On a New Construction in Group Theory.

by

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Abstract

My supervisors Ian Chiswell and Thomas Müller have found a new class of groups of functions defined on intervals of the real line, with multiplication defined by analogy with multiplication in free groups. I have extended this idea to functions defined on a densely ordered abelian group. This doesn't give rise to a class of groups straight away, but using the idea of exponentiation from a paper by Myasnikov, Remeslennikov and Serbin, I have formed another class of groups, in which each group contains a subgroup isomorphic to one of Chiswell and Müller's groups.

After the introduction, the second chapter defines the set that contains the group and describes the multiplication for elements within the set. In chapter three I define exponentiation, which leads on to chapter four, in which I describe how it is used to find my groups. Then in chapter five I describe the structure of the centralisers of certain elements within the groups.

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Chapter 1

Introduction

1.1 Introduction.

In 2004. Ian Chiswell and Thomas Müller attended a series of talks by Vladimir Remeslennikov at Queen Mary, University of London, about possibly constructing a new class of groups, $\mathcal{RF}(G)$, from the set of functions $f : [0, \alpha] \longrightarrow G$ where $\alpha \in \mathbb{R}^+ \cup \{0\}$ and G is a given group. No proofs were given in these talks, but some interesting problems were discussed and Chiswell and Müller decided to fill in the proofs to establish that these groups really did exist.

Once they had done this they went on to examine the groups in more detail and found them to be a fascinating new branch of group theory, leading them to write a book, [2].

My work was a natural extension of this class of groups. I started looking at $\mathcal{F}(G, \Lambda)$, the set of functions $f : [0, \alpha]_\Lambda \longrightarrow G$ where $\alpha \in \Lambda$, for Λ a given ordered abelian group (which is open question 21 in appendix B at the end of Chiswell and Müller's book [2]).

1.2 Ordered Abelian Groups and Lyndon Length Functions.

An ordered abelian group is an abelian group Λ, with a total ordering $\leq$ defined on it such that for all $a, b, c \in \Lambda$, $a \leq b$ implies that $a+c \leq b+c$. Here we are using the additive notation so 0 is the identity element. Let $P = \{\lambda \in \Lambda : \lambda > 0\}$. Then we have that $\Lambda = P \cup P^{-1} \cup \{0\}$, the disjoint union of P, P^{-1} and $\{0\}$, and $P + P \subset P$. We call P the positive cone of Λ (see [1]). Ordered abelian groups are useful for comparing different elements.

An ordered abelian group is called discretely ordered if it has a least positive element otherwise it is said to be densely ordered (in which case $\inf \{P\} = 0$).

Examples of such groups are given below:

(1) Additive groups $\mathbb{Q}$ and $\mathbb{R}$ are densely ordered abelian groups with their usual order.

(2) Additive group $\mathbb{Z}$ is a discretely ordered abelian group with its usual order.

(3) Given $n \in \mathbb{N}$, if Λ_n are ordered abelian groups for all n, then the direct sum $\bigoplus_n \Lambda_n$ can be made into an ordered abelian group for each n by letting $(\ldots, x_2, x_1) \leq (\ldots, y_2, y_1)$ if and only if $x_i = y_i$ for all $i > j$, and $x_j < y_j$. Hence $\mathbb{R}^m$ can be viewed as a densely ordered abelian subgroup of the densely ordered abelian group $\bigoplus_n \mathbb{R}^n$.

(4) $\mathbb{Z}[t]$ with the usual ordering, where

$$a_j t^j + \ldots + a_1 t + a_0 \geq b_{j-1} t^{j-1} + \ldots + b_1 t + b_0$$

for all $j \in \mathbb{N}$ and $a_j, \ldots, a_0, b_{j-1}, \ldots, b_0 \in \mathbb{Z}$ with $a_j > 0$, is a discretely ordered abelian group. This is isomorphic to $\bigoplus_n \mathbb{Z}$.

The next definition can be found in Ian Chiswell's book [1] on page 73.

Definition 1.2.1. *Let G be a group and Λ an ordered abelian group. A mapping $L : G \longrightarrow \Lambda$ is called a Lyndon length function if*

(1) $L(1) = 0$,

(2) For all $g \in G$, $L(g) = L(g^{-1})$,

(3) Let $c(g,h) = (L(g) + L(h) - L(g^{-1}h))/2$, *then for all* $g, h, k \in G$,

$$c(g,h) \geq \min\{c(g,k), c(h,k),$$

(4) For all $g \in G$, $L(g) \geq 0$,

(5) For all $g, h \in G$, $L(gh) \leq L(g) + L(h)$.

Note that 4 and 5 are implied by the first three.

There is an equivalence between actions of a group on Λ-trees and Λ-valued Lyndon length functions defined on the group. At the end of Chapter 4 we will prove that a length function that I define on elements of my new groups is actually a Lyndon length function, linking them to the theory of Λ-trees.

1.3 Words, Free Groups and Pregroups.

In order to describe the elements of the new sets $\mathcal{RF}(G)$ we must first examine the theory of words and free groups.

Let G be a multiplicative group generated by a subset S. The elements of S and the set of formal inverses of S, S^{-1}, are called letters and make up the alphabet $S^{\pm}$. A word in G is a finite string of letters from the alphabet $S^{\pm}$. Two words represent the same element of G if there are relations on $S^{\pm}$ such that one word can be converted to the other in a finite number of steps by using these relations. When the number of letters in the word decreases in this process we say that the word has been reduced. Since all these words are finite, we can define the length of an element $g \in G$ to be the number of letters in the shortest word that represents it. This word may not be unique, but the length of every shortest word representing the same element is. A reduced word is a word in which the subwords xx^{-1} or $x^{-1}x$ do not appear for any letter $x \in S$. The empty word (a word with no letters) is an allowed word and is the identity element. In this case G is a discrete group.

From this we can define a free group. One way of looking at a free group is as a group that is generated by a subset S with no relations other than the trivial relations on it, so that every element $g \in G$ can be written uniquely as a reduced word in the elements of S. This unique word does not contain any subword of the form $g^{-1}g$ or gg^{-1}. The definition below is a formal definition, equivalent to my definition, taken from the start of Roger Lyndon and Paul Schupp's book [5]:

Definition 1.3.1. *Let X be a subset of a group F. Then F is a* free group with basis X *provided the following holds: if ϕ is any function from the set X into a group H, then there exists a unique extension of ϕ to a homomorphism ϕ^* from F into H.*

For example, $\langle x \rangle$, the infinite cyclic group generated by x, is a free group with $S = \{x\}$. Here the length of g is the number of letters in the unique shortest word as defined above, so for example the word x^4 has length 4, the identity (the empty word) has length 0.

Multiplication of words in these groups is done in two steps: first the words are concatenated, then they are reduced. For example let $a, b \in G$ be such that $a = a_1 x a_2$ and $b = a_2^{-1} y b_1$ with a_1, a_2, b_1 words in G and $x \neq y^{-1}$ letters in S. Then

$$ab = (a_1 x a_2)(a_2^{-1} y b_1) = a_1 x y b_1$$

and $a_1 x y b_1$ is reduced.

These words can be extended to countably infinite words by not limiting the length of the words to a finite number. Ian Chiswell and Thomas Müller have gone a step further and defined uncountably infinite words by defining a function from compact intervals on the real line into a group G. These send every point of the interval to an element in G, but in this set, the only words made up of a finite number of letters are the words of length 0. Note that here they are using the whole group G as the set S.

Definition 1.3.2. *A cancellative monoid is a set, M, with a multiplication on it such that:*

(1) It is closed under this multiplication,

(2) There is an identity,

(3) The multiplication is associative,

(4) Given $a, b, c \in M$, $ab = ac$ or $ba = ca$ implies that $b = c$.

Chiswell and Müller define a multiplication analagous to concatenation on the set $\mathcal{F}(G) = \mathcal{F}(G, \Lambda)$ of uncountably infinite words using the group operation on G. This forms a cancellative monoid ([2], Section 2.1). However it doesn't necessarily contain inverses to all elements and the elements may not be reduced. Hence they use an analogue of reduction in the free group case together with the monoid operation, but since $\mathbb{R}$ is densely ordered, the two points that join the elements together must be amalgamated into one letter (their product in G).

In my work there are problems, even with these restrictions in place, so I turned to a paper by Alexei Myasnikov, Remeslennikov and Denis Serbin, [7]. Here they describe a new way of constructing a free $\mathbb{Z}[t]$-group, $F^{\mathbb{Z}[t]}$, that had been defined and studied by Roger Lyndon in [4]. Lyndon used this group to prove that only finitely many parametric words are needed to describe solutions of one-variable equations over F, see [3].

More recently this group has been linked to algebraic geometry over groups and the Tarski problem, which led to developments in fully residually free groups, proving they are embeddable into $F^{\mathbb{Z}[t]}$ and allowing one to study them using combinatorial group theory.

Myasnikov, Remeslennikov and Serbin's work uses the idea of pregroups as defined by John Robert Stallings. A pregroup is a set, defined below, that sits inside a larger group, called its universal group, which can then be studied. The definition from [8] and [9] is as follows:

Definition 1.3.3. *Let P be a set, $i : P \to P$ be an involution, denoted $x \mapsto x^{-1}$ and $1 \in P$ be a distinguished element. Then let $D \subset P \times P$ and $m : D \to P$ be a set map, denoted $(x, y) \mapsto xy$, whilst $(x_1, \ldots, x_k)_D$ means that $(x_i, x_{i+1}) \in D$ for all $1 \leq i < k$.*

Then P is a pregroup if it satisfies the following conditions:

(P1) For all $x \in P$, $(x, 1), (1, x) \in D$ and $x1 = 1x = x$,

(P2) For all $x \in P$, $(x, x^{-1}), (x^{-1}, x) \in D$ and $xx^{-1} = x^{-1}x = 1$,

(P3) For all $w, x, y \in P$, if $(w, x, y)_D$, then if one of $(w, xy)_D$ or $(wx, y)_D$ is true then they both are and (w, x, y) associates i.e. $w(xy) = (wx)y$,

(P4) For all $w, x, y, z \in P$, $(w, x, y, z)_D$ implies $(w, xy)_D$ or $(xy, z)_D$.

Pregroups are used by Myasnikov, Remeslennikov and Serbin in their paper [7] to construct the $\mathbb{Z}[t]$-exponentiation, $F^{\mathbb{Z}[t]}$, of a free group F, which, given an alphabet $X^{\pm}$ of F, can be embedded into $CDR(\mathbb{Z}[t], X)$, a group they define in their paper. In order to do this they introduce the idea of an A-group, as used by Roger Lyndon in [4]. The definition of an A-group is as follows:

Definition 1.3.4. *Let A be an associative unitary ring and let G be a group. G is an A-group if it comes with an exponentiation function $G \times A \longrightarrow G$*

$$(g, a) \mapsto g^a$$

that satisfies the following conditions, which are called Lyndon's axioms:

(E1) $g^1 = g,\ g^{\alpha+\beta} = g^\alpha g^\beta,\ g^{\alpha\beta} = (g^\alpha)^\beta$.

(E2) $g^{-1}h^\alpha g = (g^{-1}hg)^\alpha$.

(E3) If $[g, h] = 1$, *then* $(gh)^\alpha = g^\alpha h^\alpha$.

for $\alpha, \beta \in A,\ g, h \in G$.

After this they define a $\mathbb{Z}[t]$-exponentiation function on $CDR(\mathbb{Z}[t], X)$ that they prove satisfies these axioms. They then prove that Lyndon's $\mathbb{Z}[t]$-completion $F^{\mathbb{Z}[t]}$ embeds into this construction.

I have used this idea to construct a $\mathbb{Z}[t]/p(t)$-exponentiation of an extension of Chiswell and Müller's group $\mathcal{RF}(G)$, where $p(t)$ is an irreducible integral polynomial of degree n and $\mathbb{Z}[t]/p(t)$ is the ring of integral polynomials mod $p(t)$. The fact that $\mathbb{Z}[t]/p(t)$ is not an integral domain throws up some problems as does the fact that my extension of $\mathcal{RF}(G)$ is not a discrete group, but these are overcome and a new class of groups is constructed in this thesis. These groups provide a link between the work of Chiswell and Müller and that of Myasnikov, Remeslennikov and Serbin.

In the final chapter of my thesis I look at the elements of my new group and examine their centralisers. This chapter follows closely to Chapter 8 of Chiswell and Müller's book [2], but the higher dimensional elements (elements whose lengths are not a closed interval in $\mathbb{R}$) throw up some unexpected problems. I found elements with properties not found in Chiswell and Müller's group. My final theorem is therefore split into two, with the new type of element dealt with separately.

Chapter 2

$\mathcal{RF}(G, \Lambda)$

2.1 $\mathcal{F}(G, \Lambda)$

In their book [2], Ian Chiswell and Thomas Müller define a set, which they call $\mathcal{F}(G)$. It is the set of all functions $f : [0, \alpha] \longrightarrow G$, where $\alpha \in \mathbb{R}$, sending each point of $[0, \alpha]$ to an element of the group G. If $L(f) > 0$, this makes f an uncountably infinite word in letters from the alphabet of G.

In the first section of this chapter I define, for each G, the set of all functions $f : [0, \alpha]_\Lambda \longrightarrow G$, where Λ is a densely ordered abelian group and $\alpha \in \Lambda$. Λ could be $\mathbb{Q}^n$ or $\mathbb{R}^n$ for some $n \in \mathbb{N}$, with the lexicographic ordering. I decided to look at densely ordered abelian groups because these groups are not complete and yet they are not discrete either, so they are a link between the two extremes.

Definition 2.1.1. *Let G be a group and Λ a densely ordered abelian group. Consider functions $f : [0, \alpha]_\Lambda \longrightarrow G$ defined on some closed interval $[0, \alpha]_\Lambda$ with $\alpha \geq 0$ and $\alpha \in \Lambda$. Let $\mathcal{F}(G, \Lambda)$ be the collection of all of these functions for arbitrary α. α will be called the length of the function f, denoted $L(f)$.*

Definition 2.1.2. *Let G_0 be the set of all elements of $\mathcal{F}(G)$ such that $L(f) = 0$.*

I now define a multiplication on this set as it is defined in [2] at the beginning of chapter 2.

Definition 2.1.3. *For two functions $f, g \in \mathcal{F}(G, \Lambda)$ of lengths α and β respectively, let $f * g$ be the function of length $\alpha + \beta$ defined via:*

$$(f * g)(x) = \begin{cases} f(x) & 0 \leq x < \alpha \\ f(\alpha)g(0) & x = \alpha \\ g(x - \alpha) & \alpha < x \leq \alpha + \beta \end{cases}$$

where $x \in [0, \alpha + \beta]_\Lambda$.

We need $(f * g)(\alpha) = f(\alpha)g(0)$ here because Λ is densely ordered. If Λ was discrete there would be a different definition for $(f * g)$, where $(f * g)(\alpha) = f(\alpha)$.

As in [2], this set does not form a group under $*$-multiplication since inverses do not exist. For example, if $f \in \mathcal{F}(G, \Lambda)$ is such that $L(f) = \alpha > 0$, we must have $L(f * g) \geq \alpha$ for all $g \in \mathcal{F}(G, \Lambda)$, by the definitions of the length function and $*$-multiplication, but $L(\mathbf{1_G}) = 0 < \alpha$. However it does form a cancellative monoid as Chiswell and Müller prove for their set, $\mathcal{F}(G)$, at the start of Chapter 2 in [2]. Here is the proof.

Proposition 2.1.1. *The set $\mathcal{F}(G, \Lambda)$, equipped with multiplication $*$, is a cancellative monoid.*

Proof. There is an identity:

Let $\mathbf{1_G} : \{0\} \longrightarrow G$ be the function

$$\mathbf{1_G} : 0 \mapsto 1_G.$$

Then for $f : [0, \alpha]_\Lambda \longrightarrow G$

$$(\mathbf{1_G} * f)(x) = \begin{cases} \mathbf{1_G}(0)f(0) & x = 0 \\ f(x - 0) & 0 < x \leq \alpha \end{cases}$$

$$= \begin{cases} f(0) & x = 0 \\ f(x) & 0 < x \leq \alpha \end{cases}$$

$$= f(x).$$

and

$$(f * \mathbf{1_G})(x) = \begin{cases} f(x) & 0 \leq x < \alpha \\ f(\alpha)\mathbf{1_G}(0) & x = \alpha \end{cases}$$

$$= f(x).$$

Then $\mathbf{1_G}$ is a double sided identity.

Now, let f be such that $f = f_1 * f_2 = f_1 * f_2'$ with $L(f_1) = \alpha$ and $L(f_2) = \beta$, $L(f_2') = \beta'$. Then

$$L(f) = L(f_1) + L(f_2) = L(f_1) + L(f_2')$$

which implies $L(f_2) = L(f_2') = \beta$, and, for $0 \leq x \leq L(f)$, we have

$$f(x) = \begin{cases} f_1(x) & 0 \leq x < \alpha \\ f_1(\alpha)f_2(0) & x = \alpha \\ f_2(x - \alpha) & \alpha < x \leq \alpha + \beta \end{cases}$$

$$= \begin{cases} f_1(x) & 0 \leq x < \alpha \\ f_1(\alpha)f_2'(0) & x = \alpha \\ f_2'(x - \alpha) & \alpha < x \leq \alpha + \beta \end{cases}$$

Comparing values gives:

$$f_1(\alpha)f_2(0) = f_1(\alpha)f_2'(0)$$

which implies $f_2(0) = f_2'(0)$, and

$$f_2'(x-\alpha) = f_2(x-\alpha)$$

for $\alpha < x \leq \alpha + \beta$. Hence

$$f_2'(x) = f_2(x)$$

for $0 < x \leq \beta$.

Therefore $f_2' = f_2$.

Since f_1 was arbitrary, this shows that $\mathcal{F}(G, \Lambda)$ is left cancellative.

Now, let f be such that $f = f_1 * f_2 = f_1' * f_2$ with $L(f_1) = \alpha$, $L(f_2) = \beta$ and $L(f_1') = \alpha'$.

Then

$$L(f) = L(f_1) + L(f_2) = L(f_1') + L(f_2)$$

which implies $L(f_1) = L(f_1') = \alpha$, and, for $0 \leq x \leq L(f)$, we have

$$f(x) = \begin{cases} f_1(x) & 0 \leq x < \alpha \\ f_1(\alpha)f_2(0) & x = \alpha \\ f_2(x-\alpha) & \alpha < x \leq \alpha+\beta \end{cases}$$
$$= \begin{cases} f_1'(x) & 0 \leq x < \alpha \\ f_1'(\alpha)f_2(0) & x = \alpha \\ f_2(x-\alpha) & \alpha < x \leq \alpha+\beta \end{cases}$$

Comparing values gives:

$$f_1(\alpha)f_2(0) = f_1'(\alpha)f_2(0)$$

which implies $f_1(\alpha) = f_1'(\alpha)$, and

$$f_1(x) = f_1'(x)$$

for $0 \leq x < \alpha$.

Therefore $f_1 = f_1'$.

Since f_2 was arbitrary, this shows that $\mathcal{F}(G,\Lambda)$ is right cancellative.

Now the associativity needs to be checked:

Let $f, g, h \in \mathcal{F}(G,\Lambda)$, $L(f) = \alpha$, $L(g) = \beta$, $L(h) = \gamma$. Then for $0 \le x \le \alpha + \beta + \gamma$

$$(f*(g*h))(x) = \begin{cases} f(x) & 0 \le x < \alpha \\ f(\alpha)(g*h)(0) & x = \alpha \\ (g*h)(x-\alpha) & \alpha < x \le \alpha+\beta+\gamma \end{cases}$$

Now there are 2 cases to consider:

$$\text{Case } 1: L(g) \neq 0$$
$$\text{Case } 2: L(g) = 0$$

Case 1: $L(g) \neq 0$.

$$\begin{aligned}(f*(g*h))(x) &= \begin{cases} f(x) & 0 \le x < \alpha \\ f(\alpha)g(0) & x = \alpha \\ g(x-\alpha) & \alpha < x < \alpha+\beta \\ g(\beta)h(0) & x = \alpha+\beta \\ h(x-\alpha-\beta) & \alpha+\beta < x \le \alpha+\beta+\gamma \end{cases} \\ &= \begin{cases} (f*g)(x) & 0 \le x < \alpha+\beta \\ (f*g)(\alpha+\beta)h(0) & x = \alpha+\beta \\ h(x-\alpha-\beta) & \alpha+\beta < x \le \alpha+\beta+\gamma \end{cases} \\ &= ((f*g)*h)(x)\end{aligned}$$

as required.

Case 2: $L(g) = 0$.

$$(f*(g*h))(x) = \begin{cases} f(x) & 0 \leq x < \alpha \\ f(\alpha)g(0)h(0) & x = \alpha \\ h(x-\alpha) & \alpha < x \leq \alpha+\beta+\gamma \end{cases}$$

$$= \begin{cases} (f*g)(x) & 0 \leq x < \alpha \\ (f*g)(\alpha)h(0) & x = \alpha \\ h(x-\alpha) & \alpha < x \leq \alpha+\beta+\gamma \end{cases}$$

$$=((f*g)*h)(x)$$

as required.

Hence $f*(g*h) = (f*g)*h$, so $\mathcal{F}(G,\Lambda)$ is associative.

Therefore $\mathcal{F}(G,\Lambda)$ is a cancellative monoid. $\square$

Within the above proof I have described an identity element, $\mathbf{1_G}$. Define this element to be the identity element of $\mathcal{F}(G,\Lambda)$.

Following [2], Chapter 2, I now define the formal inverse of $f \in \mathcal{F}(G,\Lambda)$. This is clearly not an inverse with respect to the $*$-multiplication, but will be useful later on in this Chapter.

Definition 2.1.4. *Let $f : [0,\alpha]_\Lambda \longrightarrow G$ be an element of $\mathcal{F}(G,\Lambda)$. The formal inverse $f^{-1} : [0,\alpha]_\Lambda \longrightarrow G$ is defined via*

$$f^{-1}(x) = f(\alpha - x)^{-1}$$

for $0 \leq x \leq \alpha$.

Now, by definition

$$L(f^{-1}) = L(f).$$

Therefore

$$L((f^{-1})^{-1}) = L(f^{-1}) = L(f) = \alpha.$$

Also, for $0 \leq x \leq \alpha$

$$\begin{aligned}((f^{-1})^{-1})(x) =&((f^{-1})(\alpha - x))^{-1}\\ =&(((f(\alpha - (\alpha - x)))^{-1})^{-1}\\ =&((f(x))^{-1})^{-1}\\ =&f(x).\end{aligned}$$

Hence $(f^{-1})^{-1} = f$ for all $f \in \mathcal{F}(G, \Lambda)$.

2.2 $\mathcal{RF}(G, \Lambda)$

In this section I define the set of reduced functions, $\mathcal{RF}(G, \Lambda)$, and a second type of multiplication, called reduced multiplication. This is still following the structure of [2], Chapter 2, but with an extra condition that I had to introduce because Λ is not necessarily complete. The definition below, however, is essentially the same as Definition 2.4 in [2].

Definition 2.2.1. *Let $f : [0, \alpha]_\Lambda \longrightarrow G$ be a function in $\mathcal{F}(G, \Lambda)$. f is called reduced if for all $0 < x \leq \alpha$ with $f(x) = 1_G$ and for all $\varepsilon \in \Lambda$ such that $0 < \varepsilon \leq \min\{\alpha - x, x\}$, there exists $0 < \delta \leq \varepsilon$ such that*

$$f(x + \delta) \neq f(x - \delta)^{-1}.$$

The set of all reduced functions in $\mathcal{F}(G, \Lambda)$ will be denoted $\mathcal{RF}(G, \Lambda)$.

Remarks: (see Remark 2.5 [2])

(i) Every element in $\mathcal{F}(G, \Lambda)$ of length 0 is reduced.

(ii) If $f \in \mathcal{RF}(G, \Lambda)$, then f is not identically equal to $\mathbf{1_G}$ on any non-degenerate subinterval of its domain. Therefore we have:

$$\mathcal{RF}(1_G, \Lambda) = \{\mathbf{1_G}\}.$$

(iii) If $f \in \mathcal{RF}(G, \Lambda)$, then so does its formal inverse, f^{-1}.

(iv) If $f \in \mathcal{RF}(G, \Lambda)$ with $L(f) > 0$, then $f * f^{-1}$ is not reduced.

From (*iii*) and (*iv*), it is clear that we need a new type of multiplication where the product of $f, g \in \mathcal{RF}(G, \Lambda)$ is also reduced. I define this below, adjusting Chiswell and Müller's definition because for my set this is only a partial multiplication.

Definition 2.2.2. *Let $f, g \in \mathcal{F}(G, \Lambda)$ have lengths α, β respectively, then*

$$\mathcal{E}(f,g) := \left\{ \varepsilon \in [0, \min\{\alpha, \beta\}]_\Lambda \,\middle|\, \begin{array}{c} f(\alpha - \delta) = g(\delta)^{-1} \\ \textit{for all } \delta \in [0, \varepsilon]_\Lambda \end{array} \right\}$$

and let

$$\varepsilon_0 = \varepsilon_0(f,g) := \begin{cases} 0 & f(\alpha) \neq g(0)^{-1} \\ \sup \mathcal{E}(f,g) & f(\alpha) = g(0)^{-1} \textit{ and } \sup \mathcal{E}(f,g) \textit{ is defined in } \Lambda \\ undefined & otherwise \end{cases}$$

Now define fg on the interval $[0, (\alpha + \beta - 2\varepsilon_0)]_\Lambda$, for those functions where $\varepsilon_0(f,g)$ is defined, as

$$(fg)(x) := \begin{cases} f(x) & 0 \leq x < \alpha - \varepsilon_0 \\ f(\alpha - \varepsilon_0) g(\varepsilon_0) & x = \alpha - \varepsilon_0 \\ g(x - \alpha + 2\varepsilon_0) & \alpha - \varepsilon_0 < x \leq (\alpha + \beta - 2\varepsilon_0) \end{cases}$$

The function fg is called the reduced product of functions $f, g \in \mathcal{F}(G, \Lambda)$

Note: If Λ is not complete, $\varepsilon_0(f,g)$ can fail to exist. This is where my work differs from Chiswell and Müller's. If $\varepsilon_0(f,g)$ does not exist we say that the reduced product is not defined. However when it is defined Lemma 2.7 from [2] holds, as shown below:

Lemma 2.2.1. *The reduced product fg of functions $f, g \in \mathcal{RF}(G, \Lambda)$, if defined, is again reduced.*

Proof. Clearly, if $f(\alpha-\varepsilon_0)g(\varepsilon_0) \neq 1_G$, then the claim holds, so suppose that $f(\alpha-\varepsilon_0)g(\varepsilon_0) = 1_G$.

If fg is not reduced, there exists ε such that $0 < \varepsilon \leq \min\{(\alpha-\varepsilon_0), (\beta-\varepsilon_0)\}$ and

$$(fg)(\alpha-\varepsilon_0-\delta)(fg)(\alpha-\varepsilon_0+\delta) = 1_G, \qquad \text{for all } 0 < \delta \leq \varepsilon. \tag{2.1}$$

From Equation (2.1) and the definition of fg, we have

$$f(\alpha-\eta)g(\eta) = 1_G, \qquad \text{for all } \varepsilon_0 < \eta \leq \varepsilon_0+\varepsilon. \tag{2.2}$$

From the hypothesis, fg is defined, so $\varepsilon_0 = \sup \mathcal{E}(f,g)$ and $f(\alpha-\varepsilon_0)g(\varepsilon_0) = \mathbf{1_G}$, hence

$$f(\alpha-\eta)g(\eta) = 1_G, \qquad \text{for all } 0 \leq \eta \leq \varepsilon_0. \tag{2.3}$$

Combining Equation (2.2) and Equation (2.3), we find that

$$f(\alpha-\eta)g(\eta) = 1_G, \qquad \text{for all } 0 \leq \eta \leq \varepsilon_0+\varepsilon.$$

This implies that $\varepsilon_0+\varepsilon \in \mathcal{E}(f,g)$. But $\varepsilon_0 = \sup \mathcal{E}(f,g)$, so $\varepsilon \leq 0$, a contradiction.

Therefore fg is reduced as claimed.

□

The next Lemma shows the link between $*$-multiplication and reduced multiplication. It follows Lemma 2.8 from [2].

Lemma 2.2.2. *For $f,g \in \mathcal{RF}(G,\Lambda)$, the following are equivalent (See [2]):*

(i) $\varepsilon_0(f,g) = 0$;

(ii) $fg = f*g$;

(iii) $f*g$ *is reduced.*

Proof. $(i) \Longrightarrow (ii)$.

If $\varepsilon_0(f,g) = 0$, then

$$L(fg) = L(f) + L(g) = L(f * g),$$

and, by definition

$$(fg)(x) = \begin{cases} f(x) & 0 \leq x < L(f) \\ f(L(f))g(0) & x = L(f) \\ g(x - L(f)) & L(f) < x \leq L(f) + L(g) \end{cases}$$
$$=(f * g)(x).$$

So $fg = f * g$ as required.

$(ii) \implies (iii)$.

If $fg = f * g$, then, by Lemma 2.2.1, $f * g$ must be reduced.

$(iii) \implies (i)$.

Let $\alpha = L(f), \beta = L(g)$. Suppose $\varepsilon_0(f,g) > 0$. Then $\alpha, \beta \neq 0$, so $0 < \alpha < \alpha + \beta$ and

$$(f * g)(\alpha) = f(\alpha)g(0) = 1_G. \tag{2.4}$$

Assume that $\varepsilon_0(f,g) = \sup \mathcal{E}(f,g)$. So

$$f(\alpha - \eta)g(\eta) = 1_G \qquad \text{for } 0 < \eta \leq \varepsilon_0. \tag{2.5}$$

Using the definition of $(f * g)$, we can rewrite Equation (2.5) as:

$$(f * g)(\alpha - \eta)(f * g)(\alpha + \eta) = 1_G \qquad \text{for } 0 < \eta \leq \varepsilon_0. \tag{2.6}$$

By Equation (2.4) and Equation (2.6) we see that $(f * g)$ is not reduced, contradicting assumption (iii). Now note that if $\varepsilon_0(f,g)$ is undefined, then there exists $\varepsilon' \in \mathcal{E}(f,g)$ such that $\varepsilon' > 0$, so we can use the above argument again, using ε' instead of ε_0, to get a contradiction.

Hence $\varepsilon_0(f,g) = 0$. $\square$

Note: $(i) \Longrightarrow (ii)$ holds for $f, g \in \mathcal{F}(G,\Lambda)$.

The next operation, defined by Chiswell and Müller in [2] (Definition 2.10) emphasises this link when it occurs.

Definition 2.2.3. *For $f, g \in \mathcal{F}(G,\Lambda)$ write $f \circ g$ for $f * g$ with the extra condition that $\varepsilon_0(f,g) = 0$ so that $f \circ g = f * g = fg$ by the note above. This is another partial multiplication.*

From Proposition 2.1.1 and Lemma 2.2.2, the element $\mathbf{1_G} \in \mathcal{F}(G,\Lambda)$, as defined in Proposition 2.1.1, is a 2-sided identity element with respect to the reduced multiplication. Also, by the definition of reduced multiplication and of f^{-1}

$$f^{-1}f = \mathbf{1_G} = ff^{-1} \qquad\qquad \text{for all } f \in \mathcal{F}(G,\Lambda)$$

The following Lemma shows the relationship between the $*$-operation, the $\circ$-operation and inversion. It follows Lemma 2.12 in [2] with added parts for the cases when $\varepsilon_0(f_1, f_2)$ or $\varepsilon_0(f_2^{-1}, f_1^{-1}$ are undefined.

Lemma 2.2.3. *If $f = f_1 * f_2$ with $f_1, f_2 \in \mathcal{F}(G,\Lambda)$, then $f^{-1} = f_2^{-1} * f_1^{-1}$ and $\varepsilon_0(f_1, f_2) = \varepsilon_0(f_2^{-1}, f_1^{-1})$ or both are undefined. In particular, $f = f_1 \circ f_2$ implies $f^{-1} = f_2^{-1} \circ f_1^{-1}$.*

Proof. Let $L(f) = \alpha, L(f_1) = \beta, L(f_2) = \gamma$.

By definition of L and of f^{-1},

$$L(f^{-1}) = L(f) = L(f_1) + L(f_2) = L(f_2^{-1}) + L(f_1^{-1}) = L(f_2^{-1} * f_1^{-1})$$

Also, for $0 \leq x \leq L(f) = \alpha$

$$
\begin{aligned}
f^{-1}(x) =& (f(\alpha - x))^{-1} \\
=& \begin{cases} (f_1(\alpha - x))^{-1} & \gamma < x \le \alpha \\ (f_1(\beta) f_2(0))^{-1} & x = \gamma \\ (f_2(\gamma - x))^{-1} & 0 \le x < \gamma \end{cases} \\
=& \begin{cases} f_2^{-1}(x) & 0 \le x < \gamma \\ (f_2(0))^{-1}(f_1(\beta))^{-1} & x = \gamma \\ f_1^{-1}(x - \gamma) & \gamma < x \le \alpha \end{cases} \\
=& \begin{cases} f_2^{-1}(x) & 0 \le x < \gamma \\ f_2^{-1}(\gamma) f_1^{-1}(0) & x = \gamma \\ f_1^{-1}(x - \gamma) & \gamma < x \le \alpha \end{cases} \\
=& (f_2^{-1} * f_1^{-1})(x).
\end{aligned}
$$

So $f^{-1} = f_2^{-1} * f_1^{-1}$ as required.

Next note that

$$f_1(\beta) f_2(0) = (f_2^{-1}(\gamma) f_1^{-1}(0))^{-1}$$

or

$$f_1(L(f_1)) f_2(0) = (f_2^{-1}(L(f_2^{-1})) f_1^{-1}(0))^{-1}.$$

To prove that $\varepsilon_0(f_1, f_2) = \varepsilon_0(f_2^{-1}, f_1^{-1})$ there are 2 possibilities to possibilities.

Either $f_1(L(f_1)) f_2(0) \neq 1_G$ and hence $f_2^{-1}(L(f_2^{-1})) f_1^{-1}(0) \neq 1_G$, so that $\varepsilon_0(f_1, f_2) = 0 = \varepsilon_0(f_2^{-1}, f_1^{-1})$, or we have that $f_1(L(f_1)) f_2(0) = 1_G$, which implies $f_2^{-1}(L(f_2^{-1})) f_1^{-1}(0) = 1_G$.

In the latter case we have that $\varepsilon_0(f_1, f_2) = \sup \mathcal{E}(f_1, f_2)$ or is undefined, and

$\varepsilon_0(f_2^{-1}, f_1^{-1}) = \sup \mathcal{E}(f_2^{-1}, f_1^{-1})$ or is undefined. But then, for $\varepsilon \in [0, \min\{\beta, \gamma\}]_\Lambda$,

$$\begin{aligned}
\varepsilon \in \mathcal{E}(f_1, f_2)) &\iff f_1(\beta - \delta) f_2(\delta) = 1_G && \text{for } 0 < \delta \leq \varepsilon \\
&\iff f_2(\delta)^{-1} f_1(\beta - \delta)^{-1} = 1_G && \text{for } 0 < \delta \leq \varepsilon \\
&\iff f_2^{-1}(\gamma - \delta) f_1^{-1}(\delta) = 1_G && \text{for } 0 < \delta \leq \varepsilon \\
&\iff \varepsilon \in \mathcal{E}(f_2^{-1}, f_1^{-1}).
\end{aligned}$$

Hence $\mathcal{E}(f_1, f_2) = \mathcal{E}(f_1^{-1}, f_2^{-1})$, so either both $\sup \mathcal{E}(f_1, f_2)$ and $\sup \mathcal{E}(f_2^{-1}, f_1^{-1})$ don't exist, or they both do and

$$\varepsilon_0(f_1, f_2) = \sup \mathcal{E}(f_1, f_2) = \sup \mathcal{E}(f_2^{-1}, f_1^{-1}) = \varepsilon_0(f_2^{-1}, f_1^{-1}).$$

Hence either $\varepsilon_0(f_1, f_2) = \varepsilon_0(f_2^{-1}, f_1^{-1})$ or both are undefined as required.

The last part follows from the first two parts of the Lemma and the Definition of $\circ$.

□

2.3 Cancellation Theory of $\mathcal{RF}(G, \Lambda)$

In this section I prove that $\mathcal{RF}(G, \Lambda)$ is almost a pregroup, in the sense that it only fails on the condition $(P4)$, as in Theorem 2.3.1 (see below).

Recall from Definition 1.3.3 that a pregroup is a set with a multiplication that satisfies the following:

(P1) For all $x \in P$, $(x, 1), (1, x) \in D$ and $x1 = 1x = x$,

(P2) For all $x \in P$, $(x, x^{-1}), (x^{-1}, x) \in D$ and $xx^{-1} = x^{-1}x = 1$,

(P3) For all $w, x, y \in P$, if $(w, x, y)_D$, then if one of $(w, xy)_D$ or $(wx, y)_D$ is true then they both are and (w, x, y) associates i.e. $w(xy) = (wx)y$,

(P4) For all $w, x, y, z \in P$, $(w, x, y, z)_D$ implies $(w, xy)_D$ or $(xy, z)_D$.

Theorem 2.3.1. *For every group G the set $\mathcal{RF}(G,\Lambda)$ satisfies the conditions $(P1)$ to $(P3)$ for a pregroup with respect to the reduced multiplication.*

This says nothing about condition $(P4)$.

The results below will be crucial for the proof of Theorem 2.3.1. They follow the equivalent results of [2] in Section 2.3.

First we prove that every function $f \in \mathcal{RF}(G,\Lambda)$ can be split in two at every point $x \in [0,\alpha]_\Lambda$.

Lemma 2.3.1. *Let $f \in \mathcal{RF}(G,\Lambda)$ with length $L(f) = \alpha$ and let $\beta \in \Lambda$ be such that $0 \leq \beta \leq \alpha$. Then there exists reduced functions $f_1 : [0,\beta]_\Lambda \longrightarrow G$ and $f_2 : [0,\alpha-\beta]_\Lambda \longrightarrow G$ such that $f = f_1 \circ f_2$.*

Moreover, f_1 and f_2 with these properties are uniquely determined once one of the values $f_1(\beta), f_2(0)$ have been specified, and one of these may be arbitrarily chosen in G.

Proof. Let f_1 and f_2 be functions in $\mathcal{F}(G,\Lambda)$ with $L(f_1) = \beta, L(f_2) = \alpha - \beta$ and

$$f_1(x) = f(x) \qquad 0 \leq x < \beta$$

$$f_2(x) = f(x+\beta) \qquad 0 < x \leq \alpha - \beta$$

and let $f_1(\beta)f_2(0) = f(\beta)$.

Then $f = f_1 * f_2$, f_1 and f_2 are defined on the specified domains and they are uniquely determined once one of the values $f_1(\beta)$ or $f_2(0)$ have been chosen. Moreover, this value can be chosen arbitrarily in G.

Also, f_1 and f_2 are reduced since f is, and, since $\beta \in \Lambda$, $\varepsilon_0(f_1,f_2) = 0$ holds, as in Lemma 2.2.2. □

Next we prove that fg can be split in two at $L(f) - \varepsilon_0(f,g)$, for $f,g \in \mathcal{RF}(G,\Lambda)$ such that fg is defined, so that the two parts are equal to $f|_{[0,L(f)-\varepsilon_0(f,g)]}$ and $g|_{[\varepsilon_0(f,g),L(g)]}$.

Lemma 2.3.2. *Let $f, g \in \mathcal{RF}(G, \Lambda)$ such that fg is defined. Assume that $f_1, u, g_1 \in \mathcal{RF}(G, \Lambda)$ such that $f = f_1 \circ u, g = u^{-1} \circ g_1$, then $fg = f_1 g_1$. Also, if $L(u) = \varepsilon_0(f, g)$, then $fg = f_1 \circ g_1$ and there exists $f_1, g_1, u \in \mathcal{RF}(G, \Lambda)$ such that $fg = f_1 \circ g_1$.*

Proof. First note that if $\varepsilon_0(f, g)$ is undefined, then so is fg, so assume that $\varepsilon_0(f, g)$ is defined. Let $\varepsilon_0(f, g) = \varepsilon_0$ and $\varepsilon_0(f_1, g_1) = \varepsilon_0'$.

Now prove that $fg = f_1 g_1$.

To do this we need first to show that $L(fg) = L(f_1 g_1)$ and then to show that

$$(fg)(x) = (f_1 g_1)(x) \qquad \text{for } 0 \leq x \leq L(fg) = L(f_1 g_1)$$

But

$$\begin{aligned} L(fg) =& L(f) + L(g) - 2\varepsilon_0 \\ =& L(f_1) + L(u) + L(u^{-1}) + L(g_1) - 2\varepsilon_0 \\ =& L(f_1) + L(g_1) + 2L(u) - 2\varepsilon_0 \end{aligned}$$

and $L(f_1 g_1) = L(f_1) + L(g_1) - 2\varepsilon_0'$, so we need $-2\varepsilon_0' = 2L(u) - 2\varepsilon_0$ i.e.

$$\varepsilon_0 = L(u) + \varepsilon_0' \tag{2.7}$$

Note: By the definition of ε_0 and of u, it is clear that $0 < L(u) \leq \varepsilon_0$, so we can split this into 2 cases:

$$(i)\ L(u) = 0$$

$$(ii)\ L(u) > 0$$

Case (i) $L(u) = 0$.

Since $L(u) = 0$, we have that $L(f_1) = L(f)$ and $L(g_1) = L(g)$, so

$$\begin{aligned} (f * g)(L(f)) =& f(L(f))g(0) \\ =& f_1(L(f_1))u(0)u^{-1}(0)g_1(0) \\ =& f_1(L(f_1))g_1(0) \end{aligned}$$

Now, if $(f * g)(L(f)) \neq 1_G$, then $\varepsilon_0' = 0 = \varepsilon_0 = L(u)$, so Equation (2.7) holds.

If $(f * g)(L(f)) = 1_G$, then

$$\begin{aligned} \varepsilon_0 &= \sup \mathcal{E}(f, g) \\ \varepsilon_0' &= \sup \mathcal{E}(f_1, g_1) \end{aligned}$$

But for $\varepsilon \in [0, \min\{L(f), L(g)\}]_\Lambda$

$$\begin{aligned} \varepsilon \in \mathcal{E}(f, g) &\iff f(L(f) - \delta)g(\delta) = 1_G && 0 < \delta \leq \varepsilon \\ &\iff f_1(L(f_1) - \delta)g_1(\delta) = 1_G && 0 < \delta \leq \varepsilon \\ &\iff \varepsilon \in \mathcal{E}(f_1, g_1) \end{aligned}$$

Hence $\mathcal{E}(f, g) = \mathcal{E}(f_1, g_1)$.

Therefore $\varepsilon_0 = \sup \mathcal{E}(f, g) = \sup \mathcal{E}(f_1, g_1) = \varepsilon_0'$. So Equation (2.7) holds in Case (i).

Case (ii) $L(u) > 0$

In this case, $L(f_1) < L(f)$, $L(g_1) < L(g)$ and

$$f(L(f))g(0) = u(L(u))u^{-1}(0) = 1_G$$

by definition of u^{-1}, so $\varepsilon_0 = \sup \mathcal{E}(f, g)$.

Now,

$$f(L(f) - \delta)g(\delta) = 1_G \tag{2.8}$$

for $0 < \delta < \varepsilon_0$,

and

$$f(L(f) - \delta)g(\delta) = u(L(u) - \delta)u^{-1}(\delta) = 1_G \tag{2.9}$$

for $0 \leq \delta < L(u)$ by definition of u^{-1}, whilst

$$\begin{aligned} f(L(f) - L(u))g(L(u)) =& f(L(f_1))g(L(u)) \\ =& f_1(L(f_1))u(0)u^{-1}(L(u))g_1(0) \\ =& f_1(L(f_1))g_1(0) \end{aligned}$$

So, if $f_1(L(f_1))g_1(0) \neq 1_G$ we have $\varepsilon_0' = 0$ and $\varepsilon_0 = L(u) + \varepsilon_0'$ as required, but if $f_1(L(f_1))g_1(0) = 1_G$, then Equation (2.9) holds for $\delta = L(u)$ and by definition of ε_0',

$$f_1(L(f_1) - \eta)g_1(\eta) = 1_G \qquad 0 < \eta < \varepsilon_0'$$

Therefore Equation (2.8) holds also for the range $L(u) < \delta < \varepsilon_0' + L(u)$, which implies that Equation (2.8) holds for $0 < \delta < L(u) + \varepsilon_0'$, so that $\varepsilon_0 \geq L(u) + \varepsilon_0'$, and for $L(u) < \delta < \varepsilon_0$

$$\begin{aligned} f(L(f) - \delta)g(\delta) &= f_1(L(f_1) - (\delta - L(u)))g_1(\delta - L(u)) \\ &= f_1(L(f_1) - \eta)g_1(\eta) \end{aligned}$$

for $0 < \eta < \varepsilon_0 - L(u)$.

So $\varepsilon_0' \geq \varepsilon_0 - L(u)$, which means $\varepsilon_0 \leq \varepsilon_0' + L(u)$. Hence $\varepsilon_0 = L(u) + \varepsilon_0'$ and Equation (2.7) holds in Case (ii).

Hence Equation (2.8) holds in Case (ii)

Hence Equation (2.8) holds for $0 \leq L(u) \leq \varepsilon_0(f, g)$.

Now all that remains for the first part is to show that $(fg)(x) = (f_1g_1)(x)$ for $0 \leq x \leq L(fg) = L(f_1g_1)$.

So

$$(fg)(x) = \begin{cases} f(x) & 0 \leq x < L(f) - \varepsilon_0 \\ f(L(f) - \varepsilon_0)g(\varepsilon_0) & x = L(f) - \varepsilon_0 \\ g(L(f) - x) & L(f) - \varepsilon_0 < x < L(f) + L(g) - 2\varepsilon_0 \end{cases}$$

but $L(f)-\varepsilon_0 = L(f_1)+L(u)-L(u)-\varepsilon_0' = L(f_1)-\varepsilon_0'$, so

$$(fg)(x) = \begin{cases} f_1(x) & 0 \le x < L(f_1)-\varepsilon_0' \\ \begin{cases} f_1(L(f_1)-\varepsilon_0')g_1(\varepsilon_0') & \varepsilon_0' > 0 \text{ and } x = L(f_1)-\varepsilon_0' \\ f_1(L(f_1))u(0)u^{-1}(L(u))g_1(0) & \varepsilon_0' = 0 \text{ and } x = L(f_1)-\varepsilon_0' \end{cases} \\ g_1(L(f_1)-x) & L(f_1)-\varepsilon_0' < x \le L(f_1)+L(g_1)-2\varepsilon_0' \end{cases}$$

$$= \begin{cases} f_1(x) & 0 \le x < L(f_1)-\varepsilon_0' \\ f_1(L(f_1)-\varepsilon_0')g(\varepsilon_0') & x = L(f_1)-\varepsilon_0' \\ g_1(L(f_1)-x) & L(f_1)-\varepsilon_0' < x \le L(f_1)+L(g_1)-2\varepsilon_0' \end{cases}$$

$$= (f_1g_1)(x)$$

Now, if $L(u) = \varepsilon_0$, then by (2.7), we have that $\varepsilon_0' = 0$ and so, by Lemma 2.2.2, $fg = f_1 \circ g_1$.

For the final part there are two cases to consider:

Case 1: $\varepsilon_0 = 0$.

This means that $f * g = fg = f \circ g$ by Lemma 2.2.2, so let $f_1 = f, g_1 = g$ and $u = \mathbf{1_G}$. Then we are done.

Case 2: $\varepsilon_0 \neq 0$.

Here, $\varepsilon_0 = \sup \mathcal{E}(f,g) > 0$.

By Lemma 2.3.1 we can find

$$f_1 : [0, L(f)-\varepsilon_0]_\Lambda \longrightarrow G$$
$$u : [0, \varepsilon_0]_\Lambda \longrightarrow G$$
$$g_1 : [0, L(g)-\varepsilon_0]_\Lambda \longrightarrow G$$
$$v : [0, \varepsilon_0]_\Lambda \longrightarrow G$$

such that $f = f_1 \circ u$ and $g = v \circ g_1$.

By the definition of ε_0 and since $\varepsilon_0 > 0$, we have

$$f(L(f)-\delta)g(\delta) = 1_G \qquad 0 \le \delta < \varepsilon_0$$

hence $f(L(f)-\delta)=(g(\delta))^{-1}$ for all $0\leq\delta<\varepsilon_0$.

But $f(L(f)-\delta)=u(\varepsilon_0-\delta)$ and $g(\delta)=v(\delta)$ in this range. Hence

$$u(\varepsilon_0-\delta)=(v(\delta))^{-1} \qquad 0\leq\delta<\varepsilon_0$$

We now need $u(0)=(v(\varepsilon_0))^{-1}$, but by Lemma 2.3.1, we can choose $u(0)$ and $v(\varepsilon_0)$ arbitrarily and independently of each other, hence we can arrange for $u(0)=(v(\varepsilon_0))^{-1}$, so we have that $u=v^{-1}$.

Finally, we have

$$L(fg)=L(f)+L(g)-2\varepsilon_0=L(f_1\circ g_1)$$

and for $0\leq x\leq L(f)+L(g)-2\varepsilon_0$, using the fact that $L(f_1)=L(f)-\varepsilon_0$ and $g_1(0)=g(\varepsilon_0)$,

$$\begin{aligned}(fg)(x)&=\begin{cases}f(x) & 0\leq x<L(f)-\varepsilon_0\\ f(L(f)-\varepsilon_0)g(\varepsilon_0) & x=L(f)-\varepsilon_0\\ g(x-L(f)+2\varepsilon_0) & L(f)-\varepsilon_0<x\leq L(f)+L(g)-2\varepsilon_0\end{cases}\\ &=\begin{cases}f_1(x) & 0\leq x<L(f)-\varepsilon_0\\ f_1(L(f)-\varepsilon_0)g_1(0) & x=L(f)-\varepsilon_0\\ g_1(x-L(f)+\varepsilon_0) & L(f)-\varepsilon_0<x\leq L(f)+L(g)-2\varepsilon_0\end{cases}\\ &=\begin{cases}f_1(x) & 0\leq x<L(f_1)\\ f_1(L(f_1))g_1(0) & x=L(f_1)\\ g_1((x)-L(f_1)) & L(f_1)<x\leq L(f_1)+L(g_1)\end{cases}\\ &=(f_1\circ g_1)(x)\end{aligned}$$

since $L(u)=\varepsilon_0$, so $\varepsilon_0'=0$ by the first part.

Hence f_1, g_1 and u exist.

□

Suppose g is such that $L(g) > 0$. Then the next Lemma shows that the set $\{f | fg = f \circ g\}$ is the same set as $\{f | f(g \circ h) = f \circ (g \circ h)\}$ for all $h \in \mathcal{RF}(G, \Lambda)$ with $gh = g \circ h$, and similarly, the set $\{h | gh = g \circ h\}$ is the same set as $\{h | (f \circ g)h = (f \circ g) \circ h\}$ for all $f \in \mathcal{RF}(G, \Lambda)$ with $fg = f \circ g$. This needs a little extra work to cover the case when $f(gh)$ doesn't exist.

Lemma 2.3.3. *Let $f, g, h \in \mathcal{F}(G, \Lambda)$ with $L(g) > 0$. Then*

(a) If $\varepsilon_0(f, g) = 0$, then fg exists and

$$\varepsilon_0(fg, h) = 0 \iff \varepsilon_0(g, h) = 0$$

(b) If $\varepsilon_0(g, h) = 0$, then gh exists and

$$\varepsilon_0(f, gh) = 0 \iff \varepsilon_0(f, g) = 0$$

Proof. (a) Let $\varepsilon_0(g, h) > 0$. Then

$$g(L(g) - \delta) = (h(\delta))^{-1} \qquad 0 \leq \delta < \varepsilon_0(g, h) \tag{2.10}$$

but

$$(f * g)(L(f) + L(g) - \delta) = g(L(g) - \delta) \qquad 0 \leq \delta < L(g) \tag{2.11}$$

From the note after Lemma 2.2.2, since $\varepsilon_0(f, g) = 0$, we have that $(f * g) = fg$ and $L(fg) = L(f) + L(g)$ Therefore, using Equations (2.10) and (2.11) above,

$$\begin{aligned}(fg)(L(fg) - \delta) =& g(L(g) - \delta) && 0 \leq \delta < \min\{\varepsilon_0(g, h), L(g)\} \\ =& (h(\delta))^{-1} && 0 \leq \delta < \min\{\varepsilon_0(g, h), L(g)\}\end{aligned}$$

hence, if $(fg)h$ exists,

$$\varepsilon_0(fg, h) = \sup \mathcal{E}(fg, h) \geq \min\{\varepsilon_0(g, h), L(g)\} > 0$$

as $\varepsilon_0(g, h) > 0$ and $L(g) > 0$ by hypotheses.

If $(fg)h$ doesn't exist, then neither does $\varepsilon_0(fg, h)$, in particular, $\varepsilon_0(fg, h) \neq 0$

For the second part of (a), let $\varepsilon_0(fg,h) > 0$, so it exists. Then

$$(fg)(L(fg)-\delta) = (h(\delta))^{-1} \qquad 0 \leq \delta < \varepsilon_0(fg,h)$$

but again $\varepsilon_0(f,g) = 0$, so from Equation (2.11) we have

$$g(L(g)-\delta) = (h(\delta))^{-1} \qquad 0 \leq \delta < \min\{\varepsilon_0(fg,h), L(g)\}$$

and hence, since $(fg)h$ exists,

$$\varepsilon_0(g,h) = \sup \mathcal{E}(g,h) \geq \min\{\varepsilon_0(fg,h), L(g)\} > 0$$

as $L(g) > 0$ and $\varepsilon_0(fg,h) > 0$ by hypotheses.

If $\varepsilon_0(fg,h)$ does not exist, then there exists $0 < \varepsilon < \min\{L(g), L(h)\}$ such that

$$(fg)(L(fg)-\delta) = (h(\delta))^{-1} \qquad \text{for } 0 \leq \delta < \varepsilon$$

but since $\varepsilon_0(f,g) = 0$,

$$(fg)(L(fg)-\delta) = g(L(g)-\delta) \qquad \text{for } 0 \leq \delta < \varepsilon$$

so

$$\begin{aligned}(fg)(L(fg)-\delta) =& g(L(g)-\delta) && 0 \leq \delta < \varepsilon \\ =& (h(\delta))^{-1} && 0 \leq \delta < \varepsilon\end{aligned}$$

and hence $0 < \varepsilon \in \mathcal{E}(g,h)$ so $\mathcal{E}(g,h) \neq \emptyset$ and $\varepsilon_0(g,h) \neq 0$ as required.

Part (b) is similar.

$\square$

This implies that the $\circ$-product is associative when it exists, as shown by the following Corollary:

Corollary 2.3.1. *Let $f,g,h \in \mathcal{F}(G,\Lambda)$. Then if one of $f \circ (g \circ h)$ or $(f \circ g) \circ h$ exists, then so does the other, and*

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Proof. There are 2 cases:

$$(i) L(g) > 0$$

$$(ii) L(g) = 0$$

Case (i) $L(g) > 0$.

Assume

$$\varepsilon_0(f,g) = \varepsilon_0(f \circ g, h) = \varepsilon_0(fg, h) = 0$$

so $(fg)h$ exists and is equal to $(f \circ g) \circ h$. Then, since $L(g) > 0$, we can use part (a) of Lemma 2.3.3 to get

$$\varepsilon_0(g,h) = 0$$

Hence $gh = g \circ h$.

But now, since $L(g) > 0$ and $\varepsilon_0(f,g) = 0$, we can use part (b) of Lemma 2.3.3 to get

$$\varepsilon_0(f, g \circ h) = \varepsilon_0(f, gh) = 0.$$

Hence

$$f \circ (gh) = f \circ (g \circ h)$$

If we had assumed that $\varepsilon_0(g,h) = \varepsilon_0(f,(g \circ h)) = \varepsilon_0(f, gh) = 0$, so that $f \circ (g \circ h)$ was defined, a similar argument can be used to show that $(f \circ g) \circ h$ is also defined, using Lemma 2.3.3 part (a) where part (b) was used and part (b) where part (a) was used.

Finally, since the $*-$operation is associative, so is the $\circ-$operation, hence, once we know $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are defined, they must be equal.

Case (ii) $L(g) = 0$.

Let $\varepsilon_0(fg, h) = 0$ so that $(fg) \circ h$ is defined.

Here $\varepsilon_0(f,g) = \varepsilon_0(g,h) = 0$ since, by definition, $\varepsilon_0(f,g) \leq L(g) = 0$ and $\varepsilon_0(g,h) \leq L(g) = 0$. Hence $fg = f \circ g$, so $(f \circ g) \circ h$ is defined, and $gh = g \circ h$.

Now, if $f(L(f))(g \circ h)(0) \neq 1_G$, we have that $\varepsilon_0(f, gh) = 0$ by definition, so assume $f(L(f))(g \circ h)(0) = 1_G$ Then,

$$\varepsilon_0(f, g \circ h) = \sup \mathcal{E}(f, g \circ h)$$

and

$$\begin{aligned} f(L(f))(g \circ h)(0) =& f(L(f))g(0)h(0) \\ =& (f \circ g)(L(f))h(0) \\ =& 1_G \end{aligned}$$

Now

$$0 = \varepsilon_0(f \circ g, h) = \sup \mathcal{E}(f \circ g, h)$$

Using the fact that $(g \circ h)(\eta) = h(\eta)$ and $(f \circ g)(L(f) - \eta) = f(L(f) - \eta)$ for all $0 < \eta < \min\{L(f), L(h)\}$, we have that

$$\begin{aligned} (f \circ g)(L(f) - \eta)h(\eta) =& f(L(f) - \eta)h(\eta) \\ =& f(L(f) - \eta)(g \circ h)(\eta) \end{aligned}$$

for $0 < \eta < \min\{L(f), L(h)\}$.

But, since $(fg) \circ h$ is defined, we have that for all $0 < \eta < \min\{L(f), L(h)\}$ such that

$$(f \circ g)(L(f) - \eta)h(\eta) = 1_G$$

there exists $0 \leq \gamma < \eta$ such that

$$(f \circ g)(L(f) - \gamma)h(\gamma) \neq 1_G$$

and hence

$$f(L(f) - \gamma)(g \circ h)(\gamma) \neq 1_G$$

Therefore $\mathcal{E}(f, g \circ h) = \{0\}$, so $\sup \mathcal{E}(f, g \circ h) = \varepsilon_0(f, g \circ h) = 0$.

Hence

$$f(g \circ h) = f \circ (g \circ h)$$

So $f \circ (g \circ h)$ is defined if $(f \circ g) \circ h$ is and the two elements are equal.

Similarly, if $f \circ (g \circ h)$ is defined and $L(g) = 0$, then so is $(f \circ g) \circ h$, and hence, as before, $(f \circ g) \circ h = f \circ (g \circ h)$ if both are defined.

□

We are now in a position to prove Theorem 2.3.1.

Proof. Proof of Theorem 2.3.1.

Recall the conditions of a pregroup:

(P1) For all $x \in P$, $(x, 1), (1, x) \in D$ and $x1 = 1x = x$,

(P2) For all $x \in P$, $(x, x^{-1}), (x^{-1}, x) \in D$ and $xx^{-1} = x^{-1}x = 1$,

(P3) For all $w, x, y \in P$, if $(w, x, y)_D$, then if one of $(w, xy)_D$ or $(wx, y)_D$ is true then they both are and (w, x, y) associates i.e. $w(xy) = (wx)y$,

(P4) For all $w, x, y, z \in P$, $(w, x, y, z)_D$ implies $(w, xy)_D$ or $(xy, z)_D$.

Remember that we are not interested in $(P4)$ at this stage.

Our set is $\mathcal{RF}(G, \Lambda)$, the distinguished element is $\mathbf{1_G} : \{0\} \to G$ and the subset D is the set of pairs of elements for which reduced multiplication is defined.

(P1) Clear from the definition of $\mathbf{1_G}$.

(P2) Clear from the definition of f^{-1}.

(P3) We prove that when fg and gh are defined, either $(fg)h$ and $f(gh)$ are both defined and $(fg)h = f(gh)$ or both $(fg)h$ and $f(gh)$ are not defined.

Let $f, g, h \in \mathcal{RF}(G, \Lambda)$ be such that fg and gh exist.

Then, by Lemma 2.3.2, there exists $f_1, g_1, u \in \mathcal{RF}(G, \Lambda)$ such that $f = f_1 \circ u$, $g = u^{-1} \circ g_1$ and $fg = f_1 \circ g_1$, and there exists $g_2, h_1, v \in \mathcal{RF}(G, \Lambda)$ such that $g = g_2 \circ v, h = v^{-1} \circ h_1$ and $gh = g_2 \circ h_1$.

Here there are 3 cases to consider:

$$(i) : L(u) < L(g_2)$$
$$(ii) : L(u) > L(g_2)$$
$$(iii) : L(u) = L(g_2)$$

Case (i): $L(u) < L(g_2)$.

Since $L(u) < L(g_2)$, by Lemma 2.3.1, we can decompose g_2 into $w \circ g_3$ with $L(w) = L(u)$, $L(g_3) > 0$ and $w, g_3 \in \mathcal{RF}(G, \Lambda)$. So

$$g = u^{-1} \circ g_1 = g_2 \circ v = (w \circ g_3) \circ v$$

and hence

$$\begin{aligned} u^{-1}(x) =& g(x) & 0 \leq x < L(u) \\ =& g_2(x) & 0 \leq x < L(u) \\ =& w(x) & 0 \leq x < L(w) = L(u) \end{aligned}$$

Since we can choose $w(L(w))$ arbitrarily, we can set $w(L(w)) = u^{-1}(L(u))$ to get that $w = u^{-1}$, so we have

$$g_2 = u^{-1} \circ g_3$$

Now, using Corollary 2.3.1, we get

$$g = g_2 \circ v = (u^{-1} \circ g_3) \circ v = u^{-1} \circ (g_3 \circ v)$$

but since $g = u^{-1} \circ g_1$, we also have $g_1 = g_3 \circ v$ by Proposition 2.1.1.

So we now have

$$\begin{aligned} fg =& f_1 \circ g_1 \\ =& f_1 \circ (g_3 \circ v) \end{aligned}$$

this shows that $\varepsilon_0(f_1, g_3) = 0$.

Also, we know that $\varepsilon_0(u^{-1}, g_3) = 0 = \varepsilon_0(g_2, h_1) = \varepsilon_0(u^{-1} \circ g_3, h_1)$ so, by Lemma 2.3.3 part (a), since $L(g_3) > 0$ by the case assumption, we have that $\varepsilon_0(g_3, h_1) = 0$. Then by Lemma 2.3.3 part (a) again, since $\varepsilon_0(f_1, g_3) = 0$, we get $\varepsilon_0(f_1 \circ g_3, h_1) = 0$.

So now

$$\begin{aligned}(fg)h =&(f_1 \circ (g_3 \circ v))(v^{-1} \circ h_1)\\ =&((f_1 \circ g_3) \circ v)(v^{-1} \circ h_1) \qquad \text{by Corollary 2.3.1}\\ =&(f_1 \circ g_3)h_1\\ =&(f_1 \circ g_3) \circ h_1\end{aligned}$$

Also

$$\begin{aligned}gh =&g_2 \circ h_1\\ =&(u^{-1} \circ g_3) \circ h_1\end{aligned}$$

which shows that $\varepsilon_0(u^{-1} \circ g_3, h_1) = 0$.

We know that $\varepsilon_0(u^{-1}, g_3) = 0$, so, by Lemma 2.3.3 part (a), $\varepsilon_0(g_3, h_1) = 0$.

From before, $\varepsilon_0(f_1, g_3) = 0$, so, by Lemma 2.3.3 part (b), $\varepsilon_0(f_1, g_3 \circ h_1) = 0$. So

$$\begin{aligned}f(gh) =&(f_1 \circ u)((u^{-1} \circ g_3) \circ h_1)\\ =&(f_1 \circ u)(u^{-1} \circ (g_3 \circ h_1)) \qquad \text{by Corollary 2.3.1}\\ =&f_1(g_3 \circ h_1)\\ =&f_1 \circ (g_3 \circ h_1)\end{aligned}$$

and again by Corollary 2.3.1

$$f(gh) = f_1 \circ (g_3 \circ h_1) = (f_1 \circ g_3) \circ h_1 = (fg)h$$

as required.

Case (ii): $L(u) > L(g_2)$.

Since $L(u) > L(g_2)$, by Lemma 2.3.1, we can decompose u^{-1} into $g_4 \circ u_1$ with $u_1 \in \mathcal{RF}(G, \lambda)$, $L(g_2) = L(g_4)$ and $L(u_1) > 0$. Then we get

$$g_4(x) = g_2(x) \qquad \text{for } 0 \leq x < L(g_2)$$

and we can choose $g_4(L(g_4))$ arbitrarily to get $g_4(L(g_4)) = g_2(L(g_2))$, so that $g_4 = g_2$ and $u^{-1} = g_2 \circ u_1$.

Now, using Lemma 2.2.3, Proposition 2.1.1 and Corollary 2.3.1, we get

$$\begin{aligned} f =& f_1 \circ u \\ =& f_1 \circ (u_1^{-1} \circ g_2^{-1}) \\ =& (f_1 \circ u_1^{-1}) \circ g_2^{-1} \end{aligned}$$

which implies that $\varepsilon_0(f_1, u_1^{-1}) = 0$.

Also, by Proposition 2.1.1 and Corollary 2.3.1, we get

$$\begin{aligned} g =& u^{-1} \circ g_1 \\ =& (g_2 \circ u_1) \circ g_1 \\ =& g_2 \circ (u_1 \circ g_1) \\ =& g_2 \circ v \end{aligned}$$

hence $v = u_1 \circ g_1$, and

$$\begin{aligned} h =& v^{-1} \circ h_1 \\ =& (g_1^{-1} \circ u_1^{-1}) \circ h_1 \\ =& g_1^{-1} \circ (u_1^{-1} \circ h_1) \end{aligned}$$

so that $\varepsilon_0(u_1^{-1}, h_1) = 0$.

Therefore, by Proposition 2.1.1, Corollary 2.3.1 and the fact $\varepsilon_0(f_1, u_1^{-1}) = 0$, we get

$$
\begin{aligned}
(fg)h =& (f_1 \circ g_1)(v^{-1} \circ h_1) \\
=& (f_1 \circ g_1)((g_1^{-1} \circ u_1^{-1}) \circ h_1) \\
=& (f_1 \circ g_1)(g_1^{-1} \circ (u_1^{-1} \circ h_1)) \\
=& f_1(u_1^{-1} \circ h_1) \\
=& f_1 \circ (u_1^{-1} \circ h_1)
\end{aligned}
$$

and using Proposition 2.1.1, Corollary 2.3.1 and the fact that $\varepsilon_0(u_1^{-1}, h_1) = 0$

$$
\begin{aligned}
f(gh) =& (f_1 \circ u)(g_2 \circ h_1) \\
=& (f_1 \circ (u_1^{-1} \circ g_2^{-1}))(g_2 \circ h_1) \\
=& ((f_1 \circ u_1^{-1}) \circ g_2^{-1})(g_2 \circ h_1) \\
=& (f_1 \circ u_1^{-1})h_1 \\
=& (f_1 \circ u_1^{-1}) \circ h_1 \\
=& f_1 \circ (u_1^{-1} \circ h_1)
\end{aligned}
$$

hence

$$f(gh) = (fg)h$$

as required.

Case (iii): $L(u) = L(g_2)$.

Here we have $g = u^{-1} \circ g_1 = g_2 \circ v$ and $L(u^{-1}) = L(g_2)$, so

$$
g(x) = \begin{cases} u^{-1}(x) & 0 \leq x < L(u) \\ u^{-1}(L(u))g_1(0) & x = L(u) \\ g_1(x) & L(u) < x \leq L(g) \end{cases}
$$

$$
= \begin{cases} g_2(x) & 0 \leq x < L(u) \\ g_2(L(u))v(0) & x = L(u) \\ v(x) & L(u) < x \leq L(g) \end{cases}
$$

Without loss of generality assume that $u^{-1}(L(u)) = g_2(L(u))$ and therefore $g_1(0) = v(0)$, so $u^{-1} = g_2$ and $g_1 = v$.

Then we get

$$\begin{aligned}(fg)h =&(f_1 \circ g_1)(v^{-1} \circ h_1)\\ =&(f_1 \circ g_1)(g_1^{-1} \circ h_1)\\ =&f_1 h_1\end{aligned}$$

and

$$\begin{aligned}f(gh) =&(f_1 \circ u)(g_2 \circ h_1)\\ =&(f_1 \circ g_2^{-1})(g_2 \circ h_1)\\ =&f_1 h_1\end{aligned}$$

So $(fg)h = f_1 h_1 = f(gh)$ as required if $f_1 h_1$ exists, or if it does not exist then both $(fg)h$ and $f(gh)$ are undefined.

Hence $(fg)h = f(gh)$ in all cases, if $(fg)h$ or $f(gh)$ is defined. $\square$

For (P4) we would need to prove that if ef, fg and gh are defined, either $e(fg)$ or $(fg)h$ is defined. Looking at the last part of the proof above, $(fg)h$ can fail to be defined if $f_1 h_1$ isn't defined and also $e(fg)$ can fail to be defined if the equivalent $e_1 g_1'$ is not defined. Hence $\mathcal{RF}(G, \Lambda)$ is not necessarily a pregroup. This is proved by my example below.

Example 2.1:

Let $e, f, g, h \in \mathcal{RF}(G, \mathbb{R}^2)$ be such that $L(e) = L(f) = L(g) = L(h) = (3, 0)$ and $a, b, c, d \in G$ with none of a, b, c or d being inverses to each other and

$$e(x,y) = \begin{cases} 1_G & (x,y) = (0,0) \\ d & (0,0) < (x,y) < (1,0) \\ 1_G & (x,y) = (1,0) \\ d & (1,0) < (x,y), x = 1 \\ a & 1 < x < 2 \\ b & (x,y) < (3,0), 2 \leq x \\ 1_G & (x,y) = (3,0), \end{cases}$$

$$f(x,y) = \begin{cases} 1_G & (x,y) = (0,0) \\ b^{-1} & (0,0) < (x,y), x \leq 1 \\ a^{-1} & 2 < x < 1 \\ d^{-1} & (x,y) < (2,0), x = 2 \\ 1_G & (x,y) = (2,0) \\ c^{-1} & (2,0) < (x,y) < (3,0) \\ 1_G & (x,y) = (3,0), \end{cases}$$

$$g(x,y) = \begin{cases} 1_G & (x,y) = (0,0) \\ c & (0,0) < (x,y) < (1,0) \\ 1_G & (x,y) = (1,0) \\ d^{-1} & (1,0) < (x,y), x = 1 \\ b^{-1} & 1 < x \leq 2 \\ a^{-1} & (x,y) < (3,0), 2 < x \\ 1_G & (x,y) = (3,0), \end{cases}$$

$$h(x,y) = \begin{cases} 1_G & (x,y) = (0,0) \\ a & (0,0) < (x,y), x < 1 \\ b & 1 \leq x < 2 \\ d & (x,y) < (2,0), x = 2 \\ 1_G & (x,y) = (1,0) \\ d & (2,0) < (x,y) < (3,0) \\ 1_G & (x,y) = (3,0). \end{cases}$$

Then $(e,f,g,h)_D$ since

$$(ef)(x,y) = \begin{cases} 1_G & (x,y) = (0,0) \\ d & (0,0) < (x,y) < (1,0) \\ 1_G & (x,y) = (1,0) \\ c^{-1} & (1,0) < (x,y) < (2,0) \\ 1_G & (x,y) = (2,0), \end{cases}$$

$$(fg)(x,y) = \begin{cases} 1_G & (x,y) = (0,0) \\ b^{-1} & (0,0) < (x,y), x \leq 1 \\ a^{-1} & 1 < x < 2 \\ d^{-1} & (x,y) < (2,0), x = 2 \\ 1_G & (x,y) = (2,0) \\ d^{-1} & (2,0) < (x,y), x = 2 \\ b^{-1} & 2 < x \leq 3 \\ a^{-1} & (x,y) < (4,0), 3 < x \\ 1_G & (x,y) = (4,0) \end{cases}$$

and

$$(gh)(x,y) = \begin{cases} 1_G & (x,y) = (0,0) \\ c & (0,0) < (x,y) < (1,0) \\ 1_G & (x,y) = (1,0) \\ d & (1,0) < (x,y) < (2,0) \\ 1_G & (x,y) = (2,0). \end{cases}$$

But neither $e(fg)$ nor $(fg)h$ exists. To visualise this see I have drawn a diagram of the domains of e, f, g and h (see figure 2.1).

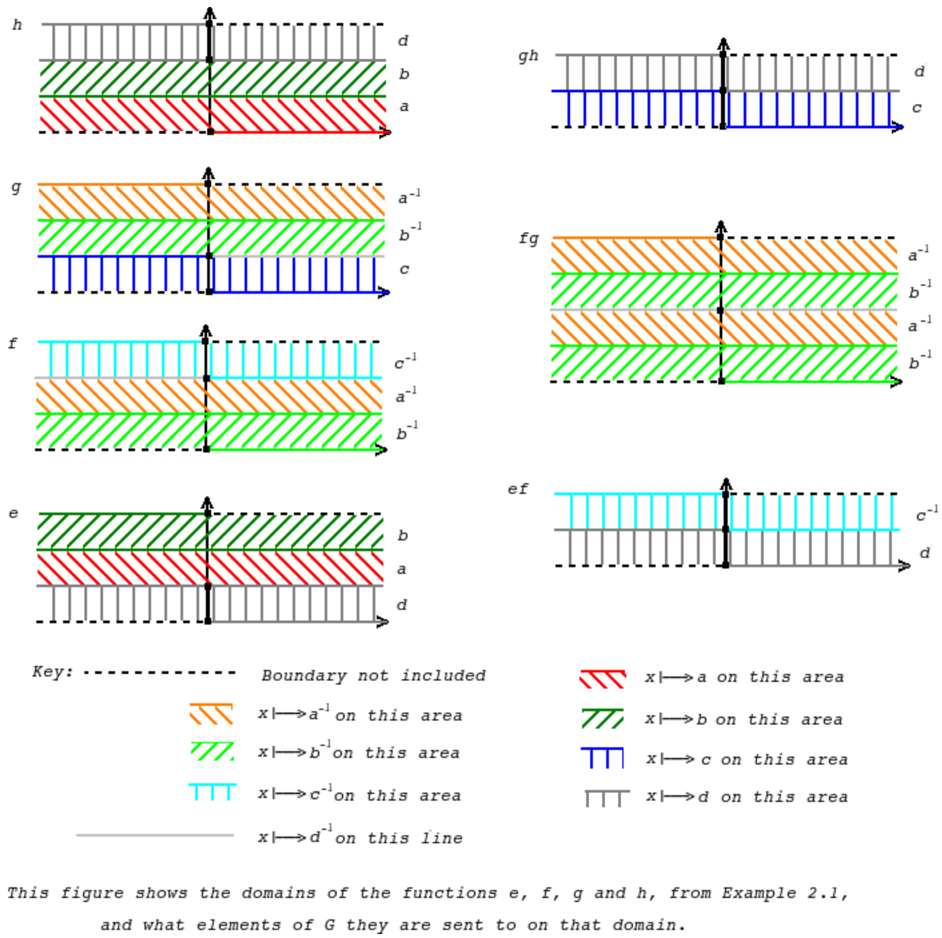

Figure 2.1: How P4 fails.

2.4 Cyclically Reduced Elements

Since multiplication in the set $\mathcal{RF}(G,\Lambda)$ is not necessarily defined for every pair of elements within it, I need to look at a subset of it where the multiplication is always defined. In this section I look at the subset of $\mathcal{RF}(G,\Lambda)$ of all those elements which have a certain property, defined below. These elements are important later on since any subgroup of $\mathcal{RF}(G,\Lambda)$ must contain only them, as proved in this section.

Definition 2.4.1. *An element $f \in \mathcal{RF}(G, \Lambda)$ is called* cyclically decomposable *if it can be written in the form $f = c \circ g \circ c^{-1}$ for some $g, c \in \mathcal{RF}(G, \Lambda)$ with $g(0) \neq g(L(g))^{-1}$ (so $\varepsilon_0(g, g) = 0$). $c \circ g \circ c^{-1}$ is the* cyclic decomposition *of f.*

In [2] all of the elements of the authors' subgroup $\mathcal{RF}(G, \Lambda)$ are cyclically decomposable, but in my subset this is not the case. f will fail to be cyclically decomposable if $\varepsilon_0(f, f)$ is undefined and I prove later that if f is not cyclically decomposable, $\varepsilon_0(f, f)$ is undefined (Lemma 2.4.2).

For example, looking at the case $\Lambda = \mathbb{R}^2$, let $h \in G$. Then $\varepsilon_0(f, f)$ is undefined for the element

$$f = \begin{cases} h & \text{for all } (x_1, x_0) \text{ such that } x_1 < m \\ h^{-1} & \text{for all } (x_1, x_0) \text{ such that } x_1 \geq m \end{cases}$$

with $L(f) = (2m, 0)$, since $\mathcal{E}(f, f) = \{(x_1, x_0) | x_1 < m\}$, which has no supremum. See the figure 2.2 below.

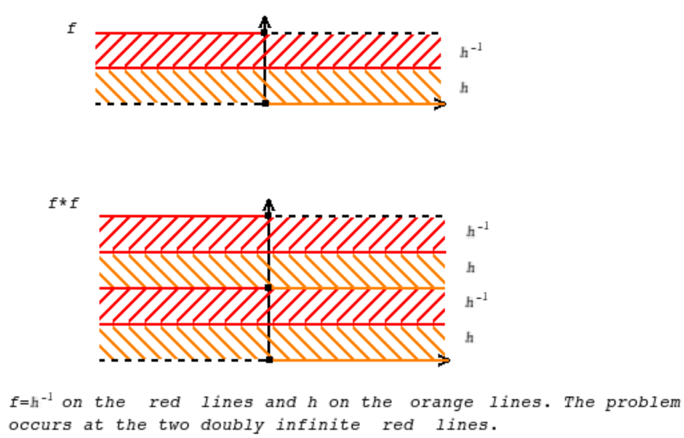

Figure 2.2: An example of how ff can be undefined.

The following definition is the same as Definition 3.5 in [2].

Definition 2.4.2. *For all $f \in \mathcal{RF}(G, \Lambda)$, f is called cyclically reduced if and only if $\varepsilon_0(f, f) = 0$.*

In the set $\mathcal{RF}(G,\Lambda)$ with reduced multiplication, this is still equivalent to the condition that $L(f^2) = 2L(f)$, since f^2 is undefined if $\varepsilon_0(f,f)$ is.

I now need to prove the following:

Lemma 2.4.1. *If f is cyclically reduced then f^{-1} is cyclically reduced.*

Proof. If $L(f) = 0$, $L(f^{-1}) = 0$ by the definition of f^{-1}. So $\varepsilon_0(f^{-1}, f^{-1}) = 0$ since $\varepsilon_0(f^{-1}, f^{-1}) \leq L(f^{-1})$. Hence f^{-1} is cyclically reduced.

So let $L(f) > 0$ and assume f is cyclically reduced, and hence $\varepsilon_0(f,f) = 0$.

We know that $f = c \circ g \circ c^{-1}$ with $L(c) = 0$, so by Lemma 2.2.3 we have that $f^{-1} = c \circ g^{-1} \circ c^{-1}$.

Since $L(c) = 0$, all we need is that $g^{-1}(0) \neq g^{-1}(L(g))^{-1}$. But by the assupmtion that f is cyclically reduced and the definition of inverses, we have that

$$g^{-1}(0) = g(L(g))^{-1} \neq g(0) = g^{-1}(L(g))^{-1}$$

so f^{-1} is also cyclically reduced. $\square$

Now I define the two sets that I will be interested in for the rest of the thesis:

Definition 2.4.3. *The set of all elements that are cyclically decomposable is called $\mathcal{CDF}(G,\Lambda)$.*

Definition 2.4.4. *The set of all cyclically reduced elements is called $\mathcal{CRF}(G,\Lambda)$.*

Clearly $\mathcal{CDF}(G,\Lambda) \supset \mathcal{CRF}(G,\Lambda)$.

In [2], the authors have that all of their group $\mathcal{RF}(G)$ is cyclically decomposable, but if $\Lambda \neq \mathbb{R}$ there may be elements in the set $\mathcal{RF}(G,\Lambda) \backslash \mathcal{CDF}(G,\Lambda)$. The reason that I am interested in the set $\mathcal{CDF}(G,\Lambda)$ is the following Lemma:

Lemma 2.4.2. *For all $f \in \mathcal{RF}(G,\Lambda) \setminus \mathcal{CDF}(G,\Lambda)$, f^2 is not defined.*

Proof. If f is not cyclically decomposable, then $\mathcal{E}(f,f) \neq \emptyset$ but $\sup \mathcal{E}(f,f)$ is undefined in Λ.

Therefore $\varepsilon_0(f,f)$ does not exist and f^2 is undefined as required. $\square$

Obviously, if f^2 is not defined, then any set containing f is not going to be a group. Hence I am not interested in these elements.

Note that, using the ordering defined in Example (3) in the introduction, for $\Lambda = \mathbb{R}^n$, if $\dim(L(f)) = 1$, $f \in \mathcal{CDF}(G, \mathbb{R}^n)$ automatically. For the rest of this thesis, $\mathbb{R}^n$ has this ordering.

The next Lemma shows that a conjugate of a cyclically decomposable element is again cyclically decomposable. It is similar to Lemma 3.7 in [2].

Lemma 2.4.3. *If $u \in \mathcal{CDF}(G, \mathbb{R}^n)$, $c \in \mathcal{RF}(G, \mathbb{R}^n)$ and $v = c^{-1}uc$ is defined, but u does not cancel completely in $c^{-1}uc$, then $v \in \mathcal{CDF}(G, \mathbb{R}^n)$.*

Proof. If $u = d^{-1} \circ u_1 \circ d$ for some $d \in \mathcal{RF}(G, \mathbb{R}^n)$, then

$$\begin{aligned} v =& c^{-1}(d^{-1} \circ u_1 \circ d)c \\ =& (dc)^{-1}u_1(dc) \end{aligned}$$

Therefore we can assume that $u \in \mathcal{CRF}(G, \mathbb{R}^n)$.

This means that in the product $c^{-1}uc$ we have that either $uc = u \circ c$ or $c^{-1}u = c^{-1} \circ u$. without loss of generality assume that $uc = u \circ c$ (otherwise we can look at $v^{-1} = c^{-1}u^{-1}c$).

Since u does not cancel completely in $c^{-1}uc$, we must have that

$$c^{-1} = c_1^{-1} \circ u_1^{-1} \text{ and } u = u_1 \circ u_2$$
$$\text{so that } c^{-1}u = c_1^{-1} \circ u_2$$

and by Lemma 2.3.3 part (a)

$$\begin{aligned} c^{-1}uc =& (c_1^{-1} \circ u_2)c \\ =& c_1^{-1} \circ u_2 \circ c \\ =& c_1^{-1} \circ u_2 \circ u_1 \circ c_1 \end{aligned}$$

but then $u_2 \circ u_1$ is a cyclic permutation of $u = u_1 \circ u_2$ and so is also cyclically reduced, hence we have found a cyclic decomposition of $v = c_1^{-1} \circ u_2 \circ u_1 \circ c_1$.

□

Now I show a link between commutativity of elements in $\mathcal{CDF}(G, \mathbb{R}^n)$ and the lengths of their common initial segments. This will be useful in Chapter 3. Recall from Definition 1.2.1 that $c(x, y) = (L(x) + L(y) - L(x^{-1}y))/2$. Also for $H \subset \mathcal{CDF}(G, \mathbb{R}^n)$, let $H \leq \mathcal{CDF}(G, \mathbb{R}^n)$ mean that H is a subgroup of $\mathcal{CDF}(G, \mathbb{R}^n)$.

Lemma 2.4.4. *Let $H \leq \mathcal{CDF}(G, \mathbb{R}^n)$ and let $f, h \in H$ be cyclically reduced. If $c(f^m, h^k) \geq L(f) + L(h)$ for some $m, k > 0$, then $[f, h] = \mathbf{1_G}$.*

Proof. Without loss of generality, assume $L(h) > L(f)$, so that $c(f^m, h^k) \geq L(h) + L(f)$ implies $h = f^l \circ h_1$ for some $h_1 \in \mathcal{RF}(G, \mathbb{R}^n)$ and $0 < l < m$, and $f = h_1 \circ f_1$.

Looking at the initial segments of f^m and h^k of length $L(f^{l+1} \circ h_1)$, we see that

$$f^{l+1} \circ h_1 = f^l \circ f \circ h_1 = f^l \circ h_1 \circ f$$

But looking at the last segments of this of length $L(f \circ h_1)$, we get that

$$h_1 \circ f = f \circ h_1$$

So $[h_1, f] = \mathbf{1_G}$. This implies that

$$h \circ f = f^l \circ h_1 \circ f = f^l \circ f \circ h_1 = f^{l+1} \circ h_1 = f \circ f^l \circ h_1 = f \circ h$$

which implies that $[h, f] = \mathbf{1_G}$ as required. $\square$

The final Lemma in this Chapter is an interesting result for elements in $\mathcal{CDF}(G, \Lambda)$.

Recall from Definition 2.1.2 that

$$G_0 = \{f \in \mathcal{F} | L(f) = 0\}.$$

Lemma 2.4.5. *For all $f \in \mathcal{CDF}(G, \Lambda)$, $f = \mathbf{1_G}$, $f^2 \neq \mathbf{1_G}$ or f is in a conjugate of G_0*

Proof. Assume $f \in \mathcal{CDF}(G, \Lambda)$ and $f \neq \mathbf{1_G}$. We have that $f = c \circ g \circ c^{-1}$ and $\varepsilon_0(g, g) = 0$, so, if $L(g) > 0$,

$$\begin{aligned} f^2 &= (c \circ g \circ c^{-1})(c \circ g \circ c^{-1}) \\ &= (c \circ g)(g \circ c^{-1}) \\ &= c \circ g \circ g \circ c^{-1} \end{aligned}$$

and $L(c \circ g \circ g \circ c^{-1}) > 0 = L(\mathbf{1_G})$, so $c \circ g \circ g \circ c^{-1} \neq \mathbf{1_G}$.

If $L(g) = 0$, then f is in a conjugate of G_0 by definition.

Hence the only possibilities are the ones listed.

□

Chapter 3

Exponentiation

3.1 Defining Exponentiation

In [4], Roger Lyndon introduced a $\mathbb{Z}[t]$-completion, $F^{\mathbb{Z}[t]}$, of a free group F using the idea of an A-group.

Alexei Myasnikov, Vladimir Remeslennikov and Denis Serbin then described in [7] how to find a $\mathbb{Z}[t]$-exponentiation of $CDR(\mathbb{Z}[t], X)$, which is a set similar to $\mathcal{CDF}(G, \mathbb{R}^n)$, the cyclically decomposable elements of $\mathcal{RF}(G, \mathbb{R}^n)$. In this Chapter I have used their ideas to find a $\mathbb{Z}[t]/p(t)$-exponentiation of $\mathcal{CDF}(G, \mathbb{R}^n)$, where $p(t) = t^n + p_{n-1}t^{n-1} + \ldots + p_1 t + p_0$ is a monic, irreducible polynomial in $\mathbb{Z}[t]$ of degree n. I need to use $\mathbb{Z}[t]/p(t)$ in order to keep the elements produced by this process within my original space, which has dimension n.

I am viewing $\mathbb{Z}[t]/p(t)$ as an ordered abelian group, using the lexicographic ordering, where higher powers of t are bigger. Addition is just the normal addition for polynomials, but multiplication is multiplication modulo the polynomial $p(t)$ defined above. From now on I am taking $\Lambda = \mathbb{R}^n$ unles I specify otherwise, for example at the beginning of Chapter 5.

Now I define $\mathbb{Z}[t]/p(t)$-exponentiation.

Definition 3.1.1. *Let* $u \in \mathcal{CRF}(G, \mathbb{R}^n)$ *be such that* $L(u) = \underline{a} = (0, \ldots, 0, a_i, \ldots, a_0)$ *where* $a_i > 0$*. Then* u *is* $(i+1)$ dimensional*. Also, any element of length zero is* zero dimensional*.*

Note that the set of one-dimensional elements, $u \in \mathcal{CRF}(G, \mathbb{R}^n)$, is isomorphic to the set of cyclically reduced elements in Ian Chiswell and Thomas Müller's group $\mathcal{RF}(G)$ in [2].

Definition 3.1.2. *For* $f(t) = f_{n-1}t^{n-1} + \ldots + f_1 t + f_0 \in \mathbb{Z}[t]/p(t)$ *and for* $\underline{a} = (0, \ldots, 0, a_0)$*, let*

$$f(t)\underline{a} = (a_0 f_{n-1}, \ldots, a_0 f_0).$$

Definition 3.1.3. *For* $u \in \mathcal{CDF}(G, \mathbb{R}^n)$ *and* $m \in \mathbb{N}\backslash\{0\}$*,* u^m *is defined as expected, so for* $m > 0$*,*

$$u^m(x) = \begin{cases} u(x) & 0 \leq x < l(u) = \underline{a} \\ u(x - k\underline{a}) & k\underline{a} < x < (k+1)\underline{a} \\ u(\underline{a})u(0) & x = k\underline{a} \\ u(\underline{a}) & x = m\underline{a} \end{cases}$$

where $0 < k < m$*,* $mL(u) \geq x \in \mathbb{R}^n$*,* $u^{-m} = (u^{-1})^m$ *and* $u^0 = \mathbf{1_G}$*.*

To visualise this for a one-dimensional u, think of the domain of u^m as being the domain u repeated m times along a line, as shown at the top of Figure 3.1. At the endpoints of each copy of u the function is sent to $u(L(u))u(0)$, except at the points $x = 0$ and $x = mL(u)$. Between the endpoints of each copy of u the function behaves like u. Now assume that u is one dimensional and define u^t as follows:

Let $L(u^t) = tL(u) = t\underline{a}$. Then for $tL(u) \geq x \in \mathbb{R}^n$,

$$
u^t(x) = \begin{cases} u(0) & x = 0 \\ u(x - k\underline{a}) & k\underline{a} < x < (k+1)\underline{a},\ k \geq 0 \\ u(\underline{a})u(0) & x = k\underline{a},\ k > 0 \\ u(x - k\underline{a} - t\underline{r}) & k\underline{a} + t\underline{r} < x < (k+1)\underline{a} + t\underline{r} \\ u(\underline{a})u(0) & x = k\underline{a} + t\underline{r} \\ u(x - (k+t)\underline{a}) & (k+t)\underline{a} < x < (k+1+t)\underline{a},\ k < 0 \\ u(\underline{a})u(0) & x = (t+k)\underline{a},\ k < 0 \\ u(\underline{a}) & x = t\underline{a} \end{cases}
$$

where $k \in \mathbb{Z}$ and $0 < \underline{r} < \underline{a}$. We now need to check that $(u^t)^m = (u^m)^t$:

$$
(u^t)^m(x) = \begin{cases} u^t(0) & x = 0 \\ u^t(x - lt\underline{a}) & lt\underline{a} < x < (l+1)t\underline{a} \\ u^t(t\underline{a})u^t(0) & x = lt\underline{a},\ l > 0 \\ u^t(t\underline{a}) & x = mt\underline{a} \end{cases}
$$

$$
= \begin{cases} u(0) & x = 0 \\ u(x - lt\underline{a} - k\underline{a}) & k\underline{a} < x - lt\underline{a} < (k+1)\underline{a},\ k \geq 0 \\ u(\underline{a})u(0) & x - lt\underline{a} = k\underline{a},\ k > 0 \\ u(x - lt\underline{a} - k\underline{a} - t\underline{r}) & k\underline{a} + t\underline{r} < x - lt\underline{a} < (k+1)\underline{a} + t\underline{r} \\ u(\underline{a})u(0) & x - lt\underline{a} = k\underline{a} + t\underline{r} \\ u(x - lt\underline{a} - (k+t)\underline{a}) & (k+t)\underline{a} < x - lt\underline{a} < (k+1+t)\underline{a},\ k < 0 \\ u(\underline{a})u(0) & x - lt\underline{a} = (k+t)\underline{a},\ k < 0 \\ u(\underline{a})u(0) & x = lt\underline{a},\ l > 0 \\ u(\underline{a}) & x = mt\underline{a} \end{cases}
$$

where $0 \leq l < m$, $k \in \mathbb{Z}$ and $0 < \underline{r} < \underline{a}$. But some of these cases are covered twice, so we can rewrite this as

$$(u^t)^m(x) = \begin{cases} u(0) & x = 0 \\ u(x - k\underline{a}) & k\underline{a} < x < (k+1)\underline{a}\ k \geq 0 \\ u(\underline{a})u(0) & x = k\underline{a},\ k > 0 \\ u(x - lt\underline{a} - k\underline{a} - t\underline{r}) & k\underline{a} + t\underline{r} < x - lt\underline{a} < (k+1)\underline{a} + t\underline{r} \\ u(\underline{a})u(0) & x - lt\underline{a} = k\underline{a} + t\underline{r} \\ u(x - lt\underline{a} - (k+t)\underline{a}) & k\underline{a} < x - (l+1)t\underline{a} < (k+1)\underline{a},\ k < 0 \\ u(\underline{a})u(0) & x - lt\underline{a} = (k+t)\underline{a},\ k < 0 \\ u(\underline{a}) & x = mt\underline{a} \end{cases}$$

where $0 < l < m$, $k \in \mathbb{Z}$ and $0 < \underline{r} < \underline{a}$, whilst

$$(u^m)^t(x) = \begin{cases} u^m(0) & x = 0 \\ u^m(x - lm\underline{a}) & lm\underline{a} < x < (l+1)m\underline{a}, l \geq 0 \\ u^m(m\underline{a})u^m(0) & lm\underline{a} = x, l > 0 \\ u^m(x - lm\underline{a} - t\underline{r}) & lm\underline{a} + t\underline{r} < x < (l+1)m\underline{a} + t\underline{r} \\ u^m(m\underline{a})u^m(0) & lm\underline{a} + t\underline{r} = x \\ u^m(x - (l+t)m\underline{a}) & (l+t)m\underline{a} < x < (l+t+1)m\underline{a},\ l < 0 \\ u^m(m\underline{a})u^m(0) & (l+t)m\underline{a} = x,\ l < 0 \\ u^m(m\underline{a}) & x = tm\underline{a} \end{cases}$$

$$= \begin{cases} u(0) & x = 0 \\ u(x - lm\underline{a} - k\underline{a}) & k\underline{a} < x - lm\underline{a} < (k+1)\underline{a},\ k \geq 0, k + lm \geq 0 \\ u(\underline{a})u(0) & k\underline{a} = x - lm\underline{a},\ k > 0, k + lm > 0, l \geq 0 \\ u(\underline{a})u(0) & lm\underline{a} = x,\ l > 0 \\ u(x - lm\underline{a} - t\underline{r} - k\underline{a}) & k\underline{a} < x - lm\underline{a} - t\underline{r} < (k+1)\underline{a} \\ u(\underline{a})u(0) & lm\underline{a} + t\underline{r} = x \\ u(\underline{a})u(0) & k\underline{a} = x - lm\underline{a} - t\underline{r} \\ u(\underline{a})u(0) & (l+t)m\underline{a} = x,\ l < 0 \\ u(x - (l+t)m\underline{a} - k\underline{a}) & k\underline{a} < x - (l+t)m\underline{a} < (k+1)\underline{a},\ l < 0, k + lm < 0 \\ u(\underline{a})u(0) & k\underline{a} = x - (l+t)m\underline{a},\ l < 0, k + lm < 0 \\ u(\underline{a}) & x = tm\underline{a} \end{cases}$$

where $k, l \in \mathbb{Z}$ and $0 < \underline{r} < \underline{a}$.

But some of these cases are covered twice, so this can be written

$$(u^m)^t(x) = \begin{cases} u(0) & x = 0 \\ u(x - lm\underline{a} - k\underline{a}) & k\underline{a} < x - lm\underline{a} < (k+1)\underline{a},\ k \geq 0, k + lm \geq 0 \\ u(\underline{a})u(0) & k\underline{a} = x - lm\underline{a},\ k \geq 0, k + l > 0 \\ u(x - lm\underline{a} - t\underline{r} - k\underline{a}) & k\underline{a} < x - lm\underline{a} - t\underline{r} < (k+1)\underline{a} \\ u(\underline{a})u(0) & k\underline{a} = x - lm\underline{a} - t\underline{r} \\ u(x - (l+t)m\underline{a} - k\underline{a}) & k\underline{a} < x - (l+t)m\underline{a} < (k+1)\underline{a},\ l < 0, k + lm < 0 \\ u(\underline{a})u(0) & k\underline{a} = x - (l+t)m\underline{a},\ l < 0, k + lm < 0 \\ u(\underline{a}) & x = tm\underline{a} \end{cases}$$

where $k, l \in \mathbb{Z}$ and $0 < \underline{r} < \underline{a}$.

With some changes of variables these two elements are equal and hence $(u^t)^m = (u^m)^t$ as required. So if $v = u^k$ for some $k \in \mathbb{Z}$, set $v^t = (u^t)^k$.

We can now define u^{t^m} inductively for $0 < m < n$, by letting $u^{t^m} = (u^{t^{m-1}})^t$ and replacing $\underline{a}$ with $t^{m-1}\underline{a}$ and t with t^m.

To see this more clearly, let

$$(u^{t^{m-1}})^t(x) = \begin{cases} u^{t^{m-1}}(0) & x = 0 \\ u^{t^{m-1}}(x - kt^{m-1}\underline{a}) & kt^{m-1}\underline{a} < x < (k+1)t^{m-1}\underline{a}, k \geq 0 \\ u^{t^{m-1}}(t^{m-1}\underline{a})u^{t^{m-1}}(0) & x = kt^{m-1}\underline{a}, k > 0 \\ u^{t^{m-1}}(x - kt^{m-1}\underline{a} - t^m\underline{r}) & kt^{m-1}\underline{a} + t^m\underline{r} < x < (k+1)t^{m-1}\underline{a} + t^m\underline{r} \\ u^{t^{m-1}}(t^{m-1}\underline{a})u^{t^{m-1}}(0) & x = kt^{m-1}\underline{a} + t^m\underline{r} \\ u^{t^{m-1}}(x - (k+t)t^{m-1}\underline{a}) & (k+t)t^{m-1}\underline{a} < x < (k+1+t)t^{m-1}\underline{a}, k < 0 \\ u^{t^{m-1}}(t^{m-1}\underline{a})u^{t^{m-1}}(0) & x = (k+t)t^{m-1}\underline{a}, k < 0 \\ u^{t^{m-1}}(t^{m-1}\underline{a}) & x = t^m\underline{a}, \end{cases}$$

where $k \in \mathbb{Z}$, $t^m\underline{a} \geq x \in \mathbb{R}^n$ and $0 < \underline{r} < \underline{a}$.

To visualise this, look at the second figure in Figure 3.1. This represents the domain of u^t. It consists of the part of the plane between the two lines $(0,x)$ and $(L(u),x)$ and includes the closed halflines from $(0,0)$ in the positive x direction and from $(L(u),0)$ in the negative x direction. This plane is built up of lines parallel to the x-axis that are copies of the top figure, extended to infinity at either end. They all equal $u(L(u))u(0)$ on the y-axis, except at $u(0)$ and $u(tL(u))$.

The final figure there is trying to show what the domain pf u^{t^2} would be like in three dimensions. This is harder to visualise. The dotted lines are again open boundaries of the domain. This time the space between the planes $z = 0$ and $z = L(u)$, the open half planes $(x, y, 0)$ with $y > 0$ and $(x, y, L(u))$ with $y < 0$ and the closed half lines $(x, 0, 0)$ with $x \geq 0$ and $(x, 0, L(u))$ with $x \leq 0$ is the space that the function is defined on. The space is built up of planes parallel to the x, y-plane, which are in turn built up of lines parallel to the x-axis.

Then make

$$u^{t^n} = u^{t^n - p(t)} = (u^{t^{n-1}})^{-p_{n-1}} \dots (u^t)^{-p_1} u^{-p_0}$$

and define u^{t^m} inductively for $m > n$.

Finally we use linearity to define $u^{f(t)}$ for $f(t) \in \mathbb{Z}[t]/p(t)$ as:

$$u^{f(t)} = (u^{t^{n-1}})^{f_{n-1}} \dots (u^t)^{f_1} u^{f_0}$$

where $f(t) = f_{n-1}t^{n-1} + \ldots + f_1 t + f_0$ for some $f_i \in \mathbb{Z}$.

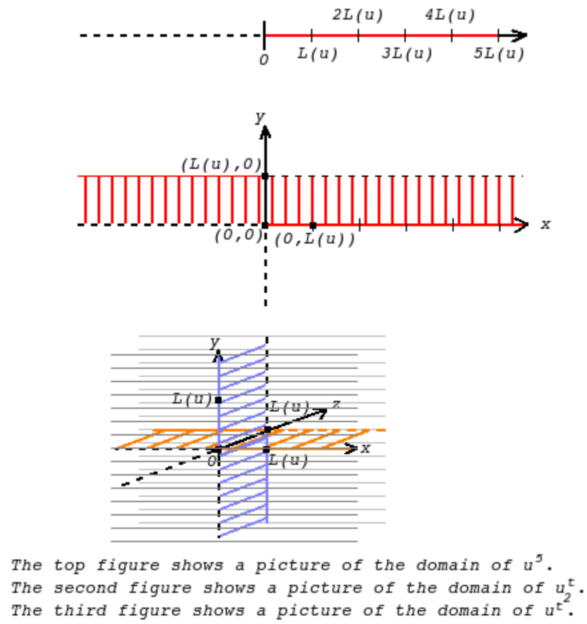

Figure 3.1: Visualisations of the $\mathbb{Z}[t]/p(t)$-exponentiation of u.

3.2 Some Results of Exponentiation

Now I need to prove that this construction satisfies Lyndon's axioms. I start by proving that for v of one dimension, $[v^{f(t)}, v] = \mathbf{1_G}$. I then prove that for u, v of one dimension and such that $u = c^{-1}vc$, $u^{f(t)} = c^{-1}v^{f(t)}c$. This then leads me to being able to prove that $u^{f(t)} = v^{g(t)}$ implies $[u, v] = \mathbf{1_G}$ and hence $[u, v] = \mathbf{1_G}$ implies $[u^{f(t)}, v] = \mathbf{1_G}$. It is then easy to prove that the axioms $(E1) - (E3)$ are satisfied. This follows what Myasnikov, Remeslennikov and Serbin did in [7].

First I need to define a subgroup of $\mathcal{RF}(G, \Lambda)$ that will be useful in the rest of the thesis.

Definition 3.2.1. *Let H_1 be the set of all cyclically decomposable elements in $\mathcal{RF}(G, \mathbb{R}^n)$ that are one dimensional.*

This group is isomorphic to the group $\mathcal{RF}(G)$ in [2] since any element of H_1 has one dimensional length and can therefore can be seen as an element of $\mathcal{RF}(G)$, whilst the whole of $\mathcal{RF}(G)$ can be embedded into $\mathcal{RF}(G, \mathbb{R}^n)$ and all of its elements are one dimensional and cyclically decomposable and hence would be embedded into H_1.

The Lemma below needs to be proved in full here, which is not done in the paper [7]

Lemma 3.2.1. *Let $v \in H_1$ have $L(v) > 0$, and $f(t) \in \mathbb{Z}[t]/p(t)$. Then we have that $v^{f(t)} \in \mathcal{CRF}(G, \mathbb{R}^n)$, $[v^{f(t)}, v] = \mathbf{1_G}$ and $L(v^{f(t)}) = f(t)L(v)$.*

Proof. We have seen that $v^{f(t)}$ begins with the initial segment, say v_1, of v and ends with the terminal segment, say v_2, of v. Since $v \in \mathcal{CRF}(G, \mathbb{R}^n)$ this means that $\varepsilon_0(v_2, v_1) = 0$ and therefore, by Lemma 2.3.3, $\varepsilon_0(v^{f(t)}, v^{f(t)}) = 0$ and hence $v^{f(t)} \in \mathcal{CDF}(\mathcal{G}, \mathbb{R}^\backslash)$.

Next, it can be seen that $v^{f(t)}v = v^{f(t)+1} = v^{1+f(t)} = vv^{f(t)}$. Hence $[v^{f(t)}, v] = \mathbf{1_G}$.

Finally, there is no cancellation between the copies of v in $v^{f(t)}$, so the length of $L(v^{f(t)})$ must be the combined length of all of them, which is, from Definition 3.1.2, $f(t)L(v)$. □

The next Lemma follows Lemma 5.2 of [7].

Lemma 3.2.2. *Let $u, v \in H_1$ be such that $u = v_1 \circ v_2$ and $v = v_2 \circ v_1$, so $u = v_2^{-1} v v_2$. Then for all $f(t) \in \mathbb{Z}[t]/p(t)$*

$$u^{f(t)} = v_2^{-1} v^{f(t)} v_2.$$

Proof. First we prove that this works for $f(t) = t$, and then we extend it to the whole of $\mathbb{Z}[t]/p(t)$:

Since $v_2 v_1 = v_2 \circ v_1$, we have that

$$(v_1 \circ v_2)^t(x) = (v_2 \circ v_1)^t(x + L(v_2))$$

for $0 < x < L(u^t) - L(u) = L(u^{t-1})$, since u is cyclically reduced.

Now

$$(v_2 \circ v_1)^t(x + L(v_2)) = (v_2^{-1}(v_2 \circ v_1)^t v_2)(x)$$

for $0 < x < L(u^{t-1})$ as conjugating by v_2 removes the initial segment of length $L(v_2)$ and can only effect the last segment of length $L(v_2) < L(u)$, which is not in the given range.

Hence

$$(v_1 \circ v_2)^t(x) = (v_2^{-1}(v_2 \circ v_1)^t v_2)(x)$$

for $0 < x < L(u^{t-1})$.

For $x = 0$ and $L(u^{t-1}) \leq x \leq L(u^t)$, we get that

$$\begin{aligned}(v_1 \circ v_2)^t(x) =& \begin{cases} v_1(0) & x = 0 \\ v_2(L(v_1))v_1(0) & x = L(u^{t-1}) \\ v_1(x - L(u^{t-1})) & L(u^{t-1}) < x < L(u^{t-1}) + L(v_1) \\ v_1(L(v_1))v_2(0) & x = L(u^{t-1}) + L(v_1) \\ v_2(x - L(u^{t-1}) + L(v_1)) & L(u^{t-1}) + L(v_1) < x < L(u^t) \\ v_2(L(v_2)) & x = L(u^t) \end{cases} \\ =& v_2^{-1}(v_2 \circ v_1)^t v_2(x)\end{aligned}$$

So we get

$$(v_1 \circ v_2)^t(x) = (v_2^{-1}(v_2 \circ v_1)^t v_2)(x) \qquad 0 \leq x \leq L(u^t)$$

and hence $(v_1 \circ v_2)^t = v_2^{-1}(v_2 \circ v_1)^t v_2$ as required.

Now, u^t and v^t are still of the form $u_1 \circ u_2$ and $u_2 \circ u_1$, so we can use this construction to prove this works for $f(t) = t^k$ for $1 \leq k < n$.

Finally, extend this to $f(t) = f_0 + f_1 t + \ldots + f_{n-1} t^{n-1}$ by the following:

$$
\begin{aligned}
u^{f(t)} =& u^{f_0} (u^t)^{f_1} \ldots (u^{t^{n-1}})^{f_{n-1}} \\
=& v_2^{-1} v^{f_0} v_2 v_2^{-1} (v^t)^{f_1} v_2 \ldots v_2^{-1} (v^{t^{n-1}})^{f_{n-1}} v_2 \\
=& v_2^{-1} v^{f_0} (v^t)^{f_1} \ldots (v^{t^{n-1}})^{f_{n-1}} v_2 \\
=& v_2^{-1} v^{f(t)} v_2
\end{aligned}
$$

as required. □

Corollary 3.2.1. *Let $u, v \in H_1$ be such that $u = c^{-1} v c$ for some $c \in \mathcal{RF}(G, \mathbb{R}^n)$. Then we have that for all $f(t) \in \mathbb{Z}[t]/p(t)$*

$$u^{f(t)} = c^{-1} v^{f(t)} c.$$

Proof. Since u and v are cyclically reduced, we get that $u = v_1 \circ v_2 = v_1 v_2$, $v = v_2 \circ v_1$ and $c = v_2$, as in Lemma 3.2.2. □

The statement of the next Lemma follows Lemma 5.3 in [7], though the proof differs considerably..

Lemma 3.2.3. *let $u, v \in H_1$ be such that there exists $f(t), g(t) \in \mathbb{Z}[t]/p(t) \setminus \{0\}$ such that $u^{f(t)} = v^{g(t)}$. Then $[u, v]$ is defined and equal to $\mathbf{1_G}$.*

Proof. By Lemma 3.2.1

$$[u, v^{g(t)}] = [u, u^{f(t)}] = \mathbf{1_G} = [v, v^{g(t)}] = [v, u^{f(t)}]$$

If $L(u) = L(v)$ then $u^{f(t)} = v^{g(t)}$ implies the initial segment of length $L(u)$ of $u^{f(t)}$ coincides with the initial segment of length $L(v)$ of $v^{g(t)}$, and therefore $u = v^{\pm 1}$. Hence

$$
\begin{aligned}
[u, v] =& \begin{cases} [u, u] & u = v \\ [u, u^{-1}] & u = v^{-1} \end{cases} \\
=& \mathbf{1_G}
\end{aligned}
$$

Therefore assume that $L(u) < L(v)$ (and use symmetry for the argument where $L(v) < L(u)$).

Without loss of generality assume that $f(t) > 0$.

There are two cases:

$$(i)\ g(t) > 0$$
$$(ii)\ g(t) < 0$$

Case (i) $g(t) > 0$

Since $u^{f(t)} = v^{g(t)}$, u and v have the same initial and terminal segments. They are also cyclically reduced and $uv^{g(t)}$ is defined, so uv and vu are both defined with $uv = u \circ v$ and $vu = v \circ u$.

Now, we know that $u, v \in H_1$, but $H_1 \cong \mathcal{RF}(G)$ from [2], which can be embedded as a group into the set $\mathcal{RF}(G, \mathbb{R}^n)$ with the partial reduced multiplication (as shown after Definition 3.2.1). Therefore $[u, v]$ is defined.

We have:

$$uv^{g(t)} = u \circ v^{g(t)} = v^{g(t)} \circ u = v^{g(t)}u$$

From this we see that the initial segments of $uv^{g(t)}$ and $v^{g(t)}u$ of length $L(v)$ are equal and the terminal segments of $uv^{g(t)}$ and $v^{g(t)}u$ are also equal. Hence

$$v = u \circ v_1 = v_2 \circ u$$

for some v_1, v_2 such that $L(v_1) = L(v_2)$. Also $v_1 = v_2$ since, if $g(t) > 1$, the terminal segments of length $L(u)+L(v)$ of $v \circ v = v_2 \circ u \circ v_2 \circ u$ and $v \circ u = u \circ v_1 \circ u$ are equal, whilst if $g(t) = 1$, then $v = u^{f(t)}$ and by Lemma 3.2.1, $[u, v] = \mathbf{1_G}$

So $v^{-1} = u^{-1} \circ v_1^{-1} = v_1^{-1} \circ u^{-1}$, and we find that

$$\begin{aligned}[u, v] =& u(v_1 \circ u)u^{-1}(v_1^{-1} \circ u^{-1}) \\ =& uv_1(v_1^{-1} \circ u^{-1}) \\ =& uu^{-1} \\ =& \mathbf{1_G}\end{aligned}$$

as required.

Case (ii) $g(t) < 0$

Once again $[u, v]$ is defined as u and v are one dimensional.

As u and v are cyclically reduced and $u^{f(t)} = v^{g(t)}$, $u = a \circ b \circ c$ and $v^{-1} = a \circ d \circ c$, where $L(c) > 0$, so $v = c^{-1} \circ d^{-1} \circ a^{-1}$ and

$$uv = (a \circ b \circ c)(c^{-1} \circ d^{-1} \circ a^{-1}) \neq u \circ v$$

Therefore $uv = u_4 \circ v_4$ where $u = u_4 \circ u_5$, $v = u_5^{-1} \circ v_4$ and $v^{-1} = v_4^{-1} \circ u_5$. Note that u and v^{-1} are cyclically reduced with the same initial and terminal segments, so $uv^{-1} = u \circ v^{-1}$ and $v^{-1}u = v^{-1} \circ u$.

From this we see that

$$uv^{g(t)} = (u_4 \circ u_5)(v_4^{-1} \circ u_5)^{-g(t)} = (u_4 \circ u_5) \circ (v_4^{-1} \circ u_5)^{-g(t)}$$

and

$$v^{g(t)}u = (v_4^{-1} \circ u_5)^{-g(t)}(u_4 \circ u_5) = (v_4^{-1} \circ u_5)^{-g(t)} \circ (u_4 \circ u_5)$$

since $g(t) < 0$.

Now, the initial segments of $uv^{g(t)}$ and $v^{g(t)}u$ of length $L(v)$ are equal as are the terminal segments of length $L(v)$, hence

$$v^{-1} = u \circ v_5 = v_6 \circ u$$

where $L(v_5) = L(v_6)$, and as above, $v_5 = v_6$.

Therefore $v^{-1} = u \circ v_5 = v_5 \circ u$ and $v = v_5^{-1} \circ u^{-1} = u^{-1} \circ v_5^{-1}$.

So

$$\begin{aligned} [u, v] =& u(u^{-1} \circ v_5^{-1})u^{-1}(u \circ v_5) \\ =& v_5^{-1} v_5 \\ =& \mathbf{1_G} \end{aligned}$$

as required $\square$

Again, here the statement of this Lemma is the same as Lemma 5.4 in [7], but the proof differs greatly.

Lemma 3.2.4. *Let $u, v \in H_1$ be such that $[u, v] = \mathbf{1_G}$. Then $[u^{f(t)}, v] = \mathbf{1_G}$ for all $f(t) \in \mathbb{Z}[t]/p(t)$, provided $[u^{f(t)}, v]$ is defined.*

Proof.

$$[u, v] = \mathbf{1_G}, \text{ so } uvu^{-1}v^{-1} = \mathbf{1_G}$$

which implies $u = vuv^{-1}$.

Let $u = c^{-1} \circ u_1 \circ c$ and $v = d^{-1} \circ v_1 \circ d$. Then set

$$\begin{aligned} v' =& (cd^{-1}) \circ v_1 \circ (dc^{-1}) \\ =& cvc^{-1} \end{aligned}$$

Now, $u = vuv^{-1}$ implies $u_1 = cvc^{-1} \circ u_1 \circ cv^{-1}c^{-1} = v'u_1v'^{-1}$, but u_1 is cyclically reduced, so by Corollary 3.2.1, $u_1^{f(t)} = v'u_1^{f(t)}v'^{-1}$ and hence $[u_1^{f(t)}, v']$ is defined. Since u_1 and v' are conjugates of u and v respectively, we have that $[u^{f(t)}, v]$ is also defined and equal to $\mathbf{1_G}$, so we are done. □

We are now in a position to prove the Theorem below, as the authors do in [7] in Theorem 5.5:

Theorem 3.2.1. *The $\mathbb{Z}[t]/p(t)$-exponentiation function $exp : (u, f(t)) \longrightarrow u^{f(t)}$ defined above satisfies:*

(E1) $u^1 = u, (u^{fg}) = (u^f)^g, u^{f+g} = u^f u^g$.

(E2) $(v^{-1}uv)^f = v^{-1}u^f v$ *if* $[u, v] = \mathbf{1_G}$ *or* $u = v \circ w$ *or* $u = w^\alpha, v = w^\beta$ *for* $w \in \mathcal{CDF}(G, \mathbb{R}^n)$ *and* $\alpha, \beta \in \mathbb{Z}[t]/p(t)$.

(E3) $[u, v] = \mathbf{1_G}$ *and* $u = w^\alpha, v = w^\beta \Longrightarrow (uv)^f = u^f v^f$ *for* $w \in \mathcal{CDF}(G, \mathbb{R}^n)$ *and* $\alpha, \beta \in \mathbb{Z}[t]/p(t)$

for $u, v \in H_1$.

Proof. (E1):

$u^1 = u$ by definition.

Let $f(t) = f_{n-1}t^{n-1} + \ldots + f_1 t + f_0$ and $g(t) = g_{n-1}t^{n-1} + \ldots + g_1 t + g_0$ be in $\mathbb{Z}[t]/p(t)$, and let $u = c^{-1} \circ u_1^k \circ c$ for some $u_1 \in \mathcal{CRF}(G, \mathbb{R}^n)$, $c \in \mathcal{RF}(G, \mathbb{R}^n)$ and $k \in \mathbb{N} \setminus \{0\}$.

Then

$$\begin{aligned}
u^{f+g} =& (c^{-1} \circ u_1^k \circ c)^{f_0+f_1t+\ldots+f_{n-1}t^{n-1}+g_0+g_1t+\ldots+g_{n-1}t^{n-1}} \\
=& c^{-1} \circ (u_1^{f_0+f_1t+\ldots+f_{n-1}t^{n-1}+g_0+g_1t+\ldots+g_{n-1}t^{n-1}})^k \circ c \\
=& c^{-1} \circ (u_1^{f_0+f_1t+\ldots+f_{n-1}t^{n-1}})^k \circ cc^{-1} \circ (u_1^{g_0+g_1t+\ldots+g_{n-1}t^{n-1}})^k \circ c \\
=& u^f u^g
\end{aligned}$$

Now note that $u^{(fg)} = (u^f)^g$ by definition, and we have proved (E1)

(E2):

If $[u, v] = \mathbf{1_G}$ we have that $u = v^{-1}uv$ and, by Lemma 3.2.4, we have that $u^f = v^{-1}u^f v$ as required.

If $u = v \circ w$ then $(v^{-1}uv)^f = (wv)^f$ and $v^{-1}u^f v = v^{-1}(v \circ w)^f v$, which are the conditions of Lemma 3.2.2, so the result follows.

If $u = w^\alpha$ and $v = w^\beta$, then α and β must be constant functions, since $u, v \in H_1$, and so the result follows from the definition of exponentiation.

(E3) Again, α and β are finite, and

$$(uv)^f = (w^\alpha w^\beta)^f = (w^{\alpha+\beta})^f = w^{(\alpha+\beta)f} = w^{\alpha f} w^{\beta f} = (w^\alpha)^f (w^\beta)^f = u^f v^f$$

$\square$

Chapter 4

A Use For Exponentiation

4.1 A Use for Exponentiation

In this section I use exponentiation to find a pregroup within $\mathcal{CDF}(G, \mathbb{R}^n)$. I start by finding a group with certain properties, analagous to properties defined in [7], within the set $\mathcal{CDF}(G, \mathbb{R}^n)$. I then define a *Lyndon's set*, R, of a group and an *R-form*, which I then use to define a pregroup, $P(H_1, R)$, as the authors do in in [7].

Here are some definitions that are analagous to the definitions at the start of section 6 in [7]. Note that when I say $[u, v] \neq \mathbf{1_G}$, I mean that when $[u, v]$ is defined it is not equal to $\mathbf{1_G}$.

Definition 4.1.1. *Let $H \leq \mathcal{CDF}(G, \mathbb{R}^n)$. Then H is* subword-closed *if for each $f : [0, \alpha] \to G$ in H, $f|_{[0,\beta]} \in H$ for all $0 \leq \beta \leq \alpha$.*

An example of a group that is not subword-closed is the group

$$H = \{g \in \mathcal{CDF}(G, \mathbb{R}^n) | L(g) \in \mathbb{Z}\}.$$

In such a group, for any $f \in H$ such that $L(f) \neq 0$, $f|_{[0, \frac{1}{2}]} \notin H$.

On the other hand a group that is subword-closed is H_1, as defined in the previous chapter. This is because, if we choose some $\alpha \in \mathbb{R}$, then for all f such that $L(f) = (0, \ldots, 0, \alpha)$, $f|_{[0,(0,\ldots,0,\beta)]} \in H_1$ for all $0 < \beta < \alpha$ since $\beta \in \mathbb{R}$.

Definition 4.1.2. *Let $a \circ_\delta b$ mean that $\varepsilon_0(a, b) \leq \delta$.*

Definition 4.1.3. *Let $u = c^{-1} \circ u_1 \circ c, v = d^{-1} \circ v_1 \circ d \in \mathcal{CDF}(G, \mathbb{R}^n)$ and $\delta = \delta(u, v) = \max\{L(c), L(d)\}$. u and v are* separated *if $u^m v^k$ is defined for any $k, m \in \mathbb{N}$ and there exists $r = r(u, v) \in \mathbb{N}$ such that for all $m, k > r$*

$$u^m v^k = u^{m-r} \circ_\delta (u^r v^r) \circ_\delta v^{k-r}.$$

For example, if $u^2 = v^{-1}$, $u^m v^k = u^m u^{-2k} = u^{m-2k}$, but for all r, if $m, 2k > r$, $c(u^m, v^{-2k} > (r+1)L(u)$ so that $c(u^{m-r}, u^{-2k+r}) > L(u) > \delta$ and u and v are not separated.

Also, if $n = 2$ and $L(u) = (3, 0)$,

$$u = \begin{cases} a & 0 \leq (x, y)|x < 2 \\ b & L(u) \geq (x, y)|x \geq 2 \end{cases}$$

and

$$v = \begin{cases} b^{-1} & 0 \leq (x, y)|x \leq 1 \\ c & L(v) \geq (x, y)|x > 1. \end{cases}$$

Then uv doesn't exist, so u and v aren't separated.

If however we have that $u(L(u)) \neq v(0)^{-1}$, we have that $uv = u \circ v$ and therefore

$$\begin{aligned} u^m v^k =& u^m \circ v^k \\ =& u^{m-r} \circ_\delta (u^r v^r) \circ_\delta v^{k-r}. \end{aligned}$$

for $r = 0$ for all $m, k \in \mathbb{N}\backslash\{0\}$. Hence u and v are separated.

Definition 4.1.4. *A subset $M \subseteq \mathcal{CDF}(G, \mathbb{R}^n)$ is an* S-set *if any two non-commuting elements of M with cyclic centralisers are separated. If M is also a group, it is an* S-subgroup.

The second example of non-separated elements shows us that we have elements in $\mathcal{CDF}(G, \mathbb{R}^n)$ that are not separated.

Recall that

$$H_1 = \{v \in \mathcal{CDF}(G, \mathbb{R}^n) \text{ such that } v \text{ is of one dimension}\} \cong \mathcal{RF}(G).$$

Lemma 4.1.1. *H_1 is an S-subgroup of $\mathcal{CDF}(G, \mathbb{R}^n)$.*

Proof. First, since H_1 is a group, if it is an S-set of $\mathcal{CDF}(G, \mathbb{R}^n)$, it must be an S-subgroup of $\mathcal{CDF}(G, \mathbb{R}^n)$. Now we prove that it is an S-set by assuming that $[u, v] \neq \mathbf{1_G}$ for some u and v that are not separated, to find a contradiction.

Let $[u, v] \neq \mathbf{1_G}$ where $u = c^{-1} \circ u_1 \circ c$, $v = d^{-1} \circ v_1 \circ d$ with u_1, v_1 cyclically reduced.

First let neither u_1 or v_1 be constant functions. Then we can assume that u_1, v_1 are not proper powers and that $L(c) \geq L(d)$.

Note that $u^m = c^{-1} \circ u_1^m \circ c$ and $v^k = d^{-1} \circ v_1^k \circ d$ by Corollary 3.2.1.

Assume that u and v are not separated, so that for all $M > 0$, $r \in \mathbb{N}$, there exists $m = m(M, r) > r$ and $k = k(M, r) > r$ such that.

$$c(u^{-m}, v^k) > M \qquad (*)$$

Let

$$M > L(u) + L(v)$$

then $c = c_1 \circ d$ for some $c_1 \in \mathcal{RF}(G, \mathbb{R}^n)$.

Note that if $c(u^m, v^k) > M$, then $c(u^{m'}, v^{k'}) > M$ for all $k' \geq k$, $m' \geq m$.

If $L(c_1) \geq L(v_1)$ let $c_1 = c_2 \circ v_1^{-l}$ for some $l \in \mathbb{N} \setminus \{0\}$, $c_2 \in \mathcal{RF}(G, \mathbb{R}^n)$ with $L(c_2) < L(v_1)$.

If $L(c_1) < L(v_1)$ take $c_1 = c_2$.

Therefore we find that $c = c_2 \circ v_1^{-l} \circ d$ for some $l \in \mathbb{N} \cup \{0\}$ and $c_2 \in \mathcal{RF}(G, \mathbb{R}^n)$ with $L(c_2) < L(v_1)$.

There are three cases:

$$\begin{aligned} (i) \quad & L(c_2) + L(u_1) = L(v_1) \\ (ii) \quad & L(c_2) + L(u_1) > L(v_1) \\ (iii) \quad & L(c_2) + L(u_1) < L(v_1) \end{aligned}$$

Case (i) : $\quad L(c_2) + L(u_1) = L(v_1)$.

$$\begin{aligned} u^m v^k =& (c^{-1} \circ u_1^m \circ c_2 \circ v_1^{-l} \circ d)(d^{-1} \circ v_1^k \circ d) \\ =& (c^{-1} \circ u_1^m \circ c_2)(v_1^{k-l} \circ d) \end{aligned}$$

so by $(*)$, letting $M > L(u) + L(v)$, we find that $v_1 = c_2^{-1} \circ u_1^{-1}$ and

$$u_1 = \begin{cases} c_{22} & L(u_1) < L(c_2),\ c_2 = c_{21} \circ c_{22}^s, L(c_{21}) < L(c_{22}) \\ c_3 \circ c_2^s & L(u_1) \geq L(c_2), L(c_3) < L(c_2) \end{cases}$$

where $c_{21}, c_{22}, c_3 \in \mathcal{RF}(G, \mathbb{R}^n)$.

If $u_1 = c_{22}$, then $v_1 = u_1^{-s} \circ c_{21}^{-1} \circ c_{22}^{-1}$. By $(*)$, letting $M > (1+s)L(u) + L(v)$,

$$\begin{aligned} u^m v^k =& (c^{-1} \circ u_1^m \circ c_{21} \circ u_1^s)((u_1^{-s} \circ c_{21}^{-1} \circ u_1^{-1})^{k-l} \circ d) \\ =& (c^{-1} \circ u_1^m)(u_1^{-1} \circ (u_1^{-s} \circ c_{21}^{-1} \circ u_1^{-1})^{k-l-1} \circ d) \\ =& (c^{-1} \circ u_1^{m-1-s})(c_{21}^{-1} \circ u_1^{-1} \circ (u_1^{-s} \circ c_{21}^{-1} \circ u_1^{-1})^{k-l-2} \circ d) \end{aligned}$$

But $L(c_{21}) < L(u_1)$, so $u_1 = c_{23} \circ c_{21}^r$ with $L(c_{23}) < L(c_{21})$, and by $(*)$, letting $M > (3+s)L(u) + L(v)$,

$$\begin{aligned} u^m v^k =& (c^{-1} \circ u_1^{m-2-s} \circ c_{23} \circ c_{21}^r)(c_{21}^- 1 \circ c_{21}^{-1} \circ c_{32}^{-1} \circ v_1^{k-l-2} \circ d) \\ =& (c^{-1} \circ u_1^{m-3-s} \circ c_{23} \circ c_{21}^r \circ c_{23})(c_{21}^{-1} \circ c_{23}^{-1} \circ v_1^{k-l-2} \circ d) \end{aligned}$$

So the terminal segments of $c^{-1} \circ u_1^{m-3-s} \circ c_{23} \circ c_{21} \circ c_{23}$ and $d^{-1} \circ v_1^{2+l-k} \circ c_{23} \circ c_{21}$ of length $L(c_{21}) + L(c_{23})$ are equal, and hence $c_{21} \circ c_{23} = c_{23} \circ c_{21}$, which means

$$[c_{21}, c_{23}] = \mathbf{1_G}$$

But then

$$\begin{aligned}[u_1, v_1] =&(c_{23} \circ c_{21}^{r})((c_{21}^{-r} \circ c_{23}^{-1})^{s} \circ c_{21}^{-1-r} \circ c_{23}^{-1})\\ &(c_{21}^{-r} \circ c_{23}^{-1})(c_{23} \circ c_{21}^{r+1} \circ (c_{23} \circ c_{21}^{r})^{s})\\ =&\mathbf{1_G}\end{aligned}$$

If $u_1 = c_3 \circ c_2^s$, then by $(*)$, letting $M > (3 + s)L(u) + L(v) \geq 2L(u) + L(v)$,

$$\begin{aligned}u^m v^k =&(c^{-1} \circ (c_3 \circ c_2^s)^m \circ c_2)((c_2^{-1} \circ (c_2^{-s} \circ c_3^{-1}))^{k-l} \circ d)\\ =&(c^{-1} \circ (c_3 \circ c_2^s)^{m-1})((c_2^{-s-1} \circ c_3^{-1})^{k-l-1} \circ d)\\ =&(c^{-1} \circ (c_3 \circ c_2^s)^{m-2} \circ c_3)(c_2^{-1} \circ c_3^{-1} \circ (c_2^{-1-s} \circ c_3^{-1})^{k-l-2} \circ d)\end{aligned}$$

and the terminal segments of $c^{-1} \circ (c_3 \circ c_2^s)^{m-2} \circ c_3$ and $d^{-1} \circ (c_3 \circ c_2^{s+1})^{k-l-2} \circ c_3 \circ c_2$ are equal, so $c_2 \circ c_3 = c_3 \circ c_2$ and therefore

$$[c_2, c_3] = \mathbf{1_G}$$

But then

$$\begin{aligned}[u_1, v_1] =&(c_3 \circ c_2^s)(c_2^{-1-s} \circ c_3^{-1})(c_2^{-s} \circ c_3^{-1})(c_3 \circ c_2^{s+1})\\ =&\mathbf{1_G}\end{aligned}$$

So if $L(c_2) + L(u_1) = L(v_1)$, we get that $v_1 = c_2^{-1} \circ u_1^{-1}$ and again $[u_1, v_1] = \mathbf{1_G}$.

Now

$$\begin{aligned}[u_1, v_1] = \mathbf{1_G} =&u_1 v_1 u_1^{-1} v_1^{-1}\\ =&u_1(c_2^{-1} \circ u_1^{-1})u_1^{-1}(u_1 \circ c_2)\\ =&u_1(c_2^{-1} \circ u_1^{-1})c_2\\ =&[u_1, c_2^{-1}]\end{aligned}$$

so $[u_1, c_2^{-1}] = \mathbf{1_G}$, which implies that

$$\begin{aligned}u =&d^{-1} \circ c_2^{-1} \circ u_1 \circ c_2 \circ d\\ =&d^{-1} \circ u_1 \circ c_2^{-1} \circ c_2 \circ d\\ =&d^{-1} \circ u_1 \circ d\end{aligned}$$

and hence

$$
\begin{aligned}
[u,v] =& uvu^{-1}v^{-1} \\
=& (d^{-1} \circ u_1 \circ d)(d^{-1} \circ v_1 \circ d)(d^{-1} \circ u_1^{-1} \circ d)(d^{-1} \circ v_1^{-1} \circ d) \\
=& (d^{-1} \circ u_1)v_1 u_1^{-1}(v_1^{-1} \circ d) \\
=& (d^{-1} \circ u_1)u_1^{-1}v_1(v_1^{-1} \circ d) \\
=& d^{-1}d \\
=& \mathbf{1_G}
\end{aligned}
$$

Contradiction.

Case (ii) : $\quad L(c_2) + L(u_1) > L(v_1)$.

Again

$$u^m v^k = (c^{-1} \circ u_1^m \circ c_2)(v_1^{k-l} \circ d)$$

and by $(*)$, letting $M > L(u) + L(v)$, the initial segment of v_1^{k-l} is equal to the initial segment of $c_2^{-1} \circ u_1^{-m}$, so we get

$$
\begin{aligned}
v_1^s =& c_2^{-1} \circ u_5^{-1} \\
u_1 =& u_4 \circ u_5 = u_4 \circ v_1^{-s+1} \circ u_3 \\
v_1 =& c_2^{-1} \circ u_3^{-1} = u_4^{-1} \circ v_2
\end{aligned}
$$

For some $u_3, u_4, u_5, v_2 \in \mathcal{RF}(G, \mathbb{R}^n)$ and some $s \in \mathbb{N} \cup \{0\}$.

So by $(*)$, letting $M > 2L(u) + (s+2)L(v)$,

$$
\begin{aligned}
u^m v^k =& (u_1^{m-1} \circ u_1 \circ c_2)(v_1^{k-l}) \\
=& (u_1^{m-1} \circ (u_4 \circ v_1^{-s+1} \circ u_3) \circ c_2)(c_2^{-1} \circ u_3^{-1})v_1^{k-l-1} \\
=& (u_1^{m-1} \circ u_4)v_1^{k-l-s} \\
=& (u_1^{m-1} \circ u_4)(u_4^{-1} \circ v_2)^{k-l-s} \\
=& u_1^{m-1}(v_2 \circ u_4^{-1})^{k-l-s-1} \circ v_2
\end{aligned}
$$

Hence

$$c(u_1^{-m+1}, (v_2 \circ u_4^{-1})^{k-l-s-1}) > c(u^m, v^k) - L(d) - L(v_1^{l+s+1}) - L(u_4)$$

So for m' and k' large enough, we find that

$$\begin{aligned}
c(u_1^{-m'}, (v_2 \circ u_4^{-1})^{k'}) >& M - L(d) - L(v_1^{l+s+1}) - L(u_4)\\
=& 2L(c_2) + 3L(d) + (2l + s + 2)L(v_1) + L(u_1) - L(v_1^{l+s+1})\\
>& L(u_1) + L(v_1)
\end{aligned}$$

Therefore, since $L(v_2 \circ u_4^{-1}) = L(v_1)$, $[u_1, v_2 \circ u_4^{-1}] = \mathbf{1_G}$ by Lemma 2.4.4.

Since u_1 and v_1 are not proper powers, and therefore neither is $v_2 \circ u_4^{-1}$, we have that

$$\begin{aligned}
u_1 =& \begin{cases} u_4 \circ v_2^{-1} \\ v_2 \circ u_4^{-1} \end{cases}\\
=& u_4 \circ u_5
\end{aligned}$$

If $u_1 = v_2 \circ u_4^{-1} = u_4 \circ u_5$ and $L(u_4) > 0$, then

$$\begin{aligned}
L(u_1^2) =& L((v_2 \circ u_4^{-1})(u_4 \circ u_5))\\
=& v_2 u_5\\
<& 2L(u_1)
\end{aligned}$$

and hence u_1 is not cyclically reduced, contradicting the definition of u_1.

If $u_1 = v_2 \circ u_4^{-1} = u_4 \circ u_5$ and $L(u_4) = 0$, then $L(v_1^s) = L(u_1) + L(c_2)$, but $L(c_2) < L(v_1) = L(u_1)$ so $s = 1$, $L(c_2) = 0$ and we are in Case (i).

So

$$\begin{aligned}
u_1 =& u_4 \circ v_2^{-1}\\
=& u_4 \circ u_5
\end{aligned}$$

and hence $u_5 = v_2^{-1}$.

Now $v_1^s = c_2^{-1} \circ u_5^{-1}$, with $L(c_2) < L(v_1)$ and $L(v_1) = L(u_1) = L(u_4) + L(u_5)$, implies that $s = 1$ and

$$\begin{aligned} v_1 =& c_2^{-1} \circ u_5^{-1} \\ =& u_4^{-1} \circ v_2 \\ =& u_4^{-1} \circ u_5^{-1} \end{aligned}$$

So by Proposition 2.1.1 and Definition 2.2.3, $c_2 = u_4$.

But then

$$\begin{aligned} u =& d^{-1} \circ v_1^l \circ c_2^{-1} \circ u_1 \circ c_2 \circ v_1^{-l} \circ d \\ =& (d^{-1} \circ v_1^l \circ c_2^{-1})(c_2 \circ u_5 \circ c_2) \circ (v_1^{-l} \circ d) \\ =& d^{-1} \circ v_1^l (u_5 \circ c_2) \circ (v_1^{-l} \circ d) \\ =& d^{-1} \circ (c_2^{-1} \circ u_5^{-1})^l (u_5 \circ c_2)(c_2^{-1} \circ u_5^{-1})^{-l} \circ d \\ =& d^{-1}(c_2^{-1} \circ u_5^{-1})^{l-1}(c_2^{-1} \circ u_5^{-1})^{-l} \circ d \\ =& d^{-1} \circ (c_2^{-1} \circ u_5^{-1})^{-1} \circ d \\ =& d^{-1} \circ v_1^{-1} \circ d \\ =& v^{-1} \end{aligned}$$

and hence $[u, v] = [v^{-1}, v] = \mathbf{1_G}$. Contradiction with the initial assumption.

Case (iii) : $\quad L(c_2) + L(u_1) < L(v_1)$.

In this case, $v_1 = c_2^{-1} \circ u_1^{-s} \circ u_2^{-1}$, with $s \geq 0$ and $u_1 = u_3 \circ u_2$.

So by $(*)$, letting $M > L(u) + L(v) + L(u_1^{s+1})$,

$$\begin{aligned} u^m v^k =& (c^{-1} \circ (u_3 \circ u_2)^m \circ c_2)((c_2^{-1} \circ (u_2^{-1} \circ u_3^{-1})^{-1} \circ u_2^{-1})^{k-l} \circ d) \\ =& (c^{-1} \circ (u_3 \circ u_2)^{m-s-1} \circ u_3)(v_1^{k-1-l} \circ d) \\ =& (c^{-1} \circ u_3 \circ (u_2 \circ u_3)^{m-s-1})(v_1^{k-1-l} \circ d) \end{aligned}$$

But

$$\begin{aligned} c((u_2 \circ u_3)^{m'}, v_1^{k'}) \geq& c(u^m, v^k) - L(c) - L(u^{s+1}) \\ >& L(u) + L(v) + L(u_1^{s+1}) - L(c) - L(u^{s+1}) \\ =& 2L(c) + 2L(d) + L(u_1) + L(v_1) - L(c) \\ >& L(u_2 \circ u_3) + L(v_1) \end{aligned}$$

as $u_1 = u_3 \circ u_2$.

Since $u_2 \circ u_3$ is a cyclic permutation of cyclically reduced u_1, it is also cyclically reduced and therefore, by Lemma 2.4.4, $[u_2 \circ u_3, v_1] = \mathbf{1_G}$.

Again, neither $u_2 \circ u_3$ or v_1 are proper powers, so $v_1 \in \{u_2 \circ u_3, u_3^{-1} \circ u_2^{-1}\}$

If $v_1 = u_2 \circ u_3 = c_2^{-1} \circ (u_3 \circ u_2)^{-s} \circ u_2^{-1} = c_2^{-1} \circ (u_2^{-1} \circ u_3^{-1})^s \circ u_2^{-1}$, then $L(u_2) = 0$ since v_1 is cyclically reduced. But then $L(v_1) = L(u_3) = L(u_1)$, so $s = 1$, $L(c_2) = 0$ and we are back in Case (i).

So $v_1 = u_3^{-1} \circ u_2^{-1} = c_2^{-1} \circ u_1^{-s} \circ u_2^{-1}$, whilst $L(u_1) = L(v_1)$. This implies that either $L(c_2) = 0 = L(u_2)$ or $s = 0$.

If $L(c_2) = L(u_2) = 0$, $L(v_1) = L(u_1) + 0 = L(u_1) + L(c_2)$, and we are back in Case (i).

Therefore $s = 0$ and

$$v_1 = c_2^{-1} \circ u_2^{-1} = u_3^{-1} \circ u_2^{-1}$$

but then, by Proposition 2.1.1 and Definition 2.2.3, $c_2^{-1} = u_3^{-1}$.

Hence

$$\begin{aligned} u =& c^{-1} \circ u_1 \circ c \\ =& d^{-1} \circ v_1^l \circ c_2^{-1} \circ u_1 \circ c_2 \circ v_1^{-l} \circ d \\ =& d^{-1} \circ (u_3^{-1} \circ u_2^{-1})^l \circ u_3^{-1} \circ (u_3 \circ u_2) \circ u_3 \circ (u_2 \circ u_3)^l \circ d \\ =& d^{-1} \circ (u_3^{-1} \circ u_2^{-1})^{l-1} \circ u_3^{-1} \circ u_2^{-1} \circ u_3^{-1} \circ (u_3 \circ u_2) \circ u_3 \circ (u_2 \circ u_3)^l \circ d \end{aligned}$$

But then we must have that

$$(u_2^{-1} \circ u_3^{-1})(u_3 \circ u_2) = (u_2^{-1} \circ u_3^{-1}) \circ (u_3 \circ u_2)$$

and so, by the definition of ε_0, $L(u_2^{-1} \circ u_3^{-1}) = L(v_1) = L(u_1) = L(u_3) = 0$, which means that $L(u_1) + L(c_2) = 0 = L(v_1)$ and we are in Case (i) again.

Hence Case (iii) cannot occur either and we have a contradiction in all three cases.

Now let u_1 be a constant function. We know that

$$u^m v^k = (c^{-1} \circ u_1^m \circ c)(d^{-1} \circ v_1^k \circ d)$$

and we can assume that $L(c) \geq L(d)$, so that $c = c_1 \circ d$. But then

$$u^m v^k = (c^{-1} \circ u_1^m \circ c_1)(v_1^k \circ d).$$

Now there exists a $q \in \mathbb{N}\backslash\{0\}$ with $qL(v_1) < L(c_1) < (q+1)L(v_1)$, so letting $M > qL(v) + L(u)$, we must have $v_1^q = c_1^{-1} \circ c_2$ and

$$\begin{aligned} u^m v^k =&(c^{-1} \circ u_1^m \circ c_1)((c_1^{-1} \circ c_2)v_1^{k-q} \circ d) \\ =&(c^{-1} \circ u_1^m)((c_2 \circ v^{k-q} \circ d) \\ =&(c^{-1} \circ u_1^m)(c_2 \circ c_1^{-1} \circ c_2 \circ v_1^{k-2q} \circ d). \end{aligned}$$

This time, let $M > 2qL(v) + L(u)$ to see that $c_2 \circ c_1^{-1}$ is a constant function, and the letter it is constant on is the inverse of the letter that u_1 is constant on. This implies that $[u_1, v_1] = \mathbf{1_G}$ and that c_1 is constant on the same letter as u_1, hence $L(d) = L(c)$, $c = d$ and

$$\begin{aligned} [u, v] =&(c^{-1} \circ u_1 \circ c)(c^{-1} \circ v_1 \circ c)(c^{-1} \circ u_1^{-1} \circ c)(c^{-1} \circ v_1^{-1} \circ c) \\ =&c^{-1} \circ u_1 v_1 u_1^{-1} v_1^{-1} \circ c \\ =&c^{-1} c \\ =&\mathbf{1_G}. \end{aligned}$$

This is a contradiction also.

Similarly if v_1 is a constant function we end up with a contradiction.

Therefore our initial assumption, that $[u, v] \neq \mathbf{1_G}$, must be wrong and we have an S-subgroup, as required.

□

Now we can define a *Lyndon's set* as they do in [7]. We do this in two parts:

Definition 4.1.5. *Let $M \subseteq \mathcal{CDF}(G, \mathbb{R}^n)$. A set $R_M \subseteq \mathcal{CDF}(G, \mathbb{R}^n)$ is a* set of representatives of M *if R_M satisfies:*

(i) R_M does not contain proper powers.

(ii) For all $u, v \in R_M$, $u \neq v^{-1}$.

(iii) For each $u \in M$ there exists $v \in R_M$, $k \in \mathbb{Z}$, $c \in \mathcal{RF}(G, \mathbb{R}^n)$ and a cyclic permutation $\pi(v)$ of v such that $u = c^{-1} \circ \pi(v)^k \circ c$ where $v, c, k, \pi(v)$ are unique.

Notes:

(a) This does not require $R_M \subset M$.

(b) (iii) implies that every element of R_M is cyclically reduced.

For Myasnikov, Remeslennikov and Serbin show in [7], these sets do exist. In my situation the constant functions, f such that $f(x) = a$ for $L(f) \geq x > 0$ and $f(0) = 1_G$, have no maximal root, so I cannot have a aet of representatives with them involved. If however my M does not contain these elements, the set R_M can be found using the steps from just after Definition 6.6 in [7] as shown below:

First, for $f \in M$, since it is cyclically decomposable, $f = c \circ g \circ c^{-1}$ for some $c \in \mathcal{RF}(G, \mathbb{R}^n)$ and $g \in \mathcal{CRF}(G, \mathbb{R}^n)$. Let $R_1 = \{g | f = c \circ g \circ c^{-1}\}$.

Next for each $g \in R_1$, let $M_g = \{g' | g'$ is a cyclic permutation of g and $g^{-1}\}$. These sets form a disjoint union of R_1. Choose a single element $\bar{g}$ from each set M_g and let $R_2 = \{\bar{g} | g \in R_1\}$.

Finally, for $\bar{g} \in R_2$ let g^* be the unique maximal root of $\bar{g}$ and let the set $R_M = \{g^* | g \in R_2\}$.

For a group H, let the set

$$K(H) = \{v \in H | C_H(v) = \langle u \rangle\}$$

This is the set of all elements in H with cyclic centralisers. The elements of these sets do not contain the constant functions as constant functions do not have cyclic centralisers. Therefore a set of representatives does exist for $K(H)$.

Recall from Definition 1.2.1 that $c(f, g) = L(f) + L(g) - L(f^{-1}g)$. This function only exists when $\varepsilon_0(^{-1}, g)$ does as otherwise the element $f^{-1}g$ is not defined. In fact it can be proved that $\varepsilon_(f, g) = c(f, g^{-1})$.

Definition 4.1.6. *Let $H \leq \mathcal{CDF}(G, \mathbb{R}^n)$, then a* Lyndon's set *of H is a set $R = R_{K(H)}$ of representatives of $K(H)$ such that:*

(i) $R \subset H$.

(ii) For all $g \in H$, $u \in R$ and $\alpha \in \mathbb{Z}[t]/p(t)$, $c(u^\alpha, g)$ exists, and we have that $c(u^\alpha, g) < kL(u)$ for some $k \in \mathbb{N}$.

(iii) No word in H contains a subword u^α where $u \in R$ and $\alpha \in \mathbb{Z}[t]/p(t)$ with $deg(\alpha) > 0$.

Notes:

(a) If H is subword closed then $(ii) \implies (iii)$.

(b) From (ii) and (iii) we see that $u^\alpha g$ and gu^α are defined for all $g \in H$, $u \in R$ and $\alpha \in \mathbb{Z}[t]/p(t)$.

Now we must prove some results about Lyndon's sets and S-subgroups of $\mathcal{CDF}(G, \mathbb{R}^n)$. The Lemma below is analagous to Lemma 6.9 in [7], but the proof is different.

Lemma 4.1.2. *Let H be an S-subgroup of $\mathcal{CDF}(G, \mathbb{R}^n)$ and let R be a Lyndon's set of H. If $u, v \in R^{\pm 1}$ and $g \in H$ are such that either $[u, v] \neq \mathbf{1_G}$ or $[u, g] \neq \mathbf{1_G}$, then there exists $r \in \mathbb{N}$ such that for all $m, k > r$ the following holds:*

$$u^m g v^k = u^{m-r} \circ (u^r g v^r) \circ v^{k-r}.$$

Proof. Assume that the Lemma fails for some $u, v \in R^{\pm 1}$ and $g \in H$. Note that if the lemma fails for (u, g, v), then it also fails for $(u, g, v^l v)$ and hence for (u, gv^l, v), where $l \in \mathbb{Z}$. Since $v \in R^{\pm 1}$, there exists some $l \in \mathbb{Z}$ such that $(gv^l)v = (gv^l) \circ v$ and therefore we can assume that $gv = g \circ v$. Similarly we can find an $l \in \mathbb{Z}$ such that $u(u^l g) = u \circ (u^l g)$.

Now if g does not cancel completely in $u^l g$, we find that

$$\begin{aligned} u^m g v^k =& u^{m-l} \circ (u^l g) v^k \\ =& u^{m-l} \circ (u^l g v^l) \circ v^{k-l} \end{aligned}$$

for any $m, k > l$, so the lemma holds. Contradiction.

Therefore the g must cancel completely in $u^l g$ so that we get $g = u^{-j} \circ g_1$ for some $j > 0$, $L(g_1) < L(u)$.

There are two cases:

$$(a)\ [u, v] \neq \mathbf{1_G}$$
$$(b)\ [u, v] = \mathbf{1_G}, \text{ but } [u, g] \neq \mathbf{1_G}$$

(a) $[u, v] \neq \mathbf{1_G}$:

Here $u = u_1 \circ g_1^{-1}$, so

$$\begin{aligned} g =& u^{-j} \circ g_1 \\ =& (u_1 \circ g_1^{-1})^{-j} \circ g_1 \end{aligned}$$

and therefore

$$\begin{aligned} u^m g v^k =& (u_1 \circ g_1^{-1})^m ((u_1 \circ g_1^{-1})^{-j} \circ g_1) v^k \\ =& (u_1 \circ g_1^{-1})^{m-j} \circ g_1 v^k \\ =& (u_1 \circ g_1^{-1})^{m-j-1} \circ u_1 v^k \\ =& u_1 \circ (g_1^{-1} \circ u_1)^{m-j-1} v^k. \end{aligned}$$

If the cancellation between $(g_1^{-1} \circ u_1)^{m-j-1}$ and v^k is long enough, beacuse m, k are sufficiently big, we get that $c((g_1^{-1} \circ u_1)^{-m}, v^k) \geq L(u) + L(v)$. Now $g_1^{-1} \circ u_1$ is a cyclic permutation of $u^{-1} = u_1 \circ g_1^{-1}$ and is therefore cyclically reduced, so by Lemma 2.4.4, $[g_1^{-1} \circ u_1, v] = \mathbf{1_G}$. This implies that a conjugate of u commutes with v. But this contradicts property (iii) of the definition of a set of representatives by implying that the $\pi(v)$ in the definition is not unique since $[u, v] \neq \mathbf{1_G}$ by the case assumption, therefore it cannot happen.

(b) $[u, v] = \mathbf{1_G}$, but $[u, g] \neq \mathbf{1_G}$.

$u, v \in R^{\pm 1}$ so u and v are not proper powers and we have that $[u, v] = \mathbf{1_G}$ implies $u = v^{\pm 1}$.

So $L(g_1) < L(u)$ and $u = u_1 \circ g_1^{-1}$. Therefore, if $m > j$

$$\begin{aligned}
u^m g v^k =& (u_1 \circ g_1^{-1})^m ((u_1 \circ g_1^{-1})^{-j} \circ g_1) v^k \\
=& ((u_1 \circ g_1^{-1})^{m-j} g_1) v^k \\
=& ((u_1 \circ g_1^{-1})^{m-j-1} \circ u_1) v^k \\
=& (u_1 \circ (g_1^{-1} \circ u_1)^{m-j-1}) v^k.
\end{aligned}$$

By our assumption that there is no $r \in \mathbb{Z}$ with $u^m g v^k = u^{m-r} \circ (u^r g v^r) \circ v^{k-r}$ for any $m, k > r$, $u^m g v^k = (u_1 \circ (g_1^{-1} \circ u_1)^{m-j-1}) v^k$ implies that the cancellation between $(g_1^{-1} \circ u_1)^{m-j-1}$ and v^k is $L(v^k)$, so, for m', k' large enough

$$c((g_1^{-1} \circ u_1)^{-m'}, v^{k'}) > L(g_1^{-1} \circ u_1) + L(v).$$

Hence, by Lemma 2.4.4, $[g_1^{-1} \circ u_1, v] = \mathbf{1_G}$ and $g_1^{-1} \circ u_1 = v^{\pm 1}$, but $v = u^{\pm 1}$, so

$$g_1^{-1} \circ u_1 = u^{\pm 1} \in \{u_1 \circ g_1^{-1}, g_1 \circ u_1^{-1}\}$$

If $g_1^{-1} \circ u_1 = u_1 \circ g_1^{-1}$, then $[u_1, g_1^{-1}] = \mathbf{1_G}$ and hence

$$\begin{aligned}
[u, g] =& (u_1 \circ g_1^{-1})((g_1 \circ u_1^{-1})^j \circ g_1)(g_1 \circ u_1^{-1})(g_1^{-1} \circ (u_1 \circ g_1^{-1})^j) \\
=& \mathbf{1_G}
\end{aligned}$$

Contradiction of the case assumption.

If $g_1^{-1} \circ u_1 = g_1 \circ u_1^{-1}$ then $g_1^{-2} \circ u_1^2 = \mathbf{1_G}$, which means that $u_1^2 = g_1^2$. But this means that $L(u_1^2) = L(g_1^2)$, which implies that $2L(u_1) = 2L(g_1)$, so that $L(u_1) = L(g_1)$, and also the initial and terminal segments of u_1 of length $\frac{2L(u_1)}{3}$ are equal to the initial and terminal segments, respectively, of g_1 of the same length. This in turn implies that $u_1 = g_1$. But then $u = u_1 \circ g_1^{-1} = u_1 \circ u_1^{-1} = \mathbf{1_G}$. Now $\mathbf{1_G} = \mathbf{1_G}^{-1}$, so this implies that $u \notin R$. Contradiction.

So this case cannot occur either. Therefore if the Lemma fails for one triple we get a contradiction.

Hence the Lemma is true for all triples (u, g, v).

$\square$

The above Lemma can be extended to the next one, as in [7]. In my proof I have expanded on what the authors of [7] wrote.

Lemma 4.1.3. *Let H be an S-subgroup of $\mathcal{CDF}(G, \mathbb{R}^n)$ and R be a Lyndon's set of H. If $u_1, \ldots, u_k \in R^{\pm 1}$ and $g_1, \ldots, g_{k+1} \in H$ are such that for any $i = 2, \ldots, k$, either $[u_{i-1}, u_i] \neq \mathbf{1_G}$ or $[u_i, g_i] \neq \mathbf{1_G}$, then there exists $r \in \mathbb{N}$ such that:*

$$g_1 u_1^{m_1} g_2 \ldots g_k u_k^{m_k} g_{k+1} = (g_1 u_1^r) \circ u_1^{m_1 - 2r} \circ (u_1^r g_2 u_2^r) \circ u_2^{m_2 - 2r} \circ \ldots \circ u_k^{m_k - 2r} \circ (u_k^r g_{k+1})$$

with $2r < m_i \in \mathbb{N}$.

Proof. We prove this by induction.

Case $k = 1$: $g_1 u_1^{m_1} g_2 = (g_1 u_1^r) \circ u_1^{m_1 - 2r} (u_1^r g_2)$.

By property (iii) of a Lyndon's set, g_1 does not contain a subword u_1^α, where $\alpha \in \mathbb{Z}[t]/p(t)$ with $\deg \alpha > 0$. By property (ii) of a Lyndon's set, $g_1 u_1^\alpha$ exists for all $\alpha \in \mathbb{Z}[t]/p(t)$. Therefore $g_1 u_1^{m_1}$ exists for all $m_1 \in \mathbb{N}$, and $g_1 = g_{11} \circ g_{12} \circ u_1^{-s}$ with $L(g_{12}) < L(u_1)$, $u_1 = g_{12}^{-1} \circ u_{12}$ so that $g_1 u_1^{s+1} = g_{11} \circ u_{12}$.

Since $u_1 \in R$, u_1 is cyclically reduced and hence $u_{12} u_1 = u_{12} \circ u_1$. Therefore $g_1 u_1^{m_1} = (g_{11} \circ u_{12}) \circ u_1^{m_1 - s - 1}$ for all $m_1 > s + 1$.

Similarly $u_1^l g_2 = u_1^{l-t-1} \circ (u'_{12} \circ g_{22})$ for all $l > t+1$ and for u'_{12} such that $u_1 = u'_{12} \circ g_{21}^{-1}$ and $g_2 = u_1^{-t-1} \circ g_{21} \circ g_{22}$. So if $m_1 - s - 1 > t + 1$, we get that

$$\begin{aligned} g_1 u_1^{m_1} g_2 =& (g_{11} \circ u_{12})(u_1^{m_1-s-1} g_2) \\ =& (g_{11} \circ u_{12}) u_1^{m_1-s-1-t-1}(u'_{12} \circ g_{22}) \\ =& (g_1 u_1^{s+1}) \circ u_1^{m_1-(s+t+2)} \circ (u_1^{t+1} g_2) \end{aligned}$$

and therefore, for $r > \max\{s+1, t+1\}$ and $m_1 > 2r$, we get that

$$g_1 u_1^{m_1} g_2 = (g_1 u_1^r) \circ u_1^{m_1-2r} \circ (u_1^r g_2)$$

as required. So the lemma is true for $k = 1$.

Now assume that this is true for $k = l$, so there exists $r_l \in \mathbb{N}$ such that

$$g_1 u_1^{m_1} g_2 \ldots g_l u_l^{m_l} g_{l+1} = (g_1 u_1^{r_l}) \circ u_1^{m_1-2r_l} \circ (u_1^{r_l} g_2 u_2^{r_l}) \circ \ldots \circ u_l^{m_l-2r_l} \circ (u_l^{r_l} g_{l+1})$$

for all $m_i \in \mathbb{N}$ with $m_i > 2r_l$, $i \in \{1, \ldots, l\}$.

Prove that the lemma is true for $k = l + 1$.

Let $w = (g_1 u_1^{r_l}) \circ u_1^{m_1-2r_l} \circ (u_1^{r_l} g_2 u_2^{r_l}) \circ \ldots \circ u_{l-1}^{m_{l-1}-2r_l} \circ (u_{l-1}^{r_l} g_l u_l^{r_l})$. Then

$$g_1 u_1^{m_1} g_2 \ldots g_l u_l^{m_l} g_{l+1} u_{l+1}^{m_{l+1}} g_{l+2} = (w \circ u_l^{m_l-2r_l} \circ (u_l^{r_l} g_{l+1})) u_{l+1}^{m_{l+1}} g_{l+2}$$

As in the case $k = 1$, by properties (ii) and (iii), there exists $r_{l+1} \in \mathbb{N}$ such that

$$(u_{l+1}^{m_{l+1}} g_{l+2}) = u_{l+1}^{m_{l+1}-r_{l+1}} \circ (u_{l+1}^{r_{l+1}} g_{l+2}) \quad \text{for all } m_{l+1} > r_{l+1}$$

So

$$g_1 u_1^{m_1} g_2 \ldots g_{l+1} u_{l+1}^{m_{l+1}} g_{l+2} = (w \circ u_l^{m_l-2r_l} \circ (u_l^{r_l} g_{l+1}))(u_{l+1}^{m_{l+1}-r_{l+1}} \circ (u_{l+1}^{r_{l+1}} g_{l+2}))$$

But by Lemma 4.1.2, there exists $r > \max\{r_l, r_{l+1}\}$ such that

$$u_l^{m_l-2r_l} \circ (u_l^{r_l} g_{l+1}) u_{l+1}^{m_{l+1}-r_{l+1}} = u_l^{m_l-2r} \circ (u_l^r g_{l+1} u_{l+1}^r) \circ u_{l+1}^{m_{l+1}-r_{l+1}-r}$$

for all $m_l > 2r, m_{l+1} > r_{l+1} + r$, so

$$\begin{aligned} g_1 u_1^{m_1} g_2 \dots u_{l+1}^{m_{l+1}} g_{l+2} =& (w \circ u_l^{m_l - 2r_l} \circ (u_l^{r_l} g_{l+1})(u_{l+1}^{m_{l+1}} g_{l+2}) \\ =& w \circ u_l^{m_l - 2r} \circ (u_l^r g_{l+1} u_{l+1}^r) \circ u_{l+1}^{m_{l+1} - 2r} \circ (u_{l+1}^r g_{l+2}) \\ =& (g_1 u_1^r) \circ u_1^{m_1 - 2r} \circ (u_1^r g_2 u_2^r) \circ u_2^{m_2 - 2r} \circ \dots \\ & \dots \circ u_l^{m_l - 2r} \circ (u_l^r g_{l+1} u_{l+1}^r) \circ u_{l+1}^{m_{l+1} - 2r} \circ (u_{l+1}^r g_{l+2}) \end{aligned}$$

for all $m_i > 2r, i \in \{1, \dots, k+1\}$, as required.

□

Now we define *R-forms*. These Definitions are analagous to Definitions 6.11 and 6.12 in [7].

Definition 4.1.7. *Let H be an S-subgroup of $\mathcal{CDF}(G, \mathbb{R}^n)$ with a Lyndon's set R. A sequence:*

$$p = (g_1, u_1^{\alpha_1}, g_2, \dots, g_m, u_m^{\alpha_m}, g_{m+1})$$

*is an R-*form *with $g_i \in H$, $u_i \in R$, $\alpha_i \in \mathbb{Z}[t]/p(t)$ and $m \geq 1$.*

Definition 4.1.8. *An R-form p is* reduced *if $deg(\alpha_i) > 0$ $(1 \leq i \leq m)$ and if $u_i = u_{i-1}$ then $[u_i, g_i] \neq \mathbf{1_G}$, or if $m = 1$ and $deg(\alpha_1) = 0$.*

Let $\mathcal{P}(H, R) = \{R\text{-forms over } H\}$.

The partial function

$$w : \mathcal{P}(H, R) \to \mathcal{RF}(G, \mathbb{R}^n)$$

sends $p = (g_1, u_1^{\alpha_1}, g_2, \dots, g_m, u_m^{\alpha_m}, g_{m+1})$ to

$$w(p) = ((\dots((g_1 u_1^{\alpha_1}) g_2) \dots g_m) u_m^{\alpha_m}) g_{m+1}$$

for $w(p)$ defined.

Definition 4.1.9. *An R-form p is* normal *if it is reduced and*

$$w(p) = g_1 \circ u_1^{\alpha_1} \circ g_2 \circ \dots \circ g_m \circ u_m^{\alpha_m} \circ g_{m+1}$$

where g_i does not have terminal segment $u_i^{\pm 1}$ for all $1 \leq i \leq m$ and $g_i \circ u_i^{\alpha_i}$ does not have initial segment $u_{i-1}^{\pm 1}$ for all $2 \leq i \leq m$.

In fact, for every R-form, p, $w(p)$ is defined and in $\mathcal{CDF}(G, \mathbb{R}^n)$, and p has a unique normal R-form, q, associated to it such that $w(p) = w(q)$, as proved in the Lemma below (similarly to that in [7]).

Lemma 4.1.4. *Let H be an S-subgroup of $\mathcal{CDF}(G, \mathbb{R}^n)$ with a Lyndon's set R. Then for every R-form p over H the following holds:*

(i) $w(p)$ is defined and does not depend on the placement of parentheses.

(ii) There exists a reduced R-form q over H such that $w(p) = w(q)$.

(iii) There exists a unique normal R-form q' over H such that $w(p) = w(q')$.

(iv) $w(p) \in \mathcal{CDF}(G, \mathbb{R}^n)$.

Proof. Let $p = (h_1, u_1^{\alpha_1}, h_2, \ldots, u_m^{\alpha_m}, h_{m+1})$. First I prove that (i) implies (ii) which implies (iii), then I prove (i) and finally I use this to prove (iv).

Assume (i) is true, so

$$w(p) = h_1 u_1^{\alpha_1} h_2 \ldots h_m u_m^{\alpha_m} h_{m+1}$$

Let $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}$ be the set of all α_i in the R-form above such that $\deg(\alpha_i) > 0$ with $i_1 < i_2 < \ldots < i_k$. Then set

$$\begin{aligned} g_1 =& h_1 u_1^{\alpha_1} h_2 \ldots u_{i_1 - 1}^{\alpha_{i_1 - 1}} h_{i_1} \\ g_{j+1} =& h_{i_j + 1} u_{i_j + 1}^{\alpha_{i_j + 1}} \ldots h_{i_{j+1}} \\ g_{k+1} =& h_{i_k + 1} u_{i_k + 1}^{\alpha_{i_k + 1}} \ldots h_{m+1} \end{aligned}$$

for $j \in \{1, \ldots, k-1\}$. Then $g_j \in H$ for all $j \in \{1, \ldots, k+1\}$, and by (i), we have that

$$\begin{aligned} w(p) =& h_1 u_1^{\alpha_1} h_2 \ldots h_m u_m^{\alpha_m} h_{m+1} \\ =& g_1 u_{i_1}^{\alpha_{i_1}} g_2 \ldots g_k u_{i_k}^{\alpha_{i_k}} g_{k+1} \end{aligned}$$

Set $u_{i_j} = v_j$ and $\alpha_{i_j} = \beta_j$ to get:

$$w(p) = g_1 v_1^{\beta_1} g_2 \dots g_k v_k^{\beta_k} g_{k+1}$$

If we set

$$p_1 = (g_1, v_1^{\beta_1}, g_2, \dots, g_k, v_k^{\beta_k}, g_{k+1})$$

we have a new R-form $p_1 \in \mathcal{P}(H, R)$ such that $w(p_1) = w(p)$. If this is not reduced, then there exists $j \in \{2, \dots, k\}$ such that $v_j = v_{j-1}$ and $[g_j, v_j] = \mathbf{1_G}$. Since $v_j \in R$, the centraliser of v_j in H, $C_H(v_j)$, is cyclic and generated by v_j, we have that $g_j = v_j^m$ for some $m \in \mathbb{Z}$. Therefore

$$v_{j-1}^{\beta_{j_1}} g_j v_j^{\beta_j} = v_j^{\beta_{j-1}+m+\beta_j}$$

Let

$$p_2 = (g_1, v_1^{\beta_1}, \dots, g_{j-1}, v_j^{\beta_{j-1}+m+\beta_j}, g_{j+1}, \dots, g_k, v_k^{\beta_k}, g_{k+1})$$

Then $p_2 \in \mathcal{P}(H, R)$, with $w(p_2) = w(p_1) = w(p)$ and the length of p_2 is less than that of p_1, which is less than or equal to that of p, where the length of p is the number of g_i in p. Now if $\beta_{j-1} + m + \beta_j \in \mathbb{Z}$, repeat the first step to get $p_3 \in \mathcal{P}(H, R)$ with length even less than p_2. Since the length of p is finite, by induction we can repeat this until there are no more j such that both $v_{j-1} = v_j$ and $[g_j, v_j] = \mathbf{1_G}$, and no β_j such that $\beta_j \in \mathbb{Z}$, leaving us with a new R-form, $q = (f_1, y_1^{\gamma_1}, f_2, \dots, f_l, y_l^{\gamma_l}, f_{l+1}) \in \mathcal{P}(H, R)$, which is now reduced and such that $w(q) = w(p)$.

So $(i) \implies (ii)$, and we can now assume that our original R-form

$$p = (h_1, u_1^{\alpha_1}, h_2, \dots, h_m, u_m^{\alpha_m}, h_{m+1})$$

is reduced. Next I show that this implies (iii).

Since $\deg(\alpha_i) > 0$ for all i, by Lemma 4.1.3 there exists $r \in \mathbb{N}$ such that

$$\begin{aligned} w(p) =& h_1 u_1^{\alpha_1} h_2 \dots h_m u_m^{\alpha_m} h_{m+1} \\ =& (h_1 u_1^{r_1}) \circ u_1^{\alpha_1 - 2r_1} \circ (u_1^{r_1} h_2 u_2^{r_2}) \circ u_2^{\alpha_2 - 2r_2} \circ \dots \\ & \dots \circ (u_{m-1}^{r_{m-1}} h_m u_m^{r_m}) \circ u_m^{\alpha_m - 2r_m} \circ (u_m^{r_m} h_{m+1}) \end{aligned}$$

where $r_j = sign(\alpha_j)r$.

Let

$$\begin{aligned} g_1 =& h_1 u_1^{r_1} \\ g_j =& u_{j-1}^{r_{j-1}} h_j u_j^{r_j} \\ g_{m+1} =& u_m^{r_m} h_{m+1} \end{aligned}$$

for $j \in \{2, \ldots, m\}$, then set $\beta_j = \alpha_j - 2r_j$ to get a new R-form

$$q_1 = (g_1, u_1^{\beta_1}, g_2, \ldots, g_m, u_m^{\beta_m}, g_{m+1})$$

which satisfies (i). Then $w(q_1) = g_1 \circ u_1^{\beta_1} \circ g_2 \circ \ldots \circ g_m \circ u_m^{\beta_m} \circ g_{m+1}$.

By properties (ii) and (iii) of a Lyndon's set, there exists a unique $s_1 \in \mathbb{Z}$ such that $g_1 = f_1 \circ u_1^{s_1}$ and f_1 does not have $u_1^{\pm 1}$ as a terminal segment.

Also by properties (ii) and (iii) of a Lyndon's set, there exists a unique $t_1 \in \mathbb{Z}$ and f_2' such that

$$g_2 \circ u_2^{\beta_2} = u_1^{t_1} \circ f_2' \circ u_2^{\beta_2}$$

where $f_2' \circ u_2^{\beta_2}$ does not have $u_1^{\pm 1}$ as an initial segment. This means that we have a new form,

$$q_2 = (f_1, u_1^{s_1+\beta_1+t_1}, f_2', u_2^{\beta_2}, g_3, \ldots, u_m^{\beta_m}, g_{m+1})$$

with $w(q_2) = w(q_1) = w(p)$.

Now if

$$q_l = (f_1, u_1^{s_1+\beta_1+t_1}, f_2, u_2^{s_2+\beta_2+t_2}, \ldots, u_{l-1}^{s_{l-1}+\beta_{l-1}+t_{l-1}}, f_l', u_l^{\beta_l}, g_{l+1}, \ldots, u_m^{\beta_m}, g_{m+1})$$

where

$$u_{i-1}^{s_{i-1}+\beta_{i-1}+t_{i-1}} f_i u_i^{s_i+\beta_i+t_i} = u_{i-1}^{s_{i-1}+\beta_{i-1}+t_{i-1}} \circ f_i \circ u_i^{s_i+\beta_i+t_i}$$

f_i does not have terminal segment $u_i^{\pm 1}$ and $f_{i+1} \circ u_{i+1}$ does not have initial segment $u_i^{\pm 1}$, for $0 < i < l < m$.

Then, by properties (ii) and (iii) of a Lyndon's set, there exists a unique $s_l \in \mathbb{Z}$ such that $f'_l = f_l \circ u_l^{s_l}$ and f_l does not have $u_l^{\pm 1}$ as a terminal segment, and there exists a unique $t_l \in \mathbb{Z}$ and f'_{l+1} such that

$$g_{l+1} \circ u_{l+1}^{\beta_{l+1}} = u_l^{t_l} \circ f'_{l+1} \circ u_{l+1}^{\beta_{l+1}}$$

and $f'_{l+1} \circ u_{l+1}^{\beta_{l+1}}$ does not have initial segment $u_l^{\pm 1}$.

So, by induction, we end up with a normal R-form

$$q_{m+1} = (f_1, u_1^{s_1+\beta_1+t_1}, f_2, \dots, f_m, u_m^{s_m+\beta_m+t_m}, f'_{m+1})$$

such that $w(q_{m+1}) = w(p)$. Uniqueness of this normal R-form follows from the uniqueness of the s_i and t_i.

Now we prove (i) by induction on m.

First let $m = 1$.

If $\alpha_1 \in \mathbb{Z}$, $h_1, u_1^{\alpha_1}, h_2 \in H$, therefore $(h_1 u_1^{\alpha_1})h_2 = h_1(u_1^{\alpha_1} h_2) \in H$, so the product $(h_1 u_1^{\alpha_1})h_2$ does not depend on the placement of parentheses.

If $\deg \alpha_1 > 0$, then, by (ii) of the definition of a Lyndon's set, we have that $c(u_1^{\alpha_1}, h_1^{-1})$ and $c(u_1^{-\alpha_1}, h_2)$ exist and that $c(u_1^{\alpha_1}, h_1^{-1}) < k_1 L(u_1)$ and $c(u_1^{-\alpha_1}, h_2) < k_2 L(u_1)$, where $k_1, k_2 \in \mathbb{Z}$. Hence

$$\begin{aligned} h_1 u_1^{\alpha_1} =& (h_1 u_1^{k_1}) \circ u_1^{\alpha_1 - k_1} \\ u_1^{\alpha_1 - k_1} h_2 =& u_1^{\alpha_1 - k_1 - k_2} \circ (u_1^{k_2} h_2) \end{aligned}$$

and therefore

$$\begin{aligned} (h_1 u_1^{\alpha_1})h_2 =& ((h_1 u_1^{k_1}) \circ u_1^{\alpha_1 - k_1})h_2 \\ =& (h_1 u_1^{k_1}) \circ (u_1^{\alpha_1 - k_1} h_2) \\ =& (h_1 u_1^{k_1}) \circ (u_1^{\alpha_1 - k_1 - k_2} \circ (u_1^{k_2} h_2)) \end{aligned}$$

But by axiom $(P3)$ of a pregroup, which holds for $\mathcal{RF}(G, \mathbb{R}^n)$, the product $(h_1 u_1^{\alpha_1})h_2$ does not depend on the placement of parentheses. Hence (i) holds for $m = 1$.

Now we can assume this holds for $m = k-1$, so for all R-forms

$$p_{k-1} = (h_1, u_1^{\alpha_1}, h_2, \ldots, u_{k-1}^{\alpha_{k-1}}, h_k)$$

$w(p_{k-1})$ is defined and doesn't depend on the placement of parentheses.

By the first part of this proof, this means that there exists a unique normal form $q_{k-1} = (f_1, v_1^{\beta_1}, f_2, \ldots, v_{l-1}^{\beta_{l-1}}, f_l)$ such that

$$w(q_{k-1}) = f_1 \circ v_1^{\beta_1} \circ f_2 \circ \ldots \circ v_{l-1}^{\beta_{l-1}} \circ f_l = w(p_{k-1})$$

Let

$$p_k = (h_1, u_1^{\alpha_1}, h_2, \ldots, u_{k-1}^{\alpha_{k-1}}, h_k, u_k^{\alpha_k}, h_{k+1})$$

We need to show that

$$w(p_k) = ([((\ldots((h_1 u_1^{\alpha_1})h_2)\ldots)u_{k-1}^{\alpha_{k-1}})h_k]u_k^{\alpha_k})h_{k+1}$$

is defined and doesn't depend on the placement of parentheses.

We know that

$$\begin{aligned} [((\ldots((h_1 u_1^{\alpha_1})h_2)\ldots)u_{k-1}^{\alpha_{k-1}})h_k] =& w(p_{k-1}) \\ =& f_1 \circ v_1^{\beta_1} \circ f_2 \circ \ldots \circ v_{l-1}^{\beta_{l-1}} \circ f_l \end{aligned}$$

which doesn't depend on the placement of parentheses, so all we need to do is prove that

$$w(p_{k-1})(u_k^{\alpha_k} h_{k+1})$$

is defined and doesn't depned on the placement of parentheses, as that would mean that the same holds for $(w(p_{k-1})u_k^{\alpha_k})h_{k+1}$.

If $\alpha_k \in \mathbb{Z}$, then $u_k^{\alpha_k} \in H$ and

$$w(p_{k-1})(u_k^{\alpha_k} h_{k+1}) = (f_1 \circ v_1^{\beta_1} \circ f_2 \circ \ldots \circ v_{l-1}^{\beta_{l-1}} \circ f_l)(u_k^{\alpha_k} h_{k+1})$$

Now $h_{k+1},\ u_k,\ f_l \in H$, therefore $f_l u_k^{\alpha_k} h_{k+1} \in H$. So, since $\deg(\beta_{l-1}) > 0$ by the construction of q_{k-1}, property (ii) of a Lyndon's set tells us that there exists $r \in \mathbb{N}$ such that $v_{l-1}^{\beta_{l-1}}(f_l u_k^{\alpha_k} h_{k+1}) = v_{l-1}^{\beta_{l-1}-r} \circ (v_{l-1}^r f_l u_k^{\alpha_k} h_{k+1})$ and hence $w(p_k) = w(p_{k-1})(u_k^{\alpha_k} h_{k+1})$ is defined and doesn't depend on the placement of parentheses.

So assume that $\deg(\alpha_k) > 0$.

If either $[v_{l-1}, u_k] \neq \mathbf{1_G}$ or $[v_{l-1}, f_l] \neq \mathbf{1_G}$, then, by conditions (ii) and (iii) of a Lyndon's set, $u_k^{\alpha_k} h_{k+1} = u_k^{\alpha_k - s_2} \circ (u_k^{s_2} h_{k+1})$ for some $s_2 \in \mathbb{N}$, and by Lemma 4.1.2,

$$\begin{aligned}(v_{l-1}^{\beta_{l-1}} \circ f_l)(u_k^{\alpha_k} h_{k+1}) =& (v_{l-1}^{\beta_{l-1}} \circ f_l)(u_k^{\alpha_k - s_2} \circ (u_k^{s_2} h_{k+1})) \\ =& v_{l-1}^{\beta_{l-1}-r} \circ (v_{l-1}^r f_l u_k^{s_1}) \circ u_k^{\alpha_k - s_1 - s_2} \circ (u_k^{s_2} h_{k+1})\end{aligned}$$

for some $r, s_1 \in \mathbb{N}$, and again we have that $w(p_k) = w(p_{k-1})(u_k^{\alpha_k} h_{k+1})$ is defined and doesn't depend on the placement of parentheses.

If both $[v_{l-1}, u_k] = \mathbf{1_G}$ and $[v_{l-1}, f_l] = \mathbf{1_G}$, then, by properties (i) and (ii) of the definition of a set of representatives, $v_{l-1} = u_k$. Also, since the R-form $(f_1, v_1^{\beta_1}, f_2, \ldots, v_{l-1}^{\beta_{l-1}}, f_l)$ is a normal form, f_l doesn't have $v_{l-1}^{\beta_{l-1}}$ as an initial segment. But $[v_{l-1}, f_l] = \mathbf{1_G}$, so by (i) of the definition of a set of representatives and the fact that $C_H(v_{l-1}) = \langle u \rangle$ for some $u \in \mathcal{RF}(G, \mathbb{R}^n)$, $f_l = \mathbf{1_G}$.

Hence

$$\begin{aligned}(v_{l-1}^{\beta_{l-1}} \circ f_l)(u_k^{\alpha_k} h_{k+1}) =& v_{l-1}^{\beta_{l-1}} v_{l-1}^{\alpha_k} h_{k+1} \\ =& v_{l-1}^{\beta_{l-1}+\alpha_k} h_{k+1} \\ =& v_{l-1}^{\beta_{l-1}+\alpha_k - s_2} \circ (v_{l-1}^{s_2} h_{k+1})\end{aligned}$$

by conditions (ii) and (iii) of a Lyndon's set, so $w(p_k) = w(p_{k-1})(u_k^{\alpha_k} h_{k+1})$ is again defined and doesn't depend on the placement of parentheses.

Hence (i) is true and therefore (ii) and (iii) are true.

Now we can prove (iv):

By Lemma 2.4.3, if $g^{-1} w(q) g \in \mathcal{CDF}(G, \mathbb{R}^n)$ for some $g \in \mathcal{RF}(G, \mathbb{R}^n)$, then so does $w(q)$ if $w(q)$ does not cancel completely in $g^{-1}w(q)g$, where

$$w(q) = f_1 \circ v_1^{\beta_1} \circ f_2 \circ \ldots \circ v_{l-1}^{\beta_{l-1}} \circ f_l$$

Let $g = f_1 \circ v_1^j$ for some $j \in \mathbb{Z}$ to get

$$(v_1^{-j} \circ f_1^{-1})w(q)(f_1 \circ v_1^j) = (v_1^{-j} \circ f_1^{-1})(f_1 \circ v_1^{\beta_1} \circ f_2 \circ \ldots \circ v_{l-1}^{\beta_{l-1}} \circ f_l)(f_1 \circ v_1^j)$$

If $v_1 \neq v_{l-1}$ or $[f_l f_1, v_1] \neq \mathbf{1_G}$, we get that

$$(v_1^{-j} \circ f_1^{-1})w(q)(f_1 \circ v_1^j) = (v_1^{\beta_1 - j} \circ f_2 \circ \ldots \circ v_{l-1}^{\beta_{l-1}} \circ f_l)(f_1 \circ v_1^j)$$

and if we choose j carefully, by Lemma 4.1.2 there exists $m \in \mathbb{Z}$ such that sign(β_1) =sign$(j-m)$ and

$$(v_{l-1}^{\beta_{l-1}} \circ f_l)(f_1 \circ v_1^j) = v_{l-1}^{\beta_{l-1}-m} \circ (v_{l-1}^m f_l f_1 v_1^m) \circ v_1^{j-m}$$

so that $(v_1^{-j} \circ f_1^{-1})w(q)(f_1 \circ v_1^j) \in \mathcal{CRF}(G, \mathbb{R}^n) \subset \mathcal{CDF}(G, \mathbb{R}^n)$ and $w(q)$ does not cancel completely in $(v_1^{-j} \circ f_1^{-1})w(q)(f_1 \circ v_1^j)$ as required.

If, however, $v_1 = v_{l-1}$ and $[f_l f_1, v_1] = \mathbf{1_G}$, we find that $f_l f_1 = v_1^s$ and

$$\begin{aligned} f_1^{-1}w(q)f_1 =& f_1^{-1}(f_1 \circ v_1^{\beta_1} \circ f_2 \circ \ldots \circ v_{l-2}^{\beta_{l-2}} \circ f_{l-1} \circ v_1^{\beta_{l-1}} \circ f_l)f_1 \\ =& v_1^{\beta_1} \circ f_2 \circ \ldots \circ v_{l-2}^{\beta_{l-2}} \circ f_{l-1} \circ v_1^{\beta_{l-1}+s} \end{aligned}$$

If sign(β_1) = sign(β_{l-1}), then

$$f_1^{-1}w(q)f_1 \in \mathcal{CRF}(G, \mathbb{R}^n)$$

and $w(q)$ does not cancel completely in $f_1^{-1}w(q)f_1$, as required.

If sign$(\beta_1) \neq$ sign(β_{l-1}), then

$$(v_1^{\beta_{l-1}+s} f_1^{-1})w(q)(f_1 v_1^{-\beta_{l-1}-s}) = v_1^{\beta_1+\beta_{l-1}+s} \circ f_2 \circ \ldots \circ v_{l-2} \circ f_{l-1}$$

Now we have reduced the number of elements with exponents of dimnesion greater than one by one and can therefore use induction on l.

If $l = 2$ and $v_1 = v_{l-1}$ whilst $[f_l, f_1] = \mathbf{1_G}$, then $w(q) = f_1 \circ v_1^{\beta_1} \circ f_2$, $f_1 f_2 = v_1^m$ and

$$\begin{aligned} f_1^{-1} w(q) f_1 =& f_1^{-1}(f_1 \circ v_1^{\beta_1} \circ f_2) f_1 \\ =& v_1^{\beta_1} \circ v_1^m \\ =& v_1^{\beta_1 + m} \end{aligned}$$

which is clearly in $\mathcal{CRF}(G, \mathbb{R}^n)$, but $w(q) = f_1 \circ v_1^{\beta_1} \circ f_2$ is not completely cancelled in $f_1^{-1}(f_1 \circ v_1^{\beta_1} \circ f_2) f_1$, so we have that $w(q) \in \mathcal{CDF}(G, \mathbb{R}^n)$. This means that $(f_1 \circ v_1^{\beta_1} \circ f_2) \in \mathcal{CDF}(G, \mathbb{R}^n)$ in all cases for $l = 2$, and therefore the process of reducing the number of exponents of dimension greater than one must end.

This proves (iv) and hence the lemma is true.

□

Let $P(H, R) = \{g u^\alpha h | \alpha \in \mathbb{Z}[t]/p(t), g, h \in H, u \in R\}$ for $H \leq \mathcal{CDF}(G, \mathbb{R}^n)$ and R a Lyndon's set of H.

I can now prove that $P(H_1, R)$ is indeed a Pregroup, as I stated at the start of this Chapter. First I prove a more general result about $P(H, R)$ as defined above.

Proposition 4.1.1. *Let H be an S-subgroup of $\mathcal{CDF}(G, \mathbb{R}^n)$ and R a Lyndon's set for H. Then $P(H, R)$ is a pregroup with respect to reduced multiplication. This follows thNote that roof of Proposition 6.14 in [7].*

Proof. Since the reduced multiplication is induced from $\mathcal{RF}(G, \mathbb{R}^n)$, we already have that it satisfies axioms $(P1)$ to $(P3)$, therefore we only need to prove that it satisfies $(P4)$, which is:

For every $u, v, w, z \in P$, if uv, vw, wz are defined, then either uvw or vwz is defined in P.

Here $P = P(H, R)$.

To prove $(P4)$, first assume that $g_i c_i^{\alpha_i} h_i \in P$ with $\deg(\alpha_i) > 0$ $i = 1, 2$ and

$$g_1 c_1^{\alpha_1} h_1 = g_2 c_2^{\alpha_2} h_2$$

Then $a = (g_1, c_1^{\alpha_1}, h_1 h_2^{-1}, c_2^{-\alpha_2}, g_2^{-1})$ is an R-form and by Lemma 4.1.4, $w(a)$ is defined, but

$$w(a) = g_1 c_1^{\alpha_1} h_1 h_2^{-1} c_2^{-\alpha_2} g_2^{-1} = \mathbf{1_G}$$

Therefore a is not reduced and so $c_1 = c_2$, whilst $[c_2, h_1 h_2^{-1}] = \mathbf{1_G}$.

Similarly, if $b = (h_2, c_2^{\alpha_2}, g_2 g_1^{-1}, c_1^{-\alpha_1}, h_1^{-1})$, then b is an R-form, and

$$w(b) = h_2 c_2^{\alpha_2} g_2 g_1^{-1} c_1^{-\alpha_1} h_1^{-1} = \mathbf{1_G}$$

so $[c_2, g_2 g_1^{-1}] = \mathbf{1_G}$. But $c_2 = c_1$ and therefore $[c_1, g_2 g_1^{-1}] = \mathbf{1_G}$. Also $(g_2 g_1^{-1})^{-1} = g_1 g_2^{-1}$, which means that $[c_1, g_1 g_2^{-1}] = \mathbf{1_G}$ and hence $g_1 g_2^{-1}$ and $h_1 h_2^{-1} \in \langle c_1 \rangle = \langle c_2 \rangle$.

Now let $p = g_p c_p^{\alpha_p} h_p, q = g_q c_q^{\alpha_q} h_q \in P$ and $x = pq \in P$. I wil prove that $c_p = c_q$ and $h_p g_q \in \langle c_p \rangle$.

First, since $x \in P$, there exists g_x, $h_x \in H$, $c_x \in R$ and $\alpha_x \in \mathbb{Z}[t]/p(t)$ such that $x = g_x c_x^{\alpha_x} h_x$.

Now

$$c = (g_p, c_p^{\alpha_p}, h_p g_q, c_q^{\alpha_q}, h_q h_x^{-1}, c_x^{-\alpha_x}, g_x^{-1})$$

is an R-form, so

$$w(c) = g_p c_p^{\alpha_p} h_p g_q c_q^{\alpha_q} h_q h_x^{-1} c_x^{-\alpha_x} g_x^{-1}$$

is defined, and $w(c) = \mathbf{1_G}$ therefore c is not reduced.

If $\deg(\alpha_x) = 0$, then by the first part of this proof, $c_p = c_q$ and $x \in H$, so

$$g_p c_p^{\alpha_p} h_p = (x h_q^{-1}) c_q^{-\alpha_q} g_q^{-1}$$

and $[c_p, h_p g_q] = \mathbf{1_G}$, hence $h_p q_q \in \langle c_p \rangle$ as required.

If $\deg(\alpha_x) > 0$ then either $\deg(\alpha_p) > 0$ or $\deg(\alpha_q) > 0$.

Let $\deg(\alpha_p) > 0$, then

$$g_p c_p^{\alpha_p} h_p = g_x c_x^{\alpha_x} (h_x h_q^{-1} c_q^{-\alpha_q} g_q^{-1})$$

So $\deg(\alpha_q) = 0$ means that $(h_x h_q^{-1} c_q^{-\alpha_q} g_q^{-1}) \in H$, so $h_p g_q c_q^{\alpha_q} h_q h_x \in \langle c_p \rangle$, but then $c_p = c_q$ and $h_q h_x, h_p g_q \in \langle c_p \rangle$, since $c_p \in R$.

Also, $\deg(\alpha_q) > 0$ means that $c_q = c_x = c_p$ and $h_x h_q^{-1} \in \langle c_p \rangle$, so $h_x h_q^{-1} = c_p^k$ for some $k \in \mathbb{Z}$. But then $g_p c_p^{\alpha_p} h_p = g_x c_p^{\alpha_x + k - \alpha_q} g_q^{-1}$ with $\deg(\alpha_x + k - \alpha_q) > 0$ (since $\deg(\alpha_p) > 0$) and hence $c_q = c_p$ and $h_p g_q \in \langle c_p \rangle$ again.

If $\deg(\alpha_p) = 0$, then $\deg(\alpha_q) > 0$, so

$$(g_p c_p^{\alpha_p} h_p g_q) c_q^{\alpha_q} h_q = g_x c_x^{\alpha_x} h_x$$

and $g_p c_p^{\alpha_p} h_p g_q \in H$. But then $c_q = c_x$ and $g_x^{-1} g_p c_p^{\alpha_p} h_p g_q \in H$, which implies $c_p = c_q$ and $h_p g_q, g_x^{-1} g_p \in \langle c_q \rangle$ as required.

Hence, if $p = g_p c_p^{\alpha_p} h_p, q = g_q c_q^{\alpha_q} h_q \in P$ with $\deg(\alpha_p) > 0$ or $\deg(\alpha_q) > 0$, and $pg \in P$, then $c_p = c_q$ and $h_p g_q \in \langle c_p \rangle$.

Now let $u = g_u c_u^{\alpha_u} h_u$, $v = g_v c_v^{\alpha_v} h_v$, $w = g_w c_w^{\alpha_w} h_w$, $z = g_z c_z^{\alpha_z} h_z$, where $uv, vw, wz \in P$.

If $\deg(\alpha_p) > 0$ for all $p \in \{u, v, w, z\}$, then by the above we must have that $c_u = c_v = c_w = c_z$ and $h_u g_v = c_u^k, h_v g_w = c_u^l$ for some $k, l \in \mathbb{Z}$. Therefore $uvw = g_u c_u^{\alpha_u + k + \alpha_v + l + \alpha_w} h_w \in P$.

If there exists a unique $p \in \{u, v, w, z\}$ such that $\deg(\alpha_p) = 0$, then either u, v, w or v, w, z has two consecutive elements r, s with $\deg(\alpha_r), \deg(\alpha_s) > 0$, whilst the last one, q, has $\deg(\alpha_q) = 0$. But then q is clearly in H and since $rs \in P$, $q \in H$, both $(rs)q$ and $q(rs) \in P$, so we are done.

If there exists $q_1, q_2 \in \{u, v, w, z\}$ such that $\deg(\alpha_{q_i}) > 0$, $i = 1, 2$, and the other two elements $p_1, p_2 \in \{u, v, w, z\}$ are such that $\deg(\alpha_{p_i}) = 0$ for $i = 1, 2$, then either one of u, v, w or v, w, z has two consecutive elements r, s with $\deg(\alpha_r), \deg(\alpha_s) > 0$ as in the case above, or one of u, v, w or v, w, z has only one element q such that $\deg(\alpha_q) > 0$, but then both of the other elements are in H, so all of $rqs, (rs)q$ and $q(rs) \in P$ and we are done.

If there exists a unique $p \in \{u, v, w, z\}$ such that $\deg(\alpha_p) > 0$, then either u, v, w or v, w, z has two consecutive elements r, s with $\deg(\alpha_r), \deg(\alpha_s) = 0$, and we can use the arguement of the second part of the case above to see that one of uvw or vwz is in P, or there are three consecutive elements in H so their product is in H and hence in P

If for all $p \in \{u, v, w, z\}$, $\deg(\alpha_p) = 0$, then $u, v, w, z \in H$ and we can use the second part of the case above to prove that both uvw and vwz are in P.

Hence $(P4)$ holds in P and we are done.

$\square$

Now, since H_1 is an S-subgroup of $\mathcal{CDF}(G, \mathbb{R}^n)$, we have found a pregroup $P(H_1, R) \subseteq \mathcal{RF}(G, \mathbb{R}^n)$ for any Lyndon's set R of H_1. The next section is about the universal group of this pregroup, which is in $\mathcal{RF}(G, \mathbb{R}^n)$, and an associated length function that I will prove is a Lyndon length function.

4.2 The Universal Group of $P(H_1, R)$ and the Λ-tree $\mathbf{X}_G$ associated with $P(H_1, R)$.

A pregroup P has a universal group, $U(P)$, associated to it, defined by

$$U(P) = \langle p | \{xy[m(x,y)^{-1}] | (x, y \in D)\} \rangle$$

In the case of $P(H_1, R)$, this group is the group generated by $P(H_1, R)$, with reduced multiplication. We have that $\langle P(H_1, R) \rangle \subset \mathcal{CDF}(G, \mathbb{R}^n) \subset \mathcal{RF}(G, \mathbb{R}^n)$, so I have found a group within $\mathcal{RF}(G, \mathbb{R}^n)$ that doesn't only have one dimensional elements, which I examine in more depth in the next chapter.

First I prove that the length function defined on $\mathcal{CDF}(G, \mathbb{R}^n)$ is a Lyndon length function, which tells us that the group acts on an $\mathbb{R}^n$-tree, giving us a link to the theory of Λ-trees (see [1]).

Proposition 4.2.1. *The length function* $L : \langle P(H_1, R)\rangle \longrightarrow \mathbb{R}^n$ *such that for* $f : [0, \alpha]_{\mathbb{R}^n} \longrightarrow G$*,* $L(f) = \alpha$*, is a* Lyndon length function. *(See [1]).*

Proof. Given a group H and an ordered abelian group Λ, recall from Definition 1.2.1 that a length function $L : H \longrightarrow \Lambda$ is a Lyndon length function if:

(i) $L(1_H) = 0$

(ii) $L(h) = L(h^{-1})$, for all $h \in H$

(iii) $c(h_1, h_3) \geq \min\{c(h_1, h_2), c(h_2, h_3)\}$, for all $h_1, h_2, h_3 \in H$

Where

$$c(g, h) := \frac{1}{2}\{L(g) + L(h) - L(g^{-1}h)\} \qquad g, h \in H$$

Let $H = \langle P(H_1, R)\rangle$ and $\Lambda = \mathbb{R}^n$. Then:

(i) is true by definition of $\mathbf{1_G}$

(ii) is true by definition of f^{-1}

For (iii), first note that $f^{-1}g$ always exists, and we have that $c(f, g) = \varepsilon_0(f^{-1}, g)$ by definition of the multiplication law in $\mathcal{RF}(G, \mathbb{R}^n)$.

Now let $f, g, h \in \mathcal{RF}(G, \mathbb{R}^n)$. If $c(f, g) = 0$ or $c(g, h) = 0$ there is nothing to prove as $c(f, h) \geq 0$ by definition.

So let $c(f, g), c(g, h) > 0$. By the above, we know that $c(f, g) = \varepsilon_0(f^{-1}, g)$ and $c(g, h) = \varepsilon_0(g^{-1}, h)$. Therefore $\varepsilon_0(f^{-1}, h) \geq \min\{\varepsilon_0(f^{-1}, g), \varepsilon_0(g^{-1}, h)\}$ implies that $c(f, h) \geq \min\{c(f, g), c(g, h)\}$.

Let $\varepsilon_0(f^{-1}, g) = c(f, g) = \varepsilon_0$, so the initial segments of f and g of length ε_0 are equal.

If $\varepsilon_0(g^{-1}, h) = c(g^{-1}, h) = \varepsilon_0'$, then the initial segments of g and h of length ε_0' are equal.

Let $\varepsilon = \min\{\varepsilon_0, \varepsilon_0'\}$. then the initial segments of f and g of length ε are equal, but then this initial segment must also be equal to the initial segment of h of length ε. Hence the initial segments of f and h of length ε are equal and therefore $c(f, h) = \varepsilon_0(f^{-1}, h) \geq \varepsilon = \min\{c(f, g), c(g, h)\}$ as required.

□

Chapter 5

Periods

5.1 Centralisers of elements of $\mathcal{CDF}(G, \mathbb{R}^n)$ with non-zero lengths.

In [2] the authors prove that the centre of $\mathcal{RF}(G)$ is trivial for all groups G. This is because elements of positive length can only commute with an element of zero length if that element is $\mathbf{1_G}$. Therefore they decide to look at the centralisers of all $f \in \mathcal{RF}(G)$ with positive length. These elements are *hyperbolic*, due to their action on the R-tree corresponding to their work. Those elements with length zero are called *elliptic*. For a detailed description of these terms see [1]. In order to look at the centralisers of hyperbolic elements they introduce periods and strong periods.

In $\mathcal{RF}(G, \mathbb{R}^n)$ the centre is also trivial, for the same reasons, so in this Chapter I define periods and strong periods for elements $f \in \mathcal{RF}(G, \mathbb{R}^n)$ with positive length, and then use them to describe the centralisers of $f \in \langle P(H_1, R) \rangle$ with positive length. I will also call these elements hyperbolic and those of length zero elliptic. I find that there are two types of hyperbolic element in this group: those with periods, ω, such that $f|_{[0,\omega]}$ has non-trivial cyclic centralisers in H_1, and those that don't. In [2] the authors touch upon these two types of elements at the end of their section on periods, but I must separate them at the start because they behave significantly differently in higher dimensions.

First I need to define the sets of periods and strong periods, Ω_f and Ω_f^0, for $f \in \mathcal{RF}(G, \Lambda)$, which I do in the following way:

Definition 5.1.1. *Let $f \in \mathcal{RF}(G, \Lambda)$ have length $L(f) = \alpha > 0$. Then*

(i) the set of periods *of $f \in \mathcal{RF}(G, \Lambda)$ is the set*

$$\Omega_f = \left\{ \begin{array}{c} \text{The points } \omega \in [0, \alpha]_\Lambda \text{ such that } f(x) = f(y) \\ \text{for all } x, y \in (0, \alpha]_\Lambda \text{ with } |x - y| = \omega \end{array} \right\}$$

(ii) the set of strong periods *of $f \in \mathcal{RF}(G, \Lambda)$ is the set*

$$\Omega_f^0 = \{\omega \in \Omega_f \text{ such that } \alpha - \omega \in \Omega_f\}$$

Notes:

(a) For $f \in \langle P(H_1, R) \rangle$, if for $\omega \in \Omega_f$ with $\omega \neq 0$, $\dim(\omega) = i$, then by the definition of exponentiation we have that for all $x_{n-1}, \ldots, x_{n-i-1} \in \mathbb{R}$ such that $\omega + (x_{n-1}, \ldots, x_{n-i-1}, 0, \ldots, 0) \leq L(f)$.

$$f|_{[0,\omega]} = f|_{[(x_{n-1},\ldots,x_{n-i-1},0,\ldots,0),\omega+(x_{n-1},\ldots,x_{n-i-1},0,\ldots,0)]}$$

(b) Let $\alpha = L(f)$. Then $0, \alpha \in \Omega_f^0$.

(c) Here we are looking at a general densely ordered abelian group again. For the rest of this thesis we will go back to the example where $\Lambda = \mathbb{R}^n$. If we let $\Lambda = \mathbb{R}$ in the above definition we get precisely Definition 8.3 from [2].

From now on assume that $L(f) = \alpha > 0$.

In the group generated by $P(H_1, R)$ there are two types of element. They are defined as follows:

Definition 5.1.2. Type 1 *elements are those* $f \in \langle P(H_1, R)\rangle$ *for which there exists* $\omega \in \Omega_f^0$ *such that* $\dim(\omega) = 1$ *and for which* $f|_{[0,\omega]}$ *has non-trivial cyclic centralisers in the group* H_1.

All other elements are of Type 2.

The Type 2 elements have centralisers like those of Chiswell and Müller's hyperbolic elements, while those of Type 1 are slightly different, as I will now prove.

The next Lemma gives an insight into where these differences occur.

Lemma 5.1.1. *(i) If* $f \in \langle P(H_1, R)\rangle$ *is of Type 1, there exists* $\omega_k \in \Omega_f^0$ *such that* $\dim(\omega_k) = k$ *for all* $1 \leq k \leq \dim(f)$

(ii) If $f \in \langle P(H_1, R)\rangle$ *is of Type 2, then* $\dim(\omega) = \dim(f)$ *for all* $\omega \in \Omega_f^0 \setminus \{0\}$

Proof. (i) Let f be of Type 1.

Since $L(f) > 0$, $\dim(f) \neq 0$.

If $\dim(f) = 1$, any $\omega \in \Omega_f^0 \setminus \{0\}$ must be of dimension 1 and $L(f) \in \Omega_f^0 \setminus \{0\}$, so there exists $\omega \in \Omega_f^0$ such that $\dim(\omega) = 1$, and we are done.

So assume that $\dim(f) = l > 1$.

By definition, there exists $\omega \in \Omega_f^0$ such that $\dim(\omega) = 1$.

By Lemma 4.1.4, f is of the form $g_1 \circ u_1^{\alpha_1} \circ g_2 \circ u_2^{\alpha_2} \ldots \circ g_m \circ u_m^{\alpha_m} \circ g_{m+1}$, where $m \geq 1$, g_i does not end in $u_i^{\pm 1}$, $\dim(\alpha_i) > 1$, $g_i, g_{m+1} \in H_1$, $u_i \in R$ for all $1 \leq i \leq m$ and either $[u_i, u_{i+1}] \neq \mathbf{1_G}$ or $[u_i, g_i] \neq \mathbf{1_G}$.

Now $\dim(\alpha_1) > 1$, so we have a segment of f, $u_1^{\alpha_1}$, such that $\dim(u_1^{\alpha_1}) > 1$ and a period $\omega \in \Omega_f^0$ such that $\dim(\omega) = 1$. Therefore

$$\omega = L(u_{11} \circ u_1^a \circ u_{12})$$

where $u_1 = u_{11}' \circ u_{11} = u_{12} \circ u_{12}'$ for some $a \in \mathbb{N}_0$ and $f|_{[\omega,2\omega]} = u_{11} \circ u_1^a \circ u_{12}$.

Let u_{112} be such that $L(u_{112}) = 0$ and $u_{112}(0) = u_{12}(L(u_{12}))$. Then

$$f|_{[0,n\omega]} = (u_{11} \circ u_1^a \circ u_{12} \circ u_{112}^{-1})^n = u_{11} \circ u_1^{na} \circ u_2$$

for some u_2 such that $L(u_2) = nL(u_{12}) + (n-1)L(u_{11})$.

Looking at the initial segments of length $L(u_{11} \circ u_1^a \circ u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_1)$, we see that

$$u_{11}u_1^a \circ u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_1 = u_{11}u_1^a \circ u_1 \circ u_3$$

where $u_3 = u_1^s \circ u_{13}$, $L(u_3) = L(u_{12} \circ u_{112}^{-1} \circ u_{11})$, $s \in \{0,1\}$ and $u_1 = u_{13} \circ u_{13}'$.

If $L(u_1) < L(u_{12} \circ u_{112}^{-1} \circ u_{11})$, we have that $s = 1$ and $u_{12} \circ u_{112}^{-1} \circ u_{11} = u_1 \circ u_{14}$, where $u_1 = u_{14} \circ u_{14}'$ and $L(u_{14}) = L(u_{13})$. But then we can choose the terminal point of u_{14} to be equal to that of u_{13} to get $u_{14} = u_{13}$ and therefore $u_{14}' = u_{13}'$.

Hence $u_1 \circ u_{14} \circ u_1 = u_1 \circ u_1 \circ u_{14}$, which implies that $[u_1, u_{14}] = \mathbf{1_G}$. But since $u_1 \in R$, $u_1u_{14} = u_1 \circ u_{14}$ and $L(u_1) \geq L(u_{14}) > 0$, we must have $u_1 = u_{14}$.

So $u_{12} \circ u_{112}^{-1} \circ u_{11} = u_1 \circ u_1$. But then $L(u_{11}), L(u_{12}) \leq L(u_1)$ and $L(u_{112}) = 0$, which implies that $L(u_{11}) = L(u_{12}) = L(u_1)$ and $u_{112}^{-1} = u_{12}' \circ u_{11}'$. So

$$\begin{aligned}
f|_{[0,2\omega]} =& u_{11} \circ (u_{11}' \circ u_{11})^a \circ u_{12} \circ u_{11} \circ (u_{11}' \circ u_{11})^a \circ u_{12} \\
=& (u_{11} \circ u_{11}')^a \circ u_{11} \circ u_{12} \circ (u_{11} \circ u_{11}')^a \circ u_{11} \circ u_{12} \\
=& (u_{11} \circ u_{11}')^{a+1} \circ u_{11}'^{-1} \circ u_{12} \circ (u_{11} \circ u_{11}')^{a+1} \circ u_{11}'^{-1} \circ u_{12} \\
=& (u_{11} \circ u_{11}')^{a+1} \circ u_{11} \circ u_{12}'^{-1} \circ (u_{11} \circ u_{11}')^{a+1} \circ u_{11} \circ u_{12}'^{-1} \\
=& (u_{11} \circ u_{11}')^{a+2} \circ u_{11}'^{-1} \circ u_{12}'^{-1} \circ (u_{11} \circ u_{11}')^{a+2} \circ u_{11}'^{-1} \circ u_{12}'^{-1} \\
=& (u_{11} \circ u_{11}')^{a+2} \circ u_{112} \circ (u_{11} \circ u_{11}')^{a+2} \circ u_{112} \\
=& u_{11} \circ u_1^{2a+3} \circ u_{112} \\
=& (u_{11} \circ u_{11}')^{2a+4} \circ u_{112}
\end{aligned}$$

But then

$$\begin{aligned}
f(\omega) =& u_{11}(L(u_{11}))u_{11}'(0)u_{112}(0)u_{11}(0) \\
=& u_{11}(L(u_{11}))u_{11}'(0)u_{11}(0)
\end{aligned}$$

and hence $u_{112} = \mathbf{1_G}$.

This in turn implies that $f = (u_{11} \circ u_{11}')^{g(t)}$ for some $g(t) \in \mathbb{Z}[t]/p(t)$, and $u_{11} \circ u_{11}')^{t^i} \in \Omega_f^0$ for all $0 \leq i \leq \dim(f)$. So we are done.

If $L(u_1) > L(u_{12} \circ u_{112}^{-1} \circ u_{11})$, $s = 0$ and $u_1 = u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_{15} = u_{15} \circ u_3$. But

$$u_1 \circ u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_1^a \circ u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_1^a \circ u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_1 = u_1^{2a+2} \circ u_4$$

for some u_4 such that $L(u_4) = 3L(u_{12} \circ u_{112}^{-1} \circ u_{11})$.

If $a \geq 1$, we have that $u_1 \circ u_1 \circ u_1 = u_1 \circ u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_1 \circ u_{16}$, which means that

$$\begin{aligned} u_1 \circ u_1 =& u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_1 \circ u_{16} \\ =& u_1 \circ u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_{17} \end{aligned}$$

If $L(u_{12} \circ u_{112}^{-1} \circ u_{11}) > 0$, $u_1(0) = (u_{12} \circ u_{112}^{-1} \circ u_{11})(0)$, so $u_{16}(0) = u_{17}(0)$, which implies that $[u_1, u_{12} \circ u_{112}^{-1} \circ u_{11}] = \mathbf{1_G}$, and since $L(u_1) = L(u_{12} \circ u_{112}^{-1} \circ u_{11})$ and $u_1 \in R$, we must therefore have that $L(u_{12} \circ u_{112}^{-1} \circ u_{11}) = 0$. Contradiction.

So let $L(u_{12} \circ u_{112}^{-1} \circ u_{11}) = 0$. Then $u_1 \circ u_1 = u_1 \circ u_{12} \circ u_{112}^{-1} \circ u_{11} \circ u_1$, and hence $u_{12} \circ u_{112}^{-1} \circ u_{11} = \mathbf{1_G}$. But then

$$\begin{aligned} f =& u_{11} \circ u_1^{g(t)} \circ u_{12} \circ u_{112}^{-1} \\ =& (u_{11} \circ u_1 \circ u_{12} \circ u_{112}^{-1})^{g(t)} \end{aligned}$$

for some $g(t) \in \mathbb{Z}[t]/p(t)$, and $L(u_{11} \circ u_1^{h(t)} \circ u_{12} \circ u_{112}^{-1}) \in \Omega_f^0$ for all $h(t) \in \mathbb{Z}[t]/p(t)$ such that $h(t) \leq g(t)$, so, as above, we are done.

If $a = 0$, $(u_{11} \circ u_{12} \circ u_{112}^{-1})^r \circ u_{18} = u_1$ for some $r \in \mathbb{N}$ and u_{18} such that $L(u_{18}) < L(u_{11} \circ u_{12} \circ u_{112}^{-1})$, and

$$\begin{aligned} u_1^2 =& (u_{11} \circ u_{12} \circ u_{112}^{-1})^{2r} \circ u_{18} \circ u_{19} \\ =& u_1 \circ (u_{11} \circ u_{12} \circ u_{112}^{-1})^r \circ u_{18} \\ =& (u_{11} \circ u_{12} \circ u_{112}^{-1})^r \circ u_1 \circ u_{19} \end{aligned}$$

So $[u_{11} \circ u_{12} \circ u_{112}^{-1}, u_1] = \mathbf{1_G}$, and with $u_1 \in R$ and $L(u_{11} \circ u_{12} \circ u_{112}^{-1}) < L(u_1)$, we have that $u_{11} \circ u_{12} \circ u_{112}^{-1} = \mathbf{1_G}$, which implies that $L(u_{18}) < 0$. Contradiction.

If $L(u_1) = L(u_{12} \circ u_{112}^{-1} \circ u_{11})$ we have $u_{12} \circ u_{112}^{-1} \circ u_{11} = u_1$. But this implies that $f = (u_{11} \circ u_{12} \circ u_{112}^{-1})^{g(t)+1}$ for some $g(t) \in \mathbb{Z}[t]/p(t)$, and also that $L(u_{11} \circ u_{12} \circ u_{112}^{-1})^{h(t)} \in \Omega_f^0$ for all $h(t) \in \mathbb{Z}[t]/p(t)$ such that $h(t) \leq g(t) + 1$.

So in each case we have $u_{11} \circ u_1^{t^{k-1}} \circ u_{12} \in \Omega_f^0$ for all $1 \leq k \leq \dim(f)$ which is what we needed.

(ii) Let f be of Type 2.

Let ω have dimension $k = \min\{\dim \omega | \omega \in \Omega_f^0 \backslash \{0\}\}$.

Again by Lemma 4.1.4, ω must be of the form $g_1 \circ u_1^{\alpha_1} \circ g_2 \circ \ldots \circ g_m \circ u_m^{\alpha_m} \circ g_{m+1}$ for some $u_i \in R$, $g_i \in H_1$, with $m \geq 1$ and $\dim(\alpha_i) > 1$ for all $1 \leq i \leq m$, unless $m = 1$.

Now, if $\dim(\alpha_1) > 1$, by the minimality of ω, $t\omega$ is not defined, so f cannot be of a higher dimension than ω and hence cannot have any higher dimension periods. Therefore the only possible periods are 0 and elements of the same dimension as ω, which are the same dimension as f.

If $m = 1$ and $\dim(\alpha_1) = 1$, then $\dim(\omega) = 1$, but f is of Type 2, so $f|_{[0,\omega]}$ doesn't have non-trivial cyclic centralisers in H_1. Hence $f|_{[0,\omega]}$ is one of the Type 2 elements in [2] and $f \notin R$, therefore $t\omega$ is not defined. Hence $\dim(f) = \dim(\omega) = 1$. So any $\omega' \in \Omega_f^0$ has dimension 0 or $1 = \dim(f)$ as required.

$\square$

I now start to follow the structure of Chapter 8 of [2].

Definition 5.1.3. *Let $\langle \Omega_f^0 \rangle$ be the subgroup of $(\mathbb{R}^n, +)$ generated by the set Ω_f^0.*

If we take $n = 1$ then this group is the same as $\langle \Omega_f^0 \rangle$ in Chapter 8 of [2], but I need to look at a different subgroup to take into account the $\mathbb{Z}[t]/p(t)$-exponentiation of one dimensional elements $f|_{[0,\omega]}$ for $\omega \in \Omega_f^0$, for f of Type 1. Therefore I define the exponential of this subgroup as follows:

Recall from Definition 3.1.2 that for one dimensional $(0, \ldots, 0, \alpha_0) = \alpha \in \mathbb{R}^n$, given $g(t) = g_{n-1}t^{n-1} + \ldots + g_1 t + g_0 \in \mathbb{Z}[t]/p(t)$, $g(t)\alpha = (g_{n-1}\alpha_0, \ldots, g_1\alpha_0, g_0\alpha_0)$.

Definition 5.1.4. *Let*

$$\langle\Omega_f^0\rangle^t = \begin{cases} \{g(t)\omega | \omega \in \Omega_f^0, \dim(\omega) = 1, g(t) \in \mathbb{Z}[t]/p(t)\}.\langle\Omega_f^0\rangle & \textit{if } f \textit{ is Type 1} \\ \langle\Omega_f^0\rangle & \textit{if } f \textit{ is Type 2} \end{cases}$$

This is a subgroup of $\mathbb{R}^n$.

By Lemma 5.1.1, all elements of $\langle\Omega_f^0\rangle^t$ are of the same dimension if f is of Type 2, whereas elements of $\langle\Omega_f^0\rangle^t$ can have different dimensions when f is of Type 1. This is where the different structures of the centralisers are seen.

The following Lemma gives us a general form for any $\omega \in \langle\Omega_f^0\rangle^t$ for both types of f, though (iii), (vi) and (vii) only apply to Type 1 elements, since $g(t)\omega$ is not defined for Type 2 elements. The non Type 1 parts are from [2], Lemma 8.5.

Lemma 5.1.2. *(i)* $\omega_1, \omega_2 \in \Omega_f$ *and* $\omega_1 + \omega_2 \in [0, \alpha] \Longrightarrow \omega_1 + \omega_2 \in \Omega_f$.

(ii) $\omega_1, \ldots, \omega_r \in \Omega_f^0$, *with* $r \geq 1$ *and* $\omega_1 + \ldots + \omega_r \in [0, \alpha] \Longrightarrow \omega_1 + \ldots + \omega_r \in \Omega_f^0$.

(iii) Let f *be of Type 1, then* $\omega \in \Omega_f^0$, $\dim(\omega) = 1$, $g(t) \in \mathbb{Z}[t]/p(t)$ *and* $g(t)\omega \in [0, \alpha] \Longrightarrow g(t)\omega \in \Omega_f^0$.

(iv) $\omega_1, \omega_2 \in \Omega_f^0$ *and* $\omega_1 - \omega_2 \in [0, \alpha] \Longrightarrow \omega_1 - \omega_2 \in \Omega_f^0$.

(v) $\omega_1, \ldots, \omega_r \in \Omega_f^0$, *with* $r \geq 1 \Longrightarrow \omega_1 + \ldots + \omega_r = k\alpha + \omega$ *for some* $k \in \mathbb{N}_0$ *and* $\omega \in \Omega_f^0 \setminus \{\alpha\}$.

(vi) Let f *be of Type 1, then for* $\omega \in \Omega_f^0$ *such that* $\dim(\omega) = 1$, *and for* $g(t) \in \mathbb{Z}[t]/p(t)$, $g(t)\omega = h(t)\alpha + \omega'$ *for some* $h(t) \in \mathbb{Z}[t]/p(t)$ *and* $\omega' \in \Omega_f^0$ *where* $\alpha = (\alpha_{n-1}, \ldots, \alpha_1, \alpha_0)$ *and* $h(t)\alpha$ *is the vector*

$$((h\alpha)_{n-1}, \ldots, (h\alpha)_1, (h\alpha)_0)$$

associated to

$$\begin{aligned} h(t)\alpha(t) =& (h_{n-1}t^{n-1} + \ldots + h_0)(\alpha_{n-1}t^{n-1} + \ldots + \alpha_0) \mod p(t) \\ =& \sum_{k=0}^{n-1} (h\alpha)_k t^k. \end{aligned}$$

(vii) For f of Type 1, $g_1(t)\omega_{11} + \ldots + g_r(t)\omega_{1r} + \omega_{21} + \ldots + \omega_{2s} = h(t)\alpha + \omega'$ where $r, s \geq 0$ and $r + s \geq 1$ for some $h(t), g_1(t), \ldots, g_r(t) \in \mathbb{Z}[t]/p(t)$, $h(t)\alpha$ as above, $\omega', \omega_{11}, \ldots, \omega_{1r}, \omega_{21}, \ldots, \omega_{2s} \in \Omega_f^0$ with $\dim(\omega_{1i}) = 1$ for all $1 \leq i \leq r$.

(viii) $\langle \Omega_f^0 \rangle^t \cap [0, \alpha] = \Omega_f^0$.

Proof. (i) $\omega_1, \omega_2 \in \Omega_f$ and $\omega_1 + \omega_2 \in [0, \alpha] \Longrightarrow \omega_1 + \omega_2 \in \Omega_f$

Consider $x, y \in (0, \alpha]$ such that $|x - y| = \omega_1 + \omega_2$.

Without loss of generality, assume that $x \leq y$. Then

$$0 < x \leq x + \omega_1 \leq x + \omega_1 + \omega_2 = y$$

But

$$\begin{aligned} f(x) =& f(x + \omega_1) && \text{since } \omega_1 \in \Omega_f \\ =& f((x + \omega_1) + \omega_2) && \text{since } \omega_2 \in \Omega_f \\ =& f(y) \end{aligned}$$

as required.

Hence $\omega_1 + \omega_2 \in \Omega_f$.

(ii) $\omega_1, \ldots, \omega_r \in \Omega_f^0$, with $r \geq 1$ and $\omega_1 + \ldots + \omega_r \in [0, \alpha] \Longrightarrow \omega_1 + \ldots + \omega_r \in \Omega_f^0$

We do this by induction on r

Case 1: $r = 1$.

$\omega_1 \in \Omega_f^0$ and $\omega_1 \in [0, \alpha]$ therefore this is true for $r = 1$.

Case 2: $r = 2$.

$\omega_1, \omega_2 \in \Omega_f^0$ implies $\omega_1, \omega_2 \in \Omega_f$ which implies $\omega_1 + \omega_2 \in \Omega_f$ by (i), so we just need to prove that $\alpha - (\omega_1 + \omega_2) \in \Omega_f$.

Let $x, y \in (0, \alpha]$ be such that $|x - y| = \alpha - (\omega_1 + \omega_2)$.

Without loss of generality assume that $x \leq y$. If $x > \omega_1$, $x - \omega_1 \in (0, \alpha]$ and therefore

$$\begin{aligned} f(y) =& f(x + \alpha - (\omega_1 + \omega_2)) \\ =& f((x - \omega_1) + (\alpha - \omega_2)) \\ =& f(x - \omega_1) && \text{since } \omega_2 \in \Omega_f^0 \\ =& f(x) && \text{since } \omega_1 \in \Omega_f^0 \end{aligned}$$

as required.

If, however $x \leq \omega_1$,

$$y = x + \alpha - \omega_1 - \omega_2 = x - \omega_1 + \alpha - \omega_2 \leq \alpha - \omega_2$$

so

$$0 < y + \omega_2 = x + \alpha - \omega_1 \leq \alpha$$

and

$$\begin{aligned} f(y) =& f(y + \omega_2) && \text{since } \omega_2 \in \Omega_f^0 \\ =& f(x + \alpha - \omega_1) \\ =& f(x) && \text{since } \omega_1 \in \Omega_f^0 \end{aligned}$$

as required. Therefore $\alpha - (\omega_1 + \omega_2) \in \Omega_f$, so $\omega_1 + \omega_2 \in \Omega_f^0$.

Case s: $r = s$.

Assume that $\omega_1, \ldots, \omega_s \in \Omega_f^0$ and $\omega_1 + \ldots + \omega_s \in [0, \alpha]$, with $s \geq 2$. Then by induction we know that

$$(\omega_1 + \ldots + \omega_{s-1}) \in \Omega_f^0$$

so by Case 2, we see that

$$(\omega_1 + \ldots + \omega_{s-1}) + \omega_s \in \Omega_f^0$$

and we are done.

(iii)Let f be of Type 1, then $\omega \in \Omega_f^0$, $\dim(\omega) = 1$, $g(t) \in \mathbb{Z}[t]/p(t)$ and $g(t)\omega \in [0, \alpha] \Longrightarrow g(t)\omega \in \Omega_f^0$.

Given $\omega \in \Omega_f^0$ with $\dim(\omega) = 1$, let $g(t)\omega \in [0, \alpha]$, and let $\dim(f) = l$. First we need to prove that, if $l > 1$, the only possible $\omega \in \Omega_f^0$ of dimension 1 are those of the form $(u_1 \circ u_2)^s$ for some $s \in \mathbb{Z}$ and $(u_2 \circ u_1) \in R$.

By the proof of Lemma 5.1.1, $f = (u_1 \circ u_2)^{h(t)}$ for some $h(t) \in \mathbb{Z}[t]/p(t)$, $(u_2 \circ u_1) \in R$.

If $\omega \neq L((u_1 \circ u_2)^s)$ for some $s \in \mathbb{N}$, then $\omega = L((u_1 \circ u_2)^r \circ u_3)$ for some $r \in \mathbb{N}_0$ and u_3 such that $u_1 \circ u_2 = u_3 \circ u_4$.

Looking at the initial segments of f and $f|_{[\omega, 2\omega]}$, we see that the initial segments of u_1 and u_3 are the same, and we have

$$(u_1 \circ u_2)^r \circ u_3 = (u_3 \circ u_4)^r \circ u_5$$

for some u_5 such that $u_3 \circ u_4 = u_5 \circ u_6$ and $L(u_5) = L(u_3)$, but looking at the terminal segments of $f|_{[0,\omega]}$ and $f|_{[0,2\omega]}$, we see that $u_5 = u_3$. Without loss of generality assume that $L(u_3) \leq L(u_1)$.

This implies that $u_1 = u_3 \circ u_3'$ for some $u_3 \in H_1$, giving

$$(u_3 \circ u_3' \circ u_2)^r \circ u_3 = (u_3 \circ u_4)^r \circ u_3$$

Looking at the initial segments of length $L(u_3 \circ u_4)$, we find $u_3 \circ u_3' \circ u_2 = u_3 \circ u_4$, so $u_3' \circ u_2 = u_4$.

Looking at the initial segment of f of length $L((u_3 \circ u_3' \circ u_2)^{r+1} \circ u_3)$, we find that

$$(u_3 \circ u_3' \circ u_2)^r \circ u_3 \circ u_3' \circ u_2 \circ u_3 = (u_3 \circ u_3' \circ u_2)^r \circ u_3 \circ u_3 \circ u_3' \circ u_2$$

But looking at the terminal segments of this of length $L(u_3' \circ u_2 \circ u_3)$, we see that

$$(u_3' \circ u_2) \circ u_3 = u_3' \circ (u_2 \circ u_3)$$

and hence $[u_3, u_3' \circ u_2] = \mathbf{1_G}$.

But then u_3 and $u_3' \circ u_2$ are powers of a common element, which makes $u_1 \circ u_2 = u_3 \circ u_3' \circ u_2$ a proper power. Hence $u_2 \circ u_1$ is a proper power, contradicting the fact that $u_2 \circ u_1 \in R$.

Therefore we must have that ω is of the form $L((u_1 \circ u_2)^s)$ for some $s \in \mathbb{N}$.

Now, by the definition of exponentiation, we have that $L(u^t) = tL(u)$, so if $\omega = L(u)$, then $g(t)\omega = L(u^{g(t)})$. All we need to prove is that $L(u^{g(t)}) \in \Omega_f^0$ for all $g(t)$ such that $L(u^{g(t)}) \leq \alpha$. By the proof of Lemma 5.1.1 (i), this is true for $l > 1$, so we only need to look at the case where $l = 1$.

In this case we must have that $g(t) \in \mathbb{Z}$ since otherwise $\dim(g(t)\omega) > l = 1$, which implies that $g(t)\omega \notin \Omega_f^0$. But $g(t)\omega = g_0\omega \in \Omega_f^0$ by (ii), so we are done.

(iv) $\omega_1, \omega_2 \in \Omega_f^0$ and $\omega_1 - \omega_2 \in [0, \alpha] \Longrightarrow \omega_1 - \omega_2 \in \Omega_f^0$

If $\omega_1, \omega_2 \in \Omega_f^0$, then

$$\alpha - (\omega_1 - \omega_2) = (\alpha - \omega_1) + \omega_2 \in \Omega_f$$

by parts (i) and (ii), so we need only prove that $\omega_1 - \omega_2 \in \Omega_f$.

Let $x, y \in (0, \alpha]$ be such that $|x - y| = \omega_1 - \omega_2$ and without loss of generality let $x \leq y$.

If $x > \omega_2$, then $x - \omega_2 \in (0, \alpha]$ and $y - \omega_1 = x - \omega_2$, so

$$\begin{aligned} f(x) =& f(x - \omega_2) && \text{since } \omega_2 \in \Omega_f^0 \\ =& f(y - \omega_1) \\ =& f(y) && \text{since } \omega_1 \in \Omega_f^0 \end{aligned}$$

as required.

If, however, $x \leq \omega_2$, then $y \leq \omega_1$ and

$$y + \alpha - \omega_1 = \alpha + y - \omega_1 = \alpha + x - \omega_2 = x + \alpha - \omega_2$$

and

$$\begin{aligned} f(x) =& f(x+\alpha-\omega_2) && \text{since } \omega_2 \in \Omega_f^0 \\ =& f(y+\alpha-\omega_1) && \\ =& f(y) && \text{since } \omega_1 \in \Omega_f^0 \end{aligned}$$

as required.

Hence $\omega_1 - \omega_2 \in \Omega_f$ and therefore $\omega_1 - \omega_2 \in \Omega_f^0$.

(v) $\omega_1, \ldots, \omega_r \in \Omega_f^0$, with $r \geq 1 \Longrightarrow \omega_1 + \ldots + \omega_r = k\alpha + \omega$ for some $k \in \mathbb{N}_0$ and $\omega \in \Omega_f^0 \setminus \{\alpha\}$.

Set

$$s_p = \sum_{1 \leq j \leq p} \omega_j \qquad 0 \leq p \leq r$$

So let $p = 1$.

$$s_1 = \omega_1 = \begin{cases} 0.\alpha + \omega_1 & \text{if } \omega_1 \neq \alpha \\ 1.\alpha + 0 & \text{if } \omega_1 = \alpha \end{cases}$$

as required.

Let $p = 2$.

$s_2 = \omega_1 + \omega_2$.

If $\omega_1 + \omega_2 \in (0, \alpha]$ then $\omega_1 + \omega_2 \in \Omega_f^0$ and we are in the case $p = 1$ above.

If $\omega_1 + \omega_2 > \alpha$, then $\alpha < \omega_1 + \omega_2 \leq 2\alpha$ with equality iff $\omega_1 = \omega_2 = \alpha$.

If $\omega_1 = \omega_2 = \alpha$, then $\omega_1 + \omega_2 = 2\alpha + 0$ as required.

If not, then $\omega_1 + \omega_2 = \alpha + \omega'$ with $\omega' > 0$. But then

$$\alpha > \omega_1 - (\alpha - \omega_2) = \omega' > 0$$

So $\omega' = \omega_1 - (\alpha - \omega_2) \in (0, \alpha]$, and since $\omega_1, \omega_2 \in \Omega_f^0$ we find that $\omega' \in \Omega_f^0 \setminus \{\alpha\}$ by part (iii).

Hence $\omega_1 + \omega_2 = s_2 = 1.\alpha + \omega'$ as required.

Now

$$
\begin{aligned}
s_p = &\sum_{1\le j\le p} \omega_j \qquad 0 \le p \le r\\
=&(\sum_{1\le j\le p-1} \omega_j) + \omega_p\\
=&s_{p-1} + \omega_p
\end{aligned}
$$

Assume $s_{p-1} = k\alpha + \omega'$ for some $k \in \mathbb{N}_0, \omega' \in \Omega_f^0 \setminus \{\alpha\}$.

Then $s_p = k\alpha + \omega' + \omega_p$.

If $\omega' = 0$, then

$$
s_p = \begin{cases} k\alpha + \omega_p & \omega_p \in \Omega_f^0 \setminus \{\alpha\} \\ (k+1)\alpha + 0 & \omega_p = \alpha \end{cases}
$$

as required.

If not then

$$
\omega' + \omega_p = \begin{cases} 0.\alpha + (\omega' + \omega_p) & \omega' + \omega_p \in \Omega_f^0 \setminus \{\alpha\} \\ 1.\alpha + \omega'_p & \omega' + \omega_p \ge \alpha \text{ and } \omega'_p = \omega' + \omega_p - \alpha \end{cases}
$$

But $\omega'_p \in \Omega_f^0 \setminus \{\alpha\}$ by case $p = 2$ since $\omega', \omega_p < \alpha$.

Hence

$$
s_p = \begin{cases} k\alpha + (\omega' + \omega_p) & \omega' + \omega_p \in \Omega_f^0 \setminus \{\alpha\}, k \in \mathbb{N}_0 \\ (k+1)\alpha + 0 & (k+1) \in \mathbb{N}_0, 0 \in \Omega_f^0 \setminus \{\alpha\} \\ (k+1)\alpha + \omega'_p & (k+1) \in \mathbb{N}_0, \omega'_p \in \Omega_f^0 \setminus \{\alpha\} \end{cases}
$$

as required.

(vi) Let f be of Type 1, then for all $\omega \in \Omega_f^0$ such that $\dim(\omega) = 1$, and for all $g(t) \in \mathbb{Z}[t]/p(t)$, $g(t)\omega = h(t)\alpha + \omega'$ for some $h(t) \in \mathbb{Z}[t]/p(t)$ and $\omega' \in \Omega_f^0$, where $\alpha = (\alpha_{n-1}, \dots, \alpha_1, \alpha_0)$ and $h(t)\alpha$ is the vector $((h\alpha)_{n-1}, \dots, (h\alpha)_1, (h\alpha)_0)$ associated to

$$
\begin{aligned}
h(t)\alpha(t) =&(h_{n-1}t^{n-1} + \ldots + h_0)(\alpha_{n-1}t^{n-1} + \ldots + \alpha_0) \mod p(t)\\
=&\sum_{k=0}^{n-1} (h\alpha)_k t^k
\end{aligned}
$$

If $0 \leq g(t)\omega \leq \alpha$, then we can take $h(t) = 0$ and $\omega' = g(t)\omega$ or $h(t) = 1$ and $\omega' = 0$, so assume that $g(t)\omega > \alpha$.

We have that f is of Type 1, hence, if $\dim(f) > 1$, by the proof of Lemma 5.1.1, $f = (u_1 \circ u_2)^{q(t)}$,for some u_1, u_2 such that $u_2 \circ u_1 \in R$, and $q(t) \in \mathbb{Z}[t]/p(t)$.

Let $\omega_1 = L(u_1 \circ u_2) \in \Omega_f^0$. Then

$$L(f) = \alpha = L((u_1 \circ u_2)^{q(t)}) = q(t)L(u_1 \circ u_2) = q(t)\omega_1$$

where $q(t) \in \mathbb{Z}[t]/p(t)$.

Since $\dim(\omega) = 1$, by the proof of part (iii) we have that $\omega = s\omega_1$ for some $s \in \mathbb{N}$. So $g(t)\omega = sg(t)\omega_1$ for all $g(t) \in \mathbb{Z}[t]/p(t)$.

By Euclid's algorithm and polynomial division

$$sg(t) = r(t)q(t) + h(t)$$

for some $r(t), h(t) \in \mathbb{Z}[t]/p(t)$ such that $h(t) < q(t)$.

This means that

$$\begin{aligned} g(t)\omega =& (r(t)q(t) + h(t))\omega_1 \\ =& r(t)q(t)\omega_1 + h(t)\omega_1 \\ =& r(t)\alpha + \omega' \end{aligned}$$

but by part (v), $\omega' = h(t)\omega_1$ and $h(t) < q(t)$, so $\omega' \in \Omega_f^0$ and we are done.

(vii) For f of Type 1, $g_1(t)\omega_{11} + \ldots + g_r(t)\omega_{1r} + \omega_{21} + \ldots + \omega_{2s} = h(t)\alpha + \omega'$ where $r, s \geq 0$ and $r + s \geq 1$ for some $h(t), g_1(t), \ldots, g_r(t) \in \mathbb{Z}[t]/p(t)$, $h(t)\alpha$ as above, and $\omega', \omega_{11}, \ldots, \omega_{1r}\omega_{21} \ldots \omega_{2s} \in \Omega_f^0$ with $\dim(\omega_{1i}) = 1$ for all $1 \leq i \leq r$.

By part (vi),

$$g_i(t)\omega_{1i} = h_i(t)\alpha + \omega'_{1i}$$

for some $\omega'_{1i} \in \Omega_f^0$, $h_i(t) \in \mathbb{Z}[t]/p(t)$.

So

$$\begin{aligned}
\sum_{i=1}^{r}(g_i(t)\omega_{1i})+\sum_{j=1}^{s}(\omega_{2j}) =&(\sum_{i=1}^{r}(h_i(t)\alpha)+\sum_{i=1}^{r}(\omega_{1i}))+\sum_{j=1}^{s}(\omega_{2j})\\
=&(\sum_{i=1}^{r}h_i(t))\alpha+(\sum_{i=1}^{r}(\omega_{1i})+\sum_{j=1}^{s}(\omega_{2j}))\\
=&(\sum_{i=1}^{r}h_i(t)+k)\alpha+\omega'
\end{aligned}$$

since by part (iv) $\sum_{i=1}^{r}(\omega_i)+\sum_{j=1}^{s}(\omega_{2j})=k\alpha+\omega'$ for some $k\in\mathbb{N},\omega'\in\Omega_f^0$.

But $\dim(h_i(t)\alpha)\le n$ for all $1\le i\le r$ and $\dim(k)=1$, so

$$\dim((\sum_{i=1}^{r}h_i(t)+k)\alpha)\le n$$

So if we let $\sum_{i=1}^{r}h_i(t)+k=h(t)$, we are done.

$(viii)$ $\langle\Omega_f^0\rangle^t\cap[0,\alpha]=\Omega_f^0$.

Let $\omega\in\langle\Omega_f^0\rangle^t\cap[0,\alpha]$. Then $\omega=g(t)\alpha+\omega'$ for some $g(t)\in\mathbb{Z}[t]/p(t)$, with $\dim(g(t)\alpha)\le n$, and $\omega'\in\Omega_f^0\setminus\{\alpha\}$.

If $\omega'=0$, $\omega=g(t)\alpha$, therefore $g(t)\in\{0,1\}$ so $\omega\in\{0,\alpha\}$.

If $\omega'\neq 0$, $\omega'>0$ and therefore $g(t)\alpha+\omega'\in[0,\alpha]$ iff $g(t)=0$. Hence $\omega=\omega'+0=\omega'$, which means that $\omega\in(0,\alpha)$ and $\omega\in\Omega_f^0$ as required.

For the other direction, let $\omega\in\Omega_f^0$. Then $\omega\in[0,\alpha]$ and hence, trivially, $\omega\in\langle\Omega_f^0\rangle^t\cap[0,\alpha]$.

□

The following Corollary explicitly defines the form of $\omega\in\langle\Omega_f^0\rangle^t$ for both types of element. It comes from Corollary 8.6 of [2], but with an extra part to account for the Type 1 elements.

Corollary 5.1.1. *Every element ω of the group $\langle\Omega_f^0\rangle^t$ can be written in the form:*

$$\omega=\begin{cases} r(t)L(u) & r(t)\in\mathbb{Z}[t]/p(t) \quad \textit{if } f=u^{s(t)} \textit{ is Type 1}\\ \sigma(k\alpha+\omega^*) & (k,\omega^*)\in\mathbb{N}_0\times\Omega_f^0\setminus\{\alpha\},\sigma=\pm1 \quad \textit{if } f \textit{ is Type 2}\end{cases}$$

Where $u = h_1 \circ h_2$ *for some* $h_1, h_2 \in \mathcal{RF}(G, \mathbb{R}^n)$ *such that* $\mathbf{1_G} \neq h_2 \circ h_1 \in R$.

Proof. If f is of Type 1, then there is a smallest positive element of $\langle \Omega_f^0 \rangle^t$, which is $L(u)$. From Lemma 5.1.2, part (vii), we have that an element of $\langle \Omega_f^0 \rangle^t$ is of the form $h(t)\alpha + \omega'$ for some $h(t)$ such that $\dim(h(t)\alpha) \leq n-1$ and $\omega' \in \Omega_f^0$. Now $f = u^{s(t)}$, so $L(f) = \alpha = s(t)L(u)$. Also, by the definition of R, we have that any $\omega \in \Omega_f^0$ is of the form $q(t)L(u)$ for some $0 \leq q(t) \leq s(t)$. Hence we have

$$\begin{aligned}\omega =& h(t)s(t)L(u) + q(t)L(u)\\ =& r(t)L(u)\end{aligned}$$

where $r(t) = h(t)s(t) + q(t) \in \mathbb{Z}[t]/p(t)$.

If f is Type 2, then all of the periods are of the same dimension as f by Lemma 5.1.1. As we have noted above, in this case $\langle \Omega_f^0 \rangle^f = \langle \Omega_f^0 \rangle$.

First assume that $\omega > 0$.

If we let $\alpha = (\alpha_{n-1}, \ldots, \alpha_1, \alpha_0)$ and $\omega = (\omega_{n-1}, \ldots, \omega_1, \omega_0)$, then we can set polynomials $\alpha(t)$ and $\omega(t)$ to be equal to $\alpha_{n-1}t^{n-1} + \ldots + \alpha_1 t + \alpha_0$ and $\omega_{n-1}t^{n-1} + \ldots + \omega_1 t + \omega_0$ respectively, with both leading coefficients positive. Since f is of Type 2, $\deg(\alpha(t)) = \deg(\omega(t))$ and so by the division algorithm we can find a unique $\omega^*(t)$ and $r(t) \in \mathbb{Z}[t]/p(t)$ such that

$$\omega(t) = r(t)\alpha(t) + \omega^*(t)$$

with $0 \leq \omega^*(t) < \alpha(t)$. In fact, because $\deg(\alpha(t)) = \deg(\omega(t))$, we have that $r(t) = r \in \mathbb{Z}$.

Let $\omega^* = (\omega_{n-1}^*, \ldots, \omega_1^*, \omega_0^*)$, where ω_i^* is the ith coefficient of $\omega^*(t)$. Then $\omega^* = \omega - r\alpha \in \langle \Omega_f^0 \rangle^t \cap [0, \alpha]$, since $\omega, r\alpha \in \langle \Omega_f^0 \rangle^t$ and $0 \leq \omega^* < \alpha$. Hence, by Lemma 5.1.2 part $(viii)$, $\omega^* \in \Omega_f^0$ and $\omega = r\alpha + \omega^*$ is of the desired form with $\sigma = +1$.

If we let $\omega < 0$, then we find that $-\omega = r\alpha + \omega^*$ for some unique $r \in \mathbb{Z}$ and $\omega^* \in \Omega_f^0$. But then $\omega = -1(r\alpha + \omega^*)$, which is of the correct form, with $\sigma = -1$.

If $\omega = 0$, then $\omega = \pm 1(0\alpha + 0)$, which are both of the correct form.

This covers all possibilities for ω, so we are done. $\square$

Note that in the above Corollary, if we take $0 = (+1)(0\alpha + 0)$ and not $(-1)(0\alpha + 0)$ for Type 2 elements, this form is unique for each $\omega \in \langle \Omega_f^0 \rangle^t$ where f is of either Type

The next Lemma is the same as Lemma 8.7 in [2].

Lemma 5.1.3. *For $\omega \in \Omega_f$, the following are equivalent (see [2]):*

(i) $\omega \in \Omega_f^0$.

(ii) For all $\omega'' \in \Omega_f^0$ such that $\omega'' \geq \omega$, $\omega'' - \omega \in \Omega_f$.

(iii) There exists $\omega' \in \Omega_f$ and $\omega'' \in \Omega_f^0$ such that $\omega + \omega' = \omega''$.

Proof. $(i) \implies (ii)$:

Let $\omega \in \Omega_f^0$. Then for all $\omega'' \in \Omega_f^0$ such that $\omega'' \geq \omega$, $\omega'' - \omega \in [0, \alpha]$, so by Lemma 5.1.2 part (iv), $\omega'' - \omega \in \Omega_f$.

$(ii) \implies (iii)$:

If for all $\omega'' \in \Omega_f^0$ such that $\omega'' \geq \omega$, $\omega'' - \omega \in \Omega_f$, let $\omega'' - \omega = \omega'$, so $\omega'' = \omega + \omega'$. Then $\omega' \in \Omega_f$ and $\omega'' \in \Omega_f^0$ as required.

$(iii) \implies (i)$:

Assume there exists $\omega' \in \Omega_f$ and $\omega'' \in \Omega_f^0$ such that $\omega + \omega' = \omega''$. Then $\alpha - \omega'' \in \Omega_f$ since $\omega'' \in \Omega_f^0$, but

$$\begin{aligned} \alpha - \omega'' &= \alpha - (\omega + \omega') \\ &= (\alpha - \omega) - \omega' \end{aligned}$$

so $\alpha - \omega = (\alpha - \omega'') + \omega'$.

But $\alpha - \omega'', \omega' \in \Omega_f$, so by lemma 5.1.2 part (i), $\alpha - \omega \in \Omega_f$, hence $\omega \in \Omega_f^0$ as required. $\square$

5.2 C_f

The following Definition comes from Definition 8.1 in [2].

Definition 5.2.1. *f is said to be* normalised *if $f(0) = 1_G$.*

This means that $f^2(L(f)) = f(L(f))$.

For the rest of this Chapter we can assume that f is cyclically reduced and normalised, with length $L(f) = \alpha$. In fact, every element $g = a \circ b \circ a^{-1}$ with b cyclically reduced such that $L(b) > 0$, is conjugate to an element of this form (See [2], Lemma 8.2).

Given f as above, we now define some more sets. These sets are the same as those defined at the top of Section 8.4 in [2].

Definition 5.2.2. *Let $\mathcal{C}_f$ be the centraliser of f in $\langle P(H_1, R)\rangle$. Then*

$C_f^- := \{g \in \mathcal{C}_f | 0 < L(g) < \alpha$ *and* $\varepsilon_0(f,g) = 0\}$

$C_f^+ := \{g \in \mathcal{C}_f | L(g) \geq \alpha$ *and* $\varepsilon_0(f,g) = 0\}$

$C_f := C_f^- \cup C_f^+ = \{g \in \mathcal{C}_f | \varepsilon_0(f,g) = 0\} \setminus \{\mathbf{1_G}\}$

We first look at C_f^-, the elements of the centraliser of $f \in \langle P(H_1, R)\rangle$ which are shorter than f but not elliptic, and for which $L(fg) = L(f) + L(g)$.

The following Lemma is the same as Lemma 8.8 of [2].

Lemma 5.2.1. *The elements $g \in C_f^-$ are in $1-1$ correspondence with the non-trivial $\omega \in \Omega_f^0$ via $g \mapsto L(g)$ and inverses $\omega \mapsto f|_{[0,\omega]}$.*

Proof. Let $g \in C_f^-$. Then $gf = fg$ and, since $\varepsilon_0(f,g) = 0$, we must have that $gf = g * f = f * g$. So, if $L(g) = \omega > 0$ and $L(f) = \alpha$

$$g(x) = f(x) \qquad 0 \leq x < \omega$$

so $g(0) = f(0) = 1_G$, whilst

$$\begin{aligned} f(\omega) =& g(\omega)f(0) \\ =& g(\omega) \qquad \text{since } f(0) = 1_G \end{aligned}$$

Therefore $g = f|_{[0,\omega]}$ and g is normalised.

Also,

$$f * g = \begin{cases} f(x) & 0 \leq x < \alpha \\ f(\alpha)g(0) & x = \alpha \\ g(x-\alpha) & \alpha < x \leq \alpha + \omega \end{cases}$$
$$= g * f$$
$$= \begin{cases} g(x) & 0 \leq x < \omega \\ g(\omega)f(0) & x = \omega \\ f(x-\omega) & \omega < x \leq \alpha + \omega \end{cases}$$

so

$$f(x) = f(x-\omega) \qquad \omega < x < \alpha$$

and

$$\begin{aligned} f(\alpha - \omega) =& f(\alpha)g(0) \\ =& f(\alpha) \qquad \text{since } g(0) = 1_G \end{aligned}$$

which implies $f(x) = f(x+\omega)$ for $0 < x \leq \alpha - \omega$ and hence $\omega \in \Omega_f$.

Furthermore, for $\alpha < x \leq \alpha + \omega$,

$$f(x-\alpha) = g(x-\alpha) = f(x-\omega) \qquad (*)$$

So, for $\alpha - \omega < x \leq \alpha$

$$\begin{aligned} f(x - (\alpha - \omega)) =& f(x + \omega - \alpha) \\ =& f(x + \omega - \omega) \qquad \text{by } (*) \\ =& f(x) \end{aligned}$$

which implies $f(x) = f(x + (\alpha - \omega))$ for all $0 < x \leq \omega$, so $(\alpha - \omega) \in \Omega_f$ and hence $\omega \in \Omega_f^0$, whilst since $\omega > 0$, ω is non-trivial.

Also, the map $g \mapsto L(g)$ is injective as $g = f|_{[0,\omega]}$, which is a unique element.

For the reverse implication, let $\omega \in \Omega_f^0$ and define $g := f|_{[0,\omega]}$.

Since $g(0) = f(0) = 1_G$, g is normalised, $L(g) = \omega$ and

$$\varepsilon_0(g, f) = 0 = \varepsilon_0(f, g)$$

since $g \in \Omega_f^0$ and f is reduced. Therefore

$$fg(x) = \begin{cases} f(x) & 0 \leq x < \alpha \\ f(\alpha)f(0) & x = \alpha \\ f(x - \alpha) & \alpha < x \leq \alpha + \omega \end{cases}$$
$$= \begin{cases} f(x) & 0 \leq x \leq \alpha \\ f(x - \alpha) & \alpha < x \leq \alpha + \omega \end{cases}$$

since $f(0) = 1_G$, whilst

$$gf(x) = \begin{cases} f(x) & 0 \leq x < \omega \\ f(\omega)f(0) & x = \omega \\ f(x - \omega) & \omega < x \leq \alpha + \omega \end{cases}$$
$$= \begin{cases} f(x) & 0 \leq x \leq \omega \\ f(x - \omega) & \omega < x \leq \alpha + \omega \end{cases}$$

since $f(0) = 1_G$ again.

But since $\omega \in \Omega_f^0$, we have

$$f(x) = f(x - \omega) \qquad \omega < x \leq \alpha$$

and

$$f(x - (\alpha - \omega)) = f(x) \qquad \alpha - \omega < x \leq \alpha$$

So, for $\alpha < x \leq \alpha + \omega$

$$\begin{aligned} f(x-\omega) =& f((x-\omega)-(\alpha-\omega)) \\ =& f(x-\alpha) \end{aligned}$$

Therefore

$$\begin{aligned} gf(x) =& \begin{cases} f(x) & 0 \leq x \leq \omega \\ f(x-\omega) & \omega < x \leq \alpha \\ f(x-\omega) & \alpha < x \leq \alpha+\omega \end{cases} \\ =& \begin{cases} f(x) & 0 \leq x \leq \omega \\ f(x) & \omega < x \leq \alpha \\ f(x-\alpha) & \alpha < x \leq \alpha+\omega \end{cases} \\ =& fg(x) \qquad \text{as required} \end{aligned}$$

Therefore $fg = gf$ and $g \in C_f^-$, which means that $g \mapsto L(g)$ is surjective and the proof is complete. $\square$

From this we prove that C_f^- does not contain the inverses of any of its elements. Again this Corollary comes directly from Corollary 8.9 of [2].

Corollary 5.2.1. *For all $g \in C_f^-$, $\varepsilon_0(f, g^{-1}) > 0$. In particular*

$$C_f^- \cap (C_f^-)^{-1} = \emptyset$$

Proof. By Lemma 5.2.1

$$C_f^- = \{f|_{[0,\omega]} | \omega \in \Omega_f^0 \setminus \{0, \alpha\}\}$$

Fix $g \in C_f^-$. Let $L(g) = \omega$. Choose $\varepsilon > 0$ such that $\omega > \varepsilon > 0$. Then, for all η

such that $0 \leq \eta \leq \varepsilon$,

$$\begin{aligned} f(\alpha-\eta)g^{-1}(\eta) = & f(\alpha-\eta)f|_{[0,\omega]}^{-1}(\eta) \\ = & f(\alpha-\eta)f(\omega-\eta)^{-1} \\ = & f(\alpha-(\alpha-\omega)-\eta)f(\omega-\eta)^{-1} \quad \text{as } \omega \in \Omega_f^0 \\ = & f(\omega-\eta)f(\omega-\eta)^{-1} \\ = & 1_G \end{aligned}$$

Hence $\varepsilon_0(f, g^{-1}) = \sup \mathcal{E}(f, g^{-1}) \geq \varepsilon > 0$ as required.

For the last bit, C_f^- contains only elements g such that $\varepsilon_0(f, g) = 0$, whilst $(C_f^-)^{-1}$ contains only elements g^{-1} such that $\varepsilon_0(f, g^{-1}) > 0$, hence their intersection is empty. $\square$

We now turn our attention to the set C_f^+. When looking at the whole set $\mathcal{RF}(G, \mathbb{R}^n)$, there are elements that commute with f, but that are not in C_f^+ or $(C_f^+)^{-1}$. For example:

Let $\dim(f) = 1$. Define a function $g \in \mathcal{RF}(G, \mathbb{R}^n)$ as follows:

$$g(x) = \begin{cases} f(0) & x = 0 \\ f(x - l\alpha) & l\alpha < x < (l+1)\alpha \qquad l \geq 0 \\ f(0)f(\alpha) & x = l\alpha \qquad l \geq 0 \\ f(x - l\alpha - t\underline{r}) & (l-1/2)\alpha + t\underline{r} < x < (l+1/2)\alpha + t\underline{r} \\ f(0)f(\alpha) & x = (l-1/2)\alpha + t\underline{r} \\ f(x-(t-l)\alpha) & (t-l)\alpha < x < (t-l-1)\alpha \qquad l > 0 \\ f(0)f(\alpha) & x = (t-l)\alpha \qquad l > 0 \\ f(\alpha) & x = \alpha t \end{cases}$$

where $l \in \mathbb{Z}$ and $\underline{r} = (0, \ldots, 0, r)$ with $0 < r < \alpha \in \mathbb{R}$.

Then we have that $fg = gf$ since α is a strong period of g, but if $f \in R$ we have that $g \notin \langle P(H_1, R) \rangle$ and hence is not in C_f^+ or $(C_f^+)^{-1}$. This does not happen in the case $n = 1$, so Chiswell and Müller do not have this problem.

The Lemma below relates the elements that are in C_f^+ to the elements of $\langle \Omega_f^0 \rangle^t \setminus \Omega_f^0$ and is split into two parts for the two distinct types of elements. Part (b) is the same as Lemma 8.11 of [2].

Let $L(f) = \alpha = (\alpha_{n-1}, \dots, \alpha_0)$ and $L(g) = \beta = (\beta_{n-1}, \dots, \beta_0)$. Then if $\dim(f) = k+1$, we must have $\alpha_{n-1}, \dots, \alpha_{k+1} = 0$ and $\alpha_k > 0$.

Recall that by the proof of Lemma 5.1.1, we have that for all $f \in \langle P(H_1, R) \rangle$ such that f is of Type 1, $f = (h_1 \circ h_2)^{s(t)}$, where $h_2 \circ h_1 \in R$ and $s(t) \in \mathbb{Z}[t]/p(t)$. Define, for $\alpha_k \neq 0$, $\lfloor \frac{\beta_k}{\alpha_k} \rfloor := \max\{z \in \mathbb{Z} | z \leq \frac{\beta_k}{\alpha_k}\}$.

Lemma 5.2.2. (a) *The elements $g \in C_f^+$, where $f = (h_1 \circ h_2)^{s(t)}$ is of Type 1, are in 1-1 correspondence with elements $q(t) \in \mathbb{Z}[t]/p(t)$ such that $q(t) \geq s(t)$, via*

$$g \mapsto q(t), \text{ where } g = (h_1 \circ h_2)^{q(t)}$$

with inverse

$$q(t) \mapsto (h_1 \circ h_2)^{q(t)}$$

(b) *The elements $g \in C_f^+$, where f is of Type 2, are in 1-1 correspondence with elements $(k, \omega) \in \mathbb{N} \setminus \{0\} \times \Omega_f^0 \setminus \{\alpha\}$ via*

$$g \mapsto (\lfloor \frac{\beta_k}{\alpha_k} \rfloor, L(g) - \lfloor \frac{\beta_k}{\alpha_k} \rfloor L(f))$$

where $\dim(f) = k+1$, *with inverse*

$$(l, \omega) \mapsto f^l \circ f|_{[0,\omega]}.$$

Note that in part (b) $\lfloor \frac{\beta_k}{\alpha_k} \rfloor$ is defined since $\alpha_k \neq 0$.

Proof. First note that $fg = gf$ so if $L(f) = L(g)$, we have that

$$(fg)(x) = \begin{cases} f(x) & 0 \le x < \alpha \\ f(\alpha)g(0) & x = \alpha \\ g(x-\alpha) & \alpha < x \le 2\alpha \end{cases}$$

$$=(gf)(x)$$

$$= \begin{cases} g(x) & 0 \le x < \alpha \\ g(\alpha)f(0) & x = \alpha \\ f(x-\alpha) & \alpha < x \le 2\alpha \end{cases}$$

so $f(x) = g(x)$ for $0 \le x < \alpha$, whilst $f(\alpha) = g(\alpha)f(0) = g(\alpha)$ since $f(0) = 1_G$. Hence $f = g$, and g is of the required form for both Types of f.

If $L(f) \neq L(g)$ then $f \in C_g^-$. By Lemma 5.2.1, this means that $f = g|_{[0,\alpha]}$ and $\alpha \in \Omega_g^0$.

I now prove part (b).

We know that $\alpha \in \Omega_g^0$, so if $\dim(f) < \dim(g)$, g is of Type 1 by Lemma 5.1.1, but g must be of Type 2 also, since f is. Contradiction.

Hence, if f is of Type 2, $\dim(f) = \dim(g)$ for all $g \in C_f^+$. So assume $\dim(f) = \dim(g) = k+1$.

Let $l := \lfloor \frac{\beta_k}{\alpha_k} \rfloor$. This is defined, since $\alpha_k \neq 0$, and $\beta_k \ge \alpha_k > 0$ since $\beta \ge \alpha$ with $\dim(\beta) = \dim(\alpha) = k+1$, hence $l > 0$.

We start, as in [2], by showing that $f^\lambda = g|_{[0,\lambda\alpha]}$ for $0 \le \lambda \le l$, using induction on λ.

Case $\lambda = 0$:

$$f^0 = f(0) = 1_G = g(0)$$

so the hypothesis is true for $\lambda = 0$.

Assume the hypothesis is true for $\lambda = j$, where $0 \le j < l$, so $f^j = g|_{[0,j\alpha]}$.

Then

$$f^{j+1}(x) = (f^j \circ f)(x) \quad \text{by Lemma 1.6, since f is cyclically reduced}$$

$$= \begin{cases} f^j(x) & 0 \le x < j\alpha \\ f^j(j\alpha)f(0) & x = j\alpha \\ f(x - j\alpha) & j\alpha < x \le (j+1)\alpha \end{cases}$$

$$= \begin{cases} f^j(x) & 0 \le x \le j\alpha \quad \text{since } f(0) = 1_G \\ f(x - j\alpha) & j\alpha < x \le (j+1)\alpha \end{cases}$$

$$= \begin{cases} g(x) & 0 \le x \le j\alpha \qquad \text{by hypothesis} \\ g(x - j\alpha) & j\alpha < x \le (j+1)\alpha \quad \text{since } f = g|_{[0,\alpha]} \end{cases}$$

But since $j\alpha \le \beta$, Lemma 5.1.2 (ii) tells us that $j\alpha \in \Omega_g$, so

$$g(x) = g(x - j\alpha) \qquad \text{for } j\alpha < x \le (j+1)\alpha \le \beta$$

and therefore $f^{j+1}(x) = g|_{[0,(j+1)\alpha]}(x)$ for $0 \le x \le (j+1)\alpha$ and $0 \le j < l$, and

$$g|_{[0,l\alpha]} = f^l$$

Next we note that $L(g) = \beta = L(f^l \circ f|_{[0,\beta - l\alpha]})$ and

$$(f^l \circ f|_{[0,\beta-l\alpha]})(x) = \begin{cases} f^l(x) & 0 \le x < l\alpha \\ f^l(x)f|_{[0,\beta-l\alpha]}(0) & x = l\alpha \\ f|_{[0,\beta-l\alpha]}(x - l\alpha) & l\alpha < x \le \beta \end{cases}$$

$$= \begin{cases} f^l(x) & 0 \le x \le l\alpha \qquad \text{since } f(0) = 1_G \\ f|_{[0,\beta-l\alpha]}(x - l\alpha) & l\alpha < x \le \beta \end{cases}$$

$$= \begin{cases} g(x) & 0 \le x \le l\alpha \qquad \text{since } f^l = g|_{[0,l\alpha]} \\ g(x - l\alpha) & l\alpha < x \le \beta \qquad \text{since } f = g|_{[0,\alpha]} \end{cases}$$

$$= g(x) \qquad 0 \le x \le \beta$$

since $l\alpha \in \Omega_g$ by Lemma 5.1.2 (ii).

Hence $g = f^l \circ f|_{[0,\beta - l\alpha]}$.

Now we have to prove that $\beta - l\alpha \in \Omega_f^0 \setminus \{\alpha\}$. But since $[g, f] = \mathbf{1_G}$,

$$\begin{aligned} f^l \circ f|_{[0,\beta-l\alpha]} \circ f =& f \circ f^l \circ f|_{[0,\beta-l\alpha]} \\ =& f^l \circ f \circ f|_{[0,\beta-l\alpha]} \end{aligned}$$

which implies that $[f, f|_{[o,\beta-l\alpha]}] = \mathbf{1_G}$, and $L(f|_{[0,\beta-l\alpha]}) < \alpha$, so we must have that $f|_{[0,\beta-l\alpha]} \in C_f^- \cup \{\mathbf{1_G}\}$.

Hence $\beta - l\alpha \in \Omega_f^0 \setminus \{\alpha\}$ by Lemma 5.2.1.

To finish part (b), all we need to do is prove that the map

$$\Phi_f^+ : C_f^+ \longrightarrow \mathbb{N} \setminus \{0\} \times \Omega_f^0 \setminus \{\alpha\}$$

is surjective.

But for $(l, \omega) \in \mathbb{N} \setminus \{0\} \times \Omega_f^0 \setminus \{\alpha\}$, let $g := f^l \circ f|_{[0,\omega]}$. Then

$$\begin{aligned} fg =& f \circ f^l \circ f|_{[0,\omega]} && \text{since } \varepsilon(f, g) = \varepsilon(f, f) = 0 \\ =& f^l \circ f \circ f|_{[0,\omega]} \\ =& f^l \circ f|_{[0,\omega]} \circ f && \text{since } f|_{[0,\omega]} \in C_f^- \text{ by Lemma 5.2.1} \\ =& gf \end{aligned}$$

and $L(g) = l\alpha + \omega \geq \alpha$ as $l \geq 1$, so $g \in C_f^+$.

Now I just need to prove part (a), so assume that f is of Type 1.

From the example given, it is clear that not all elements of $\mathcal{RF}(G, \mathbb{R}^n)$ that commute with f are of the form required. I claim that if $g \in \langle P(H_1, R)\rangle$ and $[f, g] = \mathbf{1_G}$, then g is of the form $(h_1 \circ h_2)^{q(t)}$.

Proof of claim:

We know that $g|_{[0,\alpha]} = f$, and by induction, as in the argument used above, $g|_{[0,j\alpha]} = f^j$ for all $j \in \mathbb{N}$.

We also know that $\alpha \in \Omega_g^0$, so on each copy of $\mathbb{R}$ we have that $g(x - \alpha) = g(x)$.

Now, by Lemma 4.1.4 there is a unique normal R-form p such that $g = w(p)$, so $g = g_1 \circ u_1^{\alpha_1} \circ g_2 \circ \ldots \circ g_m \circ u^m \circ g^{m+1}$.

By the fact that $g|_{[0,j\alpha]} = f^j$ for $j \in \mathbb{N}$ and that $f = (h_1 \circ h_2)^{s(t)}$, the construction of g implies that

$$u_1 = h_2 \circ h_1 \text{ and } g_1 = h_1$$

Also since $\alpha \in \Omega_g^0$, we must have that $m \leq 1$, so that

$$\begin{aligned} g =& h_1 \circ (h_2 \circ h_1)^{\alpha_1} \circ g_2 \\ =& (h_1 \circ h_2)^{\alpha_1} \circ h_1 \circ g_2 \end{aligned}$$

where $L(g_2) < L(u_1)$.

But

$$\begin{aligned} fg =& (h_1 \circ h_2)^{s(t)} \circ (h_1 \circ (h_2 \circ h_1)^{\alpha_1} \circ g_2) \\ =& (h_1 \circ h_2)^{s(t)+\alpha_1} \circ h_1 \circ g_2 \\ =& (h_1 \circ h_2)^{\alpha_1} \circ (h_1 \circ h_2)^{s(t)} \circ h_1 \circ g_2 \\ =& gf \\ =& (h_1 \circ (h_2 \circ h_1)^{\alpha_1} \circ g_2) \circ (h_1 \circ h_2)^{s(t)} \\ =& ((h_1 \circ h_2)^{\alpha_1} \circ h_1 \circ g_2) \circ (h_1 \circ h_2)^{s(t)} \end{aligned}$$

So $[h_1 \circ g_2, (h_1 \circ h_2)^{s(t)}] = \mathbf{1_G}$, but by [1], since this acts on an $\mathbb{R}^n$-tree, we must have that $h_1 \circ g_2$ and $(h_1 \circ h_2)^{s(t)}$ are common powers of the same element. Since $h_2 \circ h_1 \in R$, by definition this means that this common element must be $h_1 \circ h_2$, but as $L(g_2) < L(h_1 \circ h_2)$, this implies that

$$h_1 \circ g_2 = \begin{cases} h_1 \circ h_2 & \text{if } L(h_1 \circ g_2) > 0 \\ \mathbf{1_G} & \text{otherwise} \end{cases}$$

If $h_1 \circ g_2 = h_1 \circ h_2$, by examining the terminal segments of length h_2, we see that $g_2 = h_2$. Hence $g = (h_1 \circ h_2)^{\alpha_1+1}$.

If $h_1 \circ g_2 = \mathbf{1_G}$, then $g_2 = h_1^{-1}$, and $L(g_2) = L(h_1) = 0$ and $g = (h_1 \circ h_2)^{\alpha_1}$.

If we set

$$q(t) = \begin{cases} \alpha_1 + 1 & \text{if } L(h_1 \circ g_2) > 0 \\ \alpha_1 & \text{otherwise} \end{cases}$$

then we have proved the claim. So for every $g \in C_f^+$ we have an element $q(t) \in \mathbb{Z}[t]/p(t)$ such that $g = (h_1 \circ h_2)^{q(t)}$. Moreover, since $L(g) \geq L(f)$, we must have that $q(t) \geq s(t)$. All that is left to do is prove that

$$\Phi_f^+ : C_f^+ \longrightarrow \mathbb{Z}[t]/p(t) \backslash \{r(t) | r(t) < s(t)\}$$

is surjective.

Given $q(t) \in \mathbb{Z}[t]/p(t) \setminus \{r(t)|r(t) < s(t)\}$, set $g := (h_1 \circ h_2)^{q(t)}$.

Then

$$\begin{aligned} fg =& (h_1 \circ h_2)^{s(t)} (h_1 \circ h_2)^{q(t)} \\ =& (h_1 \circ h_2)^{s(t)+q(t)} \\ =& (h_1 \circ h_2)^{q(t)} (h_1 \circ h_2)^{s(t)} \\ =& gf \end{aligned}$$

So $[f, g] = \mathbf{1_G}$. Also we have that $\varepsilon_0(f, g) = 0$, since $\varepsilon_0(h_2, h_1) = 0$, and if we set $u(t) = q(t) - s(t) \geq 0$,

$$\begin{aligned} L(g) =& q(t) L(h_1 \circ h_2) \\ =& s(t) L(h_1 \circ h_2) + u(t) L(h_1 \circ h_2) \\ =& \alpha + u(t) L(h_1 \circ h_2) \\ \geq& \alpha \end{aligned}$$

So $g \in C_f^+$ and hence Φ_f^+ is surjective and we are done. $\square$

Note that part (a) implies that for f of Type 1, $g = f^{r(t)} \circ f|_{[0,\omega]}$, where $f^{r(t)} = ((h_1 \circ h_2)^{s(t)})^{r(t)}$ for some $0 < r(t) \in \mathbb{Z}[t]/p(t)$ and $\omega \in \Omega_f^0$, since if $g = (h_1 \circ h_2)^{q(t)}$, we can find $r(t) \geq 1$ and $0 \leq m(t) < s(t)$ such that $q(t) = r(t)s(t) + m(t)$ and $f|_{[0,\omega]} = (h_1 \circ h_2)^{m(t)}$. Also, part (b) implies that for f of Type 2, $g = f^{r(t)} \circ f|_{[0,\omega]}$ with $r(t) \in \mathbb{Z}$, so $f^{r(t)}$ is defined.

Again we can now show that C_f^+ does not contain the inverses of any of its elements. This is similar to Corollary 8.12 in [2], but contains some adjustments for Type 1 elements.

Corollary 5.2.2. *For all $g \in C_f^+$, $\varepsilon_0(f, g^{-1}) > 0$. In particular*

$$C_f^+ \cap (C_f^+)^{-1} = \emptyset$$

Proof. By the note after Lemma 5.2.2, we have that $g = f^{r(t)} \circ f|_{[0,\omega]}$ for some $0 < r(t) \in \mathbb{Z}[t]/p(t)$, $\omega \in \Omega_f^0$. So $g^{-1} = f|_{[0,\omega]}^{-1} \circ f^{-r(t)}$.

Let $\omega > 0$ and $0 < \delta < \omega$. Then, since $g|_{[0,\omega]}^{-1} = f|_{[0,\omega]}^{-1}$, $\omega \in \Omega_f^0$ and $f|_{[0,\omega]} = f$ for $0 \leq x \leq \omega$, we have that for $0 \leq x \leq \delta$,

$$\begin{aligned} f(\alpha - x)g^{-1}(x) =& f(\alpha - x)f|_{[0,\omega]}^{-1}(x) \\ =& f(\alpha - x)f|_{[0,\omega]}(\omega - x)^{-1} \\ =& f(\alpha - (\alpha - \omega) - x)f|_{[0,\omega]}(\omega - x)^{-1} \\ =& f(\omega - x)f|_{[0,\omega]}(\omega - x)^{-1} \\ =& 1_G \end{aligned}$$

If $\omega = 0$, then, since $r(t) > 1$, $g = f^{r(t)} = f^{r(t)-1} \circ f$, so $g^{-1} = f^{-1} \circ f^{-r(t)+1}$, and given $0 < \delta < \alpha$, we have that for all $0 \leq x \leq \delta$

$$\begin{aligned} f(\alpha - x)g^{-1}(x) =& f(\alpha - x)f^{-1}(x) \qquad \text{since } g|_{[0,\alpha]}^{-1} = f^{-1} \\ =& f(\alpha - x)f(\alpha - x)^{-1} \\ =& 1_G \end{aligned}$$

Hence in both cases $\varepsilon_0(f, g^{-1}) \geq \delta > 0$.

For the final bit, since for all $g \in C_f^+$, $\varepsilon_0(f, g) = 0$, and for all $g \in (C_f^+)^{-1}$, $\varepsilon_0(f, g) > 0$, if $g \in C_f^+$ then $g \notin (C_f^+)^{-1}$ and if $g \in (C_f^+)^{-1}$ then $g \notin C_f^+$. Hence $C_f^+ \cap (C_f^+)^{-1} = \emptyset$. ∎

Putting the two sets together, we see that C_f does not contain the inverses of any of its elements either. Moreover, every element of C_f is cyclically reduced, as proved below. The Corollary statement comes from Lemma 8.14 in [2], but the proof of (b) is slightly different due to the Type 1 elements.

Corollary 5.2.3. (a) $C_f \cap (C_f)^{-1} = \emptyset$.

(b) If $g_1, g_2 \in C_f$, then $\varepsilon_0(g_1, g_2) = 0$. In particular, every element of C_f is cyclically reduced.

Proof. (a) This follows from Corollaries 5.2.1 and 5.2.2

(b) Let $g_1, g_2 \in C_f$. Then

$$g_i = f^{r_i(t)} \circ f|_{[0,\omega_i]} \qquad 0 < r_i(t) \in \mathbb{Z}[t]/p(t), \omega_i \in \Omega_f^0$$

Assume that $\varepsilon_0(g_1, g_2) > 0$, so there exists $0 < \delta \leq \min\{L(g_1), L(g_2)\}$ such that

$$(*) \qquad g_1(L(g_1) - x)g_2(x) = 1_G \qquad 0 \leq x \leq \delta$$

There are two cases here:

$$(i)\ \omega_1 > 0$$

$$(ii)\ \omega_1 = 0$$

Case (i) $\omega_1 > 0$.

Let

$$\delta_0 := \min\{\delta, \omega_1, \alpha - \omega_1\}$$

Now, for all $0 \leq x < \delta_0$, we have that

$$g_1(L(g_1) - x) = f(\omega_1 - x)$$

and

$$g_2(x) = f(x)$$

so by $(*)$ we get

$$f(\omega_1 - x)f(x) = 1_G$$

But since ω_1 is a period $f(x) = f(x + \omega_1)$, so we have

$$f(\omega_1)f(0) = f(\omega_1) = 1_G$$

and

$$f(\omega_1 - x)f(\omega_1 + x) = 1_G$$

for $0 < x < \delta_0$.

This implies that f is not reduced. Contradiction.

Hence $\omega_1 \not> 0$

Case (ii) $\omega_1 = 0$.

Here $g_1 = f^{r_1(t)}$ and $g_2 = f^{r_2(t)} \circ f|_{[0,\omega_2]}$.

Now if $r_2(t) > 0$, set $\delta_0 := \min\{\delta, \alpha\}$. Then, since here we must have $r_1(t) > 0$, we find that for $0 \leq x \leq \delta_0$,

$$\begin{aligned} g_1(L(g_1) - x) =& (f^{r_1(t)-1} \circ f)(L(g_1) - x) \\ =& f(\alpha - x) \end{aligned}$$

and

$$\begin{aligned} g_2(x) =& (f \circ f^{r_2(t)-1} \circ f|_{[0,\omega_2]})(x) \\ =& f(x) \end{aligned}$$

if $r_2(t) > 0$, and therefore

$$f(\alpha - x)f(x) = 1_G \qquad 0 \leq x < \delta_0$$

which means that $\varepsilon_0(f, f) \geq \delta_0 > 0$, and hence f is not cyclically reduced. Contradiction.

So $r_2(t) \not> 0$.

If, however, $r_2(t) = 0$, we have that $\omega_2 > 0$.

Let $\delta_0 := \delta \leq \omega_2$. Then

$$g_1(L(g_1) - x) = f(\alpha - x)$$

as above, and

$$g_2(x) = f(x) \qquad \text{for } 0 \le x < \delta_0$$

Again we get that

$$f(\alpha - x) = f(x) \qquad \text{for } 0 \le x < \delta_0$$

and therefore f is not cyclically reduced. Contradiction.

Hence $\varepsilon_0(g_1, g_2) = 0$ for all $g_1, g_2 \in C_f$.

It follows that every element $g \in C_f$ is cyclically reduced because if $g \in C_f$, $\varepsilon_0(g, g) = 0$. □

5.3 $\mathcal{C}_f$, the Centraliser of f

In this Section I prove that the centrailser of f, $\mathcal{C}_f$, is partitioned into three parts: C_f, $(C_f)^{-1}$ and $\mathbf{1_G}$. This is still following Chapter 8 of [2] with changes due to the Type 1 elements.

First I prove that every element g in $\mathcal{C}_f$ for which $L(fg) < L(f) + L(g)$ is in $(C_f)^{-1}$.

Lemma 5.3.1. *For all $g \in \mathcal{C}_f$ such that $\varepsilon_0(f, g) > 0$, $g \in (C_f)^{-1}$.*

Proof. $fg = gf$ therefore $\varepsilon_0(g, f) = \varepsilon_0(f, g) := \varepsilon_0 > 0$ and

$$f(\alpha - x)g(x) = 1_G \qquad \text{for all } 0 \le x < \varepsilon_0$$

So we have that $f^{-1}|_{[0,\varepsilon_0)} = g|_{[0,\varepsilon_0)}$. Note that $\varepsilon_0 \le \alpha$.

Since $\varepsilon_0 > 0$, $g \ne \mathbf{1_G}$. There are three cases:

$$(i)\ 0 < L(g) < \alpha$$

$$(ii)\ L(g) \ge \alpha$$

$$(iii)\ L(g) = \alpha$$

Case (i) $0 < L(g) < \alpha$:

Assuming that $L(g) > \varepsilon_0$ and comparing the initial segment of fg and gf of length $L(g) - \varepsilon_0$, we find that $f(x) = g(x)$ for $0 \leq x < L(g) - \varepsilon_0$, so

$$f|_{[0,L(g)-\varepsilon_0)} = g|_{[0,L(g)-\varepsilon_0)}$$

Let $\delta_0 := \min\{\varepsilon_0, L(g) - \varepsilon_0\}$, then $\delta_0 > 0$ since $L(g) > \varepsilon_0 > 0$.

So

$$f|_{[0,\delta_0)} = f^{-1}|_{[0,\delta_0)}$$

which means that

$$f(x)f(\alpha - x) = 1_G \qquad 0 \leq x < \delta_0$$

and therefore $\varepsilon_0(f, f) \geq \delta_0 > 0$, so f is not cyclically reduced. Contradiction.

Therefore $\varepsilon_0 = L(g)$.

So

$$\begin{aligned} L(fg) =& L(f) + L(g) - 2\varepsilon_0 \\ =& L(f) - L(g) \qquad \text{since } \varepsilon_0 = L(g) \\ =& \alpha - L(g) \end{aligned}$$

But $\alpha - L(g) \in (0, \alpha)$ by the case assumption, $(fg)|_{[0,\alpha-L(g))} = f_{[0,\alpha-L(g))}$ so that $\varepsilon_0(f, fg) = 0$, and $fgf = ffg$ so $[f, fg] = \mathbf{1_G}$, all of which means $fg \in C_f^-$. Hence, by Lemma 5.2.1 $\alpha - L(g) \in \Omega_f^0 \setminus \{0, \alpha\}$.

Therefore $L(g) \in \Omega_f^0 \setminus \{0, \alpha\}$ and $f(\alpha) = f(\alpha - L(g))$.

Looking at gf, we see that

$$g(L(g))f(0) = 1_G \qquad \text{since } \varepsilon_0(g, f) > 0$$

Therefore, since f is normalised, $g(L(g)) = \mathbf{1_G}$. Hence for $0 \le x \le L(g)$,

$$\begin{aligned} g(x) =& \begin{cases} f(\alpha - x)^{-1} & 0 \le x < L(g) \\ 1_G & x = L(g) \end{cases} \\ =& \begin{cases} f(\alpha - (\alpha - L(g)) - x)^{-1} & 0 \le x < L(g), \text{ since } \alpha - L(g) \in \Omega_f^0 \\ f(0)^{-1} & x = L(g) \end{cases} \\ =& f(L(g) - x)^{-1} \\ =& f|_{[0,L(g)]}^{-1}(x) \end{aligned}$$

so $g \in (C_f^-)^{-1}$ by Corollary 5.2.1.

Case (ii) $L(g) > \alpha$:

If $\beta > \alpha$, since $[f, g] = \mathbf{1_G}$ we have that $f \in \mathcal{C}_g$. Therefore, by the first part of this proof, $\alpha \in \Omega_f^0 \backslash \{0, \beta\}$ and $f = g|_{[0,\alpha]}^{-1}$.

Let $f = (h_1 \circ h_2)^{s(t)}$ be of Type 1. Then since $\alpha \in \Omega_g^0$ and $g|_{[0,\alpha]}^{-1} = f$, we must have that $g = ((h_1 \circ h_2)^{-1})^{q(t)} \circ h_3$, where $L(h_1 \circ h_2) > L(h_3) = \beta - q(t)L(h_1 \circ h_2)$. Now $\beta, q(t)L(h_1 \circ h_2) \in \Omega_g^0$, so $\beta - q(t)L(h_1 \circ h_2) \in \Omega_g^0$. But since $L(h_3) < \alpha$ and $f = g^{-1}|_{[0,\alpha]}$ we must have that $L(h_3) \in \Omega_f^0$ and hence $L(h_3) = 0$. Also $fg = gf$ and $\alpha < \beta$, so

$$\begin{aligned} fg(L(fg)) =& g(\beta) \\ =& gf(L(fg)) \\ =& g(\beta - \alpha)f(0) \\ =& g(\beta - \alpha) \end{aligned}$$

since $f(0) = \mathbf{1_G}$. But $g(\beta - \alpha) = g((q(t) - s(t))(L(h_1 \circ h_2))) = g(L(h_1 \circ h_2))$ since $L(h_1 \circ h_2) \in \Omega_g^0$. Hence $h_3 = \mathbf{1_G}$ and

$$\begin{aligned} g =& ((h_1 \circ h_2)^{-1})^{q(t)} \\ =& ((h_1 \circ h_2)^{q(t)})^{-1} \in (C_f^+)^{-1} \subset (C_f)^{-1} \end{aligned}$$

by Lemma 5.2.2 part (a) as required.

If f is of Type 2, $\alpha \in \Omega_g^0$, so if $\dim(f) > 1$, f, and hence g, has no periods of dimension 1. Therefore g is also of Type 2 with $\dim(f) = \dim(g)$. If however $\dim(f) = 1$, assume g is of Type 1 to get a contradiction.

We know that $\alpha \in \Omega_g^0$, so, by Corollary 5.1.1 we have that $\beta = s_1(t)L(u)$ and $\alpha = q_1(t)L(u)$ for some $0 < s_1(t) \in \mathbb{Z}[t]/p(t)$, $s_1(t) > q_1(t) \in \mathbb{N} \setminus \{0\}$ and $u = h_1 \circ h_2$ such that $h_2 \circ h_1 \in R$ and $g = u^{s_1(t)}$. But then

$$\begin{aligned} f =& g^{-1}|_{[0,\alpha]} \\ =& u^{-s_1(t)}|_{[0,q_1(t)L(u)]} \\ =& u_1^{-q_1(t)} \end{aligned}$$

which is of Type 1. Contradiction.

Hence if f is of Type 2 then so is g with $\dim(g) = \dim(f) = k+1$ for some $0 \le k < n-1$, and we can use induction to prove this part of this Lemma.

Since $\beta > \alpha$ and $\dim(\beta) = \dim(\alpha)$, we can set $\beta = (0, \dots, 0, \beta_k, \dots, \beta_0)$ and $\alpha = (0, \dots, 0, \alpha_k, \dots, \alpha_0)$ with $\beta_k \neq 0 \neq \alpha_k$. Then we have $l := \lfloor \frac{\beta_k}{\alpha_k} \rfloor \in \mathbb{N} \setminus \{0\}$.

Now let $\omega := L(g) - l\alpha$. Then $0 \le \omega < \alpha$.

Claim: $L(gf^j) = L(g) - j\alpha$ for all $0 \le j \le l$.

Proof of claim by induction:

Case $j = 0$: $L(gf^0) = L(g) = L(g) - 0\alpha$ - Trivially true

Assume this is true for $j = j' < l$, so

$$\begin{aligned} L(gf^{j'}) =& L(g) - j'\alpha \\ =& L(g) + L(f^{j'}) - 2\varepsilon_0(g, f^{j'}) \\ =& L(g) + j'\alpha - 2\varepsilon_0(g, f^{j'}) \end{aligned}$$

From this we see that $\varepsilon_0(g, f^{j'}) = j'\alpha$, but $L(g) - j'\alpha \ge L(g) - l\alpha + \alpha > \alpha$ by our assumptions.

Hence $gf^{j'}|_{[0,\alpha)} = g|_{[0,\alpha)} = f|_{[0,\alpha)}^{-1}$ and

$$\varepsilon_0(f, gf^{j'}) = \alpha = \varepsilon_0(gf^{j'}, f)$$

so

$$\begin{aligned}L(gf^{j'+1}) =& L(gf^{j'}) + L(f) - 2\varepsilon_0(gf^{j'}, f)\\ =& L(g) - j'\alpha + \alpha - 2\alpha\\ =& L(g) - (j'+1)\alpha\end{aligned}$$

Hence the hypothesis holds for $j = j' + 1$ as required.

It follows that

$$\begin{aligned}L(gf^l) =& L(g) - l\alpha = \omega\\ =& L(g) + L(f^l) - 2\varepsilon_0(g, f^l)\\ =& L(g) + l\alpha - 2\varepsilon_0(g, f^l)\end{aligned}$$

which implies $\varepsilon_0(g, f^l) = l\alpha$.

If $\omega = 0$, we have that $[g, f^l] = \mathbf{1_G}$ and $L(g) = L(f^l)$, so $g = f^{-l}$. But $f^{-l} \in (C_f^+)^{-1}$, so we are done.

If $\omega > 0$ we know that $\omega < \alpha$, so

$$g^{-1}|_{[0,\omega]} = f|_{[0,\omega]}$$

and hence, since $L(gf^l) = \omega$, $\alpha > \omega \geq \varepsilon_0(f, gf^l) \geq \omega > 0$, so $\varepsilon_0(f, gf^l) = \omega$. But $[f, gf^l] = \mathbf{1_G}$, so $gf^l \in (C_f^-)^{-1}$, hence

$$g^{-1} = (f^l) \circ f|_{[0,\omega]}$$

and therefore

$$g = (f^l \circ f|_{[0,\omega]})^{-1} \in (C_f^+)^{-1}$$

by Lemma 5.2.2 part (ii) as required.

Case (iii) $L(g) = \alpha$:

Again we have that $\varepsilon_0 = \alpha$, using the arguement in Case (i), so all we need to prove is that $g(\alpha) = f^{-1}(\alpha)$. But $[f, g] = \mathbf{1_G}$, so

$$\begin{aligned} fg(0) =& f(0)g(\alpha) \\ =& gf(0) \\ =& g(0)f(\alpha) \end{aligned}$$

but $f(0) = \mathbf{1_G}$ and $g(0) = f^{-1}(0) = f(\alpha)^{-1}$, so $g(\alpha) = \mathbf{1_G} = f(0) = f^{-1}(\alpha)$ as required. Hence $g = f^{-1} \in (C_f^+)^{-1}$ and we are done. $\square$

I can now prove the main result of this Chapter. Theorem 8.16 in [2] is the special case of parts $(a_2) - (d_2)$ when $n = 1$, so we see that the elements of Chiswell and Müller's group behave like the Type 2 elements of my group.

Theorem 5.3.1. *Let $f \in \mathcal{RF}(G, \mathbb{R}^n)$ be cyclically reduced and normalised, with $L(f) = \alpha > 0$.*

Then if $f = u^{s(t)}$ is of Type 1, so $u = (h_1 \circ h_2)$ and $h_2 \circ h_1 \in R$ and $s(t) \in \mathbb{Z}[t]/p(t)$:

(a_1) The set

$$C_f = \{(h_1 \circ h_2)^{q(t)} : q(t) \in \mathbb{Z}[t]/p(t), q(t) \geq 0\}$$

forms a positive cone for $\mathcal{C}_f$, giving $\mathcal{C}_f$ the structure of an ordered abelian group.

(b_1) Every element of $\mathcal{C}_f$ is cyclically reduced; in particular, $\mathcal{C}_f$ is hyperbolic.

(c_1) The mapping $\rho_f : \mathcal{C}_f \to \langle \Omega_f^0 \rangle^t$ given by

$$(h_1 \circ h_2)^{r(t)} \mapsto r(t)L(h_1 \circ h_2)$$

where $f = (h_1 \circ h_2)^{s(t)}$, with $s(t) \in \mathbb{Z}[t]/p(t)$ and $h_1, h_2 \in \mathcal{RF}(G, \mathbb{R}^n)$ such that $\mathbf{1_G} \neq h_2 \circ h_1 \in R$, is an isomorphism of ordered abelian groups satisfying

$$L(g) = |\rho_f(g)|, \ g \in \mathcal{C}_f$$

(d_1) $\mathcal{C}_f$ *has the presentation*

$$\langle u, u^t, \ldots, u^{t^{n-1}} | [u^{t^i}, u^{t^j}] = 1 \rangle$$

where $f = u^{s_0} u^{s_1 t} \ldots u^{s_m t^m}$ *and* $m \leq n-1$.

(e_1) *Let* $\omega_0 := \inf\{\Omega_f^0 \setminus \{0\}\}$, *then*

$$(i)\ \dim(\omega_0) = 1$$

$$(ii)\ f = f_0^{k_0(t)} \text{ with } f_0 = f|_{[0,\omega_0]}$$

$$(iii)\ \alpha = |k_0(t)|\omega_0 \text{ for some } k_0(t) \in \mathbb{Z}[t]/p(t)$$

$$(iv)\ C_f = \{f_0^{k(t)} | 0 < k(t) \in \mathbb{Z}[t]/p(t)\} \text{ and } \mathcal{C}_f = \langle f_0 \rangle^t$$

If $\dim(f) = 1$, *then* Ω_f^0 *is finite.*

If f *is of Type 2:*

(a_2) *The set*

$$C_f = \{f^k \circ f|_{[0,\omega]} : (k, \omega) \in \mathbb{N}_0 \times \Omega_f^0 \setminus \{\alpha\}, k + \omega > 0\}$$

forms a positive cone for $\mathcal{C}_f$, *giving* $\mathcal{C}_f$ *the structure of an ordered abelian group.*

(b_2) *Every element of* $\mathcal{C}_f$ *is cyclically reduced; in particular,* $\mathcal{C}_f$ *is hyperbolic.*

(c_2) *The mapping* $\rho_f : \mathcal{C}_f \to \langle \Omega_f^0 \rangle^t$ *given by* $(f^k \circ f|_{[0,\omega]})^\sigma \mapsto \sigma(k\alpha + \omega)$, $\sigma = \pm 1$ *is an isomorphism of ordered abelian groups satisfying*

$$L(g) = |\rho_f(g)|, \ g \in \mathcal{C}_f$$

(d_2) $\mathcal{C}_f$ *has the presentation*

$$\langle x_\omega (\omega \in \Omega_f^0) | [x_\alpha, x_\omega] = \mathbf{1_G} (\omega < \alpha), x_{\omega_1} x_{\omega_2} = x_\alpha^{\lfloor \frac{\omega_1 + \omega_2}{\alpha} \rfloor} x_{\omega_1 \boxplus \omega_2} (\omega_1, \omega_2 < \alpha) \rangle.$$

where

$$\omega_1 \boxplus \omega_2 := \begin{cases} \omega_1 + \omega_2 & \omega_1 + \omega_2 < \alpha \\ \omega_1 + \omega_2 - \alpha & \omega_1 + \omega_2 \geq \alpha \end{cases}$$

(e_2) If f is such that $\dim(f) > 1$*, then*

(i) $\dim(\omega_0) > 1$ *and* $\omega_0 \in \Omega_f^0 \setminus \{0\}$

(ii) Ω_f^0 *is finite*

(iii) $\alpha = k_0\omega_0$ *for some* $k_0 \in \mathbb{N} \setminus \{0\}, f = f_0^{k_0}$ *with* $f_0 = f|_{[0,\omega_0]}$

(iv) $C_f = \{f_0^s | s \in \mathbb{N} \setminus \{0\}\}$ *and* $\mathcal{C}_f = \langle f_0 \rangle$*, which is cyclic*

If $\dim(f) = 1$ *then* $\omega_0 = 0$*.*

Proof. (a_1) By the construction of C_f we have that C_f is abelian. From Lemma 5.3.1 we see that $\mathcal{C}_f = C_f \cup C_f^{-1} \cup \{\mathbf{1_G}\}$, so $\mathcal{C}_f$ is abelian, and by Corollary 5.2.3, part (a), this is a partition.

Now, if $g_1, g_2 \in C_f$, then $g_1, g_2 \neq \mathbf{1_G}$ as $C_f \cap C_f^{-1} = \emptyset$ by Corollary 5.2.3, part (a). So

$$\varepsilon_0(f, g_1) = 0 = \varepsilon_0(f, g_2) = \varepsilon_0(g_1, g_2)$$

by Corollary 5.2.3, part (b) and the definition of C_f.

$L(g_1) = 0$ implies g_1 is elliptic which implies $g_1 = \mathbf{1_G} \notin C_f$, so $L(g_1) > 0$, and by Lemma 2.3.3 we have that $\varepsilon_0(f, g_1 g_2) = 0$ too. Hence $g_1 g_2 \in C_f$ and therefore C_f is closed under taking products. The explicit formula follows from Lemma 5.2.2 and Corollary 5.1.1.

(a_2) The same applies here as in (a_1), but with the different formulae as in Lemma 5.2.2 and Corollary 5.1.1.

(b_1) By Lemma 5.3.1, $\mathcal{C}_f = C_f \cup C_f^{-1} \cup \{\mathbf{1_G}\}$. Clearly $\mathbf{1_G}$ is cyclically reduced.

By Corollary 5.2.3 we have seen that every element of C_f is cyclically reduced.

By Lemma 2.4.1, since every element of C_f is cyclically reduced we must have that every element of C_f^{-1} is cyclically reduced.

Hence every element of $\mathcal{C}_f$ is cyclically reduced.

From this we see that if $g \in \mathcal{C}_f$ is such that $L(g) > 0$, it is hyperbolic

(b_2) This uses the same arguement as (b_1).

(c_1) By Corollary 5.1.1, Lemmas 5.2.1 and 5.2.2 and part (a_1) of this Theorem, every element $g \in \mathcal{C}_f$ can be written uniquely in the form

$$g = (h_1 \circ h_2)^{r(t)}$$

where $r(t) \in \mathbb{Z}[t]/p(t)$. Hence ρ_f is well-defined and, by the note after Corollary 5.1.1, it is a bijection.

By definition $L(g) = r(t)L(h_1 \circ h_2) = |\rho_f(g)|$ for $g = (h_1 \circ h_2)^{r(t)}$, and the positive cone for $\langle \Omega_f^0 \rangle^t$ is therefore identified with C_f.

If we can prove that ρ_f is a homomorphism, since it is bijective, it is an isomorphism. To do this we need

$$\rho_f(\mathbf{1_G}) = 0 \tag{5.1}$$

$$\rho_f(g_1 g_2) = \rho_f(g_1) + \rho_f(g_2) \text{ for } g_1, g_2 \in C_f \tag{5.2}$$

$$\rho_f(g_1 g_2^{-1}) = \rho_f(g_1) - \rho_f(g_2) \text{ for } g_1, g_2 \in C_f \tag{5.3}$$

By definition $\mathbf{1_G} = f^0$ and so

$$\begin{aligned} \rho_f(\mathbf{1_G}) =& 0.L(h_1 \circ h_2) \\ =& 0 \end{aligned}$$

so (4.1) is true.

For (4.2), (4.3) assume that

$$\begin{aligned} g_1 =& f^{g_1(t)} \\ g_2 =& f^{g_2(t)} \end{aligned}$$

so that $g_2^{-1} = f^{-g_2(t)}$. Then

$$g_1 g_2 = f^{g_1(t) + g_2(t)}$$

by the definition of exponentiation, so

$$\begin{aligned}\rho_f(g_1g_2) =&(g_1(t)+g_2(t))L(h_1\circ h_2)\\ =&g_1(t)L(h_1\circ h_2)+g_2(t)L(h_1\circ h_2)\\ =&\rho_f(g_1)+\rho_f(g_2)\end{aligned}$$

and

$$g_1g_2^{-1}=f^{g_1(t)-g_2(t)}$$

by the definition of exponentiation, so

$$\begin{aligned}\rho(g_1g_2^{-1}) =&(g_1(t)-g_2(t))L(h_1\circ h_2)\\ =&g_1(t)L(h_1\circ h_2)-g_2(t)L(h_1\circ h_2)\\ =&\rho_f(g_1)-\rho_f(g_2)\end{aligned}$$

as required.

So all three equations hold and ρ_f is an isomorphism.

(c_2) By Corollary 5.1.1, Lemmas 5.2.1 and 5.2.2 and part (a_2) of this Theorem, every element $g\in\mathcal{C}_f$ can be written uniquely in the form

$$g=(f^k\circ f|_{[0,\omega]})^\sigma$$

where $k\in\mathbb{N}_0$, $\omega\in\Omega_f^0\backslash\{\alpha\}$ and $\sigma=\pm1$, with the convention $\mathbf{1_G}=(f^0\circ f|_{[0,0]})^{+1}$ not $(f^0\circ f|_{[0,0]})^{-1}$. Hence ρ_f is well-defined and, by the note after Corollary 5.1.1, it is a bijection.

Now $L(f)=\alpha$, so, for g as above

$$L(g)=(k\alpha+\omega)=|\rho_f(g)|$$

by definition, and the positive cone for $\langle\Omega_f^0\rangle^t$ is therefore identified with C_f.

We prove that ρ_f is an isomorphism by proving (4.1) to (4.3) as in (c_1).

Again $\rho_f(\mathbf{1_G})=0$ so (4.1) is true.

For (4.2) and (4.3) assume that

$$\begin{aligned} g_1 =& f^{k_1} \circ f|_{[0,\omega_1]} \\ g_2 =& f^{k_2} \circ f|_{[0,\omega_2]} \end{aligned}$$

Then

$$\begin{aligned} g_1 g_2 =& f^{k_1+k_2} \circ f|_{[0,\omega_1]} \circ f|_{[0,\omega_2]} \\ =& \begin{cases} f^{k_1+k_2} \circ f|_{[0,\omega_1+\omega_2]} & \alpha > \omega_1 + \omega_2 \\ f^{k_1+k_2+1} \circ f|_{[0,\omega_1+\omega_2-\alpha]} & \alpha \leq \omega_1 + \omega_2 \end{cases} \end{aligned}$$

So

$$\rho_f(g_1 g_2) = \begin{cases} (k_1 + k_2)\alpha + \omega_1 + \omega_2 & \alpha \leq \omega_1 + \omega + 2 \\ (k_1 + k_2 + 1)\alpha + \omega_1 + \omega_2 - \alpha & \alpha > \omega_1 + \omega_2 \end{cases}$$

$$\begin{aligned} =& (k_1 + k_2)\alpha + \omega_1 + \omega_2 \\ =& (k_1\alpha + \omega_1) + (k_2\alpha + \omega_2) \\ =& \rho_f(g_1) + \rho_f(g_2) \end{aligned}$$

as required.

Meanwhile

$$g_1 g_2^{-1} = \begin{cases} \mathbf{1_G} & L(g_1) = L(g_2) \\ f^{k_3} \circ f|_{[0,\omega_3]} & L(g_1) > L(g_2) \\ (f^{k_3} \circ f|_{[0,\omega_3]})^{-1} & L(g_1) < L(g_2) \end{cases}$$

In the first case $g_1 = g_2$, so

$$\rho_f(g_1) = \rho_f(g_2)$$

as required.

In the second case

$$\begin{aligned} g_1 =& g_3 \circ g_2 \\ =& \begin{cases} f^{k_3+k_2} \circ f|_{[0,\omega_3+\omega_2]} & \alpha > \omega_3+\omega_2 \\ f^{k_3+k_2+1} \circ f|_{[0,\omega_3+\omega_2-\alpha]} & \alpha \leq \omega_3+\omega_2 \end{cases} \end{aligned}$$

so

$$(k_1, \omega_1) = \begin{cases} (k_3+k_2, \omega_3+\omega_2) & \alpha > \omega_3+\omega_2 \\ (k_3+k_2+1, \omega_3+\omega_2-\alpha) & \alpha \leq \omega_3+\omega_2 \end{cases}$$

which implies

$$(k_3, \omega_3) = \begin{cases} (k_1-k_2, \omega_1-\omega_2) & \alpha > \omega_3+\omega_2 \\ (k_1-k_2-1, \omega_1-\omega_2+\alpha) & \alpha \leq \omega_3+\omega_2 \end{cases}$$

and therefore

$$\begin{aligned} \rho_f(g_1 g_2^{-1}) =& \begin{cases} (k_1-k_2)\alpha + (\omega_1-\omega_2) & \alpha > \omega_3+\omega_2 \\ (k_1-k_2-1)\alpha + (\omega_1-\omega_2+\alpha) & \alpha \leq \omega_3+\omega_2 \end{cases} \\ =& (k_1-k_2)\alpha + \omega_1 - \omega_2 \\ =& (k_1\alpha+\omega_1) - (k_2\alpha+\omega_2) \\ =& \rho_f(g_1) - \rho_f(g_2) \end{aligned}$$

as required.

Finally, if $g_3 = (f^{k_3} \circ f|_{[0,\omega_3]})^{-1}$, then

$$\begin{aligned} g_2 =& g_3^{-1} \circ g_1 \\ =& \begin{cases} f^{k_3+k_1} \circ f|_{[0,\omega_3+\omega_1]} & \alpha > \omega_3+\omega_1 \\ f^{k_3+k_1+1} \circ f|_{[0,\omega_3+\omega_1-\alpha]} & \alpha \leq \omega_3+\omega_1 \end{cases} \end{aligned}$$

so

$$(k_2, \omega_2) = \begin{cases} (k_3 + k_1, \omega_3 + \omega_1) & \alpha < \omega_3 + \omega_1 \\ (k_3 + k_1 + 1, \omega_3 + \omega_1 - \alpha) & \alpha \leq \omega_3 + \omega_1 \end{cases}$$

which implies

$$(k_3, \omega_3) = \begin{cases} (k_2 - k_1, \omega_2 - \omega_1) & \alpha < \omega_3 + \omega_1 \\ (k_2 - k_1 - 1, \omega_2 - \omega_1 + \alpha) & \alpha \leq \omega_3 + \omega_1 \end{cases}$$

and therefore

$$\begin{aligned} \rho_f(g_1 g_2^{-1}) = & \begin{cases} -((k_2 - k_1)\alpha + \omega_2 - \omega_1) & \alpha < \omega_3 + \omega_1 \\ -((k_2 - k_1 - 1)\alpha + \omega_2 - \omega_1 + \alpha) & \alpha \leq \omega_3 + \omega_1 \end{cases} \\ = & -((k_2 - k_1)\alpha + \omega_2 - \omega_1) \\ = & (k_1\alpha + \omega_1) - (k_2\alpha + \omega_2) \\ = & \rho_f(g_1) - \rho_f(g_2) \end{aligned}$$

as required, so we are done.

(d_1) Let

$$\Gamma_f = \langle u, u^t, \ldots, u^{t^{n-1}} | [u^{t^i}, u^{t^j}] = 1 \rangle$$

The mapping

$$\{u^{t^i} : 0 \leq i \leq n-1\} \longrightarrow \langle \Omega_f^0 \rangle^t$$

defined by $u^{t^i} \mapsto t^i\omega$ where $\omega = L(h_1 \circ h_2)$, extends to a surjective homorphism

$$\phi_f : \Gamma_f \longrightarrow \langle \Omega_f^0 \rangle^t$$

Every word w $\in \Gamma_f$ can be written uniquely as $u^{a_0}u^{a_1 t} \ldots u^{a_{n-1}t^{n-1}}$, since $[u^{t^i}, u^{t^j}] = 1$, but this would be sent to

$$a_0\omega + a_1 t\omega + \ldots + a_{n-1}t^{n-1}\omega = (a_0 + a_1 t + \ldots + a_{n-1}t^{n-1})\omega$$

which is a unique element of $\langle\Omega_f^0\rangle^t$ by Corollary 5.1.1. Hence ϕ_f is injective, and therefore bijective, and we have an isomorphism. By part (c_1) this implies that $\Gamma_f \cong \mathcal{C}_f$.

(d_2) Let

$$\Gamma_f = \langle x_\omega(\omega \in \Omega_f^0) | [x_\alpha, x_\omega] = 1(\omega < \alpha), x_{\omega_1}x_{\omega_2} = x_\alpha^{\lfloor\frac{\omega_1+\omega_2}{\alpha}\rfloor} x_{\omega_1 \boxplus \omega_2}(\omega_1, \omega_2 < \alpha)\rangle$$

The mapping

$$\{x_\omega : \omega \in \Omega_f^0\} \longrightarrow \langle\Omega_f^0\rangle^t$$

defined by $x_\omega \mapsto \omega$ extends to a surjective homomorphism

$$\phi_f : \Gamma_f \longrightarrow \langle\Omega_f^0\rangle^t$$

Now, using the relators $x_{\omega_1}x_{\omega_2} = x_\alpha^{\lfloor\frac{\omega_1+\omega_2}{\alpha}\rfloor} x_{\omega_1 \boxplus \omega_2}(\omega_1, \omega_2 < \alpha)$, we see that

$$\begin{aligned} x_\omega x_0 =& x_\alpha^0 x_\omega \\ =& x_\omega \end{aligned}$$

so

$$x_0 = 1 \tag{5.4}$$

Also

$$\begin{aligned} x_{\omega_2}x_{\omega_1} =& x_\alpha^{\lfloor\frac{\omega_2+\omega_1}{\alpha}\rfloor} x_{\omega_2 \boxplus \omega_1} \\ =& x_\alpha^{\lfloor\frac{\omega_1+\omega_2}{\alpha}\rfloor} x_{\omega_1 \boxplus \omega_2} \\ =& x_{\omega_1}x_{\omega_2} \end{aligned}$$

so

$$[x_{\omega_1}, x_{\omega_2}] = 1 \tag{5.5}$$

and

$$\begin{aligned} x_{\omega_1-\omega_2}x_{\omega_2} =& x_\alpha^{\lfloor\frac{\omega_1}{\alpha}\rfloor} x_{\omega_1} \\ =& x_\alpha^0 x_{\omega_1} \\ =& x_{\omega_1} \end{aligned}$$

so

$$x_{\omega_1}x_{\omega_2}^{-1} = x_{\omega_1-\omega_2} \tag{5.6}$$

whilst

$$\begin{aligned} x_{\alpha-\omega}x_\omega =& x_\alpha^{\lfloor \frac{\alpha}{\alpha} \rfloor} x_0 \\ =& x_\alpha \end{aligned}$$

so

$$x_\alpha x_\omega^{-1} = x_{\alpha-\omega} \quad 0 < \omega < \alpha \tag{5.7}$$

Now I claim that, as in [2], any word w $\in \Gamma_f$ can be written in the form $(x_\alpha^k x_\omega)^\sigma$ with $k \in \mathbb{N}_0$, $\omega \in \Omega_f^0 \setminus \{\alpha\}$ and $\sigma = \pm 1$. Using Corollary 5.1.1, this would imply that ϕ_f is an injection and hence an isomorphism.

Proof of claim:

First, since $[x_\alpha, x_\omega] = 1$ for all $\omega < \alpha$, we can collect up all of the x_α's on the left hand side, so that we have

$$x_\alpha^l x_{\omega_1}^{\pm 1} \dots x_{\omega_s}^{\pm 1}$$

with $l \in \mathbb{Z}$ and $\omega_i \in \Omega_f^0 \setminus \{\alpha\}$.

Now we can use (5.4) and (5.6) to reduce $x_{\omega_1}^{\pm 1} \dots x_{\omega_s}^{\pm 1}$ to $x_\alpha^m x_\omega^{\pm 1}$ in finitely many steps, so we have

$$\mathrm{w} = x_\alpha^{l+m} x_\omega^\sigma$$

with $l + m \in \mathbb{Z}, \omega \in \Omega_f^0 \setminus \{\alpha\}$ and $\sigma = \pm 1$.

If $l + m \geq 0$ and $\sigma = 1$ or $l + m \leq 0$ and $\sigma = -1$ we are done, so let $l + m > 0 > \sigma$. Then

$$\begin{aligned} \mathrm{w} =& x_\alpha^{l+m} x_\omega^{-1} \\ =& x_\alpha^{l+m-1} x_\alpha x_\omega^{-1} \\ =& \begin{cases} x_\alpha^{l+m-1} x_{\alpha-\omega} & \omega > 0 \\ x_\alpha^{l+m} x_0 & \omega = 0 \end{cases} \end{aligned}$$

using (5.5) and (5.7). Since $l+m-1 \geq 0$, this is of the required form.

If $l+m<0<\sigma$, then

$$\begin{aligned}
\mathrm{w}^{-1} =& (x_\alpha^{l+m} x_\omega)^{-1} \\
=& x_\omega^{-1} x_\alpha^{-(l+m)} \\
=& x_\alpha^{-(l+m)} x_\omega^{-1} \\
=& \begin{cases} x_\alpha^{-(l+m)-1} x_{\alpha-\omega} & \omega > 0 \\ x_\alpha^{-(l+m)} x_0 & \omega = 0 \end{cases}
\end{aligned}$$

since $x_\alpha^{-(l+m)} x_\omega^{-1}$ is of the form of the case above.

Hence

$$\begin{aligned}
\mathrm{w} &= \begin{cases} (x_\alpha^{-(l+m)-1} x_{\alpha-\omega})^{-1} & \omega > 0 \\ (x_\alpha^{-(l+m)} x_0)^{-1} & \omega = 0 \end{cases} \\
&= \begin{cases} x_{\alpha-\omega}^{-1} x_\alpha^{1+(l+m)} & \omega > 0 \\ x_0^{-1} x_\alpha^{l+m} & \omega = 0 \end{cases} \\
&= \begin{cases} x_\alpha^{1+l+m} x_{\alpha-\omega}^{-1} & \omega < 0 \\ x_\alpha^{l+m} x_0^{-1} & \omega = 0 \end{cases}
\end{aligned}$$

using the case above and the fact that $[x_\alpha, x_\omega] = 1$. Now if $l+m+1=0$, then $\mathrm{w} = x_\alpha^0 x_{\alpha-\omega}^{-1} = (x_\alpha^0 x_{\alpha-\omega})^{-1}$ which is of the form required. If $l+m+1<0$, then $\mathrm{w} = (x_\alpha^{-(l+m+1)} x_{\alpha-\omega})^{-1}$, so this too is of the required form.

Hence

$$\mathrm{w} = (x_\alpha^k x_\omega)^\sigma$$

for some $k \in \mathbb{N}_0$, $\omega \in \Omega_f^0 \setminus \{\alpha\}$ and $\sigma = \pm 1$ as required.

Therefore ϕ_f is an isomorphism, and by part (c_2) this means that $\Gamma \cong \mathcal{C}_f$.

(e_1)

(i) If f is Type 1, then by Lemma 5.1.1 part (i) there exists $\omega \in \Omega_f^0$ such that $\dim(\omega) = 1$. By the proof of part (iii) of Lemma 5.1.2, we find that $f = (u_1 \circ u_2)^{h(t)}$, and for all $\omega \in \Omega_f^0$ such that $\dim(\omega) = 1$, $\omega = (u_1 \circ u_2)^s$ for some $s \in \mathbb{N}\setminus\{0\}$. But $\inf\{(u_1 \circ u_2)^s | s \in \mathbb{N} \setminus \{0\}\} = u_1 \circ u_2$ and since $u_2 \circ u_1 \in R$, we must have that $L(u_1 \circ u_2) = L(u_2 \circ u_1) > 0$. Hence $\omega_0 > 0$, so $\dim(\omega_0) = 1$.

(ii) If we let $u_1 \circ u_2 = f_0$ and $k_0(t) = h(t)$, then by Lemma 5.1.2 part (iii) we see that $f = f_0^{k_0(t)}$, where $f_0 = f|_{[0,\omega]}$, since $f|_{[0,\omega_0]} = u_1 \circ u_2$ by the proof of Lemma 5.1.1 part (i).

(iii) From the definition of exponentiation, part (ii) above implies that $\alpha = |k_0(t)|L(f_0)$, but $L(f_0) = \omega_0$, so $\alpha = |k_0(t)|\omega_0$ with $|k_0(t)| \in \mathbb{Z}[t]/p(t)$ as required.

(iv) By part (a_1) of this theorem we have that

$$\begin{aligned} C_f =& \{(h_1 \circ h_2)^{q(t)} : q(t) \in \mathbb{Z}[t]/p(t), q(t) > 0\} \\ =& \{f_0^{k(t)} | 0 < k(t) \in \mathbb{Z}[t]/p(t)\} \end{aligned}$$

since $f_0 = u_1 \circ u_2 = h_1 \circ h_2$. By part (iii) of this theorem, $\mathcal{C}_f = \langle \Omega_f^0 \rangle^t$. But $\langle \Omega_f^0 \rangle = \langle f_0 \rangle$, so $\mathcal{C}_f = \langle f_0 \rangle^t$.

If $\dim(f) = 1$, then $L(f) = k_0\omega_0$, with $k_0 \in \mathbb{N}$, so $\Omega_f^0 = \{0, \omega_0, 2\omega_0, \ldots, k_0\omega_0\}$, which is finite.

(e_2) (i) If f is Type 2 and $\dim(f) > 1$ then by Lemma 5.1.1 part (2), any $\omega \in \Omega_f^0 \setminus \{0\}$ are such that $\dim(\omega) = \dim(f) > 1$, hence $\dim(\omega_0) > 1$ and therefore $\omega_0 \in \Omega_f^0 \setminus \{0\}$.

(ii) By Lemma 4.1.4 $f = g_1 \circ u_1^{\alpha_1} \circ g_2 \circ \ldots \circ g_m \circ u_m^{\alpha_m} \circ g_{m+1}$. Since by Lemma 5.1.2 part (iv), $\omega_1 - \omega_2 \in \Omega_f^0$ if $\omega_1 - \omega_2 \in [0, \alpha]$, any $\omega \in \Omega_f^0 \setminus \{0\}$ is of the form $L(g_1 \circ u_1^{\alpha_1} \circ g_2 \circ \ldots \circ g_k \circ u_k^{\alpha_k} \circ h_{k+1})$, where $g_{k+1} = h_{k+1} \circ g_1$, and there are at most m of these periods. Hence there are only finitely many $\omega \in \Omega_f^0$.

(iii) Since $\omega_0 > 0$, using the method in Corollary 5.1.1 we can find some $k_0 \in \mathbb{N} \setminus \{0\}$ such that $\alpha = k_0\omega_0 + \omega'$ with $0 \leq \omega' < \omega_0$ and $\omega' \in \Omega_f^0$. But by the definition of ω_0, this forces $\omega' = 0$, hence $\alpha = k_0\omega_0$ for some $k_0 \in \mathbb{N} \setminus \{0\}$. If we set $f_0 = f|_{[0,\omega_0]}$, we see that $f_0^{k_0} = f$ by part (c_2) of this theorem.

(iv) By part (a_2) of this theorem

$$C_f = \{f^k \circ f|_{[0,\omega]} | (k,\omega) \in \mathbb{N}_0 \times \Omega_f^0 \setminus \{\alpha\}, k + \omega > 0\}$$

By part (c_2) and (e_2) (iii) above we see that $f^k = f_0^{kk_0}$ and $f|_{[0,\omega]} = f_0^r$ for some $0 \leq r < k_0$, so $f^k \circ f|_{[0,\omega]} = f_0^{kk_0+r}$ where $kk_0 + r \in \mathbb{N} \setminus \{0\}$. Hence

$$C_f = \{f_0^s | s \in \mathbb{N} \setminus \{0\}\}$$

and this forms a positive cone for $\mathcal{C}_f$ so $\mathcal{C}_f = \langle f_0 \rangle$.

If $\dim(f) = 1$, we see that f is an element of H_1 without cyclic centralisers in H_1 and is therefore isomorphic to elements of $\mathcal{RF}(G)$ as in [2] which do not satisfy $(i) - (iii)$ in Theorem 4.11 of [2]. Hence, by that theorem, $\omega_0 = 0$.

$\square$

Note that the centraliser of f in the whole set $\mathcal{RF}(G, \mathbb{R}^n)$ has other types of elements that do not occur in $\langle P(H_1, R) \rangle$ at all.

Chapter 6

Conclusion

In this thesis I have constructed a new group and started work on examining its properties and the properties of its elements. There is much more work to be done in this area. In particular there is a Λ-tree associated to $\mathcal{RF}(G,\Lambda)$. This comes from the fact that there is a Lyndon length function associated to it. This Λ-tree is not necessarily complete, though it should be transitive as it is strongly regular as defined in Appendix A of Chiswell and Müller's book [2], definition A31.

As can be seen from Appendix B of [2] there are already a number of open questions from their work, and as my class of groups is a generalisation of their class of groups, many of these questions could be generalised, solved and then restricted to their situation to provide the proofs they are looking for.

Finally, I have had to restrict the dimension of Λ to n for the construction of my group, but if we allowed n to go to infinity then we should find a non-discrete version of Lyndon's $F^{\mathbb{Z}[t]}$ group. This would provide a clear link between the work of Myasnikov, Remeslennikov and Serbin and Lyndon and that of Chiswell and Müller.

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Notation:

$\lfloor \frac{a}{b} \rfloor$**:** This is the largest integer z such that $z \leq \frac{a}{b}$.

$\circ$**:** A partial multiplication which shows us when reduced multiplication is no different to $*$-multiplication for given elements.

$\circ_\delta$**:** A partial multiplication that show us when reduced multiplication has at most a cancellation of length δ.

$\bigoplus$**:** The direct summand.

$\boxplus$**:** Addition modulo α.

$*$**:** Multiplication within $\mathcal{F}(G, \Lambda)$ consisting of concatenation.

0**:** The identity element of an additive ordered abelian group.

$\mathbf{1_G}$**:** The identity element of $\mathcal{F}(G, \Lambda)$ (and hence all of its subsets).

1_G**:** The identity element of the group G in $\mathcal{F}(G, \Lambda)$.

$c(f, g)$**:** $c(f, g) = (L(f) + L(g) - L(f^{-1}g))/2$.

$\mathcal{C}_f$**:** The centraliser of f in $\mathcal{CDF}(G, \Lambda)$.

C_f^-**:** Those elements,g, of $\mathcal{C}_f$ that are shorter than f but longer than $\mathbf{1_G}$ and for which $\varepsilon_0(f, g) = 0$.

C_f^+**:** Those elements g of $\mathcal{C}_f$ that are at least as long as f and for which $\varepsilon_0(f, g) = 0$.

C_f**:** $C_f = C_f^- \cup C_f^+$.

$\mathcal{E}(f, g)$**:** The set of lengths, $\varepsilon \in \Lambda$, for which $f^{-1}(x) = g(x)$ for all $x < \varepsilon$.

$\varepsilon_0(f, g)$**:** The supremum of $\mathcal{E}(f, g)$.

$F^{\mathbb{Z}[t]}$**:** Lyndon's free $\mathbb{Z}[t]$-group.

$f|_{[\alpha,\beta]}$: The function f restricted to the interval $[\alpha, \beta]$ for $0 \leq \alpha, \beta \leq L(f) \in \Lambda$. This function, once relabelled, exists in $\mathcal{RF}(G, \Lambda)$.

$f|_{[\alpha,\beta)}$: The function f restricted to the interval $[\alpha, \beta)$ for $0 \leq \alpha, \beta \leq L(f) \in \Lambda$. This function does not exist in $\mathcal{RF}(G, \Lambda)$ as its endpoints are not in Λ.

G_0: The set of all elements of $\mathcal{F}(G, \Lambda)$ that have length zero.

H_1: The one dimensional elements of $\mathcal{CDF}(G, \Lambda)$, a subgroup of $\mathcal{CDF}(G, \Lambda)$ that is isomorphic to the embedding of Chiswell and Müller's group, $\mathcal{RF}(G)$ into my set $\mathcal{RF}(G, \Lambda)$.

Λ: In this thesis this always refers to an ordered abelian group.

$L(f)$: The length of the domain of an element $f \in \mathcal{F}(G, \Lambda)$.

$\mathbb{N}_0$: The natural numbers, including 0.

$\mathbb{N}\backslash\{0\}$: The natural numbers, excluding 0.

Ω_f: Periods of f, or the points, ω, in $[0, L(f)]$ for which $f(x) = f(x+\omega)$ for all $x \in [0, L(f) - \omega]$.

Ω_f^0: Strong periods of f, or the points ω as above, but for which $L(f) - \omega$ has the same property.

$\langle \Omega_f^0 \rangle^t$: The group generated by Ω_f^0 and, if f is Type 1, its $\mathbb{Z}[t]/p(t)$-exponentiation.

$\mathbb{Q}$: The rationals.

$\mathbb{Q}^n$: The n-dimensional rational space.

$\mathbb{R}$: The reals.

$\mathbb{R}^n$: The n-dimensional real space.

$w(p)$: The result of multiplying out an R-form.

$\mathbb{Z}$**:** The integers.

$\mathbb{Z}\ [t]$**:** The ring of integral polynomials.

$\mathbb{Z}\ [t]/p(t)$**:** The ring of integral polynomials modulo $p(t)$, an integral polynomial (in this thesis p(t) must also be irreducible).

$\mathcal{F}(G,\Lambda)$**:** The set of all functions $f : [0,\alpha]_\Lambda \longrightarrow G$.

$\mathcal{RF}(G,\Lambda)$**:** The set of all reduced functions in $\mathcal{F}(G,\Lambda)$.

$\mathcal{CDF}(G,\Lambda)$**:** The set of all cyclically decomposable functions in $\mathcal{RF}(G,\Lambda)$.

$\mathcal{CRF}(G,\Lambda)$**:** The set of all cyclically reduced functions in $\mathcal{RF}(G,\Lambda)$.

$CDR\ (\mathbb{Z}[t],X))$: For X a generating set of a fre group, this is the set of cyclically decomposed elements of the $\mathbb{Z}[t]$-exponentiation of X.

$\mathcal{P}(H.R)$**:** The set of all R-forms over H.

$P(H,R)$**:** The set of elements of $\mathcal{RF}(G,\Lambda)$ of the form $gu^\alpha h$ for $\alpha \in \mathbb{Z}[t]/p(t)$, $g,h \in H$ and $u \in R$.

$P(H_1,R)$**:** As above using H_1 as the original subgroup.

A**-group:** G is an A-group if it comes with an exponentiation function that satisfies certain properties.

Cancellative monoid: A set, M, that is closed under an associative multiplication that has an identity, and such that for all $a,b,c \in M$, $ab = ac$ or $ba = ca$ inplies that $b = c$.

Cyclically decomposable functions: Functions, f, in $\mathcal{RF}(G,\Lambda)$ that can be decomposed into three parts, $c, g, c^{-1} \in \mathcal{RF}(G,\Lambda)$, such that $f = c \circ g \circ c$ and $\varepsilon_0(g,g) = 0$.

Cyclically reduced functions: Functions, f, in $\mathcal{RF}(G,\Lambda)$ which have the property that $\varepsilon_0(f,f) = 0$.

Densely ordered abelian group: An ordered abelian group, Λ, where $\inf\{P\} = 0$ for P the positive cone of Λ.

Dimension of a function, f, in $\mathcal{RF}(G, \mathbb{R}^n)$: Let $L(f) = (x_{n-1}, \ldots, x_0)$. Then the dimension of f is $i+1$ if x_i was the final non-zero entry, reading from the right.

Discretely ordered abelian group: An ordered abelian group, Λ, where the positive cone contains its least postive element.

Exponentiation: A function $G \times A \longrightarrow G$ that sends $(g, \alpha) \mapsto g^\alpha$.

Free group: A group that is generated by a set that has no relations on it other than the trivial relations.

Hyperbolic elements: Here these are the elements that have length greater than zero. They are related to hyperbolic elements in Λ-trees.

Lyndon length functions: A function $L : G \longrightarrow \Lambda$ that assigns a length to elements of an ordered abelian group in such a way that the length of the identity equals zero, an element always has the same length as its inverse and for $c(g, h)$ as above, $c(g, h) \geq \min\{c(g, k), c(h, k)\}$.

Lyndon's set: A set of representatives that also satisfies some extra conditions.

Normal function: A function, f such that $f(0) = 1_G$.

Normal R-forms: A reduced R-form, p, such that $w(p) = g_1 \circ u_1^{\alpha_1} \circ g_2 \circ \ldots \circ g_m \circ u_m^{\alpha_m} \circ g_{m+1}$ and g_i does not end in $u_i^{\pm 1}$ whilst $g_i \circ u_i^{\alpha_i}$ does not start in $u_i^{\pm 1}$.

Normalised: A function that is a normal function is said to be normalised.

Ordered abelian group: An abelian group that has a total ordering defined upon it.

Periods and strong periods: See Ω_f and Ω_f^0.

Positive cone: All elements of an ordered abelian group that are greater than zero.

Pregroup: A set P with an involution, $x \to x^{-1}$, and identity, 1, defined on it, along with a partial multiplication such that xx^{-1} and $x^{-1}x$ are always defined and equal to 1, whilst $x1 = 1x = x$ are always defined, the multiplication always associates when defined and when wx, xy and yz are defined, one of wxy or xyz are defined.

R-form: A sequence, $p = (g_1, u_1^{\alpha_1}, g_2, \ldots, g_m, u_m^{\alpha_m}, g_{m+1})$ where $g_i \in H$, $u_i \in R$, $\alpha_i \in \mathbb{Z}[t]/p(t)$ and $m \geq 1$.

Reduced function: A function in $\mathcal{F}(G, \Lambda)$ with no degenerate subintervals.

Reduced multiplication: Multiplication on $\mathcal{RF}(G, \Lambda)$ which is done in two steps, concatenation first, then reduction of the resulting element.

Reduced R-form: An R-form where $deg\{\alpha_i\} > 0$ and if u_i and u_{i+1} commute, then neither of them commute with g_i.

S-set: In this set any two non-commuting elements with cyclic centralisers are separated.

S-subgroup: An S-set that is also a group.

Separated: Two elements $u, v \in \mathcal{CDF}(G, \Lambda)$ are separated if when m and k are high enough, $u^m v^k$ starts in u and ends in v, and these two parts of $u^m v^k$ so not overlap.

Set of representatives of M: A set that does not contain proper powers or inverses, but which can generate M using permutations and conjugation.

Subword closed: For a given word in a group, every subword of that word is also in the group.

Type 1 function: Those elements $f \in \langle P(H_1, R) \rangle$ which have a one-dimensional period that has non-trivial cyclic centralisers in the group H_1.

Type 2 function: An element in $\langle P(H_1, R) \rangle$ that is not Type 1.

Universal group of a pregroup: The smallest group that the pregroup embeds into.

Words: Elements of G defined by a string of letters in the alphabet $S^{\pm}$, the generating set og G.