda^{b/N} = a\Lambda^3 \tag{5.50}$$

where $a$ is a constant which can be calculated, see the discussions in [155–157]. The formation of this condensate spontaneously breaks the discrete $\mathbb{Z}_{2N}$ symmetry since under the symmetry

$$\langle \lambda^A \lambda^A \rangle \longrightarrow e^{2i\alpha} \langle \lambda^A \lambda^A \rangle \tag{5.51}$$

which is only invariant for $\alpha = k\pi/N$ with $k = 0$ or equivalently $k = N$. Only a discrete $\mathbb{Z}_2$ symmetry is left over. The pure SYM theory has $N$ degenerate but distinct vacua.

### 5.4.4 The ADS superpotential, $F < N$

The classical moduli space of the theory with flavours $F < N$ was discussed in Section 5.4.2. In a next step we want to write down the effective superpotential representing the low energy dynamics. It is described by the gauge-invariant chiral superfields of the theory. The generated superpotential should respect all global symmetries. These are the non-anomalous ones discussed in Section 5.4.1 and additionally an anomalous $\mathsf{U}(1)_A$ symmetry. We can construct the effective superpotential out of the chiral superfields $\mathbb{W}^A$ and $M$ as well as the holomorphic scale $\Lambda$. Here, $\Lambda$ transforms under the anomalous abelian global symmetries due to the transformation of $\Theta_{\text{YM}}$. A detailed analysis of the invariance under all global symmetries yields the Affleck-Dine-Seiberg (ADS) superpotential

$$W_{\text{ADS}} = C_{N,F} \left( \frac{\Lambda^{3N-F}}{\det M} \right)^{1/(N-F)} . \tag{5.52}$$

This superpotential is generated by non-perturbative effects as one can see from the positive power of $\Lambda$. It was first discussed in [158] and studied further in [159].

One can perform various consistency checks on the ADS superpotential by deforming the original SQCD theory. The idea is here to perturb the UV limit of the theory and check for the consequence of the perturbation on the low energy theory. This can be done by 'higgsing' a squark field and adding a mass term for specific flavours such that the low energy effective theory is described by a different $N'$ and/or $F'$. For instance, by giving a vev $v$ to one squark we end up with a low energy higgsed theory with gauge group $\mathsf{SU}(N-1)$ with $(F-1)$ flavours. In the spirit of the Wilsonian renormalisation flow we can match the two theories at the scale $v$ and obtain

$$(\Lambda_{N,F})^{3N-F} = v^2(\Lambda_{N-1,F-1})^{3(N-1)-(F-1)} \ . \tag{5.53}$$

A similar approach can be used to match the scales after introducing a mass term for a single flavour into the theory. The outcome of these consistency checks is a relation between the coefficients $C_{N,F}$ and $C_{N',F'}$. To obtain an expression for the coefficient one needs to know the explicit value of $C_{N,F}$ for any particular pair of $(N,F)$. This can be achieved by considering the case of $F = N-1$ where the gauge group is completely higgsed. Here, one has $W_{\text{ADS}} \sim \Lambda^b$ such that the superpotential can be generated by instantons. A detailed analysis of this case yields $C_{N,N-1} = 1$. This theory is special since for $F < N-1$, and after introducing $F$ vevs for the squarks, the gauge group is never completely broken. Hence, the resulting SYM theory is asymptotically free and the ADS superpotential is generated by gaugino condensation. The upshot of all these consistency limits of the ADS superpotential is a result for the coefficient $C_{N,F}$. One finds that the superpotential is given by

$$W_{\text{ADS}} = (F-N)\left(\frac{\Lambda^{3N-F}}{\det\, M}\right)^{1/(N-F)} \ . \tag{5.54}$$

For further information and detailed derivations of the above results we refer the reader to the literature, for instance to the corresponding chapters in [17] or [130].

Finally, let us briefly discuss which vacuum structure the ADS superpotential induces. The scalar potential for $F < N$ is given by

$$V_{\text{ADS}} = \left|\frac{\partial W_{\text{ADS}}}{\partial M}\right|^2 \sim |M|^{\frac{-2N}{N-F}} \ . \tag{5.55}$$

Hence, the potential is minimised for $\langle M\rangle \to \infty$ and we have a so-called run-away vacuum, the quantum theory does not have a ground state. We find that the quantum effects (the dynamical $W_{\text{ADS}}$) in the SQCD theory completely lift the classical moduli space. However, by adding a tree-level mass term to the superpotential, a minimum is generated for finite $\langle M\rangle$ and we avoid the run-away vacuum. If we add a mass term

$$W_{\text{tree}} = m^i_j M^j_i \tag{5.56}$$

for all flavours and integrate them out, we end up with a pure SYM theory. A detailed analysis of the full superpotential shows that the minimum is found at a vev [158]

$$\langle M^j_i\rangle_{\min} = (m^{-1})^j_i(\det\, m\, \Lambda^{3N-F})^{1/N} \ . \tag{5.57}$$

Taking the $N$th root shows explicitly that there exist $N$ distinct vacua in the SYM theory. Although constructed in the case of $F < N$, this result holds for general $F$ since one can always integrate out enough flavours such that we end up with $F' < N$.

### 5.4.5 The special cases $F = N$ and $F = N+1$

For all theories with $F \geq N$, the ADS superpotential cannot be generated since it blows up in the weak coupling limit $\Lambda \to 0$. In general, we can distinguish four interesting regimes, namely the cases

$F = N, F = N+1, N+2 \leq F \leq \frac{3}{2}N$ and $\frac{3}{2}N < F < 3N$. For $F \geq 3N$ we lose asymptotic freedom and have an 'ordinary' weakly coupled SQCD theory which can be described perturbatively.

We begin with the special case of $F = N$. Here, the moduli space is parameterised by $F^2$ meson fields and the two baryons $B$ and $\tilde{B}$. Classically, we have the constraint (5.49). We see that at the origin of the moduli space we have a conical singularity associated with the undefined phase of the complex fields $M, B$ and $\tilde{B}$. Here, the gauge group is completely unbroken and we have massless gluons in the theory. So what happens in the quantum theory where we do not have a dynamically generated superpotential? In general, the absence of a superpotential indicates that the moduli space persists in the quantum theory. However, the classical constraint between the fields parameterising the moduli space might be modified. We can analyse the constraint by considering the massive theory where the meson vev is given by the relation (5.57). For $F = N$ we can consider the $\det\langle M\rangle$ and find

$$\det\langle M\rangle = \Lambda^{2N} \ , \tag{5.58}$$

independent of $m$. However, for det $m \neq 0$ we can integrate out all fields with non-zero baryon number, yielding $\langle B\rangle = \langle\tilde{B}\rangle = 0$. This is not fulfilled in the classical moduli space. A detailed analysis (see for instance [17] and for further details [160]) shows that the constraint of the quantum moduli space is given by

$$\det M - B\tilde{B} = \Lambda^{2N} \ . \tag{5.59}$$

Hence, the origin is not part of the quantum moduli space and there is no point on the moduli space with unbroken gauge symmetry. In addition, since all gauge-invariant operators are charged under the global symmetries (up to the R-symmetry), at a generic point on the moduli space some global symmetries are also broken. The quantum modified constraint can be used to write down a superpotential of the quantum theory which give rise to the correct equations of motion. One usually introduces a Lagrange multiplier field $X$ and writes [108]

$$W_{\text{QM}} = X\left(\det\, M - B\tilde{B} - \Lambda^{2N}\right) \tag{5.60}$$

Interestingly, this superpotential yields the correct ADS superpotential of a theory with $F' = N-1$ by introducing a mass term for the $N$th flavour of the $F = N$ theory, providing a useful consistency check.

Let us move on by considering the second special case of $F = N+1$. The classical moduli space is again given by $F^2$ mesons and $2F$ baryons (baryons + anti-baryons) under the constraints

$$M_i^j B^i = \tilde{B}_j M_i^j = 0 \ , \qquad B^{i_F}\tilde{B}_{j_F} = (M^{-1})_j^i \det\, M \ . \tag{5.61}$$

Introducing tree-level masses for the chiral superfields we have again the vev (5.57) and $\langle B\rangle = \langle\tilde{B}\rangle = 0$. It follows that the classical constraints are satisfied in the massless limit $m \to 0$ and hence, persist quantum mechanically [108]; the classical and quantum moduli space are identical. The origin is part of the quantum moduli space and additional massless degrees of freedom exist there. Classical one has massless gluons and gluinos (see also the case $F = N$) but since the quantum theory is strongly coupled and confines at the origin, it is sensible that one has massless mesons and baryons. These specific composite fields are removed anywhere else on the moduli space by the constraints (5.61). This interpretation is strongly supported by considering the so-called 't Hooft anomaly matching [161]: The anomalies of the theory's global symmetry currents computed in terms of the composite meson and baryon fields and in terms of the fundamental fields match. Since the origin lies on the quantum moduli space, the chiral symmetry is not broken at this point and gives additional constraints from the anomaly matching. An important difference to the $F = N$ case is that here, one has a dynamical superpotential which includes an interaction between the mesons and baryons. One can show that

$$W = \frac{1}{\Lambda^{2F-3}} \left( \tilde{B}_j M_i^j B^i - \det M \right) \tag{5.62}$$

is the correct superpotential, being invariant under all the symmetries [108]. The quantum (= classical) constraints (5.61) can be reproduced from this superpotential by considering the equations of motions of the composite fields. The case of $F = N+1$ is an example of a so-called s-confining theory [162], namely a theory with a dynamical superpotential and confinement without the necessity of chiral symmetry breaking.

### 5.4.6 Seiberg duality and conformal fixed points, $N + 1 < F < 3N$

If we increase the number of flavours $F$ further to $F \geq N + 2$, the classical moduli space is again given by $F^2$ mesons and a set of baryon fields. One can show that the quantum moduli space is equal to the classical one, especially the origin is part of the quantum theory. Again, we should ask ourselves how to interpret the singularity at $M = B = \tilde{B} = 0$. Unfortunately, we cannot repeat the story of the previous section: A dynamical superpotential consistent with all symmetries (such that the quantum constraints arise as equations of motion) diverges at the origin. Since also the 't Hooft anomaly matching conditions are not satisfied when one considers only the composite fields $M, B$ and $\tilde{B}$, additional light degrees of freedom must be present.

This puzzle was solved by Seiberg's proposal [103] where he suggested that the original SQCD theory is dual to another gauge theory with the same global symmetries but with a different gauge group $\mathsf{SU}(\tilde{N})$ where $\tilde{N} = F - N$. A physical interpretation for the appearance of $\tilde{N}$ is given by the fact that one can see the $F - N$ baryons $B^{i_{N+1},\ldots,i_F}$ and $\tilde{B}_{i_{N+1},\ldots,i_F}$ of the original theory as bound states of $\tilde{N}$ components which in turn are the fundamental fields in the dual theory. For reasons which become apparent shortly, Seiberg denoted these dual component fields as 'magnetic quarks' $q$ and $\tilde{q}$. Additionally, we have the meson field $M$ as a fundamental object. The dual fields have the following quantum numbers under all symmetries:

| Field | $\mathsf{SU}(F-N)$ | $\mathsf{SU}(F)$ | $\mathsf{SU}(F)$ | $\mathsf{U}(1)_B$ | $\mathsf{U}(1)_R$ |
|---|---|---|---|---|---|
| $q$ | $\square$ | $\overline{\square}$ | $\mathbf{1}$ | $\frac{N}{F-N}$ | $\frac{N}{F}$ |
| $\tilde{q}$ | $\overline{\square}$ | $\mathbf{1}$ | $\square$ | $-\frac{N}{F-N}$ | $\frac{N}{F}$ |
| $M$ | $\mathbf{1}$ | $\square$ | $\overline{\square}$ | $0$ | $2\frac{F-N}{N}$ |

In Seibergs proposal, the superpotential of the 'electric' theory vanishes in the limit of zero masses whereas the dual 'magnetic' theory has the superpotential [103]

$$W_{\text{mag}} = \frac{1}{\hat{\Lambda}} q M \tilde{q} \tag{5.63}$$

where $\hat{\Lambda}$ is some characteristic scale such that the superpotential can be written in terms of the original meson field $M$. The scale of the electric theory $\Lambda_{\text{el}}$ and the magnetic theory $\Lambda_{\text{mag}}$ matches as

$$\Lambda_{\text{el}}^{3N-F} \Lambda_{\text{mag}}^{3(F-N)-F} = (-1)^{F-N} \hat{\Lambda}^F \ . \tag{5.64}$$

The precise form is not so important for us, however, what is important is the fact that $\Lambda_{\text{el}}^b \Lambda_{\text{mag}}^{\tilde{b}} =$ const. This gives the precise reason for calling the theories 'electric' and 'magnetic' since we have a strong-weak duality here, similar to the well-know one in electro-magnetism. As an important check of Seibergs proposal, one can use the fermion content of the dual theory (the dual quarks $\psi_q, \psi_{\tilde{q}}$, the mesinos $\psi_M$ and the gaugino $\tilde{\lambda}$) to show that the anomaly matching conditions are fulfilled [103], see also the discussions in [130].

After identifying the dual theory, it is natural to ask if this description holds for arbitrary $F > N + 1$[17]. We have already mentioned that for $F > 3N$ the original theory becomes IR free, i.e. we have a theory of weakly coupled chiral (quarks) and vector (gluon) supermultiplets. To understand the behaviour of the dual theory, it is useful to consider the original theory just below the point $F = 3N$. By considering a large $N$ limit, it was shown by Banks and Zaks [163] using general gauge theory properties that there exists a non-trivial fixed point $g_*$ of small coupling. For SQCD with $F = 3N - \epsilon N$, the perturbative fixed point is given by

$$g_*^2 = \frac{8}{3}\pi^2 \frac{N}{N^2 - 1}\epsilon \tag{5.65}$$

where $\epsilon \ll 1$. Without any mass terms, the theory is scale-invariant at the fixed point $g_*$. For theories with particles of spin less than 2, this actually implies conformal invariance [164]. When considering supersymmetric theories, the conformal algebra is extended to a superconformal one[18]. For a superconformal theory, near a fixed point, all scalar gauge-invariant operators must have a dimension greater or equal to one [165]. In the large $N$ limit, the meson's dimension is given by

$$\dim(M) = \dim(Q\tilde{Q}) = 2 + \gamma = 3\frac{F - N}{F} \tag{5.66}$$

with $R[Q] = R[\tilde{Q}] = (F - N)/F$ and $\gamma$ as the meson's anomalous dimension. From this we get a lower bound on the region of flavours where we expect a fixed point in the superconformal theory, $F/N > \frac{3}{2}$. Seiberg suggested that a non-trivial conformal fixed point exists not only away from the limit of $\epsilon << 1$ but rather in the whole range of $\frac{3}{2}N < F < 3N$ [103], the so-called *conformal window*. Theories in this range are UV free, however, the coupling does not diverge in the IR but rather flows to a fixed point $g_*$. Hence, we have an interacting conformal theory with no confinement.

[17]Note that for the case $F = N + 1$ the Seiberg dual gauge group is empty. However, as we have seen, some sort of a simple duality is also happening in this case. The mesons and baryons describe the theory near the origin of the moduli space.

[18]For an introduction see for instance [17] or the lectures [141].

A similar analysis holds for the dual theory. From the coefficient $\tilde{b}$ of the one-loop beta-function one finds that the magnetic theory is IR free when $F < \frac{3}{2}N$. For $F$ slightly above $\frac{3}{2}N$ (equivalent to $F = 3\tilde{N} - \epsilon\tilde{N}$), a perturbative fixed point $\tilde{g}_*$ exists here as well in the limit of large $\tilde{N}$. Assuming that this fixed point exists away from $F \approx \frac{3}{2}N$ in the dual theory yields the following picture: The original theory flows to an interacting fixed point in the IR in the range of $\frac{3}{2}N < F < 3N$ where the coupling is weak for $F$ close to $3N$ and becomes stronger as $F$ moves down towards $\frac{3}{2}N$. In the same range of $\frac{3}{2}N < F < 3N$, the dual theory flows also to an IR interacting fixed point, however, the dual coupling is weak for $F$ close to $\frac{3}{2}N$ and gets stronger as $F$ is increased! Seiberg's conjecture states that the two conformal IR fixed points can be identified with each other, leading to a strong-weak duality in the conformal window: if one of the two theories is strongly coupled, the other one is in its weakly coupled regime. This duality only holds in the IR! Both theories are different in the UV, there is no relation between them for high energies.

We see that Seiberg's IR duality proposal holds in the range of $\frac{3}{2}N < F < 3N$. For $F \leq \frac{3}{2}$ the original theory is strongly coupled. However, in this range, asymptotic freedom in the dual theory is lost. The magnetic theory is IR free and the non-trivial IR fixed point vanishes, we are left with the trivial IR fixed point. Here, the dual theory makes sense only for energies below a UV cutoff. Therefore, in the range of $N+2 \leq F \leq \frac{3}{2}N$ the Seiberg duality proposal can be summarised as follows: The original theory is UV free and hence strongly coupled in the IR. However, the dual theory is weakly coupled in the IR and Seiberg duality tells us that both theories flow to the same IR physics but are different in the UV. Since the magnetic theory is weakly coupled for $N+2 \leq F \leq \frac{3}{2}N$, it is best to describe the physics by the IR free theory. It is this range of flavours that we will be interested in when discussing the ISS construction of metastable gauge meditation in the next section.

## 5.5 The ISS Model of supersymmetry breaking

The work which has recently invigorated the field of gauge mediated supersymmetry breaking is the work of Intriligator, Seiberg and Shih (ISS) [129]. It is based on the simple but sweeping assumption of accepting supersymmetric vacua in a phenomenological viable theory of broken supersymmetry. As we have seen in Section 5.3 this is tied to an explicitly broken R-symmetry in the theory. Models with metastable SUSY-broken states have been studied before. In the early eighties, the authors of [166] considered a metastable vacuum in a classical theory. Later on, metastable vacua were discussed in theories with pseudomoduli space that are modified by quantum corrections [167]. Further, more recent examples of models with metastable SUSY-broken states include [168, 169]. Although accepting true supersymmetric vacua in phenomenological theories made their construction much easier, many of the models had the drawback of not being under control in the IR limit. In addition, the metastable states often appeared 'accidentally' and one had to justify their existence by arguing for a long enough lifetime of the false vacua. The novelty of the ISS approach was the acceptance of metastable SUSY-broken vacua right from the beginning. Most importantly, the main phenomenological requirement of a long-lived metastable state is fulfilled by the ISS construction. The metastability of the states is controlled by a small parameter $\epsilon$ which is given by the ration of a mass and a dynamical scale, $\epsilon = \mu/\Lambda_{\rm mag}$. By taking $\epsilon \to 0$ one can 'control' the amount of metastability in the theory.

The big advantage of the ISS construction is the control of the theory in the IR limit. This is important since the metastable states occur near the origin of the field space. By the means of Seiberg duality, this macroscopic IR free theory can be identified with a microscopic UV free SQCD theory and hence, the construction is rather involved: The model relies on SQCD with the right range of flavours with an extensive number of (matrix-valued) fields transforming under the large SQCD symmetry group. We will therefore only summarise the main results of this model and focus on the parts which provide some intuition for our discussion of $\mathsf{SO}(N)$ based ISS constructions in the next two chapters. For more detailed information and derivations of results we refer the reader to the original paper [129] and the lectures [133]. We will follow mainly the discussions in [129,137] and try to stay as close as possible to the original ISS notation[19]. It is instructive to provide a brief overview of the ISS model. Its construction can be summarise in three main steps:

- Consider an IR free theory of chiral superfields only where supersymmetry is broken at tree-level by the so-called rank condition. ISS call this the *macroscopic model I*.
- Gauge one of the global symmetry groups such that gauge superfields are present in the theory. We end up with a SQCD like theory which posseses supersymmetric vacua. The gauging should happen such that the susy-broken vacua are preserved, resulting in metastable supersymmetry breaking in the *macroscopic model II*.
- Use the Seiberg duality to identify the IR free SQCD-like theory with its electric dual description which is strongly coupled in the considered range of flavours. This identification establishes dynamical metastable supersymmetry breaking in the UV free *microscopic model*.

Since we have discussed Seiberg dual theories already in Section 5.4.6, it is instructive to start directly with the microscopic theory. ISS considered a $\mathsf{SU}(N_c)$ SQCD model with flavours in the range $N_c + 1 \leq N_f < \frac{3}{2}N_c$ with massive quark superfields such that the superpotential of the electric theory is given by[20]

$$W_{\rm el} = m{\rm Tr}[Q \cdot \tilde{Q}] \ . \tag{5.67}$$

Assuming the masses to be much smaller than $\Lambda_{\rm el}$, the electric mass term arises in the dual theory as an additional term in the superpotential. Hence, the magnetic theory is described by [103]

$$W_{\rm mag} = \frac{1}{\hat{\Lambda}}{\rm Tr}[qM\tilde{q}] + m{\rm Tr}M \tag{5.68}$$

with a Kähler potential

$$K = \frac{1}{\beta}{\rm Tr}[\tilde{q}^\dagger \tilde{q} + q^\dagger q] + \frac{1}{\alpha|\Lambda_{\rm el}|}{\rm Tr}[M^\dagger M] \ . \tag{5.69}$$

[19] It is therefore sensible to slightly change notation, compared to our notation in Section 5.4: From now on we refer to the number of flavours by $N_f$ while denoting the number of colours by $N_c$ and the difference in $N_f$ and $N_c$ by $N = N_f - N_c$.

[20] For simplicity we assume equal quark masses.

The scales of the electric and magnetic theories matches as in relation (5.64), here $\Lambda_{\rm mag}$ plays the role of an UV cutoff. Setting $\beta = 1$ and identifying

$$\begin{aligned} M &= \sqrt{\alpha}\Lambda_{\rm el}\Phi \ , \qquad q = \varphi \ , \qquad \tilde{q} = \tilde{\varphi} \\ h &= \frac{\alpha\Lambda_{\rm el}}{\hat{\Lambda}} \ , \qquad \mu^2 = -m\hat{\Lambda} \end{aligned} \tag{5.70}$$

yields a canonical Kähler potential and a superpotential

$$W_{\rm mag} = h{\rm Tr}[\varphi\Phi\tilde{\varphi}] - h\mu^2{\rm Tr}\Phi \tag{5.71}$$

in terms of the new fields $\Phi, \varphi$ and $\tilde{\varphi}$. Here, $h$ is a dimensionless constant and $\mu$ has dimensions of a mass. The only free parameters of the theory are $h$ and $\mu$. Due to Seiberg duality, the new fields are charged under the symmetries as

| Field | $\mathsf{SU}(N = N_f - N_c)$ | $\mathsf{SU}(N_f)$ | $\mathsf{SU}(N_f)$ | $\mathsf{U}(1)_B$ | $\mathsf{U}(1)_A$ | $\mathsf{U}(1)_R$ |
|---|---|---|---|---|---|---|
| $\varphi$ | $\square$ | $\overline{\square}$ | $\mathbf{1}$ | 1 | 1 | 0 |
| $\tilde{\varphi}$ | $\overline{\square}$ | $\mathbf{1}$ | $\square$ | -1 | 1 | 0 |
| $\Phi$ | $\mathbf{1}$ | $\square$ | $\overline{\square}$ | 0 | -2 | 2 |

This theory is precisely the one which ISS discussed as their macroscopic model I where all symmetry groups are *global* ones [129]. We note that for $\mu \neq 0$, the global symmetries $\mathsf{SU}(N_f)\times\mathsf{SU}(N_f)\times\mathsf{U}(1)_A$ are explicitly broken to their diagonal subgroup $\mathsf{SU}(N_f)_D$.

Let us stick to this chiral model with only global symmetries for a moment. In order to check if supersymmetry is broken we have to consider the F-terms of the superpotential. For instance, the F-term of $\Phi$ is given by $-(F^\dagger_\Phi)^i_j = h\varphi^i\tilde{\varphi}_j - h\mu^2\delta^i_j$ which is a $N_f \times N_f$ matrix relation. The first term has rank $N$ whereas the second term has rank $N_f$. Since $N < N_f$, the F-term cannot vanish and hence, supersymmetry is spontaneously broken by the rank condition! If we calculate the full scalar potential of the theory, we find find that it is minimised along a classical moduli space of

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & X_0 \end{pmatrix} , \quad \varphi = \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix} , \quad \tilde{\varphi}^T = \begin{pmatrix} \tilde{\varphi}_0 \\ 0 \end{pmatrix} , \quad \text{with } \tilde{\varphi}_0\varphi_0 = \mu^2\mathbb{I}_N \ . \tag{5.72}$$

Here, $X_0$ is an arbitrary $(N_f - N) \times (N_f - N)$ matrix field whereas $\varphi_0, \tilde{\varphi}_0$ are $N \times N$ fields. The minimum occurs for

$$V_{min} = (N_f - N)|h^2\mu^4| \ . \tag{5.73}$$

The transformations of the component fields under the global symmetries for $\mu \neq 0$ are

| Field | $\mathsf{SU}(N)$ | $\mathsf{SU}(N)_f$ | $\mathsf{SU}(N_f - N)_f$ |
|---|---|---|---|
| $\Phi = \begin{pmatrix} Y_{N\text{x}N} & Z^T_{N_f\text{-}N\text{x}N} \\ \tilde{Z}_{N\text{x}N_f\text{-}N} & X_{N_f\text{-}N\text{x}N_f\text{-}N} \end{pmatrix}$ | $\begin{pmatrix} \mathbf{1} \end{pmatrix}$ | $\begin{pmatrix} \mathbf{Adj}+1 & \overline{\square} \\ \square & \mathbf{1} \end{pmatrix}$ | $\begin{pmatrix} \mathbf{1} & \square \\ \overline{\square} & \mathbf{Adj}+1 \end{pmatrix}$ |
| $\varphi = \begin{pmatrix} \chi_{N\text{x}N} \\ \rho_{N_f\text{-}N\text{x}N} \end{pmatrix}$ | $\begin{pmatrix} \square \end{pmatrix}$ | $\begin{pmatrix} \overline{\square} \\ \mathbf{1} \end{pmatrix}$ | $\begin{pmatrix} \mathbf{1} \\ \overline{\square} \end{pmatrix}$ |

where a similar decomposition holds for $\tilde{\varphi}$. In the moduli space, the vacua with the maximal unbroken subgroup of the global symmetries are given by

$$X_0 = 0 \ , \quad \varphi_0 = \tilde{\varphi}_0 = \mu \mathbb{I}_N \ . \tag{5.74}$$

Here, the global symmetries are spontaneously broken down to $\mathsf{SU}(N) \times \mathsf{SU}(N_f)_D \times \mathsf{U}(1)_B \times \mathsf{U}(1)_R \rightarrow \mathsf{SU}(N)_D \times \mathsf{SU}(N_f - N) \times \mathsf{U}(1)_{B'} \times \mathsf{U}(1)_R$. These vacua are also stable when considering quantum corrections [129], namely the one-loop corrections to the effective theory of the pseudo-moduli. Since the quantum corrections lift the moduli space elsewhere, they drive the theory into the vacua (5.74).

In order to compute the one-loop effective potential of the pseudo-moduli, one has to calculate the one-loop correction to the vacuum energy, the so-called *Coleman-Weinberg effective potential*, given by

$$V_{CW} = \frac{1}{64\pi^2} \mathrm{STr}\ \mathbf{M}^4 \log \frac{\mathbf{M}^2}{\Lambda^2} = \frac{1}{64\pi^2} \left( \mathrm{Tr}\ \mathbf{m}_B^4 \log \frac{\mathbf{m}_B^2}{\Lambda^2} - \mathrm{Tr}\ \mathbf{m}_F^4 \log \frac{\mathbf{m}_F^2}{\Lambda^2} \right) \tag{5.75}$$

where $m_B$ and $m_F$ are the mass eigenvalues of the bosonic and fermionic mass matrices of the theory. Hence, to compute $V_{CW}$ we need to know the spectrum of the theory. For the ISS model, this is quite an involved task and we refer the reader to the original paper for detailed derivations. Here, we briefly summarise the results: Firstly, most component fields get tree-level masses of the order $|h\mu|$ from the relevant terms of the scalar potential. We also have massless scalars in the spectrum, namely the Goldstone bosons from the spontaneously broken symmetries and the fields which are fluctuations around $X_0$ and $\varphi_0 - \tilde{\varphi}_0$ of the classical moduli space. Whereas the Goldstone fields stay exactly massless, the pseudo-moduli acquire masses from the one-loop effective potential. A detailed analysis shows [129]

$$\begin{aligned} m^2_{\delta X_0} &= |h^4 \mu^2| \frac{\log 4 - 1}{8\pi^2} N \ , \\ m^2_{\delta(\varphi_0 - \tilde{\varphi}_0)} &= |h^4 \mu^2| \frac{\log 4 - 1}{8\pi^2} (N_f - N) \ . \end{aligned} \tag{5.76}$$

Let us now weakly gauge the $\mathsf{SU}(N)$ group such that the macroscopic model II can be identified with the magnetic dual theory of $\mathsf{SU}(N_c)$ massive SQCD. The potential gets a D-term contribution, however, in the vacuum 5.74 the D-terms vanish. Hence, the gauging does not effect the supersymmetry broken minimum. Due to the vevs of $\varphi$ and $\tilde{\varphi}$, the gauge symmetry is completely broken and $N^2 - 1$ of the former Goldstone bosons are eaten. The gauge fields acquire mass $g\mu$. Furthermore $N^2 - 1$ of the pseudo-moduli, associated with the fluctuations $\delta(\varphi_0 - \tilde{\varphi}_0)$, get a mass $g\mu$ from the D-term potential. In addition, the gauging of $\mathsf{SU}(N)$ does not destabilise the vacua 5.74 since the effect of the gauge fields drops out in the Coleman-Weinberg calculation (5.75).

However, gauging a global symmetry introduces new supersymmetric vacua at large values of the meson vev $\langle \Phi \rangle$ of the macroscopic model II. This can be seen as follows: For any non-zero mesons vev the fields $\varphi$ and $\tilde{\varphi}$ obtain a mass $h\langle \Phi \rangle$ and one can integrate them out below that scale. Hence, the low energy theory is just a pure SYM theory with gauge group $\mathsf{SU}(N)$ which is described by an effective superpotential[21] $W = N(\Lambda_L^{3N})^{1/N}$ [103]. Matching the running gauge coupling of

[21] As usual, the holomorphic scale $\Lambda_L$ is promoted to a background chiral superfield.

the theory to energies above the mass scale but below the UV cutoff $\Lambda_{\rm mag}$ yields an additional term in the superpotential from the pure SYM theory. The low energy theory is described by the superpotential

$$W = N(h^{N_f}\Lambda_{\rm mag}^{3N-N_f}\det\,\Phi)^{1/N} - h\mu^2 {\rm Tr}\,\Phi\ . \tag{5.77}$$

By considering the corresponding field equations, we find $N_f - N = N_c$ supersymmetric vacua [129]

$$\langle\Phi\rangle = \frac{\Lambda_{\rm mag}}{h}\epsilon^{2N/(N_f-N)}\mathbb{I}_{N_f} \tag{5.78}$$

where $\epsilon = \mu/\Lambda_{\rm mag}$. For $\epsilon \ll 1$ we have a hierarchy of $|\mu| \ll |\langle h\Phi\rangle| \ll |\Lambda_{\rm mag}|$. Hence, in the limit of small $\epsilon$ (this corresponds to a large but finite $\Lambda_{\rm mag}$ ), the vacua of broken supersymmetry near the origin are far away from the supersymmetric ones and parametrically long lived. This statement can be made precise [129] by estimating the decay of the metastable vacua through a semi-classical field theory decay where the decay probability is given by $e^{-S}$ with $S$ is a 'bounce' action [170]. By analysing the potential and using a triangle potential barrier [171], the bounce action for 'pure' ISS can be estimated by

$$S \approx \frac{(\Delta\Phi)^4}{V_{peak}} \approx \left(\frac{\Lambda_{\rm mag}}{\mu}\right)^{4\frac{(N_f-3N)}{N_f-N}} \gg 1\ . \tag{5.79}$$

Here, $\Delta\Phi$ is the difference between the SUSY-broken and supersymmetric vacua in the meson vev and $V_{peak}$ is the value at the potential at its local maximum.

In a final step, one uses Seiberg duality to obtain a UV completion for the now gauged macroscopic theory such that the construction is valid for arbitrary high energy. We have already mentioned the identification of electric and magnetic degrees of freedom at the beginning of our discussion. Now, we have come full circle. The electric description, valid for energies $E > \Lambda_{\rm mag}$, is an $\mathsf{SU}(N_c)$ SQCD theory with $N_f$ flavours in the range of $N_c + 1 \leq N_f < \frac{3}{2}N_c$ which breaks supersymmetry non-perturbatively in a metastable state near the field space origin. This is dynamical supersymmetry breaking in a strongly coupled region of the $\mathcal{N} = 1$ SQCD theory! By the advances of Seiberg duality one obtains an effective description such that supersymmetry breaking can be explored in a perturbative fashion.

The ISS model was originally construction for theories with $\mathsf{SU}(N)$ groups. The authors of [129] also generalised the model to theories with $\mathsf{SO}(N)$ and $\mathsf{Sp}(N)$ groups. In the Chapters 6 and 7 we will provide a detailed discussion on the $\mathsf{SO}(N)$-ISS construction where we also consider various deformations of the 'vanilla' ISS model. Having discussed the important steps in the ISS construction we are ready to move on: In the next section we review some aspects of the gauge-mediated approach of supersymmetry breaking and how the ISS models fits into this setting.

## 5.6 Gauge mediated supersymmetry breaking

If we request that supersymmetry is broken at low energies, then gauge mediation offers an attractive framework for mediating the SUSY-broken spectrum to the supersymmetric SM. As we have already mentioned in Section 5.1.3, in this construction the gauge interactions of the SM transmit the SUSY-breaking effects. In order to separate the hidden from the visible sector, messenger superfields are introduced which inherit a SUSY-broken spectrum by directly coupling to a supersymmetry breaking theory. Since non-gravitational interactions are used for mediating between the two sectors, any gravity contributions are negligible and no flavour problems arise.

The gauge mediation setup originated with the pioneering works [172–174] in the early 1980s. Later on, in the important works [175–177] gauge mediation models with stable dynamical supersymmetry breaking[22] were constructed. Phenomenological investigations of these ideas followed in the 1990s, see the 'canonical' gauge mediation review [178]. As we have discussed in the previous section, a rather radical new approach to dynamical supersymmetry breaking was the Intriligator-Seiberg-Shih model [129]. In the years following its publication, a vast amount of different gauge mediation models using the ISS construction were developed. However, modifications of the vanilla ISS model are needed in order to obtain a phenomenological viable particle spectrum. We will return to this issue in Section 6.2.

Let us briefly review how a gauge-mediated model is constructed. One usually starts with the sector which breaks supersymmetry. The simplest approach is to consider a O'Raifeartaigh model [142] which breaks supersymmetry by a non-zero F-term. For the rest of the thesis we will consider an ISS-like theory as the hidden sector model but right now, the details of the SUSY breaking theory are not important. All we need is a chiral superfield $X$ in the hidden sector which breaks supersymmetry by a vev

$$\langle X \rangle = M + \theta^2 F \ . \tag{5.80}$$

We then couple the hidden sector with broken supersymmetry to the supersymmetric SM. More precisely, we introduce messenger fields which are charged under the gauge groups of the visible sector. Since the messengers should have a SUSY-broken spectrum, we need to couple them directly to the hidden sector. Furthermore, these newly introduced fields need to be sufficiently heavy to fulfill experimental bounds. In a minimal setup one therefore introduces two chiral superfields $\varphi$ and $\tilde{\varphi}$ in complete vector-like representations of a GUT $\mathsf{SU}(5)$ gauge group[23], i.e. the fields transform as a $\mathbf{5}$ and $\overline{\mathbf{5}}$ of $\mathsf{SU}(5)$ such that large enough (Dirac) masses are possible. The messenger fields get a SUSY-broken spectrum from the hidden sector superpotential

$$W_{\text{mess}} = X\varphi\tilde{\varphi} \tag{5.81}$$

where $X$ is the spurion whose vev breaks supersymmetry. The fermionic components of the messengers get a mass of $M$ while the scalar components have a split spectrum with a squared mass of $m^2 = M^2 \pm F$. In order to obtain positive squared masses we require $F < M^2$.

The coupling of the messenger fields $\varphi$ and $\tilde{\varphi}$ to the SM gauginos and sfermions now lifts the masses of the SM partner fields such that they can be sufficiently heavy. The gaugino masses are generated at one-loop by tree-level couplings between messenger fermion, messenger scalar and SM gaugino. The masses of the SM sfermions are only generated at two-loop order. A detailed

[22] For detailed discussions see for instance [137, 138].

[23] A $\mathsf{SU}(5)$ gauge group is not a strict requirement but merely a 'nice thing to have', also as a book-keeping tool. We will stick to the $\mathsf{SU}(5)$ language in the following.

calculation [179] yields

$$m_{\lambda_i} = \frac{\alpha_i}{4\pi} n \frac{F}{M} g(x) \ , \tag{5.82}$$

$$m_{\tilde{f}} = 2n \frac{F^2}{M^2} \sum_i \left(\frac{\alpha_i}{4\pi}\right)^2 C_i f(x) \ , \tag{5.83}$$

where $\alpha_i$ is the coupling constant for $\mathsf{SU}(3), \mathsf{SU}(2)$ and $\mathsf{U}(1)_Y$ respectively, $n$ is the number of messenger pairs $\varphi$ and $\tilde{\varphi}$ ($n = 1$ for the most minimal construction) and $C_i$ are the quadratic Casimirs of the relevant scalar representation. Furthermore, the functions $f(x)$ and $g(x)$ are given by

$$f(x) = \frac{1}{x^2} \left[(1+x)\log(1+x) + (1-x)\log(1-x)\right] \ , \tag{5.84}$$

$$g(x) = \frac{1+x}{x^2} \left[\log(1+x) - 2\mathrm{Li}_2\left(\frac{x}{1+x}\right) + \frac{1}{2}\mathrm{Li}_2\left(\frac{2x}{1+x}\right)\right] + (x \to -x) \ , \tag{5.85}$$

with $x = F/M^2 < 1$. Quite often we assume a small splitting between for the scalar masses which corresponds to the limit $F \ll M^2$ or $x \ll 1$. In that limit, the functions are approximately one, $f(x), g(x) \longrightarrow 1$ for $x \ll 1$, and we find that not only the generated scalar and gaugino masses are of the same order in the gauge couplings but we also have $m_{\tilde{f}} \sim m_\lambda$. Furthermore, we have a distinct mass hierarchy where coloured superpartners are heavier than those charged under $\mathsf{SU}(2)_L$ and $\mathsf{U}(1)_Y$. Interestingly, this is independent of the dynamics in the hidden sector or the messenger fields, at least in the limit of $F \ll M^2$. Let us also mention that the previous results (5.82) which were extracted from an explicit diagrammatic calculation [179] can also be obtained, in the limit $F \ll M^2$, from a consideration based on holomorphy and wave function renormalisation. For details on this approach see the original work [180] and the follow up publication [181].

The minimal gauge mediation model we briefly discussed above is part of a general class of so-called *ordinary gauge mediation* (OGM) models. In this construction, a hidden sector field $X$ obtains a vev through a non-specified mechanism and couples to a set of vector-like messenger fields via Yukawa interactions of the form

$$W_{\mathrm{OGM}} = \lambda_{ij} X \varphi^i \tilde{\varphi}^j \ , \tag{5.86}$$

where $\langle X \rangle = M + \theta^2 F$. Although these constructions were successful in terms of separating the SUSY-breaking dynamics from the visible sector with a low SUSY-broken scale, the messenger sector was quite arbitrarily introduced. Attempts to simplify the structure of gauge mediation models led to constructions where the messenger fields themselves participate in the supersymmetry breaking. These models are usually denoted as *direct gauge mediation* (DGM). Here, the messengers are part of the hidden sector and play a role in the breaking of supersymmetry. In order to couple them via gauge interactions to the supersymmetric SM we gauge one of the global symmetry groups of the hidden sector and identify a subgroup with the $\mathsf{SU}(3) \times \mathsf{2} \times \mathsf{U}(1)$ of the SM. In general, this is possible for hidden sectors with a large global symmetry group, like the ISS magnetic theory. However, due to additional matter fields in the now gauged flavour group, the running of the SM gauge couplings is affected. It is then a question of careful model building to avoid a running of the couplings into the strong coupling regime before the GUT scale, usually denoted as the *Landau* *pole problem* of direct mediation models. For further discussions on direct gauge mediation and models in the pre-ISS area we refer the reader to the literature, see for instance the review in [135] and reference therein.

The realisation that metastable states can be well incorporated in models of gauge mediation not only led to a vast amount of different gauge-mediated constructions utilising the ISS model and a return of the modular OGM idea, but also triggered some new approaches to gauge mediation itself. One of the ideas developed in the post-ISS days is the *semi-direct gauge mediation* model of Seiberg, Volansky and Wecht [182]. In this construction there is no separate sector of messenger fields, the messengers are part of the hidden sector. However, they do not contribute in the breaking of supersymmetry. Next, a group of authors (Cheung, Fitzpatrick and Shih) generalised [183] the OGM construction by introducing a number of singlets $X_k$ in the hidden sector that couple to the messengers to all possible renormalisable couplings, yielding the *extra-ordinary gauge mediation* (EOGM) superpotential

$$W_{\rm EOGM} = \left(\lambda_{ij}^{(k)} X_k + m_{ij}\right) \varphi^i \tilde{\varphi}^j \ . \tag{5.87}$$

From this starting point, generalised results for gaugino and sfermion masses could be derived [183]. We will return to these EOGM results for the SM superpartner masses in Chapters 6 and 7 where we utilise them for SO-based ISS models. One of the most recent developments for gauge-mediated models is a framework denoted as *general gauge mediation* (GGM). In this work, Meade, Shih and Seiberg [184] defined the most general approach to gauge mediation. Whereas before the OGM construction was only valid in the cause of weakly coupled messenger fields, the general gauge mediation framework describes also strongly interacting hidden sectors. GGM is defined as the class of models for which the hidden sector decouples in the limit of vanishing SM couplings, i.e. $\alpha_i \to 0$. The authors of GGM could describe a general SUSY-breaking sector by utilising two-point functions of gauge supercurrent correllators, yielding that SM gaugino and sfermion masses are governed by three complex and three real parameters [184]. For further details[24] see for instance the nice review in [136].

Finally, let us briefly review gauge mediation models based specifically on the ISS framework. The metastable vacua approach provides an attractive possibility for gauge mediation model-building since the phenomenologically difficult requirement of having no supersymmetric vacua in the theory is avoided. In principle, one can use the ISS model as the hidden sector theory which breaks supersymmetry and then couple it via gauge interactions to the supersymmetriy SM, for instance via modular gauge mediation (OGM) or via models of direct mediation. Because of the large flavour symmetry group of the perturbative ISS description, the direct mediation approach was mainly considered in the last years. The first works appeared quickly after the original ISS paper. In [185] the importance of metastable states to circumvent the problems related to R-symmetries are discussed, see also our related review in Section 5.3. Then Kitano, Ooguri and Ookouchi (KOO) developed a direct mediation model which is directly based on the ISS construction [186]. Another ISS-based direct mediation construction was discussed in [187] which utilised a $N_f = N_c + 1$ SQCD theory for the hidden sector. These are just some of the early attempts of using metastable states

[24] We also provide some information in Appendix C.

for gauge-mediated phenomenological constructions. For a more complete list we refer the reader to the nice review of [137].

However, all gauge mediation constructions based on the ISS model face three obstacles when one tries to build a phenomenological viable model. This forces one to modify the vanilla ISS construction, usually by deforming the superpotential of the theory. Firstly, the ISS model has an (approximate) R-symmetry which forbids gaugino masses in the SUSY-broken metastable vacuum. In order to obtain a sensible spectrum from any gauge-mediated model based on the ISS construction, one has to break the R-symmetry, either spontaneously or explicitly. We will tie in on this in Sections 6.2 and 6.3 when discussing gaugino masses and R-symmetry breaking in the context of $\mathsf{SO(N)}$ ISS models. Secondly, even for broken R-symmetry, some models of gauge mediation have, although non-zero, very small gaugino masses, leading to a hierarchy between the scalar and fermion superpartner masses of the SM. This has been observed in many direct mediation models, not only in those based on the ISS construction. The reason for this can be easily seen from the expression of the gaugino masses at leading order in the parameter $F/M^2$ [183],

$$m_\lambda = \frac{\alpha}{4\pi} F \frac{\partial}{\partial X} \log[\det\, M(X, M)] \tag{5.88}$$

where $X$ is the field whose vev breaks supersymmetry and $M$ is the fermion mass matrix of the messenger fields. If the matrix $M$ has at least one eigenvalue $m = 0$, the right-hand side of the above relation vanishes at leading order in $F/M^2$ and gaugino masses are only generated at subleading order. Models with zero entries in the fermionic messenger matrix were encountered in many occasions. The reason for this behaviour was understood recently by Komargodski and Shih [188]: If a gauge-mediated model can be described as a generalised O'Raifeartaigh model in a *stable* SUSY-broken minimum/minima, then the fermionic messenger matrix is constant, i.e. independent of the field $X$, and gaugino masses vanish at leading order in $F/M^2$. This is for instance the case for the vanilla ISS model with broken R-symmetry. In order to obtain non-zero masses the theory has to have a pseudo-moduli space which is not locally stable everywhere. For models based on the ISS construction, this can be achieved by deforming the original model such that the fermionic messenger mass matrix does not have a zero eigenvalue. A well written review on possible ISS deformations is given in [137]. Thirdly, when gauging a global flavour group of the ISS model, the additional matter fields usually ruin the unification of the gauge couplings at the GUT scale. As mentioned before, this is a general problem of direct mediation models if the mass scale of the messengers is not chosen to be fairly high. For discussions in the context of ISS direct mediation see the works [186, 189, 190].

This concludes our review of gauge-mediated models of supersymmetry breaking. In the following two chapters, we will consider an ISS construction based on $\mathsf{SO}(N)$ global symmetries. Furthermore, we discuss deformations of the basic ISS model which deal with the first two obstacles mentioned above, namely how to break the R-symmetry and how to deform the superpotential such that non-vanishing gaugino masses at leading order in $F/M^2$ are obtained.

# 6 Metastable Supersymmetry Breaking for SO(N) theories

Supersymmetry is an attractive candidate for physics beyond the SM. However, as we have realised in the previous chapter, supersymmetry can only be an approximate symmetry of nature and should be broken in a hidden sector. Especially the gauge-mediated approach to supersymmetry breaking offers the opportunity to verify the concept of supersymmetry and any signatures of a unifying 'parent gauge group' of the SM in the current LHC and other future collider experiments. Direct mediation constructions based on the ISS idea of metastable vacua are an interesting subclass of these phenomenologically viable approaches: They are succinct, perturbative and calculable thanks to Seiberg duality for $\mathcal{N}=1$ SQCD theories. The power of applying the duality in the hidden sector is that the model is perturbative in either the fundamental electric or the dual magnetic theory. In particular, one can have a UV free electric description and an IR free effective description in which supersymmetry breaking can be explored perturbatively. The supersymmetry breaking vacuum need only be metastable, allowing for R-symmetry to be broken and gaugino masses to be generated.

The hidden sector is a supersymmetric $\mathcal{N}=1$ QCD Seiberg dual theory. These have been found for $\mathsf{SU}(N)$, $\mathsf{SO}(N)$ and $\mathsf{Sp}(N)$ gauge groups. Initially $\mathsf{SU}(N)$ models were explored in which the messenger fields are in SM representations [186, 191, 192]. Later, models in which the messengers formed complete representations of a $\mathsf{SU}(5)$ GUT group were implemented [189]. Here, we extend the 'dictionary' of possible metastable constructions by implementing an $\mathsf{SO}(N)$ model in which the messengers can be in complete representations of a $\mathsf{SO}(10)$ group.

In the next two chapters, based on the author's original work [193], we identify parts of the hidden sector flavour symmetry with $\mathsf{SO}(10)$, thereby making the hidden sector dynamics compatible with visible sector GUT models based on this group. Furthermore, we will explore several deformations of the vanilla $\mathsf{SO}(N)$ construction in order to break the R-symmetry in the vacuum and to allow for non-vanishing gaugino masses at leading oder in the $F/M^2$ expansion.

In Section 6.1 we review the macroscopic model and the choice of embeddings of $SO(10)$ into a weakly gauged flavour group. We examine the field content and identify the messenger fields which will generate soft terms for the supersymmetric SM. In Section 6.2 we look at how various messengers will affect gaugino masses and use this to guide our analysis of the deformations of the ISS model. We then introduce some minimal R-symmetry breaking deformations in Section 6.3

and explore the outcomes of these deformations on the vacuum of the theory. In Section 6.4 we use these and additional deformations introduced in [186] and calculate the contributions to the messenger and gaugino masses of the supersymmetric SM. Section 6.5 explores the non-perturbative potential for the ISS model and determines the lifetime of the SUSY broken vacuum. Beginning with Chapter 7, in Section 7.1 we extend the deformation types of Section 6.4 by adding multitrace deformations to the ISS model. We then explore how these models behave when the rank of SUSY breaking magnetic quark matrices is reduced.

## 6.1 Seiberg dual pictures for SO(N) SQCD

As we have seen in the previous chapter, Seiberg duality is an electromagnetic duality in which by interchanging variables charged under the electric theory with 'magnetic' variables, one can move between different unphysical gauge groups and their strong or weak gauge coupling, preserving the physical global symmetries in both pictures. Whereas the previous discussion was focused on $\mathsf{SU}(N)$ gauge groups, in this section, we want to briefly outline the duality for $\mathsf{SO}(N)$ gauge groups. In particular we will choose the weakly coupled side of the duality to explore supersymmetry breaking. This section closely reviews the discussion for $\mathsf{SO}(N)$ groups in [129] and sets our notation.

### 6.1.1 Microscopic theory

We start by summarising the electric side of the duality in which one has a UV free electric theory when $N_f < \frac{3}{2}(N_c - 2)$ for a $\mathsf{SO}(N)$ local symmetry group. We can map this electric picture to a magnetic theory which is weakly coupled in the IR. The field content of the electric theory is given by $N_f$ flavours of quarks and squarks which are combined into complex chiral superfields $Q_i$

| Field | $\mathsf{SO}(N_c)$ | $\mathsf{SU}(N_f)$ | $\mathsf{U}(1)_R$ |
|---|---|---|---|
| $Q$ | $\square$ | $\square$ | $\frac{N_f - N_c + 2}{N_f}$ |

such that the field $Q_{ai}$ carries fundamental indices $a$ of $\mathsf{SO}(N_c)$ and $i$ of $\mathsf{SU}(N_f)$. We refer to the $Q$'s as 'electric quarks'. There is also a discrete symmetry associated with $Q$:

$$Q \to e^{\frac{2\pi i}{2N_f}} Q \quad N_c \neq 3, \quad Q \to e^{\frac{2\pi i}{4N_f}} Q \quad N_c = 3. \tag{6.1}$$

In the case of massless electric quarks the superpotential of the electric description vanishes, $W = 0$. At the non-trivial IR fixed point the duality is exact. It is insightful to note that at the scale invariant fixed point we do not have a well defined particle interpretation, for instance of the gauge bosons of the two dual gauge groups, and the duality between the different gauge groups is exact. Introducing an electric quark mass term to the superpotential

$$W = m_Q \mathrm{Tr}[Q_i \cdot Q_j] = m_Q \mathrm{Tr}[M_{ij}] \tag{6.2}$$

one moves away from the fixed point and the duality becomes effective. The term $W = m_Q \mathrm{Tr}[Q \cdot Q]$ introduces a scale and hence, the particle states are well defined and we find that the two theories have a different number of gauge bosons. Hence, in the massive case, the duality between the two gauge groups is an effective one [109].

### 6.1.2 Macroscopic theory

In this section we review the process of supersymmetry breaking for the $\mathsf{SO}(N)$ macroscopic (magnetic) theory where $N = N_f - N_c + 4$. The macroscopic theory is IR free when $N_f > 3(N-2)$. As for the case of the $\mathsf{SU}(N)$ duality, it is the effective description of an electric theory (microscopic), at energies below the scale $\Lambda_m$ where the macroscopic theory becomes strongly coupled. We may treat the $\mathsf{SO}(N)$ gauge symmetry as a global symmetry to extract the vacuum symmetries and the field representations and then later gauge this symmetry. As discussed in Section 5.5 one can redefine the 'electric' meson field such that the Kähler potential becomes canonical. We also relate the electric quark masses $m_Q$ to $\mu$ by $\mu_{ij}^2 = -m_{Q,ij}\hat{\Lambda}$ where $\hat{\Lambda}$ is introduced such that the electric meson $M$ can be used in the magnetic superpotential.

We start by considering the macroscopic theory with a *global* $\mathsf{SO}(N)$ symmetry group where $N = N_f - N_c + 4$. The field content is[1]

| Field | $\mathsf{SO}(N)$ | $\mathsf{SU}(N_f)_f$ | $\mathsf{U}(1)'$ | $\mathsf{U}(1)_R$ |
|---|---|---|---|---|
| $\Phi$ | $\mathbf{1}$ | $\square\square$ | -2 | 2 |
| $\varphi$ | $\square$ | $\overline{\square}$ | 1 | 0 |

where all fields are complex chiral superfields. The canonical Kähler potential is

$$K = \mathrm{Tr}[\varphi^\dagger \varphi] + \mathrm{Tr}[\Phi^\dagger \Phi]. \tag{6.3}$$

In the following we consider the ISS superpotential of the macroscopic theory which is given by

$$W_{\mathrm{ISS}} = h\mathrm{Tr}[\varphi^T \Phi \varphi] - h\mu^2 \mathrm{Tr}\,\Phi \tag{6.4}$$

The initial global symmetries are valid for the case of $\mu = 0$. If we have $\mu \neq 0$ the global symmetries break down as $\mathsf{SU}(N_f) \times \mathsf{U}(1)' \to \mathsf{SO}(N_f)$,

| Field | $\mathsf{SO}(N)$ | $\mathsf{SO}(N_f)_f$ | $\mathsf{U}(1)_R$ |
|---|---|---|---|
| $\Phi$ | $\mathbf{1}$ | $\square\square + 1$ | 2 |
| $\varphi$ | $\square$ | $\square$ | 0 |

As in the $\mathsf{SU}(N)$ case, supersymmetry is broken by the rank condition when $N_f > N$. For the $\mathsf{SO}(N)$ theory the scalar potential of the theory is minimised by

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & X_0 \end{pmatrix} \quad \varphi = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \quad \chi = \mu \begin{pmatrix} \cosh\theta & i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{pmatrix} \otimes \mathbb{I}_{N/2} \tag{6.5}$$

where $X_0$ is a $(N_f - N) \times (N_f - N)$ symmetric matrix and the field $\chi$ fulfills $\chi^T \chi = \mu^2 \mathbb{I}_N$. Considering the decomposition of the fields $\Phi$ and $\varphi$ suggested by the moduli space we find for their charges under the global symmetry groups

[1]Note that in the $\mathsf{SO}(N)$ there are no additional degrees of freedom associated with a field $\tilde{\varphi}$.

| Field | $\mathsf{SO}(N)$ | $\mathsf{SO}(N)_f$ | $\mathsf{SO}(N_f-N)_f$ |
|---|---|---|---|
| $\Phi = \begin{pmatrix} Y_{N\text{x}N} & Z^T_{N_f\text{-}N\text{x}N} \\ Z_{N\text{x}N_f\text{-}N} & X_{N_f\text{-}N\text{x}N_f\text{-}N} \end{pmatrix}$ | $\begin{pmatrix} \mathbf{1} \end{pmatrix}$ | $\begin{pmatrix} \square\square+\mathbf{1} & \square \\ \square & \mathbf{1} \end{pmatrix}$ | $\begin{pmatrix} \mathbf{1} & \square \\ \square & \square\square+1 \end{pmatrix}$ |
| $\varphi = \begin{pmatrix} \chi_{N\text{x}N} \\ \rho_{N_f\text{-}N\text{x}N} \end{pmatrix}$ | $\begin{pmatrix} \square \end{pmatrix}$ | $\begin{pmatrix} \square \\ \mathbf{1} \end{pmatrix}$ | $\begin{pmatrix} \mathbf{1} \\ \square \end{pmatrix}$ |

These global flavour symmetries are the symmetries into which we weakly gauge and identify with the standard model GUT "parent". The singlet $\text{Tr}X = (1,1,1)$ in the above field representation (where we use the notation where $(A,B,C)$ refers to the irreducible representations of $\mathsf{SO}(N) \times \mathsf{SO}(N)_f \times \mathsf{SO}(N_f - N)_f$) is the chiral superfield whose fermionic component is the massless Goldstino arising from the spontaneous breaking of supersymmetry. The pseudo flat directions which will be lifted by quantum corrections have a degenerate vacuum energy density

$$V_{min} = (N_f - N)|h^2\mu^4| \ . \tag{6.6}$$

Perturbative quantum effects create a local minimum at $X_0 = 0$, $\chi = \mu\mathbb{I}_N$ (up to global symmetries in the parameterisation of $\chi$). This breaks the global symmetries further down to their maximal unbroken subgroup $\mathsf{SO}(N)_D \times \mathsf{SO}(N_f - N) \times \mathsf{U}(1)_R$. As the global flavour symmetry is broken we can choose to diagonalise $\text{Tr}[\mu^2\Phi]$ according to the remaining symmetries and assign different values to parameters within each global flavour symmetry group,

$$\mu^2_{AB} = \begin{pmatrix} \mu^2\mathbb{I}_N & 0 \\ 0 & \hat{\mu}^2\mathbb{I}_{N_F-N} \end{pmatrix}_{AB} \tag{6.7}$$

with $\mu > \hat{\mu}$ and $A, B$ running over the $N_f$ flavour indices. From this, when writing the superpotential in terms of the component fields, we find

$$W = h\text{Tr}[\chi^T Y \chi + \rho^T X \rho + \chi^T Z \rho + \rho^T Z^T \chi] - h\mu^2\text{Tr}Y - h\hat{\mu}^2\text{Tr}X. \tag{6.8}$$

This choice of different values explicitly breaks the global symmetry group and would remove the Goldstone bosons of the vacuum. In the vacuum $X_0 = 0$, $\chi = \mu\mathbb{I}_N$ the charges of the component fields are

| Field | $\mathsf{SO}(N)_D$ | $\mathsf{SO}(N_f-N)_f$ |
|---|---|---|
| $\Phi = \begin{pmatrix} Y_{N\text{x}N} & Z^T_{N_f\text{-}N\text{x}N} \\ Z_{N\text{x}N_f\text{-}N} & X_{N_f\text{-}N\text{x}N_f\text{-}N} \end{pmatrix}$ | $\begin{pmatrix} \square\square+1 & \square \\ \square & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \square \\ \square & \square\square+1 \end{pmatrix}$ |
| $\varphi = \begin{pmatrix} \chi_{N\text{x}N} \\ \rho_{N_f\text{-}N\text{x}N} \end{pmatrix}$ | $\begin{pmatrix} \square\times\square \\ \square \end{pmatrix}$ | $\begin{pmatrix} 1 \\ \square \end{pmatrix}$ |

Here, $\mathsf{SO}(N)_D$ represents a so-called colour-flavour locking phase [194]. The vacuum has five sectors of fields under equivalent representations of the symmetry groups. Each sector satisfies the constraint $\text{Str}M^2 = 0$. The chiral superfields are complex matrix-valued functions with their fermionic component fields in Weyl multiplets. This gives two real boson mass eigenstates for each complex degree of freedom. In the following we give a brief overview on each of these sectors:

**Tr $X$** The trace of $X$ is the massless Goldstino of the spontaneously broken global supersymmetry and it is accompanied by two real bosons.

**$X$** This field is the classically massless pseudo-modulus which is one loop lifted by the Coleman Weinberg potential. For a pure ISS model (no deformations) its vev is lifted to the origin by quantum corrections. There are $(N_f - N)(N_f - N + 1) - 2$ real bosons and half as many Weyl fermions.

**$(\rho, Z)$** These fields give the largest contribution towards the messengers of the SUSY breaking sector. There are $2N(N_f - N)$ Weyl fermions coming from $(\psi_\rho, \psi_Z)$. For small vevs of $X$ their mass is approximately $h\mu$. We will explore their mass in more detail for different ISS models in the following sections. The Goldstone boson arising from the broken global symmetries $\mathsf{SO}(N_f)_f \to \mathsf{SO}(N_f - N)_f \times \mathsf{SO}(N)_f$ are in $Re(\rho)$. Here, the explicit breaking by high dimension operators results in pseudo Nambu-Goldstone bosons [195]. If one makes an explicit choice of different quark masses (see the relation (6.7)) these Goldstone bosons are avoided.

**$(Y, \chi_S)$** There are $N(N+1)-2$ chiral superfields whose fermion and scalar components have mass of order $h\mu$. As we explore in the next section, increasing the vevs of fields will increase the masses. Introducing explicit R-symmetry breaking terms that generate a vev for $Y$ will cause multiplet splittings of the scalar components of these fields. As $F_Y = 0$ and $F_\chi = 0$, these fields play no role as messengers.

**$\chi_A$** The antisymmetric part of $\chi$ parameterises the Goldstone bosons and pseudo-moduli of $\mathsf{SO}(N)\times \mathsf{SO}(N)_f \to \mathsf{SO}(N)_D$. If all the electric quark masses are the same $\mu \sim m_Q\Lambda$, there are $\frac{N}{2}(N-1)$ complex chiral superfields of which half are Goldstone and the other half pseudo-moduli. Using the vacuum (6.5) we can label $\theta_+ = \theta + \theta^*$ as the pseudo-modulus and $\theta_- = \theta - \theta^*$ the Goldstone boson. The Coleman-Weinberg potential will generate a mass for $\theta_+$. The gauge fields will all acquire mass from the super-Higgs mechanism when $\mathsf{SO}(N)_c$ is completely gauged, in particular the Goldstone superfields become Higgs superfields which are then eaten by the vector superfields.

Just as in the $\mathsf{SU}(N)$-model, the gauging of $\mathsf{SO}(N)_c$ does not affect the vacuum of the quantum theory: The spectrum of the added gauge superfields is supersymmetric [129] and drops out when considering in the effective potential of the pseudomoduli (5.75). The gauge fields obtain their masses from the usual scalar kinetic term whereas the mass of pseudo-moduli superfields come from the D-term potential, giving $m = gh\mu$. When one considers embedding the standard model in $\mathsf{SO}(10)$ flavours groups of the model, and $\mathsf{SO}(10)$ is broken, the correspondingly charged particles form irreducible representations of the SM gauge group, see for instance [186, 196] for related discussion in the $\mathsf{SU}(N)$ case.

### 6.1.3 Choice of embeddings

At tree-level, we have two global flavour groups available for embedding the SM via $\mathsf{SO}(10)$. Firstly, we can embed the the standard model into the global symmetry group $\mathsf{SO}(N_f - N)_f$. Contributions to the beta functions would come from the matter superfield represenations under the new $\mathsf{SO}(N_f - N)_f$ gauge group. Alternatively, we might embed the SM into the global symmetry $\mathsf{SO}(N)_f$. As we have seen, the vacuum of the quantum theory has the global symmetries $\mathsf{SO}(N)_D \times \mathsf{SO}(N_f - N)_f$. For this embedding, the beta function contributions are from matter representations under the $\mathsf{SO}(N)_f$ gauge group.

The gauged $\mathsf{SO}(N)_c$ is infrared free for $N_f > 3(N-2)$. The electric description is asymptotically free for $N_f < \frac{3}{2}(N_c - 2)$. If we identify $\mathsf{SO}(N_f - N)_f = \mathsf{SO}(10)$ and weakly gauge, then we find $N_f - N = 10$ and since $N = N_f - N_c + 4$ this leads to $N_c = 14$. These conditions are met when $12 \leq N_f \leq 18$ where the corresponding N is in the range of $8 \geq N \geq 2$.

Embedding the SM into $\mathsf{SO}(N)_f = \mathsf{SO}(10)$ results in the constraint $N_f > 24$ which yields $N_c > 18$. Of course we may also embed $\mathsf{SO}(10)$ into a subset of a larger flavour symmetry group. Identifying the full flavour group is merely the minimal choice when exploring these models. For a recent example of embedding into a subset, in particular embedding $\mathsf{SU}(5)$ into a weakly gauged $\mathsf{SU}(6)$ flavour symmetry, see for instance [197].

## 6.2 Contribution to gaugino masses

Due to its large unbroken flavour symmetries, the 'vanilla' ISS model provides a convenient framework for direct gauge mediation constructions. However, for model building purposes one has to overcome two 'obstacles' of the ISS model which are related to massless particles in the spectrum. Firstly, there is an (accidental) R-symmetry which needs to be broken in order to obtain non-zero gaugino masses and secondly, there are massless Goldstone bosons by spontaneous broken global flavour symmetries (even in the gauged theory). In order to avoid these issues one has to break the R-symmetry spontaneously or explicitly and some of the global symmetries explicitly. Following the original ISS work, a lot of deformations of the pure model were discussed in the literature, see for instance [186, 187, 196, 198–201]. Many of the studied deformed ISS constructions have a signature of heavy squarks and lighter gaugino masses. In general, these SM mass types are soft terms for the supersymmetric Standard Model which originate from gauge interactions with the messengers fields of the hidden sector. In this section we will focus on the gaugino masses, as being generally light, they pose the initial phenomenological concern.

The vector superfield of a (hidden sector) $\mathsf{SO}(10)$ gauge group is in the antisymmetric **45** (adjoint) representation which is traditionally broken at an energy scale $M_{\text{GUT}}$ that we assume to be far above the SUSY breaking scale. By the super-Higgs mechanism, the gauge superfields get a mass by eating the scalar and fermionic components of a Higgs chiral superfield. Thus the gauge bosons and gauginos of the broken hidden sector gauge group both have masses at the $M_{\text{GUT}}$ scale. We are thus left with the issue of the gaugino masses of the SM. Regardless of the super-Higgs mechanism, all the gauginos of the SM parent gauge group will get equal contributions to their masses from the messenger fields.

ISS models have multiple messengers. The fundamental messengers $(\rho, Z)$ are the major contributor to the gaugino masses $m_\lambda$. These have been the focus of much of the gauge mediation literature, for an overview see for instance [121, 178–180, 202–204]. Their contribution to gaugino and sfermion masses depends on the vev of $X$, which is non-zero only if R-symmetry breaking deformations are added to the vanilla ISS model, and also on $F_X$. Explicit examples will be discussed in the Sections 6.4.2, 7.1.1, 7.2.3 and 7.3.1 . The other contributions are either from $X$ or from $(Y, \chi_S)$. A concise method for calculating this contribution is given by the general gauge mediation approach [184]. The gaugino mass contribution is calculated from the two point function of the fermionic component of the gauge current superfield[2].

We follow this approach and start by calculating the gaugino mass contribution from the $X$ pseudo-modulus. When identifying $\mathsf{SO}(N_f - N)_f = \mathsf{SO}(10)$, $X$ is in the symmetric representation of $\mathsf{SO}(10)$. Schematically the scalar mass squared matrix is:

$$\left(X^\dagger X\right) M_X^2 \begin{pmatrix} X \\ X^\dagger \end{pmatrix} , \qquad M_X^2 \propto \frac{h^4}{64\pi^2} \begin{pmatrix} \mu^2 & \langle X \rangle^2 \\ \langle X \rangle^2 & \mu^2 \end{pmatrix} \otimes \mathbb{I}_{N_f - N} \otimes \mathbb{I}_{N_f - N} \tag{6.9}$$

The diagonal and off-diagonal terms are both found from computing the one loop Coleman Weinberg potential. The diagonal components arise from the pure ISS superpotential. The off -diagonal components are proportional to $|W_{\rho\rho}|^4$, where $W_{\rho\rho}$ is the double derivative of the superpotential with respect to $\rho$. This type of term is classically zero in the ISS model but may be non-vanishing at one loop if there are deformations to the ISS model as we will see in the following sections. The mass matrix has two mass eigenstates $m_\pm$. There is also a mass-term for the fermionic components of $X$ with a fermionic mass eigenstate $m_\psi$. The fermionic mass $m_\psi$ may found by taking the $\mathrm{STr} M^2 = 0 = m_+^2 + m_-^2 - 2m_\psi^2$ and obtain the fermionic mass from the known scalar masses. The result for the pure ISS model is

$$m_\psi^2 = \frac{h^4 \mu^2}{64\pi^2} . \tag{6.10}$$

Computing the two point function and using Ward identities (see Appendix C) we find,

$$\begin{aligned} m_{\lambda_r} &= 2 M_\psi g_r^2 R[X] C(\mathbf{r}) \left( D(x; m_+) - D(x; m_-) \right) D(x; m_\psi) \\ &= 2 M_{\psi_X} {g_r}^2 R[X] C(\mathbf{r}) \int \frac{d^4 k}{(2\pi)^4} \left( \frac{1}{k^2 + m_+^2} - \frac{1}{k^2 + m_-^2} \right) \frac{1}{k^2 + m_\psi^2} \\ &\propto \left(\frac{2\alpha_r}{4\pi}\right) R[X] C(\mathbf{r}) \frac{h^2 \langle X \rangle^2}{8\pi\mu} \qquad \text{for} \quad \langle X \rangle^2 < \mu^2 \end{aligned} \tag{6.11}$$

where $C(\mathbf{r})$ is the quadratic Casimir of the representation $\mathbf{r}$, in this case the symmetric of $\mathsf{SO}(10)$, $C(sym) = 12$. $R[X]$ is the rank of the field $X$. The subscript $r$ on the coupling $g_r$ denotes the gauge group associated to each coupling, such as $g_3$ of $SU(3)$.

Let us now look at the $(Y, \chi_s)$ sector. When identifying $\mathsf{SO}(N)_D = \mathsf{SO}(10)$, these fields are charged under the standard model GUT parent and we should expect them to behave as messengers as well. The terms depending on $Y$ and $\chi$ are of the form

$$W_{ISS} \supset h \chi^T Y \chi . \tag{6.12}$$

We see that these fields do not behave like the fundamental messengers which have a coupling of the form $W_{ISS} \supset \rho^T X \rho$ where $X$ was just a background superfield. In this case we would apply

[2] The interested reader may follow Appendix C where we review these techniques in the light of a $\mathsf{SO}(N)$-based model.

the methods [203] for multi-messengers where the fermion messenger mass matrix parameterised by the superfields $(Y, \chi_s)$ is

$$W = \left(\chi^T Y^T\right) M \begin{pmatrix} \chi \\ Y \end{pmatrix} , \qquad M = \begin{pmatrix} h\langle Y \rangle & h\mu \\ h\mu & 0 \end{pmatrix} \otimes \mathbb{I}_N \otimes \mathbb{I}_N . \tag{6.13}$$

Now, the field $\chi$ achieves a vev by requiring $F_Y = 0$ in order to minimise the scalar potential. For $Y$ to obtain a vev at the minimum requires setting $F_\chi = 0$ in general. The result is that these fields, although possibly charged under the standard model GUT group, cannot generate gaugino or sfermion contributions [203]. One may speculate that a complicated deformation of ISS with magnetic quarks may give a vev to $Y$ and achieve $F_\chi \neq 0$. In that case we would apply the methods [203] for multi-messengers and there would be no supression of gaugino masses at first order in $F$, despite a zero in the fermion mass matrix.[3]

## 6.3 Spontaneous versus explicit R-symmetry breaking

A R-symmetry of the superpotential prevents gaugino mass terms from the messengers of the magnetic description of the ISS model. For non-zero gaugino masses we have to then include R-symmetry breaking terms. The key motivation of ISS models is that they satisfy Seiberg's dual descriptions at the (trivial) IR fixed point. So any deformations should be from irrelevant operators that do not add new degrees of freedom (new fields) into the superpotential. The general approach is to add irrelevant operators to the electric description which will be parametrically suppressed.

In $\mathsf{SU}(N)$ models one has a choice between spontaneous and explicit R-symmetry breaking when adding deformations to the model[4]. For spontaneous R-symmetry breaking one requires that some of the fields in the ISS model are $R \neq 0$ or $R \neq 2$ but that the superpotential still has $R[W] = 2$. If we consider specifically the ISS model, the first term in the superpotential is

$$W_{ISS} \supset h\mathrm{Tr}[\tilde{\varphi}\Phi\varphi] \tag{6.14}$$

where $\tilde{\varphi}$ may signify the antifundamental of $\varphi$ in the $\mathsf{SU}(N)$ case, or simply transpose in the $\mathsf{SO}(N)$ and $\mathsf{Sp}(N)$ cases. If we also consider the $\mathrm{Tr}\Phi$ linear term in $W_{\mathrm{ISS}}$ we obtain two constraints

$$R[\tilde{\varphi}] + R[\Phi] + R[\varphi] = 2 \quad R[\Phi] = 2 \to R[\tilde{\varphi}] = -R[\varphi] \ . \tag{6.15}$$

In $\mathsf{SO}(N)$ and $\mathsf{Sp}(N)$ models the $\tilde{\varphi}$ signifies transpose such that the constraints can only ever be satisfied by $R[\varphi] = 0$. So we see the only explicit R-symmetry may be used for $\mathsf{SO}(N)$ and $\mathsf{Sp}(N)$ models.[5]

In $\mathsf{SO}(N)$ we can use the invariant two index Kronecker ($\delta_{\alpha\beta}$) and the Levi-Civita tensor ($\epsilon_{\alpha_1 \dots \alpha_N}$) to build terms that explicitly break R-symmetry using the dual magnetic quarks, explicitly the $\chi$ component fields of the magnetic quarks $\varphi$. For $\mathsf{SO}(N) = \mathsf{SO}(2)$ we may have a

[3] The interested reader may note that the zero in the fermion mass matrix for $(Y, \chi_s)$ can be filled by a multitrace deformation of the magnetic meson $\mathrm{Tr}\Phi^2$.

[4] See also an example of spontaneous R-symmetry breaking at two loops [148].

[5] However, see Section 4 of [145] for a spontaneous breaking of $U(1)_R$ for $\mathsf{SO}(N)$ involving $D$ terms and breaking of the $\mathsf{SO}(N)$ symmetry for an O'Raifeartaigh model.

superpotential deformation

$$\delta W = hk\delta_{st}\delta_{\alpha\beta}\varphi^s_\alpha\varphi^t_\beta + hm\epsilon_{\alpha\beta}\epsilon_{st}\varphi^s_\alpha\varphi^t_\beta \tag{6.16}$$

where the $s, t$ indices are of $\mathsf{SO}(2)_{mag}$ and the $\alpha, \beta$ are from $\mathsf{SO}(N)_f$. This is the deformation used in [199–201] for $\mathsf{SU}(5)$ models. Baryon deformations of this type will give a vev to the $Y$ field. This will effect the scalar masses of $(Y, \chi)$ but in general will have *no effect on gaugino masses* when embedding into either $\mathsf{SO}(N)_D$ or into $\mathsf{SO}(N_f - N)_f$ at leading order in $F/M^2$.

### 6.3.1 Tree level potential for a SO(2) model

Let us consider a specific example, namely the $\mathsf{SO}$-analog to the $\mathsf{SU}(5)$ models mentioned above. We first analyse the dual quark deformations from (6.16). Setting $N = 2$ and $N_f - N = 10$, we take (6.4) and (6.16) and compute the tree level potentials for $\mathsf{SO}(2)_{mag} \times \mathsf{SO}(2)_f \times \mathsf{SO}(N_f - N)_f$

$$W = hTr\varphi^T\Phi\varphi - h\mathrm{Tr}[\mu^2\Phi] + hk\delta_{st}\delta_{\alpha\beta}\varphi^T_{s\alpha}\varphi_{\beta t} + hm\epsilon_{\alpha\beta}\epsilon_{st}\varphi^T_{s\alpha}\varphi_{\beta t}. \tag{6.17}$$

Using the $\mathsf{SO}(2)_f \times \mathsf{SO}(10)_f$ symmetry we may diagonalise the matrix $\mu^2_{AB}$ as

$$\mu^2_{AB} = \begin{pmatrix} \mu^2\mathbb{I}_2 & 0 \\ 0 & \hat{\mu}^2\mathbb{I}_{10} \end{pmatrix}_{AB}. \tag{6.18}$$

The resulting tree level potential is

$$\begin{aligned} V_F = &\sum_{\alpha\beta} |h\chi^T_{s\alpha}\chi_{\beta s} - h\mu^2\delta_{\alpha\beta}|^2 + \sum_{a\beta} |2h\rho^T_{sa}\chi_{\beta s}|^2 \\ &+ \sum_{ab} |h\rho^T_{sa}\rho_{bs} - h\hat{\mu}^2\delta_{ab}|^2 + \sum_{sa} |2hZ_{a\beta}\chi_{\beta s} + 2hX_{ab}\rho_{bs}|^2 \\ &+ \sum_{s\alpha} |2hY_{\alpha\beta}\chi_{\beta t} + 2hZ^T_{\alpha b}\rho_{bs} + 2hk\delta_{\alpha\beta}\chi_{\beta s} + 2hm\epsilon_{\alpha\beta}\chi_{\beta s}|^2 . \end{aligned} \tag{6.19}$$

The indices are $A = (\alpha, a)$ and $B = (\beta, b)$ running over all $N_f$ with $\alpha, \beta$ running over the first $N$ and $a, b$ running over the $N_f - N$. We apply the rank condition of ISS and set $\rho = 0$ in a first step. The potential becomes

$$\begin{aligned} V_F = &\sum_{\alpha\beta} |h\chi^T_{s\alpha}\chi_{\beta s} - h\mu^2\delta_{\alpha\beta}|^2 + \sum_{ab} |h\hat{\mu}^2\delta_{ab}|^2 + \sum_{sa} |2hZ_{a\beta}\chi_{\beta s}|^2 \\ &+ \sum_{s\alpha} |2hY_{\alpha\beta}\chi_{\beta t} + 2hk\delta_{\alpha\beta}\chi_{\beta s} + 2hm\epsilon_{\alpha\beta}\chi_{\beta s}|^2. \end{aligned} \tag{6.20}$$

The contributions from the F-term $F_Y$ is minimised ($F_Y = 0$ for $\chi = \mu$) when

$$\chi^T_{s\alpha}\chi_{\beta s} = \mu^2\mathbb{I}_{\alpha\beta} \tag{6.21}$$

where $\alpha, \beta$ run over $N$. For $\mathsf{SO}(N)$ models $\chi^T_{s\alpha}$ is just the transpose of $\chi_{\beta s}$, they are not independent fields. To minimise, we further set $Z = 0$. Notice also that $V_F$ is independent of $X_{ab}$ and these are the pseudo-moduli and hence, the flavour group $\mathsf{SO}(10)$ is unbroken. Preliminarily we choose the local minimum to occur at a vev

$$\langle\chi\rangle = \Sigma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.22}$$

under the constraint

$$\Sigma^2 + \theta^2 = \mu^2. \tag{6.23}$$

The $\chi$ fields have excitations that are constrained to live on a circle. As expected, both the symmetric and antisymmetric piece preserve the $\mathsf{SO}(2)_D$ symmetry. We have still F-term constraints on $Y$ to minimise. Let us initially set $\langle Y_{\alpha\beta}\rangle = \eta \mathbb{I}_{\alpha\beta}$. Then, the scalar potential becomes

$$\begin{aligned} V_F = 2Nh^2|\eta\Sigma + m\Sigma + k\Sigma|^2 + Nh^2|-\eta\theta - m\theta - k\theta|^2 \\ + Nh^2|\eta\theta + m\theta + k\theta|^2 + (N_f - N)h^2\hat{\mu}^4 \end{aligned} \tag{6.24}$$

where we keep in mind that $N = 2$ and $N_f - N = 10$. Using the constraint (6.23) reduces the potential $V_F$ to

$$V_F = 2Nh^2\mu^2|\eta + k + m|^2 + (N_f - N)h^2\hat{\mu}^4 \ . \tag{6.25}$$

Minimising in $\eta$ we find $\eta = -(k + m)$ and hence,

$$\begin{aligned} \langle Y_{\alpha\beta}\rangle &= -(k+m)\ \mathbb{I}_{\alpha\beta} \\ V(\hat{\mu}) &= (N_f - N)h^2\hat{\mu}^4 \ . \end{aligned} \tag{6.26}$$

So the minimum of the scalar potential is independent of the particular choices of $\Sigma$ and $\theta$, with these fields constrained to live on a circle of radius $\mu$ in field space. Choosing a particular value of $\Sigma$ and $\theta$ will break this continuous symmetry. It is clear now that the Kronecker contracted and Levi-Civita contracted terms act equivalently to the scalar potential and we may drop one of them without loss of generality. For all values of the potential, it is positive definite and non zero in terms of the $Y(\eta)$ field.

It is useful to compare this with the $\mathsf{SU}(N)$ ISS models [199–201]. In *those* models there is a runaway direction, associated with the parameterisation of the vevs of $Y$, $\chi$ and antifundamental $\bar{\chi}$ fields, which is one loop lifted. In *those* models the deformation will be significant to the $(\rho, Z)$ messenger contributions to gaugino masses and plays an important role when embedding into both flavour groups.

## 6.4 The KOO deformation

In this section we keep the delta contracted deformation

$$\delta W = hk\delta_{st}\delta_{\alpha\beta}\varphi^s_\alpha\varphi^t_\beta \tag{6.27}$$

of the previous section, which is valid for any $\mathsf{SO}(N)$ and not just $\mathsf{SO}(2)$. We add to this a new deformation. As has previously been pointed out in [186], to obtain gaugino mass contributions from the fundamental messengers ($\rho$, $Z$) at first order in $F_X$ (the F-component of pseudo-modulus $X$) one must add a deformation that adds a mass term to the diagonal of the messenger mass matrix. The new term in the superpotential (from now on we will refer to this as the KOO deformation) is an explicit R-symmetry breaking term,

$$\delta W_{KOO} = h^2 m_z \mathrm{Tr}[Z^T Z] \ . \tag{6.28}$$

The full potential we would like to analyse is therefore

$$W = h\mathrm{Tr}[\varphi^T \Phi \varphi] - h\mathrm{Tr}[\mu^2 \Phi] + hk\delta_{st}\delta_{\alpha\beta}\varphi^T_{s\alpha}\varphi_{\beta t} + h^2 m_z \mathrm{Tr}[Z^T Z] \tag{6.29}$$

where we also apply the choice of (6.7). Hence, the scalar potential is given by

$$\begin{aligned} V_F = & \sum_{\alpha\beta} |h\chi^T_{s\alpha}\chi_{\beta s} - h\mu^2\delta_{\alpha\beta}|^2 + \sum_{a\beta} |2h\rho^T_{sa}\chi_{\beta s} + 2h^2 m_z Z^T_{\beta a}|^2 \\ & + \sum_{ab} |h\rho^T_{sa}\rho_{bs} - h\hat{\mu}^2\delta_{ab}|^2 + \sum_{sa} |2hZ_{a\beta}\chi_{\beta s} + 2hX_{ab}\rho_{bs}|^2 \\ & + \sum_{s\alpha} |2hY_{\alpha\beta}\chi_{\beta t} + 2hZ^T_{\alpha b}\rho_{bs} + 2hk\delta_{\alpha\beta}\chi_{\beta s}|^2 \ . \end{aligned} \tag{6.30}$$

We can follow the usual steps of minimising the potential. We find an ISS type minimum with an energy $V(\hat{\mu}) = (N_f - N)|h^2\hat{\mu}^4|$ at

$$\langle\rho\rangle = 0 \quad \langle Z\rangle = 0 \quad \langle Y\rangle = -k\mathbb{I}_N \ . \tag{6.31}$$

As before, $X$ is a modulus of the classical potential. The deformation introduces other metastable minima into the theory [186] which are found at the following values

$$\begin{aligned} &\chi_{\alpha s}\chi^T_{s\beta} = \mu^2\mathbb{I}_N \\ &\rho_{as}\rho^T_{sb} = \frac{h^2 m_z^2}{\mu^2} ZZ = \mathrm{diag}(\hat{\mu}^2 ... \hat{\mu}^2, 0...0)_{N_f - N} \\ &X_{ab} = -\frac{\mu^2}{m_z}\mathrm{diag}(1...1, \hat{x}...\hat{x})_{N_f - N} \\ &Y_{\alpha\beta} = -(\frac{\hat{\mu}^2}{m_z} + k)\mathbb{I}_N \\ &V_{low} = (N_f - N - n)|h\hat{\mu}^2|^2. \end{aligned} \tag{6.32}$$

The label $n$ runs from 1 to $N$. The condition for $\rho_{as}\rho^T_{sb}$ has $\hat{\mu}^2$ for the first $N$ entries corresponding to the rank condition. The remaining $N_f - 2N$ entries of the total $N_f - N$ are zero and the $\hat{\mu}^2$ of that F-term generate $V_{low}$. The $\hat{x}$ signify classical moduli. These extra SUSY broken minimum only arise because the KOO deformation gives an extra degree of freedom to fix the $Z$ minimum in the scalar potential. The deformation fills the zero of the scalar mass matrix giving gaugino masses at first order in $F_X$. It is also an explicit R-symmetry breaking term, giving a nonzero vev to the pseudo-modulus $X$ which is also crucial for non-vanishing gaugino masses.

### 6.4.1 Messenger masses with KOO deformation

As we have stated before, the deformations using the magnetic quarks (e.g. a term $hk\delta\delta\varphi\varphi$) does not effect the mass matrix of $(\rho, Z)$ and does not effect the vev of $X$. The KOO deformation of (6.28) does effect both of these quantities. In this section we examine the messenger sector of $(\rho, Z)$. We beginn by noting that for a general set of fields the fermionic mass matrix is given by [205]

$$m_{1/2} = \begin{pmatrix} W^{ab} & \sqrt{2}iD^\beta_a \\ \sqrt{2}iD^\alpha_b & 0 \end{pmatrix} \tag{6.33}$$

such that the fermionic mass squared matrix is

$$m_{1/2}^2 = \begin{pmatrix} W^{ab}W_{bc} + 2D^{\alpha a}D_c^\alpha & -\sqrt{2}W^{ab}D_b^\beta \\ -\sqrt{2}D^{\alpha b}W_{bc} & 2D^{\alpha c}D_c^\beta \end{pmatrix} \tag{6.34}$$

where $W_a = \partial W/\partial \Phi_a$, $W^a = \partial W^\dagger/\partial \Phi_a^\dagger$ and $\alpha, \beta$ are gauge group indices such that $D_a^\alpha = \partial D^\alpha/\partial \Phi_a$. The general scalar mass squared matrix is

$$m_0^2 = \begin{pmatrix} W^{ab}W_{bc} + D^{\alpha a}D_c^\alpha + D_c^{\alpha a}D^\alpha & W^{abc}W_b + D^{\alpha a}D^{\alpha c} \\ W_{abc}W^b + D_a^\alpha D_c^\alpha & W_{ab}W^{bc} + D_a^\alpha D^{\alpha c} + D_a^{\alpha c}D^\alpha \end{pmatrix} . \tag{6.35}$$

In the vacuum, the D terms are vanishing. To compute the matrices for $(\rho, Z)$, we choose to parameterise the fermion mass matrix by $\psi = (\rho_{as}, Z_{a\beta})$. The scalar mass squared matrix is parameterised by $(\rho_{as}, Z_{a\beta}, \rho_{as}^*, Z_{a\beta}^*)$. We choose the vevs to be

$$\langle X_{ab} \rangle = X_0 \mathbb{I}_{ab} \quad \langle Y_{\alpha\beta} \rangle = \eta \mathbb{I}_{\alpha\beta} \quad \langle \chi_{\beta s} \rangle = \mu \mathbb{I}_{\beta s} . \tag{6.36}$$

Here and in the following we have switched off the $\theta$-dependence of the $\chi$ vev by setting $\theta = 0$ to achieve analytic results. Hence, for both embeddings the fermion mass matrix is computed to be

$$m_{1/2} = \mathbb{I}_{N_f - N} \otimes \mathbb{I}_N \otimes 2h \begin{pmatrix} X_0 & \mu \\ \mu & hm_z \end{pmatrix} . \tag{6.37}$$

The two fermionic eigenvalues are

$$M_\pm = h(hm_z + X_0 \pm \sqrt{4\mu^2 + (-hm_z + X_0)^2}) . \tag{6.38}$$

Here, the scalar mass matrix is given by

$$m_0^2 = \begin{pmatrix} W^{ab}W_{bc} & W^{abc}W_b \\ W_{abc}W^b & W_{ab}W^{bc} \end{pmatrix} \tag{6.39}$$

where

$$W^{ab}W_{bc} = \mathbb{I}_{N_f - N} \otimes \mathbb{I}_N \otimes 4h^2 \begin{pmatrix} X_0 X_0{}^* + \mu^2 & \mu(X_0{}^* + hm_z) \\ \mu(X_0 + hm_z) & h^2 m_z^2 + \mu^2 \end{pmatrix} \tag{6.40}$$

and

$$W^{abc}W_b = W_{abc}W^b = \mathbb{I}_{N_f - N} \otimes \mathbb{I}_N \otimes 2h^2 \begin{pmatrix} -\hat{\mu}^2 & 0 \\ 0 & 0 \end{pmatrix} . \tag{6.41}$$

The four independent scalar mass squared eigenvalues are

$$\begin{aligned} m_{1,\pm}^2 &= h^2 \big[ -\hat{\mu}^2 + 4\mu^2 + 2h^2 m_z^2 + 2|X_0|^2 \\ &\quad \pm \sqrt{16\mu^2(hm_Z + X_0)(hm_Z + X_0{}^*) + (2h^2 m_Z^2 - (-\hat{\mu}^2 + 2X_0 X_0{}^*))^2} \,\big] \\ m_{2,\pm}^2 &= h^2 \big[ \hat{\mu}^2 + 4\mu^2 + 2h^2 m_z^2 + 2|X_0|^2 \\ &\quad \pm \sqrt{16\mu^2(hm_Z + X_0)(hm_Z + X_0{}^*) + (2h^2 m_Z^2 - (\hat{\mu}^2 + 2X_0 X_0{}^*))^2} \,\big] . \end{aligned} \tag{6.42}$$

Using the messenger spectrum, we can calculate the corresponding Coleman-Weinberg potential (5.75) for the messenger correction at one-loop level. We find

$$X_0 = \langle X \rangle = \frac{1}{2} hm_z, \quad M_X^2 = \frac{h^4 \hat{\mu}^2}{12\mu^2\pi^2} \begin{pmatrix} \hat{\mu}^2 & -\frac{9}{40} X_0{}^2 \\ -\frac{9}{40} X_0{}^2 & \hat{\mu}^2 \end{pmatrix} \tag{6.43}$$

where we have expanded to first order in $h, m_z$ and in $\hat{\mu}/\mu$ up to first non-vanishing order. We have suppressed factors of $N(N_f - N)$ in the expression for $M_X^2$ coming from tracing over degenerate mass eigenvalues.

### 6.4.2 SM Gaugino and sfermion masses

Having identified the mass spectrum of the messengers $(\rho, Z)$ and also $X$ we can now calculated their contributions to the gaugino and sfermion masses of the SM. Generalisations of the wavefunction renormalisation technique, in the regime that the $F$-term of the pseudo-modulus is smaller than the messenger scale, give analytic expressions for the gaugino and sfermion masses [183]. For the gauginos one has in general

$$m_{\lambda_r} = \frac{\alpha_r}{4\pi}\Lambda_G \equiv \frac{\alpha_r}{4\pi}F_X \sum_i \frac{\partial_X m_i}{m_i} \tag{6.44}$$

where $m_i$ are the eigenvalues of the corresponding fermion mass matrix . Hence, the gaugino masses from the $(\rho, Z)$ messengers are found to be

$$m_{\lambda_r,(\rho,Z)} = \frac{\alpha_r h^2 \hat{\mu}^2 m_z}{2\pi(\mu^2 - h m_z X_0)} \tag{6.45}$$

where to simplify expressions we have defined $X_0 = \langle X \rangle$. The contribution to the sfermions masses are given by

$$\begin{aligned} m_{\tilde{f}}^2 &= 2\sum_r C_{\tilde{f}}^r (\frac{\alpha_r}{4\pi})^2 \Lambda_S^2 \\ \Lambda_S^2 &= \frac{1}{2}|F_X|^2 \sum_i |\frac{\partial_X M_i}{M_i}|^2 \end{aligned} \tag{6.46}$$

where $\alpha_r$ is the gauge coupling at the messenger scale and $C_{\tilde{f}}^r$ denotes the quadratic Casimir of the irrep $\tilde{f}$ in the gauge group factor labelled by $r$. In the case of the $(\rho, Z)$ sector we find

$$\Lambda_{S,(\rho,Z)}^2 = \frac{(h^2\hat{\mu}^4)[h^4 m_z^4 + 2\mu^4 - 2h^3 m_z^3 X_0 - 2h m_z \mu^2 X_0 + h^2 m_z^2(4\mu^2 + X_0^2)]}{[(\mu^2 - h m_z X_0)^2(4\mu^2 + (-h m_z + X_0)^2)]} \tag{6.47}$$

The KOO deformation $h^2 m_z \mathrm{Tr}[Z^T Z]$ is a mass term for some of the messengers of the theory. As is highlighted in [183], the introduction of messenger masses changes the ratio

$$\frac{\Lambda_G^2}{\Lambda_S^2} = N \to N_{eff}(h, \mu, m_z, X_0) \tag{6.48}$$

where $N_{eff}$ is the effective messenger number which can continuously vary from 0 to $N$ inclusive. For the $(\rho, Z)$ messengers we find

$$\frac{\Lambda_G^2}{\Lambda_S^2} = \frac{4h^2 m_z^2[4\mu^2 + (-h m_z + X_0)^2]}{[h^4 m_z^4 + 2\mu^4 - 2h^3 m_z^3 X_0 - 2h m_z \mu^2 X_0 + h^2 m_z^2(4\mu^2 + X_0^2)]} \ . \tag{6.49}$$

In our case it ranges from 0 to 4 which is compatible with $\mathsf{SU}(N)$ models (see for instance [206]).

Let us move on by discussing the effect of the pseudo-modulus on the gaugino and sfermion masses. From (6.11) we find that the messenger field $X$ gives a contribution to the gaugino masses as (see also Appendix C)

$$m_{\lambda_r,X} = \frac{\alpha_r}{4\pi}C(\mathbf{r})R[X]\Lambda_{G,X} \ , \qquad \text{with } \Lambda_{G,X} = 2\sqrt{3}\left[\frac{3h^2\hat{\mu}^2 X_0^2}{80\pi\mu^3}\right] \ . \tag{6.50}$$

Similarly, we find for the contributions to the sfermion masses

$$m^2_{\tilde{f},X} = 2\sum_r C^r_{\tilde{f}}(\frac{\alpha_r}{4\pi})^2 C(\mathbf{r})R[X]\Lambda^2_{S,X} \ , \quad \text{with } \Lambda^2_{S,X} = \frac{27h^4\hat{\mu}^4X_0^4}{6400\mu^6\pi^2} \ . \tag{6.51}$$

### 6.4.3 Constraints on parameters

It is also useful to briefly discuss how to constrain the parameter space for comparison between differing models. More accurate constraints on the parameters would involve a more detailed phenomenological survey of the model. Here, we just review closely the constraints used in [196] where a direct mediation construction based on the KOO deformation is discussed in detail. The model discussed in the SO context has a dimensionless coupling $h$ and further five parameters $(\mu, \hat{\mu}, m_z, k, \Lambda_m)$.

- $\frac{h}{4\pi}$ is used for a perturbative expansion; we require h to be at most $\sim O(1)$.
- Cosmological bounds on the gravitino mass give

$$m_{3/2} = \frac{F}{\sqrt{3}M_{pl}} < 16 \text{ eV} \ , \tag{6.52}$$

$$h\mu^2 = \frac{F}{N_f - N} < \frac{\sqrt{3}M_{pl}16\text{eV}}{N_f - N} < (146\text{TeV})^2 \ , \tag{6.53}$$

  where the second bound follows from the F-term of the pseudo-modulus. $M_{pl}$ is the reduced Planck mass. This is of the same order as in [196].

- We can determine $\hat{\mu}$ from the ratio of $\frac{\mu}{\hat{\mu}}$ which controls the longevity of the metastable vacuum from

$$S \sim (\frac{\mu}{\hat{\mu}})^4(\frac{\mu}{m_z})^4. \tag{6.54}$$

- The scalar masses are completely equivalent to those found in Appendix A of [196]. In particular there is a 'no Tachyon' constraint

$$|\mu^2 \pm hm_zX|^2 > \hat{\mu}^2(\mu^2 + h^2m_z^2). \tag{6.55}$$

  This will give a constraint on $m_z$ dependent on the values of h, $\mu$ and $\hat{\mu}$.

- We can determine the cutoff scale $\Lambda_m$ from the longevity of the metastable vacuum to the non-perturbative SUSY vacuum. Normally we expect that

$$|\epsilon| = |\frac{\mu}{\Lambda_m}| \ll 1 \tag{6.56}$$

  is sufficient suppression of tunneling to the non-perturbative vacuum. In cases where there is no large hierarchy between $\Lambda_m$ and $\mu$, (e.g in order to avoid low energy Landau poles in SM gauge coupling constants) then it is suffices to take $\hat{\mu}$ sufficiently smaller than $\mu$ to avoid a too short lifetime of the metastable vacuum.

- The term $k$ plays no role for either gaugino masses or vacuum stability. It plays no significant role to visible sector phenomenology.

## 6.5 The non-perturbative potential

In this section we explore the non-perturbative potential of the macroscopic theory. This is perturbative in the electric theory. When gauging the $\mathsf{SO}(N)_c$ symmetry group and taking the model to be IR free when $N_f > 3(N-2)$, one finds a non-perturbative potential (compare with (5.77) in the $\mathsf{SU}$ case)

$$W_{dyn} = (N-2)(h^{N_f}\Lambda_m^{3(N-2)-N_f}\det\Phi)^{1/(N-2)} \ . \tag{6.57}$$

The supersymmetry preserving vacua are found at

$$\langle\varphi\rangle = 0 \quad \langle\Phi\rangle = \frac{\mu}{h}(\frac{\Lambda_m}{\mu})^{\frac{N_f-N+2}{3N-N_f-4}}\,\mathbb{I}_{N_f}. \tag{6.58}$$

When $\epsilon = \frac{\mu}{\Lambda_m} << 1$ the metastable vacuum will be exponentially long lived. This formula being valid when the electric quark masses are all equal. In general the vev of $X$ and $Y$ is fixed by solving $\frac{\partial W}{\partial X} = \frac{\partial W}{\partial Y} = 0, \langle\varphi\rangle = 0$ where the superpotential is the full classical one plus the dynamically generated part. We find

$$\langle X\rangle = h^{\frac{N_f-2N+4}{N-N_f-2}}\,\hat{\mu}^{\frac{4}{N-N_f-2}}\,\mu^{-\frac{2N}{N-N_f-2}}\,\Lambda_m^{-\frac{(N_f-3(N-2))}{N-N_f-2}}$$

$$\langle Y\rangle = h^{\frac{N_f-2N+4}{N-N_f-2}}\,\hat{\mu}^{\frac{2(N-N_f)}{N-N_f-2}}\,\mu^{\frac{2(-2N+N_f+2)}{N-N_f-2}}\,\Lambda_m^{-\frac{(N_f-3(N-2))}{N-N_f-2}} \ . \tag{6.59}$$

The above SUSY minimum applies to the case $N \neq 2$. For the specific embedding where $N = 2$, $N_f = 12$ one has $N_f = N_c - 2$ such that the macroscopic (IR) theory is in the coulomb phase [129]. The IR ISS superpotential should be multiplied by an arbitrary function $f(t)$ where

$$t = \det[\Phi]/\Lambda^{24} \tag{6.60}$$

subject to the boundary condition $f(0) = 1$. To leading order in $\Phi$ the SUSY broken vacuum is independent of this function. The magnetic $\mathsf{SO}(2)_c$ is Higgsed and the unbroken electric $\mathsf{SO}(2)_c$ is confined. Thus we have a metastable SUSY broken vacuum in a confining phase. For a more detailed exploration of these and other cases one can look at the original papers [129, 207].

### 6.5.1 Lifetime of the metastable vacuum

The ISS vacuum can decay into either the secondary SUSY broken minimum or into the SUSY restored non-perturbative vacuum far away in field space. In general one can apply the techniques outlined in [171]. Here we review some analytic estimates applicable to this model when tunneling into the SUSY restored minimum. At the ISS minimum

$$V_{ISS} = (N_f - N)|h\hat{\mu}|^2 \ . \tag{6.61}$$

The value of the pseudo-modulus is found from the Coleman Weinberg potential to be

$$X_0 = \langle X\rangle = \frac{hm_z}{2} \ . \tag{6.62}$$

We can estimate the value of the local maximum of the potential by expanding the superpotential around the vevs $Z = \rho = 0$. The potential has a local maximum for $\tilde{\chi}\chi = 0$,

$$V_{Peak} = N|h\mu^2|^2 + (N_f - N)|h\hat{\mu}|^2 \approx N_f|h\mu^2|^2 \ . \tag{6.63}$$

We can also estimate the value of the pseudo-modulus at this vev by use of the Tachyon constraint (6.55) and find

$$X_0 = \langle X \rangle = \frac{\mu^2 - \hat{\mu}\sqrt{\mu^2 + h^2 m_z^2}}{h m_Z} \ . \tag{6.64}$$

We use the triangle approximation [171] when $\mu \sim \hat{\mu}$ and find for the bounce action

$$S \approx \frac{(\Delta X)^4}{V_{peak}} \approx \frac{(\langle h\Phi \rangle)^4}{V_{peak}} = \frac{h^2}{N_f}(\frac{\Lambda}{\mu})^{4(\frac{N_f - N + 2}{3N - N_f - 4})} \ . \tag{6.65}$$

For the case $\mu \neq \hat{\mu}$ we approximate the action by

$$S \approx \frac{1}{N_f} h^{-\frac{(6N - 2N_f - 12)}{N - N_f - 2}} \epsilon'^{\frac{16}{N - N_f - 2}} \epsilon^{-\frac{(12N - 4N_f - 8)}{N - N_f - 2}} \ , \tag{6.66}$$

where we have defined $\epsilon' = \hat{\mu}/\Lambda_m$ and $\epsilon = \mu/\Lambda_m$. In deriving the above expression we have assumed that $\mu$ and $\hat{\mu}$ though unequal are approximately the same order so that the approximation (6.63) is still valid.

One can compare this with $S \sim 400$ for which the lifetime of the metastable state is larger than the age of the universe. Typically, the vevs of the fields are small compared to the distance from the origin to the SUSY restored vacuum. Also the $V_{peak}$ value is in general independent of the deformations used. For these reasons, this result is consistent with the results obtained in the SU-based literature. To derive the actual value of the lifetime we will need to input the various allowed values of $\Lambda_m$, $\mu$ and $\hat{\mu}$ consistent with various other conditions which we will consider later.

When using the KOO deformation one can also tunnel from the ISS metastable state to the second SUSY broken minimum. For this particular model, with the baryon deformation switched off ($k = 0$), the calculation is completely equivalent to the numerical one carried out in appendix B of [196]. In [186] there is an analytic estimate of the bounce action, using again the triangle approximation [171], given by

$$S \sim (\frac{\mu}{\hat{\mu}})^4 (\frac{\mu}{m_z})^4. \tag{6.67}$$

Similarly to the $\mathsf{SU}(N)$ models discussed in the literature, one would expect that the metastable vacua are preferred in the thermal history of the universe, see also the discussions in Section 5.5.

# 7 Multitrace Deformations of the SO(N) ISS superpotential

In the previous chapter, we saw that certain deformations of the ISS superpotential lead to different moduli spaces. We would like to follow up on these ideas by examining an introduction of multitrace deformations of the magnetic quarks $\varphi$ and the meson field $\Phi$, compare with similar $\mathsf{SU}(N)$ deformations [208–210]. The deformations of magnetic quarks mix fields of the two global symmetry groups in which one may embed the standard model GUT. These operators have been suppressed by the strong coupling scale of the magnetic picture. Whilst we do neither seek nor supply an UV completion, we find them useful as they demonstrate how the vev of the pseudo-modulus may be shifted without changing their F-term $F_X$. In contrast, the KOO deformation of Section 6.4 demonstrated a shifting of $\langle X \rangle$ and a removal of zero entries in the fermionic messenger mass matrix.

Additionally, we explore adding multitrace deformations of the meson field to the superpotential. These deformations are irrelevant operators of the electric quarks $Q$ fields of the UV electric description which are then mapped to $\Phi$ fields in the magnetic picture. These deformations will make the supersymmetry breaking order parameter $F_X$ dependent on the field $X$ itself and the KOO deformation naturally appears as part of this meson multitrace deformation. As the scalar potential is no longer independent of $X$, it has no longer a classically flat direction in $X$ . However, one loop corrections still contribute to its minimisation.

## 7.1 Multitrace deformation of magnetic quarks

We begin our discussion by considering a deformation of the superpotential of the magnetic description which is made out of magnetic quarks. A similar deformation in the context of $\mathsf{SU}(N)$ ISS constructions was also considered in [198] where the superpotential was deformed by the term

$$\delta W = \frac{h\epsilon}{\Lambda_m} \left(\mathrm{Tr}[\varphi\varphi^T]\right)^2 \ . \tag{7.1}$$

Here, we also add a single trace term and also keep the KOO deformation of the previous chapter. Therefore, after decomposing the magnetic quark fields into their components, we add to the superpotential,

$$\begin{aligned}\delta W \propto\ & h\eta\left[(\chi_{\alpha s}\chi^T_{s\beta}\chi_{\beta t}\chi^T_{t\alpha}) + (\chi_{\alpha s}\rho^T_{sb}\rho_{bt}\chi^T_{t\alpha}) + (\rho_{as}\chi^T_{s\beta}\chi_{\beta t}\rho^T_{ta}) + (\rho_{as}\rho^T_{sb}\rho_{bt}\rho^T_{ta})\right] \qquad (7.2)\\ & + h\gamma\left[(\chi_{\alpha s}\chi^T_{s\alpha})(\chi_{\gamma t}\chi^T_{t\gamma}) + (\chi_{\alpha s}\chi^T_{s\alpha})(\rho_{ct}\rho^T_{tc}) + (\rho_{as}\rho^T_{sa})(\chi_{\gamma t}\chi^T_{t\gamma}) + (\rho_{as}\rho^T_{sa})(\rho_{ct}\rho^T_{tc})\right] \ ,\end{aligned}$$

where the field indices are the same as in Section 6.4. Calculating the derivatives of the deformation gives

$$\frac{\partial\delta W}{\partial\chi_{\beta s}} = h\eta[4\chi_{\alpha s}\chi_{\beta t}\chi^T_{t\alpha} + 4\chi_{\beta s}\rho^T_{sa}\rho_{at}] + h\gamma[4\chi_{\beta s}(\chi_{\alpha t}\chi^T_{t\alpha}) + 4\chi_{\beta s}(\rho_{ct}\rho^T_{tc})] \tag{7.3}$$

and similarly

$$\frac{\partial\delta W}{\partial\rho_{bt}} = h\eta\left[4\rho_{bs}\chi^T_{s\alpha}\chi_{\alpha t} + 4\rho_{bs}\rho^T_{sa}\rho_{at}\right] + h\gamma\left[4(\chi_{\alpha s}\chi^T_{s\alpha})\rho^T_{tb} + 4\rho_{bt}(\rho_{cs}\rho^T_{sc})\right] \tag{7.4}$$

where the numerical factors arise from relabeling of the matrix indices. The rest of the F-terms are the same as for the KOO-superpotential, especially $F_X$ does not change. In order to minimise the F terms we take the vevs to be

$$\langle\rho\rangle = \langle Z\rangle = 0 \quad \chi^T_{s\alpha}\chi_{\beta s} = \mu^2\mathbb{I}_{\alpha\beta} \quad \langle Y_{\alpha\beta}\rangle = -2(\eta+\gamma)\mu^2\mathbb{I}_{\alpha\beta}\ . \tag{7.5}$$

The minimum of the scalar potential has then an energy of

$$V_{min} = (N_F - N)|h\hat{\mu}^2|^2. \tag{7.6}$$

There is a local maximum of the potential with $\chi = 0$ and Y undetermined at

$$V = (N_F - N)|h\hat{\mu}^2|^2 + N|h\mu^2|^2\ . \tag{7.7}$$

We would like to investigate how the deformation affects the masses of the fundamental messengers $(\rho, Z)$. To that extent we follow the same reasoning as in Section 6.4.1. For this model, the fermion mass matrix is given by

$$m_{1/2} = \mathbb{I}_{N_f-N}\otimes\mathbb{I}_N\otimes 2h\begin{pmatrix} X_0 + 2\mu^2\Delta & \mu \\ \mu & hm_z\end{pmatrix} \tag{7.8}$$

where we have defined $\Delta = (\eta+\gamma)$. In particular we have $F^\dagger_\chi = 0$ and $\frac{\partial W}{\partial\rho^3}\propto\rho = 0$. For the scalar mass matrix we use (6.39) and find for the sub-blocks

$$W^{ab}W_{bc} = \mathbb{I}_{N_f-N}\otimes\mathbb{I}_N\otimes 4h^2\begin{pmatrix} |X_0{+}2\mu^2\Delta|^2{+}|\mu|^2 & \mu(X_0{}^*{+}2\mu^2\Delta{+}hm_z) \\ \mu(X_0{+}2\mu^2\Delta{+}hm_z) & |\mu|^2{+}|hm_z|^2\end{pmatrix} \tag{7.9}$$

and

$$W^{abc}W_b = W_{abc}W^b = \mathbb{I}_{N_f-N}\otimes\mathbb{I}_N\otimes 2h^2\begin{pmatrix} -\hat{\mu}^2 & 0 \\ 0 & 0\end{pmatrix}\ . \tag{7.10}$$

From these mass matrices the independent mass eigenvalues for the scalar fields are given by

$$m^2_{1,\pm} = h^2(-\hat{\mu}^2 + G) \pm \sqrt{(G-\hat{\mu}^2)^2 + H}$$

$$m^2_{2,\pm} = h^2(\hat{\mu}^2 + G) \pm \sqrt{(G+\hat{\mu}^2)^2 + H} \tag{7.11}$$

where the functions $G$ and $H$ are defined as

$$\begin{aligned} G &= 8\Delta^2\mu^4 + 2X_0X_0{}^* + 2h^2m_z^2 + 4\mu^2(1+\Delta X_0 + \Delta X_0{}^*) \\ H &= -8(2\mu^4(1-2\Delta hm_z)^2 - h^2m_z^2(\hat{\mu}^2 - 2X_0X_0{}^*) \\ &\quad +\mu^2(-\hat{\mu}^2 + 2hm_z(-1+2\Delta hm_z)(X_0+X_0{}^*)))\ . \end{aligned} \tag{7.12}$$

Expanding the Coleman-Weinberg potential we find the vev of $X$ to first order in $(\eta+\gamma)$ and first order in $m_z$, to be

$$X_0 = \frac{1}{2}hm_z - 2\mu^2(\eta+\gamma)\ . \tag{7.13}$$

Hence, we find a shift in the vev of $X$ compared to the KOO-deformed theory of the order of $\mu^2$ although the F-term $F_X$ has *not* changed. The mass matrix for $X$ is given by

$$M_X^2 = \frac{h^4}{\pi^2}\begin{pmatrix} \frac{\hat\mu^4}{12\mu^2} & -\frac{3\Delta\hat\mu^4}{40\mu^2}X_0 - \frac{3\hat\mu^4}{160\mu^4}{X_0}^2 \\ -\frac{3\Delta\hat\mu^4}{40\mu^2}X_0 - \frac{3\hat\mu^4}{160\mu^4}{X_0}^2 & \frac{\hat\mu^4}{12\mu^2} \end{pmatrix}\ . \tag{7.14}$$

These expression are sufficient to determine the contributions to gaugino and sfermion masses as was discussed in Section 6.4.2.

### 7.1.1 Gaugino and squark masses from multitrace of magnetic quarks

For this model the fermion mass matrix differs from the KOO case. However, we can still use (6.44) and find the gaugino masses to be

$$m_{\lambda_r,(\rho,Z)} = \frac{\alpha_r h^2\hat\mu^2 m_z}{4\pi(\mu^2(1-2hm_z\Delta) - hm_zX_0)}\ . \tag{7.15}$$

In exactly the same way we find the sfermion masses from (6.46) where we have in this case

$$\Lambda^2_{S,(\rho,Z)} = \frac{h^2\hat\mu^4}{2[\mu^2(-1+2hm_z\Delta)+hm_zX_0]^2[4\mu^2+(-hm_z+2\Delta\mu^2+X_0)^2]} \tag{7.16}$$
$$\times\, [(h^4m_z^4+2\mu^4)-(2h^3m_z^3+2hm_z\mu^2)(2\mu^2\Delta+X_0)+h^2m_z^2(4\Delta^2\mu^4+X_0^2+4\mu^2(1+\Delta X_0)]\ .$$

Equally, the effective messenger number is given by

$$N_{eff}^{(\rho,Z)} = 2h^2m^2(4u^2+[-hm+2\Delta u^2+X_0]^2)$$
$$\times\, \frac{1}{h^4m^4+2u^4-(2h^3m^3+2hmu^2)[2\Delta u^2+X_0]+h^2m^2(4\Delta^2u^4+X_0^2+4u^2[1+\Delta X_0])}\ . \tag{7.17}$$

In addition, we have contributions to the gaugino and sfermion masses from the messenger field $X$. By applying the techniques of Section 6.2 we find

$$m_{\lambda_r,X} = \frac{2\alpha_r}{4\pi}C(\mathbf{r})\frac{3\sqrt{3}h^2\hat\mu^2\mu^2X_0[4(\eta+\gamma)\mu^2+X_0]}{80\mu^3\pi} \tag{7.18}$$

for the gauginos and

$$m^2_{\tilde f,X} = 2\sum_r C^r_{\tilde f}(\frac{\alpha_r}{4\pi})^2 C(\mathbf{r})R[X]\Lambda^2_{S,X}\ ,\quad \Lambda^2_{S,X} = \frac{27h^4\hat\mu^4X_0^2[4\Delta\mu^2+X_0]^2}{6400\mu^6\pi^2} \tag{7.19}$$

as the contribution to the sfermion masses.

## 7.2 Multitrace deformation of the meson field

In this section we explore the multitrace deformations suggested in [208–210]. First we consider the case in which all the electric quark masses are the same. In the second case we again explicitly split the electric quark masses with the hierarchy $m << \hat{\mu} < \mu << \Lambda$ where $m$ is playing the role of $m_z$ of the previous, KOO-based models. The superpotential we explore is

$$\begin{aligned} W &= h\text{Tr}[\varphi^T \Phi \varphi] - h\text{Tr}[\mu^2 \Phi] + \frac{h^2 m}{2}\text{Tr}[\Phi^2] + \frac{h^2 m\gamma}{2}\text{Tr}[\Phi]^2 \\ &= h[\chi^T Y \chi + \rho^T X \rho + \chi^T Z^T \rho + \rho^T Z \chi] - h\mu^2 \text{Tr}Y - h\hat{\mu}^2 \text{Tr}X + h^2 m \text{Tr}[Z^T Z] \\ &+ \frac{h^2 m}{2}[(\text{Tr}[Y^2] + \gamma\text{Tr}[Y]^2) + (\text{Tr}[X^2] + \gamma\text{Tr}[X]^2)] + \frac{h^2 m\gamma}{2}\text{Tr}X\text{Tr}Y \ . \end{aligned} \tag{7.20}$$

In order to minimise the tree-level potential we take the vevs of the fields to be

$$\langle \rho \rangle = \langle Z \rangle = 0, \quad \langle Y \rangle = 0, \quad \langle \chi \rangle = q_0, \quad \langle X \rangle = X_0 \tag{7.21}$$

and find for the F-terms of the fields $X$ and $Y$

$$F_X^\dagger = -h\hat{\mu}^2 + h^2 m(1 + \gamma(N_f - N))X_0 \tag{7.22}$$

$$F_Y^\dagger = hq_0^2 - h\mu^2 + h^2 \frac{m\gamma}{2}(N_f - N)X_0 \ . \tag{7.23}$$

This determines $q_0$ to be of the form $hq_0^2 = h\mu^2 - h^2 \frac{m\gamma}{2}(N_f - N)X_0$. As before, we investigate the influence on the masses of $(\rho, Z)$. From the superpotential we find the fermionic mass matrix to be

$$m_{1/2} = \mathbb{I}_{N_f - N} \otimes \mathbb{I}_N \otimes 2h \begin{pmatrix} X_0 & q_0 \\ q_0 & hm \end{pmatrix} \tag{7.24}$$

with the independent eigenvalues

$$M_\pm = h(hm + X_0 \pm \sqrt{4q_0^2 + (-hm + X_0)^2}) \ . \tag{7.25}$$

Computing derivatives of the scalar potential one finds the scalar mass matrix to be

$$m_s^2 = 4h^2 \mathbb{I}_{N_f - N} \otimes \mathbb{I}_N \otimes \begin{pmatrix} q_0 q_0^* + X_0 X_0{}^* & hm q_0^* + q_0 X_0{}^* & F_X^\dagger/2h & 0 \\ hm q_0 + q_0^* X_0 & h^2 m^2 q_0 q_0^* & 0 & 0 \\ F_X/2h & 0 & q_0 q_0^* + X_0 X_0{}^* & hm q_0 + q_0^* X_0 \\ 0 & 0 & hm q_0^* + q_0 X_0{}^* & h^2 m^2 + q_0 q_0^* \end{pmatrix} . \tag{7.26}$$

The independent eigenvalues are given by

$$m_{1\pm}^2 = h^2 \Big\{ 2h^2 m^2 + 2X_0 X_0{}^* + 4q_0 q_0^* \pm |\hat{\mu}^2 - hmX_0(1 + \gamma(N_f - n))| + \tag{7.27}$$
$$+ \sqrt{16|hmq_0 + X_0 q_0^*|^2 + (2X_0 X_0{}^* - 2h^2 m^2 \pm |\hat{\mu}^2 - hmX_0(1 + \gamma(N_f - n))|)^2}, \Big\}$$

$$m_{2\pm}^2 = h^2 \Big\{ 2h^2 m^2 + 2X_0 X_0{}^* + 4q_0 q_0^* \pm |\hat{\mu}^2 - hmX_0(1 + \gamma(N_f - n))| - \tag{7.28}$$
$$+ \sqrt{16|hmq_0 + X_0 q_0^*|^2 + (2X_0 X_0{}^* - 2h^2 m^2 \pm |\hat{\mu}^2 - hmX_0(1 + \gamma(N_f - n))|)^2} \Big\} \ .$$

The meson deformation results in a scalar potential $V$, dependent on the field $X$ and hence, $X$ is no longer a classically flat direction. In [208] the vev of $X$ is found by considering the scalar potential plus $\mathrm{Tr}[XX^\dagger]$-terms from the one-loop Coleman-Weinberg contribution. Initially expanding the Coleman-Weinberg potential in $X$ and $X^*$ around 0 we find linear terms in the expansion as

$$V(X) = V^{\text{tree}}(X) + V^{\text{1-loop}}(X) = |F_X|^2 + BXX^* + C|X| + D|X^*| \tag{7.29}$$

where we take $F_Y = 0$ for our choices of vevs. We note that the B term (one loop mass term) will correspond to the b term of [129, 208, 209].

### 7.2.1 Equal electric quark masses

We first consider the case of equal electric quark masses, $\hat{\mu} = \mu$, and discuss two scenarios. We start by considering the deformation with switched off multitrace-terms ($\gamma = 0$) and turn then to the case with $\gamma \neq 0$. The classical plus one-loop Coleman-Weinberg potential to first order in $m$ and second order in $X$ is given by (taking $X$ to be real)

$$V(X) = V^{\text{tree}}(X) + V^{\text{1-loop}}(X) = |F_X|^2 + b\,\frac{h^4\mu^2}{\pi^2}X^2 + c\,2\frac{h^5\mu^2 m}{\pi^2}X \tag{7.30}$$

where $F_X, b$ and $c$ are in general $\gamma$-dependent factors. Furthermore, we note that there is an overall factor of $N \times (N_f - N)$ for the scalar potential coming from the degeneracy of the mass matrices.

**The $\gamma = 0$ case:** Switching off the $\mathrm{Tr}[\Phi]^2$ deformation, the coefficients $b$ of the mass term and $c$ of the linear term are given by [1]

$$\begin{aligned} b &= \log[(\frac{3^9}{256})^{\frac{1}{4}}] - 1, &(7.31)\\ c &= \frac{1}{48}\left(51 + 21\log[2h^2\mu^2] + 48\log[4h^2\mu^2] - 63\log[6h^2\mu^2]\right) . &(7.32)\end{aligned}$$

We note that we have scaled the coefficients in the full scalar potential by a factor of $1/\pi^2$ compared to [208, 209]. This potential gives

$$X_0 = \langle X\rangle = \frac{m\mu^2 - c\,h^2 m\mu^2}{h(b\mu^2 + m^2)}, \quad M_X^2 = \frac{h^4}{\pi^2}\begin{pmatrix} b\mu^2 & X_0 f + {X_0}^2 g \\ X_0 f + {X_0}^2 g & b\mu^2 \end{pmatrix}. \tag{7.33}$$

We remind the reader that again overall factors of $N(N_f - N)$ coming from tracing over degenerate mass eigenvalues are omitted. Also, we have not included tree level mass terms in the diagonal components of $M_X^2$ since they are sub leading of order $\mathcal{O}(m^2)$. The off-diagonal components of the mass matrix are given by the two functions

$$\begin{aligned} f &= \frac{mh}{24}(172 + 87\log[2h^2\mu^2] + 192\log[4h^2\mu^2] - 279\log[6h^2\mu^2]) &(7.34)\\ g &= \frac{1}{24}(86 + 30\log[2h^2\mu^2] + 132\log[4h^2\mu^2] - 162\log[6h^2\mu^2]) . &(7.35)\end{aligned}$$

Hence we find that the vev of X gets a correction proportional to $c$ due to the inclusion of the linear term into the full potential.

[1] Note that in the SU-based ISS case, $b = \log 4 - 1$ but in both SO and SU cases $b > 0$.

**The $\gamma \neq 0$ case:** We now turn on the $\gamma$-deformation and find that to first order in $m$, the mass term of $X$ does not depend on $\gamma$. The coefficient of the linear term now has a $\gamma$ dependence,

$$\begin{aligned} c &= -\frac{1}{16}\Big(17 - \gamma(N_f - N) + 7\log[2h^2\mu^2] + 4(4 - \gamma(N_f - N))\log[4h^2\mu^2] \\ &\qquad + (-21 + 6\gamma(N_f - N))\log[6h^2\mu^2]\Big) \ . \end{aligned} \tag{7.36}$$

We find for the vev of $X$ and the mass matrix $M_X$

$$X_0 = \langle X\rangle = \frac{m\mu^2(1 + \gamma(N_f - N) - c\,h^2)}{h(b\mu^2 + m^2(1 + \gamma(N_F - N))^2)}, \tag{7.37}$$

$$M_X^2 = \frac{h^4}{\pi^2}\begin{pmatrix} b\mu^2 & X_0 f_\gamma + {X_0}^2 g_\gamma \\ X_0 f_\gamma + {X_0}^2 g_\gamma & b\mu^2 \end{pmatrix}, \tag{7.38}$$

where $f_\gamma$ is given by

$$\begin{aligned} f_\gamma &= \frac{hm}{24}\Big(172 - 3(-29 + \gamma(N_f - N))\log[2h^2\mu^2] + 12(16 + \gamma(N_f - N))\log[4h^2\mu^2] \\ &\qquad - 9(31 + \gamma(N_f - N))\log[6h^2\mu^2]\Big) \end{aligned} \tag{7.39}$$

and $g_\gamma = g$. Again, $X_0$ is shifted by a correction proportional to $c$.

### 7.2.2 Unequal electric quark masses

We may now make use of the hierarchy between $\hat{\mu}$ and $\mu$ and expand in $\hat{\mu}/\mu$. The classical plus one loop potential is given by

$$V(X) = |F_X|^2 + \tilde{b}\,\frac{h^4\mu^2}{\pi^2}X^2 + \tilde{c}\,\frac{2h^5\mu^2 m}{\pi^2}X \tag{7.40}$$

and again as before $F_X$, $\tilde{b}$ and $\tilde{c}$ are $\gamma$ dependent.

**The $\gamma = 0$ case:** Switching the multitrace parts off, the coefficients of the potential are given by

$$\tilde{b} = \frac{\hat{\mu}^4}{12\mu^4} \quad \tilde{c} = -\frac{\hat{\mu}^2}{48\mu^4}(9\mu^2 + 2\hat{\mu}^2 + 6\mu^2\log[4h^2\mu^2]) \ . \tag{7.41}$$

From this potential we find

$$X_0 = \frac{m\hat{\mu}^2 - \tilde{c}\,h^2 m\mu^2}{h(b\mu^2 + m^2)} \quad M_X^2 = \frac{h^4}{\pi^2}\begin{pmatrix} \tilde{b}\mu^2 & X_0\tilde{f} + {X_0}^2\tilde{g} \\ X_0\tilde{f} + {X_0}^2\tilde{g} & \tilde{b}\mu^2 \end{pmatrix} \tag{7.42}$$

where we have for the functions $\tilde{f}$ and $\tilde{g}$

$$\tilde{f} = \frac{hm\hat{\mu}^4}{40\mu^4} - \frac{hm\hat{\mu}^2}{12\mu^2} \quad \tilde{g} = -\frac{3\hat{\mu}^4}{160\mu^4} \ . \tag{7.43}$$

Also in the case of unequal electric quark masses the tree-level mass terms of the $X$ field are sub leading and we do not include them in the mass matrix.

**The $\gamma \neq 0$ case:** The coefficient of the linear term in the considered potential is given by

$$\tilde{c} = -\frac{\hat{\mu}^2}{48\mu^4}\Big(9\mu^2[1+\gamma(N_f-N)]+\hat{\mu}^2[2+\frac{3}{2}\gamma(N_f-N)] \\ +6\mu^2[1+\gamma(N_f-N)]\log(4h^2\mu^2)\Big) \quad (7.44)$$

whereas the mass term $\tilde{b}$, stays the same to first order in $m$. Using the full scalar potential we find for the vev and the mass matrix

$$X_0 = \frac{m(\hat{\mu}^2[1+\gamma(N_f-N)]-\tilde{c}\,h^2\mu^2)}{h(\tilde{b}\mu^2+m^2[1+\gamma(N_F-N)]^2)}, \quad (7.45)$$

$$M_X^2 = \frac{h^4}{\pi^2}\begin{pmatrix} \tilde{b}\mu^2 & X_0\tilde{f}_\gamma + {X_0}^2\tilde{g}_\gamma \\ X_0\tilde{f}_\gamma + {X_0}^2\tilde{g}_\gamma & \tilde{b}\mu^2 \end{pmatrix} \quad (7.46)$$

where we have for the functions $\tilde{g}_\gamma = \tilde{g}$ and $\tilde{f}_\gamma$ is given by

$$\tilde{f}_\gamma = \frac{hm\hat{\mu}^4}{40\mu^4} - \frac{hm\hat{\mu}^2}{12\mu^2}[1+\gamma(N_f-N)] + \frac{hm\hat{\mu}^4}{48\mu^4}\gamma(N_f-N). \quad (7.47)$$

We see that also in the case of unequal electric quark masses, the vev of $X$ is shifted by a term proportional to the coefficient $\tilde{c}$.

### 7.2.3 Gaugino and squark masses from multitrace of meson

The multitrace model has the same fermion mass matrix as the generic model with KOO deformation in the previous sections. However, the $F_X$ terms are different. Using again (6.44) the gaugino masses from the fundamental messengers are given by

$$m_{\lambda_r,(\rho,Z)} = \frac{\alpha_r h^2 m(\hat{\mu}^2 + hm[-1+\gamma(N-N_f)]X_0)}{4\pi(q_0^2 - hmX_0)} \quad (7.48)$$

Similarly, using (6.46) we find the masses of the sfermions to be proportional to

$$\Lambda^2_{S,(\rho,Z)} = h^2(\hat{\mu}^2 + hm[\gamma(N-N_f)-1]X_0)^2 \\ \times \frac{[h^4m^4 + 2{q_0}^4 - 2h^3m^3X_0 - 2hm{q_0}^2X_0 + h^2m^2(4q_0^2+X_0^2)]}{2[(q_0^2 - hmX_0)^2(4q_0^2 + (-hm+X_0)^2)]}\,. \quad (7.49)$$

Hence, the effective messenger number is given by

$$N_{\text{eff}}^{(\rho,Z)} = \frac{\Lambda_G^2}{\Lambda_S^2} = \frac{2h^2m^2[(-hm+X_0)^2+4q_0^2)]}{[h^4m^4 + 2q_0^4 - 2h^3m^3X_0 - 2hmq_0^2X_0 + h^2m^2(X_0^2+4q_0^2)]}\,. \quad (7.50)$$

The gaugino masses from the $X$ messenger are

$$m_{\lambda_r,X} = C(\mathbf{r})R[X](\frac{\alpha_r}{4h^2\pi^2})\frac{[X_0f + X_0^2g]}{\sqrt{b\mu^2}} \quad (7.51)$$

and the sfermion contribution can be calculated by using

$$\Lambda^2_{S,X} = (\frac{h^4}{\pi^2})\frac{[X_0f+X_0^2g]^2}{b\mu^2}\,. \quad (7.52)$$

For the different cases one may scale $(b,f,g)$ to $(\tilde{b},\tilde{f},\tilde{g})$ and switch $\gamma$ on or off as appropriate.

## 7.3 Uplifted vacuum

A recent suggestion [211] to improve the viability of SU-based ISS models is to reduce the rank of the magnetic quark matrix: $\text{rank}(\varphi^T\varphi) = k < N$. This reduction in rank leads to new vacua which are higher in energy than the ISS vacuum of full rank and are metastable with respect to decay to the ISS vacuum. In particular, new minimal fundamental messengers (labeled $\omega$) are formed that are tachyonic in some range of the parameter space. Their contribution to gaugino masses will somewhat alleviate the problem of light gauginos and heavy sfermion (quantified by the ratio $N_{eff}$) usually found in the ISS model building literature. Here, in order to stabilise $\langle X\rangle$, deformations are added to the basic 'ISS with reduced rank' model. In this section we use meson multitrace operators. To stay away from these tachyonic directions it is important that we construct a hierarchy as

$$m \ll \hat{\mu} < \langle X\rangle < \mu \ll \Lambda\ . \tag{7.53}$$

We now apply the uplifting procedure to our SO-based ISS models. Firstly, let us focus again on the field content of the basic SO-model in (6.4). However, now we break the global symmetry groups by a reduced rank. This means we can choose to parameterise the matrix $\mu^2$ by

$$\mu^2_{AB} = \begin{pmatrix} \mu^2\mathbb{I}_k & 0 \\ 0 & \hat{\mu}^2\mathbb{I}_{N_f-k} \end{pmatrix}_{AB}\ . \tag{7.54}$$

In this case the rank $k < N$ of the magnetic quark matrix will break both the $\mathsf{SO}(N)_c$ and $\mathsf{SO}(N_f)_f$ where the magnetic quark matrix which comes from the F-term of the meson field $\Phi$ is $(\varphi^T\varphi)_{N_f\times N_f}$. The matrices are contracted on their $\mathsf{SO}(N)_c$ index. From breaking supersymmetry by the rank condition we get the following constraints from the F-term equations:

$$\begin{aligned} &(\chi^T\chi + \rho^T\rho)_{k\times k} = \mu^2\mathbb{I}_{kk} \\ &(\chi^T\sigma + \rho^T\omega)_{k\times N-k} = 0\times\mathbb{I}_{k\times N-k} \\ &(\sigma^T\sigma + \omega^T\omega)_{N-k\times N-k} = 0\times\mathbb{I}_{N-k\times N-k}. \end{aligned} \tag{7.55}$$

Using the usual vevs, the middle condition implies that $\langle\sigma\rangle = 0$. The vacuum energy without deformations is

$$V = (N_f - k)|h^2\hat{\mu}^4| \tag{7.56}$$

with $X$ again a classically flat direction. To obtain the full rank breaking and return to an ISS *type* vacuum, one would set[2]

$$(\omega^T\omega)_{N-k\times N-k} = \hat{\mu}^2\mathbb{I}_{N-k\times N-k}. \tag{7.57}$$

which is lower in energy by

$$\Delta V = (N-k)|h^2\hat{\mu}^4|. \tag{7.58}$$

As such, this reduced rank breaking is metastable with regard to the lower minimum. We keep the traditional $(\chi^T\chi)_{\alpha\beta} = \mu^2\mathbb{I}_{\alpha\beta}$ and find under all symmetry groups

[2] One should not confuse this ISS *type* vacuum with the ISS vacuum that has full rank breaking with the energy V= $(N_f - N)|h^2\hat{\mu}^4|$ and only $(\rho, Z)$ fundamental messengers. In the former the gauge symmetry that is completely Higgsed is actually $\mathsf{SO}(N-k)\times\mathsf{SO}(k)$.

| Field | $\mathsf{SO}(k)_c$ | $\mathsf{SO}(N-k)_c$ |
|---|---|---|
| $\Phi = \begin{pmatrix} Y_{\text{k x k}} & Z^T_{\text{k x(Nf-k)}} \\ Z_{\text{(Nf-k)xk}} & X_{\text{(Nf-k)x(Nf-k)}} \end{pmatrix}$ | 1 | 1 |
| $\varphi = \begin{pmatrix} \chi_{\text{k x k}} & \sigma_{\text{k x N-k}} \\ \rho_{\text{Nf-k x k}} & \omega_{\text{Nf-k x N-k}} \end{pmatrix}$ | $\begin{pmatrix} \square & 1 \\ \square & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \square \\ 1 & \square \end{pmatrix}$ |

| Field | $\mathsf{SO}(k)_f$ | $\mathsf{SO}(N_f-k)_f$ |
|---|---|---|
| $\Phi = \begin{pmatrix} Y_{\text{k x k}} & Z^T_{\text{k x(Nf-k)}} \\ Z_{\text{(Nf-k)xk}} & X_{\text{(Nf-k)x(Nf-k)}} \end{pmatrix}$ | $\begin{pmatrix} \square\square & \square \\ \square & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \square \\ \square & \square\square+1 \end{pmatrix}$ |
| $\varphi = \begin{pmatrix} \chi_{\text{k x k}} & \sigma_{\text{k x N-k}} \\ \rho_{\text{Nf-k x k}} & \omega_{\text{Nf-k x N-k}} \end{pmatrix}$ | $\begin{pmatrix} \square & \square \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ \square & \square \end{pmatrix}$ |

The vacuum of unbroken global symmetries have the following field content ($\chi = \mu\mathbb{I}$):

| Field | $\mathsf{SO}(N-k)_c$ | $\mathsf{SO}(k)_D$ | $\mathsf{SO}(N_f-k)_f$ |
|---|---|---|---|
| $\Phi = \begin{pmatrix} Y_{\text{k x k}} & Z^T_{\text{k x(Nf-k)}} \\ Z_{\text{(Nf-k)xk}} & X_{\text{(Nf-k)x(Nf-k)}} \end{pmatrix}$ | 1 | $\begin{pmatrix} \square\square & \square \\ \square & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \square \\ \square & \square\square+1 \end{pmatrix}$ |
| $\varphi = \begin{pmatrix} \chi_{\text{k x k}} & \sigma_{\text{k x N-k}} \\ \rho_{\text{Nf-k x k}} & \omega_{\text{Nf-k x N-k}} \end{pmatrix}$ | $\begin{pmatrix} 1 & \square \\ 1 & \square \end{pmatrix}$ | $\begin{pmatrix} \square\times\square & \square \\ \square & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ \square & \square \end{pmatrix}$ |

Here, $\mathsf{SO}(k)_c$ is completely Higgsed by the $\chi$ vevs and forms again part of a colour flavour locking phase. The superpotential in the component fields is then given by

$$\begin{aligned} W &= h\mathrm{Tr}[\varphi^T\Phi\varphi] - h\mathrm{Tr}[\mu^2\Phi] + \frac{h^2m}{2}\mathrm{Tr}[\Phi^2] + \frac{h^2m\gamma}{2}\mathrm{Tr}[\Phi]^2 \\ &= h[\chi^TY\chi + \rho^TX\rho + \chi^TZ^T\rho + \rho^TZ\chi] - h\mu^2\mathrm{Tr}Y - h\hat{\mu}^2\mathrm{Tr}X + h^2m\mathrm{Tr}[Z^TZ] \\ &\quad + h[\sigma^TY\sigma + \omega^TZ\sigma + \sigma^TZ^T\omega + \omega^TX\omega] \\ &\quad + \frac{h^2m}{2}[(\mathrm{Tr}[Y^2] + \gamma\mathrm{Tr}[Y]^2) + (\mathrm{Tr}[X^2] + \gamma\mathrm{Tr}[X]^2)] + \frac{h^2m\gamma}{2}\mathrm{Tr}X\mathrm{Tr}Y. \end{aligned} \tag{7.59}$$

In the above we have also included the meson multi-trace deformation terms $[\mathrm{Tr}\Phi]^2$ in addition to $\mathrm{Tr}[\Phi^2]$. With $\langle X\rangle = X$ and $\langle Y\rangle = Y$ the F-terms are given by

$$\begin{aligned} F_X^\dagger &= h(\rho^T\rho - \hat{\mu}^2 + \omega^T\omega) + h^2mX(1+\gamma(N_f-k)) + \frac{h^2m}{2}Y\gamma k\ , \\ F_Y^\dagger &= h(\chi^T\chi - \mu^2 + \sigma^T\sigma) + h^2mY(1+\gamma k) + \frac{h^2m}{2}X\gamma(N_f-k)\ , \\ F_\omega^\dagger &= h(2Z\sigma + 2X\omega)\ , \\ F_\sigma^\dagger &= h(2Y\sigma + 2\omega^TZ)\ . \end{aligned} \tag{7.60}$$

Minimising under the condition that $\langle\omega\rangle = 0$, the vevs are

$$\langle\rho\rangle = \langle Z\rangle = 0, \quad \langle Y\rangle = 0, \quad \langle\sigma\rangle = 0, \quad \langle\chi\rangle = q_0 \tag{7.61}$$

where we have $hq_0^2 = \mu^2 - \frac{h^2m}{2}\gamma X(N_f-k)$. We now expect two separate sectors to contribute to the one loop scalar potential in addition to the tree level potential.

$$V(X) = V_{\text{tree}} + V^{(\rho,Z)}_{\text{1-loop}} + V^{(\omega)}_{\text{1-loop}}. \tag{7.62}$$

**The $(\rho, Z)$ sector** From the uplifted superpotential we find that the fermion mass matrix is given by

$$m_f = \mathbb{I}_{N_f-k} \otimes \mathbb{I}_k \otimes 2h \begin{pmatrix} X_0 & q_0 \\ q_0 & hm \end{pmatrix} \tag{7.63}$$

whereas the bosonic mass matrix squared is

$$m_s^2 = \mathbb{I}_{N_f-N} \otimes \mathbb{I}_N \otimes 4h^2 \begin{pmatrix} q_0 q_0^* + X_0 {X_0}^* & hm q_0^* + q_0 {X_0}^* & F_X^\dagger/2h & 0 \\ hm q_0 + q_0^* X_0 & h^2 m^2 q_0 q_0^* & 0 & 0 \\ F_X/2h & 0 & q_0 q_0^* + X_0 {X_0}^* & hm q_0 + q_0^* X_0 \\ 0 & 0 & hm q_0^* + q_0 {X_0}^* & h^2 m^2 + q_0 q_0^* \end{pmatrix}. \tag{7.64}$$

They are identical to the mass matrices of the deformation by a meson field. This is as expected because the uplifted model does not introduce any new mixing between $(\rho, Z)$ and the other fields. Note that now we include a tree level mass term for $X$. The tree level term is roughly of the same order as the one loop contribution. The mass matrix is therefore given by

$$M_X^2 = \frac{h^4}{\pi^2} \begin{pmatrix} \tilde{b}\mu^2 + m^2\pi^2(1+\gamma(N_f-k))^2 & X_0 \tilde{f}_\gamma + {X_0}^2 \tilde{g}_\gamma \\ X_0 \tilde{f}_\gamma + {X_0}^2 \tilde{g}_\gamma & \tilde{b}\mu^2 + m^2\pi^2(1+\gamma(N_f-k))^2 \end{pmatrix} \tag{7.65}$$

for the general case of $\hat{\mu} \neq \mu$ and $\gamma \neq 0$. Here we have used again

$$\begin{aligned} \tilde{b} &= \frac{\hat{\mu}^4}{12\mu^4}, \\ \tilde{c} &= -\frac{\hat{\mu}^2}{48\mu^4}\Big(9\mu^2[1+\gamma(N_f-N)] + \hat{\mu}^2[2+\frac{3}{2}\gamma(N_f-N)] \\ &\qquad +6\mu^2[1+\gamma(N_f-N)]\log[4h^2\mu^2]\Big), \\ \tilde{f}_\gamma &= \frac{hm\hat{\mu}^4}{40\mu^4} - \frac{hm\hat{\mu}^2}{12\mu^2}[1+\gamma(N_f-N)] + \frac{hm\hat{\mu}^4}{48\mu^4}\gamma(N_f-N), \\ \tilde{g}_\gamma &= -\frac{3\hat{\mu}^4}{160\mu^4}. \end{aligned}$$

Results for the other cases can be found in the corresponding sections of 7.2. As before, we keep the quadratic and linear terms in $X$ for the scalar potential. The result is

$$V_{\text{1-loop}}^{(\rho,Z)}(X) = \tilde{b}\,\frac{h^4\mu^2}{\pi^2}X^2 + \tilde{c}\,\frac{2h^5\mu^2 m}{\pi^2}X\ , \tag{7.66}$$

yielding the same one-loop contribution from the $(\rho, Z)$ messengers as for meson deformation without reduced rank.

**The $\omega$ sector** The reduced rank produces new messengers $\omega$ where the field $\omega$ does not mix with the other messengers. The fermion mass is given by

$$m_f = 2hX_0 \tag{7.67}$$

and parameterising by $(\omega, \omega^*)$ we find the scalar mass squared matrix to be

$$m_\omega^2 = \mathbb{I}_{N_f-k} \otimes \mathbb{I}_{N-k} \otimes 4h^2 \begin{pmatrix} X_0 X_0{}^* & \frac{F_X^\dagger}{2h} \\ \frac{F_X}{2h} & X_0 X_0{}^* \end{pmatrix} . \tag{7.68}$$

The bosonic eigenvalues are

$$m_{\omega\pm}^2 = 2h^2(2X_0 X_0{}^* \pm |\hat{\mu}^2 - hmX_0(1+\gamma(N_f-k))|) \ . \tag{7.69}$$

In order to express the contribution from the $\omega$ messenger to the full potential we make use of the hierarchy $\hat{\mu} < X_0 < \mu$. We expand the Coleman Weinberg potential for the $\omega$ sector and keep only the leading logarithm terms (as in [211]) ignoring terms higher than quadratic order in $X$. Expanding the potential and considering appropriate cancellations we find

$$V_{\text{1-loop}}^{(\omega)} = \frac{h^4\hat{\mu}^4}{8\pi^2} \log[4h^2X^2] \tag{7.70}$$

for the contribution to the one-loop potential from the $\omega$ messengers.

**The scalar potential at one loop** Combing the results from the $(\rho, Z)$ and $\omega$ sectors, the full scalar potential is given by

$$V(X) = |F_X|^2 + k\tilde{b}\,\frac{h^4\mu^2}{\pi^2}X^2 + k\tilde{c}\,\frac{2h^5\mu^2 m}{\pi^2}X + (N-k)\frac{h^4\hat{\mu}^4}{8\pi^2}\log[4h^2X^2] \ . \tag{7.71}$$

There are $(N_f - k)$ copies of this potential coming from the trace on $X$. This plays no role on the minimisation as it is an overal factor. The term $|F_X|^2$ is obtained from

$$F_X^\dagger = -h\hat{\mu}^2 + h^2mX(1+\gamma(N_f-k)) \ . \tag{7.72}$$

Minimising the full potential we find two stationary points for real $X$

$$\begin{aligned} X_0^\pm = \frac{1}{4h^2(\frac{1}{12}k\hat{\mu}^4 + m^2\mu^2\pi^2\Gamma^2)} \Big( & -2ch^3km\hat{\mu}^2 + 2m\mu^2\hat{\mu}^2\pi^2\Gamma^2 \pm \sqrt{2\hat{\mu}^4} \\ & \times \sqrt{(2m^2(ch^3k-\mu^2\pi^2\Gamma)^2 - h^4\mu^2(N-k)(\frac{1}{12}k\hat{\mu}^4 + m^2\mu^2\pi^2\Gamma^2))} \Big), \end{aligned} \tag{7.73}$$

where we defined $c = \tilde{c}\mu^4/\hat{\mu}^2$ and $\Gamma = 1 + \gamma(N_f - k)$. One may derive simplifications for the case $\gamma = 0$ by setting $\Gamma = 1$. We have two conditions for a stationary point to be a minimum

$$\begin{aligned} \left(\frac{X}{\mu}\right)^2 &> \frac{\hat{\mu}^4(N-k)}{8(\frac{1}{12}k\hat{\mu}^4 + m^2\mu^2\pi^2\Gamma^2)} \ , \\ 0 &< \hat{\mu}^4\left(2m^2(ch^3k-\mu^2\pi^2\Gamma)^2 - h^4\mu^2(N-k)(\frac{1}{12}k\hat{\mu}^4 + m^2\mu^2\pi^2\Gamma^2)\right) \ , \end{aligned} \tag{7.74}$$

where the second one is the reality condition of the solutions. From this we may derive a lower bound on m which is set to be the smallest mass scale in this model. We get

$$m^2 > \frac{\frac{1}{12}h^4k\mu^2\hat{\mu}^4(N-k)}{2c^2h^6k^2 - 4ch^3k\mu^2\pi^2\Gamma + \mu^4\pi^2\Gamma^2(2\pi^2 - h^4(N-k))} \ . \tag{7.75}$$

The above is a generalization of a similar lower bound found in [211] (referred to as GKK from now on) but includes the additional multitrace deformation related to $\gamma$ and has also included a tree level $|X|^2$ term and a 1-loop linear term (proportional to $c$).[3]

[3] In order to compare with the results in GKK the parameter $\epsilon$ defined by the latter is related to our mass scale $m$ via $m = \epsilon\hat{\mu}$ and $\mu_i, i = 1, 2$ in the GKK notation corresponds to our $\mu, \hat{\mu}$. Finally the field $Z$ in GKK is the field $X$ here.

As a result, we encounter a more complex expression for the lower bound in $m$ which requires us to also demand that the denominator of this bound on $m^2$ is positive. We may approximate the coefficient $c \sim 9/48\mu^2\Gamma + \dots$ since the term proportional to $\mu^2 \log[\mu^2]$ does not change the general analysis. Taking $h \sim 1$ we find

$$9k^2 + 224k\pi^2 - 128\pi^2(N - 2\pi^2) > 0 \tag{7.76}$$

in order to have a positive denominator in (7.75). For a given value of $N$, this constraint puts a lower bound on the allowed values of $k$.

Using these approximations and the condition on $m^2$ we get

$$\left(\frac{X}{\mu}\right)^2 > \frac{3(N-k)}{2(k+J)} \tag{7.77}$$

with

$$J = \frac{128\pi^2(h^4k(N-k))}{(9k^2 + 224k\pi^2 - 128\pi^2(N - 2\pi^2))} \ . \tag{7.78}$$

The lower bound in the relation (7.77) is an extension of the simpler expression of GKK, namely $(N-k)/k$. It is interesting that this lower bound on the vev of $X$ is independent of $\gamma$ (though the lower bound on $m$ does depend on both these parameters). Due to the hierarchy condition we must also demand that

$$\frac{3(N-k)}{2(k+J)} < 1 \ . \tag{7.79}$$

As such there will be constraints on the allowed values of $k$ for a given $N$ in which all bounds are satisfied in the same spirit as those found in GKK. It is not difficult to find values of $N, k$ that satisfy this.

Let us discuss the scaling behaviour of the vev of $X$. Since we have $c \sim \mu^2$ we see that $m \gtrsim \hat{\mu}^2/\mu$. From this we find that the vev of $X$ generally scales as

$$\langle X \rangle \sim \mu \ . \tag{7.80}$$

Using the tree level part of the full scalar potential we may approximate the SUSY restored vacuum of this model and get

$$\langle X \rangle_{\text{SUSY}} \sim \frac{\hat{\mu}^2}{m} \ . \tag{7.81}$$

The difference between the two vacua of the theory is $\Delta X \sim \mu$. Hence, we find for the bounce action

$$S \sim \frac{(\Delta X)^4}{\Delta V} \sim \frac{\mu^4}{\hat{\mu}^4} = \left(\frac{\mu}{\hat{\mu}}\right)^4 \tag{7.82}$$

which is parametrically large as required. The minimum of full rank arises when $\omega^T\omega = \hat{\mu}^2$. Again, the bounce action scales as $S \sim \frac{(\Delta X)^4}{\Delta V} \sim \frac{\mu^4}{\hat{\mu}^4}$ and tunneling is suppressed. The main results of GKK are therefore reproduced in our SO-based ISS deformed model even if we include the additional meson single-trace deformation $\text{Tr}[\Phi^2]$ in the superpotential.

### 7.3.1 Gaugino and sfermion masses from the uplifted model

As before we begin with the contributions to the mass terms from the fundamental $(\rho, Z)$ messenger fields. The gaugino masses are

$$m_{\lambda_r,(\rho,Z)} = \frac{\alpha_r h^2 m[\hat{\mu}^2 + hm(-1+\gamma(k-N_f)X_0)]}{4\pi[\mu^2 - hmX_0]}. \tag{7.83}$$

The sfermion masses contribution arises from

$$\Lambda_S^2 = \frac{h^2[h^4m^4+2\mu^4-2h^3m^3X_0-2hm\mu^2X_0+h^2m^2(4\mu^2+{X_0}^2)][F_{X_0}/h]^2}{2[\mu^2-hmX_0]^2[4\mu^2+(-hm+X_0)^2)]}. \tag{7.84}$$

Finally the effective messenger number is

$$N_{eff}^{(\rho,Z)} = \frac{2h^2m^2[4\mu^2+(-hm+X_0)^2]}{h^4m^4+2\mu^4-2h^3m^3X_0-2hm\mu^2X_0+h^2m^2[4\mu^2+{X_0}^2]} . \tag{7.85}$$

A novel feature of the uplifted model is that a new messenger is naturally introduced. The fundamental $\omega$ can be a messenger when embedding into $\mathsf{SO}(N_f-k)_f$ and is the sole messenger if one attempts to embed into into $\mathsf{SO}(N-k)_c$ ! The contributions to gaugino and sfermion masses can be easily determined from Appendix C

$$m_{\lambda_r,\omega} = C(\mathbf{r})R[\omega](\frac{\alpha_r}{4\pi})\frac{F_X}{X_0}. \tag{7.86}$$

$$m_{\tilde{f}}^2 = 2\sum_r C_{\tilde{f}}^r(\frac{\alpha_r}{4\pi})^2 C(\mathbf{r})R[\omega]\Lambda_{S,\omega}^2 \tag{7.87}$$

where

$$\Lambda_{S,\omega}^2 = \frac{|F_X|^2}{|X_0|^2} . \tag{7.88}$$

These expressions are valid when the off-diagonal terms of the $w$ bosonic mass squared matrix are smaller than the diagonal terms (compare with Appendix C for $a > b$).

The gaugino masses from the $X$ messenger are

$$m_{\lambda_r,X} = C[\mathbf{r}]R[X](\frac{\alpha_r}{4h^2\pi^2})\frac{X_0\tilde{f}_\gamma + X_0^2\tilde{g}_\gamma}{\sqrt{\tilde{b}\mu^2+m^2\pi^2(1+\gamma(N_f-k))^2}} . \tag{7.89}$$

The contribution to the sfermion masses can be calculated using

$$\Lambda_{S,X}^2 = (\frac{h^4}{\pi^2})\frac{[X_0\tilde{f}_\gamma + X_0^2\tilde{g}_\gamma]^2}{\tilde{b}\mu^2+m^2\pi^2(1+\gamma(N_f-k))^2} . \tag{7.90}$$

# 8 | Summary and Conclusions

In this thesis, we have taken the reader onto a journey on applications of supersymmetry, both in terms of more mathematical explorations as well as phenomenological questions. Overall, the thesis is a testament to the power of supersymmetry. In this last chapter we would like to give a concluding overview on all the topics covered and the findings developed in the previous discussions.

In Chapter 2, after a short introduction, we began by considering supersymmetry as an extension to the Poincaré symmetries of spacetime. After introducing the basic ideas of supersymmetry, we discussed spinors in various dimensions and concentrated on the maximally supersymmetric theories in four and six dimensions which were of importance for the rest of the thesis. Furthermore, we considered the off-shell construction of an $\mathcal{N} = 1$ non-extended superspace and its corresponding superfields which are of phenomenological importance for the second half of the thesis.

After this quite general introduction we moved to the topic of scattering amplitudes, both in four as well as six dimensions. In Chapter 3 we saw the importance of the concept of scattering amplitudes, namely as a link between theoretical considerations and experimental measurements. We discussed the general features of amplitudes with external gauge boson states and introduced efficient techniques for their calculation like the BCFW recursion relations. Then, we realised the importance of supersymmetry in this context by considering the maximally supersymmetric $\mathcal{N} = 4$ theory in four dimensions. Here, an on-shell superspace can be introduced where the full particle content of the $\mathcal{N} = 4$ supermultiplet can be combined in a compact super-wavefunction. Furthermore, we introduced the concept of superamplitudes in the maximally supersymmetric theory. A superamplitude is a scattering amplitude of super-wavefunctions and contains all amplitudes of that specific theory with a fixed amount of external states. This concept proved to be very powerful since we could apply similar techniques for efficient computations as for the usual bosonic amplitudes, however, had a much richer particle content covered. Also, this approach could be nicely combined with the unitarity method for computing scattering amplitudes at loop-level. At the end of Chapter 3, we discussed the calculation of one-loop superamplitudes by so-called double and quadruple cuts via the unitarity method.

This approach was taken to the next level in Chapter 4. It is intriguing that the concepts of the four-dimensional spinor helicity formalism, superamplitudes and unitarity method can be applied to the six-dimensional maximally supersymmetric $\mathcal{N} = (1, 1)$ SYM theory. After a review of the recently introduced 6D spinor helicity formalism, we discussed the construction of superamplitudes with three, four and five external states. The basic ideas of the four-dimensional theory can be imported to the six-dimensional one, however, the main difference is the non-chiral nature of the $\mathcal{N} = (1,1)$ superspace. This fact makes the construction of superamplitudes more complicated, yet also more powerful due to the explicit transformation of the particle states under the $\mathsf{SU}(2)\times\mathsf{SU}(2)$ little group in six dimensions. Then, for the first time, we considered double as well as quadruple cuts for one-loop superamplitudes with four and five external particles. Especially the structure of the five-point one-loop superamplitude is interesting since it can be expressed in terms of a linear pentagon integral function in six dimensions. It would be highly interesting to extend this research to amplitudes with more external states, both at tree- and loop-level, as well as one-loop amplitude with more cut-propagators. For instance, a five-particle cut of the five-point superamplitude has not been performed so far. However, due to the structure of the spinor helicity formalism, these calculations are expected to become quite involved.

In the second part of the thesis, beginning with Chapter 5, we focused on phenomenological implications of non-extended supersymmetry in four dimensions. More specifically, we were interested in supersymmetric theories for which the vacuum state possesses no supersymmetry. Especially important are supersymmetric $\mathcal{N} = 1$ QCD-like theories due to their rich moduli space structure. For these theories, Seiberg duality can be used to related theories with different number of flavours and colours with each other. One approach which strongly invigorated the interest in models of gauge mediated supersymmetry breaking was the ISS construction which we discussed in Chapter 5. In these models, metastable states of broken supersymmetry are accepted right from the beginning which offers interesting phenomenological applications. We concluded our discussion of $\mathcal{N} = 1$ theories with broken supersymmetry with an overview on recent developments in gauge mediation.

Finally, in the last two Chapters 6 and 7, we discussed our work on ISS like constructions with $\mathsf{SO}(N)$ gauge and flavour groups. Firstly, we had a brief discussion on the general ideas of Seiberg duality for $\mathsf{SO}(N)$ SQCD theories where we found some important distinctions compared to $\mathsf{SU}(N)$ models. We then moved on and considered gaugino masses in those $\mathsf{SO}(N)$ models before specifically discussing several approaches of how to construct a phenomenologically viable model of supersymmetry breaking. Here, the main idea was to introduce deformations to the basic $\mathsf{SO}(N)$-ISS superpotential in order to allow for non-zero gaugino masses. We systematically investigated the deformations suggested in the $\mathsf{SU}$-based literature, namely that of baryonic deformations of magnetic quarks, the KOO deformation and multitrace operators of magnetic quarks and the magnetic meson. Furthermore, we extended the multitrace meson deformation by looking at reducing the rank of the magnetic squark matrices of supersymmetry breaking. Here, we could confirm that this approach helps to make ISS models more viable by generating a vev for the $X$ meson field at tree level by destabilising the origin, for details see the corresponding sections. Our findings can be extended in several ways, one quite important one would be to extract a detailed picture of the phenomenology of $\mathsf{SO}(N)$ ISS models which has been carried out in the literature for particular $\mathsf{SU}(N)$ deformations. In addition, it would be interesting to see how a string and brane construction can realise our $\mathsf{SO}(N)$ ISS setup which was also discussed before for the $\mathsf{SU}(N)$ case.

# A | Notations and Conventions

In this section we give - for the reader's convenience - an overview of notations and conventions used throughout this thesis. The discussed material can be found in various place, see standard textbooks and reviews about quantum field theory and supersymmetry.

## A.1 Basic spinor notation

Throughout this thesis we work in $D$-dimensional Minkowski spacetime with metric

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, \dots, -1) \ . \tag{A.1}$$

This establishes the usual field theory conventions for the metric for which the Lorentz invariant mass squared $p^\mu p_\mu = m^2$ of a particle is positive. However, we face the tradeoff that the determinant of the metric changes sign when adding a spatial dimension. Since we are dealing with four and six spacetime dimensions only this is not relevant for our discussions.

In many places throughout the thesis, we face two-component spinor indices, both in the four- as well as six-dimensional context. In order to raise and lower two-component spinor indices we introduce the $\mathsf{SL}(2, \mathbb{C})$ invariant matrices

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot\alpha\dot\beta} \tag{A.2}$$

where we raise and lower indices with respect to the second index of the tensor. Here, $\sigma^2$ is one of the $2 \times 2$ Pauli matrices. These objects are often used in two-component spinorial expressions. We use the form

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.3}$$

From this, the sigma matrices in four dimensions are defined as

$$(\sigma^\mu)_{\alpha\dot\alpha} = (\mathbf{1}, \sigma^i)_{\alpha\dot\alpha} \tag{A.4}$$

and $\sigma^i = -\sigma_i$ are the $2 \times 2$ Pauli matrices introduced above. We see that the sigma matrices naturally have a undotted-dotted index. We can raise these indices and find

$$(\bar\sigma^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (\mathbf{1}, -\sigma^i)_{\beta\dot\beta} \ . \tag{A.5}$$

This can be explicitly seen by considering the individual components of $\sigma^\mu$. Note that the $\tilde{\sigma}^\mu$ have a dotted - undotted index structure.

Further information can be found in the corresponding sections in the Chapters 3 and 4, namely in Sections 3.1.2 and 4.1. In general, for spinors in four dimensions as well as Grassmann variables, we follow the conventions of [2] and [17].

## A.2 Some Group Theory Notation

For completeness we collect some basic group theory results for $\mathsf{SU(N)}$ and $\mathsf{SO(N)}$ groups which were used throughout this thesis and which might be useful for the reader's reference. Also, we set our conventions for the group generators used in various sections. We have tried to be consistent with the normalisations throughout the text and whenever we have used different conventions we have tried to make the change visible to the reader.

Throughout this thesis, we have used different representations of various Lie groups. In general, a representation of a group $G$ is a correspondence between elements of $G$ and the set of linear operators which act on a vector space $V$ over the group $G$. For each group element $g \in G$ we have a linear operator $D(g)$ with

$$D(g) : V \longrightarrow V \tag{A.6}$$

and conventionally $D(g)$ is called the representation of $G$. The linear operators satisfy the operations

$$D(g_1)D(g_2) = D(g_1 g_2) \; , \qquad D(g_1^{-1}) = D(g_1)^{-1} \qquad \forall \, g_1, g_2 \in G \; . \tag{A.7}$$

In particular, we are interested in irreducible representations. These can be classified by Schur's lemma: A representation $D(g)$ is irreducible if and only if the only class of operators $A : V \to V$ commuting with the other elements of the representation are proportional to the identity operator,

$$[D(g), A] = 0 \quad \forall \, g \in G \quad \Leftrightarrow \quad A = c\,\mathbf{1} \quad \text{with } c \in \mathbb{C} \; . \tag{A.8}$$

In this thesis we consider representations of classical Lie groups. Their elements are labelled by a number of continuous parameters. In important additional property of Lie groups is the fact that in a neighbourhood of the identity element one can obtain a representation of the Lie group in terms of the corresponding Lie algebra generators $T^a$ where $a$ is the group index with $a = 1, \ldots, \dim(G)$. We have

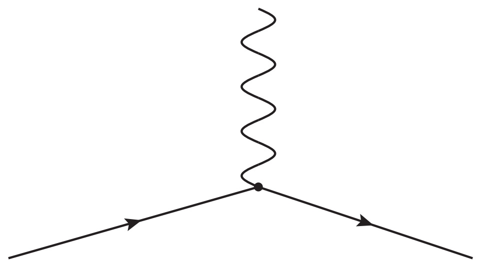
$$D(g) = e^{-i\alpha_a T^a} \tag{A.9}$$

with $\alpha_a$ as a set of coordinates in the neighbourhood of the $\mathbf{1}$. The generators satisfy a commutation relation, relating them to the structure constants $f^{abc}$ as

$$[T^a, T^b] = i f^{abc} T^c \qquad \text{with } f^{abc} \in \mathbb{C} \; . \tag{A.10}$$

This relation together with the set of generators $T^a$ form the corresponding Lie algebra. Therefore, a representation of the Lie algebra generators yields a representation of the Lie group by simply exponentiating the generators $T^a$ around the identity element.

We now turn to the application of Lie groups in quantum field theory. The basic group theory structure of Lie groups has a nice diagrammatic interpretation in the general case of non-abelian groups $G$ [212]. One identifies the generators[1] $T^a$ as part of an interaction vertex between a gauge boson and two fermions, as in the following diagram for the generator $(T^a)^m_n$:

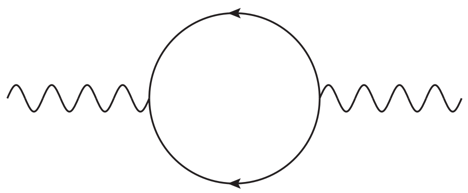


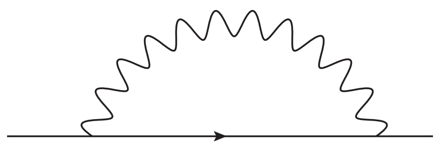

In principle, the generator keeps track of the colour indices, denoted by $m$ and $n$ in the diagram. Having identified the $T^a$ we now consider products of generators in specific irreducible representations $\mathbf{r}$ of the gauge group $G$. The generators obey the relations

$$\mathrm{Tr}[T^a_{\mathbf{r}} T^b_{\mathbf{r}}] \equiv \left(T^a_{\mathbf{r}}\right)^m_n \left(T^b_{\mathbf{r}}\right)^n_m = C(\mathbf{r})\delta^{ab} \tag{A.11}$$

$$\sum_a T^a_{\mathbf{r}} T^a_{\mathbf{r}} \equiv \left(T^a_{\mathbf{r}}\right)^m_n \left(T^a_{\mathbf{r}}\right)^n_l = C_2(\mathbf{r})\delta^m_l \tag{A.12}$$

which define the quadratic invariant $C(\mathbf{r})$ and the quadratic Casimir $C_2(\mathbf{r})$. For both relations we can give a diagrammatic interpretation by using the identifitcation of the group generators $T^a$:




Furthermore, by contracting the relation (A.11) with $\delta_{ab}$ and evaluating the LHS using the second relation (A.12) yields

$$\dim(\mathbf{r})C_2(\mathbf{r}) = \dim(\mathbf{Ad})C(\mathbf{r}) \tag{A.13}$$

Here, $\dim(\mathbf{r})$ is the dimension of the specific representation and $\dim(\mathbf{Ad})$ is the dimension of the adjoint representation. This result can also be obtained from the two diagrams shown above. Closing the first diagram corresponds to setting $a = b$. Summing over $a$ gives the dimension of the adjoint representation. Closing the second diagram means setting $m = l$ and then summing over $m$ leads to the dimension of the representation. Since closing both diagrams yields a identical diagram the identity (A.13) follows.

We can use this relation to calculate the quadratic casimir $C_2(\mathbf{r})$ for any representation $\mathbf{r}$. Following the usual field theory conventions for the invariants of the fundamental and adjoint representation, we have for groups $\mathsf{SU}(N)$, $\mathsf{SO}(N)$ and $\mathsf{Sp}(2N)$:

[1] Here, we follow the discussion of [17].

| group $G$ | rank($G$) | dim(**Ad**) | $C(\mathbf{Ad})$ | dim($\Box$) | $C(\Box)$ |
|---|---|---|---|---|---|
| $\mathsf{SU}(N)$ | $N-1$ | $N^2-1$ | $N$ | $N$ | $\frac{1}{2}$ |
| $\mathsf{SO}(N)$ | $[\frac{N}{2}]$ | $\frac{1}{2}N(N-1)$ | $N-2$ | $N$ | 1 |
| $\mathsf{Sp}(2N)$ | $N$ | $N(2N+1)$ | $N+1$ | $2N$ | $\frac{1}{2}$ |

Here, we have denoted the fundamental representation by $\Box$ and have used the standard convention for $\mathsf{SU}(N)$ groups,

$$C(\Box) = \frac{1}{2}\ , \qquad C(\mathbf{Ad}) = N\ , \tag{A.14}$$

and using the relation (A.13) yield the quadratic Casimirs

$$C_2(\Box) = \frac{N^2-1}{2N}\ , \quad C_2(\mathbf{Ad}) = N\ . \tag{A.15}$$

Finally, one can also define the index of a representation which is defined as the quadratic Casimir of a representation $\mathbf{r}$, normalised with respect to the fundamental representation,

$$T(\mathbf{r}) \equiv \frac{C(\mathbf{r})}{C(\Box)}\ . \tag{A.16}$$

One should be careful about different conventions in the literature, especially when considering one-loop beta-functions for theories with particles in different representations (i.e. $\mathcal{N}=1$ super-QCD with $N_f$ flavours). For instance, the reference [17] normalises the group generators with respect to the index $T(\mathbf{r})$ and not with respect to the quadratic invariant $C(\mathbf{r})$. Denoting the index of [17] by $T'$ we have

$$T'(\mathbf{r}) = C(\Box)\, T(\mathbf{r}) = C(\mathbf{r})\ . \tag{A.17}$$

For $\mathsf{SU}(N)$ groups this results in a factor of $\frac{1}{2}$ whereas for $\mathsf{SO}(N)$ we have a trivial factor of 1. An overview on dimensions on quadratic invariants for some representations of $\mathsf{SU}(N)$, $\mathsf{SO}(N)$ and $\mathsf{Sp}(2N)$ can be found in Appendix B of [17] where one can simply replace $2T(\mathbf{r}) \to T(\mathbf{r})$ in order to obtain results compatible with our conventions.

## A.3 Representations of the Poincaré Algebra

As we have discussed in Section 2.2, the isometries of spacetime are given by the Poincaré transformations, namely the elements of the Poincaré group. They act as

$$x^\mu \to x'^{\,\mu} = \Lambda^\mu_{\ \nu}\, x^\nu + a^\mu\ , \tag{A.18}$$

with $\Lambda^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ as the elements of the Lorentz group and $a^\mu$ as the parameter of spacetime translations. The Lorentz transformations preserve the spacetime metric $\eta_{\mu\nu}$ since

$$\Lambda^{T\ \sigma}_{\mu}\eta_{\sigma\rho}\Lambda^\rho_{\ \nu} = \eta_{\mu\nu}\ . \tag{A.19}$$

Furthermore, any Lorentz transformation leaves the origin of spacetime unchanged and preserves the infinitesimal line element

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu\ . \tag{A.20}$$

The Lorentz group in $D$ spacetime dimensions is $\mathsf{O(p,q)}$ where $p$ stands for the spatial and $q$ for the time directions (in a standard notation for our metric convention). Therefore, for $D$-dimensional Minkowski spacetime the Lorentz group is $\mathsf{O(D-1,1)}$. This group includes also parity and time-reversal transformations. Due to physical reasons we restrict ourselves to transformation matrices $\Lambda$ with determinant $\det(\Lambda) = +1$ and request that $\Lambda^0_{\ 0} \geq 0$. These matrices form a subgroup of $\mathsf{O(d-1,1)}$ whose elements preserve the direction of time-flow and have positive determinant. This subgroup is the *proper orthochronus Lorentz group* and is denoted by $\mathsf{SO^+(D-1,1)}$. In the literature, this is conventionally written just as $\mathsf{SO(D-1,1)}$. Focusing on four-dimensional spacetime we have the transformation group $\mathsf{SO(3,1)}$ which we simply call the Lorentz group. The semi-direct product of the group $\mathsf{O(D-1,1)}$ and the spacetime translations forms the so called inhomogeneous Lorentz group $\mathsf{IO}(D-1,1)$. Restricting ourselves to transformations with $\det(\Lambda) = +1$ and $\Lambda^0_{\ 0} \geq 0$ yields the subgroup $\mathsf{SIO}(D-1,1)$, namely the Poincaré group with its generators $P^\mu$ and $M^{\mu\nu}$.

In the previous Appendix A.2 we have seen that one obtains an element of a group by exponentiating the generator of the corresponding algebra. Hence, in order or obtain representations of the Poincaré group, we need to find representations of the Poincaré generators $P^\mu$ and $M^{\mu\nu}$. The representations depend on the vector space which the group elements act on. In general, we are interested in vector spaces of scalar, (four-)vector and spinor fields.

Let us begin with spacetime translations which are generated by $P^\mu$. For a general field $\Phi^a$ as a function of spacetime, the translations act as spacetime derivatives. One can represent the generator by

$$(P^\mu)^a_{\ b} = i\frac{\partial}{\partial x^\mu}\delta^a_b \ , \tag{A.21}$$

where $a, b$ are generic field indices. For instance, in the case of four-vectors $a$ and $b$ are Lorentz indices, e.g. $a = \mu$ and $b = \nu$. The translation elements of the Poincaré group are then given by

$$D(g) = T^b_c = \exp\big[-ia^\mu(P_\mu)\big]^b_{\ c} = \exp\left[a^\mu \frac{\partial}{\partial x^\mu}\delta^a_b\right] . \tag{A.22}$$

Since translations and Lorentz transformations commute, this is all to say for the translations and their representation as differential operators when acting on spacetime fields.

The case of Lorentz transformations is a bit more involved. In general, a field $\Phi^a(x)$ will transform as (active transformation)

$$\Phi^a(x) \to \Phi'^a(x) = D(\Lambda)^a_b \Phi^b(\Lambda^{-1}x) \ . \tag{A.23}$$

The coordinate transformation $x^\mu \to \Lambda^\mu_\nu x^\nu$ results in a shifted field argument $\left(\Lambda^{-1}\right)^\mu_{\ \nu} x^\nu$. The elements $D(\Lambda)$ of the Lorentz group are given by the exponentiation of the algebra generators $M^{\mu\nu}$, i.e. $D(\Lambda) = \Lambda^a_b = \exp\big[-iw_{\mu\nu}M^{\mu\nu}\big]^a_{\ b}$. The form of the generators $\left(M^{\mu\nu}\right)^a_{\ b}$ depends on the nature of the field $\Phi^a$. Scalar fields transform with the trivial representation $M^{\mu\nu}_S \equiv 0$ and hence $D(\Lambda) = 1$ (one-dimensional vector space) yields

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) \ . \tag{A.24}$$

Vector fields $A^\mu(x)$ carry a spacetime index and therefore, they transform with the fundamental representation of the Lorentz group. The generators take the form

$$M_V^{\mu\nu} \equiv (M^{\mu\nu})^\rho_{\ \sigma} = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma) \tag{A.25}$$

for this four-dimensional vector space (vector fields in $D = 4$) and we find

$$A^\mu(x) \to A'^\mu(x) = D(\Lambda)^\mu_{\ \nu} A^\nu(\Lambda^{-1}x) = \exp\big[-\frac{i}{2}w_{\rho\sigma}M_V^{\rho\sigma}\big]^\mu_{\ \nu} A^\nu(\Lambda^{-1}x)\ . \tag{A.26}$$

In the case of spinor fields $\psi^A(x)$ where $A$ is a spinor index we have also have a four-dimensional vector space (Dirac spinors in $D = 4$). In the case of four-dimensional Dirac spinors the generators take the form

$$M_D^{\mu\nu} \equiv (\Sigma^{\mu\nu})^A_{\ B}\ , \qquad \text{where } \Sigma^{\mu\nu} \equiv \frac{i}{4}\,[\gamma^\mu, \gamma^\nu]\ , \tag{A.27}$$

is the commutator of the usual $\gamma$-matrices satisfying the Dirac algebra (2.5) in four dimensions. A particularly useful representation of the $\gamma$-matrices in four dimensions is the Weyl representation,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\alpha} \\ \bar\sigma^{\mu\alpha\dot\alpha} & 0 \end{pmatrix}\ , \qquad \text{and } \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}\ , \tag{A.28}$$

where $\gamma^5$ is the chirality operator. We see that the $\gamma$-matrices can be decomposed into a $2 \times 2$ block form which represents the fact that a Dirac spinor in four dimension is reducible into two irreducible Weyl spinors. This is true for any even dimension. From this, the Lorentz generators of a Dirac spinor are also block-diagonal,

$$\Sigma^{\mu\nu} = \begin{pmatrix} (\sigma^{\mu\nu})_\alpha^{\ \beta} & 0 \\ 0 & (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \end{pmatrix}\ , \tag{A.29}$$

where $(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\mu\bar\sigma^\nu)_\alpha^{\ \beta}$ and $(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\mu\sigma^\nu)^{\dot\alpha}_{\ \dot\beta}$. Here, $\Sigma_L^{\mu\nu} \equiv (\sigma^{\mu\nu})_\alpha^{\ \beta}$ is a representation of the generator for a left-handed Weyl spinor whereas $\Sigma_R^{\mu\nu} \equiv (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta}$ is similarly acting on a right-handed Weyl spinor.

Let us return to the transformation of a general spinor field under the Lorentz transformations. In this case, the spinor field transforms as

$$\psi^A(x) \to \psi'^A(x) = D(\Lambda)^A_{\ B}\psi^B(\Lambda^{-1}x) = \exp\big[-\frac{i}{2}w_{\rho\sigma}M_D^{\rho\sigma}\big]^A_{\ B}\psi^B(\Lambda^{-1}x)\ . \tag{A.30}$$

We note that without any spacetime dependence of the fields, the above representations are finite-dimensional when acting on finite-dimensional vector spaces (all fields have a finite number of components). However, since the quantum fields ultimately depend on the four-vector $x^\mu$, we have to take into account the change of the field argument. The dependence on $x^\mu$ is the same for all the different types of fields and hence, in all cases the active coordinate transformation (resulting in the changed field argument $(\Lambda^{-1})^\mu_{\ \nu}\, x^\nu$) can be implemented in the same way. Here, $\Lambda^{-1}$ is the inverse of a Lorentz transformation $\Lambda$ in the fundamental representation, i.e.

$$\Lambda \equiv \big(\Lambda_V\big)^\mu_{\ \nu} = \exp\big[-\frac{i}{2}w_{\rho\sigma}M_V^{\rho\sigma}\big]^\mu_{\ \nu}\ . \tag{A.31}$$

By considering an infinitesimal Lorentz transformation in the fundamental representation we can Taylor-expand the generic field $\Phi^a(\Lambda^{-1}x)$ and find

$$\Phi^a(\Lambda^{-1}x) = \exp\big[-\frac{i}{2}w_{\rho\sigma}(M_V^{\rho\sigma})^\mu_{\ \nu}\, x^\nu\partial_\mu\big]\Phi^a(x) = \exp\big[-\frac{i}{2}w_{\rho\sigma}L^{\rho\sigma}\big]\Phi^a(x)\ . \qquad \text{(A.32)}$$

Using the explicit form (A.25), the generators $L^{\mu\nu}$ are given by differential operators which can be written as

$$L^{\mu\nu} = i\,(x^\mu\partial^\nu - x^\nu\partial^\mu)\ . \qquad \text{(A.33)}$$

Furthermore, we not that the $L^{\mu\nu}$ act on an infinite-dimensional vector space (the space of the function $\Phi^a$) and hence, it is an infinite-dimensional representation of the Lorentz algebra.

In a final step we can combine the exponential factors of the individual finite-dimensional representations (corresponding to the generators $M_S^{\mu\nu}, M_V^{\mu\nu}$ and $M_D^{\mu\nu}$) of the different fields with the infinite-dimensional representation of the $L^{\mu\nu}$, yielding the transformation

$$\Phi^a(x) \to \exp\big[-\frac{i}{2}w_{\rho\sigma}M_i^{\rho\sigma}\big]^a_{\ b}\exp\big[-\frac{i}{2}w_{\rho\sigma}L^{\rho\sigma}\big]\Phi^b(x)\ , \qquad \text{(A.34)}$$

where $i = S, V, D$ stand for the finite-dimensional generators of the Lorentz group for scalar, vector and spinor objects. Since the generators $M_i^{\mu\nu}$ are finite and constant, the exponential factors can be combined into a single exponential, resulting in

$$\Phi^a(x) \to \exp\big[-\frac{i}{2}w_{\rho\sigma}M^{\rho\sigma}\big]^a_{\ b}\Phi^b(x) \qquad \text{(A.35)}$$

where $M^{\mu\nu} = M_i^{\mu\nu} + L^{\mu\nu}$ are the generators of an infinite-dimensional representation of the Lorentz algebra. Whereas the form of the $M_i^{\mu\nu}$ depend on the type of field the transformation is acting on, the form of the generators $L^{\mu\nu}$ is always the same.

# B Calculations for Six-Dimensional Superamplitudes

In this appendix we would like to present some details on derivations and calculations that are used in the context of the six-dimensional unitarity method. We first present some basic conventions and provide then a proof of the supersymmetry invariance of the three-point superamplitude for the six-dimensional $(1,1)$ SYM theory. Furthermore, spinor manipulations in six dimensions are discussed as well as a brief overview on the PV reduction of the linear pentagon integral function which was found in the one-loop five-point superamplitude.

## B.1 Some conventions

In this appendix we collect some details on the conventions of the six-dimensional spinors and related constructions.

We can raise and lower $\mathsf{SU}(2)$ indices with the $\mathsf{SU}(2)$-invariant epsilon tensor defined in (A.2). The Grassmann integration measure is defined as $d^2\eta = (1/2)d\eta^a d\eta_a = d\eta_2 d\eta_1$, such that

$$\int d^2\eta \left[\lambda^{Aa}\eta_a\, \lambda^{Bb}\eta_b\right] = -\left(\lambda^{Aa}\lambda^B_a\right) . \tag{B.1}$$

The Clebsch-Gordan symbols are normalised as

$$\tilde{\sigma}^{AB}_\mu := \frac{1}{2}\epsilon^{ABCD}\,\sigma_{\mu,CD}\,, \tag{B.2}$$

with

$$\mathrm{Tr}(\sigma^\mu\tilde{\sigma}^\nu) \;=\; 4\,\eta^{\mu\nu}\ . \tag{B.3}$$

Using these relations, the scalar product of two vectors $p$ and $q$ can equivalently be expressed as

$$p\cdot q = -\frac{1}{4}p^{AB}q_{AB} \;=\; -\frac{1}{8}\epsilon_{ABCD}p^{AB}q^{CD}\ , \tag{B.4}$$

where $p^{AB} := p^\mu\tilde{\sigma}^{AB}_\mu$ and $p_{AB} := p^\mu\sigma_{\mu,AB}$.
Momentum conservation for three-point amplitudes implies that $p_i\cdot p_j = 0$, $i,j = 1,2,3$. In six dimensions, this condition is equivalent to [85]

$$\det\langle i|j]_{a\dot{a}} \;=\; 0 \tag{B.5}$$

where $\lambda^A_{ia}\tilde{\lambda}_{Aja} := \langle i_a|j_{\dot{a}}]$ and we used $p^{AB}_i = \lambda^A_{ia}\lambda^{Ba}_i$ and $p_{iAB} = \tilde{\lambda}^{\dot{a}}_{iA}\tilde{\lambda}_{iB\dot{a}}$. Hence, (B.5) allows to recast the matrix $\langle i_a|j_{\dot{a}}]$ as a product of two spinors, as [85]

$$\langle i_a|j_{\dot{b}}] = (-)^{\mathcal{P}_{ij}}u_{ia}\tilde{u}_{j\dot{b}}\ , \tag{B.6}$$

where we choose $(-)^{\mathcal{P}_{ij}} = +1$ for $(i,j) = (1,2),(2,3),(3,1)$, and $-1$ for $(i,j) = (2,1),(3,2),(1,3)$. Hence, for a generic three-point vertex with all momenta defined to be incoming (see Figure 3.1) we have a positive sign when rewriting Lorentz contracted spinor combinations in a clockwise ordering.

## B.2 Invariance of the three-point superamplitudes

Here we provide an explicit proof of the fact that the three-point superamplitude (4.25) is supersymmetric. To show this, we choose to decompose each variable $\eta_i$ as

$$\eta_i^a = u_i^a \eta_i^{\parallel} + w_i^a \eta_i^{\perp} \ , \tag{B.7}$$

which is a convenient choice since $u_i^a w_{ia} = 1$. We also notice that, using this decomposition, we can recast the quantities $W$ and $\tilde{W}$ defined in (4.27) entering the expression of the three-point superamplitude, as

$$W = \sum_{i=1}^{3} \eta_i^{\parallel} \ , \qquad \tilde{W} = \sum_{i=1}^{3} \tilde{\eta}_i^{\parallel} \ . \tag{B.8}$$

The supersymmetry generators can then be written as

$$Q^A = \sum_i \lambda_i^{Aa} u_{ia} \eta_i^{\parallel} + \sum_i \lambda_i^{Aa} w_{ia} \eta_i^{\perp} \ . \tag{B.9}$$

A direct consequence of six-dimensional momentum conservation is the fact that the quantities $\lambda_i^{Aa} u_{ia}$ are $i$-independent, therefore we can rewrite (B.9) in several equivalent ways, one of which is

$$Q^A = (\lambda_1^{Aa} u_{1a}) W + (\lambda_1^{Aa} w_{1a})(\eta_2^{\perp} - \eta_1^{\perp}) + (\lambda_2^{Aa} w_{2a})(\eta_3^{\perp} - \eta_1^{\perp}) \ , \tag{B.10}$$

where $W$ is given in (B.8), and the constraint on the $w$'s (4.22) is used. Using the decomposition (B.10) it is very easy to prove that $Q^A A_3 = 0$. To this end, we first observe that the presence of a factor $\delta(W)\delta(\tilde{W})$ in (4.25) effectively removes the first term from the expression of (B.10), and we are left to prove that $Q_{\perp}^A := (\lambda_1^{Aa} w_{1a})(\eta_2^{\perp} - \eta_1^{\perp}) + (\lambda_2^{Aa} w_{2a})(\eta_3^{\perp} - \eta_1^{\perp})$ annihilates the amplitude. Specifically, we will show that

$$Q_{\perp}^A \left[ \delta(Q^A)\delta(\tilde{Q}_A) \right]^2 = 0 \, . \tag{B.11}$$

To begin with, we observe that

$$\begin{aligned} \delta(Q^A)\delta(\tilde{Q}_A) &= \sum_{i,j=1}^{3} \langle i_a | j_{\dot{a}} ] \eta_i^a \tilde{\eta}_j^{\dot{a}} = \sum_{i,j=1}^{3} (-)^{\mathcal{P}_{ij}} u_{ia} \tilde{u}_{j\dot{a}} (1 - \delta_{ij}) \eta_i^a \tilde{\eta}_j^{\dot{a}} \\ &= \sum_{i,j=1}^{3} (-)^{\mathcal{P}_{ij}} (1 - \delta_{ij}) \eta_i^{\perp} \tilde{\eta}_j^{\perp} \\ &= \eta_1^{\perp}\tilde{\eta}_2^{\perp} - \eta_1^{\perp}\tilde{\eta}_3^{\perp} - \eta_2^{\perp}\tilde{\eta}_1^{\perp} + \eta_2^{\perp}\tilde{\eta}_3^{\perp} + \eta_3^{\perp}\tilde{\eta}_1^{\perp} - \eta_3^{\perp}\tilde{\eta}_2^{\perp} \ , \end{aligned} \tag{B.12}$$

where we have used (B.6). Using (B.12), one then finds (we drop the superscript $\perp$ in the following)

$$\begin{aligned} \left[ \delta(Q^A)\delta(\tilde{Q}_A) \right]^2 = &- \eta_1\tilde{\eta}_2\eta_2\tilde{\eta}_1 + \eta_1\tilde{\eta}_2\eta_2\tilde{\eta}_3 + \eta_1\tilde{\eta}_2\eta_3\tilde{\eta}_1 + \eta_1\tilde{\eta}_3\eta_2\tilde{\eta}_1 - \eta_1\tilde{\eta}_3\eta_3\tilde{\eta}_1 + \eta_1\tilde{\eta}_3\eta_3\tilde{\eta}_2 \\ &- \eta_2\tilde{\eta}_1\eta_1\tilde{\eta}_2 + \eta_2\tilde{\eta}_1\eta_1\tilde{\eta}_3 + \eta_2\tilde{\eta}_1\eta_3\tilde{\eta}_2 + \eta_2\tilde{\eta}_3\eta_1\tilde{\eta}_2 + \eta_2\tilde{\eta}_3\eta_3\tilde{\eta}_1 - \eta_2\tilde{\eta}_3\eta_3\tilde{\eta}_2 \\ &+ \eta_3\tilde{\eta}_1\eta_1\tilde{\eta}_2 - \eta_3\tilde{\eta}_1\eta_1\tilde{\eta}_3 + \eta_3\tilde{\eta}_1\eta_2\tilde{\eta}_3 + \eta_3\tilde{\eta}_2\eta_1\tilde{\eta}_3 + \eta_3\tilde{\eta}_2\eta_2\tilde{\eta}_1 - \eta_3\tilde{\eta}_2\eta_2\tilde{\eta}_3 \ . \end{aligned} \tag{B.13}$$

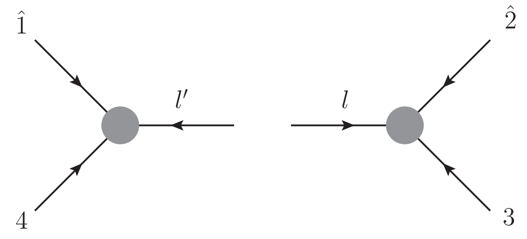



Figure B.1: *The recursive construction of a four-point tree-level amplitude. The shifted legs are* 1 *and* 2 *and we have* $l' = -l$ *for the internal propagator.*

Next, we calculate

$$\eta_1 \Big[\delta(Q^A)\delta(\tilde{Q}_A)\Big]^2 \quad = \quad 2\eta^1\eta^2\eta^3(\tilde{\eta}^1\tilde{\eta}^3 - \tilde{\eta}^1\tilde{\eta}^2 - \tilde{\eta}^2\tilde{\eta}^3) \ , \tag{B.14}$$

and furthermore we find that

$$\eta_2 \Big[\delta(Q^A)\delta(\tilde{Q}_A)\Big]^2 = \eta_3 \Big[\delta(Q^A)\delta(\tilde{Q}_A)\Big]^2 = \eta_1 \Big[\delta(Q^A)\delta(\tilde{Q}_A)\Big]^2 . \tag{B.15}$$

Inspecting the form of $Q^A$ in (B.10) and using (B.15), we conclude that (B.11) holds, and therefore the three-point superamplitude is invariant under supersymmetry.

## B.3 Useful spinor identities in six dimensions

In this appendix we collect identities between six-dimensional spinor variables that we have frequently used in the calculations presented in this thesis.

We begin by quickly stating two basic relations for three-point spinors $u_{i,a}$ and $w_{i,a}$. For a general three-point amplitude in six dimensions we have [85]

$$u_i^a|i_a\rangle = u_j^b|j_b\rangle \ , \qquad \tilde{u}_i^{\dot{a}}|i_{\dot{a}}] = \tilde{u}_j^{\dot{b}}|j_{\dot{b}}] \ . \tag{B.16}$$

We also have the constraints (4.22) on the $w$'s and their $\tilde{w}$ counterparts, which are essentially a consequence of momentum conservation.

Next, we make use of relations between two three-point amplitudes, connected by an internal propagator, just as in the BCFW construction of the four-point amplitude. We give a pictorial representation of this in Figure B.1. We have defined the internal momenta $l$ and $l'$ to be incoming for the three-point amplitudes, giving the relation $l' = -l$. Since six-dimensional momenta are products of two spinors we can define

$$|l'_i\rangle = i|l_i\rangle \ , \qquad |l_i\rangle = (-i)|l'_i\rangle \ , \tag{B.17}$$

and similarly for $\tilde{\lambda}$-spinors. Also note that we can normalise the spinors $u_a, w_b$ of one three-point subamplitudes in Figure B.1 such that they are related to the spinors of the other subamplitude, yielding (see Appendix B.3.2)

$$w_{l'_i a} = \frac{u_{l_i a}}{\sqrt{-s}} \ , \qquad w_{l_i a} = -\frac{u_{l'_i a}}{\sqrt{-s}} \ . \tag{B.18}$$

Similar expressions hold for the spinors $\tilde{u}_{\dot{a}}, \tilde{w}_{\dot{b}}$. In the following we will be discussing several relations in the cases of the four- and five-point amplitudes.

### B.3.1 Product of two $u$-spinors

In the calculation of the five-point cut-expression we encounter $u$-spinors belonging to the same external state and would like to remove them from the expression. Consider the object $u_{ia}\tilde{u}_{i\dot{a}}$ with states $p_i$ and $p_j$ belonging to the same three-point amplitude. We can write [87]

$$\begin{aligned} u_{ia}\tilde{u}_{i\dot{a}} &= u_{ia}\tilde{u}_{i\dot{b}}\delta^{\dot{b}}_{\dot{a}} = u_{ia}\tilde{u}_{i\dot{b}}\frac{\langle P_b|i^{\dot{b}}]}{\langle P_b|i^{\dot{a}}]} \\ &= u_{ia}\tilde{u}_{i\dot{b}}\langle P_b|i^{\dot{b}}]\langle P_b|i^{\dot{a}}]^{-1} = u_{ia}\tilde{u}_{i\dot{b}}[i^{\dot{b}}|P_b\rangle\langle P^b|i_{\dot{a}}]\frac{1}{s_{iP}} \\ &= u_{ia}\tilde{u}_{j\dot{b}}[j^{\dot{b}}|P_b\rangle\langle P^b|i_{\dot{a}}]\frac{1}{s_{iP}} = (-)^{\mathcal{P}_{ij}}\langle i_a|j_{\dot{b}}][j^{\dot{b}}|P_b\rangle\langle P^b|i_{\dot{a}}]\frac{1}{s_{iP}} \\ &= \frac{(-)^{\mathcal{P}_{ij}}}{s_{iP}}\langle i_a|\hat{p}_j\hat{p}_P|i_{\dot{a}}] \ , \end{aligned} \tag{B.19}$$

where we have $(-)^{\mathcal{P}_{ij}} = +1$ for clockwise ordering of the states $(i, j)$ for the three-point amplitude. Also, $P$ is an arbitrary momentum. By the same series of manipulations we can show that

$$u_{ia}\tilde{u}_{i\dot{a}} = u_{ib}\tilde{u}_{i\dot{a}}\delta^b_a = \frac{(-)^{\mathcal{P}_{ji}}}{s_{iP}}\langle i_a|\hat{p}_P\hat{p}_j|i_{\dot{a}}] \ . \tag{B.20}$$

Note that the difference between (B.19) and (B.20) is just a sign since $(-)^{\mathcal{P}_{ji}} = -(-)^{\mathcal{P}_{ji}}$.

### B.3.2 The relation $w_l \cdot w_{l'}\ \tilde{w}_l \cdot \tilde{w}_{l'} = -s_{ij}^{-1}$

Here we provide an expression for the contraction between $w$- and $\tilde{w}$-spinors of two three-point amplitudes, connected by an internal propagator, originally encountered in the recursive calculation of the four-point tree amplitude in [85].

We start with expression B.19 and choose $i = 1, j = 4$ and $P = 2$, following Figure B.1. This yields

$$u_{1a}\tilde{u}_{1\dot{a}}s_{\hat{1}\hat{2}} = -\langle\hat{1}_a|\hat{p}_4\hat{p}_{\hat{2}}|\hat{1}_{\dot{a}}] \ . \tag{B.21}$$

However, we can also write

$$\begin{aligned} \langle\hat{1}_a|\hat{p}_4\hat{p}_{\hat{2}}|\hat{1}_{\dot{a}}] &= -u_{\hat{1}a}\tilde{u}^{\dot{d}}_4[4_{\dot{d}}|\hat{2}^b\rangle\langle\hat{2}_b|\hat{1}_{\dot{a}}] = -u_{\hat{1}a}\tilde{u}^{\dot{d}}_{l'}[l'_{\dot{d}}|\hat{2}^b\rangle\langle\hat{2}_b|\hat{1}_{\dot{a}}] \\ &= (-i)u_{\hat{1}a}\tilde{u}^{\dot{d}}_{l'}[l_{\dot{d}}|\hat{2}^b\rangle\langle\hat{2}_b|\hat{1}_{\dot{a}}] = iu_{\hat{1}a}\tilde{u}^{\dot{d}}_{l'}\tilde{u}_{l\dot{d}}u^b_{\hat{2}}\langle\hat{2}_b|\hat{1}_{\dot{a}}] \\ &= iu_{\hat{1}a}\tilde{u}^{\dot{d}}_{l'}\tilde{u}_{l\dot{d}}u^b_l\langle l_b|\hat{1}_{\dot{a}}] = u_{\hat{1}a}\tilde{u}^{\dot{d}}_{l'}\tilde{u}_{l\dot{d}}u^b_l\langle l'_b|\hat{1}_{\dot{a}}] \\ &= -u_{\hat{1}a}\tilde{u}^{\dot{d}}_{l'}\tilde{u}_{l\dot{d}}u^b_l u_{l'b}\tilde{u}_{\hat{1}\dot{a}} = -u_{\hat{1}a}\tilde{u}_{\hat{1}\dot{a}}\tilde{u}_{l'}\cdot\tilde{u}_l\ u_l\cdot u_{l'} \ . \end{aligned} \tag{B.22}$$

Comparing (B.21) and (B.22) we conclude

$$\tilde{u}_{l'}\cdot\tilde{u}_l\ u_{l'}\cdot u_l = -s_{12} \ , \tag{B.23}$$

since $s_{\hat{1}\hat{2}} = s_{12}$. Now we express the contractions of $u$-spinors in terms of $w$-spinors. As discussed in [85] we can deduce from (B.23) that

$$u_l\cdot w_{l'} = \tilde{u}_l\cdot\tilde{w}_{l'} = w_l\cdot u_{l'} = \tilde{w}_l\cdot\tilde{u}_{l'} = 0 \ , \tag{B.24}$$

by using the redundancy of the $w$-spinors under a shift $w_{la} \to w_{la} + b_l u_{la}$. Exploiting the defining relation between a spinor $u_l$ and its inverse $w_l$ and multiplying by $u_{l',a}$ and $w_{l',b}$ we have

$$u_l^a u_{l',a} w_l^b w_{l',b} - u_l^b w_{l',b} w_l^a w_{l',a} = u_{l',a} w_{l',b} \epsilon^{ab} \ . \tag{B.25}$$

Now, the second term on the RHS vanishes as stated in (B.24). Since $u_l \cdot u_{l'} \neq 0$, we have the relation

$$u_l \cdot u_{l'} \ w_l \cdot w_{l'} = 1 \quad \Leftrightarrow \quad u_l \cdot u_{l'} = \frac{1}{w_l \cdot w_{l'}} \ . \tag{B.26}$$

From this we can deduce that a spinor $w_{la}/w_{l'b}$ is related to the spinor $u_{l'a}/u_{lb}$, respectively, and we can choose to normalise as in (B.18)

$$w_{l_i'a} = \frac{u_{l_i a}}{\sqrt{-s_{ij}}} \ , \qquad\qquad w_{l_i a} = -\frac{u_{l_i' a}}{\sqrt{-s_{ij}}} \ . \tag{B.27}$$

### B.3.3 Spinor identities for the one-loop five-point calculation

Here we would like to outline some steps of the calculation which takes us from (4.89) to (4.91).

The basic idea is to express the result of the Grassmann integration as a sum of coefficients of factors $\tilde{\eta}_{i\dot{c}}\eta_{jc}$ with $i,j = 1,2,5$ for the (3,4)-cut. It is then a matter of algebra to rewrite the coefficient of $\tilde{\eta}_{i\dot{c}}\eta_{jc}$ in such a way that any dependence on the three-point quantities $w_{l_i}$, $w_{l_i'}$ and their counterparts in $\tilde{\eta}_{l_i}$ is removed. In the following we provide some explicit terms as examples.

Let us consider one of the terms of the product in (4.91), e.g.

$$\begin{aligned}\tilde{\eta}_{1\dot{c}}\eta_{1c}&\Big\{\langle 1^c|l_3]\cdot\tilde{w}_{l_3'}\tilde{w}_{l_2'}\cdot[l_2|\hat{l}_1|l_4]\cdot\tilde{w}_{l_4} - \langle 1^c|l_2]\cdot\tilde{w}_{l_2'}\tilde{w}_{l_3'}\cdot[l_3|\hat{l}_1|l_4]\cdot\tilde{w}_{l_4}\Big\}\\ \times&\Big\{[1^{\dot{c}}|l_3\rangle\cdot w_{l_3'}w_{l_2'}\cdot\langle l_2|\hat{l}_1|l_4\rangle\cdot w_{l_4} - [1^{\dot{c}}|l_2\rangle\cdot w_{l_2'}w_{l_3'}\cdot\langle l_3|\hat{l}_1|l_4\rangle\cdot w_{l_4}\Big\} \ .\end{aligned} \tag{B.28}$$

The first thing one realises is that the two factors in the brackets antisymmetrise among themselves. This can be seen by applying the normalisation relations for the $w$-spinors related to the internal momenta

$$\begin{aligned}[1^{\dot{c}}|l_3\rangle\cdot w_{l_3'}w_{l_2'}\cdot\langle l_2|\hat{l}_1|l_4\rangle\cdot w_{l_4} &= [1^{\dot{c}}|l_3\rangle\cdot w_{l_3'}\frac{u_{l_2}^a}{\sqrt{-s_{12}}}\langle l_{2a}|\hat{l}_1|l_4\rangle\cdot w_{l_4}\\ &= \frac{1}{\sqrt{-s_{12}}}u_{l_3'}^a w_{l_3'}^b[1^{\dot{c}}|l_{3b}\rangle\langle l_{3a}'|\hat{l}_1|l_4\rangle\cdot w_{l_4}\end{aligned} \tag{B.29}$$

where a similar relation is used for the second term of each bracket factor. Since

$$u_{l_3'}^a w_{l_3'}^b - u_{l_3'}^b w_{l_3'}^a = \epsilon^{ab} \ , \tag{B.30}$$

we can write (B.28) as

$$\begin{aligned}&\tilde{\eta}_{1\dot{c}}\eta_{1c}\Big(\frac{1}{\sqrt{-s_{12}}}\Big)^2[1^{\dot{c}}|l_{3b}\rangle\epsilon^{ab}\langle l_{3a}'|\hat{l}_1|l_4\rangle\cdot w_{l_4}\ \langle 1^c|l_{3\dot{b}}]\epsilon^{\dot{a}\dot{b}}[l_{3\dot{a}}'|\hat{l}_1|l_4]\cdot\tilde{w}_{l_4}\\ &=\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{i^2}{-s_{12}}[1^{\dot{c}}|l_3^a\rangle\langle l_{3a}|\hat{l}_1|l_4\rangle\cdot w_{l_4}\ \langle 1^c|l_3^{\dot{a}}][l_{3\dot{a}}|\hat{l}_1|l_4]\cdot\tilde{w}_{l_4}\\ &=\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{(-1)}{-s_{12}}[1^{\dot{c}}|\hat{l}_3\hat{l}_1|l_4\rangle\cdot w_{l_4}\ \langle 1^c|\hat{l}_3\hat{l}_1|l_4]\cdot\tilde{w}_{l_4}\\ &=\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{l}_1|l_4\rangle\cdot w_{l_4}\ \tilde{w}_{l_4}\cdot[l_4|\hat{l}_1\hat{p}_2|1^c\rangle \ ,\end{aligned} \tag{B.31}$$

where we have used momentum conservation at the second corner, $l_3 = l_2 + p_2 = l_1 + p_1 + p_2$, in the last line.

The next step is to remove the dependence on the $w$-spinors. The following relation holds:

$$\begin{aligned}\hat{l}_1|l_{4a}\rangle w^a_{l_4}\tilde{w}^{\dot{a}}_{l_4}[l_{4\dot{a}}|\hat{l}_1 &= \hat{p}_5|l_{4a}\rangle w^a_{l_4}\tilde{w}^{\dot{a}}_{l_4}[l_{4\dot{a}}|\hat{p}_5 = \hat{p}_5|l'_{1a}\rangle w^a_{l'_1}(-1)^2\tilde{w}^{\dot{a}}_{l'_1}[l'_{1\dot{a}}|\hat{p}_5 \\ &= \Big(\frac{1}{\sqrt{-s_{15}}}\Big)^2\hat{p}_5|l'_{1a}\rangle u^a_{l_1}\tilde{u}^{\dot{a}}_{l_1}[l'_{1\dot{a}}|\hat{p}_5 = \frac{1}{s_{15}}\hat{p}_5|1_a\rangle u^a_1\tilde{u}^{\dot{a}}_1[1_{\dot{a}}|\hat{p}_5\ .\end{aligned} \tag{B.32}$$

Using the result of (B.19) we arrive at the following string of momenta,

$$\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}s_{15}s_{1P}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{p}_1\hat{l}_1\hat{P}\hat{p}_1\hat{p}_5\hat{p}_2|1^c\rangle\ . \tag{B.33}$$

Choosing now $P = p_5$, after some rearrangement of the momenta we arrive at

$$\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}s_{15}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{p}_1\hat{l}_1\hat{p}_5\hat{p}_2|1^c\rangle\ . \tag{B.34}$$

This expression can be further simplified as follows: Since $l_1 = p_5 + l_4$ we have

$$\hat{l}_1\hat{p}_5 = \hat{l}_1(\hat{l}_1 - \hat{l}_4) = -(\hat{l}_4 + \hat{p}_5)\hat{l}_4 = -\hat{p}_5\hat{l}_1\ . \tag{B.35}$$

Permuting now the string of external momenta the final result for the coefficient becomes

$$-\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{l}_1\hat{p}_2|1^c\rangle = \tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{l}_1\hat{p}_5\hat{p}_2|1^c\rangle \tag{B.36}$$

by rearranging the order of $\hat{p}_5$ and $\hat{l}_1$ again.

This algebraic procedure can then be similarly repeated to simplify all the other coefficients in the cut expression (4.89).

## B.4 PV reduction of the linear pentagon $I^{\mu}_{5,l_1}$

We have found in Section 4.5 that the one-loop five-point superamplitude can be expressed in terms of just a single function, namely a linear pentagon,

$$I^{\mu}_{5,l_1}(1,\ldots,5) := \int \frac{d^Dl}{(2\pi)^D}\frac{l_1^{\mu}}{l_1^2l_2^2l_3^2(p_3+l_3)^2l_5^2}\ . \tag{B.37}$$

This can be decomposed on a basis of four independent momenta, as

$$I^{\mu}_{5,l_1}(1,\ldots,5) = Ap_1^{\mu} + Bp_2^{\mu} + Cp_3^{\mu} + Dp_5^{\mu}\ . \tag{B.38}$$

The choice of the basis vectors is most convenient one due to the kinematical structure of the cut expression in (4.91). Contracting with the basis momenta yields

$$\begin{aligned}2p_1\cdot I_{5,l_1} &= \int\frac{d^Dl}{(2\pi)^D}\frac{2p_1\cdot l_1}{\prod_{i=1}^5 l_i^2} = I_{4,1} - I_{4,5} \overset{!}{=} Bs_{12} + Cs_{13} + Ds_{15}\ , \\ 2p_2\cdot I_{5,l_1} &= \int\frac{d^Dl}{(2\pi)^D}\frac{2p_2\cdot l_1}{\prod_{i=1}^5 l_i^2} = I_{4,2} - I_{4,1} - s_{12}I_5 \overset{!}{=} As_{12} + Cs_{23} + Ds_{25}\ , \\ 2p_3\cdot I_{5,l_1} &= \int\frac{d^Dl}{(2\pi)^D}\frac{2p_3\cdot l_1}{\prod_{i=1}^5 l_i^2} = I_{4,3} - I_{4,2} - (s_{12}+s_{23})I_5 \overset{!}{=} As_{13} + Bs_{23} + Ds_{35}\ , \\ 2p_5\cdot I_{5,l_1} &= \int\frac{d^Dl}{(2\pi)^D}\frac{2p_5\cdot l_1}{\prod_{i=1}^5 l_i^2} = I_{4,4} - I_{4,4} \overset{!}{=} As_{15} + Bs_{25} + Cs_{35}\ .\end{aligned} \tag{B.39}$$

Solving the set of linear equations in (B.39), one obtains the desired coefficients $A, B, C$ and $D$, used in Section 4.5.3.

## B.5 Four-dimensional reduction

In this appendix we consider the four-dimensional limit of the four- and five-point tree-level amplitudes in pure Yang Mills theory, and provide detailed information of how the calculations of Section 4.5.5 are carried out.

We begin with the four-point amplitude of [85], given by (4.28), and reduce down to a four-dimensional amplitude with helicities $(1^-, 2^-, 3^+, 4^+, 5^+)$. Using (4.103), the four-dimensional reduction of (4.28) yields

$$A_4^{(4d)} = -\frac{i}{st}\langle 12\rangle^2[34]^2 = i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \tag{B.40}$$

Next, we consider the five-point amplitude (4.31) and reduce to a four-dimensional helicity configuration $(1^+, 2^+, 3^+, 4^-, 5^-)$. For this case, only a few terms in (4.31) survive. The $\mathcal{A}$-tensor becomes

$$\begin{aligned}\mathcal{A}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} = &\langle 1_a|\hat{p}_2\hat{p}_3\hat{p}_4\hat{p}_5|1_{\dot{a}}]\langle 2_b 3_c 4_d 5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}]\\ &+\langle 2_b|\hat{p}_3\hat{p}_4\hat{p}_5\hat{p}_1|2_{\dot{b}}]\langle 3_c 4_d 5_e 1_a\rangle[3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}1_{\dot{a}}]\\ &+\langle 3_c|\hat{p}_4\hat{p}_5\hat{p}_1\hat{p}_2|3_{\dot{c}}]\langle 4_d 5_e 1_a 2_b\rangle[4_{\dot{d}}5_{\dot{e}}1_{\dot{a}}2_{\dot{b}}]\ ,\end{aligned} \tag{B.41}$$

which in the four-dimensional limit takes the form

$$\begin{aligned}\mathcal{A}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} \xrightarrow{4d}\ &-[12]\langle 23\rangle[34]\langle 45\rangle[51]\times[23]^2\langle 45\rangle^2\\ &-[23]\langle 34\rangle[45]\langle 51\rangle[12]\times[31]^2\langle 45\rangle^2\\ &-[34]\langle 45\rangle[51]\langle 12\rangle[23]\times[12]^2\langle 45\rangle^2\,.\end{aligned} \tag{B.42}$$

For our specific helicity choice, the non-zero parts of the $\mathcal{D}$-tensor are those involving the Lorentz invariant brackets

$$\langle 2_b 3_c 4_d 5_e\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}]\ , \tag{B.43}$$

and

$$[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}]\langle 1_a 3_c 4_d 5_e\rangle\ , \tag{B.44}$$

where both reduce to $[23]\langle 45\rangle[13]\langle 45\rangle$ in four dimensions. Each factor multiplies $\langle 1_a(2.\tilde{\Delta}_2)_{\dot{b}}]$ and $[1_{\dot{a}}(2.\Delta_2)_b\rangle$. The quantities $\Delta_i$'s that are of interest here take the form:

$$\Delta_2 = \langle 2|\hat{p}_3\hat{p}_4\hat{p}_5 - \hat{p}_5\hat{p}_4\hat{p}_3|2\rangle\ , \quad \text{and} \quad \tilde{\Delta}_2 = [2|\hat{p}_3\hat{p}_4\hat{p}_5 - \hat{p}_5\hat{p}_4\hat{p}_3|2]\,. \tag{B.45}$$

Expanding the expression of the first non-vanishing $\mathcal{D}$-term yields,

$$\begin{aligned}\langle 1_a(2.\tilde{\Delta}_2)_{\dot{b}}] &= \langle 1_a|2^{\dot{b}}][2_{\dot{b}}|3^c\rangle\langle 3_c|4^{\dot{d}}][4_{\dot{d}}|5^e\rangle\langle 5_e|2_{\dot{b}}] - \langle 1_a|2^{\dot{b}}][2_{\dot{b}}|5^e\rangle\langle 5_e|4^{\dot{d}}][4_{\dot{d}}|3^c\rangle\langle 3_c|2_{\dot{b}}]\\ &\xrightarrow{4d} [12]\langle 23\rangle[34]\langle 45\rangle[52] - [12]\langle 25\rangle[54]\langle 43\rangle[32]\ .\end{aligned} \tag{B.46}$$

Using similar manipulations one can reduce the $[1_{\dot{a}}(2.\Delta_2)_b\rangle$ term. This yields:

$$2\mathcal{D}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} \xrightarrow{4d} 2\left([12]\langle 23\rangle[34]\langle 45\rangle[52] - [12]\langle 25\rangle[54]\langle 43\rangle[32]\right)\times[13][23]\langle 45\rangle^2. \tag{B.47}$$

Combining (B.42) and (B.47), one finds, after a little algebra, the expected Parke-Taylor result.

# C | Gaugino Masses from General Gauge Mediation

In this section we give a brief review on the techniques of general gauge mediation [184]. In particular this section generalises, to adjoint representation, the results obtained for fundamental messengers in Appendix B of [213]. We derive the expression for gaugino masses which was used in Section 6.4.2 Further literature on general gauge mediation relevant for our discussion can be found in [214–216].

## C.1 A brief overview

We start by considering the gauge current superfield (suppressing a group index $A$)

$$\begin{aligned}\mathcal{J} =& J + i\theta_\alpha j^\alpha + i\bar{\theta}^{\dot{\alpha}} j_{\dot{\alpha}} - \theta_\alpha \sigma^{\mu\alpha\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} j_\mu + \frac{1}{2}\theta_\alpha\theta^\alpha \bar{\theta}^{\dot{\alpha}} \bar{\sigma}^\mu_{\dot{\alpha}\alpha} \partial_\mu j^\alpha \\ & - \frac{1}{2}\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}}\theta^\alpha \bar{\sigma}^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{j}^{\dot{\alpha}} - \frac{1}{4}\theta_\alpha\theta^\alpha\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}} \Box J \; . \end{aligned} \tag{C.1}$$

Here, $\mathcal{J}$ is a real linear superfield defined by the current conservation relation

$$\bar{D}^2 \mathcal{J} = D^2 \mathcal{J} = 0. \tag{C.2}$$

One can derive each component of the current multiplet by looking at the kinetic terms that couple the chiral superfields to the gauge vector supermultiplet. Two parts are necessary to generate these currents. Firstly, the vector super field in Wess Zumino gauge

$$V^{WZ}_{ij} = V^A T^A_{ij} = \theta\sigma^\mu\bar{\theta} A_\mu + \theta\theta\bar{\theta}\bar{\lambda} + \bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta} D \tag{C.3}$$

in matrix notation where the gauge index $A$ runs from 1 to the dimension of the gauge group. And secondly the fully expanded chiral superfield

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) = & \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} \\ & - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^\mu\partial_\mu\phi(x) + (\theta\theta)F(x) \; .. \end{aligned} \tag{C.4}$$

Once one has the gauge current supermultiplet, one may take two point functions of the component currents. These two point functions are related to the soft terms for sfermions and gauginos as shown in [184]. For example to calculate the gaugino contribution one first looks at the fermionic current of the full gauge current supermultiplet. In general this has the form

$$j^A_\alpha(x) = -i\sqrt{2}(\phi^* T^A \psi - \tilde{\phi}^* T^A \tilde{\psi}) \; . \tag{C.5}$$

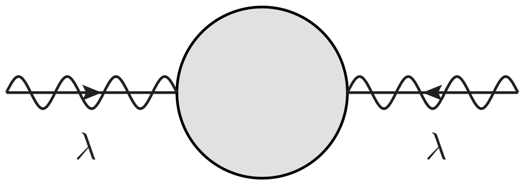



Figure C.1: *A general gauge mediation diagram for the gaugino mass at one loop. The grey object in the diagram represents the current correlator.*

The scalar and fermionic fields in this three vertex source term are the interaction eigenstates of the gauge messengers fields $\phi(x)$ and $\psi(x)$. Furthermore, the external legs of the source current are the gauginos $\lambda$ in a non-abelian representation. The external fields have been amputated but the corresponding generators $T^A$ are still part of the source. The gauge index $A$ runs from 1 to the dimension of the gauge group. The tilde represents the possibility of opposite charge conjugation of the field. The two point function of two fermionic currents is then given by [184]

$$\langle j^A_\alpha(x) j^B_\beta(0)\rangle = \epsilon_{\alpha\beta}\delta^{AB}\frac{C(\mathbf{r})}{x^5}B_{1/2}(x^2M^2) \tag{C.6}$$

where $M$ is a characteristic mass scale of the theory and $B_{1/2}$ is a complex function. We normalise the gauge group generators as $\mathrm{Tr}[T^AT^B] = C(\mathbf{r})\delta^{AB}$, with $C(\mathbf{r})$ as the quadratic invariant of the group represenation $\mathbf{r}$. Applying a Fourier transformation in order to write the current in momentum space,

$$M\tilde{B}_{1/2}(p^2/M^2) = \int d^4x e^{ip.x}\frac{1}{x^5}B_{1/2}(x^2M^2) \tag{C.7}$$

we obtain

$$\langle j^A_\alpha(p) j^B_\beta(-p)\rangle = \epsilon_{\alpha\beta}MC(\mathbf{r})\delta^{AB}\tilde{B}_{1/2}(p^2/M^2)\ . \tag{C.8}$$

The gaugino mass arrises as the term proportional to $\lambda_r\lambda_r$ in the effective Lagranian and is proportional to $\tilde{B}_{1/2}(0)$, explicitly we have

$$M_{\lambda_r} = g_r^2 MC(\mathbf{r})\tilde{B}_{1/2}(0) \tag{C.9}$$

for a couple constant $g_r$ for the gauge index $r$. Here, we have removed the $\delta^{AB}$ on both sides for clarity. We then use Ward identities to contract the fields in the two point function. In particular, we have for scalar fields

$$\langle \phi(x)_m\phi(0)_m\rangle = D(x,m) = \int \frac{d^4p}{(4\pi)^4}\frac{ie^{ip.x}}{p^2-m^2}. \tag{C.10}$$

Similar expressions for the sfermion contributions may be found in [184].

## C.2 Contributions to gaugino masses from SO(N) fields

We now demonstrate the utility for the the case of $\mathsf{SO}(10)$ for fields in a symmetric representation. These results would be similarly obtained for $\mathsf{SU}(5)$ for adjoint representation fields and other higher representations. We start with the Lagrangian

$$\mathcal{L}_{\mathrm{X}} = \int d^4\theta\left(X^\dagger e^{V^AT^A}X\right) \tag{C.11}$$

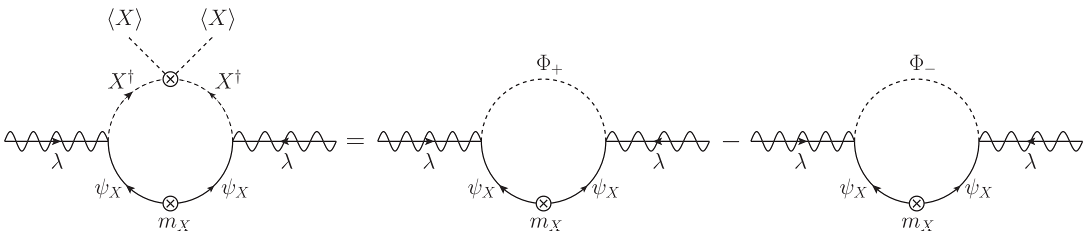



Figure C.2: *One loop gaugino masses from the symmetric X messenger field.*

The gauge superfield $V$ is in the antisymmetric (adjoint) and chiral superfield $X$ is in the symmetric representation of $\mathsf{SO}(N)$. We can amputate the component fields $(D^A, \lambda^A, \bar{\lambda}^A, A^A_\mu)$ of the full gauge supermultiplet and leave the generators as part of the source currents. In the following we write for the components of the chiral superfield $X = (\phi, \psi, F)$. In this notation we obtain for the currents

$$\begin{aligned} J^A &= (\phi^\dagger_{ij} T^A_{jk} \phi_{ki}) \ , \\ j^A_\alpha &= -i\sqrt{2}(\phi^\dagger_{ij} T^A_{jk} \psi_{ki}) \ , \\ \bar{j}^A_{\dot\alpha} &= i\sqrt{2}(\bar{\psi}_{ij} T^A_{jk} \phi_{ki}) \ , \\ j^A_\mu &= i(\partial_\mu \phi^\dagger_{ij} T^A_{jk} \phi_{ki} - \phi^\dagger_{ij} T^A_{jk} \partial_\mu \phi_{ki}) - \bar{\psi}_{ij} \sigma^\mu T^A_{jk} \psi_{ki} \ , \end{aligned} \tag{C.12}$$

where we have written out all gauge indices. In addition to the kinetic Lagrangian there is a mass term for the complex scalar field $\phi$,

$$\mathcal{L} \supset \left(\phi^\dagger \phi\right) M_0^2 \begin{pmatrix} \phi \\ \phi^\dagger \end{pmatrix} \ , \tag{C.13}$$

with a scalar squared mass matrix

$$M_0^2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \ , \tag{C.14}$$

and a mass term for the fermion with mass $m_\psi^2 = a$. We might diagonalise the scalar mass matrix to find the two real eigenvalues and their mass eigenstates,

$$\phi_+ = \frac{1}{\sqrt{2}}(\phi + \phi^\dagger) \ , \quad i\phi_- = \frac{1}{\sqrt{2}}(\phi - \phi^\dagger) \ . \tag{C.15}$$

Hence, the interaction eigenstates can be written as

$$\phi = \frac{1}{\sqrt{2}}(\phi_+ + i\phi_-) \ , \quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_+ - i\phi_-) \ . \tag{C.16}$$

The masses are

$$m_\pm^2 = a \pm b \quad m_\psi = \sqrt{a} \ . \tag{C.17}$$

We can rewrite the fermionic current which will couple to the gauginos in terms of mass eigenstates

$$j^A_\alpha = -(\phi_- + i\phi_+) T^A \psi_\alpha \ . \tag{C.18}$$

Considering the corresponding two point function and using Ward identities yields

$$\langle j^A_\alpha j^B_\beta \rangle = C(\mathbf{r}) \delta^{AB} \langle \psi_\alpha \psi_\beta \rangle [\langle \phi_- \phi_- \rangle + (i)^2 \langle \phi_+ \phi_+ \rangle] \ , \tag{C.19}$$

and we have for the fermion two-point function

$$\langle \psi_\alpha(x)\psi_\beta(0)\rangle = \epsilon_{\alpha\beta} m_\psi D(x, M_\psi) \ . \tag{C.20}$$

Now, we can substitute the corresponding propagators of the scalars and fermion which yields a diagrams shown in Figure C.2. We evaluate the mass from the symmetric X field to be

$$\begin{aligned} m_{\lambda_r} &= 2m_\psi g_r^2 R[X] C(\mathbf{r}) \left(D(x;m_+) - D(x;m_-)\right) D(x;m_\psi) \\ &= 2m_\psi g_r^2 R[X] C(\mathbf{r}) \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2+m_+^2} - \frac{1}{k^2+m_-^2}\right) \frac{1}{k^2+m_\psi^2} \ , \end{aligned} \tag{C.21}$$

where $R[X]$ is the rank of the representation of the $X$ field. The resulting gaugino masses are

$$m_{\lambda_r} = C(\mathbf{r}) R[X] \frac{\alpha_r}{4\pi} \frac{b}{\sqrt{a}} \times 2g(x) \tag{C.22}$$

where $x = b/a$ and the function $g(x)$ is given by

$$g(x) = \frac{(1-x)\text{Log}(1-x) + (1+x)\text{Log}(1+x)}{x^2} \ . \tag{C.23}$$

Similar reasoning results in an expression for the sfermions

$$m_{\tilde{f}}^2 = 2\sum_r C_{2\tilde{f}}^r (\frac{\alpha_r}{4\pi})^2 \text{T}(\mathbf{r}) R[X] \Lambda_S^2 \tag{C.24}$$

$$\Lambda_S^2 = \frac{b^2}{a} f(x) \tag{C.25}$$

where we have for $x = \frac{b}{a}$

$$f(x) = \frac{1+x}{x^2}\left[\ln(1+x) - 2\text{Li}_2\left(\frac{x}{1+x}\right) + \text{Li}_2\left(\frac{2x}{1+x}\right)\right] + (x \to -x) \ . \tag{C.26}$$

Here, $\alpha_r$ is the gauge coupling at the messenger scale and $C_{\tilde{f}}^r$ denotes the quadratic Casimir of the irrep $\tilde{f}$ of the gauge group labeled $r$.

For the fundamental messengers the calculation is analogous. In the $\mathsf{SU}(N)$ there is a fundamental ($\phi_i$) and antifundamental ($\tilde{\phi}_i$) messenger. For $\mathsf{SO}(N)$ the real and imaginary components of the fundamental messenger play these roles. The source for fundamental messenger fields $(\rho, Z)$ are computed using

$$\begin{aligned} J^A &= (\phi_i^\dagger T^A \phi_i) \ , \\ j_\alpha^A &= -i\sqrt{2}(\phi_i^\dagger T^A \psi_i) \ , \\ \bar{j}_{\dot\alpha}^A &= i\sqrt{2}(\bar\psi_i T^A \phi_i) \ , \\ j_\mu^A &= i(\partial_\mu \phi_i^\dagger T_i^A \phi_i - \phi_i^\dagger T^A \partial_\mu \phi_i) - \bar\psi_i \sigma^\mu T^A \psi_i \ . \end{aligned} \tag{C.27}$$

The label $i$ is a flavour index. The resulting gaugino and sfermions formulas are the same as in the $\mathsf{SU}(N)$ case of [213].

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