On Black Holes in String Theory

A thesis submitted for the degree of Doctor of Philosophy

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Abstract

This thesis investigates black holes in string theory through string amplitudes and through gauge-gravity duality. The research presented in this thesis supports the claim that string theory is capable of a consistent quantum-mechanical description of black holes and develops techniques which may prove useful in testing this claim in new scenarios.

The thesis comprises two parts. Part I describes novel disk amplitudes which derive the supergravity fields sourced by a D-brane with a travelling wave, and Part II describes free particle structures arising in a matrix model which is related through gauge-gravity duality to asymptotically anti-de Sitter black holes.

The disk amplitudes calculated in Part I provide a direct connection between the microscopic worldsheet description of a D-brane with a travelling wave and its macroscopic supergravity description. A D-brane carrying a travelling wave can be mapped via string dualities to the two-charge D1-D5 black hole and this research opens up the possibility to use these techniques to study the three-charge D1-D5-P black hole.

Part II of the thesis identifies free particle descriptions of non-holomorphic operators in a complex matrix model derived from dimensional reduction of $\mathcal{N} = 4$ Super-Yang-Mills theory. This research generalizes the free particle description in the half-BPS sector of this theory which was realized in supergravity and enabled studies of the microscopics of singular geometries. The free particle descriptions have been derived at zero gauge coupling; if these or similar structures are also present at strong coupling this research could be used to study the microscopics of non-extremal asymptotically anti-de Sitter black holes.

Declaration

This thesis is my own work. It is based on research carried out at Queen Mary, University of London and reported in the joint-authored papers [1] and [2].

This thesis has not been submitted previously in whole or in part for a degree examination at this or any other institution.

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Table of Contents

Introduction

Black holes are among the most fascinating objects in Nature; they are abundant in our universe, however certain fundamental aspects of their physics remain poorly understood. Black hole physics includes phenomena whose descriptions require the use of both of the two main pillars of theoretical physics, quantum mechanics and general relativity. Famously, as currently formulated, quantum mechanics and general relativity are mutually inconsistent theories. The challenge of constructing a consistent quantum theory of gravity is one of the major outstanding problems of theoretical physics today.

String theory is the leading candidate for a theory of quantum gravity and is thus a natural arena in which to explore outstanding problems in black hole physics. One such problem is the information paradox: black hole formation and evaporation as described by quantum field theory on curved spacetime leads to a violation of unitary or the formation of exotic remnant objects, either of which would require modifications to basic principles of physics. We shall explore this in more detail in Chapter 1.

A black hole is, roughly speaking, an extremely dark and compact object (we will be more careful about terminology shortly). In string theory, models of such objects may be constructed from strings and D-branes. We shall study these objects directly in Part I of this thesis.

D-branes are fundamental objects on which open strings can end [3], and have classical descriptions as solutions of the supergravity low-energy effective action. Mixed open/closed string amplitudes give a way to connect these two descriptions, in particular to derive information about the classical solution from the microscopic description [4, 5].

The research presented in Part I of this thesis investigates the gravitational description of bound states of strings and D-branes by calculating amplitudes for closed string emission. We first review the calculation for a flat D-brane [6, 7, 8, 9] and then describe research generalizing this to a wrapped D-brane carrying a travelling wave. These amplitudes directly probe the physics of bound states of D-branes and this research may lead to an improvement of our understanding of the three-charge D1-D5-P black hole.

A second major outstanding problem in the physics of black holes is to explain their entropy microscopically. The entropy of a black hole is proportional to its area, while more familiar systems in physics have entropy proportional to their volume. This suggests that black hole physics may have a holographic aspect. Over the past 14 years, much research in string theory has been devoted to investigating conjectured holographic dualities, such as the duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions [10, 11, 12], which we shall refer to as the AdS_5/CFT_4 duality.

The research presented in Part II of this thesis investigates the physics of a matrix model which plays an important role in the AdS_5/CFT_4 duality. Free particle structures in the half-BPS sector, derived in [13] and investigated in [14], were realized in supergravity in the LLM geometries [15] and this enabled studies of obtaining singular geometries from coarse-graining over a family of smooth geometries [16, 17] (see also [18, 19]). In Part II we derive free particle structures in non-holomorphic sectors of this matrix model using the Brauer algebra basis [20]. This generalizes the free particle description in the half-BPS sector.

The operators we study in Chapter 5 are in general not expected to be protected by non-renormalization theorems, however it has been conjectured that certain heavy operators may not receive large corrections [17]. If these or similar free particle structures are also present at strong coupling, this research could be used to study the microscopics of non-extremal asymptotically anti-de Sitter black holes.

Part I

Disk Amplitudes for Black Holes in String Theory

Chapter 1

Black Holes in Nature, General Relativity and String Theory

1.1 Black holes

When we think of black holes and talk about black holes, we sometimes mean rather different things. It can be helpful to draw distinctions between the black holes we talk about in different contexts - without trying to be precise about definitions, here we observe the following three distinctions:

- A '*classical black hole*': a geometrical solution to the equations of motion of a classical gravity theory, with a horizon causally dividing the spacetime manifold into the external universe and compact 'black hole' regions. For example: "Black holes have no hair" [21].

- A '*quantum black hole*': a model of a bound state of matter in a theory of quantum gravity which has properties of being very heavy, compact, and extremely dark. For example: "As one increases the string coupling, the size of a highly excited string state becomes less than its Schwarzschild radius, so it must become a black hole" [22].

- A '*physical black hole*': an object observed in nature which is very heavy, compact, and extremely dark; in particular, an object whose physics is well described by models based on classical black hole solutions. For example: "Our Galaxy's supermassive black hole" [23].

In what follows, we shall endeavour to be clear about what kind of black hole we are referring to in each context. In particular, we shall often speak of "a quantum mechanical model of a physical black hole".

1.2 Dark, compact objects in nature

Most observations of physical black holes fall into two mass ranges, stellar mass black holes and supermassive black holes of order 10^6-10^{10} solar masses (for reviews, see e.g. [24, 25]). The best evidence we have for a physical black hole is given by the observations of the compact radio source Sagittarius A* at the centre of our galaxy [26, 27, 28, 29, 30, 23, 31].

There does not yet appear to be a consensus over whether a horizon is a necessary feature of a model of a physical black hole; this would appear to depend on what exactly is meant by horizon. The observations appear to rule out any physically reasonable 'surface' where the classical event horizon should be located, favouring a model in which matter is accreted extremely efficiently onto the central body [31], however it has been argued that causality prevents a definitive detection of an event horizon as defined in GR [32]. It appears fair to say that the Schwarzschild and Kerr solutions form the basis of the best descriptions we have to date of physical black holes; for further discussion see e.g. [33].

1.3 Black hole solutions in general relativity

We briefly review some features of the Schwarzschild, Reissner-Nordstrom and Kerr solutions to general relativity (GR) highlighting only the features which are of relevance to this thesis. We use units where the speed of light $c = 1$ and we follow in places [34, 35, 36, 37, 38].

1.3.1 The Schwarzschild black hole

The Schwarzschild metric is the unique static, spherically symmetric solution to the vacuum Einstein equations in four dimensions, our conventions for which are given in Section A.1 of the Appendix. The line element is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) , \tag{1.3.1}$$

where

$$f(r) = 1 - \frac{2GM}{r} . \tag{1.3.2}$$

There is a coordinate singularity at $f(r) = 0$, i.e. $r = 2GM$. This does not present a problem when describing an astronomical body of the size and mass of the Earth, since the surface $r = 2GM$ is far inside the body. To put this another way, well before an object in free-fall towards the Earth encounters the surface $r = 2GM$, it will interact with the Earth's atmosphere and we require more than geodesic motion to describe this physics.

If we suppose that the matter content of a body was confined well within $r = 2GM$, and that the physics at $r = 2GM$ was well described by geodesic motion, the surface $r = 2GM$ would be the location of an event horizon and no timelike observer who fell beyond this surface could ever return. This is the Schwarzschild black hole.

There is also a curvature singularity at $r = 0$ signalling that the Schwarzschild metric is not a good description of any physics at this point. Thus, at some non-zero critical radius $r = r_{crit}$, geodesic motion on the Schwarzschild metric ceases to be a good description of physics (see e.g. [39]). It is an open question whether r_{crit} should be the Planck length, the string length, the horizon, or some other lengthscale; we will return to this question.

The Schwarzschild black hole has thermodynamic properties; by examining the near-horizon region, one obtains a temperature (denoting by κ the surface gravity at the horizon)

$$T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi GM} . \tag{1.3.3}$$

If the black hole satisfies the first law of thermodynamics in the form

$$dE = TdS \tag{1.3.4}$$

with the mass M being identified with the energy E, then we deduce that the black hole must have non-zero entropy. This is the Bekenstein-Hawking entropy [40] which we denote S_{Bek}, and is given by

$$S_{Bek} = \frac{A}{4G\hbar} = \frac{4\pi G M^2}{\hbar} . \tag{1.3.5}$$

As an aside, restoring units of the speed of light c and Boltzmann's constant k_B, the Schwarzschild radius becomes $r = 2GM/c^2$ and we obtain the formulae

$$T = \frac{\kappa}{2\pi}\frac{\hbar}{k_B c} = \frac{\hbar}{8\pi G M}\frac{c^3}{k_B} , \tag{1.3.6}$$

$$S_{Bek} = \frac{A}{4G\hbar}c^3 k_B = \frac{4\pi G M^2}{\hbar}\frac{k_B}{c} . \tag{1.3.7}$$

Historically, the conjecture that black holes should have entropy proportional to their area was first made by Bekenstein [40] following from the result of Hawking that the area of a black hole does not decrease with time [41], and by analogy with the second law of thermodynamics.

This analogy was strengthened with the proposal of a 'generalized second law of thermodynamics' [42], which says that the total entropy of black holes plus the total entropy of matter external to black holes does not decrease with time. The analogy between surface gravity and temperature was put on a firmer physical footing by the discovery of Hawking that semiclassically, black holes radiate with temperature T [43].

As a result, one should consider the above temperature and entropy to indeed be the physical temperature and entropy of the physical object described by the classical black hole solution. Since classical black solutions appear to describe physical black holes extremely well, any quantum mechanical model of a physical black hole must reproduce these properties in the classical limit. One of the challenges for any such quantum model is to obtain the entropy of black holes statistically, i.e. as the logarithm of a degeneracy of quantum states.

1.3.2 The Reissner-Nordstrom and Kerr black holes

The Reissner-Nordstrom line element describes an electrically charged solution to the Einstein-Maxwell equations in four dimensions, our conventions for which are given in Section A.2 of the Appendix. The line element is

$$ds^2 = -f_{RN}(r)dt^2 + \frac{dr^2}{f_{RN}(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) , \qquad (1.3.8)$$

where

$$f_{RN}(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} . \qquad (1.3.9)$$

As in the Schwarzschild solution, there is a coordinate singularity when $f_{RN}(r) = 0$, i.e. at

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \qquad (1.3.10)$$

which (for a body confined within r_-) form two horizons, the *outer* and *inner* horizons respectively. There is again a curvature singularity at $r = 0$.

In order to avoid a solution with a naked singularity (invoking the 'cosmic censorship hypothesis' [44]), we consider only the ranges of parameters which satisfy the bound

$$M \geq |Q| \qquad (1.3.11)$$

which is saturated for the 'extremal' choice of parameters $M = \pm Q$.

The temperature of the Reissner-Nordstrom black hole is given by

$$T = \frac{\kappa}{2\pi} = \frac{\sqrt{M^2 - Q^2}}{8\pi M\,(r_+ - Q^2)} \qquad (1.3.12)$$

and the entropy is

$$S = \frac{A}{4} = \pi r_+^2 . \qquad (1.3.13)$$

Note that in the extremal limit we obtain

$$T = 0 , \qquad S = \pi M^2 . \qquad (1.3.14)$$

Since there is a non-zero entropy in the limit of zero temperature, the extremal Reissner-Nordstrom black hole is a system which violates the third law of thermodynamics (which says that $S \to 0$ as $T \to 0$).

Since the extremal Reissner-Nordstrom black hole has zero temperature, it is a stable state in isolation. The three-charge D1-D5-P black hole in string theory which we will review shortly has an extremal Reissner-Nordstrom black hole as its classical limit.

The Kerr solution is a rotating, stationary solution to the vacuum Einstein equations and it describes the gravitational field sourced by a rotating astrophysical body [45]. We shall not need its explicit form in this thesis; we simply note that the causal structure of a Kerr black hole is the same as that of a Reissner-Nordstrom black hole and so physics which depends only on causal structure is common to both the Kerr and Reissner-Nordstrom solutions. For such questions it is usually easier to work with the Reissner-Nordstrom solution.

Since a black hole has a temperature and an entropy, it is natural to ask whether one can build a quantum mechanical model of a black hole which explains these properties via statistical mechanics. String theory is the leading theory of quantum gravity, and we next review examples of quantum black holes in string theory.

1.4 Black holes in string theory

In this section we will review selected aspects of black holes in string theory, focusing on examples which have relevance to this thesis.

In order to model a physical black hole, we are interested in constructing solutions to string theory which reduce to classical black holes in an appropriate classical limit. As a result we are interested in constructing bound states of matter with large degeneracies, in order to give a statistical explanation of the entropy of classical black holes. These bound states will be built from the fundamental building-blocks of matter in the theory, namely strings and D-branes.

As we shall review, the extra dimensions of string theory allow one to construct objects which are localized from the point of view of physics in lower dimensions, and we shall construct states which are BPS in order to extrapolate certain quantities from zero string coupling to large string coupling.

Since we are interested in BPS bound states, the configurations we study are bound states at threshold, i.e. the energy of the bound state is the same as the sum of the energies of its constituent parts. A threshold bound state is distinguished from a configuration which is a simple superposition of non-bound constituents in the following way: if the constituents solve the equations of the theory for arbitrary relative separation then the solution is not a bound state; otherwise, one is indeed dealing with a bound state.

1.4.1 Two-charge black holes in five dimensions

A simple example of a BPS configuration of string theory with a large degeneracy of states is a fundamental string carrying large winding number n_w and left (or right) moving momentum n_p in a compact direction [46, 47].

Since there are no longitudinal oscillations of a fundamental string, the momentum must be carried in the form of a transverse travelling wave along the string. The wave travels at the speed of light and is thus described by an arbitrary profile function $f(v)$ depending on a light-cone coordinate v.

For a heterotic string, to leading order in the large charges, such a state has a degeneracy (see e.g. [46])

$$d_{micro} \;\sim\; e^{4\pi\sqrt{n_w n_p}} \tag{1.4.1}$$

and so to leading order the microscopic entropy of this system is

$$S_{micro} \;=\; \log d_{micro} \;\sim\; 4\pi\sqrt{n_w n_p}\;. \tag{1.4.2}$$

The supergravity solutions sourced by such a string were first written down for the heterotic string [48, 49] by solving the supergravity equations in the presence of a delta-function source at the location of the string profile.

For our purposes, we will work with the analogous solutions of type IIB supergravity, our conventions for which are given in Section A.3 of the Appendix.

We study solutions with five non-compact directions, more specifically solutions on $\mathbb{R}^{4,1}\times S^1\times T^4$ using the light-cone coordinates $u=(t+y)\,,\ v=(t-y)$ constructed from the time and S^1 directions. The indices $(I,J,\ldots)$ refer collectively to the other eight directions which we then split into the $\mathbb{R}^4$ directions $x^1,\ldots,x^4$ labelled by $(i,j,\ldots)$ and the T^4 directions $x^5,\ldots,x^8$ labelled by $(a,b,\ldots)$.

We take the string to be wrapped n_w times around y, and smeared along the T^4 and y directions [50, 51]. Letting the length of the y direction be $2\pi R$, the brane then has overall extent $L_T = 2\pi n_w R$ and we use $\hat{v}$ for the corresponding world-volume coordinate on the D-brane, having periodicity L_T. The non-trivial fields are the metric, B-field and dilaton:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}} dv \left(- du + K dv + 2 A_I dx^I \right) + H^{\frac{1}{2}} dx^I dx^I \ , \\ e^{2\Phi} &= g_s^2 H \ , \quad B^{(2)}_{uv} = \tfrac{1}{2}(H^{-1} - 1) \ , \quad B^{(2)}_{vI} = H^{-1} A_I \ , \end{aligned} \tag{1.4.3}$$

where the harmonic functions take the form

$$\begin{aligned} H &= 1 + \frac{Q_{F1}}{L_T} \int_0^{L_T} \frac{d\hat{v}}{|x_i - f_i(\hat{v})|^2} \ , \qquad A_I = -\frac{Q_{F1}}{L_T} \int_0^{L_T} \frac{d\hat{v} \dot{f}_I(\hat{v})}{|x_i - f_i(\hat{v})|^2} \ , \\ K &= \frac{Q_{F1}}{L_T} \int_0^{L_T} \frac{d\hat{v} |\dot{f}_I(\hat{v})|^2}{|x_i - f_i(\hat{v})|^2} \ , \end{aligned} \tag{1.4.4}$$

where $f_i(\hat{v} + L_T) = f_i(\hat{v})$ and where $\dot{f}$ denotes the derivative of f with respect to $\hat{v}$. In the above we have used the abuse of notation

$$|x_i - f_i(\hat{v})|^2 = \sum_i (x_i - f_i(\hat{v}))^2 \ , \qquad |\dot{f}_I(\hat{v})|^2 = \sum_I (x_I - f_I(\hat{v}))^2 \ . \tag{1.4.5}$$

The functions f_I describe classically the null travelling wave on the fundamental string. Q_{F1} is proportional to g_s and to the string winding number n_w and is given by

$$Q_{F1} = \frac{(2\pi)^4 n_w g_s^2 (\alpha')^3}{V_4} \ . \tag{1.4.6}$$

One can use S and T dualities to dualize these solutions to the D1-D5 duality frame, as follows (for more details see e.g. [52]):

$$F1\text{-}P \xrightarrow{S} D1\text{-}P \xrightarrow{T5678} D5\text{-}P \xrightarrow{S} NS5\text{-}P \xrightarrow{Ty5} NS5\text{-}F1 \xrightarrow{S} D1\text{-}D5 \ . \tag{1.4.7}$$

In the D1-D5 duality frame, the corresponding supergravity solutions become everywhere smooth, non-singular horizonless geometries. This illustrates that the property of whether or not a supergravity solution is everywhere smooth is not a duality-invariant property. This observation was highlighted recently [53] and we shall discuss this further in Chapter 3.

1.4.2 Three-charge black holes in five dimensions

A three-charge black hole in string theory may be constructed by adding left-moving (or right-moving) momentum to a D1-D5 bound state and considering the limit of large charges. We refer to this as a D1-D5-P bound state. There is an extremal Reissner-Nordstrom black hole with a macroscopic horizon in the reduced five-dimensional supergravity with these charges, whose Bekenstein-Hawking entropy agrees in the large charge limit with the microscopic entropy of the low energy degrees of freedom of the D1-D5-P system [54].

This agreement between macroscopic and microscopic entropy is possible because the degeneracy of BPS states is a protected quantity, meaning that it can be calculated at weak coupling and extrapolated to strong coupling. This provided the first example of the entropy of a black hole with a macroscopic horizon being reproduced from a microscopic string theory calculation.

A general remark is in order at this point: given a bound state of N Dp-branes without a momentum charge, we will see in the next chapter that the characteristic size of the bound state of D-branes is set by the lengthscale R_p, which is given (for $p < 7$) by

$$R_p^{7-p} \;\sim\; g_s N \sqrt{\alpha'}^{7-p} \;. \tag{1.4.8}$$

For the D1-D5 system, if there are n_1 D1 branes and n_5 D5 branes, then the effective value of N is $n_1 n_5$.

When one considers a momentum charge this also enters into the size of the bound state. For the D1-D5-P black hole, the horizon size is [54]

$$R \;\sim\; \sqrt{g_s}(n_1 n_5 n_p)^{1/6}\sqrt{\alpha'} \;. \tag{1.4.9}$$

So we see that it is only for the regime of parameters where $g_s^3(n_1 n_5 n_p) \gg 1$ that the horizon is large in string units and the curvature is small at the horizon, in order that the classical black hole solution is a valid supergravity solution at the horizon scale. We shall come back to this discussion in the next chapter when we discuss the regime of validity of the disk amplitude calculations that we shall present in this thesis.

1.4.3 Asymptotically anti-de Sitter Black Holes

The conjectured AdS/CFT duality [10, 11, 12] has been a major research theme in string theory over the last 14 years, and has led to many studies of asymptotically anti-de Sitter black holes. As for the examples mentioned above, one would like to construct consistent quantum mechanical models which explain the entropy of these black holes statistically.

We will focus on asymptotically AdS_5 black holes, which are related to the research presented in Part II of this thesis. Examples of such black holes are the 'large' [55] and 'small' [56] Schwarzschild-AdS_5 black holes, and the supersymmetric 1/16-BPS black hole [57, 58] whose entropy remains to be fully understood.

Of particular interest in this thesis are the R-charged asymptotically $AdS_5 \times S^5$ black holes [59, 60] obtained by uplifting asymptotically AdS_5 solutions to $\mathcal{N} = 2$ $U(1)^3$ gauged 5D supergravity [61, 62]. The extremal limits of these black holes produce supergravity solutions which have naked null singularities at the two-derivative supergravity level. These solutions are known as 'superstars' [60] and depending on the number of independent R-charges there are 1/2-BPS, 1/4-BPS, and 1/8-BPS solutions. They have been interpreted as 'incipient' black holes, in the sense that any small energy added above extremality produces a non-zero size classical horizon [17].

Using AdS/CFT duality and an explicit free particle description [13, 14], the microscopic entropy of the 1/2-BPS superstar has been studied quantitatively both in the dual field theory and in gravity [17] where there is a family of smooth supergravity solutions [15]. The dependence on N of the entropy of the large Schwarzschild-AdS_5 black hole [17] and near-extremal R-charged black holes [63] has also been understood qualitatively using AdS/CFT, as we shall describe in more detail in Chapter 4.

1.5 The information paradox

Having briefly reviewed some examples of black holes in string theory we now review the information paradox. This has been a stubborn outstanding problem in black hole physics for over three decades, since the seminal papers of Hawking [43, 64]. As we shall see, the information paradox imposes constraints on the properties of any quantum model of a physical black hole. In order to meaningfully discuss the paradox, it is necessary to be rather specific in places and we ask for the reader's patience in this respect.

1.5.1 Mixed states and remnants

Before we discuss the precise statement of paradox, we introduce some of the possible consequences of black hole formation and evaporation, namely evolution into *mixed states* and production of *remnants*. We follow in places the treatment in [35].

A *mixed state* arises in quantum mechanics when a system consists of two subsystems, A and B, which have previously been in contact but are no longer interacting. For our purposes we think of A as the external region to the black hole and B the black hole region. The combined system has a wavefunction $\Psi(\alpha, \beta)$ where α and β are commuting variables for the subsystems A and B. A is described by a density matrix ρ_A, defined by

$$\rho_A(\alpha, \alpha') = \sum_{\beta} \Psi^*(\alpha, \beta)\Psi(\alpha', \beta) \tag{1.5.1}$$

however if the black hole system B evaporates completely without the information content of the black hole escaping into region A, the combined system will have evolved in a non-unitary fashion from a pure state to a mixed state [64] (see also [65]).

A *remnant* is an object with finite bounded mass and size which may have an arbitrarily large entanglement with systems far away from itself. Such remnants are problematic for physics for the following reasons. Firstly, having an arbitrarily large entanglement entropy means that remnants would necessarily violate the Bekenstein entropy bound [66] and thus mergers of remnants with black holes would violate the generalized second law of thermodynamics [67].

Secondly, if remnants are Planck-sized objects, in order to keep track of all the possible states that can form an arbitrarily large black hole, the number of distinct species of remnant must be infinite, leading to possibly infinite rates of production of remnants [68, 69].

In this thesis we shall take the point of view that non-unitarity and remnants are unacceptable features of a physical theory, and investigate the alternative resolution of the information paradox offered by the fuzzball proposal.

1.5.2 The Hawking theorem

Similarly to the term 'black hole', sometimes people mean different things when they use the term 'information paradox'. Hawking's original paradox is of the following form:

(a) The formation and evaporation of a black hole as described by semi-classical gravity (quantum field theory in the background of a black hole) leads to mixed states or remnants.

However the following (related) statement of the problem is also often encountered:

(b) Supposing that black hole formation and evaporation is unitary, how does the information about the matter which went into making up the black hole actually get out? (See e.g. [70, 71]).

The first of these, (a), is in my opinion far more serious and we shall focus on this, following the treatment in [72].

The precise statement of the Hawking theorem in the language of [72], is as follows:

To formulate the theorem, we assume that:

(A) There exists a 'solar system limit' in which physics can be described by known, local, semiclassical physics up to Planck scale corrections. This limit is described by 'niceness conditions' such as all curvatures being small compared to the Planck length and all matter satisfying appropriate energy conditions.

(B) There is a configuration of matter in the theory whose physics is well described by a classical black hole solution with an 'information-free horizon', where a point on the horizon is called 'information-free' if the evolution of fields in the neighbourhood of the point is given by the semiclassical evolution of quantum fields on 'empty' curved space, up to corrections controlled by the Planck scale.

Then it follows that:

1. To leading order, the emission of each Hawking quantum increases the entanglement entropy of the black hole with the exterior by a fixed amount, so the total entanglement entropy increases linearly for the lifetime of the black hole for which the assumptions of the theorem hold.

2. The evaporation of the black hole then leads either to a mixed final state (in the case that the black hole evaporates completely) or to a remnant.

3. Importantly, *allowing for small corrections* to Hawking's calculation, the results are robust [72].

This statement of the Hawking theorem forces us to either:

(i) accept loss of unitarity/remnants in black hole physics as a feature of quantum gravity, and revise our physical laws accordingly;

(ii) violate the assumptions of the Hawking theorem by

- violating assumption A and revising the 'solar system limit' assumption in some way, within the constraints of experimental tests of general relativity to date.

- violating assumption B, either by modifying local quantum field theory (see e.g. [73]) or by constructing a more refined description of a physical black hole.

Resolving the information paradox by constructing a more refined description of a physical black hole would avoid having to modify our current formulations of quantum mechanics or local quantum field theory. This seems to me the more conservative option, the option more likely to be correct, and a topic very much worth pursuing. However it very much remains to be shown whether or not this is the correct answer. In the next section we describe one such attempt to resolve the information paradox within string theory.

1.6 The fuzzball proposal for black holes

The fuzzball proposal [74, 52, 75] is the conjecture that:

1. Black hole formation and evaporation is unitary, and the Hawking theorem is avoided by violating assumption B, i.e. claiming that the classical horizon does not provide a good description of all physics in this region;

2. Physics at the location of the horizon should be affected by the bound state of matter making up the black hole having a non-trivial size.

Note that the fuzzball proposal does not make any statement about whether the classical black hole is a good description of the physics experienced by an infalling classical observer; it is a proposal for the physics of Hawking radiation. This has been described as "the separation of the *information paradox* and the *infall problem*" [76]. It is an open question as to whether or not an explicit fuzzball model of a black hole reproduces the gentle experience of a classical observer falling into a large black hole; this is an important question for the program to address (see also [77]).

If the fuzzball proposal gives the correct quantum mechanical model of a black hole, then the awkward theoretical consequences of the Hawking theorem are avoided, and the information in the black hole appears to come out in a similar way to when a bowl of water evaporates, or a lump of coal is burned [78] (see also [79]).

Having stated the fuzzball proposal and its consequences if correct, we now briefly review its status.

The fuzzball proposal grew out of studies of two-charge black holes in string theory [50, 51, 80, 81, 74]. Recalling the model of the wrapped oscillating fundamental string reviewed in Section 1.4.1, we saw that the string carried a large degeneracy of states by vibrating in the transverse directions; this means that the system occupies a non-trivial size.

Using the convenient fact that *classical* vibrations of the string can be well described by the (horizonless) supergravity solutions in Section 1.4.1, an estimate was obtained for the length scale at which geometries describing classical vibration profiles start to differ from one another. This length scale was then extrapolated to the *generic* vibration profile of the string to obtain an estimate for the length scale at which generic states of the string give different physics. There is no classical black hole geometry at the two-derivative level, however if one places a stretched horizon (see e.g. [82]) at this length scale, the area of this horizon reproduces the entropy coming from the microscopic count of states of the system [51].

It is important to note that the generic state of the system is not expected to be well described by supergravity. Indeed, the name '*fuzzball*' is intended to give a sense of the fact that certain physics around the horizon of the classical black hole solution should require descriptions *beyond* smooth geometry. Here, the smooth supergravity solutions are used as a (coarse) tool to probe the physics of the generic, very quantum, state using the technically convenient description of states in the Hilbert space which happen to have good classical descriptions.

The next main challenge is to extend this program to the three-charge D1-D5-P extremal black hole reviewed in Section 1.4.2, and from there to other classes of black holes, in particular non-extremal black holes. Following the progress in the two-charge system, and since supergravity is often easier to deal with than string theory, much effort has been focused on constructing smooth horizonless three-charge supergravity solutions.

Various classes of smooth horizonless three-charge supergravity solutions are now known [83, 84, 85, 86, 87, 88], and a class of non-extremal three-charge solutions has also been constructed [89]. The ergoregion emission from these geometries has been interpreted as Hawking radiation [90, 91, 92, 93, 94, 95], suggesting that even non-extremal black holes may admit fuzzball descriptions.

The construction of these supergravity solutions then leads to the question: given a horizonless supergravity solution with appropriate charges, how do we know whether or not it corresponds to a microstate of a black hole? (See e.g. [53]). As we shall see, the research presented in Chapter 3 addresses exactly this question in the two-charge system, and opens the possibility to address this question also in the three-charge system.

In summary, the fuzzball proposal is a promising program which holds the potential to resolve the information paradox by constructing a more refined model of a physical black hole. It offers the possibility of doing so without sacrificing solar system physics or even the physics of the infall of a classical observer into a black hole, although this remains to be investigated. The fuzzball program is still at an early stage and many questions remain, especially with regard to non-extremal black holes; this is an exciting and active area of theoretical physics and will likely continue to be so for some time.

Chapter 2

The Supergravity Fields for a flat D-brane from String Amplitudes

Outline of Chapter 2

In this chapter we review the derivation of the asymptotic supergravity fields sourced by a flat D-brane from world-sheet disk amplitudes, using the boundary state formalism. This allows us to set up the technology which forms the basis of the research presented in Chapter 3.

The physics behind this calculation is that the regime of being at weak coupling and at large distance from the D-brane is a regime in which both perturbative string theory and supergravity are valid descriptions; this is because the interaction of a D-brane with a probe far from the D-brane is dominated by the exchange of massless closed strings, which can be thought of as an interaction between the probe and the background fields generated by the D-brane [4].

The structure of this chapter is as follows:

In Section 2.1 we set out our conventions for the type IIB superstring world-sheet theory, and derive the boundary conditions appropriate for a flat Dp-brane.

In Section 2.2 we review the construction of the boundary state for a flat D-brane, and in Section 2.3 we review the application of the boundary state to the derivation of the supergravity fields sourced by such a D-brane.

2.1 World-sheet CFT and conventions

In this section we set out our notation and conventions for the type IIB superstring world-sheet CFT, working in the Ramond-Neveu-Schwarz formalism.

2.1.1 Closed superstring

We start with a closed superstring in which the world-sheet metric has Euclidean signature and we use complex coordinates $z = \exp(\tau + i\sigma)$ where $\tau \in \mathbb{R}$ and σ is periodic with period 2π. We use the following action:

$$S = \frac{1}{2\pi\alpha'} \int d^2z \, \left(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu \right) . \tag{2.1.1}$$

Varying the action gives boundary terms which can be solved by imposing periodicity conditions on the fields as follows: considering for simplicity the case of a non-compact direction (and so ignoring winding modes) the bosons are periodic,

$$X^\mu(e^{2\pi i}\, z, e^{-2\pi i}\, \bar{z}) \;=\; X^\mu(z, \bar{z}) \tag{2.1.2}$$

while the left and right-moving fermions may be periodic or antiperiodic:

$$\begin{aligned} \psi^\mu(e^{2\pi i}\, z) &= e^{2\pi i \nu} \psi^\mu(z) \\ \widetilde{\psi}^\mu(e^{-2\pi i}\, \bar{z}) &= e^{2\pi i \widetilde{\nu}} \widetilde{\psi}^\mu(\bar{z}) \end{aligned} \tag{2.1.3}$$

where for the left-moving fields $\nu = 0$ gives the Ramond (R) sector and $\nu = \frac{1}{2}$ gives the Neveu-Schwarz (NS) sector, and similarly for $\widetilde{\nu}$, giving rise to four sectors: NS-NS, NS-R, R-NS, R-R.

The action (2.1.1) is invariant under the supersymmetry transformations

$$\delta X^\mu = \varepsilon \psi^\mu + \tilde{\varepsilon} \widetilde{\psi}^\mu \;, \quad \delta \psi^\mu = -\varepsilon \partial X^\mu \;, \quad \delta \widetilde{\psi}^\mu = -\tilde{\varepsilon} \bar{\partial} X^\mu \;. \tag{2.1.4}$$

The equations of motion and boundary conditions are solved by the mode expansions

$$X^\mu(z,\bar z) \;=\; x^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \ln z - i\sqrt{\frac{\alpha'}{2}}\widetilde\alpha_0^\mu \ln \bar z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m}+\frac{\widetilde\alpha_m^\mu}{\bar z^m}\right),$$
$$\psi^\mu(z) \;=\; \sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}+\nu}\frac{\psi_r^\mu}{z^{r+\frac12}}\,, \qquad \widetilde\psi^\mu(\bar z) \;=\; \sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}+\nu}\frac{\widetilde\psi_r^\mu}{\bar z^{r+\frac12}}\,, \tag{2.1.5}$$

where we set $\alpha_0^\mu = \widetilde\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$.

After canonical quantization, the non-zero commutation relations for the left-moving fields and zero modes are:

$$\left[\alpha_m^\mu,\alpha_n^\nu\right] \;=\; m\,\delta_{m+n,0}\,\eta^{\mu\nu}\,, \qquad \left[\psi_r^\mu,\psi_s^\nu\right] \;=\; \delta_{r+s,0}\,\eta^{\mu\nu}\,,$$
$$\left[x^\mu,p^\nu\right] \;=\; i\,\eta^{\mu\nu} \tag{2.1.6}$$

and similar commutation relations hold for the right-moving fields.

The vacuum of the bosonic fields $|0;k^\mu\rangle$ is defined by

$$\alpha_m\,|0;k^\mu\rangle \;=\; \widetilde\alpha_m\,|0;k^\mu\rangle \;=\; 0 \qquad \text{for} \quad m\geq 1\,,$$
$$p^\mu\,|0;k^\mu\rangle \;=\; k^\mu\,|0;k^\mu\rangle\,. \tag{2.1.7}$$

The vacuum of the left-moving fermions is defined in the NS sector by

$$\psi_r^\mu\,|0\rangle_{\rm NS} \;=\; 0 \qquad \text{for} \quad r>0\,, \tag{2.1.8}$$

and similarly in the R sector; in the R sector the ground state is degenerate due to the fermion zero modes which satisfy the Clifford algebra with

$$\Gamma^\mu \;\longleftrightarrow\; \sqrt{2}\psi_0^\mu \qquad \Rightarrow \qquad \{\Gamma^\mu,\Gamma^\nu\} \;=\; 2\eta^{\mu\nu}\,. \tag{2.1.9}$$

We shall return to the R sector zero modes after introducing our conventions for the open string.

2.1.2 Open superstring

For the open superstring, we again consider a Euclidean world-sheet and use complex coordinates $z = \exp(\tau + i\sigma)$ where $\tau \in \mathbb{R}$ and now $\sigma \in [0, \pi]$. The action is otherwise unchanged; varying the action gives rise to boundary conditions at $\sigma = 0$ and $\sigma = \pi$, i.e. at $z = \bar{z}$. If we assume Neumann boundary conditions on all spacetime coordinates, the boundary conditions on the world-sheet fields take the form

$$\begin{aligned} (\partial X^\mu - \bar{\partial} X^\mu)\Big|_{z=\bar{z}} &= 0, \\ (\psi^\mu - \eta\widetilde{\psi}^\mu)\Big|_{z=\bar{z}} &= 0. \end{aligned} \qquad (2.1.10)$$

By convention we set $\eta = +1$ at $\sigma = 0$; then at $\sigma = \pi$, setting $\eta = +1$ ($\eta = -1$) gives the Neveu-Schwarz (Ramond) sector.

The presence of the world-sheet boundary breaks half of the world-sheet supersymmetry, such that the action is now invariant only under the subset of transformations (2.1.4) for which

$$\tilde{\varepsilon} = \eta\varepsilon\ . \qquad (2.1.11)$$

We can rewrite the fields appearing in (2.1.10) in modes using the expansions in the closed string mode expansions (2.1.5), in terms of which the (Neumann) boundary conditions identify the right and left-moving oscillators as follows:

$$\widetilde{\alpha}^\mu_n = \alpha^\mu_n\ , \qquad \widetilde{\psi}^\mu_r = \eta\,\psi^\mu_r \qquad (2.1.12)$$

and p^μ is unconstrained for Neumann boundary conditions.

The spectrum of the open string is the same as that of one side (e.g. left-movers) of the closed string, reviewed in the previous section.

From these expressions one can derive the boundary conditions for a Dp-brane by T-dualizing along the $(9-p)$ transverse directions, which we label by x^i. This sends

$$\widetilde{\alpha}^i_n \to -\widetilde{\alpha}^i_n\ , \qquad \widetilde{\psi}^i_r \to -\widetilde{\psi}^i_r\ . \qquad (2.1.13)$$

We work only with strings of zero winding number; focusing on the $\sigma = 0$ endpoint at attached to a Dp-brane located at y^i in the transverse directions we have

$$X^i|_{\sigma=0} \;=\; y^i \;. \tag{2.1.14}$$

Introducing the reflection matrix R to keep track of Neumann and Dirichlet directions,

$$R^{\mu\nu} \;=\; (\eta^{\alpha\beta}, -\delta^{ij}) \;, \tag{2.1.15}$$

we thus have the following boundary conditions for an open string with the end-point at $\sigma = 0$ attached to a flat Dp-brane:

$$\widetilde{\alpha}^{\mu}_{n} \;=\; R^{\mu}{}_{\nu}\, \alpha^{\nu}_{n} \;, \qquad \widetilde{\psi}^{\mu}_{r} \;=\; \eta\, R^{\mu}{}_{\nu}\, \psi^{\nu}_{r} \;, \qquad x^i \;=\; y^i \;. \tag{2.1.16}$$

We shall be interested in mixed open/closed string amplitudes and a useful tool is open-closed string duality. In order to explain this, consider the cartoon of the process of emission of a closed string from a D-brane shown in Fig. 2.1.

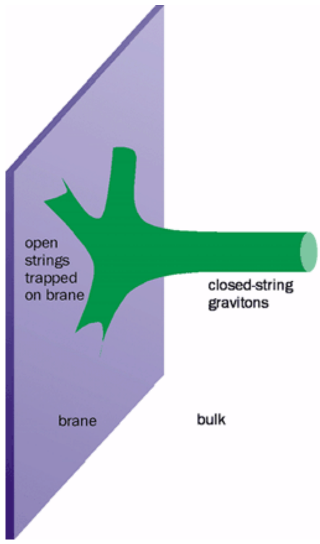


Figure 2.1: Closed string emission from a D-brane. Source: *physicsworld.com* [96]

From the open string point of view, one can describe this process as the emission of an on-shell closed string from an open string world-sheet, as sketched in Fig. 2.2. Alternatively, one can think of this process as a closed string being created by the D-brane and instead formulate the calculation in the closed string picture. The technology of the boundary state enables us to do exactly this, as we shall see.

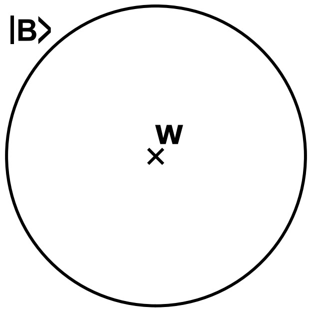


Figure 2.2: Corresponding string Feynman diagram for disk amplitude.

Changing from the open string picture to the closed string picture involves exchanging the world-sheet coordinates σ and τ; this is realized as a transformation on the string oscillators as (see e.g. [9]):

$$\alpha_n^\mu \;\rightarrow\; -\,\alpha_{-n}^\mu \,, \qquad\qquad \psi_r^\mu \;\rightarrow\; i\psi_{-r}^\mu \qquad\qquad \forall\,\mu,n,r\;. \tag{2.1.17}$$

This means that in the closed string picture, the boundary conditions for a closed string being emitted from a flat Dp-brane at $\tau = 0$ take the form

$$\widetilde{\alpha}_n^\mu \;=\; -\,R^\mu{}_\nu \alpha_{-n}^\nu \,, \qquad \widetilde{\psi}_r^\mu \;=\; i\eta\, R^\mu{}_\nu \psi_{-r}^\nu \,, \qquad p_\alpha \;=\; 0 \,, \qquad x^i \;=\; y^i \;. \tag{2.1.18}$$

2.1.3 Spinor conventions

In order to deal with the Ramond sector zero modes, we here describe our spinor conventions. As reviewed at the end of Section 2.1.1 the Ramond sector zero modes realize the 10D Clifford algebra via

$$\Gamma^\mu \;\longleftrightarrow\; \sqrt{2}\psi_0^\mu \qquad \Rightarrow \qquad \{\Gamma^\mu,\Gamma^\nu\} \;=\; 2\eta^{\mu\nu} \;. \tag{2.1.19}$$

When required we use the basis used in [6], constructed as follows.

Let γ^i be the eight 16×16 γ-matrices of $SO(8)$. We use these to construct a chiral representation for the 32×32 Γ-matrices of $SO(1,9)$, via

$$\Gamma^i \;=\; \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix} \;=\; \sigma^1 \otimes \gamma^i \,,$$

$$
\begin{aligned}
\Gamma^9 &= \begin{pmatrix} 0 & \gamma^1\cdots\gamma^8 \\ \gamma^1\cdots\gamma^8 & 0 \end{pmatrix} = \sigma^1 \otimes \left(\gamma^1\cdots\gamma^8\right) \,, \qquad (2.1.20)\\
\Gamma^0 &= \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix} = i\,\sigma^2 \otimes 1\!\!1 \,,
\end{aligned}
$$

where σ^a are the standard Pauli matrices. The chirality matrix and charge conjugation matrix are then

$$
\begin{aligned}
\Gamma_{11} &= \Gamma^0 \ldots \Gamma^9 = \Gamma^t\Gamma^y\Gamma^1 \ldots \Gamma^8 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} = \sigma^3 \otimes 1\!\!1 \,, \qquad (2.1.21)\\
C &= \begin{pmatrix} 0 & -i1\!\!1 \\ i1\!\!1 & 0 \end{pmatrix} = \sigma^2 \otimes 1\!\!1 \,,
\end{aligned}
$$

where C satisfies

$$
\left(\Gamma^\mu\right)^T = -\,C\,\Gamma^\mu\,C^{-1} \qquad (2.1.22)
$$

and we note that in our conventions $C^T = -C$ and $C^{-1} = C$.

In order to define the R sector vacuum, we next introduce the reparameterization ghosts and superghosts.

2.1.4 Ghost and superghost fields

We now briefly review the ghost fields b, c of world-sheet reparameterization invariance and the superghost fields β, γ of world-sheet supersymmetry which arise in the BRST quantization of the superstring. For the sake of brevity, we discuss only the holomorphic ghosts; analogous expressions hold for the antiholomorphic ghosts $\widetilde{b}$, $\widetilde{c}$ and superghosts $\widetilde{\beta}$, $\widetilde{\gamma}$. We follow in places [9, 97, 98].

The fields b and c are fermionic fields with conformal dimension equal to 2 and -1 respectively and have the action

$$
\mathcal{S}_{\rm g} = \frac{1}{2\pi\alpha'}\int d^2z\, b\,\bar{\partial}c \,, \qquad (2.1.23)
$$

giving the equations of motion

$$
\bar{\partial}b = 0 \,, \qquad \bar{\partial}c = 0 \,. \qquad (2.1.24)
$$

The fields have the mode expansions

$$b(z) \;=\; \sum_{n\in\mathbb{Z}} \frac{b_n}{z^{n+2}}\;, \qquad c(z) \;=\; \sum_{n\in\mathbb{Z}} \frac{c_n}{z^{n-1}} \tag{2.1.25}$$

and canonical quantization gives the anticommutation relations

$$\{b_n, c_m\} \;=\; \delta_{n+m,0}\;. \tag{2.1.26}$$

The superghosts β and γ are bosonic fields with conformal dimension equal to $\frac{3}{2}$ and $-\frac{1}{2}$ respectively and their action has the same form as the bc ghosts,

$$\mathcal{S}_{\rm sg} \;=\; \int d^2z\, \beta\, \bar{\partial}\gamma\;. \tag{2.1.27}$$

They are thus holomorphic with mode expansions

$$\beta(z) \;=\; \sum_{r\in\mathbb{Z}+\nu} \frac{\beta_r}{z^{r+3/2}}\;, \qquad \gamma(z) \;=\; \sum_{r\in\mathbb{Z}+\nu} \frac{\gamma_r}{z^{r-1/2}} \tag{2.1.28}$$

where $\nu = 0$ gives the R sector and $\nu = \frac{1}{2}$ gives the NS sector, and canonical quantization gives the commutation relations

$$[\gamma_r, \beta_s] \;=\; \delta_{r+s,0}\;. \tag{2.1.29}$$

The $SL(2,\mathbb{R})$ invariant vacuum $|0\rangle_{\rm sg}$ of the superghost Hilbert space is annihilated by

$$\beta_r \quad \text{for} \quad r \geq -\tfrac{1}{2}\;, \qquad\qquad \gamma_r \quad \text{for} \quad r \geq \tfrac{3}{2}\;. \tag{2.1.30}$$

The superghosts may be bosonized by introducing

$$\gamma(z) \;=\; e^{\phi(z)}\,\eta(z)\;, \qquad \beta(z) \;=\; e^{-\phi(z)}\,\partial\xi(z) \tag{2.1.31}$$

where η, ξ are introduced because β and γ are bosonic; one finds that the $\eta\xi$ CFT decouples from the ϕ CFT (see e.g. [98]).

Due to the presence of superghost zero modes in the R sector which do not annihilate the vacuum, the vacuum has infinite degeneracy which is accounted for by introducing different 'pictures' in which one may work. Schematically we can write the left-moving superghost vacuum in the P picture by [99, 100]

$$|0\rangle_P \;\equiv\; \lim_{z\to 0} :e^{P\phi(z)}: |0\rangle_{\rm NS} \tag{2.1.32}$$

which is annihilated by

$$\beta_m \quad \text{for} \quad m \geq -(P+\tfrac{1}{2})\,, \qquad\qquad \gamma_m \quad \text{for} \quad m \geq (P+\tfrac{3}{2})\,. \tag{2.1.33}$$

Since the disk has a background superghost charge of 2, we must take the right-moving sector to be in the $\widetilde{P}$ picture, where

$$\widetilde{P} \;=\; -2-P\,. \tag{2.1.34}$$

In the NS-NS sector we shall work in the $(-1,-1)$ picture, while in the R-R sector we shall work in the $(-\frac{1}{2},-\frac{3}{2})$ picture, in which the superghost vacuum is annihilated by the zero modes β_0 and $\widetilde{\gamma}_0$.

In order to define the $(-\frac{1}{2},-\frac{3}{2})$ picture R vacuum more carefully we require spin fields, as follows. Let $A, B, ...$ be 32-dimensional indices for spinors in ten dimensions, and let S^A, $\widetilde{S}^B$ be left and right-moving spin fields. Then the Ramond vacuum in the left-moving sector is defined by

$$|A\rangle_\ell \;\equiv\; \lim_{z\to 0} : S^A(z) e^{\ell\phi(z)} : |0\rangle_{\rm NS} \qquad \text{for} \quad \ell \;=\; -\tfrac{1}{2} \text{ or } -\tfrac{3}{2} \tag{2.1.35}$$

and similarly for the right-moving sector. In the above formula, the condition $\ell = -\frac{1}{2}$ or $-\frac{3}{2}$ ensures that the operator acting on the NS vacuum has weight one. In the $(-\frac{1}{2},-\frac{3}{2})$ picture we thus have the Ramond vacuum

$$|A\rangle_{-\frac{1}{2}}\, |\widetilde{B}\rangle_{-\frac{3}{2}}\,. \tag{2.1.36}$$

The action of the Ramond oscillators ψ^μ_n and $\widetilde{\psi}^\mu_n$ on the above state is given for $r > 0$ by (suppressing temporarily the subscripts):

$$\psi^\mu_r\, |A\rangle|\widetilde{B}\rangle \;=\; \widetilde{\psi}^\mu_r\, |A\rangle|\widetilde{B}\rangle \;=\; 0 \tag{2.1.37}$$

and for $r = 0$ by

$$\begin{aligned} \psi_0^\mu\,|A\rangle|\widetilde{B}\rangle &= \frac{1}{\sqrt{2}}\,(\Gamma^\mu)^A{}_C\,(1\!\!1)^B{}_D\,|C\rangle\,|\widetilde{D}\rangle \\ \widetilde{\psi}_0^\mu\,|A\rangle|\widetilde{B}\rangle &= \frac{1}{\sqrt{2}}\,(\Gamma_{11})^A{}_C\,(\Gamma^\mu)^B{}_D\,|C\rangle|\widetilde{D}\rangle\;. \end{aligned} \tag{2.1.38}$$

One can check that this action correctly reproduces the anticommutation properties of the ψ-oscillators, in particular that $\{\psi_0^\mu\,,\,\psi_0^\nu\} = \{\widetilde{\psi}_0^\mu\,,\,\widetilde{\psi}_0^\nu\} = \eta^{\mu\nu}$, and $\{\psi_0^\mu\,,\,\widetilde{\psi}_0^\nu\} = 0$.

2.2 Boundary state for a flat D-brane

The boundary state $|\mathrm{D}p\rangle$ is a state of the closed string that inserts a boundary on the world-sheet and enforces the boundary conditions appropriate for a Dp-brane. Boundary states were studied originally in the context of world-sheet disk amplitudes [101, 102, 103] before this physics was described in terms of D-branes [3]. Other applications of boundary states may be found in [104, 105, 106, 107, 108, 109, 110, 111, 112, 113].

In this section we review the construction of the boundary state for a flat D-brane, following the treatment in [6, 7, 9].

2.2.1 Oscillator part of D-brane Boundary state

For both the NS-NS and R-R sectors of the closed superstring, $|\mathrm{D}p\rangle$ can be written as the product of matter and ghost parts,

$$|\mathrm{D}p\rangle \;=\; |\mathrm{D}p_{\mathrm{mat}}\rangle\,|\mathrm{D}p_{\mathrm{gh}}\rangle\;, \tag{2.2.1}$$

where the matter part can be written as a product of bosonic and fermionic matter boundary states and the ghost part can be written as a product of reparameterization ghost and superghost parts:

$$|\mathrm{D}p_{\mathrm{mat}}\rangle \;=\; \frac{\kappa\tau_p}{2}\,|\mathrm{D}p_X\rangle\,|\mathrm{D}p_\psi\rangle\;, \qquad |\mathrm{D}p_{\mathrm{gh}}\rangle \;=\; |\mathrm{D}p_{\mathrm{g}}\rangle\,|\mathrm{D}p_{\mathrm{sg}}\rangle\;. \tag{2.2.2}$$

In the above, the constant κ is related to the 10D gravitational constant $G_N^{(10)}$ via

$$2\kappa^2 \;=\; 16\pi G_N^{(10)} \qquad \Rightarrow \quad \kappa \;=\; \frac{1}{2\sqrt{\pi}}(2\pi\sqrt{\alpha'})^4 g_s \tag{2.2.3}$$

and τ_p is the tension of a Dp-brane,

$$\tau_p \;=\; \frac{1}{(2\pi\sqrt{\alpha'})^p\sqrt{\alpha'}g_s} \;. \tag{2.2.4}$$

Our applications involve saturating the boundary state with a physical on-shell state and so we shall not need the ghost boundary state for these calculations; we will however use the superghost boundary state to make the GSO projection shortly.

The matter part $|\mathrm{D}p_{\mathrm{mat}}\rangle$ is defined to be the state which solves the boundary conditions of the closed superstring (2.1.18), i.e.

$$\widetilde{\alpha}^\mu_n \;=\; -\,R^\mu{}_\nu\alpha^\nu_{-n}\;, \qquad \widetilde{\psi}^\mu_r \;=\; i\eta\, R^\mu{}_\nu\psi^\nu_{-r}\;, \qquad p_\alpha \;=\; 0\;, \qquad x^i \;=\; y^i\;. \tag{2.2.5}$$

The oscillator part of the boundary state in each sector is a coherent state: for the bosonic coordinates we have

$$|\mathrm{D}p_X\rangle \;=\; \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\,\widetilde{\alpha}_{-n}\cdot R\cdot\alpha_{-n}\right]\,|\mathrm{D}p_X\rangle^{(0)}\;, \tag{2.2.6}$$

where we shall solve for the zero mode part $|\mathrm{D}p_X\rangle^{(0)}$ in the next section.

For the fermionic coordinates, in the NS-NS sector we have

$$|\mathrm{D}p_\psi,\eta\rangle_{\mathrm{NS}} \;=\; \exp\left[i\eta\sum_{m=1/2}^{\infty}\widetilde{\psi}_{-m}\cdot R\cdot\psi_{-m}\right]\,|0\rangle_{NS}\;, \tag{2.2.7}$$

and in the R-R sector we have

$$|\mathrm{D}p_\psi,\eta\rangle_{\mathrm{R}} \;=\; \exp\left[i\eta\sum_{m=1}^{\infty}\widetilde{\psi}_{-m}\cdot R\cdot\psi_{-m}\right]\,|\mathrm{D}p_\psi,\eta\rangle^{(0)}_{\mathrm{R}} \tag{2.2.8}$$

where we shall solve for the zero mode part $|\mathrm{D}p_\psi,\eta\rangle^{(0)}_{\mathrm{R}}$ in Section 2.2.4.

2.2.2 Zero mode part of bosonic boundary state

The zero mode part of the bosonic D-brane boundary state, $|\mathrm{D}p_X\rangle^{(0)}$, is determined by the zero mode boundary conditions

$$(\hat{x}^i - y^i)\,|\mathrm{D}p_X\rangle^{(0)} \;=\; 0\;, \qquad \hat{p}^\alpha\,|\mathrm{D}p_X\rangle^{(0)} \;=\; 0 \tag{2.2.9}$$

which are solved by a combination of zero mode states localized in position or momentum,

$$|\mathrm{D}p_X\rangle^{(0)} \;=\; |0\rangle_{\alpha,\widetilde{\alpha}}\,|\hat{x}^i = y^i\rangle\,|p^\alpha = 0\rangle \tag{2.2.10}$$

where $|0\rangle_{\alpha,\widetilde{\alpha}}$ is the ground state of the $\alpha, \widetilde{\alpha}$ oscillators.

While the solution above is fairly trivial for the case for a flat D-brane, the zero-mode bosonic boundary state will play an important role in the next chapter when we construct the boundary state for a D-brane with a travelling wave.

2.2.3 Superghost boundary state

As previously mentioned, for the amplitudes we are interested in calculating we will not need the explicit form of the reparameterization ghost and superghost boundary states. However in order to make the GSO projection in the next section, we will use the superghost boundary state which we review here following [7], in which the reparameterization ghost boundary state may also be found.

The boundary conditions for the superghosts follow from imposing that the boundary state be BRST invariant and are [7]

$$\Big(\gamma_r + i\eta\,\widetilde{\gamma}_{-r}\Big)\,|\mathrm{D}p_{\mathrm{sg}},\eta\rangle \;=\; 0\;, \qquad \Big(\beta_r + i\eta\,\widetilde{\beta}_{-r}\Big)\,|\mathrm{D}p_{\mathrm{sg}},\eta\rangle \;=\; 0\;. \tag{2.2.11}$$

In the NS-NS sector in the $(-1,-1)$ picture, the superghost boundary conditions are solved by

$$|\mathrm{D}p_{\mathrm{sg}},\eta\rangle_{\mathrm{NS}} \;=\; \exp\left[i\eta\sum_{r=1/2}^{\infty}(\gamma_{-r}\widetilde{\beta}_{-r} - \beta_{-r}\widetilde{\gamma}_{-r})\right]\,|0\rangle_{-1,-1} \tag{2.2.12}$$

and in the R-R sector in the $(-\frac{1}{2},-\frac{3}{2})$ picture, recalling that the superghost vacuum in this picture is annihilated by β_0 and $\widetilde{\gamma}_0$, we have

$$|\mathrm{D}p_{\mathrm{sg}},\eta\rangle_{\mathrm{R}} = \exp\left\{ i\eta\left[\gamma_0\widetilde{\beta}_0 + \sum_{r=1}^{\infty}(\gamma_{-r}\widetilde{\beta}_{-r} - \beta_{-r}\widetilde{\gamma}_{-r})\right]\right\}\, |A\rangle_{-\frac{1}{2}}\, |\widetilde{B}\rangle_{-\frac{3}{2}}\ . \qquad (2.2.13)$$

2.2.4 Zero mode part of fermionic boundary state

For the fermionic part of the D-brane boundary state, there is a non-trivial zero mode state $|\mathrm{D}p_\psi,\eta\rangle^{(0)}_{\mathrm{R}}$ in the R-R sector, which requires some work to derive. We now review this, following [6].

We now use these definitions to derive the R-R sector zero mode boundary state $|\mathrm{D}p_\psi,\eta\rangle^{(0)}_{\mathrm{R}}$ for a Dp-brane. Let us write

$$|\mathrm{D}p_\psi,\eta\rangle^{(0)}_{\mathrm{R}} \;=\; \mathcal{M}_{AB}\, |A\rangle_{-\frac{1}{2}}\, |\widetilde{B}\rangle_{-\frac{3}{2}}\ . \qquad (2.2.14)$$

Then the fermion boundary conditions (2.1.18) for $r=0$ and the action of the zero mode fields (2.1.38) imply that the 32×32 matrix $\mathcal{M}$ must satisfy the following equation

$$(\Gamma^\mu)^T\,\mathcal{M} - i\eta\, R^\mu{}_\nu\, \Gamma_{11}\,\mathcal{M}\,\Gamma^\nu \;=\; 0\ . \qquad (2.2.15)$$

Using our previous definitions, one finds that a solution is[1]

$$\mathcal{M} \;=\; i\,C\,\Gamma^0\cdots\Gamma^p\,\frac{1+i\eta\,\Gamma_{11}}{1+i\eta}\ . \qquad (2.2.16)$$

2.2.5 GSO projected boundary states

Before using the boundary state to compute amplitudes involving D-branes we must perform the GSO projection, which projects out the tachyon states of the open string and closed string spectra and ensures space-time supersymmetry.

In the NS-NS sector the GSO projected boundary state is (see e.g. [7, 9]):

$$|\mathrm{D}p\rangle_{\mathrm{NS}} \;\equiv\; \frac{1-(-1)^{F+G}}{2}\,\frac{1-(-1)^{\widetilde{F}+\widetilde{G}}}{2}\,|\mathrm{D}p,+\rangle_{\mathrm{NS}}\ , \qquad (2.2.17)$$

[1] The overall phase of M is a matter of convention; see also [8].

where F and G are the fermion and superghost number operators

$$F = \sum_{r=1/2}^{\infty} \psi_{-r} \cdot \psi_r \ , \qquad G = -\sum_{r=1/2}^{\infty} (\gamma_{-r}\beta_r + \beta_{-r}\gamma_r) \tag{2.2.18}$$

and similarly for $\widetilde{F}$, $\widetilde{G}$. Their action on the NS-NS fermionic matter boundary state (2.2.7) and superghost boundary state (2.2.12) gives:

$$\begin{aligned} (-1)^F |\mathrm{D}p_\psi, \eta\rangle_{\mathrm{NS}} &= |\mathrm{D}p_\psi, -\eta\rangle_{\mathrm{NS}} \ , \qquad & (-1)^{\widetilde{F}} |\mathrm{D}p_\psi, \eta\rangle_{\mathrm{NS}} &= |\mathrm{D}p_\psi, -\eta\rangle_{\mathrm{NS}} \ , \\ (-1)^G |\mathrm{D}p_{sg}, \eta\rangle_{\mathrm{NS}} &= |\mathrm{D}p_{sg}, -\eta\rangle_{\mathrm{NS}} \ , \qquad & (-1)^{\widetilde{G}} |\mathrm{D}p_{sg}, \eta\rangle_{\mathrm{NS}} &= |\mathrm{D}p_{sg}, -\eta\rangle_{\mathrm{NS}} \ . \end{aligned}$$

Using these expressions the NS-NS GSO projection (2.2.17) simplifies to

$$|\mathrm{D}p\rangle_{\mathrm{NS}} = \frac{1}{2}\Big(|\mathrm{D}p, +\rangle_{\mathrm{NS}} - |\mathrm{D}p, -\rangle_{\mathrm{NS}} \Big) \ . \tag{2.2.19}$$

In the R-R sector the GSO projected boundary state is

$$|\mathrm{D}p\rangle_{\mathrm{R}} \equiv \frac{1 + (-1)^p (-1)^{F+G}}{2} \, \frac{1 - (-1)^{\widetilde{F}+\widetilde{G}}}{2} \, |\mathrm{D}p, +\rangle_{\mathrm{R}} \ . \tag{2.2.20}$$

where p is even for Type IIA and odd for Type IIB, and where now

$$(-1)^F = \Gamma_{11} (-1)^{\sum_{m=1}^{\infty} \psi_{-m} \cdot \psi_m} \ , \qquad G = -\gamma_0 \beta_0 - \sum_{m=1}^{\infty} (\gamma_{-m}\beta_m + \beta_{-m}\gamma_m) \ . \tag{2.2.21}$$

The action of these operators on the R-R fermionic boundary state given by (2.2.8), (2.2.14) and the superghost boundary state (2.2.13) gives

$$\begin{aligned} (-1)^F |\mathrm{D}p_\psi, \eta\rangle_{\mathrm{R}} &= (-1)^p |\mathrm{D}p_\psi, -\eta\rangle_{\mathrm{R}} \ , \qquad & (-1)^{\widetilde{F}} |\mathrm{D}p_\psi, \eta\rangle_{\mathrm{R}} &= |\mathrm{D}p_\psi, -\eta\rangle_{\mathrm{R}} \\ (-1)^G |\mathrm{D}p_{sg}, \eta\rangle_{\mathrm{R}} &= |\mathrm{D}p_{sg}, -\eta\rangle_{\mathrm{R}} \ , \qquad & (-1)^{\widetilde{G}} |\mathrm{D}p_{sg}, \eta\rangle_{\mathrm{R}} &= -|\mathrm{D}p_{sg}, -\eta\rangle_{\mathrm{R}} \end{aligned}$$

and so the R-R GSO projection (2.2.20) simplifies to

$$|\mathrm{D}p\rangle_{\mathrm{R}} = \frac{1}{2}\Big(|\mathrm{D}p, +\rangle_{\mathrm{R}} + |\mathrm{D}p, -\rangle_{\mathrm{R}} \Big) \ . \tag{2.2.22}$$

It is natural to decompose the spinors of the R-R zero mode boundary state into chiral and antichiral components ($A = (\alpha, \dot{\alpha})$) with sixteen-dimensional indices

α and $\dot{\alpha}$ respectively. Following [6], we now illustrate this for type IIB chirality where $\mathcal{M}$ is non-trivial only in the off-diagonal blocks, that is in the antichiral-chiral sector and in the chiral-antichiral one. In the sixteen-dimensional notation, we first introduce the state

$$|\alpha\rangle_\ell \;\equiv\; \lim_{z\to 0} :S^\alpha(z)\mathrm{e}^{\ell\phi(z)}:\,|0\rangle_{\rm NS} \qquad \text{for} \quad \ell \;=\; -\tfrac{1}{2} \text{ or } -\tfrac{3}{2} \tag{2.2.23}$$

We then define

$$M_{AB} \equiv \begin{pmatrix} M_{\alpha\beta} & M_{\alpha\dot{\beta}} \\ M_{\dot{\alpha}\beta} & M_{\dot{\alpha}\dot{\beta}} \end{pmatrix} = \left(C\,\Gamma^0\cdots\Gamma^p\right)_{AB}\ , \tag{2.2.24}$$

and so the R-R zero mode boundary state (2.2.14) becomes

$$|\mathrm{D}p_\psi,\eta\rangle^{(0)}_{\rm R} = |\Omega_R\rangle^{(1)} - i\,|\Omega_R\rangle^{(2)} \tag{2.2.25}$$

where

$$|\Omega_R\rangle^{(1)} = M_{\dot{\alpha}\beta}\,|\dot{\alpha}\rangle_{-\frac{1}{2}}|\widetilde{\beta}\rangle_{-\frac{3}{2}} \tag{2.2.26}$$

and

$$|\Omega_R\rangle^{(2)} = M_{\alpha\dot{\beta}}\,|\alpha\rangle_{-\frac{1}{2}}|\widetilde{\dot{\beta}}\rangle_{-\frac{3}{2}}\ . \tag{2.2.27}$$

We therefore find that for type IIB theory (where p is odd) the R-R matter boundary state is

$$\begin{aligned}|\mathrm{D}p_{\rm mat}\rangle_{\rm R} \;=\; -i\frac{\kappa\tau_p}{2}\,|\mathrm{D}p_X\rangle\,\Big\{&\left(C\Gamma^0\Gamma^1\ldots\Gamma^p\right)_{\dot{\alpha}\beta}\cos\left[\Theta\right]\,|\dot{\alpha}\rangle_{-\frac{1}{2}}\,|\widetilde{\beta}\rangle_{-\frac{3}{2}} \\ &+\left(C\Gamma^0\Gamma^1\ldots\Gamma^p\right)_{\alpha\dot{\beta}}\sin\left[\Theta\right]\,|\alpha\rangle_{-\frac{1}{2}}\,|\widetilde{\dot{\beta}}\rangle_{-\frac{3}{2}}\Big\}\ ,\end{aligned} \tag{2.2.28}$$

where

$$\Theta = \sum_{m=1}^{\infty}\left(\psi_{-m}\cdot R\cdot\widetilde{\psi}_{-m}\right) \tag{2.2.29}$$

and where we recall

$$|\mathrm{D}p_X\rangle = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\,\alpha_{-n}\cdot R\cdot\widetilde{\alpha}_{-n}\right]|0\rangle_{\alpha,\widetilde{\alpha}}\,|\hat{x}^i = y^i\rangle\,|p^\alpha = 0\rangle\ . \tag{2.2.30}$$

In order to have a complete summary of the matter boundary states, we also record here the NS-NS matter boundary state:

$$|\mathrm{D}p_{\mathrm{mat}}\rangle_{\mathrm{NS}} \;=\; -i\frac{\kappa\tau_p}{2}\,|\mathrm{D}p_X\rangle\,\sin\!\left[\sum_{m=1/2}^{\infty}\psi_{-m}\cdot R\cdot\widetilde{\psi}_{-m}\right]|0\rangle_{NS} \qquad (2.2.31)$$

with $|\mathrm{D}p_X\rangle$ as above.

2.2.6 Example: wrapped D5-brane bosonic zero modes

For later convenience we next write out explicitly the bosonic zero mode boundary state for a single wrapped D5 brane.

As in Section 1.4.1, we work on $\mathbb{R}\times S^1\times\mathbb{R}^4\times T^4$ parameterized by t, y, x^i, x^a. and use light-cone coordinates $u=(t+y)\,,\; v=(t-y)\,.$ The D5-brane is taken to be wrapped around $S^1\times T^4$.

For an S^1 direction with radius R we normalize the zero-mode momentum eigenstates as $\langle n|m\rangle = 2\pi R\,\delta_{nm}$ and the position eigenstates as $\langle x|y\rangle=\delta(x-y)$.

The bosonic zero-mode boundary state for a such a D5-brane is then

$$|\mathrm{D}5_X\rangle^{(0)}_{T^4} \;=\; |\hat{x}^i=y^i\rangle\,|\hat{p}^\alpha=0\rangle\,|\hat{p}^u=0\rangle\,|\hat{p}^v=0\rangle\;. \qquad (2.2.32)$$

We will find it convenient to write the Neumann directions u, v in position space and the Dirichlet directions i in momentum space, as follows:

$$|\mathrm{D}5_X\rangle^{(0)}_{T^4} \;=\; \int du\,dv\,\frac{dp^i}{(2\pi)^4}e^{-ip^ix^i}\,|p^i\rangle\,|u\rangle\,|v\rangle\,|\hat{p}^\alpha=0\rangle\;. \qquad (2.2.33)$$

We will refer to this form of the bosonic zero mode boundary state in the next chapter.

2.3 Supergravity fields for a flat D-brane from disk amplitudes

2.3.1 The calculation and its regime of validity

We next give an overview of the calculation of the long distance behaviour of the classical massless fields generated by a generic D-brane bound state by computing the amplitude for emission of the relevant string states from a disk with appropriate boundary conditions.

The procedure for calculating the spacetime amplitude for a supergravity field at a given point in the transverse directions is:

(i) Calculate the momentum-space amplitude $\mathcal{A}(k)$ for the emission of a massless closed string, as sketched for a flat D-brane in Fig. 2.1 and Fig. 2.2;

(ii) Extract the field of interest, e.g. graviton;

(iii) Multiply by a free propagator;

(iv) Fourier transform to obtain the spacetime amplitude.

The boundary state allows us to perform the calculation in step (i) in the closed string picture, since it inserts a boundary on the closed string world-sheet and acts as a source for all closed string fields.

Since the fields we are interested in are massless they have non-zero momentum only in the four non-compact directions of the $\mathbb{R}^4$, i.e. they have spacelike momentum. The amplitude in step (i) above is defined by analytically continuing k to complex values such that we impose $k^2 = 0$, i.e. the emitted closed string is treated as on-shell [6].

One can ask whether this procedure fails to capture any physics relevant to the calculation. For example, one could add to the amplitude $\mathcal{A}(k)$ a contribution proportional to any positive power of k^2, which would vanish if $k^2 = 0$. Suppose we add a term proportional to k^2; then multiplying by a free propagator $1/k^2$ and Fourier transforming gives a Dirac delta-function in position space. Similarly, higher powers of k^2 correspond to derivatives of the delta-function in position space. This signifies that these terms are relevant for physics very close to the location of the D-brane and thus do not affect the large distance behaviour of the supergravity fields.

We next discuss the regime of parameters for which our calculation can be trusted. The calculation we consider is a disk level calculation, so we are working in perturbation theory and neglecting higher order diagrams in both open string and closed string perturbation theory.

The next order in closed string perturbation theory corresponds to adding handles to the closed string propagator, which we suppress by working at $g_s \ll 1$. The next order in open string perturbation theory corresponds to adding an extra border to the string worldsheet; introducing r for the radial coordinate in the Dirichlet directions, for our calculation adding an extra border brings a factor of

$$\epsilon \;=\; g_s N \left(\frac{\alpha'}{r^2}\right)^{\frac{7-p}{2}} \tag{2.3.1}$$

as we discuss below. Thus we work in the following regime of parameters:

$$g_s \;\ll\; 1 \;, \qquad g_s N \left(\frac{\alpha'}{r^2}\right)^{\frac{7-p}{2}} \;\ll\; 1 \;. \tag{2.3.2}$$

One can rephrase the second of these as saying that disk amplitudes give the leading contribution to the fields at lengthscales r where $r^{7-p} \gg g_s N \sqrt{\alpha'}^{7-p}$, i.e. lengthscales greater than the characteristic size of the D-brane bound state. We shall see in the next section that this matches the large distance expansion of the corresponding supergravity solutions. An analogous structure appeared long ago in the field theory version of our calculation [114].

One can see that the quantity ϵ controls the open string perturbation expansion as follows. Adding an extra border to the string worldsheet gives a factor of $g_s N$ since there are N D-branes on which the open string can end. It also introduces a loop momentum integral, two extra propagators, and also reduces the background superghost charge by two units, requiring us to increase the picture of the vertex operators into a picture two units higher.

Qualitatively, each of these contributes as follows: At large distances, the loop momentum integral is dominated by the closed string channel, effectively resulting in an integral over the Dirichlet directions, $\int d^{9-p}k$. The two propagators bring two factors of $1/k^2$, and the picture-changing procedure brings a factor of k^2 as we describe below. Thus all together we have an additional integral of the form

$$\int d^{9-p}k \frac{1}{k^2} \;\sim\; \frac{1}{r^{7-p}} \tag{2.3.3}$$

and so restoring units of α' we indeed find that ϵ is the appropriate dimensionless expansion parameter.

The factor of k^2 from the change of picture arises as follows. Without entering into the full details of the picture-changing procedure, this involves acting with the BRST charge [99]

$$Q_{\rm BRST}(z) \;\sim\; \oint \frac{dz}{2\pi i}\Big(c(z)\,T(z) + \frac{1}{2}\gamma(z)\,j(z) + \ldots\Big) \tag{2.3.4}$$

where '...' indicates additional terms involving only the ghost and superghost fields not relevant for our purposes. The BRST charge contains the worldsheet supercurrent

$$j(z) \;\sim\; \partial X(z)\cdot\psi(z) \tag{2.3.5}$$

and since this is multiplied by $\gamma = \eta\, e^{\phi}$, increasing the picture by one unit introduces into the amplitude an additional factor of momentum. Since we need to increase the picture by two units one gets an extra factor of k^2.

Since our calculation requires us to take $\epsilon \ll 1$, there is the possibility to simultaneously consider $g_s N \gg 1$ which in the three-charge case is related to the regime where there is a classical black hole solution with low curvature at the horizon, as discussed in the previous chapter. Strictly speaking, there is the possibility that there may be non-perturbative effects in $g_s N$ that we might missing when considering this regime, but modulo this potential subtlety the disk amplitudes we consider should give the leading contribution to the supergravity fields at distances where $r^{7-p} \gg g_s N \sqrt{\alpha'}^{7-p}$, even if $g_s N \gg 1$.

If this reasoning is correct, and if the calculations we present in this chapter and the next can be successfully generalized to the three-charge D1-D5-P system, this offers the possibility to calculate the multipole moments of a D1-D5-P bound state in the regime of parameters where there is a classical black hole.

A calculation of non-trivial multipole moments in this regime of parameters would lend weight to the conjecture that a classical black hole solution should be regarded as a thermodynamic average over microscopic states, where individual states would have non-trivial long range supergravity multipole moments and the ensemble average would erase these moments and obtain the 'unique' classical black hole solution with horizon [52, 17, 115, 116].

In this context, it is expected that typical states should have very small moments [52] and it has been proposed that distinguishing typical states from the ensemble average requires measurements of Planck-scale precision [115, 116]. As we have discussed, in our calculation there is a potential subtlety regarding non-perturbative effects but we regard our approach as a promising one which could potentially be used to explore these ideas further.

We now move on to reviewing the calculation for a flat D-brane, before describing its generalization to two-charge bound states in the D-brane/momentum duality frame in the next chapter.

2.3.2 Dp-brane supergravity solutions

We now review the extremal p-brane solutions in supergravity, in order to show in the next section how they are produced by the disk one-point functions.

The supergravity fields sourced by a flat Dp-brane localized in the Dirichlet directions x_i include non-trivial metric (written in string frame), dilaton and the R-R $(p+1)$-form gauge potential:

$$ds^2 \;=\; H^{-\frac{1}{2}}\left(\eta_{\alpha\beta}\, dx^\alpha dx^\beta\right) + H^{\frac{1}{2}}\left(\delta_{ij}\, dx^i dx^j\right) , \tag{2.3.6}$$

$$e^{2\Phi} \;=\; g_s^2 H^{\frac{3-p}{2}} , \qquad C^{(p+1)}_{01\ldots p} \;=\; -\left(H^{-1}-1\right) \tag{2.3.7}$$

where the harmonic function H is:

$$H \;=\; 1 + \frac{Q_p}{|x_i|^{7-p}} . \tag{2.3.8}$$

and where for n_p coincident Dp branes, the charge Q_p is given (for $p < 7$) by

$$Q_p = g_s n_p \frac{(2\pi\sqrt{\alpha'})^{7-p}}{(7-p)\omega_{8-p}} . \tag{2.3.9}$$

where ω_n is the volume of the unit n-sphere (the unit sphere in $\mathbb{R}^{n+1}$), given in terms of the Euler Γ-function via

$$\omega_n = \frac{2\,\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} . \tag{2.3.10}$$

For later convenience we express the charge Q_p in terms of the constant κ and the Dp-brane tension τ_p introduced in (2.2.3) and (2.2.4):

$$Q_p = \frac{2\kappa^2\tau_p}{(7-p)\omega_{8-p}} \tag{2.3.11}$$

The disk amplitudes we now review reproduce the canonically normalized, linearized form of these fields. In order to compare with the amplitudes we therefore canonically normalize the metric, dilation and R-R fields so that they have kinetic terms that lead to propagators of the form $1/p^2$:

$$g = \eta + 2\kappa\hat{h} , \qquad \Phi = \sqrt{2}\kappa\hat{\Phi} , \qquad C = \sqrt{2}\kappa\hat{C} \tag{2.3.12}$$

and then expand the fields to linear order in ϵ as defined in (2.3.1), giving:

$$h_{\mu\nu} = \frac{Q_p}{2\kappa}\frac{1}{|x_i|^{7-p}}\,\mathrm{diag}\left(-\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2},-\tfrac{1}{2},\ldots,-\tfrac{1}{2}\right) \tag{2.3.13}$$

$$\hat{\Phi} = (3-p)\frac{Q_p}{\sqrt{2}\kappa}\frac{1}{|x_i|^{7-p}} \tag{2.3.14}$$

$$\hat{C}^{(p+1)}_{01\ldots p} = \frac{Q_p}{\sqrt{2}\kappa}\frac{1}{|x_i|^{7-p}} . \tag{2.3.15}$$

2.3.3 Classical fields from world-sheet disk amplitudes

We now describe the calculation of the long distance behaviour of the classical massless fields generated by a Dp-brane as described in Section 2.3.1.

The NS-NS one-point function thus takes the form (before the GSO projection)

$$\mathcal{A}_{\rm NS}(k;\eta) \;\equiv\; \langle p_i = k_i|\,\langle k_a = 0|\,\mathcal{G}_{\mu\nu}\psi^{\mu}_{\frac{1}{2}}\,\widetilde{\psi}^{\nu}_{\frac{1}{2}}\,|{\rm D}p;\eta\rangle_{\rm NS} \tag{2.3.16}$$

$$= i\eta\, V_{p+1}\,\frac{\kappa\tau_p}{2}\,\mathcal{G}_{\mu\nu}R^{\nu\mu} \tag{2.3.17}$$

where V_{p+1} is the (divergent) volume of the D-brane, which we divide by to ensure a finite amplitude. As reviewed in (2.2.19) the GSO projection has the effect of

$$\mathcal{A}_{\rm NS}(k) \;=\; \frac{1}{2}\Big(\mathcal{A}_{\rm NS}(k;+) - \mathcal{A}_{\rm NS}(k;-)\Big) \tag{2.3.18}$$

which leaves

$$\mathcal{A}_{\rm NS}(k) \;=\; i\,\frac{\kappa\tau_p}{2}\,\mathcal{G}_{\mu\nu}R^{\nu\mu}\;. \tag{2.3.19}$$

Expanding in terms of canonically normalized supergravity fields, $\mathcal{G}_{\mu\nu}$ is given by

$$\mathcal{G}_{\mu\nu} \;=\; \hat{h}_{\mu\nu} + \frac{1}{\sqrt{2}}\hat{b}_{\mu\nu} + \frac{\hat{\Phi}}{2\sqrt{2}}\left(\eta_{\mu\nu} - k_\mu l_\nu - k_\nu l_\mu\right)\,, \tag{2.3.20}$$

where k_μ and l_ν satisfy

$$k^2 \;=\; l^2 \;=\; 0\,, \qquad\qquad k\cdot l \;=\; 1\;. \tag{2.3.21}$$

We thus read off the canonically normalized fields of interest via

$$\hat{h}_{\mu\nu}(k) \;=\; \frac{1}{2}\frac{\delta\mathcal{A}_{\rm NS}}{\delta\hat{h}^{\mu\nu}} \qquad \text{for } \mu<\nu\,, \tag{2.3.22}$$

$$\hat{h}_{\mu\mu}(k) \;=\; \frac{\delta\mathcal{A}_{\rm NS}}{\delta\hat{h}^{\mu\mu}} \qquad (\text{no sum over } \mu)\,, \tag{2.3.23}$$

$$\hat{b}_{\mu\nu}(k) \;=\; \frac{\delta\mathcal{A}_{\rm NS}}{\delta\hat{b}^{\mu\nu}} \qquad \text{for } \mu<\nu\,, \tag{2.3.24}$$

$$\hat{\Phi}(k) \;=\; \frac{\delta\mathcal{A}_{\rm NS}}{\delta\hat{\Phi}}\;. \tag{2.3.25}$$

The space-time configuration associated with a closed string emission amplitude is obtained by multiplying the momentum-space amplitude for emission of the relevant supergravity field by a free propagator and taking the Fourier transform [6].

In general for a field $a_{\mu_1\ldots\mu_n}$ we have

$$a_{\mu_1\ldots\mu_n}(x) = \int \frac{d^4k}{(2\pi)^4} \left(-\frac{i}{k^2}\right) a_{\mu_1\ldots\mu_n}(k)\, \mathrm{e}^{ikx} \ , \tag{2.3.26}$$

with $a_{\mu_1\ldots\mu_n}(k)$ given in terms of derivatives of $\mathcal{A}$ as in (2.3.22)-(2.3.25). Using the identity

$$\int \frac{d^{9-p}k}{(2\pi)^{9-p}} \frac{e^{ik_ix^i}}{|k_i|^2} = \frac{1}{(7-p)\omega_{8-p}} \frac{1}{|x^i|^{7-p}} \tag{2.3.27}$$

and the relation (2.3.11), i.e.

$$Q_p = \frac{2\kappa T_p}{(7-p)\omega_{8-p}} \ , \tag{2.3.28}$$

we obtain the non-zero fields

$$h_{\mu\nu} = \frac{Q_p}{2\kappa} \frac{1}{|x_i|^{7-p}} \,\mathrm{diag}\left(-\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}, \ldots, -\tfrac{1}{2}\right) \tag{2.3.29}$$

$$\hat{\Phi} = (3-p)\frac{Q_p}{\sqrt{2}\kappa} \frac{1}{|x_i|^{7-p}} \tag{2.3.30}$$

in agreement with (2.3.13) and (2.3.14).

We next calculate the coupling between the R-R zero mode boundary state and the on-shell R-R potential state [8, 6, 7]:

$$\langle \hat{C}_{(n)}| = {}_{-\frac{1}{2}}\langle \widetilde{B}, \tfrac{k}{2}| \; {}_{-\frac{3}{2}}\langle A, \tfrac{k}{2}| \left[C\Gamma^{\mu_1\ldots\mu_n} \frac{1\!\!1 - \Gamma_{11}}{2}\right]_{AB} \frac{(-1)^n}{4\sqrt{2}\, n!}\, \hat{C}_{\mu_1\ldots\mu_n} \tag{2.3.31}$$

where the numerical factor contains an extra factor of $\frac{1}{2}$ to account for the fact that we are not using the full superghost expression (for the full expression, see e.g. [8]). Using the fact (see e.g. [7]) that

$$\left(\langle A| \, \langle \widetilde{B}|\right) \left(|\mathrm{D}\rangle \, |\widetilde{E}\rangle\right) = -\langle A|D\rangle \, \langle \widetilde{B}|\widetilde{E}\rangle = -(C^{-1})^{AD}(C^{-1})^{BE} \ , \tag{2.3.32}$$

we find the coupling of the R-R potential to the (GSO projected) boundary state to be

$$\begin{aligned} \mathcal{A}_{\mathrm{R}}^{(0)} &= \langle \hat{C}_{(n)}|\mathrm{D}p_{\psi}\rangle_{\mathrm{RR}}^{(0)} \\ &= \frac{-i}{4\sqrt{2}\, n!} \,\mathrm{tr}\left[\Gamma_{\mu_n\cdots\mu_1}\Gamma^{01\cdots p} \frac{1+\Gamma_{11}}{2}\right]_{AB} \hat{C}^{\mu_1\ldots\mu_n} \ . \end{aligned} \tag{2.3.33}$$

We then extract the gauge field profile via

$$\hat{C}^{(n)}_{\mu_1\ldots\mu_n}(k) \;=\; \frac{\delta\mathcal{A}_{\rm R}}{\delta\hat{C}^{(n)\mu_1\ldots\mu_n}} \quad (\mu_1 < \mu_2 \ldots < \mu_n)\,, \tag{2.3.34}$$

and as for the NS-NS calculation we insert the propagator and perform the Fourier transform. The only non-trivial potential is then

$$\hat{C}^{(p+1)}_{01\ldots p} \;=\; \frac{Q_p}{\sqrt{2}\kappa}\frac{1}{|x_i|^{7-p}} \tag{2.3.35}$$

in agreement with (2.3.15). The result is consistent with the fact that a Dp-brane is charged only under the $(p+1)$ gauge field of the R-R sector.

This completes the link between the microscopic and macroscopic descriptions of a Dp-brane via disk amplitudes. In the next chapter we describe research generalizing these results to the derivation of the supergravity fields for a D-brane with a travelling wave.

Chapter 3

The Supergravity Fields for a D-brane with a travelling wave from Disk Amplitudes

In this chapter we derive the supergravity fields for a D-brane with a travelling wave from disk amplitudes with appropriate boundary conditions. This chapter is based on research first presented in [1].

3.1 Introduction and discussion of results

We calculate the supergravity fields sourced by a D-brane carrying momentum charge in the form of a null right (or left) moving wave, and show that the fields sourced by this bound state reproduce the non-trivial features of the supergravity solutions which are U-dual to the fundamental string solution of [48, 49]. In particular we describe in detail the calculation in the D5-P duality frame.

The world-sheet calculation employs the fact that these D-brane configurations admit an exact CFT description [117] in which the travelling wave on the D-brane can be included in the world-sheet action for the open strings in a tractable way. We use the boundary state describing a D-brane with a travelling wave [118, 119, 120] to compute the disk one-point functions for emission of massless closed string states, and we read off the various supergravity fields.

In the D1-D5 duality frame [121], the analogous calculation has reproduced the leading order terms in the (large distance) $1/r$ expansion of the supergravity fields, while here the world-sheet calculation yields the full integrals over the D-brane profile appearing in the classical solutions. This is possible because the profile function parameterizing the solutions arises as a condensate of massless open strings related to the physical shape of the D-brane, which can be included exactly in the string world-sheet action.

The direct link between microscopic D-brane configurations and supergravity solutions might also shed further light on the entropy of two charge systems in string theory. It was recently proposed [53] that the macroscopic entropy of a two-charge configuration should be defined to be the sum of the contributions of small black hole solutions and horizonless smooth classical solutions (see also [122]).

In this language the term 'smooth classical solutions' does not include solutions which are singular due to delta-function sources, and the scaling arguments of [53] applied to the D-brane/momentum duality frame show that α'-corrections to the supergravity action cannot produce small black holes with a non-zero horizon area.

Here we observe that the supergravity solutions which are sourced by the microscopic D-brane bound states are necessarily singular at the two-derivative level: the one-point functions on the disk discussed in this chapter provide the asymptotic behaviour of the solutions, and the nonlinear part of the standard supergravity equations of motion determines the background in the interior, leading to the singular backgrounds obtained by dualizing the fundamental string solution. Of course, it might still be possible to recover a fully smooth field configuration starting from the same data provided by the disk one-point functions if one includes α'-corrections to the supergravity equations of motion.

In Section 3.5 we give an overview of research following on from the results presented in section 3.3.

3.2 Two-charge system in D1-P and D5-P duality frames

In this section we describe the two-charge supergravity solutions in the D1-P and D5-P duality frame. As in Section 1.4.1, we work in type IIB string theory on $\mathbb{R}^{4,1} \times S^1 \times T^4$ using coordinates

$$u \;=\; (t+y) \;, \qquad v \;=\; (t-y) \;, \tag{3.2.1}$$

$$i,j \;=\; 1,\ldots 4 \;\; (\mathbb{R}^4) \;, \qquad a,b \;=\; 5,\ldots 8 \;\; (T^4) \;, \qquad I,J \;=\; 1,\ldots 8 \;.$$

The family of classical supergravity solutions in which we are interested describe two-charge D-brane bound states [50, 51, 81, 123, 124] and are connected through S and T dualities to the multi-wound fundamental string solution reviewed in Section 1.4.1, as described in (1.4.7).

In the D1-P duality frame, we have a D1-brane wrapped n_w times around y with overall extent $L_T = 2\pi n_w R$ and world-volume coordinate $\hat{v}$. The non-trivial fields are the metric, the dilaton and the R-R 2-form gauge potential:

$$\begin{aligned} ds^2 \;&=\; H^{-\frac{1}{2}} dv \left(- du + K dv + 2 A_I dx^I \right) + H^{\frac{1}{2}} dx^I dx^I \;, \\ e^{2\Phi} \;&=\; g_s^2 H \;, \quad C^{(2)}_{uv} \;=\; -\tfrac{1}{2}(H^{-1} - 1) \;, \quad C^{(2)}_{vI} \;=\; -\, H^{-1} A_I \;, \end{aligned} \tag{3.2.2}$$

where the harmonic functions take the form

$$\begin{aligned} H \;=\; 1 + \frac{Q_1}{L_T} \int_0^{L_T} \frac{d\hat{v}}{|x_i - f_i(\hat{v})|^2} \;, \qquad A_I \;=\; -\frac{Q_1}{L_T} \int_0^{L_T} \frac{d\hat{v}\, \dot{f}_I(\hat{v})}{|x_i - f_i(\hat{v})|^2} \;, \\ K \;=\; \frac{Q_1}{L_T} \int_0^{L_T} \frac{d\hat{v} |\dot{f}_I(\hat{v})|^2}{|x_i - f_i(\hat{v})|^2} \;, \end{aligned} \tag{3.2.3}$$

where as before $f_i(\hat{v} + L_T) = f_i(\hat{v})$ and we use the same abuse of index notation described in (1.4.5). The D1-brane charge Q_1 is proportional to g_s and to the D1-brane winding number n_w and is given by

$$Q_1 \;=\; g_s n_w \frac{(2\pi)^4 (\alpha')^3}{V_4} \;. \tag{3.2.4}$$

T-dualizing to the D5-P duality frame and using the symmetry of the IIB equations of motion to reverse the sign of B and $C^{(4)}$, we obtain the fields:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}dv\Big(-du + \big(K - H^{-1}|A_a|^2\big)\,dv + 2A_i dx_i\Big) \\ &\quad + H^{\frac{1}{2}}dx^i dx^i + H^{-\frac{1}{2}}dx^a dx^a\ , \\ e^{2\Phi} &= (g_s')^2 H^{-1}\ , \qquad\qquad B_{va} \;=\; -\,H^{-1}A_a\ , \\ C^{(4)}_{vbcd} &= -H^{-1}A_a\epsilon_{abcd}\ , \\ C^{(6)}_{vi5678} &= -H^{-1}A_i\ , \qquad C^{(6)}_{uv5678} \;=\; -\,\tfrac{1}{2}\left(H^{-1}-1\right), \end{aligned} \tag{3.2.5}$$

where g_s' is the string coupling in the new duality frame and ϵ_{abcd} is the alternating symbol with $\epsilon_{5678} = 1$. The effect of rewriting the functions in (3.2.3) in terms of D5-P frame quantities is to substitute the D1 with the D5 charge,

$$Q_1 \to Q_5 = g_s' n_w \alpha'\ . \tag{3.2.6}$$

From now on, we drop the prime and refer to the string coupling in the D5-P frame as g_s.

From the large distance behaviour of the g_{vv} component of the metrics above, one can read off how the momentum charge is related to the D-brane profile function f. For instance, in the D1-P frame we have

$$\frac{n_w}{L_T}\int_0^{L_T} |\dot{f}|^2 d\hat{v} = \frac{g_s n_p \alpha'}{R^2}\ , \tag{3.2.7}$$

where n_p is the Kaluza-Klein integer specifying the momentum along the compact y direction. From a statistical point of view [52], the typical two-charge bound state with fixed D1 and momentum charges has a profile f consisting of Fourier modes of average frequency $\sqrt{n_w n_p}$. Then (3.2.7) implies that the typical profile wave has an amplitude of order $\sqrt{g_s}$. Despite this potential g_s dependence, we always keep track of f exactly and expand in the D-brane charges Q_i. From the point of view of the string amplitudes, this means that we are resumming all diagrams with open string insertions describing the D-brane profile, but that we are considering only the disk level contribution.

From now on, for concreteness we present the calculation in the D5-P frame and we focus on the field components that vanish in the absence of a wave; the calculations of the remaining components are analogous. We canonically normalize the metric, B-field and R-R fields:

$$g = \eta + 2\kappa \hat{h} \, , \qquad B = \sqrt{2}\kappa \hat{b} \, , \qquad C = \sqrt{2}\kappa \hat{C} \, . \tag{3.2.8}$$

We then expand the relevant components of (3.2.5) for small ϵ as defined in (2.3.1), keeping only linear order terms, which yields the field components that we shall reproduce from the disk amplitudes:

$$\begin{aligned} \hat{h}_{vi} &= \frac{Q_5}{2\kappa L_T} \int_0^{L_T} \frac{-\dot{f}_i \, d\hat{v}}{|x_i - f_i(\hat{v})|^2} \, , \qquad \hat{h}_{vv} = \frac{Q_5}{2\kappa L_T} \int_0^{L_T} \frac{|\dot{f}|^2 \, d\hat{v}}{|x_i - f_i(\hat{v})|^2} \, , \\ \hat{b}_{va} &= \frac{Q_5}{\sqrt{2}\kappa L_T} \int_0^{L_T} \frac{\dot{f}_a \, d\hat{v}}{|x_i - f_i(\hat{v})|^2} \, , \\ \hat{C}^{(4)}_{vbcd} &= \frac{Q_5}{\sqrt{2}\kappa L_T} \int_0^{L_T} d\hat{v} \, \frac{\dot{f}_a \epsilon_{abcd}}{|x_i - f_i(\hat{v})|^2} \, , \qquad \hat{C}^{(6)}_{vi5678} = \frac{Q_5}{\sqrt{2}\kappa L_T} \int_0^{L_T} d\hat{v} \, \frac{\dot{f}_i}{|x_i - f_i(\hat{v})|^2} \, . \end{aligned} \tag{3.2.9}$$

Similar expressions are easily derived in the D1-P frame from (3.2.2).

3.3 World-sheet boundary conditions for a D-brane with a travelling wave

The key ingredients of our string computation are the boundary conditions which must be imposed upon the world-sheet fields of a string ending on a D-brane with a travelling wave, which we now review. As in the previous chapter we consider a Euclidean world-sheet with complex coordinate $z = \exp(\tau + i\sigma)$ with $\tau \in \mathbb{R}$ and $\sigma \in [0, \pi]$. We first review the boundary conditions applicable for a D-brane wrapped once around y and later account for higher wrapping numbers.

We begin with the following world-sheet action for the superstring coupled to a background gauge field A^μ on a D9-brane following [102, 125, 119]:

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 \, , \tag{3.3.1}$$

where $\mathcal{S}_0$ and $\mathcal{S}_1$ are the world-sheet bulk and boundary actions respectively,

$$\mathcal{S}_0 = \frac{1}{2\pi\alpha'}\int\limits_{M} d^2z \left(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu\right), \tag{3.3.2}$$

$$\mathcal{S}_1 = i\int\limits_{\partial M} dz \left(A_\mu(X)\left(\partial X^\mu + \bar{\partial} X^\mu\right) - \frac{1}{2}\left(\psi^\mu + \widetilde{\psi}^\mu\right) F_{\mu\nu}\left(\psi^\nu - \widetilde{\psi}^\nu\right)\right) \tag{3.3.3}$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the abelian field strength.

As reviewed in the previous chapter, the action $\mathcal{S}_0$ is invariant under the supersymmetry transformations

$$\delta X^\mu = \varepsilon\psi^\mu + \tilde{\varepsilon}\widetilde{\psi}^\mu \ , \quad \delta\psi^\mu = -\varepsilon\partial X^\mu \ , \quad \delta\widetilde{\psi}^\mu = -\tilde{\varepsilon}\bar{\partial} X^\mu \tag{3.3.4}$$

for which

$$\tilde{\varepsilon} = \eta\varepsilon \ . \tag{3.3.5}$$

When we include $\mathcal{S}_1$, the total action $\mathcal{S}_0 + \mathcal{S}_1$ preserves $\mathcal{N} = 1$ supersymmetry only up to appropriate boundary conditions at $z = \bar{z}$ [125]. Defining

$$E_{\mu\nu} = \eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} \ , \tag{3.3.6}$$

varying the above action yields the boundary conditions [125]

$$\left[E_{\mu\nu}\widetilde{\psi}^\nu = \eta E_{\nu\mu}\psi^\nu\right]_{z=\bar{z}} \ , \tag{3.3.7}$$

$$\left[E_{\mu\nu}\bar{\partial} X^\nu - E_{\nu\mu}\partial X^\nu - \eta E_{\nu\rho,\mu}\widetilde{\psi}^\nu\psi^\rho - E_{\mu\nu,\rho}\psi^\nu\psi^\rho + E_{\nu\mu,\rho}\widetilde{\psi}^\nu\widetilde{\psi}^\rho\right]_{z=\bar{z}} = 0 \ ,$$

where η takes the value 1 or -1 corresponding to the NS and R sectors respectively.

For the systems under consideration the gauge field takes a plane-wave profile and so A^μ will be a function only of the bosonic field $V = (X^0 - X^9)$, where X^0 is the string coordinate along time and X^9 indicates the compact y direction. A physical gauge field can be written as $A^I(V)$, where we set to zero the light-cone components. Then the non-vanishing components of $E_{\mu\nu}$ take the form

$$E_{uv} = E_{vu} = -\frac{1}{2}, \qquad E_{IJ} = \delta_{IJ} \ , \qquad E_{Iv} = -E_{vI} = \dot{f}_I(V) \ , \tag{3.3.8}$$

where we have defined $f_I = -2\pi\alpha' A_I$.

We again write the fields appearing in (3.3.7) in modes by using the expansions

$$\begin{aligned} X^\mu(z,\bar z) &= x^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \ln z - i\sqrt{\frac{\alpha'}{2}}\widetilde{\alpha}_0^\mu \ln \bar z + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m}+\frac{\widetilde{\alpha}_m^\mu}{\bar z^m}\right), \\ \psi^\mu(z) &= \sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}+\nu}\frac{\psi_r^\mu}{z^{r+\frac12}}, \qquad \widetilde{\psi}^\mu(\bar z) = \sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbb{Z}+\nu}\frac{\widetilde{\psi}_r^\mu}{\bar z^{r+\frac12}}, \end{aligned} \tag{3.3.9}$$

where $\nu = 0$ and $\frac{1}{2}$ for R and NS respectively. In this setup, the presence of a non-constant field strength $F_{\mu\nu}$ makes the boundary conditions nonlinear in the oscillators. We will see however that for the amplitudes in which we are interested, only the linear terms contribute.

As usual, we can change from the open string picture to the closed string picture, and derive the boundary conditions describing a closed string emitted or absorbed by the D-brane. This has the effect of

$$\alpha_n^\mu \to -\alpha_{-n}^\mu, \qquad \psi_r^\mu \to i\psi_{-r}^\mu \qquad \forall\,\mu,n,r\,. \tag{3.3.10}$$

We can then obtain the boundary conditions for a lower dimensional D-brane by performing a series of T-dualities; after these transformations, the components of f along the dualized coordinates describe the profile of the brane. We perform four or eight T-dualities in order to obtain the boundary conditions appropriate for a D5 or a D1-brane, for instance in order to move from the D9 frame to the D5-P frame we T-dualize along each x^i which sends

$$\widetilde{\alpha}_n^i \to -\widetilde{\alpha}_n^i\,, \qquad \widetilde{\psi}_r^i \to -\widetilde{\psi}_r^i\,. \tag{3.3.11}$$

By following the procedure outlined above, we can summarize the boundary conditions for the closed string oscillators as follows

$$\widetilde{\psi}_r^\mu = i\eta\, R^\mu{}_\nu \psi_{-r}^\nu + \dots\,, \qquad \widetilde{\alpha}_n^\mu = -R^\mu{}_\nu \alpha_{-n}^\nu + \dots\,, \tag{3.3.12}$$

where '...' indicates that we ignore terms which are higher than linear order in the oscillator modes. We shall justify this below (3.4.7). The reflection matrix R is obtained from (3.3.7) by performing the transformations (3.3.10) and (3.3.11) and replacing V by its zero-mode v:

$$R^{\mu}{}_{\nu}(v) = T^{\mu}{}_{\rho}(E^{-1})^{\rho\sigma}E_{\nu\sigma} \ , \tag{3.3.13}$$

where the matrix T performs the T-duality (3.3.11), i.e. it is diagonal with values -1 in the x^i directions and 1 otherwise. R has the lowered-index form

$$R_{\mu\nu}(v) \;=\; \eta_{\mu\rho}R^{\rho}{}_{\nu}(v) \;=\; \begin{pmatrix} -2|\dot{f}(v)|^2 & -\frac{1}{2} & 2\dot{f}^i(v) & 2\dot{f}^a(v) \\ -\frac{1}{2} & 0 & 0 & 0 \\ 2\dot{f}^i(v) & 0 & -\mathbb{1} & 0 \\ -2\dot{f}^a(v) & 0 & 0 & \mathbb{1} \end{pmatrix} . \tag{3.3.14}$$

We refer the reader to [118, 119, 120] for a detailed discussion of the boundary state describing a D-brane with a travelling wave. For our purposes it is sufficient to know the linearized boundary conditions for the non-zero modes (3.3.12) that the boundary state must satisfy, and to construct explicitly only the zero-mode structure of the boundary state. Addressing firstly the bosonic sector, the boundary conditions on the zero modes are

$$p_v + \dot{f}^i(v)\, p_i \;=\; 0\,, \qquad p_u \;=\; 0\,, \qquad p_a \;=\; 0\,, \qquad x^i \;=\; f^i(v) \tag{3.3.15}$$

where the first three equations follow directly from (3.3.12) and the fourth equation must be included to account for the T-duality transformations. The first equation in (3.3.15) may be represented as $i\frac{\partial}{\partial v} = \dot{f}^i(v)p_i$ and similarly the last constraint may be represented as $i\frac{\partial}{\partial p_i} = f^i(v)$. Then generalizing the bosonic zero-mode boundary state for a flat D5-brane derived in (2.2.33), the boundary state zero-mode structure in the t, y and x^i direction is

$$\int dv\, du \int \frac{d^4p_i}{(2\pi)^4} e^{-ip_i f^i(v)}\, |p_i\rangle\, |u\rangle\, |v\rangle \ . \tag{3.3.16}$$

So far we have essentially discussed a D-brane with a travelling wave in a non-compact space; we next generalize this description to the case of compact y and higher wrapping number. One may view a D-brane wrapped n_w times along the y-direction as a collection of n_w different D-brane strands with a non-trivial holonomy gluing these strands together. This approach was developed in [126, 127] for the case of branes with a constant magnetic field.

In the presence of a null travelling wave with arbitrary profile $f(V)$, the individual boundary states of each strand will differ in their oscillator part and not just in their zero-mode part described above. However, we are interested in the emission of massless closed string states, which have zero momentum and winding along all compact directions. In this sector the full boundary state is simply the sum of the boundary states for each constituent, along with the condition that the value of the function f at the end of one strand must equal the value of f at the beginning of the following strand. We label the strands of the wrapped D-brane with the integer s; then restricting to the sector of closed strings with trivial winding (m) and Kaluza-Klein momentum (k), the boundary state takes the following form:

$$|\mathrm{D5}, f\rangle^{k,m=0} = -\frac{\kappa\,\tau_5}{2}\sum_{s=1}^{n_w}\int du\int_0^{2\pi R} dv\int\frac{d^4p_i}{(2\pi)^4}e^{-ip_if^i_{(s)}(v)}\,|p_i\rangle\,|u\rangle\,|v\rangle\,|\mathrm{D5}, f_{(s)}\rangle^{k,m=0}_{\mathrm{rem}} \tag{3.3.17}$$

We have written explicitly only the bosonic zero-modes along t, y and the x^i directions and we denote by $|\mathrm{D5}, f_{(s)}\rangle^{k,m=0}_{\mathrm{rem}}$ the remaining part of the matter boundary state for the strand with profile $f_{(s)}$. The range of integration over $v = t - y$ follows from the periodicity condition of the space-time coordinate y.

We next address the fermion zero modes in the R-R sector. Using the same conventions as in Section 2.2.4, the R-R zero mode boundary state in the $(-\frac{1}{2}, -\frac{3}{2})$ picture (before the GSO projection) takes the form

$$|\mathrm{D5}_\psi, f; \eta\rangle^{(0)}_R \;=\; \mathcal{M}_{AB}\;|A\rangle_{-\frac{1}{2}}\,|\widetilde{B}\rangle_{-\frac{3}{2}} \tag{3.3.18}$$

where $\mathcal{M}$ satisfies the following equation [6],

$$\Gamma_{11}\,\mathcal{M}\,\Gamma^{\mu} - i\eta\,R^{\mu}_{\;\;\nu}\,(\Gamma^{\nu})^T\,\mathcal{M} \;=\; 0\,. \tag{3.3.19}$$

A solution to this equation for the case of our reflection matrix R (3.3.14) is given by

$$\mathcal{M} \;=\; i\,C\left(\frac{1}{2}\Gamma^{vu} + \dot{f}^I(v)\Gamma^{Iv}\right)\Gamma^{5678}\left(\frac{1\!\!1 - i\eta\Gamma_{11}}{1 - i\eta}\right). \tag{3.3.20}$$

where C is the charge conjugation matrix. As reviewed in (2.2.22) the R-R GSO projection has the effect of

$$|\mathrm{D5}, f\rangle_R \;=\; \frac{1}{2}\Big(|\mathrm{D5}, f; +\rangle_R + |\mathrm{D5}, f; -\rangle_R \Big) \tag{3.3.21}$$

and so the fermionic zero mode part of the D5-P R-R boundary state for the strand with profile $f_{(s)}$ is

$$|\mathrm{D5}_\psi, f_{(s)}\rangle_R^{(0)} \;=\; i\left[C\left(\frac{1}{2}\Gamma^{vu} + \dot{f}^I_{(s)}(v)\Gamma^{Iv}\right)\Gamma^{5678}\frac{1+\Gamma_{11}}{2}\right]|A\rangle_{-\frac{1}{2}}\,|\widetilde{B}\rangle_{-\frac{3}{2}} \tag{3.3.22}$$

which we can insert into the relevant part of the boundary state (3.3.17).

3.4 Disk amplitudes for the supergravity fields

We now calculate the fields sourced by the D5-P bound state by computing the disk one-point functions for emission of a massless state, starting with the NS-NS fields. Since the states are massless they have non-zero momentum only in the four non-compact directions of the $\mathbb{R}^4$, i.e. they have spacelike momentum (see also [6]). The NS-NS one-point function thus takes the form (before the GSO projection)

$$\mathcal{A}_{\mathrm{NS}}(k;\eta) \;\equiv\; \langle p_i = k_i|\,\langle p_v = 0|\,\langle p_u = 0|\,\langle n_a = 0|\,\mathcal{G}_{\mu\nu}\psi^\mu_{\frac{1}{2}}\,\widetilde{\psi}^\nu_{\frac{1}{2}}\,|\mathrm{D5}, f;\eta\rangle^{k,m=0}_{\mathrm{NS}} \tag{3.4.1}$$

where for an S^1 direction with radius R we normalize the momentum eigenstates as $\langle n|m\rangle = 2\pi R\,\delta_{nm}$ and the position eigenstates as $\langle x|y\rangle = \delta(x-y)$. As in (2.3.20), $\mathcal{G}_{\mu\nu}$ is given in terms of canonically normalized fields via

$$\mathcal{G}_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{\sqrt{2}}\hat{b}_{\mu\nu} + \frac{\hat{\Phi}}{2\sqrt{2}}\left(\eta_{\mu\nu} - k_\mu l_\nu - k_\nu l_\mu\right)\,, \tag{3.4.2}$$

where $k^2 = l^2 = 0$, $k\cdot l = 1$. The contribution to the zero mode part of the amplitude from a single strand with profile $f_{(s)}(v)$ is

$$V_4 V_u \frac{\kappa\,\tau_5}{2}\int_0^{2\pi R} dv\, e^{-ik_i f^i_{(s)}(v)}\,, \tag{3.4.3}$$

where V_u represents the infinite volume of the D-brane in the u direction. Since we have used a delocalized probe ($p_v = 0$), the string amplitude contains an integral over the length of the strand of the D-brane.

Importantly, the supergravity fields obtained from this amplitude can only be trusted when the curvatures are small compared to the string and Planck scales. Aside from the singularity at the location of the D-brane (when the geometrical description breaks down in all cases), to obtain supergravity fields with small curvatures we must consider a classical profile of oscillation, i.e. a long wavelength profile. This should be contrasted with the typical profile of oscillation which would lead to a supergravity solution with string scale fluctuations [52]; such a supergravity solution must be discarded and we interpret this as the statement that such quantum states of the D-brane do not have good classical descriptions.

In the classical limit n_w is very large, the wavelength of the profile is much bigger than R, and so f is almost constant over each strand [50, 51]. The contribution to the value of each supergravity field is thus (3.4.3) divided by the volume of the strand:

$$\mathcal{A}^{(0)}_{X,(s)}(k) \;=\; \frac{\kappa\,\tau_5}{2}\frac{1}{2\pi R}\int_0^{2\pi R} dv\, e^{-ik_i f^i_{(s)}(v)}\;. \tag{3.4.4}$$

The contribution from the n_w different strands of the brane is therefore

$$\mathcal{A}^{(0)}_{X}(k) \;=\; \frac{\kappa\,\tau_5}{2}\frac{1}{2\pi R}\sum_{s=1}^{n_w}\int_0^{2\pi R} dv\, e^{-ik_i f^i_{(s)}(v)}\;, \tag{3.4.5}$$

and we combine the integrals over each strand to give the integral over the full world-volume coordinate $\hat{v}$, giving

$$\mathcal{A}^{(0)}_{X}(k) \;=\; \frac{\kappa\,\tau_5}{2}\frac{n_w}{L_T}\int_0^{L_T} d\hat{v}\, e^{-ik_i f^i(\hat{v})}\;. \tag{3.4.6}$$

Adding in the non-zero modes, the coupling of the boundary state to the NS-NS fields is

$$\mathcal{A}_{\rm NS}(k;\eta) \;=\; -\,i\eta\frac{\kappa\,\tau_5\, n_w}{2L_T}\int_0^{L_T} d\hat{v}\, e^{-ik_i f^i(\hat{v})}\mathcal{G}_{\mu\nu}R^{\nu\mu}(\hat{v}) \tag{3.4.7}$$

where $R(\hat{v})$ is the obvious strand-by-strand extension of the reflection matrix (3.3.14).

We can now observe why we were justified in ignoring terms higher than linear order in the oscillator boundary conditions (3.3.12). To arrive at the above result we substitute $\widetilde{\psi}^{\nu}_{\frac{1}{2}}$ for an expression involving only creation modes using (3.3.12), and only the linear term can contract with the remaining annihilation mode to give a non-zero result. A similar argument holds for the R-R amplitude.

As reviewed in (2.2.19) the GSO projection has the effect of

$$\mathcal{A}_{\mathrm{NS}}(k) \;=\; \frac{1}{2}\Big(\mathcal{A}_{\mathrm{NS}}(k;+) - \mathcal{A}_{\mathrm{NS}}(k;-)\Big) \tag{3.4.8}$$

and we read off the canonically normalized fields of interest via

$$\hat{h}_{vi}(k) \;=\; \frac{1}{2}\frac{\delta\mathcal{A}_{\mathrm{NS}}}{\delta\hat{h}^{vi}}\,, \qquad \hat{h}_{vv}(k) \;=\; \frac{\delta\mathcal{A}_{\mathrm{NS}}}{\delta\hat{h}^{vv}}\,, \qquad \hat{b}_{va}(k) \;=\; \frac{\delta\mathcal{A}_{\mathrm{NS}}}{\delta\hat{b}^{va}}\,. \tag{3.4.9}$$

The space-time configuration associated with a closed string emission amplitude is obtained by multiplying the derivative of the amplitude with respect to the closed string field by a free propagator and taking the Fourier transform [6]. In general for a field $a_{\mu_1\ldots\mu_n}$ we have

$$a_{\mu_1\ldots\mu_n}(x) = \int \frac{d^4k}{(2\pi)^4}\left(-\frac{i}{k^2}\right) a_{\mu_1\ldots\mu_n}(k)\,\mathrm{e}^{ikx}\,, \tag{3.4.10}$$

with $a_{\mu_1\ldots\mu_n}(k)$ given in terms of derivatives of $\mathcal{A}$ as in (3.4.9). Using the identity

$$\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik^i(x^i-f^i)}}{k^2} \;=\; \frac{1}{4\pi^2}\frac{1}{|x^i-f^i|^2} \tag{3.4.11}$$

and the relation

$$Q_5 \;=\; \frac{2\kappa^2\,\tau_5\,n_w}{4\pi^2}\,, \tag{3.4.12}$$

we obtain

$$\hat{h}_{vi} \;=\; \frac{Q_5}{2\kappa L_T}\int_0^{L_T}\frac{-\dot{f}_i\,d\hat{v}}{|x_i-f_i(\hat{v})|^2}\,, \qquad \hat{h}_{vv} \;=\; \frac{Q_5}{2\kappa L_T}\int_0^{L_T}\frac{|\dot{f}|^2\,d\hat{v}}{|x_i-f_i(\hat{v})|^2}\,,$$

$$\hat{b}_{va} = \frac{Q_5}{\sqrt{2}\kappa L_T} \int_0^{L_T} \frac{\dot{f}_a \, d\hat{v}}{|x_i - f_i(\hat{v})|^2} \,, \tag{3.4.13}$$

in agreement with (3.2.9).

We next calculate the coupling between the R-R zero mode boundary state and the on-shell R-R potential state

$$\langle \hat{C}_{(n)}| = {}_{-\frac{1}{2}}\langle \widetilde{B}, \tfrac{k}{2}| \; {}_{-\frac{3}{2}}\langle A, \tfrac{k}{2}| \left[C\Gamma^{\mu_1 \cdots \mu_n} \frac{1\!\!1 - \Gamma_{11}}{2} \right]_{AB} \frac{(-1)^n}{4\sqrt{2}\, n!} \hat{C}_{\mu_1 \ldots \mu_n} \tag{3.4.14}$$

where as in (2.3.31) the numerical factor contains an extra factor of $\frac{1}{2}$ to account for the fact that we are not using the full superghost expression. Using again the fact that

$$\Big(\langle A| \, \langle \tilde{B}| \Big) \Big(|\mathrm{D}\rangle \, |\tilde{E}\rangle \Big) = -\langle A|D\rangle \, \langle \tilde{B}|\tilde{E}\rangle = -(C^{-1})^{AD} (C^{-1})^{BE} \,, \tag{3.4.15}$$

we find the coupling of the R-R potential to the (GSO projected) boundary state for an individual strand (3.3.22) to be

$$\begin{aligned} \mathcal{A}^{(0)}_{\mathrm{R},(s)} &= \langle \hat{C}_{(n)} | \mathrm{D}5_\psi, f_{(s)} \rangle^{(0)}_R \\ &= \frac{-i}{4\sqrt{2}\, n!} \, \mathrm{tr} \left[\Gamma_{\mu_n \cdots \mu_1} \left(\frac{1}{2} \Gamma^{vu} + \dot{f}^I_{(s)}(v) \Gamma^{Iv} \right) \Gamma^{5678} \frac{1 + \Gamma_{11}}{2} \right]_{AB} \hat{C}^{\mu_1 \ldots \mu_n} \,. \end{aligned} \tag{3.4.16}$$

This then combines with the bosonic zero mode part of the amplitude $\mathcal{A}^{(0)}_{X,(s)}$ given in (3.4.4) and we sum over strands to obtain the full R-R amplitude $\mathcal{A}_{\mathrm{R}}$. We then extract the gauge field profile via

$$\hat{C}^{(n)}_{\mu_1 \ldots \mu_n}(k) = \frac{\delta \mathcal{A}_{\mathrm{R}}}{\delta \hat{C}^{(n)\mu_1 \ldots \mu_n}} \qquad (\mu_1 < \mu_2 \ldots < \mu_n) \,, \tag{3.4.17}$$

and we insert the propagator and perform the Fourier transform. The fields which are non-trivial only in the presence of a travelling wave are then

$$\hat{C}^{(4)}_{vbcd} = \frac{Q_5}{\sqrt{2}\kappa L_T} \int_0^{L_T} d\hat{v} \, \frac{\dot{f}_a \epsilon_{abcd}}{|x_i - f_i(\hat{v})|^2} \,, \qquad \hat{C}^{(6)}_{vi5678} = \frac{Q_5}{\sqrt{2}\kappa L_T} \int_0^{L_T} d\hat{v} \, \frac{\dot{f}_i}{|x_i - f_i(\hat{v})|^2} \tag{3.4.18}$$

which agrees with (3.2.9). This completes the link between the microscopic and macroscopic descriptions of a D5-brane with a travelling wave.

3.5 Summary and future research

One of the main motivations for studying disk amplitude for a D-brane with a travelling wave was to understand how to go about extending this program to three-charge D1-D5-P bound states. The three-charge system is of great interest because of the macroscopic black hole reviewed in Section 1.4.2 and the outstanding questions over the various families of smooth supergravity solutions reviewed in Section 1.6.

At the time of writing this thesis, work is in progress in this direction which we now briefly describe.

As mentioned in the introduction to this chapter, disk amplitudes for D1-D5 bound states [121] have reproduced the leading order terms in the $1/r$ expansion of the supergravity fields. Building on these results and the research presented in this thesis on D5-P bound states, there are two ways to proceed for extending this work to D1-D5-P bound states.

The first approach is to include the momentum of the D-branes perturbatively by inserting the Wilson line (3.3.3) as a vertex operator on the boundary of the world-sheet:

$$V_f \;\sim\; \int\limits_{\partial M} dz \left(f_\mu(V)\partial X^\mu + \dot{f}_\mu(V)\psi^v\psi^\mu \right) . \qquad (3.5.1)$$

Since the term involving the fermions in the above expression does not bring any power of momentum, inserting this term onto the boundary of a D1-D5 disk will produce an amplitude which contributes at the same order in $1/r$ as the original amplitude. The simplest non-zero amplitude of this kind involves inserting two copies of V_f onto a D1-D5 disk.

The second approach is to analyze exactly the boundary conditions appropriate D1-D5-P bound state, as done in this chapter for the D5-P case. As we saw in this chapter, this approach yields the full functional form of the supergravity solutions rather than the $1/r$ expansion, so one would expect that if this technique can be applied to three-charge bound states it would be more powerful than a perturbative approach. Ultimately of course, the two approaches must give the same terms in the $1/r$ expansion.

In conclusion, successfully applying the research presented in this chapter to the D1-D5-P system will hopefully provide valuable further insight into the physics of black holes in string theory.

Part II

Free Particles from Brauer Algebras in Matrix Models and Gauge-Gravity Duality

Chapter 4

Review of Free Particles and the Brauer Algebra basis

Outline of Chapter 4

In this chapter we provide background material and motivations for the research presented in Chapter 5 and review some necessary technical preliminaries.

The structure of this chapter is as follows:

In Section 4.1 we review the half-BPS sector of the AdS_5/CFT_4 duality, describing the half-BPS bubbling geometries and their applications to the singular 'superstar' geometries and near-extremal black holes.

In Section 4.2 we review examples of the emergence of free particle descriptions in hermitian, unitary and complex matrix models which arise in string theory.

In Section 4.3 we introduce from first principles the Brauer algebra and its representations, and in Section 4.4 we review the construction of the Brauer basis for complex matrix models.

4.1 Superstars, black holes, and coarse-graining

4.1.1 Gauge-gravity duality and bubbling geometries

The conjectured AdS_5/CFT_4 duality [10, 11, 12] has been a major research theme in string theory over the last 14 years. The strong version of the conjecture states that type IIB string theory on $AdS_5 \times S^5$ with radius R_{AdS} and $\mathcal{N} = 4$, $SU(N)$ supersymmetric Yang-Mills theory in four dimensions with coupling g_{YM} are equivalent at all values of the respective parameters, which are related by

$$\begin{aligned} g_s &\longleftrightarrow g_{YM}^2 \\ \left(\frac{R_{AdS}}{\sqrt{\alpha'}}\right)^4 &\longleftrightarrow 4\pi\lambda\,, \qquad \lambda = g_{YM}^2 N \end{aligned} \tag{4.1.1}$$

In Section 1.4.3 we reviewed asymptotically AdS_5 black holes whose entropy and microscopic structure we would like to understand quantum mechanically.

The half-BPS sector of the AdS_5/CFT_4 duality is one in which the relatively high levels of technical control have led to many interesting results. In the field theory, the degrees of freedom are encoded in a theory of free fermions [13] whilst in the bulk there is an explicit family of supergravity solutions [15], as we now review.

We shall not distinguish between gauge groups $SU(N)$ and $U(N)$ and for convenience we work with $\mathcal{N} = 4$ Super Yang-Mills (SYM) with gauge group $U(N)$. $\mathcal{N} = 4$ super Yang-Mills contains six real scalar fields in the adjoint of the gauge group. The half-BPS operators constructed from these scalars lie in $(0, l, 0)$ representations of the $SO(6)$ R-symmetry (see e.g. [13]). These may be combined into three complex scalar fields X, Y, Z. Focusing on one of these fields, say Z, each multi-trace holomorphic operator built from Z belongs to a distinct half-BPS multiplet [13], and so these operators may be used to study the half-BPS sector of the theory.

The truncation of the $\mathcal{N} = 4$ Super Yang-Mills to this sector, when dimensionally reduced onto $S^3 \times \mathbb{R}$, can be mapped to a theory of N free fermions in a one-dimensional simple harmonic oscillator potential [13, 14, 128]. The allowed energies for the fermions are $E_n = (n + \frac{1}{2})\hbar$, and the ground state is the Fermi sea, when the fermions occupy the N lowest energy levels.

Each fermion occupies an area $\hbar$ in phase space, and no two fermions may occupy the same region of phase space due to Fermi statistics. The system therefore takes up an area $N\hbar$ of phase space, and so its state can be described by a colouring of the plane into regions of black and white, representing a fermion or a hole respectively. The radius (squared) corresponds to energy, and so the ground state is a black disk centred on the origin. Young diagrams label operators corresponding to excited states of the system.

The dual supergravity solutions are determined by a single scalar function $u_0(x_1, x_2)$ defined on the x_1-x_2 plane in the ten dimensional geometry. and the solutions are regular if and only if u_0 is piecewise 0 and 1. The map between the field theory and the bulk consists of identifying this plane with the phase space of the fermions, where the black and white regions correspond to $u_0 = 1$ and $u_0 = 0$ respectively.

4.1.2 Superstars and coarse graining of geometries

In the class of LLM geometries, there are no black hole solutions but there are geometries which have naked singularities whenever u_0 differs from being piecewise 0 and 1. If at any point $u_0 < 0$ or $u_0 > 1$, the resulting geometry has closed timelike curves [129] and so we ignore these cases. For $0 < u_0 < 1$, the geometries have naked null singularities at $y = 0$.

Although there are no true black hole solutions in this sector of IIB supergravity, it is a useful toy model since the holographic duality enables us to make precise calculations using the dual field theory. In particular, one can study statistical ensembles of Young diagrams, and take a thermodynamic limit, defined by taking $\hbar \to 0$ with $\hbar N$ fixed [17]. In this limit the overall size of the Young diagram stays the same but the size of the boxes tends to zero and the Young diagram approaches a continuous a 'limit curve'. It has been shown that almost all Young diagrams in the ensemble will be arbitrarily close to one particular limit curve [17], suggesting a universal thermodynamic description of the underlying microscopic states. Related studies were carried out in [18, 19, 130, 131, 132, 133].

4.1.3 Near-extremal and non-extremal black holes

Although the high levels of technical control the half-BPS sector enabled detailed studies of coarse-graining, ultimately one would like to apply these ideas to black holes with macroscopic horizons in supergravity. One way to do this is to study near-extremal R-charged black holes [63]. This motivates studying the field theory in near-BPS and even non-BPS regimes.

In the half-BPS sector, the auxiliary description in terms of free particles plays a key role in the holographic duality, and so it is natural to ask whether there may be free particle descriptions also in non-BPS sectors. Non-BPS operators are not protected by non-renormalization theorems and so there is no guarantee that the results we find working at zero Yang Mills coupling will survive at strong coupling; nevertheless, it has been conjectured that certain heavy non-BPS operators should not renormalize strongly [17]. For an investigation of the renormalization of heavy operators at one-loop and two-loop, see [134].

The research described in the next chapter finds free particle descriptions in non-holomorphic sectors of complex matrix models. If renormalization effects indeed do not spoil this free particle description for certain large operators, our findings may be applicable to the microscopic physics of non-extremal asymptotically $AdS_5 \times S^5$ black holes.

4.2 Free particle descriptions in matrix models and gauge-gravity duality

In this section we review examples of hermitian, unitary and complex matrix models[2] which arise in the context of string theory, in particular string theory in two dimensions. This review is not intended to be complete in any sense but rather to provide the reader with context for the research presented in Chapter 5.

[2]By 'matrix models' in D spacetime dimensions, we include random matrix models ($D = 0$), matrix quantum mechanics ($D = 1$) or field theories.

4.2.1 Hermitian matrix quantum mechanics

Let us consider the hermitian matrix quantum mechanics defined by the Lagrangian

$$\mathcal{L} = \mathrm{tr}\left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\Phi^2\right) \tag{4.2.1}$$

which is invariant under the $U(N)$ action

$$\Phi \to g\Phi g^\dagger \ , \qquad\qquad g \in U(N) \ . \tag{4.2.2}$$

We follow the treatment in [135, 136, 137] restricting attention to the theory with quadratic potential. The Hamiltonian of this model is

$$H \quad = \quad \mathrm{tr}\left(-\frac{1}{2}\frac{\partial^2}{\partial\Phi\partial\Phi} + \frac{1}{2}\Phi^2\right) \ . \tag{4.2.3}$$

Introducing the annihilation and creation operators

$$A = \frac{1}{\sqrt{2}}\left(\Phi + \frac{\partial}{\partial\Phi}\right) \quad A^\dagger = \frac{1}{\sqrt{2}}\left(\Phi - \frac{\partial}{\partial\Phi}\right) \tag{4.2.4}$$

and using the usual convention for matrix indices

$$\left(\frac{\partial}{\partial\Phi}\right)^i_{\ j} = \frac{\partial}{\partial\Phi^j_{\ i}} \tag{4.2.5}$$

we have $[A^i_{\ j}, A^{\dagger k}_{\ \ l}] = \delta^k_{\ j}\delta^i_{\ l}$ and the Hamiltonian can be rewritten as

$$H = \mathrm{tr}(A^\dagger A) + \frac{N^2}{2} \ . \tag{4.2.6}$$

The ground state has energy $\frac{N^2}{2}$ and its wavefunction is

$$\Phi_0 = \langle\Phi|0\rangle = e^{-\frac{1}{2}\mathrm{tr}\,\Phi^2} \ . \tag{4.2.7}$$

$U(N)$ singlet excited states are obtained by acting on Φ_0 with $U(N)$ invariant functions of $A^\dagger$. Equivalently, if we absorb appropriate factors of $\sqrt{2}$, excited states are obtained by multiplying the ground state by $U(N)$ invariant functions of Φ. A basis for such functions is given by the Schur polynomials, which are polynomials of degree n labelled by a representation R of S_n,

$$\chi_R(\Phi) = \sum_{\sigma\in S_n} \chi_R(\sigma) \Phi^{i_1}{}_{i_{\sigma_1}} \cdots \Phi^{i_n}{}_{i_{\sigma_n}} \ , \tag{4.2.8}$$

where $\chi_R(\sigma)$ is the character of σ in the representation R. The associated wave-function

$$\Psi_R = \chi_R(\Phi) e^{-\frac{1}{2}\operatorname{tr}\Phi^2} \tag{4.2.9}$$

has energy $\frac{N^2}{2} + n$.

A hermitian matrix Φ may be decomposed as

$$\Phi = U\Lambda U^\dagger, \qquad \Lambda = diag(\lambda_1, \ldots, \lambda_N) \ , \qquad U \in U(N) \tag{4.2.10}$$

under which one obtains

$$\operatorname{tr}(\dot{\Phi}^2) = \operatorname{tr}(\dot{\Lambda}^2) + \operatorname{tr}[\Lambda, U^\dagger \dot{U}]^2 \ . \tag{4.2.11}$$

The anti-hermitian matrix $U^\dagger \dot{U}$ may be expanded in generators of $U(N)$. Introducing the variables α_i, α_{ij}, β_{ij} and their time derivatives $\dot{\alpha}_i$, $\dot{\alpha}_{ij}$, $\dot{\beta}_{ij}$, we expand

$$U^\dagger \dot{U} = \sum_i \dot{\alpha}_i H_i + \frac{i}{\sqrt{2}} \sum_{j<k} (\dot{\alpha}_{jk} T_{jk} + \dot{\beta}_{jk} \widetilde{T}_{jk})$$

where H_i are the diagonal generators of the Cartan subalgebra, T_{jk} is the matrix M such that $M_{jk} = M_{kj} = 1$ and all other entries are 0, and $\widetilde{T}_{ij}$ is the matrix M such that $M_{ij} = -M_{ji} = -i$ and all other entries are 0. This gives

$$\operatorname{tr}\,[\Lambda, U^\dagger \dot{U}]^2 = \sum_{i<j} (\lambda_i - \lambda_j)^2 (\dot{\alpha}_{ij}^2 + \dot{\beta}_{ij}^2)$$

and so the Lagrangian becomes

$$L = \sum_i \left(\frac{1}{2} \dot{\lambda_i}^2 + \frac{1}{2} \lambda_i^2 \right) + \frac{1}{2} \sum_{i<j} (\lambda_i - \lambda_j)^2 (\dot{\alpha}_{ij}^2 + \dot{\beta}_{ij}^2) \ . \tag{4.2.12}$$

Under the transformation (4.2.2) the measure becomes

$$\mathcal{D}\Phi = \mathcal{D}\Omega \prod_i d\lambda_i \Delta^2(\lambda) \tag{4.2.13}$$

where $\Delta(\lambda)$ is the Vandermonde determinant $\prod_{i<j}(\lambda_i - \lambda_j)$ and where $\mathcal{D}\Omega$ is the Haar measure on $U(N)$. The kinetic term for the eigenvalues becomes

$$-\frac{1}{2}\sum_i \frac{1}{\Delta^2(\lambda)}\frac{\partial}{\partial\lambda_i}\Delta^2(\lambda)\frac{\partial}{\partial\lambda_i} \;=\; -\frac{1}{2\Delta(\lambda)}\sum_i \frac{d^2}{d{\lambda_i}^2}\Delta(\lambda) \tag{4.2.14}$$

and so the Hamiltonian is

$$H \;=\; \frac{1}{2}\sum_i\left(-\frac{1}{\Delta(\lambda)}\frac{\partial^2}{\partial\lambda_i^2}\Delta(\lambda)+\lambda_i^2\right) - \frac{1}{2}\sum_{i<j}\frac{1}{(\lambda_i-\lambda_j)^2}\left(\frac{\partial^2}{\partial\alpha_{ij}^2}+\frac{\partial^2}{\partial\beta_{ij}^2}\right) . \tag{4.2.15}$$

Wavefunctions which are singlet under (4.2.2) are symmetric functions of the eigenvalues, $\chi_{sym}(\lambda)$. On these wavefunctions the Hamiltonian simplifies to

$$H \;=\; \frac{1}{2}\sum_i\left(-\frac{1}{\Delta(\lambda)}\frac{\partial^2}{\partial\lambda_i^2}\Delta(\lambda)+\lambda_i^2\right) . \tag{4.2.16}$$

One may simplify further the analysis by defining the antisymmetric wavefunction

$$\Psi^f(\lambda) = \Delta(\lambda)\chi_{sym}(\lambda) \tag{4.2.17}$$

and the modified Hamiltonian

$$H^f \;=\; \Delta(\lambda)H\frac{1}{\Delta(\lambda)} \;=\; \frac{1}{2}\sum_i\left(-\frac{d^2}{d{\lambda_i}^2}+\lambda_i^2\right) \tag{4.2.18}$$

which is a sum of one particle harmonic oscillator Hamiltonians. Then H^f has eigenstates $\Psi^f(\lambda)$ with the same eigenvalues as H:

$$H\Psi(\lambda) \;=\; E\,\Psi(\lambda) \tag{4.2.19}$$

$$\Rightarrow H^f\Psi^f(\lambda) \;=\; E\,\Psi^f(\lambda) . \tag{4.2.20}$$

The ground state wavefunction of H^f is

$$\Psi_0^f = \Delta e^{-\frac{1}{2}\operatorname{tr}\Phi^2} , \tag{4.2.21}$$

excited states are given by Slater determinants

$$\Psi_{\vec{\mathcal{E}}}^f = \det_{i,j} \lambda_i^{\mathcal{E}_j} e^{-\frac{1}{2}\operatorname{tr}\Phi^2} = \Delta(\lambda)\Psi_R(U) \tag{4.2.22}$$

and so the $U(N)$ singlet sector is equivalent to N non-interacting fermions in a harmonic oscillator potential, where the fermion energies $\mathcal{E}_i$ are related to the integer row lengths r_i of R by

$$\mathcal{E}_i = r_i + (N - i) . \tag{4.2.23}$$

4.2.2 Unitary matrix quantum mechanics

We next review the unitary matrix quantum mechanics which arises in the study of two-dimensional Yang-Mills, which is given by the Hamiltonian [138, 139]:

$$H = \operatorname{tr}\left(U\frac{\partial}{\partial U}\right)^2 = \sum_a E^a E^a \tag{4.2.24}$$

where E^a generate left rotations of U and are defined in terms of the generators t^a of the fundamental representation:

$$E^a = \operatorname{tr} t^a U \frac{\partial}{\partial U} \tag{4.2.25}$$

The form of H means that acting on a wavefunction which is a matrix element of an irreducible representation R,

$$(\psi_R)_{ij}(U) = D^R_{ij}(U) \tag{4.2.26}$$

it measures the quadratic Casimir of the representation R,

$$H\psi_R(U) = C_2(R) D^R_{ij}(U) . \tag{4.2.27}$$

Representations are classified by their characters, the Schur polynomials

$$\chi_R(U) = \mathrm{tr}\, D^R(U) \tag{4.2.28}$$

which form an orthonormal basis for wavefunctions invariant under the $U(N)$ action

$$U \to gUg^\dagger, \qquad g \in U(N)\ . \tag{4.2.29}$$

This may be used to express any unitary matrix U as

$$U = gDg^\dagger, \qquad D = diag(e^{i\theta_1}, \ldots, e^{i\theta_N}), \qquad g \in U(N)\ . \tag{4.2.30}$$

On functions invariant under (4.2.29), performing the change of variables (4.2.30) the Hamiltonian becomes [138]:

$$H \;=\; -\sum_i \left[\frac{1}{\tilde{\Delta}}\frac{d^2}{d\theta_i^2}\tilde{\Delta}\right] - \frac{1}{12}N(N^2-1) \tag{4.2.31}$$

where denoting the eigenvalues by $u_i = e^{i\theta_i}$,

$$\tilde{\Delta} \;=\; \prod_{i<j}\sin\frac{\theta_i-\theta_j}{2} \;=\; \frac{\Delta(u)}{\prod_i u_i^{\frac{N-1}{2}}} \;=\; \frac{\Delta(u)}{(\det U)^{\frac{N-1}{2}}} \tag{4.2.32}$$

and where

$$\Delta(u) = \prod_{i<j}(u_i - u_j)\ . \tag{4.2.33}$$

Absorbing $\tilde{\Delta}$ into the wavefunctions and the Hamiltonian,

$$\psi_f = \tilde{\Delta}\psi\ , \qquad H_f \;=\; \tilde{\Delta}H\frac{1}{\tilde{\Delta}} \;=\; \sum_i \frac{\partial}{\partial\theta_i^2} - \frac{1}{12}N(N^2-1) \tag{4.2.34}$$

the wavefunctions become antisymmetric under exchange of any pair $\theta_i \leftrightarrow \theta_j$. The one-particle wavefunctions with quantized momentum p are $\psi_p = e^{ip\theta}$ and the Slater determinants

$$\psi_{\vec{p}} \;=\; \det_{i,j} u_i^{p_j} \tag{4.2.35}$$

are eigenfunctions of H_f with energy $E = \sum_i p_i^2 - N(N^2-1)/12$, so the sector of this theory invariant under (4.2.29) is equivalent to a theory of N free fermions on a circle. The ground state has fermions with momenta distributed symmetrically about $n = 0$, and energy zero, so the Fermi energy is $n_F = \frac{N-1}{2}$ and there are Fermi surfaces at $\pm n_F$. The Slater determinants are related to the Schur polynomials via

$$\psi_{\vec{p}} \ = \ \Delta(u)\chi_R(U) \tag{4.2.36}$$

where the momenta p_i are related to the integer row lengths r_i of R by

$$p_i = r_i + (n_F + 1 - i) \ . \tag{4.2.37}$$

4.2.3 Complex matrix models

Previous studies of complex matrix models have centred on models in which there is enough symmetry to diagonalize the matrix. This can be achieved by studying a normal matrix ($[Z, Z^\dagger] = 0$) with $U(N)$ symmetry (see e.g. [140, 141])

$$Z \to gZg^\dagger \ , \qquad g \in U(N) \tag{4.2.38}$$

or by studying an unrestricted complex Z with $U(N) \times U(N)$ symmetry (see e.g. [142])

$$Z \to gZh^\dagger \ , \qquad g, h \in U(N) \ . \tag{4.2.39}$$

In the following chapter, motivated by gauge-gravity duality we study an unrestricted complex matrix Z with a single $U(N)$ symmetry (4.2.38). Unlike in the hermitian and unitary single matrix models, the unitary group action (4.2.38) is insufficient to diagonalize the matrix; the best one can do is use the Schur decomposition

$$Z = UTU^\dagger \tag{4.2.40}$$

where T is upper triangular, which we shall describe in more detail in Section 5.2.

Due to the off-diagonal degrees of freedom one would not expect a straightforward transformation to a description in terms of free particles for complex matrix models with unitary symmetry. Indeed, the free particle descriptions we shall describe in Chapter 5 are emergent degrees of freedom arising from combinations of eigenvalues and off-diagonal elements.

In passing we note that a complex matrix model with $U(N)$ symmetry (4.2.38) may be written as a two-Hermitian matrix model [143] using

$$X = \frac{1}{2}\left(Z + \bar{Z}\right), \qquad Y = -\frac{i}{2}\left(Z - \bar{Z}\right) \tag{4.2.41}$$

where $\bar{Z}$ denotes complex conjugate of Z. Studies of complex matrix models in terms of two hermitian matrices are done in [144, 145].

Many of the results presented in the next chapter are applicable to any Gaussian complex matrix model[3] in D spacetime dimensions x^μ, where the two-point function of the matrix $Z(x^\mu)$, up to a trivial spacetime dependence, is

$$\langle Z^i_{\ j}\, Z^{\dagger k}_{\ \ l}\rangle = \delta^i_{\ l}\, \delta^k_{\ j} \ . \tag{4.2.42}$$

As we have mentioned in our motivations, we are particularly interested in the complex matrix harmonic oscillator quantum mechanics which arises from dimensional reduction of $\mathcal{N} = 4$ Super Yang-Mills on $R \times S^3$ in the zero coupling limit, which we next review.

4.2.4 Complex matrix harmonic oscillator quantum mechanics

We now review the complex matrix harmonic oscillator quantum mechanics which arises from dimensional reduction of $\mathcal{N} = 4$ Super Yang-Mills on $R \times S^3$ in the zero coupling limit. As reviewed in Section 4.1, in order to study the half-BPS sector we focus on a single complex scalar field Z.

Dimensional reduction of $\mathcal{N} = 4$ SYM onto $\mathbb{R}_t \times S^3$ yields a $U(N)$ gauged matrix quantum mechanics involving a complex matrix $Z(t)$ in the adjoint coupled to a gauge field $A_0(t)$. In radial quantization, a half-BPS operator corresponds to the S-wave state on the S^3, i.e. the constant mode [14].

The dimensionally reduced action takes the form

$$\mathcal{S} = \int dt \ \mathrm{tr}\left(D_0 Z (D_0 Z)^\dagger - Z Z^\dagger\right) \tag{4.2.43}$$

[3] By 'Gaussian' we mean a theory with a quadratic Lagrangian.

where $D_0 Z = \partial_0 Z + i[A_0, Z]$.

One may choose the $A_0 = 0$ gauge while imposing Gauss's Law, yielding the quantum mechanics for the matrix $Z(t)$ defined by the following action [146]:

$$\mathcal{S} = \int dt \, \mathrm{tr} \left(\dot{Z}\dot{Z}^\dagger - ZZ^\dagger \right) \tag{4.2.44}$$

It is well known that the holomorphic sector of the theory is equivalent to a system of non-interacting fermions in a one-dimensional harmonic oscillator potential [13, 14, 128]. As a subsector of $\mathcal{N} = 4$ Super Yang-Mills extremal correlators in this sector are protected by supersymmetry [147, 148] and the states of this sector are dual to the LLM supergravity geometries [15].

Going beyond the holomorphic sector, we no longer have non-renormalization theorems so the connection to supergravity is not straightforward. Based on the research presented in Chapter 5, we will infer properties of any candidate string dual of the complex matrix model sector at zero coupling in Section 5.7.

We first review the previous analysis of the above theory [13]. The momenta conjugate to $Z^i{}_j$ and $Z^{\dagger i}{}_j$ are

$$\Pi^j{}_i \equiv \Pi_{Z^i{}_j} = \frac{\partial L}{\partial \dot{Z}^i{}_j} = \dot{Z}^{\dagger j}{}_i, \qquad \Pi^{\dagger j}{}_i \equiv \Pi_{Z^{\dagger i}{}_j} = \frac{\partial L}{\partial \dot{Z}^{\dagger i}{}_j} = \dot{Z}^j{}_i. \tag{4.2.45}$$

The equal time canonical commutation relations are

$$\left[Z^p{}_q, \Pi^j{}_i\right] = i\,\delta^j{}_q \delta^p{}_i \qquad \left[Z^{\dagger p}{}_q, \Pi^{\dagger j}{}_i\right] = i\,\delta^j{}_q \delta^p{}_i \tag{4.2.46}$$

so we can identify the conjugate momenta with matrix derivatives in the usual way using (4.2.5). We define the creation and annihilation operators:

$$\begin{aligned} A^\dagger &= \frac{1}{\sqrt{2}}(Z - i\Pi^\dagger) = \frac{1}{\sqrt{2}}\left(Z - \frac{\partial}{\partial Z^\dagger}\right) & A &= \frac{1}{\sqrt{2}}(Z^\dagger + i\Pi) = \frac{1}{\sqrt{2}}\left(Z^\dagger + \frac{\partial}{\partial Z}\right) \\ B^\dagger &= \frac{1}{\sqrt{2}}(Z^\dagger - i\Pi) = \frac{1}{\sqrt{2}}\left(Z^\dagger - \frac{\partial}{\partial Z}\right) & B &= \frac{1}{\sqrt{2}}(Z + i\Pi^\dagger) = \frac{1}{\sqrt{2}}\left(Z + \frac{\partial}{\partial Z^\dagger}\right) \end{aligned} \tag{4.2.47}$$

Importantly, the dagger on $A^\dagger$ does **not** signify hermitian conjugate of A. It signifies purely that this is a creation operator. The hermitian conjugate of ${A^\dagger}^i{}_j$ is $A^j{}_i$. The canonical commutation relations become

$$[A^i{}_j, {A^\dagger}^k{}_l] = \delta^i_l \delta^k{}_j \qquad\qquad [B^i{}_j, {B^\dagger}^k{}_l] = \delta^i_l \delta^k{}_j. \tag{4.2.48}$$

The Hamiltonian and $U(1)$ current take the form

$$\begin{aligned} \hat{H} &= \operatorname{tr}\left(-\frac{\partial^2}{\partial Z \partial Z^\dagger} + ZZ^\dagger\right) &= \operatorname{tr}(A^\dagger A + B^\dagger B) + N^2 \\ \hat{J} &= \operatorname{tr}\left(Z\frac{\partial}{\partial Z} - Z^\dagger \frac{\partial}{\partial Z^\dagger}\right) &= \operatorname{tr}(A^\dagger A - B^\dagger B) \end{aligned} \tag{4.2.49}$$

where N^2 is the zero point energy for N^2 harmonic oscillators in two dimensions.

The ground state of this system satisfies $A|0\rangle = B|0\rangle = 0$. The corresponding (non-normalized) wavefunction $\Psi_0 = \langle Z, \bar{Z}|0\rangle$ is

$$\Psi_0(Z, Z^\dagger) = e^{-\operatorname{tr}(ZZ^\dagger)}. \tag{4.2.50}$$

Holomorphic gauge invariant excitations of this system are defined by the constraint $B|\mathcal{O}\rangle = 0$ and consist of operators built from $A^\dagger$ acting on the ground state. These may be written as

$$\operatorname{tr}_n(\sigma(A^\dagger)^{\otimes n})|0\rangle \tag{4.2.51}$$

where σ is an element of S_n, and controls how the indices are contracted to form either a single or multi-trace operator as we shall illustrate in the next section. A more convenient basis for operators of the form (4.2.51) is the Schur polynomial basis [13] as introduced for the hermitian matrix model in Section 4.2.1:

$$|\Psi_R\rangle = \chi_R(A^\dagger)|0\rangle \tag{4.2.52}$$

where χ_R is the character of the $U(N)$ representation R. Since

$$A^\dagger e^{-\operatorname{tr}(ZZ^\dagger)} = \sqrt{2} Z e^{-\operatorname{tr}(ZZ^\dagger)}, \tag{4.2.53}$$

we may write

$$\Psi_R(Z, Z^\dagger) \;=\; \chi_R(\sqrt{2}Z)e^{-\operatorname{tr}(ZZ^\dagger)}. \tag{4.2.54}$$

This state has $E = m + N^2$ and $J = m$ and is holomorphic in Z up to the exponential factor.

Using the Schur decomposition (4.2.40) and redefining the wavefunction by absorbing the Jacobian of the transformation into the definition of the wavefunction, it becomes a wavefunction for N fermions in one dimension, analogous to the Lowest Landau Level of the Quantum Hall system [13, 14, 128, 149].

4.3 Introduction to the Brauer algebra

4.3.1 The Brauer algebra

In this section we introduce the Brauer algebra and its application to the construction of gauge invariant functions of a complex matrix Z and its complex conjugate $Z^\dagger$.

The Brauer algebra was first constructed as the commuting algebra of the action of $GL(N, \mathbb{C})$ on certain tensor product representations [150]. More specifically, following the exposition in [151], let V_R be a vector space furnishing a representation R of some group G. Then one can consider the tensor product representation on the space

$$V_R^{\otimes n} = V_R \otimes V_R \otimes \cdots \otimes V_R \tag{4.3.1}$$

and ask how this representation decomposes into irreducible representations of G; in particle physics we are used to doing this e.g. with spin and $SU(2)$.

For our purposes we will use $U(N)$ rather than $GL(N, \mathbb{C})$. Let the fundamental representation of $U(N)$ act on V and the complex conjugate of the fundamental representation act on $\bar{V}$, the dual space of V. Let us consider the tensor product space

$$V^{\otimes m} \otimes \bar{V}^{\otimes n}\ . \tag{4.3.2}$$

The (walled) Brauer algebra $B_N(m,n)$ is then the commuting algebra of the action of $U(N)$ on this space.

The Brauer algebra may be represented diagrammatically as we now briefly review. Full details of this construction may be found in the original paper [150].

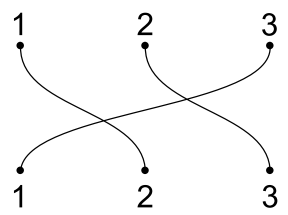
We first introduce a diagrammatic representation of S_n, the symmetric group on n objects, which is the commuting algebra of the action of $U(N)$ on $V^{\otimes n}$. Let $\{\,|e_i\rangle\}$ be a basis of V. Then we define the action of $\sigma \in S_n$ on a vector in $V^{\otimes n}$ via:

$$\sigma\,|e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}\rangle \;=\; |e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \cdots \otimes e_{i_{\sigma(n)}}\rangle \tag{4.3.3}$$

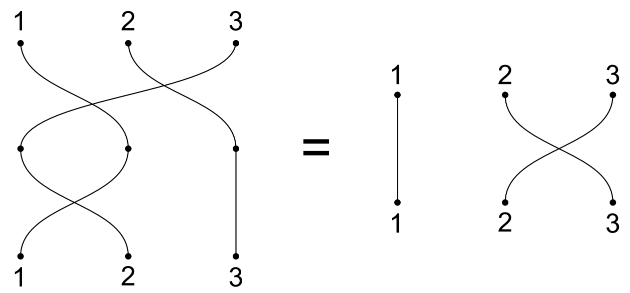
so for example, the action of (123) on a vector in $V^{\otimes 3}$ is

$$(123)\,|e_{i_1} \otimes e_{i_2} \otimes e_{i_3}\rangle \;=\; |e_{i_2} \otimes e_{i_3} \otimes e_{i_1}\rangle\;. \tag{4.3.4}$$

The above action may be represented by diagrams: for example (123) is represented by the diagram



and products are obtained by stacking diagrams: e.g. the product $(12)(123) = (23)$ is represented by:



The Brauer algebra contains the group algebra of $S_m \times S_n$ along with 'contraction' elements $C_{i\bar{j}}$ where $i \in 1,\ldots,m$ and $j \in 1,\ldots,n$, to be defined below, and is generated by the generators of $S_m \times S_n$ along with a single contraction element.

To define the contraction $C_{i\bar{j}}$, let $\{\,|\bar{e}^i\rangle\}$ be the basis of $\bar{V}$ dual to the basis $\{\,|e_i\rangle\}$ of V. Then the action of the contraction $C_{i\bar{j}}$ on $V^{\otimes m}\otimes\bar{V}^{\otimes n}$ is defined by (suppressing $\otimes$ symbols below)

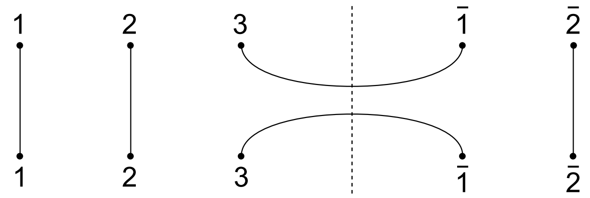
$$\begin{aligned}&C_{i\bar{j}}\,|e_1\cdots e_{i-1}\,e_i\,e_{i+1}\cdots e_n\,\bar{e}^1\cdots\bar{e}^{j-1}\,\bar{e}^j\,\bar{e}^{j+1}\cdots\bar{e}^m\rangle\\&=\delta^j_{\ i}\,|e_1\cdots e_{i-1}\,e_p\,e_{i+1}\cdots e_n\,\bar{e}^1\cdots\bar{e}^{j-1}\,\bar{e}^p\,\bar{e}^{j+1}\cdots\bar{e}^m\rangle\end{aligned}\tag{4.3.5}$$

where there is summation over p.

For example, the contraction $C_{1\bar{1}}\in B_N(1,1)$ has the following action on $V\otimes\bar{V}$:

$$C_{1\bar{1}}\,|e_i\otimes\bar{e}^j\rangle \;=\; \delta^j_{\ i}\,|e_p\otimes\bar{e}^p\rangle\ . \tag{4.3.6}$$

Diagrammatically, contraction elements are represented by lines crossing a wall separating the dots representing the vectors of $V^{\otimes n}$ from those of $\bar{V}^{\otimes m}$, e.g. the contraction element $C_{3\bar{1}}\in B_N(3,2)$ is represented by:



In a product, a closed loop is replaced by multiplication by the parameter N, as can be seen from

$$\begin{aligned}C_{1\bar{1}}C_{1\bar{1}}\,|e_i\otimes\bar{e}^j\rangle &= \delta^j_{\ i}C_{1\bar{1}}\,|e_p\otimes\bar{e}^p\rangle\\&=\delta^j_{\ i}\delta^p_{\ p}\,|e_r\otimes\bar{e}^r\rangle\\&=N\,\delta^j_{\ i}\,|e_r\otimes\bar{e}^r\rangle\\&=N\,C_{1\bar{1}}\,|e_i\otimes\bar{e}^j\rangle\end{aligned}\tag{4.3.7}$$

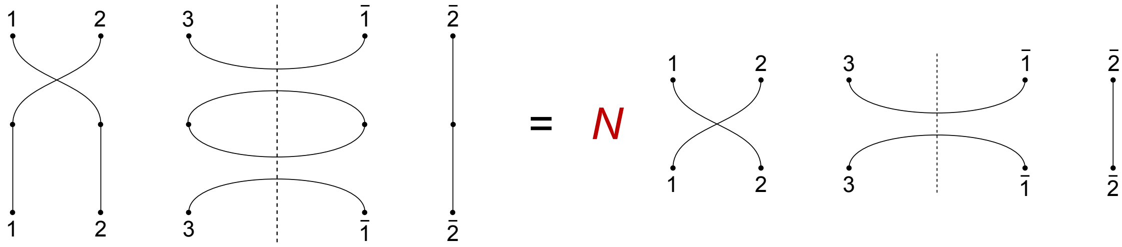
which establishes that

$$C_{1\bar{1}}C_{1\bar{1}} \;=\; N\,C_{1\bar{1}}\ . \tag{4.3.8}$$

As a final example, the product

$$C_{3\bar{1}}\big[(12)C_{3\bar{1}}\big]=N\,(12)C_{3\bar{1}} \tag{4.3.9}$$

takes the diagrammatic form:



4.3.2 Gauge invariant operators

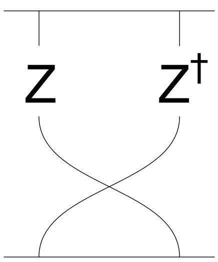
The two gauge invariant operators $\operatorname{tr} Z \operatorname{tr} Z^\dagger$ and $\operatorname{tr} ZZ^\dagger$ can both be constructed by considering the action of $Z \otimes Z^\dagger$ on $V \otimes V$ [13]. Taking the trace[4] in $V \otimes V$ yields the operator $\operatorname{tr} Z \operatorname{tr} Z^\dagger$ while a trace with an insertion of the permutation element (12) yields $\operatorname{tr} ZZ^\dagger$, as follows:

$$\begin{aligned} \operatorname{tr}_{V\otimes V}\left(Z \otimes Z^\dagger\right) &= \langle e^i \otimes e^j |\, Z \otimes Z^\dagger\, | e_i \otimes e_j \rangle \\ &= \langle e^i \otimes e^j |\, Z^p{}_i\, Z^{\dagger q}{}_j\, | e_p \otimes e_q \rangle \\ &= \delta^i{}_p \delta^j{}_q Z^p{}_i Z^{\dagger q}{}_j \;=\; \operatorname{tr} Z \operatorname{tr} Z^\dagger \end{aligned} \tag{4.3.10}$$

and

$$\begin{aligned} \operatorname{tr}_{V\otimes V}\left((12) Z \otimes Z^\dagger\right) &= \langle e^i \otimes e^j |\, (12)\, Z \otimes Z^\dagger\, | e_i \otimes e_j \rangle \\ &= \langle e^i \otimes e^j |\, (12)\, Z^p{}_i\, Z^{\dagger q}{}_j\, | e_p \otimes e_q \rangle \\ &= \langle e^i \otimes e^j |\, Z^p{}_i\, Z^{\dagger q}{}_j\, | e_q \otimes e_p \rangle \\ &= \delta^i{}_q \delta^j{}_p Z^p{}_i Z^{\dagger q}{}_j \;=\; \operatorname{tr} ZZ^\dagger \;. \end{aligned} \tag{4.3.11}$$

The above manipulations may be represented diagrammatically as follows. Focusing on the case of $\operatorname{tr} ZZ^\dagger$, consider the diagram:



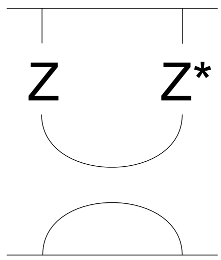
[4]Our notation for traces is that tr is the trace in V, while the trace in another space W is denoted tr_W. We use the shorthands tr_n for $\operatorname{tr}_{V^{\otimes n}}$ and $\operatorname{tr}_{m,n}$ for $\operatorname{tr}_{V^{\otimes m} \otimes \bar{V}^{\otimes n}}$.

The line coming out of the top of Z represents the first index of Z and the line coming out of the bottom of Z represents the second index. The crossed lines below Z and $Z^\dagger$ represent the permutation (12), and the horizontal lines above and below are identified, representing the trace. Following the lines we see that the first index of Z becomes identified with the second index of $Z^\dagger$ (and summed over) exactly as in the above manipulations, so this diagram indeed represents the gauge invariant operator $\mathrm{tr}\, ZZ^\dagger$.

Alternatively, $\mathrm{tr}\, ZZ^\dagger$ can be constructed by considering the action of Z and Z^* on $V \otimes \bar{V}$, and inserting a Brauer algebra contraction [20]. Similarly to the discussion above, we have:

$$\begin{aligned}
\mathrm{tr}_{V\otimes\bar{V}} \left(C_{1\bar{1}} Z \otimes Z^\dagger\right) &= \langle e^i \otimes \bar{e}_j | \, C_{1\bar{1}}\, Z \otimes Z^* \, | e_i \otimes \bar{e}^j \rangle \\
&= \langle e^i \otimes \bar{e}_j | \, C_{1\bar{1}}\, Z^p{}_i \, Z^*{}_q{}^j \, | e_p \otimes \bar{e}^q \rangle \\
&= \langle e^i \otimes \bar{e}_j | \, \delta^q{}_p Z^p{}_i \, Z^*{}_q{}^j \, | e_r \otimes \bar{e}^r \rangle \\
&= \delta^i{}_r \delta^r{}_j \delta^q{}_p Z^p{}_i Z^*{}_q{}^j \\
&= Z^p{}_i Z^*{}_p{}^i \\
&= Z^p{}_i Z^{\dagger i}{}_p \;=\; \mathrm{tr}\, ZZ^\dagger \;.
\end{aligned} \tag{4.3.12}$$

The above manipulation is then represented diagramatically as:

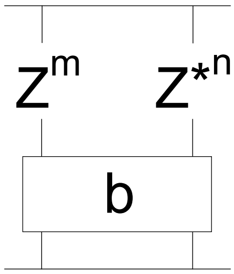


where we see that by following the lines, the first index of Z is identified with the first index of Z^* as in the penultimate line of (4.3.12).

Having treated both transpositions and contractions, it follows that any gauge invariant operator may be written using Z, Z^* and a Brauer algebra element $b \in B_n(m,n)$ as

$$\mathrm{tr}_{m,n} \left(b\, \mathbf{Z} \otimes \mathbf{Z}^*\right) \tag{4.3.13}$$

which one may represent diagrammatically as follows:



4.3.3 Representations of the Brauer algebra

In this section we quote from the mathematics literature some facts about the representations of the Brauer algebra. Details may be found in the references [152, 153, 154].

A representation of the Brauer algebra $B_N(m,n)$ is labelled by $\gamma = (k, \gamma_+, \gamma_-)$, where k is an integer in the range $0 \leq k \leq \min(m,n)$ and γ_+, γ_- are Young diagrams with $m-k$ and $n-k$ boxes respectively, with $c_1(\gamma_+) + c_1(\gamma_-) \leq N$.

Since the Brauer algebra is the maximally commuting algebra of the action of $U(N)$, using Schur-Weyl duality the tensor product representation decomposes into a sum over tensor products of all irreducible representations γ of $U(N)$ and $B_N(m,n)$, i.e.

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_{\gamma} V_{\gamma}^{U(N)} \otimes V_{\gamma}^{B_N(m,n)} \ . \tag{4.3.14}$$

The Brauer representation $\gamma = (k, \gamma_+, \gamma_-)$ has an associated $U(N)$ representation labelled by a composite Young diagram γ_c[5] which is defined as follows: Using the usual notation in which a Young diagram with row lengths r_i is written $[r_1, r_2, \ldots, r_N]$, let

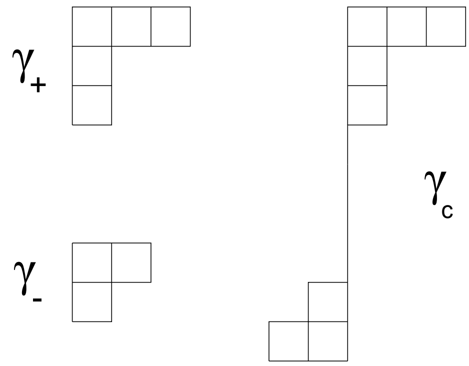
$$\gamma_+ = [r_1, r_2, \ldots, r_p], \qquad \gamma_- = [s_1, s_2, \ldots, s_q] \tag{4.3.15}$$

then providing $p + q \leq N$, γ_c is given by

$$\gamma_c = [r_1, r_2, \ldots, r_p, 0, 0, \ldots, 0, -s_q, -s_{q-1}, \ldots, -s_1] \tag{4.3.16}$$

[5]For ease of notation we will often refer to the $U(N)$ representation γ_c simply as γ, as above in $V_{\gamma}^{U(N)}$. There should hopefully be no confusion in this respect.

where there are $N-(p+q)$ zeroes inserted. In the mathematics literature γ_c has been referred to as an N-staircase with positive part γ_+ and negative part γ_- [153, 152]; diagramatically, we have:



As we have seen, the Brauer algebra $B_N(m,n)$ contains the subalgebra $\mathbb{C}[S_m \times S_n]$. As a consequence, an irrep γ of $B_N(m,n)$ may be decomposed into irreps

$$A = (\alpha, \beta) \tag{4.3.17}$$

of $S_m \times S_n$, where α is an irrep of S_m (given by a Young diagram with m boxes) and similarly β is an irrep of S_n.

In this decomposition the irrep A of $S_m \times S_n$ may appear in this decomposition with non-trivial multiplicity. We write this integer multiplicity as $M_A^{\gamma;N}$ and we can express this decomposition as [20, 2]:

$$V_\gamma^{B_N(m,n)} = \bigoplus_A V_A^{\mathbb{C}(S_m\times S_n)} \otimes V_{\gamma\to A}^{B_N(m,n)\to\mathbb{C}(S_m\times S_n)} , \tag{4.3.18}$$

where

$$\dim V_{\gamma\to A}^{B_N(m,n)\to\mathbb{C}[S_m\times S_n]} = M_A^{\gamma;N} . \tag{4.3.19}$$

4.4 The Brauer algebra basis for complex matrix models

In this section we introduce the Brauer algebra basis for gauge invariant polynomials in a complex matrix $Z, Z^\dagger$, following the original paper [20].

The Brauer basis is one of many orthogonal bases of operators in $\mathcal{N} = 4$ Super-Yang-Mills that has been developed in recent years. Following the half-BPS Schur polynomials [13], a basis was constructed which diagonalizes 1/4 and 1/8-BPS operators using the global $U(3)$ symmetry [155]. This research was generalized to multi-matrix operators with arbitrary global symmetry group in [156]. Recently these bases have been applied to the problem of finding the 1/4 and 1/8-BPS operators at one-loop level [157, 158, 159].

Another basis composed of operators constructed to study strings ending on giant gravitons (spherical D3-branes [160]) called 'restricted Schur polynomials' [161, 162, 163] also provides a diagonal basis for multi-matrix models [164, 165]. This basis has enabled many interesting studies of the physics of giant gravitons and LLM geometries using correlators in $\mathcal{N} = 4$ Super-Yang-Mills [166, 167, 168, 169, 170, 171].

The Brauer basis is constructed using the decomposition of irreducible representations of the Brauer algebra into irreps of the $\mathbb{C}[S_m \times S_n]$ subalgebra. The reason for this is that we are interested in building multi-trace operators built from m Z's and n $Z^\dagger$'s, and these operators will be left invariant under permutations of the Z fields amongst themselves, and similarly the $Z^\dagger$ fields.

The construction of the Brauer basis uses the technology of projectors and generalizations thereof, as follows. Consider an irrep γ of $B_N(m,n)$ which reduces onto an irrep A of $\mathbb{C}[S_m \times S_n]$ with multiplicity $M_A^{\gamma;N}$. Let i run from 1 to the multiplicity $M_A^{\gamma;N}$, and take an orthonormal set of vectors in the irrep γ which transform in the ith copy of the state m_A of the irrep A, denoted by

$$\Big\{ |\gamma; A, m_A; i\rangle \Big\} . \tag{4.4.1}$$

A projector to the ith copy of the irrep A of the subalgebra $\mathbb{C}[S_m \times S_n]$ is therefore

$$P^\gamma_{A,i} = \sum_{m_A} |\gamma; A, m_A; i\rangle\langle\gamma; A, m_A; i| \ . \tag{4.4.2}$$

The projector above belongs to a more general class of operators which commute with the $\mathbb{C}[S_m \times S_n]$ subalgebra, called symmetric branching operators: these map a vector which transforms in the jth copy of the state m_A to a vector which transforms in the ith copy of the state m_A, and are defined as

$$Q^{\gamma}_{A,ij} = \sum_{m_A} |\gamma; A, m_A; i\rangle\langle\gamma; A, m_A; j| \ . \tag{4.4.3}$$

A Brauer basis operator is constructed by viewing $Z^{\otimes m} \otimes (Z^*)^{\otimes n}$ as operators on $V^{\otimes m} \otimes \bar{V}^{\otimes n}$, acting with a symmetric branching operator $Q^{\gamma}_{\alpha,\beta;i,j}$ and taking a trace [20], as described in the diagram below (4.3.13). Expanding out $A = (\alpha, \beta)$ as in (4.3.17), this definition is written as:

$$\mathcal{O}^{\gamma}_{\alpha,\beta;i,j}(Z, Z^{\dagger}) \;\; = \;\; \mathrm{tr}_{m,n}\left(Q^{\gamma}_{\alpha,\beta;i,j}(\mathbf{Z} \otimes \mathbf{Z}^*)\right) \tag{4.4.4}$$

These operators diagonalize the two-point function for $Z, Z^{\dagger}$ at zero Yang-Mills coupling [20].

The same construction can be done with the creation operators of the matrix quantum mechanics by replacing Z with $A^{\dagger}$ and Z^* with $(B^{\dagger})^T$ where T denotes matrix transpose. These operators diagonalize the Fock space inner product for the states created by the $A^{\dagger}, B^{\dagger}$ reviewed in Section 4.2.4.

To provide a simple example of the Brauer basis operators, taking $(m, n) = (1, 1)$ and suppressing non-essential labels, the Brauer basis is

$$\mathcal{O}^{k=0}_{[1],[\bar{1}]}(Z, Z^{\dagger}) \;\; = \;\; \mathrm{tr}\, Z \,\mathrm{tr}\, Z^{\dagger} - \frac{1}{N}\,\mathrm{tr}\, Z Z^{\dagger} \tag{4.4.5}$$

$$\mathcal{O}^{k=1}_{[1],[\bar{1}]}(Z, Z^{\dagger}) \;\; = \;\; \frac{1}{N}\,\mathrm{tr}\, Z Z^{\dagger} \ . \tag{4.4.6}$$

Details on the explicit calculation of these and other Brauer basis operators may be found in [20].

Here we have suppressed γ_+ and γ_- since for a $k = 0$ operator it is always the case that $\alpha = \gamma_+$ and $\beta = \gamma_-$, and since for the above $k = 1$ operator, γ_+ and γ_- are both the empty diagram. The multiplicity indices i, j are not relevant for this example. Further examples of Brauer basis operators may be found in Appendix A.4 of [20].

Since we discuss in particular the label k throughout the rest of this paper, we make the following comment. In the construction of the Brauer basis, a term with a single '$ZZ^\dagger$' inside the same trace, such as $\mathrm{tr}\, ZZZ^\dagger$, involves a single 'Brauer contraction'. Terms such as $\mathrm{tr}\, ZZ^\dagger\, \mathrm{tr}\, ZZ^\dagger Z^\dagger$ or $\mathrm{tr}\, ZZ^\dagger ZZZ^\dagger$ involve two such Brauer contractions, etc.

The label k is related to the number of contractions as follows. If one writes a Brauer basis operator as a sum of terms in order of increasing contractions, as the two operators above are written, an operator with label k begins with a term involving k Brauer contractions. We have not proved this, but we believe it to be true from all the examples we know. Thus the leading term in a $k = 0$ operator is the product of a purely holomorphic operator and a purely anti-holomorphic operator, while all terms in a $m = n = k$ operator involve k contractions.

The $k = 0$ operators are of particular interest in this thesis. When $k = 0$ the i, j labels are trivial and we have $\alpha = \gamma_+,\ \beta = \gamma_-$. Thus γ is given by $(k = 0, \alpha, \beta)$ and so the $k = 0$ operators are thus determined by two Young diagrams α and β, which we sometimes denote (α, β) by (R, S) to make contact with the notation of the string theory of two-dimensional Yang-Mills theory [172].

For later use, we record that in the $k = 0$ case the projector (4.4.2) becomes simply (with $A = (R, S)$):

$$P_{R\bar{S}} \;=\; P^{\gamma}_{A} \;=\; \sum_{m_A} |\gamma; A, m_A\rangle\langle\gamma; A, m_A| \;. \tag{4.4.7}$$

where $R\bar{S}$ is the composite Young diagram formed from (R, S) in the same way that γ_c was formed from (γ_+, γ_-) above.

We also note that there is an isomorphism between the $k = 0$ sector and the states of the unitary matrix model [20]:

$$\mathcal{O}^{k=0}_{R,S}(Z, Z^\dagger) \;\longleftrightarrow\; \chi_{R\bar{S}}(U) \tag{4.4.8}$$

which is obtained by replacing Z with a unitary matrix:

$$\mathcal{O}^{k=0}_{R,S}(U, U^\dagger) \;=\; d_R d_S \chi_{R\bar{S}}(U) \tag{4.4.9}$$

where d_R is the dimension of the S_m representation R.

The two point functions of both sets of operators are diagonal; up to a choice of normalization,

$$\langle\, \mathcal{O}^{\dagger\, k=0}_{R,S}(Z,Z^\dagger)\,|\,\mathcal{O}^{k=0}_{R',S'}(Z,Z^\dagger)\,\rangle \quad = \quad \langle\, \chi^\dagger_{R\bar{S}}(U)\,|\,\chi_{R'\bar{S}'}(U)\,\rangle \quad = \quad \delta_{RR'}\delta_{\bar{S}\bar{S}'}\ . \tag{4.4.10}$$

As we shall see in Section 5.3.4, when both R and S are nontrivial, the leading order term in the expansion of $\mathcal{O}^{k=0}$ begins with the product of the holomorphic and antiholomorphic Schur polynomials:

$$\mathcal{O}^{k=0}_{R,S}(Z,Z^\dagger) \;=\; \chi_R(Z)\chi_S(Z^\dagger) + \cdots\ , \tag{4.4.11}$$

where the dots denote terms with at least one $ZZ^\dagger$ inside a trace. The reader familiar with the 'coupled characters' studied in two-dimensional Yang-Mills will notice that the structure of (4.4.11) is of the same form as the coupled character $\chi_{R\bar{S}}$ (see e.g. [173]).

As a result of this isomorphism between the $k = 0$ sector and the states of the unitary matrix model, the $k = 0$ states are in turn isomorphic to the states of N free fermions on a circle via the map given in Section 4.2.2. This provides the technical motivation to explore free particle descriptions in the Brauer basis; we will return to this fact in the next chapter.

Chapter 5

Free Particles from Brauer Algebras in Complex Matrix Models

Outline of Chapter 5

In this chapter we identify free particle descriptions of non-holomorphic operators in the complex matrix models, in particular the complex matrix model derived from dimensional reduction of $\mathcal{N} = 4$ Super-Yang-Mills theory. We also present results on counting of Brauer basis operators. This chapter is based on research first reported in [2].

The structure of this chapter is as follows:

In Section 5.1 we introduce the research contained in this chapter; in Section 5.2 we review the Schur decomposition, and describe our parameterizations of the matrix coordinates.

In Section 5.3 we describe free particle operators as functions of differential operators at $N = 2$, including the conjectured $k = 0$ sector free fermion momenta on a circle. In Section 5.4 we describe free particle operators at general N, both in the $k = 0$ sector and in the $m = n = k$ sector.

In Section 5.5 we present a conjecture and numerical evidence for the counting of states of the Brauer basis at $N = 2$. Section 5.6 deals with the matrix quantum mechanics obtained by dimensional reduction of $\mathcal{N} = 4$ Super-Yang-Mills and draws a connection with Ginibre's $D = 0$ matrix model [174].

5.1 Introduction to Chapter 5

In the previous chapter we presented physical motivations, based on gauge-gravity duality and applications to black hole physics, for investigating free particle descriptions in non-BPS sectors of $\mathcal{N} = 4$ Super-Yang-Mills and the associated complex matrix model derived from dimensional reduction on S^3.

Using the Schur decomposition $Z = UTU^\dagger$ introduced in Section 4.2.3, the space $gl(N;\mathbb{C})$ may be decomposed into a parameter space of inequivalent orbits $\mathcal{M}_N$ and the orbits of the $U(N)$ action. $\mathcal{M}_N$ has real dimension N^2+1 and is a fibration over the symmetric product $Sym^N(\mathbb{C})$:

$$\begin{array}{c} \mathcal{M}_N \\ \downarrow \\ Sym^N(\mathbb{C}) \end{array} = \mathbb{C}^N/S_N \tag{5.1.1}$$

The eigenvalues are however coupled to the off-diagonal triangular entries and so cannot represent positions of free particles.

We show that free particle descriptions arise the $k = 0$ sector of the Brauer basis by exploiting the map (4.4.9) between the $k = 0$ sector and the unitary matrix model, providing in turn a map to N free fermions on a circle. In this chapter we will give evidence for the following conjecture: that the N free fermions of the $k = 0$ sector can be constructed from degrees of freedom which are composed of eigenvalues as well as off-diagonal elements of the matrix Z.

We also observe a different emergence of free particles in the $m = n = k$ sector[6]. While we start with the gauge invariant sector of a Gaussian complex matrix model, which is a system of N^2 particles *constrained* by the gauge invariance condition, the emergent particles are N free fermions without constraints.

In this chapter we work at finite N. There is an important distinction between

$$N \geq m+n \qquad \text{and} \qquad N < m+n\ . \tag{5.1.2}$$

[6] As the paper containing these results [2] was being written up, we became aware of [175] which studies this sector and the associated free fermions using a matrix polar decomposition.

The condition $N \geq m+n$ may be read as a condition that N be larger than the lengths of operators one wishes to discuss; for example taking the planar limit $N \to \infty$ achieves this trivially. The opposite regime $N < m+n$ is relevant to studies of heavy operators in $\mathcal{N}=4$ Super Yang-Mills such as conjectured duals of black holes in $AdS_5 \times S^5$, where one is interested in m and n scaling like N^2 [63].

The representation theory of Brauer algebras, and thus the construction of the Brauer basis, is well understood for $N \geq m+n$ however there are interesting subtleties for $N < m+n$ (see for example [176]); in this chapter we make various studies at $N=2$ where we can access the regime $N < m+n$ with full explicit control. We also present results valid for general N - in particular the key point of free particles emerging from matrix model from degrees of freedom *beyond eigenvalues* is valid for any N.

As mentioned in Section 4.2.3, many of our results are applicable to any Gaussian complex matrix model with two-point function

$$\langle Z^i_{\ j} \, Z^{\dagger k}_{\ \ l} \rangle \;=\; \delta^i_{\ l} \delta^k_{\ j} \tag{5.1.3}$$

and with the adjoint $U(N)$ action

$$Z \;\to\; gZg^\dagger, \qquad\qquad g \in U(N). \tag{5.1.4}$$

In particular we shall make connections to the $D=0$ Gaussian complex matrix model considered by Ginibre [174]; For some earlier works on complex matrix models, see for example [177, 178, 179, 180].

5.2 Orbits and parameter spaces

In this section we review the Schur decomposition, describe our parameterizations of the matrix coordinates, and review the matrix Gauss's Law in local and global form. We also describe in detail the the ring of gauge invariant polynomials at $N = 2$.

5.2.1 Orbits and the structure of $\mathcal{M}_N$

The relation between $gl(N, \mathbb{C})$, the space of complex matrices Z and the space $\mathcal{M}_N$, of orbits under the adjoint action (5.1.4) is given by the Schur decomposition (see e.g. [181]), which allows one to write any complex matrix Z as

$$Z = UTU^\dagger \tag{5.2.1}$$

where $U \in U(N)$ and T is upper triangular. It has been used previously in the context of the complex matrix model in [182, 128]. The eigenvalues z_i of Z become the diagonal entries (and hence the eigenvalues) of T. There are also off-diagonal elements t_{ij} for $i < j$. The equation (5.2.1) can be viewed as describing a map from the pair (U, T) to complex matrices. The map is onto, but not one-to-one. Pairs (U, T) and $(e^{i\theta}U, T)$ describe the same Z. There is a $U(1)^N$ action

$$\begin{aligned} U &\to U' = UH, \qquad H = diag(e^{i\theta_1}, \ldots, e^{i\theta_N}) \\ T &\to T' = H^\dagger TH \end{aligned} \tag{5.2.2}$$

which leaves Z unchanged. The diagonal $e^{i\theta}$ acts trivially on T but the $U(1)^{N-1}$ part defined by $\sum \theta_j = 0$ mixes non-trivially with the angles in T.

We can parameterize the coset $U(N)\backslash U(1)^N$ using the variable L and decomposing $U = LH$ (as for example in [183]) leading to

$$Z = L(HTH^\dagger)L^\dagger = L\tilde{T}L^\dagger \tag{5.2.3}$$

where $\tilde{T} \equiv HTH^\dagger$. It is also convenient to use the $U(1)^{N-1}$ part of (5.2.2) to set the $N - 1$ entries on the superdiagonal of T (namely $t_{j,j+1}$) to be real, and to use

(U, T).

There is also the freedom, for fixed Z, to rearrange the eigenvalues in any order on the diagonal of T by altering U. This freedom exists because there is a Schur decomposition for each possible ordering of eigenvalues on the diagonal of T. Given

$$Z = U_1 T_1 U_1^\dagger = U_2 T_2 U_2^\dagger \tag{5.2.4}$$

where T_1 and T_2 have different orderings of diagonal entries, we have

$$T_2 = \left(U_2^\dagger U_1\right) T_1 \left(U_2^\dagger U_1\right)^\dagger = U_{12} T_1 U_{12}^\dagger \tag{5.2.5}$$

where $U_{12} \equiv U_2^\dagger U_1$.

We have thus derived the construction mentioned in the introduction of $\mathcal{M}_N$ as a fibration over the symmetric product $Sym^N(\mathbb{C})$:

$$\begin{array}{c} \mathcal{M}_N \\ \downarrow \\ Sym^N(\mathbb{C}) \end{array} = \mathbb{C}^N / S_N \tag{5.2.6}$$

The set of eigenvalues $z_1, z_2, \ldots, z_N$ of Z modulo permutations in S_N forms the space $Sym^N(\mathbb{C})$. Local coordinates on the fibre of $\mathcal{M}_N$ over $Sym^N(\mathbb{C})$ are obtained from the upper triangular elements t_{ij}, with $i < j$, appearing in T.

The space of $N \times N$ complex matrices $gl(N, \mathbb{C})$ consists of orbits generated by the $U(N)$ action $Z \to UZU^\dagger$. Due to the trivial $U(1)$ action the real dimension of the parameter space of orbits $\mathcal{M}_N$ is $N^2 + 1 = 2N^2 - (N^2 - 1)$.

This suggests that the number of generators of ring of functions on $\mathcal{M}_N$ should be $N^2 + 1$. This works in a straightforward way at $N = 2$, but in a nontrivial way at $N = 3$. We will come back to this in Section 5.4.

Local coordinates on $\mathcal{M}_N$ are given by z_i and variables t_{ij}. At generic z_i, t_{ij} the orbits are topologically $U(N)/U(1) = SU(N)/Z_N$. At $t_{ij} = 0$, the parameter space $\mathcal{M}_N$ becomes $Sym^N(\mathbb{C})$. The orbit is then generically $SU(N)/U(1)^{N-1}$. Note that, when $U(N)$ acts on its Lie algebra, the adjoint orbits are always Kähler (and hence even dimensional) [184]. This is no longer the case for orbits in the complexified Lie algebra $gl(N, \mathbb{C})$.

5.2.2 Differential Gauss's law

As reviewed in Section 4.2.4, dimensional reduction of $\mathcal{N} = 4$ SYM onto $\mathbb{R}_t \times S^3$ yields a $U(N)$ gauged matrix quantum mechanics involving a complex matrix $Z(t)$ in the adjoint coupled to a gauge field $A_0(t)$. The action takes the form

$$\mathcal{S} \;=\; \int dt \;\mathrm{tr}\left(D_0 Z (D_0 Z)^\dagger - Z Z^\dagger \right) \tag{5.2.7}$$

where $D_0 Z = \partial_0 Z + i[A_0, Z]$.

We next review remarks contained in [185] and introduce notation we shall use later. A convenient gauge fixing choice is to set $A_0 = 0$. The equation of motion for A_0 must still be imposed, leading to Gauss's Law:

$$Z^\dagger \dot{Z} + Z \dot{Z}^\dagger - \dot{Z} Z^\dagger - \dot{Z}^\dagger Z \;=\; 0 \,. \tag{5.2.8}$$

Upon canonical quantization this leads to the differential form of Gauss's Law, which can be written as

$$G \;=\; G_1 + G_2 + G_3 + G_4 \;=\; 0 \tag{5.2.9}$$

where G_i are defined as:

$$\begin{aligned} (G_1)^i_{\;j} &= Z^{\dagger i}{}_k \left(\frac{\partial}{\partial Z^\dagger}\right)^k_{\;j} & \qquad (G_2)^i_{\;j} &= Z^i_{\;k} \left(\frac{\partial}{\partial Z}\right)^k_{\;j} \\ (G_3)^i_{\;j} &= -Z^{\dagger k}{}_j \left(\frac{\partial}{\partial Z^\dagger}\right)^i_{\;k} & \qquad (G_4)^i_{\;j} &= -Z^k_{\;j} \left(\frac{\partial}{\partial Z}\right)^i_{\;k} \end{aligned} \tag{5.2.10}$$

and we use the usual convention for matrix indices given in (4.2.5). Note that in G_1 and G_2 the ordering of indices is that of usual matrix multiplication, while for G_3 and G_4 the opposite is the case. The G_i correspond respectively to each of the terms in (5.2.8). The operator G is the infinitesimal generator of the adjoint action

$$Z \to U Z U^\dagger, \qquad Z^\dagger \to U Z^\dagger U^\dagger \tag{5.2.11}$$

and invariance under this action restricts gauge invariant operators to be products of traces of the matrices Z and $Z^\dagger$.

5.2.3 Geometry of $\mathcal{M}_2$: coordinates

In this section and in Section 5.3 we perform explicit calculations at $N = 2$. The motivation for considering small values of N is to perform explicit calculations which shed light on the harder (and more interesting) task of obtaining results at arbitrary finite N, a task we return to in Section 5.4.

We start from the Schur decomposition as discussed in Section 5.2.1,

$$Z \;=\; UTU^\dagger \;=\; L\tilde{T}L^\dagger. \tag{5.2.12}$$

In the $N = 2$ case $U(2)/U(1) \cong SU(2)/\mathbb{Z}_2 \cong SO(3)$. We can specify explicit coordinates

$$U \;=\; \begin{pmatrix} \cos\frac{\theta}{2}\, e^{\frac{i}{2}(\phi+\psi)} & \sin\frac{\theta}{2}\, e^{\frac{i}{2}(\phi-\psi)} \\ -\sin\frac{\theta}{2}\, e^{-\frac{i}{2}(\phi-\psi)} & \cos\frac{\theta}{2}\, e^{-\frac{i}{2}(\phi+\psi)} \end{pmatrix} \tag{5.2.13}$$

$$T \;=\; \begin{pmatrix} z_1 & t_0 \\ 0 & z_2 \end{pmatrix}. \tag{5.2.14}$$

The angles θ, ϕ, ψ are the Euler angles of $SU(2)/\mathbb{Z}_2 \cong SO(3)$. With these coordinates L and $\tilde{T}$ take the form

$$L \;=\; \begin{pmatrix} \cos\frac{\theta}{2}\, e^{\frac{i}{2}\phi} & \sin\frac{\theta}{2}\, e^{\frac{i}{2}\phi} \\ -\sin\frac{\theta}{2}\, e^{-\frac{i}{2}\phi} & \cos\frac{\theta}{2}\, e^{-\frac{i}{2}\phi} \end{pmatrix} \tag{5.2.15}$$

$$\tilde{T} \;=\; \begin{pmatrix} z_1 & t_0 e^{i\psi} \\ 0 & z_2 \end{pmatrix}. \tag{5.2.16}$$

The ranges of the coordinates are

$$z_1,\, z_2 \in \mathbb{C}, \qquad 0 \le t_0 < \infty, \tag{5.2.17}$$

$$0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi, \qquad 0 \le \psi < 2\pi. \tag{5.2.18}$$

The Jacobian for the change of variables from Z_{ij} to those above is

$$J \;=\; |z_1 - z_2|^2\, t_0 \sin\theta \tag{5.2.19}$$

and so we have

$$\int \prod_{i,j} dZ_{ij} d\bar{Z}_{ij} \;=\; \int dz_1 dz_2 dt_0 t_0 dU\, |z_1 - z_2|^2. \tag{5.2.20}$$

Note the factor of t_0 here which is analogous to the $\int rdr$ one gets when using plane polar coordinates. Here dU is the Haar measure on $SU(2)$ which we integrate out and normalize to 1 in the definition of the measure.

The invariant line element on $gl(2,\mathbb{C})$ is given by

$$ds^2 \;=\; \mathrm{tr}\, dZ dZ^\dagger. \tag{5.2.21}$$

We introduce the notation

$$\omega \;=\; U^{-1}dU \;=\; \begin{pmatrix} \omega_{11} & \omega_{12} \\ -\bar{\omega}_{12} & -\omega_{11} \end{pmatrix}, \tag{5.2.22}$$

and using $\omega^\dagger = -\omega$ we expand $dZ = U\left(dT + [\omega, T]\right)U^\dagger$.

The line element is then expressible as

$$ds^2 \;=\; \mathrm{tr}\left(dT + [\omega, T]\right)\left(dT^\dagger + [\omega, T^\dagger]\right) \tag{5.2.23}$$

$$= \left|dz_1 + t_0\bar{\omega}_{12}\right|^2 + \left|dz_2 - t_0\bar{\omega}_{12}\right|^2 + \left|dt_0 + 2t_0\omega_{11} - (z_1 - z_2)\omega_{12}\right|^2 + \left|(z_1 - z_2)\omega_{12}\right|^2. \tag{5.2.24}$$

Using the Cartan one-forms ω_i on $SU(2)$ (see e.g. [186]),

$$\omega = U^{-1}dU = -\omega_i T_i, \qquad T_j = \frac{i}{2}\sigma_j, \tag{5.2.25}$$

one may read off the metric on the orbit; we shall do this in the next section.

As an aside, we note that U_{12} defined below (5.2.5) is not a standard permutation matrix in $U(N)$ (the reader may check that the standard permutation matrices in

$U(N)$ do not preserve the triangular form). For concreteness we now exhibit this at $N = 2$. Consider the two matrices

$$T_1 = \begin{pmatrix} z_1 & t_0 \\ 0 & z_2 \end{pmatrix} \tag{5.2.26}$$

$$T_2 = \begin{pmatrix} z_2 & t_0 \\ 0 & z_1 \end{pmatrix} \tag{5.2.27}$$

where we have chosen $t_0 \in \mathbb{R}$.

Defining $D = \sqrt{t_0^2 + |z_1 - z_2|^2}$, we then have $T_2 = U_{12} T_1 U_{12}^\dagger$ with

$$U_{12} = \frac{1}{D} \begin{pmatrix} t_0 & -(\bar{z}_1 - \bar{z}_2) \\ z_1 - z_2 & t_0 \end{pmatrix} . \tag{5.2.28}$$

Clearly this is not the standard permutation matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, but it performs the permutation transformation $z_1 \leftrightarrow z_2$ while preserving the triangular structure. For $N > 2$ the analogous transformation does not just permute the z_i entries but transforms the t_{ij} nontrivially.

5.2.4 Differential Gauss's law and orbits at $N = 2$

Using a change of variables, one may express the Gauss Law operator G (5.2.9-5.2.10) in the coordinates defined in (5.2.13-5.2.14). This results in the following form of the Gauss's Law operator:

$$G = \begin{pmatrix} -i\frac{\partial}{\partial\phi} & ie^{i\psi}(-\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi} + i\csc\theta\frac{\partial}{\partial\psi}) \\ ie^{-i\psi}(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi} + i\csc\theta\frac{\partial}{\partial\psi}) & i\frac{\partial}{\partial\phi} \end{pmatrix} . \tag{5.2.29}$$

This must vanish on gauge invariant wavefunctions, which must therefore be functions only of z_i, t_0 as expected. We will show in Section 5.2.5 that the algebra of gauge invariant polynomials has five generators.

The Gauss's Law reduces the 8D space $gl(2, \mathbb{C})$ to the 5D space parametrized by (z_1, z_2, t_0). We shall find it convenient to define

$$z_c = z_1 + z_2, \qquad z = z_1 - z_2. \tag{5.2.30}$$

As we have seen, we can exchange z_1, z_2 while leaving t_0 invariant; this means mapping $z \to -z$, and so the space of inequivalent orbits is

$$\mathcal{M}_2 = \mathbb{C} \times (\mathbb{C}/\mathbb{Z}_2) \times \mathbb{R}^+. \tag{5.2.31}$$

From the metric (5.2.24) expressed in terms of z_i, t_0 we see that the nature of the orbits changes as we move in the space $(\mathbb{C}/\mathbb{Z}_2) \times \mathbb{R}^+$. The centre of mass coordinate z_c does not affect the nature of the orbits and so we restrict our attention to a $\mathbb{Z}_2$ quotient of the z, t_0 space. Let us define

$$\begin{aligned} X &= (\mathbb{C}/\mathbb{Z}_2) \times \mathbb{R}^+ \\ &= X_0 \cup X_1 \cup X_2 \cup X_3 \end{aligned} \tag{5.2.32}$$

where X is the region in (z, t_0) space where $t_0 \geq 0$, $Re(z) \geq 0$, and the subregions X_i are defined as follows:

- X_0 is the subregion $t_0 > 0, z \neq 0$
- X_1 is the subregion $t_0 > 0, z = 0$
- X_2 is the subregion $t_0 = 0, z \neq 0$
- X_3 is the point $t_0 = 0, z = 0$.

The metric on the gauge orbit is determined by fixing z_i, t_0 in (5.2.24). On X_0 and X_1 the orbit is topologically $SO(3)$; the metric is complicated in general but on X_1 it qualitatively resembles the round three-sphere metric. On X_2 the orbit is a round S^2, while on X_3 the orbit is a point. This completes the global description of the parameter space and the orbits. Note that on X_0 the metric is regular but on X_1, X_2 and X_3, the determinant of the metric is zero.

5.2.5 The algebra of functions on $\mathcal{M}_2$

The algebra of functions on $\mathcal{M}_N$ is generated by single trace polynomials in $Z, Z^\dagger$. In the $N \to \infty$ limit any word in the two letters $Z, Z^\dagger$, up to cyclic permutations, corresponds to a single-trace gauge-invariant function and hence to a function on

$\mathcal{M}_\infty$. At finite N, traces of long words can be expressed in terms of products of traces of shorter words and so the algebra of gauge invariant functions has a finite set of generators.

In [20] this truncation of the generators was discussed in terms of degenerations of Brauer algebra projectors. Here we investigate these finite N truncations in detail at $N = 2$ and find that it suffices to apply the Cayley-Hamilton theorem to obtain the necessary relations.

The Cayley-Hamilton theorem states that a matrix satisfies its own characteristic polynomial. At $N = 2$ this means that

$$Z^2 - (\operatorname{tr} Z)Z + (\det Z)\,\mathbb{1}_2 \;=\; 0. \tag{5.2.33}$$

Taking the trace of this equation gives a relation between $\operatorname{tr} Z^2$, $\operatorname{tr} Z$ and $\det Z$, only two of which are thus algebraically independent as polynomials in the matrix entries. We choose $\operatorname{tr} Z^2$ and $\operatorname{tr} Z$ to be independent, and write

$$\det Z \;=\; \frac{1}{2}\left[\operatorname{tr} Z \operatorname{tr} Z - \operatorname{tr} Z^2\right]. \tag{5.2.34}$$

We also have the corresponding equation for $Z^\dagger$.

We claim that the algebra of multi-trace gauge invariant operators in $Z, Z^\dagger$ at $N = 2$ is the polynomial ring in the five variables

$$\mathcal{B} \;=\; \left\{\operatorname{tr} Z, \quad \operatorname{tr} Z^2, \quad \operatorname{tr} Z^\dagger, \quad \operatorname{tr} Z^{\dagger 2}, \quad \operatorname{tr} Z Z^\dagger\right\}. \tag{5.2.35}$$

In order to prove this, it is enough to show that all other *single trace* operators are algebraically dependent on the operators above, i.e. can be expressed as polynomials in the above five variables.

We prove this in an inductive fashion. Let W to denote any matrix word made from Z and $Z^\dagger$, e.g. $W = ZZZ^\dagger Z$. Multiply (5.2.33) by W and take the trace. This yields the relation

$$\operatorname{tr}(Z^2 W) - (\operatorname{tr} Z)\operatorname{tr}(ZW) + \frac{1}{2}\left[\operatorname{tr} Z \operatorname{tr} Z - \operatorname{tr} Z^2\right]\operatorname{tr} W = 0. \tag{5.2.36}$$

This shows that $\operatorname{tr}(Z^2W)$ is algebraically dependent on $\operatorname{tr}(ZW)$, $\operatorname{tr} W$ and the operators in $\mathcal{B}$, and similarly, $\operatorname{tr}(Z^{\dagger 2}W)$ is algebraically dependent on $\operatorname{tr}(Z^\dagger W)$, $\operatorname{tr} W$ and the operators in $\mathcal{B}$.

Replacing Z by $ZZ^\dagger$ in (5.2.33) and using $\det ZZ^\dagger = \det Z \det Z^\dagger$ gives

$$\operatorname{tr}(ZZ^\dagger)^2 = (\operatorname{tr} ZZ^\dagger)^2 - \frac{1}{2}\left[\operatorname{tr} Z \operatorname{tr} Z - \operatorname{tr} Z^2\right]\left[\operatorname{tr} Z^\dagger \operatorname{tr} Z^\dagger - \operatorname{tr} Z^{\dagger 2}\right]. \tag{5.2.37}$$

This shows us that $\operatorname{tr}(ZZ^\dagger)^2$ is algebraically dependent on the operators in the set $\mathcal{B}$. Similarly, for any word W_2 of length at least two, $\operatorname{tr} W_2^2$ is algebraically dependent on $\operatorname{tr} W_2$ and the operators in the set $\mathcal{B}$.

We conclude that a single trace operator consisting of the trace of a word made from Z and $Z^\dagger$ is algebraically dependent on single trace operators of shorter length iff it contains one of the following combinations as part of the word:

$$Z^2W, \qquad Z^{\dagger 2}W, \quad \text{or} \quad W_2^2 \tag{5.2.38}$$

where as above W stands for any (non-zero length) word in Z and $Z^\dagger$, and W_2 stands for such a word of length at least two.

Iterating the above results, a single trace operator containing one of the combinations in (5.2.38) can be expressed as sums of products of shorter and shorter single trace operators until it is expressed as a sum of products of single trace operators containing none of the combinations in (5.2.38). A maximal set of algebraically independent operators is therefore given by those single trace operators which do not contain any of the expressions in (5.2.38). As claimed this is the set $\mathcal{B}$.

It is worth remarking that we start with a description of the space $gl(2,\mathbb{C})$ in terms of polynomials in $z_1, z_2, t_0, \theta, \phi, \psi$. The differential Gauss Law (5.2.29) removes the angular variables leaving the polynomial ring in the remaining variables, which we denote

$$\langle z_1, z_2, \bar{z}_1, \bar{z}_2, t_0 \rangle. \tag{5.2.39}$$

Invariance under large gauge transformations reduces the algebra of gauge invariant polynomials to the polynomial ring generated by $\mathcal{B}$. Recalling the definitions

$z_c = z_1 + z_2,\ z = z_1 - z_2$ and defining

$$\mathcal{Z} = z^2, \quad \bar{\mathcal{Z}} = \bar{z}^2, \quad T_0 = t_0^2 + \frac{z\bar{z}}{2}, \tag{5.2.40}$$

the algebra of gauge invariant polynomials is equivalently the polynomial ring

$$\langle z_c, \bar{z}_c, \mathcal{Z}, \bar{\mathcal{Z}}, T_0 \rangle. \tag{5.2.41}$$

This is analogous to $U(N)$ gauged Hermitian matrix quantum mechanics where the differential Gauss Law reduces to polynomials in the eigenvalues

$$\langle x_1, x_2, \ldots, x_N \rangle \tag{5.2.42}$$

and invariance under the S_N residual Weyl transformations reduces the gauge invariant polynomials to symmetric polynomials in $x_1, x_2, \cdots, x_N$, equivalently polynomials in the variables

$$\langle (x_1 + x_2 + \cdots + x_N), (x_1^2 + x_2^2 + \cdots + x_N^2), \ldots, (x_1^N + x_2^N + \cdots + x_N^N) \rangle. \tag{5.2.43}$$

In the hermitian case, we are going from a ring to a sub-ring, which corresponds to going from the space $\mathbb{R}^N$ to its quotient space $\mathbb{R}^N/S_N$. In our model, we are going from the ring (5.2.39) to the sub-ring (5.2.41), and correspondingly from the $\mathbb{R}^4 \times \mathbb{R}^+ = \mathbb{C}^2 \times \mathbb{R}^+$ parametrized by the five coordinates z_i, t_0 to $\mathcal{M}_2$. Because of the off-diagonal degrees of freedom, $\mathcal{M}_2$ is not a straightforward quotient of $\mathbb{R}^4 \times \mathbb{R}^+$.

A full investigation of finite N relations for $N > 2$ is left for the future. We expect it will be useful to combine the Cayley-Hamilton approach with the the vanishing of the Brauer projectors, such as in equation (8.16) of [20].

5.3 Free particle structures at $N = 2$

In this section we find evidence of a 'free fermions on a circle' structure in the $k = 0$ sector at $N = 2$. This generalizes to any N, as discussed in Section 5.4.1.

In these developments a crucial role is played by the structure of the ring of

Casimirs; we also show that the momenta of the free fermions of the $k = 0$ sector can be constructed from differential operators in variables which include both eigenvalues and off-diagonal elements of Z. This leads us to observe that the complex matrix model contains free fermions arising in a novel way, different from the way they arise in hermitian or unitary models.

5.3.1 Casimir operators and a ring of degree-preserving differential operators

The differential operators introduced in equation (5.2.10) were studied in [185] as generalized Casimirs commuting with the scaling operator for $Z, Z^\dagger$, which is the Hamiltonian for zero coupling SYM. This ring is analogous to the ring generated by $\mathcal{B}$ in Section 5.2.5; at $N = 2$ the generating set is

$$\mathcal{D} \;=\; \left\{ \operatorname{tr} G_2, \quad \operatorname{tr} G_2^2, \quad \operatorname{tr} G_3, \quad \operatorname{tr} G_3^2, \quad \operatorname{tr} G_2 G_3 \right\} \tag{5.3.1}$$

where G_2, G_3 were defined in (5.2.10)

$$(G_2)^i_j \;=\; Z^i_k \left(\frac{\partial}{\partial Z} \right)^k_j \qquad\qquad (G_3)^i_j \;=\; -\, Z^{\dagger k}{}_j \left(\frac{\partial}{\partial Z^\dagger} \right)^i_k . \tag{5.3.2}$$

Defining

$$G_L \;=\; G_2 + G_3 \;, \tag{5.3.3}$$

we introduce the Hamiltonians

$$\begin{aligned} &H_1 = \operatorname{tr} G_2 && H_2 = \operatorname{tr} G_2^2 && \\ &\bar{H}_1 = \operatorname{tr} G_3 && \bar{H}_2 = \operatorname{tr} G_3^2 && H_L = \operatorname{tr} G_L^2 \;. \end{aligned} \tag{5.3.4}$$

Each of these operators commutes with the scaling operator for Z and $Z^\dagger$, which is $H = H_1 + \bar{H}_1$. The operators in $\mathcal{D}$ generate a ring of commuting Hamiltonians related to the integrability of the system. We have defined H_L for later convenience; its name derives from the fact that the operator $G_2 + G_3$ is the infinitesimal generator of the left action of $U(N)$ [185]:

$$Z \to UZ, \qquad Z^\dagger \to Z^\dagger U^\dagger. \tag{5.3.5}$$

It was shown in [185] that the five operators defined in (5.3.4),

$$\mathcal{H}_A = \left\{ H_1, \quad \bar{H}_1, \quad H_2, \quad \bar{H}_2, \quad H_L \right\} \tag{5.3.6}$$

measure respectively the Casimirs

$$\mathcal{C}_A = \left\{ C_1(\alpha), \quad C_1(\beta), \quad C_2(\alpha), \quad C_2(\beta), \quad C_2(\gamma) \right\} . \tag{5.3.7}$$

Generalized Casimir operators such as $\mathrm{tr}(G_2^2 G_3)$ were investigated in [185] and were shown to be sensitive to the labels i, j of the Brauer basis. Since the matrix elements of G_2 and G_3 commute, we may regard G_2 and G_3 as matrices of c-numbers and apply the Cayley-Hamilton theorem as in Section 5.2.5 to show that the set $\mathcal{D}$ is a maximal algebraically independent set of degree-preserving gauge invariant differential operators.

5.3.2 The Casimirs as differential operators in z_i, t_0

In this section we express the Casimir operators from the previous section as differential operators on $\mathcal{M}_2$.

Below are calculated expressions in the coordinates z_i, t_0 for the Hamiltonians defined in (5.3.4). For convenience define

$$L_1 = z_1 \frac{\partial}{\partial z_1} \qquad \bar{L}_1 = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \tag{5.3.8}$$

$$L_2 = z_2 \frac{\partial}{\partial z_2} \qquad \bar{L}_2 = \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \qquad L_t = \frac{t_0}{2} \frac{\partial}{\partial t_0} \tag{5.3.9}$$

and recall the notation $z_c = z_1 + z_2,\ z = z_1 - z_2$.

Recalling the definition $G_L = G_2 + G_3$ from above (5.3.4), we find the following expressions:

$$H_1 = \mathrm{tr}\, G_2 = L_1 + L_2 + L_t \tag{5.3.10}$$

$$\bar{H}_1 = -\mathrm{tr}\, G_3 = \bar{L}_1 + \bar{L}_2 + \bar{L}_t \tag{5.3.11}$$

$$\begin{aligned} H_2 &= \operatorname{tr} G_2^2 = L_1^2 + L_2^2 + \left(1 - \frac{2 z_1 z_2 \bar{z}}{z t_0^2}\right) L_t^2 \\ &+ \frac{2}{z}\left(z_1 L_1 - z_2 L_2\right) L_t + \frac{z_c}{z}\left(L_1 - L_2\right) + L_t \end{aligned} \tag{5.3.12}$$

$$H_3 = \operatorname{tr} G_3^2 = \overline{\operatorname{tr}(G_2^2)} \tag{5.3.13}$$

$$\begin{aligned} H_L = \operatorname{tr} G_L^2 &= (L_1 - \bar{L}_1)^2 + (L_2 - \bar{L}_2)^2 + \frac{z_c}{z}\left(L_1 - L_2\right) + \frac{\bar{z}_c}{\bar{z}}\left(\bar{L}_1 - \bar{L}_2\right) \\ &- \frac{2}{|z|^2}\Big\{ t_0^2 (L_1 - L_2)(\bar{L}_1 - \bar{L}_2) + \frac{1}{t_0^2}(z_1 \bar{z}_1 - z_2 \bar{z}_2)^2 L_t^2 \\ &-(z_1 \bar{z}_1 - z_2 \bar{z}_2)\left[(L_1 - L_2) + (\bar{L}_1 - \bar{L}_2)\right] L_t - (z_1 \bar{z}_1 + z_2 \bar{z}_2) L_t \Big\} \end{aligned} \tag{5.3.14}$$

Some useful formulae in doing these calculations are now given. Recalling the decomposition

$$Z = L \tilde{T} L^\dagger \tag{5.3.15}$$

and defining $V = L^\dagger dL$, one obtains the expression

$$dZ = L\left(d\tilde{T} + [V, \tilde{T}]\right) L^\dagger \ . \tag{5.3.16}$$

Defining

$$d\tilde{X} = d\tilde{T} + [V, \tilde{T}] \qquad \text{and} \qquad (\tilde{G}_2)^i_{\ j} = \tilde{T}^i_{\ p} \left(\frac{\partial}{\partial \tilde{X}}\right)^p_{\ j} \tag{5.3.17}$$

one may derive

$$\begin{aligned} dZ^i_{\ j} &= L^i_{\ p} d\tilde{X}^p_{\ q} L^{\dagger q}_{\ \ j} \\ \left(\frac{\partial}{\partial Z}\right)^i_{\ j} &= L^i_{\ p} L^{\dagger q}_{\ \ j} \left(\frac{\partial}{\partial \tilde{X}}\right)^p_{\ q} \\ (G_2)^i_{\ j} &= L^i_{\ p} L^q_{\ j} (\tilde{G}_2)^p_{\ q} \ . \end{aligned} \tag{5.3.18}$$

The computation of $(\tilde{G}_2)^p_{\ q}$ shows that it contains angular derivatives. When we calculate

$$\operatorname{tr} G_2^2 = L^i_{\ p} L^q_{\ j} (\tilde{G}_2)^p_{\ q} L^i_{\ r} L^s_{\ j} (\tilde{G}_2)^r_{\ s} \tag{5.3.19}$$

it is important not to neglect the terms obtained from the action of these angular derivatives from $(\tilde{G}_2)^p{}_q$ on $L^i{}_r L^s{}_j$.

The Casimirs as operators on polynomial rings

We observed in equation (5.2.41) that the multi-trace operators built from $Z, Z^\dagger$ form a polynomial ring whose generators we may take to be

$$z_c, \qquad \bar{z}_c, \qquad \mathcal{Z} = z^2, \qquad \bar{\mathcal{Z}} = \bar{z}^2, \qquad T_0 = t_0^2 + \frac{z\bar{z}}{2} \ . \tag{5.3.20}$$

The above differential operators H_2, H_3, H_L map polynomials in these variables to polynomials; we can make this manifest by changing to these variables. Defining

$$L \ = \ \mathcal{Z}\frac{\partial}{\partial \mathcal{Z}} \ , \qquad L_0 \ = \ T_0\frac{\partial}{\partial T_0} \ , \qquad L_c \ = \ z_c\frac{\partial}{\partial z_c} \tag{5.3.21}$$

we obtain the expression

$$\begin{aligned} H_2 \ = \ & 2L\left(L+\frac{1}{2}\right) + \frac{1}{2}L_c(L_c+3) + L_0(L_0+1) + \frac{2z_c^2}{\mathcal{Z}}L + \frac{z_c^2}{\mathcal{Z}}(2L-1)L \\ & + \frac{\mathcal{Z}}{2z_c^2}L_c(L_c-1) + \frac{\bar{\mathcal{Z}}}{8T_0^2}(z_c^2-\mathcal{Z})L_0(L_0-1) \\ & + 2\left(1+\frac{z_c^2}{\mathcal{Z}}\right)LL_0 + 2L_0L_c + 4LL_c \ . \end{aligned} \tag{5.3.22}$$

H_3 is obtained by complex conjugation and the same exercise can also be done for H_L to illustrate that they are operators that map polynomials to polynomials.

5.3.3 Eigenvalues of the Casimir operators

As reviewed in Chapter 4.2, a Young diagram R with non-negative row lengths r_i labels energies $\mathcal{E}_i$ of N fermions in a one-dimensional harmonic oscillator potential, given by

$$\mathcal{E}_i \ = \ r_i + (N-i) \tag{5.3.23}$$

and a Young diagram (N-staircase) R with arbitrary integer r_i labels momenta p_i of N free fermions on a circle given in terms of the Fermi energy $n_F = \frac{N-1}{2}$ by

$$p_i \ = \ r_i + (n_F + 1 - i) \ . \tag{5.3.24}$$

In this section we review the fact that equivalent data is contained in the values of

- the N independent $U(N)$ Casimirs $C_i(R)$ of the representation R
- the N row lengths r_i, and
- the N corresponding fermion momenta p_i.

The same remark holds for non-negative r_i with p_i replaced by $\mathcal{E}_i$.

In Section 5.3.1 we introduced differential operators studied in [185] which when acting on a Brauer basis function $\mathcal{O}^{\gamma}_{\alpha\beta}(Z, Z^{\dagger})$ measure the quadratic Casimir of the Young diagrams α, β, γ. Given a $U(N)$ Young diagram R, its linear and quadratic Casimirs are

$$C_1(R) = \sum_i r_i = n \tag{5.3.25}$$

$$C_2(R) = nN + \sum_i r_i(r_i - 2i + 1). \tag{5.3.26}$$

Using the definition of p_i (5.3.24) we can write C_2 as

$$C_2(R) = \sum_{i=1}^{N} p_i^2 - \frac{N}{12}(N^2 - 1) \tag{5.3.27}$$

which agrees with (4.2.34). Using the definition of $\mathcal{E}_i$ (5.3.23) we can also write C_2 as

$$C_2(R) = \sum_{i=1}^{N} \mathcal{E}_i^2 - (N-1)n - \frac{N}{6}(N-1)(2N-1)\ . \tag{5.3.28}$$

For general N, knowledge of the values of the N independent Casimir invariants C_i determine the values of the power sum symmetric polynomials

$$\mathcal{P}_a = p_1^a + p_2^a + ... + p_N^a \tag{5.3.29}$$

which in turn for $a = 1, \ldots, N$ enables us to solve for p_i or respectively $\mathcal{E}_i$ (see e.g. [187]).

We now demonstrate this in the $N = 2$ theory. The free fermions on a circle have ground state with energy $p_1 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$ and in general we have

$$p_1 = r_1 + \frac{1}{2}, \qquad p_2 = r_2 - \frac{1}{2} \; . \tag{5.3.30}$$

Setting $N = 2$ in (5.3.26) gives

$$C_2 \;=\; r_1(r_1 + 1) + r_2(r_2 - 1) \tag{5.3.31}$$

and so we may express C_1 and C_2 in terms of p_i as

$$\begin{aligned} C_1 &= p_1 + p_2 \\ C_2 &= p_1^2 + p_2^2 - \frac{1}{2} \; . \end{aligned} \tag{5.3.32}$$

The resulting quadratic equations for p_i in terms of C_1 and C_2 have solution

$$\begin{aligned} p_1 &= \frac{C_1}{2} + \sqrt{\frac{C_2}{2} - \frac{C_1^2}{4} + \frac{1}{4}} \\ p_2 &= C_1 - p_1. \end{aligned} \tag{5.3.33}$$

5.3.4 The $k = 0$ sector

Recall that in the $k = 0$ sector the Brauer basis labels are $\gamma = (0, \alpha, \beta)$ so operators are labelled simply by α and β which are representations of S_m and S_n respectively. To connect with the notation of the unitary matrix model, we write $\alpha = R$ and $\beta = S$. If $S = \emptyset$, then the $k = 0$ operator is the holomorphic Schur polynomial corresponding to the representation R:

$$\mathcal{O}^{k=0}_{R,\emptyset}(Z, Z^\dagger) = \chi_R(Z) \; . \tag{5.3.34}$$

If $R = \emptyset$, then the $k = 0$ operator is the anti-holomorphic Schur polynomial corresponding to the representation $\bar{S}$:

$$\mathcal{O}^{k=0}_{\emptyset,S}(Z, Z^\dagger) \;=\; \chi_S(Z^\dagger) \tag{5.3.35}$$

and if both α and β are nontrivial, the leading order term in the expansion of $\mathcal{O}^{k=0}$ begins with the product of the holomorphic and antiholomorphic Schur polynomials:

$$\mathcal{O}^{k=0}_{R,S}(Z,Z^\dagger) \;=\; \chi_R(Z)\chi_S(Z^\dagger)+\cdots\ , \tag{5.3.36}$$

where the dots denote terms with at least one $ZZ^\dagger$ inside a trace as discussed at the start of Section 5.3.

As mentioned in Section 4.4 there is an isomorphism between the $k=0$ sector and the states of the unitary matrix model [20]:

$$\mathcal{O}^{k=0}_{R,S}(Z,Z^\dagger) \;\longleftrightarrow\; \chi_{R\bar{S}}(U) \tag{5.3.37}$$

obtained by replacing Z with a unitary matrix:

$$\mathcal{O}^{k=0}_{R,S}(U,U^\dagger) \;=\; d_R d_S \chi_{R\bar{S}}(U)\ . \tag{5.3.38}$$

At $N=2$, the label γ_c as defined in (4.3.16) may have at most two rows, r_1^γ, r_2^γ and so the integers $(k=0, r_1^\gamma, r_2^\gamma)$ are enough to specify an operator. By enumerating all $N=2$ operators for given (m,n), we next show that:

- If $r_1^\gamma > 0, r_2^\gamma \geq 0$, then $\beta=\emptyset$ and we have a holomorphic Schur polynomial.
- If $r_1^\gamma \leq 0, r_2^\gamma < 0$ then $\alpha=\emptyset$ and we have an antiholomorphic Schur polynomial.
- If $r_1^\gamma > 0, r_2^\gamma < 0$ then the operator is of the form (4.4.11). At $N=2$ there is a unique such operator.

Using the Young diagram notation introduced in Section 4.3.3, and using the shorthand $C_2(\gamma)$ for the $U(N)$ quadratic Casimir of the representation labelled by γ_c, the $N=2$ operators are as follows:

List of γ_+ and γ_- when $m \geq n$ using $d = m - n$

γ_+	γ_-	γ_c	k	$C_2(\gamma)$
$[m]$	$[n]$	$[m, -n]$	0	$m(m+1) + n(n+1)$
$[m-1]$	$[n-1]$	$[m-1, -(n-1)]$	1	$(m-1)(m) + (n-1)(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$[d+1]$	$[1]$	$[d+1, -1]$	$n-1$	$(d+1)(d+2) + 2$
$[d]$	$\emptyset$	$[d, 0]$	n	$d(d+1)$
$[d-1, 1]$	$\emptyset$	$[d-1, 1]$	n	$(d-1)(d)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\left[\, \lceil \frac{d}{2} \rceil, \lfloor \frac{d}{2} \rfloor \,\right]$	$\emptyset$	$\left[\, \lceil \frac{d}{2} \rceil, \lfloor \frac{d}{2} \rfloor \,\right]$	n	$\lceil \frac{d}{2} \rceil \left(\lceil \frac{d}{2} \rceil + 1 \right) + \lfloor \frac{d}{2} \rfloor \left(\lfloor \frac{d}{2} \rfloor - 1 \right)$

List of γ_+ and γ_- when $m < n$ using $\widetilde{d} = n - m$

γ_+	γ_-	γ_c	k	$C_2(\gamma)$
$[m]$	$[n]$	$[m, -n]$	0	$m(m+1) + n(n+1)$
$[m-1]$	$[n-1]$	$[m-1, -(n-1)]$	1	$(m-1)(m) + (n-1)(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$[1]$	$[\widetilde{d}+1]$	$[1, -(\widetilde{d}+1)]$	$m-1$	$(\widetilde{d}+1)(\widetilde{d}+2) + 2$
$\emptyset$	$[\widetilde{d}]$	$[0, -\widetilde{d}]$	m	$\widetilde{d}(\widetilde{d}+1)$
$\emptyset$	$[\widetilde{d}-1, 1]$	$[-1, -(\widetilde{d}-1)]$	m	$(\widetilde{d}-1)(\widetilde{d})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\emptyset$	$\left[\left\lceil \frac{\widetilde{d}}{2} \right\rceil, \left\lfloor \frac{\widetilde{d}}{2} \right\rfloor \right]$	$\left[-\left\lfloor \frac{\widetilde{d}}{2} \right\rfloor, -\left\lceil \frac{\widetilde{d}}{2} \right\rceil \right]$	m	$\left\lceil \frac{\widetilde{d}}{2} \right\rceil \left(\left\lceil \frac{\widetilde{d}}{2} \right\rceil + 1 \right) + \left\lfloor \frac{\widetilde{d}}{2} \right\rfloor \left(\left\lfloor \frac{\widetilde{d}}{2} \right\rfloor - 1 \right)$

This establishes the classification of all Brauer operators at $N = 2$. Since row lengths and fermion momenta are equivalent data in specifying a state, the above classification may be rewritten in terms of fermion momenta p_i^γ. In the next section we use this to find explicit expressions for free particle momentum operators.

5.3.5 Free particle momenta as functions of differential operators

As noted in (5.3.6), when applied to an $N = 2$ Brauer basis operator $\mathcal{O}^{\gamma}_{\alpha,\beta}$, the differential operators

$$\mathcal{H}_A = \Big\{ H_1, \quad \bar{H}_1, \quad H_2, \quad \bar{H}_2, \quad H_L \Big\} \tag{5.3.39}$$

measure the values of the Casimirs

$$\mathcal{C}_A = \Big\{ C_1(\alpha), \quad C_1(\beta), \quad C_2(\alpha), \quad C_2(\beta), \quad C_2(\gamma) \Big\} \tag{5.3.40}$$

respectively. We also have the fact that $C_1(\gamma)$ is measured by $H_1 - \bar{H}_1$. We define fermion momentum operators

$$\hat{p}_A = \Big\{ \hat{p}_1, \quad \hat{p}_2, \quad \hat{\bar{p}}_1, \quad \hat{\bar{p}}_2, \quad \hat{p}^{\gamma}_1, \quad \hat{p}^{\gamma}_2 \Big\} \tag{5.3.41}$$

whose eigenvalues are $p_1,\ p_2,\ \bar{p}_1,\ \bar{p}_2,\ p^{\gamma}_1,\ p^{\gamma}_2$ respectively. We now repeatedly apply (5.3.33) to each of α, β, γ in turn which enables us to derive expressions for these operators in terms of the basic gauge invariant operators $\mathcal{H}_A$.

Applying (5.3.33) to the label α and promoting to an operator equation we obtain

$$\begin{aligned} \hat{p}_1 &= \frac{H_1}{2} + \sqrt{\frac{H_2}{2} - \frac{H_1^2}{4} + \frac{1}{4}} \\ \hat{p}_2 &= H_1 - \hat{p}_1. \end{aligned} \tag{5.3.42}$$

Applying (5.3.33) to the label β we obtain analogous expressions for $\hat{\bar{p}}_1, \hat{\bar{p}}_2$ in terms of $\bar{H}_1, \bar{H}_2$.

Applying (5.3.33) to the label γ, promoting to an operator equation and defining $\hat{d} = H_1 - \bar{H}_1$ we obtain

$$\begin{aligned} \hat{p}^{\gamma}_1 &= \frac{\hat{d}}{2} + \sqrt{\frac{H_L}{2} - \frac{\hat{d}^2}{4} + \frac{1}{4}} \\ \hat{p}^{\gamma}_2 &= \hat{d} - \hat{p}^{\gamma}_1 \end{aligned} \tag{5.3.43}$$

As noted in Section 5.3.4, in the $k = 0$ sector a state is specified simply by the values of the row lengths r_1^γ, r_2^γ, or equivalently by the values of the fermion momenta p_1^γ, p_2^γ and so we now identify $\hat{p}_1^\gamma, \hat{p}_2^\gamma$ as formal expressions for the momenta of the $k = 0$ fermions on a circle. We shall extend this result to arbitrary N in the next section.

Comparing to the explicit expressions for $\mathcal{H}_A$ obtained in Section 5.3.2, we see that these fermion momenta are functions of differential operators in both the eigenvalues z_i and the off-diagonal element t_0. In hermitian matrix models and unitary matrix models, the emergent fermions are the eigenvalues of the relevant matrix. Here, however, the $k = 0$ emergent fermions have no such direct connection to eigenvalues of Z.

5.4 Free particle structures at general N

In this section we extend aspects of our $N = 2$ discussion of the algebra of gauge invariant functions and the rings of scale invariant and gauge invariant differential operators to the case of general N.

Following our considerations for the $k = 0$ sector from Section 5.3, we show that the momenta of the free fermions are determined in terms of differential operators on $\mathcal{M}_N$.

We also study the $m = n = k$ sector. This is the maximum possible value of k, in contrast to our studies of $k = 0$ which is the minimum possible value. This sector consists of traces and multi-traces of $Z^\dagger Z$ and we show that it may be mapped to N free fermions in a one-dimensional harmonic oscillator potential. This is a second, distinct appearance of free particles in complex matrix models.

5.4.1 The $k = 0$ sector at general N

We first observe that our construction of free fermion momenta as functions of differential operators in z_i, t_{ij} may be extended to general N in a slightly weaker form, as follows.

1. The construction in the previous section may be carried out for general N

by identifying differential operators which measure higher order Casimirs. These will be traces of higher powers of the G_i.

2. We have not found closed form expressions analogous to (5.3.32) for higher N since this would require us to solve arbitrary order polynomials;

3. However, since the p_i are integer or half-integer, they may always be determined in terms of the eigenvalues of the Hamiltonians [187], and hence implicitly in terms of differential operators in z_i, t_{ij}.

We have thus identified an implicit map from $k = 0$ operators to fermions on a circle for all finite N.

We next conjecture that the $k = 0$ sector may be described as the kernel of the differential operator $\operatorname{tr} G_2 G_3$. Let us recall from Section 5.3.1 that the differential operator

$$\operatorname{tr}(G_2 + G_3)^2 = \operatorname{tr}(G_2^2 + 2G_2G_3 + G_3^2) \tag{5.4.1}$$

measures $C_2(\gamma)$, and so $\operatorname{tr} G_2 G_3$ measures

$$\frac{1}{2}\left(C_2(\gamma) - C_2(\alpha) - C_2(\beta)\right). \tag{5.4.2}$$

Since for a $k = 0$ operator $\gamma = (0, \alpha, \beta)$, we have that

$$C_2(\gamma) = C_2(\alpha) + C_2(\beta) \tag{5.4.3}$$

and so

$$(\operatorname{tr} G_2 G_3)\; O^{k=0}(Z, Z^\dagger) = 0. \tag{5.4.4}$$

As a brief aside, note that the action of the Brauer contraction element $C_{1\bar{1}}$ on $Z^i{}_j Z^{\dagger k}{}_l$ is as follows [20]:

$$C_{1\bar{1}}\; \left(Z^i{}_j Z^k{}_l\right) = \delta^i_l (Z^\dagger Z)^k{}_j. \tag{5.4.5}$$

Since

$$(G_2)^p{}_q Z^i{}_j = \delta^i{}_q Z^p{}_j \qquad \text{and} \qquad -(G_3)^q{}_p Z^{\dagger k}{}_l = \delta^q{}_l Z^{\dagger k}{}_p \tag{5.4.6}$$

we have

$$-\operatorname{tr} G_2 G_3 \left(Z^i{}_j Z^{\dagger k}{}_l\right) = \delta^i_l (Z^\dagger Z)^k_j \tag{5.4.7}$$

and since $\operatorname{tr} G_2 G_3$ acts via the Leibniz rule, the action of $-\operatorname{tr} G_2 G_3$ on

$$\mathcal{O} = Z^{i_1}{}_{j_1} Z^{i_2}{}_{j_2} \cdots Z^{i_m}{}_{j_m} Z^{\dagger p_1}{}_{q_1} Z^{\dagger p_2}{}_{q_2} \cdots Z^{\dagger p_n}{}_{q_n} \tag{5.4.8}$$

is that of the sum over all individual contractions

$$C = \sum_{r=1}^{m} \sum_{s=1}^{n} C_{r\bar{s}}. \tag{5.4.9}$$

Similarly the action of the laplacian

$$\Box = \operatorname{tr}\left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}\right) \tag{5.4.10}$$

on $Z^i_{\ j} Z^{\dagger k}{}_l$ is given by

$$\Box\left(Z^i_{\ j} Z^{\dagger k}{}_l\right) = \delta^i_l \delta^k_j. \tag{5.4.11}$$

which is a Wick contraction using the two point function (4.2.42), and as before extends via the Leibniz rule. It was noted in [20] that the $k = 0$ operators have no self Wick contractions and so we have

$$\Box\, O^{k=0}(Z, Z^\dagger) = 0\ , \tag{5.4.12}$$

a result we shall use later in Section 5.6.

While it is possible to construct simple examples which show that the $k = 0$ operators do not comprise the full kernel of $\Box$, we conjecture that the $k = 0$ sector is the kernel of the differential operator $\operatorname{tr} G_2 G_3$, i.e. that the converse of (5.4.4) is true for any N:

$$\operatorname{tr}(G_2 G_3)\,\mathcal{O} = 0 \qquad \Rightarrow \qquad \mathcal{O} = \mathcal{O}^{k=0} \tag{5.4.13}$$

As a differential operator, $\operatorname{tr}(G_2 G_3)$ can be viewed as a modification of the laplacian which is invariant under scalings of Z and $Z^\dagger$.

It is instructive to try and construct a counterexample to (5.4.13). From (5.4.2) we know that $\operatorname{tr}(G_2 G_3)\,\mathcal{O} = 0$ is equivalent to

$$C_2(\gamma) = C_2(\alpha) + C_2(\beta). \tag{5.4.14}$$

One could consider for example the operator with labels

$$\alpha = [1,1], \qquad \beta = [1,1], \qquad \gamma = (k = 1, \gamma_+ = [1], \gamma_- = [1]) \tag{5.4.15}$$

which has Casimirs

$$C_2(\alpha) = 2, \qquad C_2(\beta) = 2, \qquad C_2(\gamma) = 4 \tag{5.4.16}$$

and so is a candidate counterexample since it appears to be a $k = 1$ operator satisfying $\mathrm{tr}(G_2 G_3)\,\mathcal{O} = 0$.

We shall see in Section 5.5 that the labels above fail to satisfy the constraint

$$c_1(\alpha) + c_1(\beta) \leq N + k \tag{5.4.17}$$

which we shall conjecture to be necessary at $N = 2$. If this constraint is indeed correct, the operator considered above in fact does not exist. This example shows that the conjecture (5.4.13) is sensitive to finite N constraints of the Brauer basis.

5.4.2 The $m = n = k$ sector: Operators and free fermions

We recall from the discussion at the start of Section 5.3 that the integer k is directly related to the minimum number of Brauer contractions involved in the terms which are summed to make up an operator in the Brauer basis.

For $m = n = k$, all terms in an operator involve the maximum number of contractions, which translates into the fact that these operators are multi-traces of the matrix $Y = Z^\dagger Z$. Since Y is hermitian we find the N fermions of the hermitian matrix model emerging in this sector, as follows.

In this sector we have $\gamma = (k = m, \gamma_+ = \emptyset, \gamma_- = \emptyset)$ and $\alpha = \beta$, so the projectors $Q^\gamma_{\alpha,\beta}$ (defined in Section 4.4) are in this sector labelled by α alone. We write

$$P^{k=m}_\alpha = Q^\gamma_{\alpha,\alpha} \qquad \text{with} \quad \gamma \;=\; (k = m, \gamma_+ = \emptyset, \gamma_- = \emptyset)\;. \tag{5.4.18}$$

The projector is written in terms of the k-contraction operator $C_{(k)}$ defined by

$$C_{(k)} \quad = \quad \sum_{\sigma\in S_k} C_{\sigma(1)\bar{1}} \cdots C_{\sigma(k)\bar{k}}, \tag{5.4.19}$$

and the projector p_α which projects the holomorphic half of $V^{\otimes k} \otimes \bar{V}^{\otimes k}$ to the representation α.

Introducing the notation $Dim\alpha$ for the dimension of the $U(N)$ representation α, and recalling the notation d_α for the dimension of the S_k representation α, it is proved in Section 5.4.3 that the projector takes the form

$$P_\alpha^{k=m} = \frac{d_\alpha}{k!Dim\alpha} C_{(k)} p_\alpha \tag{5.4.20}$$

and that the operator satisfies the following required properties:

$$(P_\alpha^{k=m})^2 = P_\alpha^{k=m} \qquad \text{and} \qquad \mathrm{tr}_{k,k}(P_\alpha^{k=m}) = (d_\alpha)^2 \ . \tag{5.4.21}$$

The operators in the $m = n = k$ sector therefore take the explicit form:

$$\begin{aligned}
& \mathrm{tr}_{k,k}(P_\alpha^{k=m} Z^{\otimes k} \otimes Z^{*\otimes k}) \\
= \ & \frac{d_\alpha}{k!Dim\alpha} \mathrm{tr}_{k,k}(C_{(k)} p_\alpha Z^{\otimes k} \otimes Z^{*\otimes k}) \\
= \ & \frac{d_\alpha}{k!Dim\alpha} \sum_{\sigma\in S_k} \mathrm{tr}_{k,k}(\sigma C_{1\bar{1}} \cdots C_{k\bar{k}} \sigma^{-1} p_\alpha Z^{\otimes k} \otimes Z^{*\otimes k}) \\
= \ & \frac{d_\alpha}{Dim\alpha} \mathrm{tr}_{k,k}(C_{1\bar{1}} \cdots C_{k\bar{k}} p_\alpha Z^{\otimes k} \otimes Z^{*\otimes k}) \\
= \ & \frac{d_\alpha}{Dim\alpha} \mathrm{tr}_{k}(p_\alpha Y^{\otimes k})
\end{aligned} \tag{5.4.22}$$

where $Y = Z^\dagger Z$. So operators in the $m = n = k$ sector are Schur polynomials constructed from Y.

We may understand these results in the following way. First observe that H_L annihilates $(Z^\dagger Z)^i_{\ j}$, since $H_L = G_2 + G_3$ generates the $U(N)$ action on the lower index of $Z^\dagger$ and the upper index of Z,

$$Z \to UZ, \qquad Z^\dagger \to Z^\dagger U^\dagger \tag{5.4.23}$$

and that the product $(Z^\dagger Z)^i_{\ j}$ is invariant under this action. Traces of powers of Y

are thus also invariant under (5.4.23).

H_L measures $C_2(\gamma)$ which implies that $C_2(\gamma) = 0$ for all operators built from $Y^i{}_j$. This is consistent with the fact that in the $m = n = k$ sector $\gamma = (k = m, \gamma_+ = \emptyset, \gamma_- = \emptyset)$ and so $C_2(\gamma) = 0$. We can consider a Casimir of the form $\mathrm{tr}(Y\frac{\partial}{\partial Y})^2$ which measures the labels of the Young diagram.

By the map discussed in Section 4.2.1, Schur polynomials in a hermitian matrix correspond to the states of N free fermions in a harmonic oscillator potential. The harmonic oscillator fermions observed here are a second emergence of free particles, distinct from those of the $k = 0$ sector.

5.4.3 Proofs for $m = n = k$ projectors

In this section, we shall show the operator (5.4.20) satisfies the following properties:

$$(P_\alpha^{k=m})^2 = P_\alpha^{k=m} \tag{5.4.24}$$

and

$$\mathrm{tr}_{k,k}(P_\alpha^{k=m}) = (d_\alpha)^2. \tag{5.4.25}$$

The second equation follows from the Schur-Weyl duality;

$$\begin{aligned} V^{\otimes k} \otimes \bar{V}^{\otimes k} &= \bigoplus_\gamma V_\gamma^{U(N)} \otimes V_\gamma^{B_N(k,k)} \\ &= \bigoplus_{\gamma,A} V_\gamma^{U(N)} \otimes V_A^{\mathbb{C}[S_k \times S_k]} \otimes V_{\gamma\to A}^{B_N(k,k)\to\mathbb{C}(S_k\times S_k)} \ . \end{aligned} \tag{5.4.26}$$

In the second line, we have decomposed each irreducible representation γ of the Brauer algebra into irreducible representations A of the group algebra of $S_m \times S_n$. Acting with the projector $P_\alpha^{k=m}$ on this equation and taking a trace in $V^{\otimes k} \otimes \bar{V}^{\otimes k}$, we get

$$\mathrm{tr}_{k,k}(P_\alpha^{k=m}) = d_{(\alpha,\alpha)} = (d_\alpha)^2 \tag{5.4.27}$$

where we have used $Dim\gamma = 1$ and $M_A^\gamma = 1$ for $\gamma = (\emptyset, \emptyset, k = m)$.

The k-contraction operator $C_{(k)}$ can be written in many ways, for example

$$\begin{aligned} C_{(k)} &= \sum_{\sigma\in S_k} C_{\sigma(1)\bar{1}}\cdots C_{\sigma(k)\bar{k}} \\ &= \sum_{\sigma\in S_k} \sigma C_{1\bar{1}}\cdots C_{k\bar{k}}\sigma^{-1} \\ &= \sum_{\bar{\sigma}\in \bar{S}_k} \bar{\sigma} C_{1\bar{1}}\cdots C_{k\bar{k}}\bar{\sigma}^{-1} \end{aligned} \tag{5.4.28}$$

The second equality follows from

$$\sigma C_{i\bar{j}} = C_{\sigma(i)\bar{j}}\sigma \tag{5.4.29}$$

In order to show (5.4.24), we first calculate $(C_{(k)})^2$:

$$\begin{aligned} (C_{(k)})^2 &= \sum_{\rho,\sigma\in S_k} \rho C_{1\bar{1}}\cdots C_{k\bar{k}}\rho^{-1}\sigma C_{1\bar{1}}\cdots C_{k\bar{k}}\sigma^{-1} \\ &= \sum_{\rho,\sigma\in S_k} \mathrm{tr}_k(\rho^{-1}\sigma)\rho C_{1\bar{1}}\cdots C_{k\bar{k}}\sigma^{-1} \\ &= \sum_{\rho,\sigma\in S_k} N^{C_{\rho^{-1}\sigma}}\rho C_{1\bar{1}}\cdots C_{k\bar{k}}\sigma^{-1} \\ &= \sum_{\tau,\sigma\in S_k} N^{C_{\tau}}\tau\sigma C_{1\bar{1}}\cdots C_{k\bar{k}}\sigma^{-1} \\ &= N^k\Omega_k C_{(k)} \end{aligned} \tag{5.4.30}$$

where Ω_k is the Omega factor defined by

$$\Omega_k = \sum_{\sigma\in S_k} N^{C_\sigma - k}\sigma \tag{5.4.31}$$

where C_σ is the number of cycles in σ. Using the equation (5.4.30), we can easily show that the projector (5.4.20) satisfies (5.4.24).

We also have another interesting equation for $C_{(k)}$:

$$C_{(k)}p_\alpha = C_{(k)}\bar{p}_\alpha, \tag{5.4.32}$$

which is a consequence of

$$C_{1\bar{1}}\cdots C_{k\bar{k}}\sigma = C_{1\bar{1}}\cdots C_{k\bar{k}}\bar{\sigma}^{-1}. \tag{5.4.33}$$

We finally prove (5.4.25):

$$\begin{aligned} \mathrm{tr}_{k,k}(P_\alpha^{k=m}) &= \frac{d_\alpha}{k!Dim\alpha}\,\mathrm{tr}_{k,k}(C_{(k)}p_\alpha) \\ &= \frac{d_\alpha}{k!Dim\alpha}\sum_{\sigma\in S_k}\mathrm{tr}_{k,k}(\sigma C_{1\bar{1}}\cdots C_{k\bar{k}}\sigma^{-1}p_\alpha) \\ &= \frac{d_\alpha}{Dim\alpha}\,\mathrm{tr}_{k,k}(C_{1\bar{1}}\cdots C_{k\bar{k}}p_\alpha) \\ &= \frac{d_\alpha}{Dim\alpha}\,\mathrm{tr}_{k}(p_\alpha) \\ &= \frac{d_\alpha}{Dim\alpha}d_\alpha Dim\alpha \\ &= (d_\alpha)^2. \end{aligned} \tag{5.4.34}$$

5.5 Counting of operators

In this section we study the counting of the operators of the Brauer basis. This counting was known already at $m+n<N$ [154], however for physical applications one may be interested in ranges of parameters for which m and n are order N or even order N^2. We conjecture a solution for the counting of operators at $N=2$, for which we provide numerical evidence.

5.5.1 The Brauer basis labels at $N=2$ in terms of five integers

In Section 5.3.1 we observed that the generalized Casimir operators such as $\mathrm{tr}\, G_2^2G_3$ do not yield independent information about the wavefunctions at $N=2$, i.e. that all the information in the labels $\{\alpha,\beta,\gamma,i,j\}$ is in fact contained only in $\{\alpha,\beta,\gamma\}$. We can interpret this fact in terms of Brauer algebra representation theory as follows.

As reviewed in Section 4.3, when decomposing an irrep γ of the Brauer algebra into irreps $A=(\alpha,\beta)$ of $\mathbb{C}[S_m\times S_n]$, we denote the integer multiplicity by $M_A^{\gamma;N}$, i.e.

$$\dim V_{\gamma\to A}^{B_N(m,n)\to\mathbb{C}[S_m\times S_n]} = M_A^{\gamma;N}\ . \tag{5.5.1}$$

For large N, i.e. $m+n<N$, we denote this multiplicity by M_A^γ and using $\delta\vdash k$

to denote that δ is a partition of k, M^γ_A is given by the formula [154]

$$M^\gamma_A \equiv M^\gamma_{\alpha,\beta} = \sum_{\delta\vdash k}\sum_{\delta} g(\gamma_+,\delta;\alpha)g(\gamma_-,\delta,\beta) \qquad (5.5.2)$$

where $g(\gamma_+,\delta;\alpha)$ is a Littlewood-Richardson coefficient.

As reviewed in Section 4.4 the indices i,j on a Brauer operator range over the values $\{1,\ldots,M^{\gamma;N}_A\}$, and so the redundancy of the i,j labels at $N=2$ means that $M^{\gamma;N=2}_A$ is either 0 or 1 for all γ, A.

A direct proof of this by using the finite N constraints on the states of the Brauer representation in [152] would be interesting to obtain. At this point we will take a more pragmatic perspective, assume it is true, and will find that it leads to a consistent counting of states of the complex matrix model at $N=2$.

In Section 5.2.5 we described the states of the $N=2$ theory as generated by a finite set of traces. In this section we will obtain the corresponding description in terms of the Brauer basis for multi-traces. For general N, we give a review of the Brauer basis states in Section 4.4. For ease of notation we denote $r_i = r_i(\alpha)$ and $\bar{r}_i = r_i(\beta)$.

We can choose different sets of five integers to parameterize the states, such as

$$r_1, r_2, \bar{r}_1, \bar{r}_2, r^\gamma_1 \qquad (5.5.3)$$

$$r^\gamma_1, r^\gamma_2, k, r_1, \bar{r}_1 \qquad (5.5.4)$$

$$r^\gamma_1, r^\gamma_2, k, r_1, \bar{r}_2. \qquad (5.5.5)$$

We will show that each of the above sets of five integers determines a state uniquely, and we will give the constraints on the integers.

A state is determined uniquely at $N=2$ by α,β,γ, containing the set of integers

$$\{r_1, r_2;\ \bar{r}_1, \bar{r}_2;\ k, r^\gamma_1, r^\gamma_2\}\,. \qquad (5.5.6)$$

From the Brauer algebra representation theory briefly reviewed in Section 4.4, we have the following relations :

$$\sum_i r_i = m, \qquad \sum_i \bar{r}_i = n, \tag{5.5.7}$$

$$\sum_i r_i(\gamma_+) = m - k, \qquad \sum_i r_i(\gamma_-) = n - k. \tag{5.5.8}$$

Using the relationship between $r_i(\gamma)$, $r_i(\gamma_+)$ and $r_i(\gamma_-)$ we have

$$\sum_i r_i(\gamma) \;=\; \sum_i r_i(\gamma_+) - \sum_i r_i(\gamma_-) \;=\; m - n \tag{5.5.9}$$

which at $N = 2$ reads

$$r_1^\gamma + r_2^\gamma \;=\; m - n. \tag{5.5.10}$$

Adding the two expressions in (5.5.8) we find that

$$\sum_i |r_i(\gamma)| \;=\; \sum_i r_i(\gamma_+) + \sum_i r_i(\gamma_-) \;=\; m + n - 2k \tag{5.5.11}$$

which at $N = 2$ gives

$$k = \frac{1}{2}\left(m + n - |r_1^\gamma| - |r_2^\gamma|\right). \tag{5.5.12}$$

We now show that each of (5.5.3)-(5.5.5) are enough to determine the state via (5.5.6):

1. Starting from the five integers in (5.5.3), we deduce m, n from (5.5.7), r_2^γ from (5.5.10) and k from (5.5.12).

2. Starting from (5.5.4) we read off $r_i(\gamma_+)$ and $r_i(\gamma_-)$ by inspecting whether r_1^γ and r_1^γ are positive or negative. We then deduce m and n from (5.5.8) and r_2 and $\bar{r}_2$ from (5.5.7).

3. Starting from (5.5.5) we proceed as in point 2 above.

This shows that each of the three sets of five integers identified are sufficient to identify any state.

5.5.2 Counting of states at $N = 2$ and Brauer basis labels

The ring of gauge invariant operators at $N = 2$ is generated by five single trace operators (5.2.35). Hence the number of linearly independent multi-trace operators $Q^{N=2}_{mt}(m,n)$ for fixed (m,n) is counted by the generating function

$$\frac{1}{(1-x)(1-y)(1-x^2)(1-y^2)(1-xy)} = \sum_{m,n} Q^{N=2}_{mt}(m,n)x^m y^n. \tag{5.5.13}$$

This is the Plethystic Exponential [188, 189] of the single trace generating function

$$\sum_{m,n} Q^{N=2}_{st}(m,n)\, x^m y^n \;=\; 1 + x + y + x^2 + y^2 + xy \tag{5.5.14}$$

derived from the independent single traces in the basis $\mathcal{B}$ (5.2.35).

Having found the $N = 2$ counting of multi-traces, we can express it in terms of constraints on the large N Brauer counting. The constraint $c_1(\gamma_+) + c_1(\gamma_-) \le N$ turns out not to be sufficient. We have argued above that the multiplicities $M^{\gamma;N=2}_{\alpha,\beta}$ are either 0 or 1. We first set

$$M^{\gamma;N=2}_{\alpha,\beta} \;=\; \begin{cases} 1 & \text{if } M^{\gamma}_{\alpha,\beta} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.5.15}$$

where $M^{\gamma}_{\alpha,\beta}$ is given by (5.5.2). Having done this we also find it necessary to impose extra constraints on the labels α, β for agreement with (5.5.13).

The constraints on α, β are as follows. Denoting the length of the p^{th} column of a Young diagram R by $c_p(R)$, we constrain:

1. $c_1(\alpha) + c_1(\beta) \;\le\; N + k$
2. $[c_1(\alpha) + c_1(\beta)] + [c_2(\alpha) + c_2(\beta)] \;\le\; 2N + k$

   $\vdots$

and in general for each $p = 1, 2, \ldots, \min(m, n)$, constrain

$$\sum_{r=1}^{p}(c_r(\alpha) + c_r(\beta)) \;\leq\; pN + k. \tag{5.5.16}$$

We used the mathematics computer software SAGE[7] to enumerate all possible Brauer basis operators subject to the constraint (5.5.16) and to compare with the $N = 2$ trace basis generating function (5.5.13). The two agree up to $(m, n) = (15, 15)$ which is the practical limit for a desktop computer. This conjecture generalizes the 'Non-chiral Stringy Exclusion Principle' introduced in [20]. This counting of operators at $N = 2$ implies a result for the reduction multiplicities $M_A^{\gamma,N=2}$, namely that

$$M_{\alpha,\beta}^{\gamma;N=2} \;=\; \begin{cases} 1 & \text{if } M_{\alpha,\beta}^{\gamma} > 0 \text{ and (5.5.16) holds} \\ 0 & \text{otherwise} \end{cases} \tag{5.5.17}$$

We will re-state this result after simplifying the condition (5.5.16).

5.5.3 $N = 2$ constraints in terms of five integers

Let us consider the case where k is one of our five integers. We rewrite the $N = 2$ constraint (5.5.16) as a lower bound on k:

$$k \geq \sum_{r=1}^{p}(c_r(\alpha) + c_r(\beta)) - 2p \qquad \text{for each } p = 1, \ldots \min(m, n). \tag{5.5.18}$$

Note that as p increases the lower bound on k gets stronger only when

$$c_p(\alpha) + c_p(\beta) > 2. \tag{5.5.19}$$

Before presenting a general expression for the lower bound on k we examine in detail the case

$$0 < r_2 < \bar{r}_2 < r_1 < \bar{r}_1. \tag{5.5.20}$$

We observe that

[7] www.sagemath.org

- For $1 \leq p \leq r_2$ we have $c_p(\alpha) + c_p(\beta) = 4$
- For $r_2 < p \leq \bar{r}_2$ we have $c_p(\alpha) + c_p(\beta) = 3$
- For $p > \bar{r}_2$ we have $c_p(\alpha) + c_p(\beta) \leq 2$

The strongest lower bound on k is therefore at $p = \bar{r}_2$ where we have

$$\begin{aligned} k &\geq 4r_2 + 3(\bar{r}_2 - r_2) - 2\bar{r}_2 \\ \Rightarrow k &\geq r_2 + \bar{r}_2. \end{aligned} \tag{5.5.21}$$

Proceeding similarly we find a general expression for the lower bound on k. For simplicity, wlog suppose $r_2 \leq \bar{r}_2$. There are three cases to consider:

1. $r_2 \leq r_1 \leq \bar{r}_2 \leq \bar{r}_1 \quad \Rightarrow \quad k \geq r_1 + r_2$
2. $r_2 \leq \bar{r}_2 \leq r_1 \leq \bar{r}_1 \quad \Rightarrow \quad k \geq r_2 + \bar{r}_2$
3. $r_2 \leq \bar{r}_2 \leq \bar{r}_1 \leq r_1 \quad \Rightarrow \quad k \geq r_2 + \bar{r}_2.$

Combining these we obtain the lower bound

$$k \;\geq\; \min(r_2, \bar{r}_2) + \min\left(\; \min(r_1, \bar{r}_1),\, \max(r_2, \bar{r}_2)\; \right) \tag{5.5.22}$$

which is equivalent to (5.5.16). We can also express the constraint (5.5.22) in terms of the five integers in (5.5.3) by substituting for k from (5.5.12) to find

$$\frac{1}{2}\left(m + n - |r_1^\gamma| - |m - n - r_1^\gamma|\right) \;\geq\; \min(r_2, \bar{r}_2) + \min\left(\min(r_1, \bar{r}_1), \max(r_2, \bar{r}_2)\right)\ . \tag{5.5.23}$$

We can now re-state the result (5.5.17) for the $N = 2$ reduction multiplicities:

$$M_{\alpha,\beta}^{\gamma;N=2} \;=\; \begin{cases} 1 & \text{if } M_{\alpha,\beta}^{\gamma} > 0 \;\text{ and (5.5.22) holds} \\ 0 & \text{otherwise.} \end{cases} \tag{5.5.24}$$

5.6 Applications to harmonic oscillator quantum mechanics

5.6.1 Non-holomorphic sector of harmonic oscillator QM

In this section we show that the results obtained in this chapter so far, presented in terms of $Z, Z^\dagger$, apply equally to the matrix harmonic oscillator quantum mechanics obtained from dimensional reduction of $\mathcal{N} = 4$ Super-Yang-Mills reviewed in Section 4.2.4, in terms of the creation operators $A^\dagger$ and $B^\dagger$. For convenience we recall their definitions here:

$$A^\dagger = \frac{1}{\sqrt{2}}(Z - i\Pi^\dagger) = \frac{1}{\sqrt{2}}\left(Z - \frac{\partial}{\partial Z^\dagger}\right) \quad A = \frac{1}{\sqrt{2}}(Z^\dagger + i\Pi) = \frac{1}{\sqrt{2}}\left(Z^\dagger + \frac{\partial}{\partial Z}\right)$$
$$B^\dagger = \frac{1}{\sqrt{2}}(Z^\dagger - i\Pi) = \frac{1}{\sqrt{2}}\left(Z^\dagger - \frac{\partial}{\partial Z}\right) \quad B = \frac{1}{\sqrt{2}}(Z + i\Pi^\dagger) = \frac{1}{\sqrt{2}}\left(Z + \frac{\partial}{\partial Z^\dagger}\right) \ . \tag{5.6.1}$$

A generic eigenstate of the harmonic oscillator quantum mechanics is constructed by acting on the ground state with a generic gauge invariant operator $\mathcal{O}(A^\dagger, B^\dagger)$ constructed from m $A^\dagger$'s and n $B^\dagger$'s, i.e.

$$|\Psi_{\mathcal{O}}\rangle \;=\; \mathcal{O}(A^\dagger, B^\dagger)\,|0\rangle \ . \tag{5.6.2}$$

The wavefunction of such a state may be written as

$$\Psi_{\mathcal{O}}(Z, Z^\dagger) \;=\; \langle Z, Z^\dagger|\Psi_{\mathcal{O}}\rangle \;=\; \mathcal{O}(A^\dagger, B^\dagger)\, e^{-\operatorname{tr}(ZZ^\dagger)}. \tag{5.6.3}$$

The Brauer Algebra may be used to organize the states above. Such states are analogous to those used in Section 5.3 and take the form

$$|\Psi^{\gamma}_{\alpha,\beta;i,j}\rangle \;=\; \mathcal{O}^{\gamma}_{\alpha,\beta;i,j}(A^\dagger, B^\dagger)|0\rangle \tag{5.6.4}$$

where the labels are explained in Section 4.4. This state has $E = m + n + N^2$ and $J = m - n$.

Unlike for the holomorphic sector wavefunctions, we have

$$\mathcal{O}(A^\dagger, B^\dagger)e^{-\operatorname{tr}(ZZ^\dagger)} \neq \mathcal{O}(\sqrt{2}Z,\, \sqrt{2}Z^\dagger)\, e^{-\operatorname{tr}(ZZ^\dagger)} \tag{5.6.5}$$

because the derivative of Z inside $A^\dagger$ acts on Z which comes from the action of $B^\dagger$ on the exponential factor. For example we have

$$\operatorname{tr}(A^\dagger B^\dagger)e^{-\operatorname{tr}(ZZ^\dagger)} = \left(2\operatorname{tr} ZZ^\dagger - N^2\right) e^{-\operatorname{tr}(ZZ^\dagger)} \tag{5.6.6}$$

and in general the correct relation is

$$\boxed{\Psi_{\mathcal{O}}(Z,Z^\dagger) = \mathcal{O}(A^\dagger, B^\dagger)e^{-\operatorname{tr}(ZZ^\dagger)} = \left[e^{-\frac{\Box}{2}}\mathcal{O}(\sqrt{2}Z,\, \sqrt{2}Z^\dagger)\right] e^{-\operatorname{tr}(ZZ^\dagger)}} \tag{5.6.7}$$

where $\Box$ is the laplacian $\operatorname{tr} \frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}$ and the brackets indicate that the derivatives in $\Box$ act only on $\mathcal{O}(\sqrt{2}Z,\, \sqrt{2}Z^\dagger)$ and not on the exponential. $e^{-\frac{\Box}{2}}$ is defined by its series expansion; it was observed in (5.4.11) that the laplacian generates Wick contractions and so here $e^{-\frac{\Box}{2}}$ performs a normal ordering, subtracting terms in which pairs of $\sqrt{2}Z$ and $\sqrt{2}Z^\dagger$ have been contracted (c.f. [190]).

Note however that in a $k = 0$ operator we have from (5.4.12) that

$$\Box\, \mathcal{O}^{k=0} = 0 \tag{5.6.8}$$

and so we can replace $A^\dagger$ and $B^\dagger$ with $\sqrt{2}Z$ and $\sqrt{2}Z^\dagger$ respectively without worrying about the above subtlety.

We can define operators corresponding to the G_i in (5.2.10) as follows.

$$\begin{aligned} (\hat{G}_1)^i_{\ j} &= (B^\dagger B)^i_{\ j} & (\hat{G}_2)^i_{\ j} &= (A^\dagger A)^i_{\ j} \\ (\hat{G}_3)^i_{\ j} &= -\, B^{\dagger k}_{\ \ j} B^i_{\ k} & (\hat{G}_4)^i_{\ j} &= -\, A^{\dagger k}_{\ \ j} A^i_{\ k} \end{aligned} \tag{5.6.9}$$

Defining $|A^i_{\ j}\rangle = A^i_{\ j}|0\rangle$ and so on, using the commutation relations we find

$$\begin{aligned} (\hat{G}_1)^i_{\ j}\, |B^{\dagger p}_{\ \ q}\rangle &= \delta^p_{\ j}\, |B^{\dagger i}_{\ \ q}\rangle & (\hat{G}_2)^i_{\ j}\, |A^{\dagger p}_{\ \ q}\rangle &= \delta^p_{\ j}\, |A^{\dagger i}_{\ \ q}\rangle \\ (\hat{G}_3)^i_{\ j}\, |B^{\dagger p}_{\ \ q}\rangle &= -\, \delta^i_{\ q}\, |B^{\dagger p}_{\ \ j}\rangle & (\hat{G}_4)^i_{\ j}\, |A^{\dagger p}_{\ \ q}\rangle &= -\, \delta^i_{\ q}\, |A^{\dagger p}_{\ \ j}\rangle \end{aligned} \tag{5.6.10}$$

which is the same as the adjoint action of the operators G_i defined in (5.2.10) on the matrices $Z, Z^\dagger$ (see equation (11) of [185]).

The result is that we can define harmonic oscillator Casimir operators

$$\hat{\mathcal{H}}_A = \left\{ \hat{H}_1, \quad \hat{H}_2, \quad \hat{\hat{H}}_1, \quad \hat{\hat{H}}_2, \quad \hat{H}_L \right\} \tag{5.6.11}$$

by replacing G_i in (5.3.4) with $\hat{G}_i$. The eigenvalues of hatted Casimirs acting on $\mathcal{O}^\gamma_{\alpha,\beta;i,j}(A^\dagger, B^\dagger)|0\rangle$ are the same as those of the corresponding unhatted Casimirs acting on $\mathcal{O}^\gamma_{\alpha,\beta;i,j}(Z, Z^\dagger)$. This is because the same commutator manipulations can be done to evaluate both, and the arguments which prove that $\mathcal{O}^\gamma_{\alpha,\beta;i,j}(Z, Z^\dagger)$ are eigenstates of the Casimirs in (5.3.4) also prove that $\mathcal{O}^\gamma_{\alpha,\beta;i,j}(A^\dagger, B^\dagger)|0\rangle$ are eigenstates of the hatted versions.

We can take this one step further. Noting that

$$\left[Z^i_{\ j}, -\frac{\Box}{2} \right] = \frac{1}{2}\left(\frac{\partial}{\partial Z^\dagger} \right)^i_{\ j} \tag{5.6.12}$$

$$\Rightarrow \left[Z^i_{\ j}, e^{-\frac{\Box}{2}} \right] = \frac{1}{2}\left(\frac{\partial}{\partial Z^\dagger} \right)^i_{\ j} e^{-\frac{\Box}{2}} \tag{5.6.13}$$

and similarly

$$\left[Z^{\dagger i}_{\ \ j}, e^{-\frac{\Box}{2}} \right] = \frac{1}{2}\left(\frac{\partial}{\partial Z} \right)^i_{\ j} e^{-\frac{\Box}{2}} \tag{5.6.14}$$

then using (5.6.7) we derive

$$\begin{aligned} A^{\dagger i}_{\ \ j}\, \Psi_{\mathcal{O}}(Z, Z^\dagger) &= A^{\dagger i}_{\ \ j}\, \mathcal{O}(A^\dagger, B^\dagger)\, e^{-\operatorname{tr}(ZZ^\dagger)} \\ &= A^{\dagger i}_{\ \ j} \left[e^{-\frac{\Box}{2}} \mathcal{O}(\sqrt{2}Z,\, \sqrt{2}Z^\dagger) \right] e^{-\operatorname{tr}(ZZ^\dagger)} \\ &= \left[e^{-\frac{\Box}{2}} \left(\sqrt{2} Z^i_{\ j} \right) \mathcal{O}(\sqrt{2}Z,\, \sqrt{2}Z^\dagger) \right] e^{-\operatorname{tr}(ZZ^\dagger)} \end{aligned} \tag{5.6.15}$$

where again the brackets indicate that the derivatives act only on $\mathcal{O}(\sqrt{2}Z,\, \sqrt{2}Z^\dagger)$ and not on the exponential. Similarly

$$A^i_{\ j}\, \Psi_{\mathcal{O}}(Z, Z^\dagger) = \left[e^{-\frac{\Box}{2}} \left(\frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial Z} \right)^i_{\ j} \right) \mathcal{O}(\sqrt{2}Z,\, \sqrt{2}Z^\dagger) \right] e^{-\operatorname{tr}(ZZ^\dagger)} \tag{5.6.16}$$

implying the following relation between $\hat{G}_2$ and G_2:

$$(\hat{G}_2)^i_{\ j}\,\Psi_{\mathcal{O}}(Z,Z^\dagger) \;=\; \left[e^{-\frac{\Box}{2}}\,(G_2)^i_{\ j}\,\mathcal{O}(\sqrt{2}Z,\,\sqrt{2}Z^\dagger)\right]e^{-\operatorname{tr}(ZZ^\dagger)} \tag{5.6.17}$$

Similar results apply to the remaining $\hat{G}_i$, the Hamiltonians $\hat{H}_i$ as well as the canonical Hamiltonian

$$\hat{H} \;=\; \hat{H}_1 + \hat{\bar{H}}_1 + N^2 \;=\; \operatorname{tr}(A^\dagger A + B^\dagger B) + N^2 \tag{5.6.18}$$

whose action on wavefunctions $\Psi(Z,Z^\dagger)$ can be written in terms of the (first-order) scaling operator H:

$$H \;=\; H_1 + \bar{H}_1 + N^2 \;=\; \operatorname{tr}\left(Z\frac{\partial}{\partial Z} + Z^\dagger\frac{\partial}{\partial Z^\dagger}\right) + N^2. \tag{5.6.19}$$

Applying (5.6.17) and the corresponding relation for $\hat{G}_3$ we find that

$$\boxed{\hat{H}\Psi_{\mathcal{O}}(Z,Z^\dagger) = \left[e^{-\frac{\Box}{2}}\,H\,\mathcal{O}(\sqrt{2}Z,\,\sqrt{2}Z^\dagger)\right]e^{-\operatorname{tr}(ZZ^\dagger)}.} \tag{5.6.20}$$

A similar manipulation in the holomorphic sector was performed in Appendix A of [191]. Note that for a $k=0$ operator we have $\Box\,\mathcal{O}^{k=0}=0$ and so the above analysis gives

$$\hat{H}\left[\mathcal{O}^{k=0}(A^\dagger,B^\dagger)\,e^{-\operatorname{tr}(ZZ^\dagger)}\right] = \left[H\,\mathcal{O}^{k=0}(\sqrt{2}Z,\,\sqrt{2}Z^\dagger)\right]e^{-\operatorname{tr}(ZZ^\dagger)}. \tag{5.6.21}$$

The inner product on wavefunctions may be derived using

$$\int [dZdZ^\dagger]\;|Z,Z^\dagger\rangle\langle Z,Z^\dagger| \;=1 \tag{5.6.22}$$

where $[dZdZ^\dagger]=\prod_{i,j} dZ_{ij}dZ^\dagger_{ij}$, as follows:

$$\begin{aligned}\langle\Psi_{\mathcal{O}_1}|\Psi_{\mathcal{O}_2}\rangle \;&=\; \frac{1}{\pi^{N^2}}\int [dZdZ^\dagger]\,\langle\mathcal{O}_1(A^\dagger,B^\dagger)|Z,Z^\dagger\rangle\langle Z,Z^\dagger|\mathcal{O}_2(A^\dagger,B^\dagger)\rangle\\ &=\; \frac{1}{\pi^{N^2}}\int [dZdZ^\dagger]\,\overline{\Psi_{\mathcal{O}_1}(Z,Z^\dagger)}\Psi_{\mathcal{O}_2}(Z,Z^\dagger)\end{aligned} \tag{5.6.23}$$

where the factor of π^{N^2} compensates for using non-normalized wavefunctions, and is found by imposing

$$\langle \Psi_0 | \Psi_0 \rangle = 1. \tag{5.6.24}$$

Using (5.6.7), the above expression (5.6.23) becomes

$$\begin{aligned}
\langle \Psi_{\mathcal{O}_1} | \Psi_{\mathcal{O}_2} \rangle &= \frac{1}{\pi^{N^2}} \int [dZdZ^\dagger]\, \overline{\mathcal{O}_1(A^\dagger, B^\dagger) e^{-\operatorname{tr} ZZ^\dagger}}\, \mathcal{O}_2(A^\dagger, B^\dagger) e^{-\operatorname{tr} ZZ^\dagger} \\
&= \frac{1}{\pi^{N^2}} \int [dZdZ^\dagger]\, \overline{\left(e^{-\frac{\Box}{2}} \mathcal{O}_1(\sqrt{2}Z, \sqrt{2}Z^\dagger) \right)} \left(e^{-\frac{\Box}{2}} \mathcal{O}_2(\sqrt{2}Z, \sqrt{2}Z^\dagger) \right) e^{-2\operatorname{tr} ZZ^\dagger}
\end{aligned}$$

and rescaling factors of two we have the result

$$\langle \Psi_{\mathcal{O}_1} | \Psi_{\mathcal{O}_2} \rangle = \frac{1}{(2\pi)^{N^2}} \int [dZdZ^\dagger]\, \overline{\left(e^{-\Box}\, \mathcal{O}_1(Z, Z^\dagger) \right)} \left(e^{-\Box}\, \mathcal{O}_2(Z, Z^\dagger) \right) e^{-\operatorname{tr} ZZ^\dagger} \tag{5.6.25}$$

which is the non-holomorphic generalization of (A.12) of [191].

Thus we have shown that our general analysis of complex matrix models applies equally well to the matrix harmonic oscillator quantum mechanics obtained from dimensional reduction of $\mathcal{N} = 4$ Super-Yang-Mills, with our analysis of Z, $Z^\dagger$ carrying over to the states built from the creation operators $A^\dagger$, $B^\dagger$.

5.7 Summary and outlook

In this chapter we described free particle structures in matrix models of an $N \times N$ complex matrix Z, and related these structures to the geometry of the configuration space $\mathcal{M}_N$ of gauge-inequivalent configurations, a space of dimension $N^2 + 1$.

For any N the $k = 0$ sector has states in one-to-one correspondence with those of N free fermions on a circle. For $N = 2$ we expressed the momenta of free fermions on a circle as algebraic functions of differential operators, and discussed the generalization of this result to general N. Importantly, while the usual emergence of free fermions in matrix models can be seen from a change of variables to eigenvalues, here the $k = 0$ sector depends on combinations of eigenvalues and off-diagonal elements.

We also found a description in terms of free particles in the $m = n = k$ sector, this time in terms of free fermions in a one-dimensional harmonic oscillator potential.

We observe that k appears to interpolate between radial and angular free particle systems on a plane. It would be interesting to investigate this possibility further.

We studied in detail the Brauer basis operators at $N = 2$ and presented a conjecture for their counting, or equivalently for the reduction multiplicities of representations of $B_N(m, n)$ to $S_m \times S_n$ for $N = 2$. We also presented computational evidence for this conjecture.

We now discuss open questions and opportunities for further research.

An important question from the point of view of this thesis is whether the free particle structures described in this chapter can be realized in the supergravity side of the AdS_5/CFT_4 duality in a sector which has an $SO(4) \times SO(4)$ isometry, and if so whether the free particle structures can be used to study non-supersymmetric asymptotically AdS black holes in terms of heavy non-BPS operators in the field theory.

Non-BPS deformations of LLM geometries have been studied in supergravity, and the existence of smooth horizonless 'solitonic AdS bubbles' has been demonstrated numerically [192]. In the most optimistic scenario, one could anticipate a non-BPS generalization of the LLM [15] discovery of supergravity geometries corresponding to the free fermions of the holomorphic sector of the complex matrix model [13, 14].

Aside from the applications to black hole physics there are a number of questions, interesting in their own right, which arise from this research:

1. We obtained explicit expressions for the free fermion momenta for the $k = 0$ sector of the $N = 2$ matrix theories in terms of the original matrix variables. It is an open problem to find explicit expressions for the dual coordinates of the fermions, and the wavefunctions as Slater determinants. It is also interesting to explore whether this would be useful for the computation of field theory correlators.

2. We presented results on the counting of complex matrix model operators in terms of Brauer algebras at $N = 2$. These are related to reduction multiplicities for $B_{N=2}(m, n)$ irreps into $S_m \times S_n$ irreps. What are these finite N reduction multiplicities for general m, n, N, in particular for $N < m + n$ at

large N?

3. There is a substantial literature discussing *consistent truncations* of the Maldacena duality. For example, it is known that the $SU(2)$ sector defines a consistent truncation to all orders in perturbation theory [193]. The $Z, Z^\dagger$ sector is a well-defined truncation at zero coupling. Assuming the strong form of the Maladacena conjecture, and making the plausible assumption that consistent quantum truncations of a quantum field theory with a string dual have a string dual, we are led to ask: What is the gauge-string theory dual of one free complex matrix in four dimensions? Similarly, what is the dual of the quantum mechanics obtained from reduction on $\mathbb{R} \times S^3$?

   Such dualities are known for the large N Gaussian Hermitian matrix model [136], double scaling limits of complex matrix models [142], the large N hermitian matrix oscillator quantum mechanics [194] and the BFSS matrix model for M-theory [195].

We do not have a clear answer to the last question, but based on the research in this chapter we make the following tentative conjecture. We conjecture that there exists a string dual of the matrix harmonic oscillator quantum mechanics discussed in Section 5.6 which has a $2+1$ dimensional space-time and whose physics involves interacting strings and branes. The z_i coordinates are positions of N branes in 2 space dimensions. By analogy to the treatment in [196] we expect the off-diagonal variables t_{ij} to describe strings connecting brane i to j; here the triangular constraint ($t_{ij} = 0$ for $i > j$) will make the dual qualitatively different from the standard system of strings and branes at weak coupling.

The Hamiltonian H contains terms $t\frac{\partial}{\partial t}$ along with $z_i\frac{\partial}{\partial z_i}$. Excitations involving polynomials in z_i have energies comparable to excitations involving t. This means that if this picture is correct the strings and branes have comparable masses. Usually string states have masses of order 1 (with $l_s = 1$) whereas branes have masses of order $1/g_s$. In this sense, the conjectured model appears to have $g_s \sim 1$. It would be interesting to see if such a model can be constructed and investigate the physical interpretation of the Brauer algebra basis labels, in particular k, and their constraints at finite N.

Conclusions and Outlook

In this thesis we have presented research on black holes in string theory in two different contexts.

In Part I we used disk amplitudes to derive the supergravity fields sourced by a D-brane with a travelling wave, presenting the calculation in the D5-P duality frame. We saw that this provided a direct link between a microscopic bound state in string theory representing a very dark, compact, heavy object and its gravitational description in supergravity.

We noted that, as described by the fuzzball program, only classical vibration profiles on the D-brane have reliable supergravity descriptions in the region close to the D-brane, and that for a generic vibration profile the microscopic bound state is not geometrical in this sense. For states which have good classical descriptions, the known two-charge supergravity solutions were identified with the microscopic bound states.

We discussed the fact that the D5-P duality frame is one in which the supergravity solutions sourced by the D-brane bound states are singular, and the scaling arguments of [53] show that α'-corrections to the supergravity action cannot produce small black holes with a non-zero horizon area. One can then ask whether this is in contradiction with the proposal of [53] that the macroscopic entropy of a two-charge configuration should be defined to be the sum of the contributions of small black hole solutions and horizonless, everywhere smooth classical solutions. We noted that a way out of such a contradiction is offered by the possibility that α'-corrections lead to a family of everywhere smooth horizonless geometries.

We also briefly mentioned work in progress on applying these techniques to the three-charge D1-D5-P black hole. We discussed the importance of this development because of the presence of an extremal black hole with a macroscopic horizon in this setup, and the question of the interpretation of the many smooth horizonless three-charge supergravity solutions reviewed in Chapter 1.

In Part II we presented free particle descriptions in non-holomorphic sectors of complex matrix models, in particular the matrix quantum mechanics obtained by dimensional reduction of $\mathcal{N} = 4$ Super-Yang-Mills theory. We motivated this research from the detailed investigations of black hole physics in the half-BPS sector, and a desire to extend these results to near-extremal black holes.

Since the publication of the research contained in Part II of this thesis, there have been further developments in applying the Brauer algebra basis to matrix models. Firstly, the Brauer algebra basis was extended to multi-complex matrix models [197]. Secondly, the Brauer basis has been applied to the $SU(2)$ sector of $\mathcal{N} = 4$ Super-Yang-Mills (consisting of holomorphic operators in two complex matrices X, Y) and the $k = 0$ operators in this sector were shown to lie in the kernel of the $SU(2)$ one-loop dilatation operator, and so are 1/4-BPS operators at one-loop [158].

The connection to black hole physics is less well developed in this context, and there remain gaps to bridge before one can draw any conclusions about black holes from this research. We described opportunities to make progress in this respect and we look forward to understanding more about the physics of near-extremal asymptotically anti-de Sitter black holes in the future.

In summary, string theory promises to provide a consistent quantum mechanical description of physical black holes, an essential and non-trivial test of any theory of quantum gravity, and the research contained in this thesis provides further evidence that this promise can be realized. There are many interesting open questions which remain to be investigated through string world-sheet amplitudes, through the construction of supergravity solutions, and through gauge-gravity duality.

Appendix A

Notation and Conventions

In this appendix we record our conventions for four-dimensional Einstein gravity and Einstein-Maxwell theory, and ten-dimensional type IIB supergravity. In all cases we use signature $(-++\cdots+)$ and we follow in places [36, 198, 199, 200, 98].

A.1 General relativity in four dimensions

In general relativity, spacetime is a differentiable manifold $(\mathcal{M}, g)$ and the line element is given by

$$ds^2 \;=\; g_{\mu\nu}dx^\mu dx^\nu \;. \tag{A.1.1}$$

We denote by ∇ the Levi-Civita connection, which is metric preserving ($\nabla g = 0$) and torsion-free, i.e. its components $\Gamma^{\;\mu}_{\nu\rho}$ are symmetric, $\Gamma^{\;\mu}_{\nu\rho} = \Gamma^{\;\mu}_{(\nu\rho)}$.

The Riemann curvature tensor is defined by the commutator of ∇ acting on an arbitrary vector field V:

$$[\nabla_\mu, \nabla_\nu]V_\rho = R_{\mu\nu\rho\sigma}V^\sigma \;. \tag{A.1.2}$$

The Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R are defined by

$$R_{\mu\nu} \;=\; R^\rho{}_{\mu\rho\nu} \;, \qquad R \;=\; R^\mu{}_\mu \tag{A.1.3}$$

where indices are lowered and raised with $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ respectively.

The Einstein equations, without cosmological constant, are

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} \;=\; 8\pi T_{\mu\nu} \tag{A.1.4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. In vacuum $T_{\mu\nu} = 0$ which gives the equations of motion

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \;=\; 0 \tag{A.1.5}$$

which may be derived from the four-dimensional Einstein-Hilbert action

$$\mathcal{S}_{EH} \;=\; \frac{1}{16\pi G} \int d^4x\, \sqrt{-g}\, R\ . \tag{A.1.6}$$

The Schwarzschild line element takes the form (1.3.1)

$$ds^2 \;=\; -\, f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2)\ , \tag{A.1.7}$$

where

$$f(r) \;=\; 1 - \frac{2GM}{r}\ . \tag{A.1.8}$$

A.2 Einstein-Maxwell theory in four dimensions

Einstein-Maxwell theory describes gravity coupled to classical electromagnetism, described by a $U(1)$ vector potential A_μ with field strength

$$F_{\mu\nu} \;=\; \nabla_\mu A_\nu - \nabla_\nu A_\mu\ . \tag{A.2.1}$$

The energy-momentum tensor is then

$$T_{\mu\nu} \;=\; \frac{1}{4\pi} \left(F_{\mu\rho} F_\nu^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \tag{A.2.2}$$

and the Einstein equations (again without cosmological constant)

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \;=\; 8\pi T_{\mu\nu} \tag{A.2.3}$$

may be derived from the four-dimensional Einstein-Maxwell action

$$\mathcal{S}_{EM} \;=\; \frac{1}{16\pi G} \int d^4x\, \sqrt{-g}\, (R - F_{\mu\nu} F^{\mu\nu}) \tag{A.2.4}$$

where the Maxwell term is normalized to measure charge in 'geometrized units', i.e. the magnitude of the Coulomb force between point charges Q_1, Q_2 at separation

r in flat space is [199]

$$\frac{G\,|Q_1 Q_2|}{r^2}\ . \tag{A.2.5}$$

The equations of motion for A_μ are, in the absence of sources,

$$\nabla_\mu F^{\mu\nu} \;=\; 0 \tag{A.2.6}$$

$$\nabla_{[\mu} F_{\nu\rho]} \;=\; 0\ . \tag{A.2.7}$$

The Reissner-Nordstrom solution is obtained by solving for a point charge field configuration,

$$A_t \;=\; \frac{Q}{r} \qquad \Rightarrow \qquad E_t \;\equiv\; F_{rt} \;=\; -\frac{Q}{r^2} \tag{A.2.8}$$

giving rise to the Reissner-Nordstrom line element (1.3.8)

$$ds^2 \;=\; -f_{RN}(r)dt^2 + \frac{dr^2}{f_{RN}(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ , \tag{A.2.9}$$

where

$$f_{RN}(r) \;=\; 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\ . \tag{A.2.10}$$

A.3 Type IIB supergravity

We next introduce our conventions for ten-dimensional type IIB supergravity. The bosonic fields of the theory are the metric g, NS two-form B with field strength $H^{(3)}$, dilaton ϕ, and RR potentials $C^{(0)}$, $C^{(2)}$, $C^{(4)}$ with corresponding field strengths $F^{(p+2)} = dC^{(p+1)}$.

Following the conventions of [98] we also introduce the modified field strengths

$$\tilde{F}^{(3)} \;=\; F^{(3)} - C^{(0)} \wedge H^{(3)}\ , \tag{A.3.1}$$

$$\tilde{F}^{(5)} \;=\; F^{(5)} - \frac{1}{2}C^{(2)} \wedge H^{(3)} + \frac{1}{2}B^{(2)} \wedge F^{(3)} \tag{A.3.2}$$

The five-form field strength $F^{(5)}$ must satisfy the self-duality constraint

$$F^{(5)} \;=\; *\,F^{(5)} \tag{A.3.3}$$

(one could also choose instead $F^{(5)}$ to be anti-self-dual), which obstructs the construction of a covariant action. The constraint (A.3.3) must be imposed in addition to the equations of motion resulting from the action

$$\mathcal{S}_{IIB} \;=\; \mathcal{S}_{NS} + \mathcal{S}_R + \mathcal{S}_{CS} \;, \tag{A.3.4}$$

where

$$\mathcal{S}_{NS} \;=\; \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\; e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H^{(3)}|^2\right) \;, \tag{A.3.5}$$

$$\mathcal{S}_R \;=\; \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(-\frac{1}{2}|F^{(1)}|^2 - \frac{1}{2}|\tilde{F}^{(3)}|^2 - \frac{1}{4}|\tilde{F}^{(5)}|^2\right) \;, \tag{A.3.6}$$

$$\mathcal{S}_{CS} \;=\; \frac{1}{2\kappa^2}\int\left(-\frac{1}{2}C^{(4)}\wedge H^{(3)}\wedge F^{(3)}\right) \tag{A.3.7}$$

and where for each n,

$$|F^{(n)}|^2 \;=\; \frac{1}{n!}\,F^{(n)}_{\mu_1\cdots\mu_n}\,F^{(n)\mu_1\cdots\mu_n} \;. \tag{A.3.8}$$

References

[1] W. Black, R. Russo, and D. Turton, "The supergravity fields for a D-brane with a travelling wave from string amplitudes," *Phys. Lett.* **B694** (2010) 246–251, 1007.2856.

[2] Y. Kimura, S. Ramgoolam, and D. Turton, "Free particles from Brauer algebras in complex matrix models," *JHEP* **05** (2010) 052, 0911.4408.

[3] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, hep-th/9510017.

[4] M. R. Garousi and R. C. Myers, "Superstring Scattering from D-Branes," *Nucl. Phys.* **B475** (1996) 193–224, hep-th/9603194.

[5] A. Hashimoto and I. R. Klebanov, "Scattering of strings from D-branes," *Nucl. Phys. Proc. Suppl.* **55B** (1997) 118–133, hep-th/9611214.

[6] P. Di Vecchia *et al.*, "Classical p-branes from boundary state," *Nucl. Phys.* **B507** (1997) 259–276, hep-th/9707068.

[7] M. Billo *et al.*, "Microscopic string analysis of the D0-D8 brane system and dual R-R states," *Nucl. Phys.* **B526** (1998) 199–228, hep-th/9802088.

[8] P. Di Vecchia, M. Frau, A. Lerda, and A. Liccardo, "(F,Dp) bound states from the boundary state," *Nucl. Phys.* **B565** (2000) 397–426, hep-th/9906214.

[9] P. Di Vecchia and A. Liccardo, "D branes in string theory. I," *NATO Adv. Study Inst. Ser. C. Math. Phys. Sci.* **556** (2000) 1–59, hep-th/9912161.

[10] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, hep-th/9711200.

[11] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, hep-th/9802150.

[12] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett.* **B428** (1998) 105–114, hep-th/9802109.

[13] S. Corley, A. Jevicki, and S. Ramgoolam, "Exact correlators of giant gravitons from dual N = 4 SYM theory," *Adv. Theor. Math. Phys.* **5** (2002) 809–839, hep-th/0111222.

[14] D. Berenstein, "A toy model for the AdS/CFT correspondence," *JHEP* **07** (2004) 018, hep-th/0403110.

[15] H. Lin, O. Lunin, and J. M. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," *JHEP* **10** (2004) 025, hep-th/0409174.

[16] V. Balasubramanian, V. Jejjala, and J. Simon, "The library of Babel," *Int. J. Mod. Phys.* **D14** (2005) 2181–2186, hep-th/0505123.

[17] V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon, "The library of Babel: On the origin of gravitational thermodynamics," *JHEP* **12** (2005) 006, hep-th/0508023.

[18] A. Buchel, "Coarse-graining 1/2 BPS geometries of type IIB supergravity," *Int. J. Mod. Phys.* **A21** (2006) 3495–3502, hep-th/0409271.

[19] N. V. Suryanarayana, "Half-BPS giants, free fermions and microstates of superstars," *JHEP* **01** (2006) 082, hep-th/0411145.

[20] Y. Kimura and S. Ramgoolam, "Branes, Anti-Branes and Brauer Algebras in Gauge-Gravity duality," *JHEP* **11** (2007) 078, 0709.2158.

[21] R. Ruffini and J. A. Wheeler, "Introducing the black hole," *Physics Today* **24** (1971) 30–36.

[22] G. T. Horowitz and J. Polchinski, "A correspondence principle for black holes and strings," *Phys. Rev.* **D55** (1997) 6189–6197, hep-th/9612146.

[23] A. M. Ghez *et al.*, "Measuring Distance and Properties of the Milky Way's Central Supermassive Black Hole with Stellar Orbits," *Astrophys. J.* **689** (2008) 1044–1062, 0808.2870.

[24] A. Mueller, "Experimental evidence of black holes," *PoS* **P2GC** (2006) 017, astro-ph/0701228.

[25] A. Celotti, J. C. Miller, and D. W. Sciama, "Astrophysical evidence for the existence of black holes," *Class. Quant. Grav.* **16** (1999) A3, astro-ph/9912186.

[26] R. Narayan, R. Mahadevan, J. E. Grindlay, R. G. Popham, and C. Gammie, "Advection-dominated accretion model of sagittarius a*: evidence for a black hole at the galactic center," *Astrophys.J.* **492** (1998) 554–568, astro-ph/9706112.

[27] A. M. Ghez, B. L. Klein, M. Morris, and E. E. Becklin, "High Proper Motion Stars in the Vicinity of Sgr A*: Evidence for a Supermassive Black Hole at the Center of Our Galaxy," *Astrophys. J.* **509** (1998) 678–686, astro-ph/9807210.

[28] R. Schodel *et al.*, "A Star in a 15.2 year orbit around the supermassive black hole at the center of the Milky Way," *Nature* **419** (2002) 694–696.

[29] A. E. Broderick and R. Narayan, "On The Nature of the Compact Dark Mass at the Galactic Center," *Astrophys. J.* **638** (2006) L21–L24, astro-ph/0512211.

[30] R. Narayan and J. E. McClintock, "Advection-Dominated Accretion and the Black Hole Event Horizon," *New Astron. Rev.* **51** (2008) 733–751, 0803.0322.

[31] A. E. Broderick, A. Loeb, and R. Narayan, "The Event Horizon of Sagittarius A*," *Astrophys. J.* **701** (2009) 1357–1366, 0903.1105.

[32] M. A. Abramowicz, W. Kluzniak, and J.-P. Lasota, "No observational proof of the black-hole event-horizon," *Astron. Astrophys.* **396** (2002) L31–L34, astro-ph/0207270.

[33] M. Visser, C. Barcelo, S. Liberati, and S. Sonego, "Small, dark, and heavy: But is it a black hole?," *PoS* **BHSGRANDSTRINGS2008** (2008) 010, 0902.0346.

[34] R. M. Wald, "General Relativity,". Chicago Univ. Pr. (1984) 491p.

[35] L. Susskind and J. Lindesay, *An introduction to black holes, information and the string theory revolution: The holographic universe*. Hackensack, USA: World Scientific, 2005.

[36] M. Perry, "Part III Lectures on Black Holes, Cambridge, 2007." Unpublished.

[37] V. Balasubramanian, "Lectures at Black Objects in Supergravity School, Frascati, 2011." Unpublished.

[38] A. Dabholkar, "Lectures at Black Objects in Supergravity School, Frascati, 2011." Unpublished.

[39] R. M. Wald, *General Relativity*, p. 212. Chicago Univ. Pr., 1984.

[40] J. D. Bekenstein, "Black holes and entropy," *Phys. Rev.* **D7** (1973) 2333–2346.

[41] S. W. Hawking, "Gravitational radiation from colliding black holes," *Phys. Rev. Lett.* **26** (1971) 1344–1346.

[42] J. D. Bekenstein, "Generalized second law of thermodynamics in black hole physics," *Phys. Rev.* **D9** (1974) 3292–3300.

[43] S. W. Hawking, "Particle Creation by Black Holes," *Commun. Math. Phys.* **43** (1975) 199–220.

[44] R. Penrose, "Gravitational collapse: The role of general relativity," *Riv. Nuovo Cim.* **1** (1969) 252–276.

[45] R. P. Kerr, "Gravitational field of a spinning mass as an example of algebraically special metrics," *Phys. Rev. Lett.* **11** (1963) 237–238.

[46] A. Sen, "Black hole solutions in heterotic string theory on a torus," *Nucl. Phys.* **B440** (1995) 421–440, hep-th/9411187.

[47] A. Sen, "Extremal black holes and elementary string states," *Mod. Phys. Lett.* **A10** (1995) 2081–2094, hep-th/9504147.

[48] A. Dabholkar, J. P. Gauntlett, J. A. Harvey, and D. Waldram, "Strings as Solitons & Black Holes as Strings," *Nucl. Phys.* **B474** (1996) 85–121, hep-th/9511053.

[49] C. G. Callan, J. M. Maldacena, and A. W. Peet, "Extremal Black Holes As Fundamental Strings," *Nucl. Phys.* **B475** (1996) 645–678, hep-th/9510134.

[50] O. Lunin and S. D. Mathur, "Metric of the multiply wound rotating string," *Nucl. Phys.* **B610** (2001) 49–76, hep-th/0105136.

[51] O. Lunin and S. D. Mathur, "AdS/CFT duality and the black hole information paradox," *Nucl. Phys.* **B623** (2002) 342–394, hep-th/0109154.

[52] S. D. Mathur, "The fuzzball proposal for black holes: An elementary review," *Fortsch. Phys.* **53** (2005) 793–827, hep-th/0502050.

[53] A. Sen, "Two Charge System Revisited: Small Black Holes or Horizonless Solutions?," 0908.3402.

[54] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Phys.Lett.* **B379** (1996) 99–104, hep-th/9601029.

[55] S. W. Hawking and D. N. Page, "Thermodynamics of Black Holes in anti-De Sitter Space," *Commun. Math. Phys.* **87** (1983) 577.

[56] G. T. Horowitz and V. E. Hubeny, "Note on small black holes in AdS(p) x S(q)," *JHEP* **06** (2000) 031, hep-th/0005288.

[57] J. B. Gutowski and H. S. Reall, "Supersymmetric AdS(5) black holes,"

JHEP **02** (2004) 006, hep-th/0401042.

[58] J. B. Gutowski and H. S. Reall, "General supersymmetric AdS(5) black holes," *JHEP* **04** (2004) 048, hep-th/0401129.

[59] M. Cvetic *et al.*, "Embedding AdS black holes in ten and eleven dimensions," *Nucl. Phys.* **B558** (1999) 96–126, hep-th/9903214.

[60] R. C. Myers and O. Tafjord, "Superstars and giant gravitons," *JHEP* **11** (2001) 009, hep-th/0109127.

[61] K. Behrndt, A. H. Chamseddine, and W. A. Sabra, "BPS black holes in N = 2 five dimensional AdS supergravity," *Phys. Lett.* **B442** (1998) 97–101, hep-th/9807187.

[62] K. Behrndt, M. Cvetic, and W. A. Sabra, "Non-extreme black holes of five dimensional N = 2 AdS supergravity," *Nucl. Phys.* **B553** (1999) 317–332, hep-th/9810227.

[63] V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon, "Entropy of near-extremal black holes in AdS5," *JHEP* **05** (2008) 067, 0707.3601.

[64] S. W. Hawking, "Breakdown of Predictability in Gravitational Collapse," *Phys. Rev.* **D14** (1976) 2460–2473.

[65] S. W. Hawking, "The Unpredictability of Quantum Gravity," *Commun. Math. Phys.* **87** (1982) 395.

[66] J. D. Bekenstein, "A Universal Upper Bound on the Entropy to Energy Ratio for Bounded Systems," *Phys. Rev.* **D23** (1981) 287.

[67] J. D. Bekenstein, "Entropy bounds and black hole remnants," *Phys. Rev.* **D49** (1994) 1912–1921, gr-qc/9307035.

[68] L. Susskind, "Trouble for remnants," hep-th/9501106.

[69] S. B. Giddings, "The Black hole information paradox," hep-th/9508151.

[70] S. L. Braunstein and A. K. Pati, "Quantum information cannot be

completely hidden in correlations: Implications for the black-hole information paradox," *Phys. Rev. Lett.* **98** (2007) 080502, gr-qc/0603046.

[71] S. L. Braunstein, H.-J. Sommers, and K. Zyczkowski, "Entangled black holes as ciphers of hidden information," 0907.0739.

[72] S. D. Mathur, "The information paradox: A pedagogical introduction," *Class. Quant. Grav.* **26** (2009) 224001, 0909.1038.

[73] S. B. Giddings, "Models for unitary black hole disintegration," 1108.2015.

[74] S. D. Mathur, "Where are the states of a black hole?," hep-th/0401115.

[75] S. D. Mathur, "Fuzzballs and the information paradox: a summary and conjectures," 0810.4525.

[76] S. D. Mathur, "The information paradox and the infall problem," 1012.2101.

[77] D. Marolf, "The dangers of extremes," *Gen. Rel. Grav.* **42** (2010) 2337–2343, 1005.2999.

[78] S. D. Mathur and C. J. Plumberg, "Correlations in Hawking radiation and the infall problem," 1101.4899.

[79] D. N. Page, "Information in black hole radiation," *Phys. Rev. Lett.* **71** (1993) 3743–3746, hep-th/9306083.

[80] O. Lunin and S. D. Mathur, "Statistical interpretation of Bekenstein entropy for systems with a stretched horizon," *Phys. Rev. Lett.* **88** (2002) 211303, hep-th/0202072.

[81] O. Lunin, S. D. Mathur, and A. Saxena, "What is the gravity dual of a chiral primary?," *Nucl. Phys.* **B655** (2003) 185–217, hep-th/0211292.

[82] L. Susskind, L. Thorlacius, and J. Uglum, "The Stretched horizon and black hole complementarity," *Phys. Rev.* **D48** (1993) 3743–3761, hep-th/9306069.

[83] O. Lunin, "Adding momentum to D1-D5 system," *JHEP* **04** (2004) 054, hep-th/0404006.

[84] S. Giusto, S. D. Mathur, and A. Saxena, "Dual geometries for a set of 3-charge microstates," *Nucl. Phys.* **B701** (2004) 357–379, hep-th/0405017.

[85] I. Bena and N. P. Warner, "One ring to rule them all ... and in the darkness bind them?," *Adv.Theor.Math.Phys.* **9** (2005) 667–701, hep-th/0408106.

[86] S. Giusto and S. D. Mathur, "Geometry of D1-D5-P bound states," *Nucl. Phys.* **B729** (2005) 203–220, hep-th/0409067.

[87] J. Ford, S. Giusto, and A. Saxena, "A class of BPS time-dependent 3-charge microstates from spectral flow," *Nucl. Phys.* **B790** (2008) 258–280, hep-th/0612227.

[88] I. Bena and N. P. Warner, "Black holes, black rings and their microstates," *Lect. Notes Phys.* **755** (2008) 1–92, hep-th/0701216.

[89] V. Jejjala, O. Madden, S. F. Ross, and G. Titchener, "Non-supersymmetric smooth geometries and D1-D5-P bound states," *Phys. Rev.* **D71** (2005) 124030, hep-th/0504181.

[90] B. D. Chowdhury and S. D. Mathur, "Radiation from the non-extremal fuzzball," *Class. Quant. Grav.* **25** (2008) 135005, 0711.4817.

[91] B. D. Chowdhury and S. D. Mathur, "Pair creation in non-extremal fuzzball geometries," *Class. Quant. Grav.* **25** (2008) 225021, 0806.2309.

[92] B. D. Chowdhury and S. D. Mathur, "Non-extremal fuzzballs and ergoregion emission," *Class. Quant. Grav.* **26** (2009) 035006, 0810.2951.

[93] S. G. Avery, B. D. Chowdhury, and S. D. Mathur, "Emission from the D1D5 CFT," *JHEP* **10** (2009) 065, 0906.2015.

[94] S. G. Avery, B. D. Chowdhury, and S. D. Mathur, "Deforming the D1D5 CFT away from the orbifold point," *JHEP* **06** (2010) 031, 1002.3132.

[95] S. G. Avery, B. D. Chowdhury, and S. D. Mathur, "Excitations in the

deformed D1D5 CFT," *JHEP* **06** (2010) 032, 1003.2746.

[96] S. Abel and J. March-Russell, "The search for extra dimensions," *Physics World* (Nov, 2000). http://physicsworld.com/cws/article/print/403.

[97] J. Polchinski, *String theory. Vol. 1: An Introduction to the Bosonic String.* Cambridge Univ. Pr., 1998.

[98] J. Polchinski, *String theory. Vol. 2: Superstring Theory and Beyond.* Cambridge Univ. Pr., 1998.

[99] D. Friedan, E. J. Martinec, and S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," *Nucl. Phys.* **B271** (1986) 93.

[100] S. A. Yost, "Bosonized Superstring Boundary States and Partition Functions," *Nucl. Phys.* **B321** (1989) 629.

[101] C. G. Callan, Jr., C. Lovelace, C. R. Nappi, and S. A. Yost, "Adding Holes and Crosscaps to the Superstring," *Nucl. Phys.* **B293** (1987) 83.

[102] C. G. Callan, Jr., C. Lovelace, C. R. Nappi, and S. A. Yost, "Loop Corrections to Superstring Equations of Motion," *Nucl. Phys.* **B308** (1988) 221.

[103] N. Ishibashi, "The Boundary and Crosscap States in Conformal Field Theories," *Mod. Phys. Lett.* **A4** (1989) 251.

[104] C. G. Callan, Jr. and I. R. Klebanov, "D-Brane Boundary State Dynamics," *Nucl. Phys.* **B465** (1996) 473–486, hep-th/9511173.

[105] A. Recknagel and V. Schomerus, "D-branes in Gepner models," *Nucl. Phys.* **B531** (1998) 185–225, hep-th/9712186.

[106] A. Recknagel and V. Schomerus, "Boundary deformation theory and moduli spaces of D- branes," *Nucl. Phys.* **B545** (1999) 233–282, hep-th/9811237.

[107] D.-E. Diaconescu and J. Gomis, "Fractional branes and boundary states in orbifold theories," *JHEP* **10** (2000) 001, hep-th/9906242.

[108] J. Frohlich, O. Grandjean, A. Recknagel, and V. Schomerus, "Fundamental strings in Dp-Dq brane systems," *Nucl. Phys.* **B583** (2000) 381–410, hep-th/9912079.

[109] A. Rajaraman and M. Rozali, "Boundary states for D-branes in AdS(3)," *Phys. Rev.* **D66** (2002) 026006, hep-th/0108001.

[110] M. R. Gaberdiel, A. Recknagel, and G. M. T. Watts, "The conformal boundary states for SU(2) at level 1," *Nucl. Phys.* **B626** (2002) 344–362, hep-th/0108102.

[111] A. Sen, "Rolling tachyon boundary state, conserved charges and two dimensional string theory," *JHEP* **05** (2004) 076, hep-th/0402157.

[112] D. Blakeley and A. Recknagel, "Symmetry-breaking boundary states for WZW models," *Nucl. Phys.* **B806** (2009) 636–655, 0705.1068.

[113] M. Kiermaier, Y. Okawa, and B. Zwiebach, "The boundary state from open string fields," 0810.1737.

[114] M. J. Duff, "Quantum Tree Graphs and the Schwarzschild Solution," *Phys. Rev.* **D7** (1973) 2317–2326.

[115] V. Balasubramanian, B. Czech, K. Larjo, and J. Simon, "Integrability vs. information loss: A simple example," *JHEP* **11** (2006) 001, hep-th/0602263.

[116] V. Balasubramanian, B. Czech, K. Larjo, D. Marolf, and J. Simon, "Quantum geometry and gravitational entropy," *JHEP* **12** (2007) 067, 0705.4431.

[117] C. Bachas, "Relativistic string in a pulse," *Ann. Phys.* **305** (2003) 286–309, hep-th/0212217.

[118] Y. Hikida, H. Takayanagi, and T. Takayanagi, "Boundary states for D-branes with traveling waves," *JHEP* **04** (2003) 032, hep-th/0303214.

[119] J. D. Blum, "Gravitational radiation from travelling waves on D- strings," *Phys. Rev.* **D68** (2003) 086003, hep-th/0304173.

[120] C. P. Bachas and M. R. Gaberdiel, "World-sheet duality for D-branes with travelling waves," *JHEP* **03** (2004) 015, hep-th/0310017.

[121] S. Giusto, J. F. Morales, and R. Russo, "D1D5 microstate geometries from string amplitudes," *JHEP* **03** (2010) 130, 0912.2270.

[122] J. Simon, "Small Black holes vs horizonless solutions in AdS," *Phys. Rev.* **D81** (2010) 024003, 0910.3225.

[123] O. Lunin, J. M. Maldacena, and L. Maoz, "Gravity solutions for the D1-D5 system with angular momentum," hep-th/0212210.

[124] I. Kanitscheider, K. Skenderis, and M. Taylor, "Fuzzballs with internal excitations," *JHEP* **06** (2007) 056, 0704.0690.

[125] P. Haggi-Mani, U. Lindstrom, and M. Zabzine, "Boundary conditions, supersymmetry and A-field coupling for an open string in a B-field background," *Phys. Lett.* **B483** (2000) 443–450, hep-th/0004061.

[126] D. Duo, R. Russo, and S. Sciuto, "New twist field couplings from the partition function for multiply wrapped D-branes," *JHEP* **0712** (2007) 042, arXiv:0709.1805.

[127] P. Di Vecchia, A. Liccardo, R. Marotta, I. Pesando, and F. Pezzella, "Wrapped magnetized branes: two alternative descriptions?," *JHEP* **0711** (2007) 100, arXiv:0709.4149.

[128] Y. Takayama and A. Tsuchiya, "Complex matrix model and fermion phase space for bubbling AdS geometries," *JHEP* **10** (2005) 004, hep-th/0507070.

[129] M. M. Caldarelli, D. Klemm, and P. J. Silva, "Chronology protection in anti-de Sitter," *Class. Quant. Grav.* **22** (2005) 3461–3466, hep-th/0411203.

[130] P. G. Shepard, "Black hole statistics from holography," *JHEP* **10** (2005) 072, hep-th/0507260.

[131] S. Giombi, M. Kulaxizi, R. Ricci, and D. Trancanelli, "Half-BPS

geometries and thermodynamics of free fermions," *JHEP* **01** (2007) 067, hep-th/0512101.

[132] L. D'Errico, W. Mueck, and R. Pettorino, "Stretched Horizon and Entropy of Superstars," *JHEP* **05** (2007) 063, hep-th/0703223.

[133] N. Ogawa and S. Terashima, "Coarse-graining of bubbling geometries and fuzzball conjecture," 0805.1405.

[134] M. Berkooz and D. Reichmann, "Weakly Renormalized Near 1/16 SUSY Fermi Liquid Operators in N = 4 SYM," *JHEP* **10** (2008) 084, 0807.0559.

[135] I. R. Klebanov, "String theory in two-dimensions," hep-th/9108019.

[136] P. H. Ginsparg and G. W. Moore, "Lectures on 2-D gravity and 2-D string theory," hep-th/9304011.

[137] S. Alexandrov, "Matrix quantum mechanics and two-dimensional string theory in non-trivial backgrounds," hep-th/0311273.

[138] M. R. Douglas, "Conformal field theory techniques for large N group theory," hep-th/9303159.

[139] M. R. Douglas, "Conformal field theory techniques in large N Yang-Mills theory," hep-th/9311130.

[140] L.-L. Chau and Y. Yu, "Unitary polynomials in normal matrix models and Laughlin's wave functions for the fractional quantum Hall effect,". UCDPHYS-PUB-91-13.

[141] L.-L. Chau and O. Zaboronsky, "On the structure of correlation functions in the normal matrix model," *Commun. Math. Phys.* **196** (1998) 203–247, hep-th/9711091.

[142] I. R. Klebanov, J. M. Maldacena, and N. Seiberg, "Unitary and complex matrix models as 1-d type 0 strings," *Commun. Math. Phys.* **252** (2004) 275–323, hep-th/0309168.

[143] C. Itzykson and J. B. Zuber, "The Planar Approximation. 2," *J. Math.*

Phys. **21** (1980) 411.

[144] J. P. Rodrigues, "Large N spectrum of two matrices in a harmonic potential and BMN energies," *JHEP* **12** (2005) 043, hep-th/0510244.

[145] J. P. Rodrigues and A. Zaidi, "Non supersymmetric strong coupling background from the large N quantum mechanics of two matrices coupled via a Yang-Mills interaction," *Phys. Rev.* **D82** (2010) 085030, 0807.4376.

[146] A. Hashimoto, S. Hirano, and N. Itzhaki, "Large branes in AdS and their field theory dual," *JHEP* **08** (2000) 051, hep-th/0008016.

[147] B. U. Eden, P. S. Howe, A. Pickering, E. Sokatchev, and P. C. West, "Four-point functions in N = 2 superconformal field theories," *Nucl. Phys.* **B581** (2000) 523–558, hep-th/0001138.

[148] B. U. Eden, P. S. Howe, E. Sokatchev, and P. C. West, "Extremal and next-to-extremal n-point correlators in four- dimensional SCFT," *Phys. Lett.* **B494** (2000) 141–147, hep-th/0004102.

[149] A. Ghodsi, A. E. Mosaffa, O. Saremi, and M. M. Sheikh-Jabbari, "LLL vs. LLM: Half BPS sector of N = 4 SYM equals to quantum Hall system," *Nucl. Phys.* **B729** (2005) 467–491, hep-th/0505129.

[150] R. Brauer, "On Algebras Which are Connected with the Semisimple Continuous Groups," *Annals of Mathematics* **38** (Oct, 1937) 857–872.

[151] H. Wenzl, "On the Structure of Brauer's Centralizer Algebras," *Annals of Mathematics* **128** (Jul, 1988) 173–193.

[152] G. Benkart, M. Chakrabarti, T. Halverson, R. Leduc, C. Lee, and J. Stroomer, "Tensor product Representations of General Linear Groups and Their Connections with Brauer Algebras," *Journal of Algebra* **166** (1994) 529–567.

[153] J. R. Stembridge, "Rational tableaux and the tensor algebra of gl(n)," *Journal of Combinatorial Theory Series A* **46** (1987) 79–120.

[154] K. Koike, "On the Decomposition of Tensor Products of the

Representations of the Classical Groups: By Means of the Universal Characters," *Advances in Mathematics* **74** (1989) 57–86.

[155] T. W. Brown, P. J. Heslop, and S. Ramgoolam, "Diagonal multi-matrix correlators and BPS operators in N=4 SYM," *JHEP* **02** (2008) 030, 0711.0176.

[156] T. W. Brown, P. J. Heslop, and S. Ramgoolam, "Diagonal free field matrix correlators, global symmetries and giant gravitons," *JHEP* **04** (2009) 089, 0806.1911.

[157] T. W. Brown, "Cut-and-join operators and N=4 super Yang-Mills," *JHEP* **05** (2010) 058, 1002.2099.

[158] Y. Kimura, "Quarter BPS classified by Brauer algebra," *JHEP* **05** (2010) 103, 1002.2424.

[159] J. Pasukonis and S. Ramgoolam, "From counting to construction of BPS states in N=4 SYM," *JHEP* **02** (2011) 078, 1010.1683.

[160] J. McGreevy, L. Susskind, and N. Toumbas, "Invasion of the giant gravitons from anti-de Sitter space," *JHEP* **06** (2000) 008, hep-th/0003075.

[161] R. de Mello Koch, J. Smolic, and M. Smolic, "Giant Gravitons - with Strings Attached (I)," *JHEP* **06** (2007) 074, hep-th/0701066.

[162] R. de Mello Koch, J. Smolic, and M. Smolic, "Giant Gravitons - with Strings Attached (II)," *JHEP* **09** (2007) 049, hep-th/0701067.

[163] D. Bekker, R. de Mello Koch, and M. Stephanou, "Giant Gravitons - with Strings Attached (III)," *JHEP* **02** (2008) 029, 0710.5372.

[164] R. Bhattacharyya, S. Collins, and R. d. M. Koch, "Exact Multi-Matrix Correlators," *JHEP* **03** (2008) 044, 0801.2061.

[165] R. Bhattacharyya, R. de Mello Koch, and M. Stephanou, "Exact Multi-Restricted Schur Polynomial Correlators," *JHEP* **06** (2008) 101, 0805.3025.

[166] R. d. M. Koch, “Geometries from Young Diagrams,” *JHEP* **11** (2008) 061, 0806.0685.

[167] R. d. M. Koch, N. Ives, and M. Stephanou, “Correlators in Nontrivial Backgrounds,” *Phys. Rev.* **D79** (2009) 026004, 0810.4041.

[168] R. de Mello Koch, T. K. Dey, N. Ives, and M. Stephanou, “Correlators Of Operators with a Large R-charge,” *JHEP* **08** (2009) 083, 0905.2273.

[169] R. de Mello Koch, T. K. Dey, N. Ives, and M. Stephanou, “Hints of Integrability Beyond the Planar Limit,” *JHEP* **01** (2010) 014, 0911.0967.

[170] R. d. M. Koch, G. Mashile, and N. Park, “Emergent Threebrane Lattices,” *Phys. Rev.* **D81** (2010) 106009, 1004.1108.

[171] W. Carlson, R. d. M. Koch, and H. Lin, “Nonplanar Integrability,” *JHEP* **03** (2011) 105, 1101.5404.

[172] D. J. Gross and W. Taylor, “Twists and Wilson loops in the string theory of two- dimensional QCD,” *Nucl. Phys.* **B403** (1993) 395–452, hep-th/9303046.

[173] D. J. Gross and W. Taylor, “Two-dimensional QCD is a string theory,” *Nucl. Phys.* **B400** (1993) 181–210, hep-th/9301068.

[174] J. Ginibre, “Statistical Ensembles of Complex, Quaternion, and Real Matrices,” *J. Math. Phys.* **6** (1965) 440.

[175] M. Masuku and J. P. Rodrigues, “Laplacians in polar matrix coordinates and radial fermionization in higher dimensions,” 0911.2846.

[176] A. Cox, M. De Visscher, S. Doty, and P. Martin, “On the blocks of the walled Brauer algebra,” *Journal of Algebra* **320** (2007) 169, 0709.0851.

[177] I. K. Kostov, M. Staudacher, and T. Wynter, “Complex matrix models and statistics of branched coverings of 2D surfaces,” *Commun. Math. Phys.* **191** (1998) 283–298, hep-th/9703189.

[178] C. Kristjansen, J. Plefka, G. W. Semenoff, and M. Staudacher, “A new

double-scaling limit of N = 4 super Yang-Mills theory and PP-wave strings," *Nucl. Phys.* **B643** (2002) 3–30, hep-th/0205033.

[179] G. Akemann and G. Vernizzi, "Characteristic polynomials of complex random matrix models," *Nucl. Phys.* **B660** (2003) 532–556, hep-th/0212051.

[180] A. Alexandrov, A. Mironov, and A. Morozov, "BGWM as Second Constituent of Complex Matrix Model," 0906.3305.

[181] C. D. Meyer, *Matrix analysis and applied linear algebra*, pp. 508–514. SIAM, 2000.

[182] M. L. Mehta, *Random matrices.* Academic Press, 2004 (3rd edition).

[183] L. Castellani, "On G/H geometry and its use in M-theory compactifications," *Annals Phys.* **287** (2001) 1–13, hep-th/9912277.

[184] A. A. Kirillov, *Elements of the Theory of Representations.* Springer, 1972.

[185] Y. Kimura and S. Ramgoolam, "Enhanced symmetries of gauge theory and resolving the spectrum of local operators," *Phys. Rev.* **D78** (2008) 126003, 0807.3696.

[186] T. Ortín, *Gravity and Strings*, p. 603. Cambridge University Press, 2007.

[187] D. P. Zelobenko, *Compact Lie Groups and their Representations*, pp. 156–175. American Mathematical Society, 1973.

[188] S. Benvenuti, B. Feng, A. Hanany, and Y.-H. He, "Counting BPS operators in gauge theories: Quivers, syzygies and plethystics," *JHEP* **11** (2007) 050, hep-th/0608050.

[189] B. Feng, A. Hanany, and Y.-H. He, "Counting Gauge Invariants: the Plethystic Program," *JHEP* **03** (2007) 090, hep-th/0701063.

[190] J. Polchinski, *String theory. Vol. 1: An Introduction to the Bosonic String*, p. 39. Cambridge Univ. Pr., 1998.

[191] T. Yoneya, "Extended fermion representation of multi-charge 1/2-BPS operators in AdS/CFT: Towards field theory of D-branes," *JHEP* **12** (2005) 028, hep-th/0510114.

[192] J. T. Liu, H. Lu, C. N. Pope, and J. F. Vazquez-Poritz, "Bubbling AdS Black Holes," *JHEP* **10** (2007) 030, hep-th/0703184.

[193] J. A. Minahan, "The SU(2) sector in AdS/CFT," *Fortsch. Phys.* **53** (2005) 828–838, hep-th/0503143.

[194] N. Itzhaki and J. McGreevy, "The large N harmonic oscillator as a string theory," *Phys. Rev.* **D71** (2005) 025003, hep-th/0408180.

[195] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, "M theory as a matrix model: A conjecture," *Phys. Rev.* **D55** (1997) 5112–5128, hep-th/9610043.

[196] E. Witten, "Bound states of strings and p-branes," *Nucl. Phys. B* **460** (1995) 335.

[197] Y. Kimura, "Non-holomorphic multi-matrix gauge invariant operators based on Brauer algebra," *JHEP* **12** (2009) 044, 0910.2170.

[198] J. Stewart, "Part III Lectures on General Relativity, Cambridge, 2007." Unpublished.

[199] P. K. Townsend, "Black holes," gr-qc/9707012.

[200] M. Durkee, "New approaches to higher-dimensional general relativity," 1104.4414.