Game Semantics for Interface Middleweight Java *

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Abstract
We consider an object calculus in which open terms interact with the environment through interfaces. The calculus is intended to capture the essence of contextual interactions of Middleweight Java code. Using game semantics, we provide fully abstract models for the induced notions of contextual approximation and equivalence. These are the first denotational models of this kind.

Categories and Subject Descriptors D.3.1 [Formal Definitions and Theory]: Semantics; F.3.2 [Semantics of Programming Languages]: Denotational semantics

Keywords Full Abstraction, Game Semantics, Contextual Equivalence, Java

1. Introduction
Denotational semantics is charged with the construction of mathematical universes (denotations) that capture program behaviour. It concentrates on compositional, syntax-independent modelling with the aim of illuminating the structure of computation and facilitating reasoning about programs. Many developments in denotational semantics have been driven by the quest for full abstraction [21]: a model is fully abstract if the interpretations of two programs are the same precisely when the programs behave in the same way (i.e. are contextually equivalent). A faithful correspondence like this opens the path to a broad range of applications, such as compiler optimisation and program transformation, in which the preservation of semantics is of paramount importance.

Recent years have seen game semantics emerge as a robust denotational paradigm [4, 6, 12]. It has been used to construct the first fully abstract models for a wide spectrum of programming languages, previously out of reach of denotational semantics. Game semantics models computation as an exchange of moves between two players, representing respectively the program and its computational environment. Accordingly, a program is interpreted as a strategy in a game corresponding to its type. Intuitively, the plays that game semantics generates constitute the observable patterns that a program produces when interacting with its environment, and this is what underlies the full abstraction results. Game semantics is compositional: the strategy corresponding to a compound program phrase is obtained by canonical combinations of those corresponding to its sub-phrases. An important advance in game semantics was the development of nominal games [3, 17, 26], which underpinned full abstraction results for languages with dynamic generative behaviours, such as the $\nu$-calculus [3], higher-order concurrency [18] and ML references [24]. A distinctive feature of nominal game models is the presence of names (e.g. memory locations, references names) in game moves, often along with some abstraction of the store.

The aim of the present paper is to extend the range of the game approach towards real-life programming languages, by focussing on Java-style objects. To that end, we define an imperative object calculus, called Interface Middleweight Java (IMJ), intended to capture contextual interactions of code written in Middleweight Java (MJ) [9], as specified by interfaces with inheritance. We present both equational (contextual equivalence) and inequational (contextual approximation) full abstraction results for the language. To the best of our knowledge, these are the first denotational models of this kind.

Related Work While the operational semantics of Java has been researched extensively [7], there have been relatively few results regarding its denotational semantics. More generally, most existing models of object-oriented languages, such as [8, 15], have been based on global state and consequently could not be fully abstract.

On the other hand, contextual equivalence in Java-like languages has been studied successfully using operational approaches such as trace semantics [2, 13, 14] and environmental bisimulation [16]. The trace-based approaches are closest to ours and the three papers listed also provide characterizations of contextual equivalence. The main difference is that traces are derived operationally through a carefully designed labelled transition system and, thus, do not admit an immediate compositional description in the style of denotational semantics.

However, similarities between traces and plays in game semantics indicate a deeper correspondence between the two areas, which also manifested itself in other cases, e.g. [20] vs [19]. At the time of writing, there is no general methodology for moving smoothly between the two approaches, but we believe that there is scope for unifying the two fields in the not so distant future.

In comparison to other game models, ours has a quite lightweight structure. For the most part, playing consists of calling the opponent’s methods and returning results to calls made by the opponent. In particular, there are no justification pointers between moves. This can be attributed to the fact that Java does not feature first-class higher-order functions and that methods in Java objects cannot be updated. On the other hand, the absence of pointers makes definitions of simple notions, such as well-bracketing, less direct, since the dependencies between moves are not given explicitly any
more and need to be inferred from plays. The latter renders strategy composition non-standard. Because it is impossible to determine statically to which arena a move belongs, the switching conditions (cf. [6]) governing interactions become crucial for determining the strategy responsible for each move. Finally, it is worth noting that traditional copycat links are by definition excluded from our setting: a call/return move for a given object cannot be copycapped by the other player, as the move has a fixed polarity, determined by the ownership of the object. In fact, identity strategies contain plays of length at most two!

Further Directions In future work, we would like to look for automata-theoretic representations of fragments of our model in order to use them as a foundation for a program verification tool for Java programs. Our aim is to take advantage of the latest developments in automata theory over infinite alphabets [10], and fresh-register automata in particular [23, 27], to account for the nominal features of the model.

2. The language IMJ

We introduce an imperative object calculus, called Interface Middleware Java (IMJ), in which objects are typed using interfaces. The calculus is a stripped down version of Middleweight Java (MJ), expressive enough to expose the interactions of MJ-style objects with the environment.

Definition 1. Let Ints, Flds and Meths be sets of interface, field and method identifiers. We range over them respectively by \( I, f, m \) and variants. The types \( \theta \) of IMJ are given below, where \( \theta \) stands for a sequence \( \theta_1, \ldots, \theta_n \) of types (for any \( n \)). An interface definition \( \Theta \) is a finite set of typed fields and methods. An interface table \( \Delta \) is a finite assignment of interface definitions to interface identifiers.

\[
\begin{align*}
\text{Types} & ::= \text{void} \mid \text{int} \mid I \\
\text{IDfns} & ::= \Theta ::= \emptyset \mid \{ (f : \theta), \Theta \} \mid \{ (m : \theta \rightarrow \theta), \Theta \} \\
\text{ITblds} & ::= \Delta ::= \emptyset \mid (I : \Theta), \Delta \mid (I(I) : \Theta), \Delta
\end{align*}
\]

We write \( I(I') : \Theta \) for interface extension: interface \( I \) extends \( I' \) with fields and methods from \( \Theta \). We stipulate that the extension relation must not lead to circular dependencies. Moreover, each identifier \( f, m \) can appear at most once in each \( \Theta\) and each \( I \) can be defined at most once in \( \Delta \) (i.e. there is at most one element of \( \Delta \) of the form \( I : \Theta \) or \( I(I') : \Theta \)). Thus, each \( \Theta \) can be seen as a finite partial function \( \Theta : \text{Fields} \cup \text{Meths} \rightarrow \text{Types} \). We write \( \Theta(f) \) for \( \Theta(f) \) and \( \Theta(m) \) for \( \Theta(m) \). Similarly, \( \Delta \) defines a partial function \( \Delta : \text{Ints} \rightarrow \text{IDfns} \) given by

\[
\Delta(I) = \begin{cases} \\
\Theta & (I : \Theta) \in \Delta \\
\Delta(I') \cup \Theta & (I(I') : \Theta) \in \Delta \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

An interface table \( \Delta \) is well-formed if, for all interface types \( I, I' \):

- if \( I' \) appears in \( \Delta(I) \) then \( I' \in \text{dom}(\Delta) \),
- if \( (I(I') : \Theta) \in \Delta \) then \( \text{dom}(\Delta(I')) \cap \text{dom}(\Theta) = \emptyset \).

Henceforth we assume that interface tables are well-formed. Interface extensions yield a subtyping relation. Given a table \( \Delta \), we define \( \Delta \vdash \theta_1 \leq \theta_2 \) by the following rules.

\[
\begin{align*}
(\Delta(I') : \Theta), \Delta & \vdash I' \leq I' \\
\Delta & \vdash \theta_1 \leq \theta_2 \\
\Delta & \vdash \theta_1 \leq \theta_3 \\
\Delta & \vdash \theta_2 \leq \theta_3
\end{align*}
\]

We might omit \( \Delta \) from subtyping judgements for economy.

Definition 2. Let \( h \) be a countably infinite set of object names, which we range over by \( o \) and variants. IMJ terms are listed below, where we let \( x \) range over a set of variables \( \text{Vars} \), and \( i \) over \( I \). Moreover, \( \oplus \) is selected from some set of binary numeric operators. \( M \) is a method-set implementation. Again, we stipulate that each \( m \) appear in each \( M \) at most once.

\[
M ::= x \mid a \mid \text{skip} \mid \text{null} \mid i \mid M \oplus M \mid \text{let } x = M \text{ in } M \\
\mid M = M \mid \text{if } M \text{ then } M \text{ else } M \mid (I)M \mid \text{new(x : I ; M)} \\
\mid M.f \mid M.f : M \mid M.m(M) \\
\]

The terms are typed in contexts comprising an interface table \( \Delta \) and a variable context \( \Gamma = \{ x_1 : \theta_1, \ldots, x_n : \theta_n \} \cup \{ a_1 : I_1, \ldots, a_m : I_m \} \) such that any interface in \( \Gamma \) occurs in \( \text{dom}(\Delta) \). The typing rules are given in Figure 1.
between terms-in-state, presented in Figure 2. The transition relation uses evaluation contexts $E$ that are defined as follows.

$E ::= \text{let } x = _\epsilon \text{ in } M | _\epsilon + M | i + _\epsilon | _\epsilon = M | a = _\epsilon$

$| _\epsilon \text{ then } M \text{ else } M' | (\overline{E}) | _\epsilon = _\epsilon | _\epsilon . f := _\epsilon | _\epsilon . m(\overline{M})$.

Given $\Delta[0] \vdash M : \emptyset$, we write $M \Downarrow$ if there exists $S$ such that $(\emptyset, M) \rightarrow^{*} (S, \text{skip})$.

**Definition 3.** Given $\Delta \Gamma \vdash M : \emptyset$ contextually approximates $\Delta \Gamma \vdash M' : \emptyset$ if, for all $\Delta' \supseteq \Delta$ and all contexts $C$ such that $\Delta[0] \vdash C[M_1] : \emptyset$, we can see that $\Delta[0] \vdash C[M_2] : \emptyset$. We then write $\Delta \Gamma \vdash M_1 \approx M_2 : \emptyset$. Two terms are contextually equivalent (written $\Delta \Gamma \vdash M_1 \approx M_2 : \emptyset$) if they approximate each other.

For technical convenience, IMJ features the let construct, even though it is definable: given $\Delta \Gamma \vdash x : \emptyset$ and $\Delta \Gamma \vdash M : \emptyset$ and $\Delta \Gamma \vdash M' : \emptyset'$, consider $\text{new}(x : \overline{I}; \lambda x. M).m(M')$, where $\overline{I}$ is a fresh interface with a single method $\theta \rightarrow \emptyset$. As usual, we write $M : M'$ for let $x = M$ in $M'$, where $x$ is not free in $M'$.

Although IMJ does not have explicit local variables, they could easily be introduced by taking let $x = \text{new}(y : \overline{I}_0); \ldots$ where $\overline{I}_0$ has a single field of type $\emptyset$. In the same manner, one can define variables and methods that are private to objects, and invisible to the environment through interfaces.

**Example 1 ([16]).** Let $\Delta = \{ \emptyset : \text{Cell} ; \text{get : void } \rightarrow \text{Empty} ; \text{set : Empty } \rightarrow \text{void} \}$, $\overline{V} : \{ \text{val : Empty} , \text{Var} : \{ \text{val : int} \} \}$ and consider the terms $\Delta[0] \vdash M_1 : \text{Cell} (i = 1, 2)$ defined by:

$M_1 \equiv \text{let } v = \text{new}(x : \overline{V}_1) \text{ in } \text{new}(x : \text{Cell} ; M_1)$

$M_2 \equiv \text{let } b = \text{new}(x : \overline{V}_1) \text{ in } \text{new}(x : \text{Cell} ; M_2)$

with

$M_1 = (\text{get : } \lambda().v.\text{val}, \text{set : } \lambda y.(v.\text{val} := y))$

$M_2 = (\text{get : } \lambda().(\text{if } (b.\text{val}) \text{ then } (b.\text{val} := 0; v_1.\text{val}) \text{ else } (b.\text{val} := 1; v_2.\text{val}), \text{set : } \lambda y.(v_1.\text{val} := y; v_2.\text{val} := y))$.

We have $\Delta[0] \vdash M_1 \approx M_2 : \text{Cell}$. Intuitively, each of the two implementations of Cell corresponds to recording a single value of type Empty (using set) and providing access to it via get. The difference lies in the way the value is stored: a single private variable is used in $M_1$, while two variables are used in $M_2$. However, in the latter case the variables always hold the same value, so it does not matter which of the variables is used to return the value.

**Figure 2.** Operational semantics of IMJ.

The game semantics of the two terms will turn out to consist of plays of the shape $s^n \ast 0 G_0 S_1 \cdots S_k G_k$, where

$\begin{align*}
G_1 &= \text{call } n.\text{get}(\ast)^{G_0} \text{ ret } n.\text{get}(\ast)^{G_0} n^0 i = 0 \\
G_2 &= \text{call } n.\text{get}(\ast)^{G_0} \text{ ret } n.\text{get}(n^1)^{G_1} n^0 i > 0 \\
S_1 &= \text{call } n.\text{set}(n^1)^{G_0} \text{ ret } n.\text{set}(\ast)^{G_1}
\end{align*}$

and $\Sigma_i = \{ n \mapsto (\text{Cell}, \emptyset) \} \cup \{ n_j \mapsto (\text{Empty}, \emptyset) \mid 0 < j \leq i \}$. Intuitively, the plays describe all possible interactions of a Cell object. The first two moves $s^n \ast 0 G_0$ correspond to object creation. After that, the $G_i$ segments represent the environment reading the current content (initially having null value), while the $S_i$ segments correspond to updating the content with a reference name provided by the environment. The stores $\Sigma_i$ attached to moves consist of all names that have been introduced during the interaction so far.

It is worth noting that, because IMJ has explicit casting, a context can always guess the actual interface of an object and extract any information we may want to hide through casting.

**Example 2.** Let $\Delta = \{ \emptyset : \text{Empty} ; \text{Point(Empty) : (x : int, y : int)} \}$ and consider the terms $\Delta[0] \vdash M_i : \text{Empty} (i = 1, 2)$ defined by:

$M_1 \equiv \text{new}(x : \text{Empty} ; \ldots)$

$M_2 \equiv \text{let } p = \text{new}(x : \text{Point} ; \ldots) \text{ in } p.x := 0; p.y := 1 ; \text{Empty} . p$

In our model they will be interpreted by the strategies $\sigma_1 = \{ \epsilon, \ast \ast n^0 (\emptyset, \ast 0 (\text{Empty}, \emptyset)) \}$ and $\sigma_2 = \{ \epsilon, \ast \ast n^0 (\ast 0 (\text{Point}, \ast 0 (\text{Empty}, \emptyset))) \}$ respectively.

On the whole, IMJ is a compact calculus that stripes down Middleweight Java to the essentials needed for interface-based interaction. Accordingly, we suppressed the introduction of explicit class hierarchy, as it would remain invisible to the environment anyway and any class-based internal computations can be represented using standard object encodings [1].

At the moment the calculus allows for single inheritance for interfaces only, but extending it to multiple inheritance is not problematic. The following semantic developments only rely on the assumption that $\leq$ must not give rise to circularities.

### 3. The game model

In our discussion below we assume a fixed interface table $\Delta$.

The game model will be constructed using mathematical objects (moves, plays, strategies) that feature names drawn from the set $\mathcal{H}$. Although names underpin various elements of our model, we do not want to delve into the precise nature of the sets containing them. Hence, all of our definitions preserve name-invariance, i.e. our objects are (strong) nominal sets [11, 26]. Note that we do not
Definition 4. An arena is a pair \( A = (M_A, \xi_A) \) where:

- \( M_A \) is a nominal set of moves;
- \( \xi_A : A \to (A \to \text{Ints}) \) is a nominal typing function;

such that, for all \( m \in M_A \), \( \text{dom}(\xi_A(m)) = \nu(m) \).

We start by defining the following basic arenas,

\[ I = \{ (\ast), (\ast, \emptyset), \{ (\ast, \emptyset) \} \}, \quad \mathcal{I} = (A \cup \{ \text{nul} \}, \{ (\text{nul}, \emptyset), \{ (a, a), I \} \} ) \]

for all interfaces \( I \). Given arenas \( A \) and \( B \), we can form the arena \( A \times B \) by:

\[
M_{A \times B} = \{ (m, n) \in M_A \times M_B \mid a \in \nu(m) \cap \nu(n) \}
\]

\[ \xi_{A \times B}((m, n), a) = \begin{cases} 
\xi_A(m, a) & \text{if } a \in \nu(n) \cup \xi_B(n, a) \\
\xi_B(n, a) & \text{otherwise}
\end{cases} \]

Another important arena is \( \#(I_1, \ldots, I_n) \), with:

\[
M_{\#(I)} = \{ (a_1, \ldots, a_n) \in A^n \mid a_i \text{ 's distinct} \}
\]

\[ \xi_{\#(I)}((a_1, \ldots, a_n), a_i) = I_i \]

for all \( n \in \mathbb{N} \). In particular, \( A^{\emptyset} = 1 \).

For each type \( \theta \), we set \( \text{Val}_\theta \) to be the set of semantic values of type \( \theta \), given by:

\[
\text{Val}_{\text{void}} = M_1, \quad \text{Val}_{\ast} = M_{\ast}, \quad \text{Val}_{\mathcal{I}} = M_{\mathcal{I}}.
\]

For each type sequence \( \theta = \theta_1, \ldots, \theta_n \), we set \( \text{Val}_\theta = \text{Val}_{\theta_1} \times \cdots \times \text{Val}_{\theta_n} \).

We let a store \( \Sigma \) be a type-preserving finite partial function from names to object types and field assignments, that is, \( \Sigma : A \to \text{Ints} \times (\text{Fields} \to \text{Val}) \) such that \( |\Sigma| \) is finite and

\[
\Sigma(a) : \mathcal{I} \land \Delta(\mathcal{I}), f = \emptyset \implies \Sigma(a), f = v \land \Sigma' \vdash v \leq \theta,
\]

where the new notation is explained below. First, assuming \( \Sigma(a) = (I', \phi) \), the judgement \( \Sigma(a) : I \) holds iff \( \mathcal{I} = I' \) and \( \Sigma(a), f \) stands for \( \phi(f) \). Next we define typing rules for values in store contexts:

\[
v \in \text{Val}_{\text{void}} \quad v \in \text{Val}_{\ast} \quad \Sigma(v) : \mathcal{I} \lor v = \text{nu1}
\]

\[ \Sigma' \vdash v \leq \theta \quad \Sigma \vdash v \leq \theta \]

and write \( \Sigma \vdash v \leq \theta \) for \( \Sigma' \vdash v : \emptyset \lor (\Sigma' \vdash v : \mathcal{I} \lor I' \leq \theta) \).

We let \( \text{Sto} \) be the set of all stores. We write \( \text{dom}(\Sigma(a)) \) for the set of all \( f \) such that \( \Sigma(a), f \) is defined. We let \( \text{Sto}_0 \) contain all stores \( \Sigma \) such that:

\[ \forall a \in \text{dom}(\Sigma), f \in \text{dom}(\Sigma(a)). \Sigma(a), f \in \{ \ast, 0, \text{nu1} \} \]

and we call such a \( \Sigma \) a default store.

Given arenas \( A \) and \( B \), plays in \( AB \) will consist of sequences of moves (with store) which will be either moves from \( M_A \cup M_B \), or moves representing method calls and returns. Formally, we define:

\[
M_{AB} = M_A \cup M_B \cup \text{Calls} \cup \text{Retns}
\]

where we set \( \text{Calls} = \{ \text{call } a.m(\bar{v}) \mid a \in A \land \bar{v} \in \text{Val} \} \) and \( \text{Retns} = \{ \text{ret } a.m(v) \mid a \in A \land v \in \text{Val} \} \).

Definition 5. A legal sequence in \( AB \) is a sequence of moves from \( M_{AB} \) that adheres to the following grammar (Well-Bracketing), where \( m_A \) and \( m_B \) range over \( M_A \) and \( M_B \) respectively.

\[
L_{AB} ::= \epsilon \mid m_A X \mid m_A Y m_B X \quad X ::= Y \mid Y (\text{call } a.m(\bar{v})) \quad Y ::= \epsilon \mid YY \mid (\text{call } a.m(\bar{v})) Y \quad \text{ret } a.m(v)
\]

We write \( L_{AB} \) for the set of legal sequences in \( AB \). In the last clause above, we say that \( \text{call } a.m(\bar{v}) \) justifies \( \text{ret } a.m(v) \).

To each \( s \in L_{AB} \) we assign a polarity \( p \) from move occurrences in \( s \) to the set \( \text{Pol}_s = \{ O, P \} \). Polarieties represent the two players in our game reading of programs: \( O \) is the Opponent and \( P \) is the Proponent in the game. The latter corresponds to the modelled program, while the former models the possible computational environments surrounding the program. Polarieties are complemented via \( \overline{O} = \{ P \} \) and \( \overline{P} = \{ O \} \). In addition, the polarity function must satisfy the condition:

- For all \( m_X \in M_X (X = A, B) \) occurring in \( s \) we have \( p(m_X) = O \) and \( p(m_B) = P \) (O-starting).

- If \( mn \) are consecutive moves in \( s \) then \( p(n) = p(m) \). (Alternation)

It follows that there is a unique \( p \) for each legal sequence \( s \), namely the one which assigns \( O \) precisely to those moves appearing in odd positions in \( s \).

A move-with-store in \( AB \) is a pair \( m^\Sigma \) with \( \Sigma \in \text{Sto} \) and \( m \in M_{AB} \). For each sequence \( s \) of moves-with-store we define the set of available names of \( s \) by:

\[
\text{Av}(s) = \emptyset, \quad \text{Av}(sm^\Sigma) = \Sigma^s(\text{Av}(s) \cup \nu(m))
\]

where, for each \( X \subseteq A \), we let \( \Sigma^s(X) = \bigcup_i \Sigma^i(X) \), with \( \Sigma^0(X) = X \), \( \Sigma^{i+1}(X) = \nu(\Sigma^i(X)) \).

That is, a name is available in \( s \) just if it appears inside a move in \( s \), or it can be reached from an available name through some store in \( s \). We write \( s \) for the underlying sequence of moves of \( s \) (i.e. \( \pi_1(s) \)), and let \( \Sigma \) denote the prefix relation between sequences. If \( s^m X \subseteq s \) and \( a \in \nu(m) \cup \nu(s') \) then we say \( a \) is introduced by \( m^\Sigma \) in \( s \). In such a case, we define the owner of the name \( a \) in \( s \), written \( o(a) \), to be \( p(m) \) (where \( p \) is the polarity associated with \( s \)). For each polarity \( X \in \{ O, P \} \) we let

\[
X(s) = \{ a \in \nu(s) \mid o(a) = X \}
\]

be the set of names in \( s \) owned by \( X \).

Definition 6. A play in \( AB \) is a sequence of moves-with-store \( s \) such that \( s \) is a legal sequence and, moreover, for all \( s' \subseteq s \):

- It holds that \( \text{dom}(\Sigma) = \text{Av}(s'm^\Sigma) \). (Fragility)

- If \( a \in \text{dom}(\Sigma) \) with \( \Sigma(a) : I \) then:
  - if \( m \in M_X, X \in \{ A, B \}, \text{then } I \subseteq \xi_X(m, a) \);
  - for all \( n \in \text{Sto} \), if \( a \in \text{dom}(T) \) then \( T(a) : I \);
  - if \( \Delta(\mathcal{I}), m = \emptyset \) then:
    - if \( m = \text{call } a.m(\bar{v}) \) then \( I \vdash \bar{v} : \bar{\theta} \) for some \( \theta \leq \bar{\theta} \);
    - if \( m = \text{ret } a.m(v) \) then \( I \vdash v : \emptyset \) for some \( \theta' \leq \theta \).

(Well-clasS)
Note above that, because of well-bracketing and alternation, if \( m = \text{ret} a.m(v) \) then well-calling implies \( o(a) = p(m) \). Thus, the frugality condition stipulates that names cannot appear in a play in unreachable parts of a store (cf. [17]). Moreover, well-calling ensures that the typing information in stores is consistent and adheres to the constraints imposed by \( \Delta \) and the underlying arenas. Finally, well-calling implements the specification that each player need only call the other player’s methods. This is because calls to each player’s own methods cannot in general be observed and so should not be accounted for in plays.

Given arenas \( A, B, C \), next we define interaction sequences, which show how plays from \( AB \) and \( BC \) can interact to produce a play in \( AC \). The sequences will rely on moves with stores, where the moves come from the set:

\[
M_{ABC} = M_A \cup M_B \cup M_C \cup \text{Calls} \cup \text{Retns}.
\]

The moves will be assigned polarities from the set:

\[
Pol_2 = \{ O_L, P_L, P_L O_R, P_L O_R, O_R, P_R \}.
\]

The index \( L \) stands for “left”, while \( R \) means “right”. The indices indicate which part of the interaction (\( A, B \) or \( C \)) a move comes from, and what polarity it has therein. We also consider an auxiliary notion of pseudo-polarities:

\[
OO = \{ O_L, O_R \}, \quad PO = \{ P_L, P_L O_R \}, \quad OP = \{ P_R, O_L P_R \}.
\]

Each polarity has an opposite pseudo-polarity determined by:

\[
\overline{\text{OO}} = O_L O_R, \quad \overline{PO} = P_L P_L O_R, \quad \overline{OP} = P_R O_L P_R.
\]

Finally, each \( X \in \{ AB, BC, AC \} \) has a designated set of polarities given by:

\[
p(AB) = \{ O_L, P_L, P_L P_R, P_L O_R \},
p(BC) = \{ O_R, P_R, P_R O_L, P_R O_R \},
p(AC) = \{ O_L, P_L, P_L O_R, P_R \}.
\]

Note the slight abuse of notation with \( p_0 \), as it is also used for move polarities.

Suppose \( X \in \{ AB, BC, AC \} \). Consider a sequence \( s \) of moves-with-store from \( ABC \) (i.e. a sequence with elements \( m^S \) with \( m \in M_{ABC} \)) along with an assignment \( \Sigma \) of polarities from \( Pol_2 \) to moves of \( s \). Let \( s \upharpoonright X \) be the subsequence of \( s \) containing those moves-with-store \( m^S \) of \( s \) for which \( p(m) \in p(X) \). Additionally, we define \( s \upharpoonright X \) to be \( \gamma(s \upharpoonright X) \), where the function \( \gamma \) acts on moves-with-store by restricting the domains of stores to available names:

\[
\gamma(\epsilon) = \epsilon, \quad \gamma(s^S \upharpoonright X) = \gamma(s) m^S_{\text{Av}(s^S \upharpoonright X)}.
\]

**Definition 7.** An interaction sequence in \( ABC \) is a sequence \( s \) of moves-with-store in \( ABC \) satisfying the following conditions:

- For each \( s^S \subseteq s \) and \( a \in \text{dom}(\Sigma) \) with \( \Sigma(a) : T \): then:
  - if \( m \in M_X \), for \( X \in \{ A, B, C \} \), then \( \Sigma \subseteq \chi(m, a) \);  
  - for all \( n_T \) in \( s^S \), if \( a \in \text{dom}(T) \) then \( T(a) : T \);  
  - if \( \Delta(T), m = \theta' \rightarrow \theta \) then:
    - if \( m = \text{call} a.m(\bar{v}) \) then \( \Sigma \vdash \bar{v} : \theta' \);  
    - if \( m = \text{ret} a.m(v) \) then \( \Sigma \vdash v : \theta' \) for some \( \theta' \leq \theta \).  

(Well-calling)

- There is a polarity function \( p \) from move occurrences in \( s \) to \( Pol_2 \) such that:
  - For all \( m \in M_X \) (\( X = A, B, C \) occurring in \( s \)) we have \( p(m_A) = O_L, p(m_B) = P_L O_R \) and \( p(m_C) = P_R \);  
  - If \( m_n \) are consecutive moves in \( s \) then \( p(n) \in p(m) \).  

(Alternation)

Figure 3. Interaction diagram for \( \text{Int}(ABC) \). The diagram specifies the alternation of polarities in interaction sequences. Transitions are labelled by move polarities, while \( OO \) is the initial state.

- If \( s^S \subseteq s \) then \( m = \text{call} a.m(v) \) implies \( o(a) \in \overline{p(m)} \).  
- For each \( X \in \{ AB, BC, AC \} \), \( s \upharpoonright X \in L_X \).  
- If \( s^S \subseteq s \) and \( m = \text{ret} a.m(v) \) then there is a move \( n_T \) in \( s^S \) such that, for all \( X \) such that \( p(m) \in p(X) \), \( n \) is the justification of \( m \) in \( s \upharpoonright X \).  
- Laird’s conditions [17]:  
  - \( P(s \upharpoonright X, AB) \cap P(s \upharpoonright X, BC) = \emptyset \);  
  - \( P(s \upharpoonright X, AB) \cup P(s \upharpoonright X, BC) \cap O(s \upharpoonright X, AC) = \emptyset \);  
  - For each \( s^S \subseteq s \) ending in \( m^S \upharpoonright n_T \) and each \( a \in \text{dom}(T) \), if \( p(m) \in PO \) and \( a \notin s^S \upharpoonright X \), \( AB \), \( BC \), then \( \Sigma(a) = T(a) \).

We write \( \text{Int}(ABC) \) for the set of interaction sequences in \( ABC \).

Note that, by projecting and well-returning, each return move in \( s \) has a unique justifier. Next we show that the polarities of moves inside an interaction sequence are uniquely determined by the interaction diagram of Figure 3. The diagram can be seen as an automaton accepting \( s' \), for each \( s \in \text{Int}(ABC) \). The edges represent moves by their polarities, while the labels of vertices specify the polarity of the next (outgoing) move. For example, from \( OO \) we can only have a move \( m \) with \( p(m) \in \{ O_L, O_R \} \), for any \( p \).

**Lemma 1.** Each \( s \in \text{Int}(ABC) \) has a unique polarity function \( p \).

**Proof.** Suppose \( s \in \text{Int}(ABC) \). We claim that the alternation, well-calling, projecting and well-returning conditions uniquely specify \( p \). Consider the interaction diagram of Figure 3, which we read as an automaton accepting \( s' \), call it \( A \). The edges represent moves by their polarities, while the labels of vertices specify the polarity of the next (outgoing) move. By projecting we obtain that the first element of \( s \) is some \( m_A \) and, by alternation, its polarity is \( O_L \). Thus, \( OO \) is the initial state.

We now use induction on \( |s| \) to show that \( A \) has a unique run on \( s \). The base case is trivial, so suppose \( s = s' \). By induction hypothesis, \( A \) has a unique run on \( s' \), which reaches some state \( X \). We do a case analysis on \( m \). If \( m \in M_A \cup M_B \cup M_C \), then there is a unique edge accepting \( m \) and, by alternation, this edge must depart from \( X \). If, on the other hand, \( m = \text{call} a.m(\bar{v}) \) then the fact that \( o(a) \in \overline{p(m)} \) gives two possible edges for accepting \( m \). But observe that no combination of such edges can depart from \( X \). Finally, let \( m = \text{ret} a.m(v) \) be justified by some \( n \) in \( s' \). Then, by well-bracketing, \( n \) is the justifier of \( m \) in all projections, and hence the edge accepting \( m \) must be the opposite of the one accepting \( n \) (e.g. if \( m \) is accepted by \( O_L \) then \( n \) is accepted by \( P_L \)).

Next we show that interaction sequences project to plays. The projection of interaction sequences in \( ABC \) on \( AB, BC \) and \( AC \)
Lemma 2. Let $s \in \text{Int}(ABC)$. Then, for each $X \in \{AB, BC, AC\}$ and each $m^\Sigma$ in $s$, if $p(m) \in p(X)$ then $\pi_X(p(m)) = p_X(m)$, where $p_X$ is the polarity function of $s \mid X$.

Proof. We show this for $X = AB$, the other cases are proven similarly, by induction on $|s| \geq 0$; the base case is trivial. For the inductive case, if $m$ is the first move in $s$ with polarity in $p(AB)$ then, by projecting, $m \in M_A$ and therefore $p(m) = O_L$ and $p_A(m) = O$, as required. Otherwise, let $n$ be the last move in $s$ with polarity in $p(AB)$ before $m$. By IH, $p_A(b) = p_B(b)$ and observe that, for all $X \in p(n)$, $p_A(X) = p_B(b(n))$, so in particular $p_A(b(m)) = p_B(b(n)) = p_A(b(m))$.

The following lemma formulates a taxonomy on names appearing in interaction sequences.

Lemma 3. Let $s \in \text{Int}(ABC)$. Then,

1. $\nu(s) = O(s \mid AC) \cup P(s \mid AB) \cup P(s \mid BC)$;
2. if $s = tm^\Sigma$ and:
   - $p(m) \in O_O$ and $s \mid AC = t'm^\Sigma$,
   - or $p(m) \in P_O$ and $s \mid AB = t'm^\Sigma$,
   - or $p(m) \in P_O$ and $s \mid BC = t'm^\Sigma$,
then $\nu(t) \cap \nu(m^\Sigma) \subseteq \nu(t)$ and, in particular, if $m$ introduces a name in $t'm^\Sigma$ then $m$ introduces a name in $s$.

Proof. For 1, by definition of interactions we have that these sets are disjoint. It therefore suffices to show the left-to-right inclusion. Suppose that $a \in \nu(s)$ is introduced in some $m^\Sigma$ in $s$, with $p(m) \in P_O$ and let $s \mid AB = \cdots m^\Sigma \cdots$. If $a \in \nu(m^\Sigma)$ then $a \in P(s \mid AB)$, as required. Otherwise, by Laird’s last set of conditions, $a$ is copied from the store of the move preceding $m^n$ in $s$, a contradiction to its being introduced at $m^n$. Similarly if $p(m) \in O_O$. Finally, if $p(m) \in O_O$ then we work similarly, considering $O(s \mid AC)$.

For 2, we show the first case, and the other cases are similar. It suffices to show that $(\nu(m^\Sigma) \setminus \nu(t)) \cap \nu(t') = \emptyset$. So suppose $a \in \nu(m^\Sigma) \setminus \nu(t')$, therefore $a \in O(s \mid AC)$. But then we cannot have $a \in \nu(t)$ as the latter, by item 1, would imply $a \in P(s \mid AB) \cup P(s \mid BC)$.

Proposition 4. For all $s \in \text{Int}(ABC)$, the projections $s \mid AB$, $s \mid BC$ and $s \mid AC$ are plays in $AB$, $BC$ and $AC$ respectively.

Proof. By frugality of $s$ and application of $\gamma$, all projections satisfy frugality. Moreover, well-classing is preserved by projections. For well-calling, let $m = \text{call}_A.m(s)$ be a move in $s$ and let $n^\Sigma$ be the move introducing $a$ in $s$. Suppose $p(m) \in P(AB)$ and let us assume $p_{AB}(m) = O$. We need to show that $o_{AB}(m) = P$. By $p_{AB}(m) = O$ we obtain that $p(m) \in \{O_L, O_P, O\}$, and by well-calling of $s$, we have that $o_A(a) \in O$. Thus, $p(n) \in O_O$ and, by Lemma 3, $a$ introduces a in $s \mid AB$ and therefore $o_{AB}(m) = P$, as required. If, on the other hand, $p_A(m) = P$ then we obtain $p(n) \in O_O \cup O_P$ and therefore, by Lemma 3, $a \in P(s \mid BC) \cup O(s \mid AC)$. Thus, by the same lemma, $a \notin P(s \mid AB)$ and hence $o_{AB}(a) = O$. The cases for the other projections are shown similarly.

In our setting programs will be represented by strategies between arenas. We shall introduce them next after a few auxiliary definitions. Intuitively, strategies capture the observable computational patterns produced by a program. Let us define the following notion of subtyping between stores. For $\Sigma, \Sigma' \in \text{Sto}$, $\Sigma \subseteq \Sigma'$ holds if, for all names $a$,

$\Sigma'(a) : T' \Rightarrow \Sigma(a) \subseteq T' \land \forall f \in \text{dom}(\Sigma'(a)), (\Sigma(a).f = \Sigma'(a).f)

In particular, if $a$ is in the domain of $\Sigma'$, $\Sigma$ may contain more information about $a$ because of assigning to $a$ a larger interface. Accordingly, for plays $s, s' \in P_{AB}$, we say that $s$ is an $O$-extension of $s'$ if $s$ and $s'$ agree on their underlying sequences, while their stores may differ due to subtyping related to O-names. Where such subtyping leads to $s$ having stores with more fields than those in $s'$, $P$ is assumed to copy the values of those fields. Formally, $s \leq_O s'$ is defined by the rules:

$$
\begin{align*}
\epsilon \leq_O \epsilon' & \quad \text{if } \epsilon \leq_O \epsilon' \\
\epsilon s & \leq_O \epsilon s' & \text{if } \epsilon s' & \leq_O \epsilon s' \\
\epsilon p & \leq_O \epsilon p' & \text{if } \epsilon p' & \leq_O \epsilon p'
\end{align*}
$$

where $\Sigma$ extends $\Sigma'$ by $T$ if:

$\begin{align*}
\text{for all } a & \in \text{dom}(\Sigma') \setminus \text{dom}(\Sigma), \Sigma(a) = T(a); \\
\text{for all } a & \in \text{dom}(\Sigma(a)) \setminus \text{dom}(\Sigma'(a)), \Sigma(a).f = T(a).f.
\end{align*}$

The utility of O-extension is to express semantically the fact that the environment of a program may use up-casting to inject in its objects additional fields (and methods) not accessible to the program.

Definition 8. A strategy $\sigma$ in $AB$ is a non-empty set of even-length plays from $P_{AB}$ satisfying the conditions:

- If $\text{sm}^\Sigma \cdot \text{sn}^T \in \sigma$ then $s \in \sigma$. (Even-prefix closure)
- If $\text{sm}^\Sigma, \text{sn}^T \in \sigma$ then $\text{sm}^\Sigma \sim \text{sn}^T$. (Determinacy)
- If $s \in \sigma$ and $s \sim t$ then $t \in \sigma$. (Equivariance)$^2$
- If $s \in \sigma$ and $t \leq_O s$ then $t \in \sigma$. (O-extension)

We write $\sigma : A \rightarrow B$ when $\sigma$ is a strategy in $AB$. If $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$, we define their composition $\sigma ; \tau$ by:

$$
\sigma ; \tau = \{ s \mid AC \mid s \in \sigma[\tau]\}
$$

where $\sigma[\tau] = \{ s \mid Int(ABC) \mid s \mid AB \in \sigma \land s \mid AC \in \tau \}$.

In definitions of strategies we may often leave the presence of the empty sequence implicit, as the latter is a member of every strategy. For example, for each arena A, we define the strategy:

$$
\text{id}_A : A \rightarrow A = \{ m_A s_A m_A \in P_{AA} \}
$$

The next series of lemmata allow us to show that strategy composition is well defined.

Lemma 5. If $\text{sm}^\Sigma \cdot \text{sn}^T \in \sigma[\sigma]$ with $p(m) \notin \text{OO}$ then $\text{sm}^\Sigma \sim \text{sn}^T$. Hence, if $s_1, s_2, s_t \in \sigma[\sigma]$ with $p(m) \notin \text{OO}$ and $s_1 \sim s_2$ then $s_1 m^\Sigma \sim s_2 m^T$.

Proof. For the latter part, if $s_1 = \pi \cdot s_2$ then, since $\pi \cdot (s_2 m^T) \in \sigma[\sigma]$ then $s_1 m^\Sigma \sim s_2 m^T$. Now, for the former part, suppose WLOG that $p(m) \in \text{PO}$. Then, by the interaction diagram, we also have $p(n) \in P_A$. As $\text{sm}^\Sigma, \text{sn}^T \parallel AB \in \sigma$, by determinacy of $\sigma$ we get $s^m \sim s^m$, $^2$ Recall that, for any nominal set $X$ and $x, y \in X$, we write $x \sim y$ just if there is a permutation $\pi$ such that $x = \pi \cdot y$.}
with $s'm^{'\Sigma} = sm^\Sigma \vdash \gamma AB$ and $s'n^{'\Sigma} = sn^\Sigma \vdash \gamma AB$. We therefore have $(s', m^{'\Sigma}) \sim (s', n^{'\Sigma})$ and, trivially, $(s, s') \sim (s, s')$. Moreover, by Lemma 3, $\nu(m^{'\Sigma}) \cap \nu(s) \subseteq \nu(s')$ and $\nu(n^{'\Sigma}) \cap \nu(s) \subseteq \nu(s')$ hence, by Strong Support Lemma [26], $sm^{'\Sigma} \sim sn^{'\Sigma}$. By Laird’s last set of conditions, the remaining values of $\Sigma, T$ are determined by the last store in s, hence $sm^\Sigma \sim sn^\Sigma$. 

Lemma 6. If $s_1, s_2 \in \sigma||\tau$ end in moves with polarities in $p(AC)$ and $s_1 \vdash AC s_2 \vdash AC$ then $s_1 \sim s_2$.

Proof. By induction on $[s_1, AC] > 0$. The base case is encompassed in $s_i \equiv s_i'm^{'\Sigma} + \nu(p(m)) \in OO, i = 1, 2$, where note that by IH $m$ will have the same polarity in $s_1, s_2$. Then, by IH we get $s_1 = \pi \cdot s_2$, for some $\pi$. Let $s_i'm^{'\Sigma} = s_i \vdash \gamma AC$, for $i = 1, 2$, so in particular $s_i'o^{'\Sigma} = \pi \cdot s_i^o$ and therefore $(s_1, s_1^{'\Sigma}) \sim (s_1^o, s_2)$. Moreover, by hypothesis, we trivially have $(m^{'\Sigma}, s_1^{'\Sigma}) \sim (m^{'\Sigma}, s_2)$ and hence, by Lemma 3 and Strong Support Lemma [26], we obtain $s_1'm^{'\Sigma} \sim s_2'm^{'\Sigma}$ which implies $s_1 \sim s_2$ by Laird’s conditions. Suppose now $s_1 = s_1'm^{'\Sigma}, s_2 = s_2, i = 1, 2$ with $(p(m)) \in P(AC) \setminus OO$ and the last move in $s_1$ being the last move in $s_1'm$ having polarity in $p(AC)$. By IH, $s_1 \sim s_2$. Then, by consecutive applications of Lemma 5, we obtain $s_1 \sim s_2$.

Proposition 7. If $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$ then $\sigma; \tau : A \rightarrow C$.

Proof. We show that $\sigma; \tau$ is a strategy. Even-prefix closure and equivariance are clear. Moreover, since each $s \in \sigma||\tau$ has even-length projections in $AB$ and $BC$, we can show that its projection in $AC$ is even-length too. For O-extension, if $s \in \sigma \land \tau \land t \subseteq O \subseteq s$ with $s = u \vdash \gamma AC$ and $u \in \sigma||\tau$, then we can construct $v \vdash \gamma AC$ such that $t = v \vdash \gamma AC$ and $v \subseteq O$, where $O$ is defined for intersection sequences in an analogous way as for plays (with condition $p(m) = O$ replaced by $p(m) \in OO$, and $p(m) = P$ by $p(m) \in PO \cup OP$). Moreover, $v \vdash \gamma AB \subseteq O \subseteq v \vdash \gamma AC$, and $v \vdash \gamma BC \subseteq O \subseteq v \vdash \gamma BC$, so $t \subseteq v \vdash \gamma AC$. Finally, for determinacy, let $sm^{'\Sigma}, sn^{'\Sigma} \in \sigma||\tau$ be due to $s_1s_1'm^{'\Sigma}, s_2s_2'n^{'\Sigma} \in \sigma||\tau$ respectively, where $s_1, s_2$ both end in the last move of $s$. By Lemma 6, we have $s_1 \sim s_2$ and thus, by consecutive applications of Lemma 5, we get $s_1's_1'm^{'\Sigma} \sim s_2's_2'n^{'\Sigma}$, so $sm^{'\Sigma} \sim sn^{'\Sigma}$.

The above result shows that strategies are closed under composition. We can prove that composition is associative and, consequently, obtain a category of games.

Proposition 8. For all $\rho : A \rightarrow B, \sigma : B \rightarrow C$ and $\tau : C \rightarrow D$, $(\rho; \sigma; \tau) = \rho(\sigma; \tau)$.

Definition 9. Given a class table $\Delta$, we define the category $G_D$ having arenas as objects and strategies as morphisms. Identity morphisms are given by $\text{id}_A$, for each arena $A$.

Note that neutrality of identity strategies easily follows from the definitions and, hence, $G_D$ is well defined. In the sequel, when $\Delta$ can be inferred from the context, we shall write $G_D$ simply as $G$. As a final note, for class tables $\Delta \subseteq \Delta'$, we can define a functor $\Delta/\Delta' : G_D \rightarrow G_{\Delta'}$ which acts as the identity map on arenas, and sends each $s : A \rightarrow B$ of $G_D$ to:
$$\Delta/\Delta')(\sigma) = \{ s \in P\Delta' | \exists t \in \sigma, s \subseteq O \}$$

where $P\Delta'$ refers to plays in $G_{\Delta'}$. In the other direction, we can define a strategy transformation:
$$\Delta'(\Delta/\Delta')(\sigma) = \sigma \cap P\Delta_B$$

which satisfies $\Delta'(\Delta/\Delta')(\sigma) = \sigma$.

4. Soundness

Here we introduce constructions that will allow us to build a model of IMJ. We begin by defining a special class of strategies. A strategy $\sigma : A \rightarrow B$ is called evaluated if there is a function $f_\sigma : MA \rightarrow MB$ such that:
$$\sigma = \{ m_A^\Sigma m_B^\Sigma \in PAB | m_B = f_\sigma(m_A) \}$$

Note that equivariance of $\sigma$ implies that, for all $m_A \in MA$ and permutations $\pi$, it holds that $\pi \cdot f_\sigma(m_A) = f_\sigma(\pi \cdot m_A)$. Thus, in particular, $f_\sigma(m_A) \subseteq f_\sigma(m_A)$.

Recall that, for arenas $A$ and $B$, we can construct a product arena $A \times B$. We can also define projection strategies:

$$\pi_1 : A \times B \rightarrow A = \{ (m_A, m_B) \subseteq \Sigma m_A^\Sigma m_B^\Sigma \in P(AXB)A \}$$

and, analogously, $\pi_2 : A \times B \rightarrow B$. Note that the projections are evaluated. Moreover, for each object $A$,
$$!_A = \{ m_A^\Sigma \subseteq \Sigma m_A^\Sigma \in P\Sigma A \}$$

is the unique evaluated strategy of type $A \rightarrow 1$.

Given strategies $\sigma : A \rightarrow B$ and $\tau : A \rightarrow C$, with $\tau$ evaluated, we define:
$$\langle \sigma, \tau \rangle : A \rightarrow B \times C = \{ m_A^\Sigma m_B^\Sigma m_C^\Sigma \in P(AXB)A \}$$

where we write $s[m^\Sigma/m_B]$ for the sequence obtained from $s$ by replacing any occurrences of $m_B$ in it by $m'$ (note that there can be at most one occurrence of $m_B$ in $s$).

The above structure yields products for evaluated strategies.

Lemma 9. Evaluated strategies form a wide subcategory of $G$ which has final products, given by the above constructions.

Moreover, for all $\sigma : A \rightarrow B$ and $\tau : A \rightarrow C$ with $\tau$ evaluated, $\langle \sigma, \tau \rangle, \pi_1 : \sigma \rightarrow \pi_1(\sigma, \tau)$ and $\pi_2 : \sigma \rightarrow \pi_2(\sigma, \tau)$.

Using the above result, we can extend pairings to general $\sigma : A \rightarrow B$ and $\tau : A \rightarrow C$ by:
$$\langle \sigma, \tau \rangle = A^{(\sigma, \text{id}_A)} B \times A^{(\tau, \text{id}_A)} C \times B \rightarrow B \times C$$

where $\text{id}$ is the isomorphism $\langle \pi_2, \pi_1 \rangle$. The above represents a notion of left-pairing of $\sigma$ and $\tau$, where the effects of $\sigma$ precede those of $\tau$. We can also define a left-tensor between strategies:
$$\sigma \times \tau = A \times B \langle \sigma_1, \tau_1 \rangle A' \times B \langle \sigma_2, \tau_2 \rangle A' \times B'$$

for any $\sigma : A \rightarrow A'$ and $\tau : B \rightarrow B'$.

Lemma 10. Let $\tau' : A' \rightarrow A, \sigma : A \rightarrow B_1, \tau : A \rightarrow B_2, \sigma_1 : B_1 \times B_2 \rightarrow C_1$ and $\sigma_2 : B_2 \rightarrow C_2$, with $\tau$ and $\tau'$ evaluated. Then $\tau' : \langle \sigma, \tau \rangle, \langle \sigma, \tau_2 \rangle, \langle \sigma_1, \tau_1 \rangle, \langle \sigma_2, \tau_2 \rangle$

Proof. The result follows from the simpler statements:
$$\tau : \langle \text{id}, \text{id} \rangle, \langle \sigma, \text{id} \rangle, \langle \sigma, \pi_2 \rangle, \langle \sigma', \pi_2 \rangle, \langle \sigma', \text{id} \rangle$$

for all appropriately typed $\sigma, \sigma', \tau$, with $\tau$ evaluated, and Lemma 9.

An immediate consequence of the above is:
$$A^{(\sigma, \tau)} B \times B_2^{\sigma_1 \times \sigma_2} C_1 \times C_2 \rightarrow A^{(\sigma, \sigma_1, \tau_2)} C_1 \times C_2$$

More generally, Lemma 10 provides us with naturality conditions similar to those present in Freyd categories [25] or, equivalently, categories with monadic products [22].

We also introduce the following weak notion of coproduct.

Given strategies $\sigma, \tau : A \rightarrow B$, we define:
$$\{ [\sigma, \tau] : Z \times A \rightarrow B = \{ (1, m_A^\Sigma s) | m_A^\Sigma s \in \sigma \} \cup \{ (0, m_B^\Sigma s) | m_B^\Sigma s \in \tau \}$$
Setting $i : 1 \to \mathbb{Z} = \{ *, i \}$, for each $i \in \mathbb{Z}$, we can show the following.

**Lemma 11.** For all strategies $\sigma' : A' \to A$ and $\sigma, \tau : A \to B$,
- if $\langle \xi, \mu \rangle \vdash [\sigma, \tau] = \tau$ and $\langle \xi, \mu \rangle \vdash [\sigma, \tau] = \sigma$;
- if $\sigma'$ is evaluated then $\langle \xi, \mu \rangle \vdash [\sigma, \tau] = \tau$.

Method definitions in IMJ induce a form of exponentiation:
\[
\Delta_n^{\mu}(\Delta_\eta \vdash [\xi, \mu] : \eta) \vdash \Delta^{\mu}(\eta) \vdash \Delta^{\mu}(\Delta_n \vdash [\xi, \mu] : \eta) \land \Delta_n \vdash [\xi, \mu] : \eta
\]
the modelling of which requires some extra semantic machinery. Traditionally, in call-by-value game models, exponentiation leads to 'effectless' strategies, corresponding to higher-order value terms. In our case, higher-order values are methods, manifesting themselves via the objects they may inherit. Hence, exponentiation necessarily passes through generation of fresh object names containing these values. These considerations give rise to two classes of strategies introduced below.

We say that an even-length play $s \in P_{AB}$ is total if it is either empty or $s = m_1 \Sigma m_2 \ldots s'$ and:
- $T \in St0$ and $\nu(m_B) \cup \nu(\Sigma) \subseteq \nu(m_A)$,
- if $s = s''m_1 \Sigma m_2 t''$ and $a \in dom(\Sigma'') \cup \nu(\gamma(\Sigma'' \cup \nu(s''h)))$, for $\Sigma_0 \in St0$ such that $\gamma(\Sigma'' \cup \nu(s''h)) \in P_{AB}$, then $\phi \notin \nu(n)$ and $T' \vdash a \Rightarrow \Sigma(a) \equiv \Sigma'(a)$.

We write $P_{\Sigma, AB}$ for the set of total plays in $AB$. Thus, in total plays, the initial move $m_0$ is immediately followed by a move $m_1$, and the initial store $\Sigma$ is invisible to $P$ in the sense that $P$ cannot use its names nor their values. A strategy $\phi : A \to B$ is called single-threaded if it consists of total plays and satisfies the conditions:

- for all $m_1 \Sigma m_2 \ldots s' \in P_{AB}$ there is $m_1 \Sigma m_2 \ldots s' \in \phi$;
- if $m_1 \Sigma m_2 \ldots s' \in \phi$ then $\gamma(\Sigma'' \cup \nu(s''h)) \in \phi$, for $\Sigma_0 \in St0$;
- if $m_1 \Sigma m_2 \ldots s' \in \phi$, $a \in \nu(T)$ then $s = e$. Thus, single-threaded strategies reply to every initial move $m_1 \Sigma m_2 \ldots s'$ with a move $m_1 \Sigma m_2 \ldots s''$, which depends only on $m_1 \Sigma m_2 \ldots s'$ (i.e. $P$ does not read before playing). Moreover, $m_1 \Sigma m_2 \ldots s''$ does not change the values of $\Sigma$ ($P$ does not write) and may introduce some fresh objects, albeit with default values. Finally, plays of single-threaded strategies consist of just one thread, where a thread is a total play in which there can be at most one call to names introduced by its second move.

Conversely, given a total play starting with $m_1 \Sigma m_2 \ldots s'$, we can extract its threads by tracing back for each move in $s$ the method call of the object $a \in \nu(T)$ (it is related to $T$). Formally, for each total play $s = m_1 \Sigma m_2 \ldots s'$ with $|s'| > 0$, the thread move of $s$, written $\text{thr}(s)$, is given by induction:

- if $\text{thr}(s') = \text{thr}(s)$, then $\text{thr}(s') = \text{thr}(s')$;
- if $\text{thr}(s') = \text{ THR } a \in m(\Sigma')$ then $\text{thr}(s') = \text{ THR } a$.

where the restriction retains only those moves $n'' \Sigma s'$ such that $\text{thr}(n'' \Sigma s') = m'' \Sigma s'$. We extend this to the case of $|s| \leq 2$ by setting $\nu(s) = s$. Finally, we call a total play $s \in P_{AB}$ thread-independent if for all $s' \Sigma m_2 \Sigma s''$ with $|s''| > 2$:
- if $\gamma(\Sigma'' \cup \nu(s''h)) \land \nu(\gamma(\Sigma'' \cup \nu(s''h))) \text{ then } \Sigma' = T'(a)$;
- if $s''$ ends in some $n'' \Sigma s''$ and $a \in \nu(\gamma(\Sigma'' \cup \nu(s''h)))$ then $\Sigma''(a) = T''(a)$.

We write $P_{\Sigma, AB}$ for the set of thread-independent plays in $AB$. We can now define strategies which occur as interleavings of single-threaded ones. Let $\phi : A \to B$ be a single-threaded strategy. We define:
\[
\phi^1 = \{ s \in P_{\Sigma, AB} \mid \forall s' \Sigma m_2 \Sigma s'' \in \phi \}.
\]

**Lemma 12.** $\phi^1$ is a strategy, for each single-threaded $\phi$.

Proof. Equivariance, Even-prefix closure and O-extension follow from the corresponding conditions on $\phi$. For determinacy, if $s m_2 \Sigma s'' \in \phi \cup \nu(s) > 0$ then, using determinacy of $\phi$ and the fact that $P$-moves do not change the current thread, nor do they modify or use names from other threads, we can show that $s m_2 \Sigma \nu(s'')$.

We say that a strategy $\sigma$ is thread-independent if $\sigma = \tau^1$ for some single-threaded strategy $\tau$. Thus, thread-independent strategies do not depend on initial stores and behave in each of their threads in an independent manner. Note in particular that evaluated strategies are thread-independent (and single-threaded).

**Lemma 13.** Let $\sigma : A \to B$ and $\tau : A \to C$ be strategies with $\tau$ thread-independent. Then, $(\sigma ; \tau) ; \tau_1 = \sigma$ and:
\[
\langle \sigma, \tau \rangle = A \times (\sigma ; \tau) \times C \times B \times C.
\]

Proof. The former claim is straightforward. For the latter, we observe that the initial effects of $\sigma$ and $\tau$ commute: on initial move $m_1 \Sigma \tau_0$ $\tau$ does not read the store updates that $\sigma$ includes in its response $m_1 \Sigma \tau_0$, while $\sigma$ cannot access the names created by $\tau$ in its second move $m_1 \Sigma \tau_0$.

It is worth noting that the above lemma does not suffice for obtaining categorical products. Allowing thread-independent strategies to create fresh names in their second move breaks universality of pairings. Considering, for example, the strategy:
\[
\sigma : 1 \to I \times I = \{ * ; (a, b) \Sigma \}$
\]
we can see that $\sigma \notin \langle (\sigma ; \sigma_1) \Sigma \sigma_2 \rangle$ for the right-hand-side contains plays of the form $s (a, b)^T$ with $a \neq b$. We can now define an appropriate notion of exponential for our games. Let us assume a translation assigning an arena $\bar{\theta}$ to each type sequence $\theta$. Moreover, let $I$ be an interface such that $\Delta(I) \vdash \text{ Meths} = \{ m_0 : \theta_0 \to \theta_1, \ldots, m_n : \theta_n \to \theta_n \}$ where $\theta_i \equiv \theta_i, \ldots, \theta_m$, for each $i$. For any arena $A$, given single-threaded strategies $\phi_1, \ldots, \phi_n : A \to I$ such that, for each $i$, if $m_1 \Sigma \phi_i \Sigma s \in \phi_i$ then $a \notin \nu(\sigma) \land T(a) ; I \bigl( \text{ call.a.m(\bar{\theta}) } \bigr) \Rightarrow m = m_a$.

We can group them into one single-threaded strategy:
\[
\langle \phi_1, \ldots, \phi_n \rangle : A \to I = \bigcup_{i=1}^n \phi_i.
\]
Note that the $a$ above is fresh for each $m_1 \Sigma \phi_i$ (i.e. $a \notin \nu(m_1 \Sigma \phi_i)$).
Let now $\sigma_1, \ldots, \sigma_n$ be strategies with $\sigma_i : A \times [\vec{b}_i] \to \{0,1\}$. For each $i$, we define the single-threaded strategy $\Lambda(\sigma_i) : A \to I$: 

$$
\Lambda(\sigma_i) = \{(a, m, \vec{b}_i, \vec{s}) \in P_{\Delta_1}| \gamma((a, m, \vec{b}_i, \vec{s})) \in \sigma_i\} 
\cup \{(a, m, \vec{b}_i, \vec{s}) \in P_{\Delta_1}| \gamma((a, m, \vec{b}_i, \vec{s})) \in \sigma_i\} 
\cup \{(a, m, \vec{b}_i, \vec{s}) \in P_{\Delta_1}| \gamma((a, m, \vec{b}_i, \vec{s})) \in \sigma_i\}
$$

where $a \notin \nu(\Sigma, \vec{b}_i, \vec{s}, \vec{\Sigma}'', T')$ and $T(a) : I$. By definition, $\Lambda(\sigma_i)$ is single-threaded. Therefore, setting 

$$
\Lambda(\sigma_1, \ldots, \sigma_n) = (\Lambda(\sigma_1), \ldots, \Lambda(\sigma_n))^T : A \to I,
$$

we obtain a thread-independent strategy implementing a simultaneous currying of $\sigma_1, \ldots, \sigma_n$. In particular, given translations $[M_i]$ for each method in a method-set implementation $M$, we can construct: 

$$
[M] : [I]\to I = \Lambda([M_1], \ldots, [M_n]).
$$

Finally, we define evaluation strategies $ev_{m_i} : I \times [\vec{b}_i] \to [\theta_i]$ by (taking remaining prefixes of): 

$$
ev_{m_i} = \{(a, v)^\Sigma \text{call a.m.(v)}^{\mathsf{ret}\ a.m.(v)^T}, v^T \in P_{\Delta_1} | \sigma_i(a) \leq I\}
$$

where $\Delta_1 = (I \times [\vec{b}_i])\{[\theta_i]\}$. We can now show the following natural mapping from groups of strategies in $A \times [\vec{b}_i] \to [\theta_i]$ to thread-independent ones in $A \to I$.

**Lemma 14.** Let $\sigma_1, \ldots, \sigma_n$ be as above, and let $\tau : A' \to A$ be evaluated. Then,

- $\Lambda(\sigma_1, \ldots, \sigma_n) \times \mathsf{id} : ev_{m_i} = \sigma_i$,
- $\tau; \Lambda(\sigma_1, \ldots, \sigma_n) = \Lambda((\tau \times \mathsf{id}); \sigma_1, \ldots, (\tau \times \mathsf{id}; \sigma_n)$.

Apart from dealing with exponentials, in order to complete our strategy $\tau \in I \times \mathsf{I}$, we are faced with the necessity of invoking a method $m$ on input $\vec{v}$, the Opponent would have to call $\text{b.m.(v)}$. The remaining challenge is to merge the two occurrences of $I$ in (1). We achieve this as follows. Let us assume a well-formed extension $\Delta'$ of $\Delta$:

$$
\Delta' = (I' : (f' : I'), I),
$$

that is, $I'$ contains a single field $f'$ of type $I$. We next define the strategy $\kappa_{\Delta'} : 1 \to I' \times I$ of $G_{\Delta'}$:

$$
\kappa_{\Delta'} = \{i' \to [M_i]_{\Delta'}, \sigma_i \}
$$

where $m \in \text{dom}(\Delta(I'))$, $h = \Sigma(a)\overrightarrow{f}$, and $\Sigma_0 \in S\Delta_0$ is such that $\Sigma_0(a) = I$ and $\Sigma_0(a) : I'$. We let $\text{new}(x : I; M)$ be the strategy:

$$
\text{new}(x : I; M) = \{i' \to [M_i]_{\Delta'}, \sigma_i \}
$$

and $\text{asnr}_i$ is the assignment strategy:

$$
\text{asnr}_i : I \times [0] \to \{1 \in (a, v)^\Sigma \cdot \Sigma(a, f \to v) \in P_{\langle I \times [0] \rangle_{\Delta'}} \}
$$

Note that $x$ may appear free in $M_i$; it stands for the keyword this of Java.

3) Here we omit wrapping $[M]$ inside $\Delta'/\Delta$, as well as wrapping the whole $[\text{new}(x : I; M)]$ in $\Delta'/\Delta$, for conciseness.

for each field $f$. Thus, object creation involves creating a pair of names $(a', a)$ with $a : I$ and $a' : I'$, where $a$ is the name of the object we want to return. The name $a'$ serves as a store where the handle of the method implementations, that is, the name created by the second move of $[M_i]$, will be passed. The strategy $\kappa\Delta_i$, upon receiving a request call $\text{a.m.(v)}$, simply forwards it to the respective method of $a', f$ and, once it receives a return value, copies it back as the return value of the original call.

Let $\#(I) : I \to \#(I) = \{a^2 \cdot a^2 | a, s \text{ distinct}\}$, for each sequence of interfaces $I$. The latter has a right inverse $\#(I)^{-1} : \#(I) \to I$ with the same plays. We can now define the semantic translation of terms.

**Definition 10.** The semantic translation is given as follows.

- Contexts $\Gamma = \{x_1 : \theta_1, \ldots, x_n : \theta_n\}$ are translated into arenas by 
  $$
  [\Gamma] = \{[\theta_1] \times \cdots \times [\theta_n] \times \#(I_1, \ldots, I_m)\},
  $$
  where $\text{void} = 1$, $\text{int} = \mathbb{Z}$ and $\mathbb{Z} = I$.
- Terms are translated as in Figure 4 (top part).

In order to prove that the semantics is sound, we will also need to interpret terms inside state contexts. Let $\Gamma \vdash M : \theta$, with $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ contains only variables and $\text{dom}(\Gamma_2) = \text{dom}(S)$. A term-in-state-context $(S, M)$ is translated into the strategy:

$$
[(S, M)] = \{[\Gamma_1] | [\Gamma_1] \times \mathcal{T} \vdash \mathsf{id}(\#(I), [\Gamma]) \vdash [M] \theta\}.
$$

The semantic translation of states (Figure 4, lower part), comprises two stages:

$$
[S] = \{[\Gamma_1] | [\Gamma_1] \times \mathcal{T} \vdash \mathsf{id}(\#(I), [\Gamma]) \vdash [M] \theta\}.
$$

The first stage, $[S]_1$, creates the objects in $\text{dom}(S)$ and implements their methods. The second stage of the translation, $[S]_2$, initialises the fields of the newly created objects.

In the rest of this section we show soundness of the semantics. Let us call $\text{NEW}, \text{FIELDUP}, \text{FIELDAC}$ and $\text{METHODCL}$ respectively the transition rules in Figure 2 which involve state. Given a rule $r$, we write $(S, M) \longrightarrow (S', M')$ if the transition $(S, M) \longrightarrow (S', M')$ involves applying $r$ and context rules.

**Proposition 15 (Correctness).** Let $(S, M)$ be a term-in-state-context and suppose $(S, M) \longrightarrow (S', M')$.

1. If the transition $r$ is not stateful then $[M] = [M']$.
2. If $r$ is one of $\text{FIELDAC}$ or $\text{FIELDUP}$ then $[S'] \vdash (\mathsf{id} \times \#(I)) ; [M] = [S'] \vdash (\mathsf{id} \times \#(I)) ; [M']$.
3. If $r$ is one of $\text{METHODCL}$ or $\text{NEW}$ then $[(S, M)] = [(S', M')]$.

**Proof.** Claim 1 is proved by using the naturality results of this section. For the let construct, we show by induction on $M$ that $[M'[x]] = \{i \to [\vec{v}]\}; [M']$. For we use the following properties of field assignment and access:

$$
\{a_{\vec{v}}, \pi_1\} : \mathsf{drf}_r = \{a_{\vec{v}}, \pi_2\} : I \times [\theta] \to [\theta] \{a_{\vec{v}}, \pi_1 \times \mathsf{id}; \pi_2 : I \times [\theta] \times [\theta] \to [\theta] \}
$$

which are easily verifiable (the former one states that assigning a field value and accessing it returns the same value; the latter that two assignments in a row have the same effect as just the last one). The final claim follows by showing that the diagrams below...
- \( \Gamma \vdash x_i : \theta_i = [\Gamma] \stackrel{\pi_i}{\rightarrow} [\theta_i] \);
- \( \Gamma \vdash \text{skip} : \text{void} = [\Gamma] \leftarrow 1 \);
- \( \Gamma \vdash i : \text{int} = [\Gamma] \stackrel{1}{\rightarrow} \mathbb{Z} \);
- \( \Gamma \vdash \text{let } x = M' \in M : \theta = [\Gamma] \stackrel{(\lambda \theta \cdot M')}{\rightarrow} [\Gamma] \times [\theta] \rightarrow [M_1, \cdots, [\theta]] \);
- \( \Gamma \vdash (T)M : I = [\Gamma] \stackrel{\text{stęp}_T}{\rightarrow} I \), where \( \text{stęp}_T : T' \rightarrow I = \{ \text{null} \} \cup \{ a^{\Delta} a^\gamma \in P_{T'XZ} | \gamma(a) \leq I \} \);
- \( \Gamma \vdash M \oplus M' : \text{int} = [\Gamma] \stackrel{(\text{stęp}_T, \text{stęp}_T)}{\rightarrow} I \times I \rightarrow \mathbb{Z} = \{ (i, j) \in (i, j) \}; \)
- \( \Gamma \vdash M = M' : \text{int} = [\Gamma] \stackrel{(\text{stęp}_T, \text{stęp}_T)}{\rightarrow} I \times I \rightarrow \mathbb{Z} \), where eq = \( \{ (a, b)^{\Delta} 1^2 \in P_{T'XZ} \} \cup \{ (a, b)^{\Delta} 0^2 \in P_{(T'XZ)} \mid a \neq b \} \);
- \( \Gamma \vdash \text{if } M \text{ then } M' : \theta = [\Gamma] \stackrel{(\text{stęp}_T, \text{stęp}_T)}{\rightarrow} Z \times \Gamma \rightarrow [\Gamma][M_1, \cdots, [\theta]] \rightarrow [\theta] \);
Var $f$ is an interface with a sole field $f$ : int. Observe that each $s \in [\Gamma + N]$ induces some $s' \in [\Gamma_a + N_a]$ such that $a$ appears in $s'$ only in stores (and in a single place in the initial move) and $O$ never changes the value of $a.f$, while $P$ never decreases the value of $a.f$. We write $[\Gamma_a + N_a]$ for the subset of $[\Gamma_a + N_a]$ containing precisely these play. Then, take $M_0$ to be the term let $s = \text{new}(x : \text{Var}_f)$ in $(M_0[x/a];x.f)$, where $x$ a fresh variable. Because $\ast \in \{M_0\}$, we get $s \in [M_0]$ for some $j \in \mathbb{Z}$. Consider now the infinite reduction sequence of $(\theta, M)$. It must have infinitely many METHOD steps, so suppose $(\theta, M) \rightarrow \rightarrow (S, M') \rightarrow \rightarrow (j+1)$ such steps. Then, we obtain $(\theta, M) \rightarrow \rightarrow (S_a, (M')_a; a.f)$, with $S_a(a.f) = j + 1$. By Correctness, we have that $s \in [S_a]$. Let $\Gamma \rightarrow \rightarrow (\Gamma')_a; a.f$. Since in $[(M')_a]_x$, the value of $a$ cannot decrease, and its initial value is $j + 1$ as stipulated by $S_a$, we reach a contradiction. □

5. Full Abstraction

Recall that, given plays $s, s'$, we call $s$ an O-extension of $s'$ (written $s \triangleleft O s'$) if $s, s'$ are identical except the type information regarding O-names present in stores: the types of O-names in $s$ may be subtypes of those in $s'$. We shall write $s \not\triangleleft O s'$ for the dual notion involving P-names, i.e., $s \not\triangleleft O s'$ if $s, s'$ are the same, but the types of P-names in $s'$ may be subtypes of those in $s$. Then, given $X \in \{O, P\}$ and fixed $A, B$, we define $\text{cl}_X(s) = \{s' \in P_{A,B} | s \triangleleft X s' \land \text{cl}_X(s') \}$ for $P_{A,B}[\leq]$. A play will be called complete if it is of the form $m_1 Y m_2 Y$.

Next we establish a definability result stating that any complete play (together with other plays implied by O-closure) originates from a term.

Lemma 18 (Definability). Let $s \in P_{\Delta|\Gamma| \triangleright \theta}$ be a complete play. There exists $\Delta' \supseteq \Delta$ and $\Delta'|\Gamma| \triangleright M : \theta$ such that $[(\Delta'|\Gamma| \triangleright M) : \theta] = \text{cl}_O(s)$.

Proof. The argument proceeds by induction on $|s|$. For $s = \epsilon$, any divergent term suffices. For example, one can take $\Delta' = \Delta \oplus (\text{Div} \mapsto (m : \text{void} \mapsto \text{void})), and pre-compose any term of type $\theta$ with new($x : \text{Div} : m : \Lambda(); m()$).

Suppose $s \neq \epsilon$. Then the second move can be a question or an answer. We first show how to reduce the former case to the latter, so that only the latter needs to be attacked directly.

Suppose
\[ s = q_s \cdot \text{call} \cdot \text{m}(\overline{u}) \cdot \Sigma_1 \cdot s_1 \cdot \text{ret} \cdot \text{m}(\overline{v}) \cdot \Sigma_2 \cdot s_2 \cdot \Sigma_3 \cdot s_3, \]
where \( o : \Gamma' \) and $\Delta'(\overline{m}) : \Gamma \rightarrow \mathcal{R}$. Consider $\Delta' = \Delta \oplus (\overline{\text{I}}' \mapsto (\overline{t} : \overline{I}, \overline{m} : \overline{\text{I}} : \theta))$ and the following play from $P_{\Delta'|\Gamma| \triangleright \theta}$:
\[ s' = q_{s'} \cdot p \cdot s'_1 \cdot \text{call} \cdot \text{p}(\overline{m}(\overline{v})) \cdot \Sigma_{s'} \cdot s'_2 \cdot \text{ret} \cdot \text{p}(\overline{m}(\overline{v})) \cdot \Sigma_{s'} \cdot s'_3, \]
where $p \not\in \overline{v}(s)$, $\Sigma' = \Sigma_1 \oplus \Sigma_2$, $\Sigma = \{ p \rightarrow (\overline{t'}, \overline{f'} \mapsto \overline{u}) \}$ and $s'_2$ is the same as $s_2$ except that each store is extended by $\Sigma$. If $\Delta'|\Gamma| \triangleright M : \Gamma'$ satisfies the Lemma for $s'$ then, for $s$, one can take let $x_p = M'_p$ in $x_p.m(y.m(x_p.f'))$, where $y$ refers to $o$, i.e., $y$ is of the shape $x.f'$, where $x \in \text{dom} \Gamma$ and $f'$ is a sequence of fields that points to $o$ in $\Sigma_2$.

Thanks to the reduction given above we can now assume that $s \in P_{\Delta|\Gamma| \triangleright \theta}$ is non-empty and
\[ s = q_s \cdot \Sigma_0 \cdot m_0 \cdot \Sigma_1 \cdots \Sigma_{2k} \cdot m_k, \]
where $m_0$ is an answer. We are going to enrich $s$ in two ways so that it is easier to decompose. Ultimately, the decomposition of $s$ will be based on the observation that the $m_i^2 \cdots m_{2k}^2$ segment can be viewed as an interleaving of threads, each of which is started by a move of the form call $p$ for some P-name $p$. A thread consists of the starting move and is generated according to the following two rules: $m_0$ belongs to the thread of $m_{2i+1}$ and every answer-move belongs to the same thread as the corresponding question-move.

- The first transformation of $s$ brings forward the point of P-name creation to the second move. In this way, threads will never create objects and, consequently, it will be possible to compose them without facing the problem of object fusion.

Suppose $P(s) = \overline{p}_i$ and $p_i : I_{p_i}$. Let $\Delta' = \Delta \oplus (I_p \mapsto f_i : I_{p_i})$. Consider $s' = (\overline{n}, q_{s'}) m_0 \cdots m_{2k}^2$, where $\Sigma_{s'} = \Sigma_n \oplus \{ n \rightarrow (I_{p_i}, \overline{p}_i) \}$ and $\Sigma_{s'_i} = \Sigma_n \oplus \{ n \rightarrow (I_{p_i}, \overline{p}_i) | (I_{p_i}, \overline{p}_i) \in P(s) | \Sigma_n \} \cup \{ (I_{p_i}, \text{null}) \}$. Let $\Gamma' = \{ x_n : I_{p_i} \} \rightarrow \theta$. Observe that $s' \in P_{\Delta'|\Gamma| \triangleright \theta}$.

- The second transformation consists in storing the unfolding play in a global variable. It should be clear that the recursive structure of types along with the ability to store names is sufficient to store plays in objects. Let $I_{p_{play}}$ be a signature that makes this possible. This will be used to enforce the intended interleaving of threads after their composition (in the style of Innocent Factorization [5]). Let $\Delta'' = \Delta \oplus (\text{History} \mapsto \text{play} : I_{p_{play}})$ and $\Gamma'' = \{ x_n : \text{History} \} \rightarrow \theta$. Consider

Now we shall decompose $m_1^2 \cdots m_{2k}^2$ into threads. Recall that each of them is a subsequence of $s''$ of the form
\[ \text{call} \cdot \text{p}(\overline{m}(\overline{v})) \cdot t \cdot \text{ret} \cdot \text{p}(\overline{m}(\overline{v})) \cdot \Sigma_t, \]
where the segment $t$ contains moves of the form call $o$ or ret $o$ for some $o \in \overline{O}(s)$. We would now like to invoke the IH for each thread but, since a thread is not a play, we do so for the closely related play $(h, n, q, \overline{u}) \cdot \Sigma_t \mapsto \Sigma_t \cdot \Sigma_t^{2k}$. Let us call the resultant term $M_{p.m.\overline{u}.\Sigma_t}$. Next we combine terms related to the same $p : I_p$ into an object definition by
\[ M_p \equiv \text{new}(x : I_p ; m : \lambda \overline{u} . \text{case} (\overline{u}, \Sigma_c) [M_{p.m.\overline{u}.\Sigma_c}]). \]

The case statement, which can be implemented in IMJ using nested ifs, is needed to recognize instances of $\overline{u}$ and $\Sigma_c$ that really occur in threads related to $p$. In such cases the corresponding term $M_{p.m.\overline{u}.\Sigma_c}$ will be run. Otherwise, the statement leads to divergence.

The term $M$ for $s$ can now be obtained by taking
\begin{align*}
\text{let } x_n &= \text{new}(x : I_p) ; \text{in} \\
\text{let } x_h &= \text{new}(x : \text{History}) ; \text{in} \\
\text{let } x_{p_i} &= M_{p_i} ; \text{in} \\
\text{assert}(q_{s''} ; \overline{x}_{n.f_i} = x_{p_i}) ; \text{make}(\Sigma_0'' ; ) ; \text{play}(m_0)
\end{align*}
where $x_{p_i} = M_{p_i}$ represents a series of bindings (one for each P-name $p_i \in P(s)$), assert$(h, n, q_{s''}^p)$ is a conditional that converges if and only if the initial values of free $\overline{I}$ identifiers as well as values accessible through them are consistent with $q$ and $\Sigma_c$ respectively, make$(\Sigma_0'')$ is a sequence of assignments that set values to those specified in $\Sigma_0''$ (up-casts need to be performed to ensure typability) and play$(m_0)$ is skip, i, null or, if $m_0$ is a name,
Proof. The Lemma holds because, on the one hand, it relies on contexts of a specific shape and, on the other hand, any closing context \( C[-] \) for \( M_i \) can be presented in the above form with test \( \equiv C[z, \lambda x_1, \ldots, x_k] \).

Given a term \( \Delta \vdash M : \theta \), let us write \( \langle \Delta \mid \vdash M : \theta \rangle_{\text{comp}} \) for the set of complete plays from \( \langle \Delta \mid \vdash M : \theta \rangle \). In what follows, we shall often omit \( \Delta \), \( \vdash \), and \( \theta \) for brevity.

**Theorem 20** (Inequational full abstraction). Given \( \Delta \vdash M : \theta \) (\( i = 1, 2 \)), we have \( \langle \Delta \mid \vdash M_i : \theta \rangle_{\text{comp}} \) is equal to if and only if
\[
\llbracket \Delta \mid \vdash M_1 : \theta \rrbracket_{\text{comp}} \subseteq \llbracket \Delta \mid \vdash M_2 : \theta \rrbracket_{\text{comp}}.
\]

Proof. The proof uses the following play transformation. Given \( t = q^{\ast}s_1a^{\ast}s_2 \in P_{\Delta \vdash n} \), we define \( t = P_{\Delta \vdash \text{Wrap}_P X} \) as
\[
n^{\Sigma_n} \text{call} \; n.f(q)^{\Sigma_n} \text{ret} \; n.f(a)^{\Sigma_n} \text{store} \; n.f(a)^{\Sigma_n} \text{store} \; n.f(a)^{\Sigma_n},
\]
where \( \text{Wrap}_P X \) are the same as in the above Lemma. \( \Sigma_n = (n \mapsto \text{Wrap}_P Y) \).

**Example 3.** Let us revisit Example 2. We have \( \llbracket \sigma_1 \rrbracket = \sigma_2 \cup \{ \ast, n^{\ast}(\text{Empty}, \theta) \} \), i.e. \( \llbracket \sigma_1 \rrbracket \subseteq \llbracket \sigma_2 \rrbracket \). Thus, it follows from Theorem 20 that \( \Delta \vdash \text{M}_1 \cong \text{M}_2 \) and \( \Delta \vdash \theta \cong \text{M}_1 \cong \text{M}_2 \).

**Theorem 21** (Equational full abstraction). Given \( \Delta \vdash M_i : \theta \) (\( i = 1, 2 \)), \( \Delta \vdash \text{M}_1 \cong \text{M}_2 \) if and only if
\[
\llbracket \Delta \mid \vdash \text{M}_1 : \theta \rrbracket_{\text{comp}} = \llbracket \Delta \mid \vdash \text{M}_2 : \theta \rrbracket_{\text{comp}}.
\]

Proof. The preceding result implies that \( \text{M}_1 \cong \text{M}_2 \) if and only if
\[
\llbracket \Delta \mid \vdash \text{M}_1 \rrbracket_{\text{comp}} = \llbracket \Delta \mid \vdash \text{M}_2 \rrbracket_{\text{comp}}.
\]

**References**


