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Game Semantics for Interface Middleweight Java

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Abstract
We consider an object calculus in which open terms interact with the environment through interfaces. The calculus is intended to capture the essence of contextual interactions of Middleweight Java code. Using game semantics, we provide fully abstract models for the induced notions of contextual approximation and equivalence. These are the first denotational models of this kind.

Categories and Subject Descriptors D.3.1 [Formal Definitions and Theory]: Semantics; F.3.2 [Semantics of Programming Languages]: Denotational semantics

Keywords Full Abstraction, Game Semantics, Contextual Equivalence, Java

1. Introduction
Denotational semantics is charged with the construction of mathematical universes (denotations) that capture program behaviour. It concentrates on compositional, syntax-independent modelling with the aim of illuminating the structure of computation and facilitating reasoning about programs. Many developments in denotational semantics have been driven by the quest for full abstraction [21]: a model is fully abstract if the interpretations of two programs are the same precisely when the programs behave in the same way (i.e. are contextually equivalent). A faithful correspondence like this opens the path to a broad range of applications, such as compiler optimisation and program transformation, in which the preservation of semantics is of paramount importance.

Recent years have seen game semantics emerge as a robust denotational paradigm [4, 6, 12]. It has been used to construct the first fully abstract models for a wide spectrum of programming languages, previously out of reach of denotational semantics. Game semantics models computation as an exchange of moves between two players, representing respectively the program and its computational environment. Accordingly, a program is interpreted as a strategy in a game corresponding to its type. Intuitively, the plays that game semantics generates constitute the observable patterns that a program produces when interacting with its environment, and this is what underlies the full abstraction results. Game semantics is compositional: the strategy corresponding to a compound program phrase is obtained by canonical combinations of those corresponding to its sub-phrases. An important advance in game semantics was the development of nominal games [3, 17, 26], which underpinned full abstraction results for languages with dynamic generative behaviours, such as the $\nu$-calculus [3], higher-order concurrency [18] and ML references [24]. A distinctive feature of nominal game models is the presence of names (e.g. memory locations, references names) in game moves, often along with some abstraction of the store.

The aim of the present paper is to extend the range of the game approach towards real-life programming languages, by focusing on Java-style objects. To that end, we define an imperative object calculus, called Interface Middleweight Java (IMJ), intended to capture contextual interactions of code written in Middleweight Java (MJ) [9], as specified by interfaces with inheritance. We present both equational (contextual equivalence) and inequational (contextual approximation) full abstraction results for the language. To the best of our knowledge, these are the first denotational models of this kind.

Related Work While the operational semantics of Java has been researched extensively [7], there have been relatively few results regarding its denotational semantics. More generally, most existing models of object-oriented languages, such as [8, 15], have been based on global state and consequently could not be fully abstract.

On the other hand, contextual equivalence in Java-like languages has been studied successfully using operational approaches such as trace semantics [2, 13, 14] and environmental bisimulation [16]. The trace-based approaches are closest to ours and the three papers listed also provide characterizations of contextual equivalence. The main difference is that traces are derived operationally through a carefully designed labelled transition system and, thus, do not admit an immediate compositional description in the style of denotational semantics.

However, similarities between traces and plays in game semantics indicate a deeper correspondence between the two areas, which also manifested itself in other cases, e.g. [20] vs [19]. At the time of writing, there is no general methodology for moving smoothly between the two approaches, but we believe that there is scope for unifying the two fields in the not so distant future.

In comparison to other game models, ours has quite lightweight structure. For the most part, playing consists of calling the opponent’s methods and returning results to calls made by the opponent. In particular, there are no justification pointers between moves. This can be attributed to the fact that Java does not feature first-class higher-order functions and that methods in Java objects cannot be updated. On the other hand, the absence of pointers makes definitions of simple notions, such as well-bracketing, less direct, since the dependencies between moves are not given explicitly any
more and need to be inferred from plays. The latter renders strategy composition non-standard. Because it is impossible to determine statically to which arena a move belongs, the switching conditions (cf. [6]) governing interactions become crucial for determining the strategy responsible for each move. Finally, it is worth noting that traditional copycat links are by definition excluded from our setting: a call/return move for a given object cannot be copycated by the other player, as the move has a fixed polarity, determined by the ownership of the object. In fact, identity strategies contain plays of length at most two!

Further Directions In future work, we would like to look for automata-theoretic representations of fragments of our model in order to use them as a foundation for a program verification tool for Java programs. Our aim is to take advantage of the latest developments in automata theory over infinite alphabets [10], and fresh-register automata in particular [23, 27], to account for the nominal features of the model.

2. The language IMJ

We introduce an imperative object calculus, called Interface Middleware Java (IMJ), in which objects are typed using interfaces. The calculus is a stripped down version of Middleware Java (MJ), expressive enough to expose the interactions of MJ-style objects with the environment.

Definition 1. Let Ints, Flds and Meths be sets of interface, field and method identifiers. We range over them respectively by I, f, m and variants. The types θ of IMJ are given below, where θ stands for a sequence θ₁,...,θᵣ of types (for any n). An interface definition Θ is a finite set of typed fields and methods. An interface table Δ is a finite assignment of interface definitions to interface identifiers.

Types ⊨ θ := void | int | Θ

IDfns ⊨ Θ ::= Θ | (f : θ), Θ | (m : θ → θ), Θ

ITbls ⊨ Δ ::= Δ | (I : Θ), Δ | (I(I) : Θ), Δ

We write I(I') : Θ for interface extension: interface I extends I' with fields and methods from Θ. We stipulate that the extension relation must not lead to circular dependencies. Moreover, each identifier f, m can appear at most once in each Θ, and each I can be defined at most once in Δ (i.e. there is at most one element of Δ of the form I : Θ or I(I') : Θ). Thus, each Θ can be seen as a finite partial function Θ : (Fields ∪ Meths) → Types. We write Θ(f) and Θ(m) for Θ(f) and Θ(m). Similarly, Δ defines a partial function Δ : Ints → IDfns given by

Δ(I) = Θ(I) ∪ Θ(I(I)) ∪ Θ(I(I')) ∪ Θ(I(I'))

An interface table Δ is well-formed if, for all interface types I, I' :

• if I' appears in Δ(I) then I' ∈ dom(Δ),
• if (I(I') : Θ) ∈ Δ then dom(Δ(I')) ∩ dom(Θ) = ∅.

Henceforth we assume that interface tables are well-formed. Interface extensions yield a subtyping relation. Given a table Δ, we define Δ ⊨ θ₁ ≤ θ₂ by the following rules.

\[ \Delta(I(I') : \Theta), \Delta', \Delta' \leq I' \]

\[ \Delta ⊨ \theta₁ \leq \theta₂ \]

We might omit Δ from subtyping judgements for economy.

Definition 2. Let k be a countably infinite set of object names, which we range over by a and variants. IMJ terms are listed below, where we let x range over a set of variables Vars, and ι over I. Moreover, ⊕ is selected from some set of binary numeric operators. M is a method-set implementation. Again, we stipulate that each m appear in each M at most once.

M ::= x | a | skip | null | i | M ⊕ M | let x = M in M [M = M | if M then M else M] (I)M | new(x : I ; M) | M.f | M.f := M := M.m(\overline{M})

The terms are typed in contexts comprising an interface table Δ and a variable context Γ = \{x₁ : θ₁,...,xₙ : θₙ\} ∪ \{a₁ : I₁,...,aₘ : Iₘ\} such that any interface in Γ occurs in dom(Δ). The typing rules are given in Figure 1.

For the operational semantics, we define the sets of term values, heap configurations and states by:

TVals ⊨ v ::= skip | i | null | a

HCnfss ⊨ V ::= \emptyset | (f : v), V

States ⊨ S : A → Ints × (HCnfss × HCnfss)

If S(a) = (I, (V,M)) then we write S(a) : I, while S(a).f and S(a).m stand for V.f and M.m respectively, for each f and m. Given an interface table Δ such that I ∈ dom(Δ), we let the default heap configuration of type I be

V_I = I(f : v_I | Δ(I).f = θ),

where v_{void} = skip, v_{null} = 0 and v_{x} = null. The operational semantics of IMJ is given by means of a small-step transition relation
Given $\Delta | M : \theta$, we write $M \Downarrow$ if there exists $S$ such that $(0, M) \rightarrow^* (S, \text{skip})$.

**Definition 3.** Given $\Delta | M : \theta$ contextually approximates $\Delta | M : \theta$ if, for all $\Delta \supseteq \Delta$ and all contexts $C$ such that $\Delta | 0 \vdash C[M] : \theta$, void, if $C[M] \downarrow$ then $C[M] \Downarrow$. We then write $\Delta | M \equiv M' : \theta$. Two terms are contextually equivalent (written $\Delta | M \equiv M' : \theta$) if they approximate each other.

For technical convenience, IMJ features the let construct, even though it is definable: given $\Delta | x:: \theta \vdash M : \theta$ and $\Delta | M : \theta$ contextually approximates $\Delta | M : \theta$, consider new$(x : I; m : \lambda x.M), m(M')$, where $I$ is a fresh interface with a single method $m : \theta \rightarrow \theta$. As usual, we write $M : M'$ for let $x = M$ in $M'$, where $x$ is not free in $M'$.

Although IMJ does not have explicit local variables, they could easily be introduced by taking let $x = \text{new}(y : I_0); i \cdots$, where $I_0$ has a single field of type $\theta$. In the same manner, one can define variables and methods that are private to objects, and invisible to the environment through interfaces.

**Example 1 ([16]).** Let $\Delta =$ \{empty : \emptyset, cell : \{get : void \rightarrow \emptyset, set : \emptyset \rightarrow \text{void}\}\}, $\forall E : \{\text{val : Empty}, \text{Var}_E : \{\text{val : int}\}\}$ and consider the terms $\Delta | M_1 : \text{cell} (i = 1, 2)$ defined by:

\[
M_1 \equiv \text{let } x = \text{new}(x : \text{Var}_E; i) \text{ in new}(x : \text{cell}; M_1)
\]

\[
M_2 \equiv \text{let } b = \text{new}(x : \text{Var}_E; i) \text{ in }
\]

with

\[
M_1 = (\text{get} : \lambda().x.\text{val}, \text{set} : \lambda y.(y.\text{val} := y))
\]

\[
M_2 = (\text{get} : \lambda().\text{if}(b.\text{val} = 0; v_1.\text{val}), \text{else} (b.\text{val} = 1; v_2.\text{val}), \text{set} : \lambda y.(v_1.\text{val} := y; v_2.\text{val} := y)).
\]

We have $\Delta | M_1 \equiv M_2 : \text{cell}$. Intuitively, each of the two implementations of Cell corresponds to recording a single value of type $\text{Empty}$ (using set) and providing access to it via get. The difference lies in the way the value is stored: a single private variable is used in $M_1$, while two variables are used in $M_2$. However, in the latter case the variables always hold the same value, so it does not matter which of the variables is used to return the value.

Figure 2. Operational semantics of IMJ.

The game semantics of the two terms will turn out to consist of plays of the shape $s^0 n^{\Sigma_0} G^*_0 S_1 G^*_1 \cdots S_k G^*_k$, where

\[
G_i = \begin{cases} 
\text{call } n.\text{get}()^{\Sigma_0} & \text{if } i = 0 \\
\text{call } n.\text{get}()^{\Sigma_1} & \text{if } i > 0
\end{cases}
\]

and $\Sigma_i = \{n \mapsto (\text{Cell}, \emptyset) \cup \{n_i \mapsto (\text{Empty}, \emptyset) | 0 < j \leq i\}$. Intuitively, the plays describe all possible interactions of a Cell object. The first two moves $s^0 n^{\Sigma_0}$ correspond to object creation. After that, the $G_i$ segments represent the environment reading the current context (initially having null value), while the $S_i$ segments correspond to updating the content with a reference name provided by the environment. The stores $\Sigma_i$ attached to moves consist of all names that have been introduced during the interaction so far.

It is worth noting that, because IMJ has explicit casting, a context can always guess the actual interface of an object and extract any information we may want to hide through casting.

**Example 2.** Let $\Delta =$ \{empty : \emptyset, point : \emptyset\}, \{get : \text{void} \rightarrow \text{Empty}, \text{set} : \text{Empty} \rightarrow \text{void}\}\}, $\forall E : \{\text{val : Empty}, \text{Var}_E : \{\text{val : int}\}\}$ and consider the terms $\Delta | M_i : \text{empty} (i = 1, 2)$ defined by:

\[
M_1 \equiv \text{new}(x : \text{Empty} ;)
\]

\[
M_2 \equiv \text{new}(x : \text{point} ; p.x := 0; p.y := 1; (\text{Empty}) p.
\]

In our model they will be interpreted by the strategies $\sigma_1 = (\epsilon, s^0 n^{\text{null}})$ and $\sigma_2 = (\epsilon, s^0 n^{(\text{Point}, 0 \mapsto \text{Point}, 0 \mapsto \text{Point})})$ respectively. Using e.g. the casting context $C \equiv (\text{Point}) \cdot \text{skip}$, we can see that $\Delta | M_2 \not
\Delta | M_1 : \text{Empty}$. On the other hand, Theorem 20 will imply $\Delta | M_1 \equiv M_2 : \text{Empty}$.

On the whole, IMJ is a compact calculus that strips down Middle-level Java to the essentials needed for interface-based interaction. Accordingly, we suppressed the introduction of explicit class hierarchy, as it would remain invisible to the environment anyway and any class-based internal computations can be represented using standard object encodings [1].

At the moment the calculus allows for single inheritance for interfaces only, but extending it to multiple inheritance is not problematic. The following semantic developments only rely on the assumption that $\leq$ must not give rise to circularities.

**3. The game model**

In our discussion below we assume a fixed interface table $\Delta$.

The game model will be constructed using mathematical objects (moves, plays, strategies) that feature names drawn from the set $\Delta$. Although names underpin various elements of our model, we do not want to delve into the precise nature of the sets containing them. Hence, all of our definitions preserve name-invariance, i.e. our objects are (strong) nominal sets [11, 26]. Note that we do not
need the full power of the theory but mainly the basic notion of name-permutation. For an element $x$ belonging to a (nominal) set $X$ we write $\nu(x)$ for its name-support, which is the set of names occurring in $x$. Moreover, for any $x, y \in X$, we write $x \sim y$ if there is a permutation $\pi$ such that $x = \pi \cdot y$.

We proceed to define a category of games. The objects of our category will be arenas, which are nominal sets carrying specific type information.

**Definition 4.** *An arena* is a pair $A = (M_A, \xi_A)$ where:
- $M_A$ is a nominal set of moves;
- $\xi_A : M_A \rightarrow (A \rightarrow \text{Ints})$ is a nominal typing function;

such that, for all $m \in M_A$, $\text{dom}(\xi_A(m)) = \nu(m)$.

We start by defining the following basic arenas,

$1 = \{\ast\}, \{\ast, 0\}, \mathbb{Z} = \{\mathbb{Z}, \{\ast, 0\}\},$

$I = (A \cup \{\text{null}1\}, \{(\text{null}1, 0), \ast, (a, a, I)\})$, for all interfaces $I$. Given arenas $A$ and $B$, we can form the arena $A \times B$ by:

$M_{A \times B} = \{(m, n) \in M_A \times M_B \mid a \in \nu(m) \cap \nu(n)\}$

$\xi_{A \times B}((m, n), a) = \begin{cases} \xi_A(m, a) & \text{if } a \notin \nu(n) \lor \xi_B(n, a) \leq \xi_A(m, a) \\ \xi_B(n, a) & \text{otherwise} \end{cases}$

Another important arena is $\#(I_1, \ldots, I_n)$, with:

$M_{\#(I)} = \{(a_1, \ldots, a_n) \in A^n \mid a_i \text{'s distinct}\}$

$\xi_{\#(I)}((a_1, \ldots, a_n), a_i) = I_i$

for all $n \in \mathbb{N}$. In particular, $A^{\ast\emptyset} = 1$.

For each type $\theta$, we set $\text{Val}_{\theta}$ to be the set of semantic values of type $\theta$, given by:

$\text{Val}_{\text{void}} = M_1, \text{Val}_{\text{int}} = M_z, \text{Val}_{I} = M_X.$

For each type sequence $\theta = \theta_1, \ldots, \theta_n$, we set $\text{Val}_{\vec{\theta}} = \text{Val}_{\theta_1} \times \cdots \times \text{Val}_{\theta_n}.$

We let a store $\Sigma_\vec{\theta}$ be a type-preserving finite partial function from names to object types and field assignments, that is, $\Sigma : \vec{\theta} \rightarrow \text{Ints} \times (\text{Fields} \rightarrow \text{Val})$ such that $|\Sigma|$ is finite and

$\Sigma(a) : I \land \Delta(I).f = \theta \implies \Sigma(a).f = v \land \Sigma \vdash v \leq \theta,$

where the new notation is explained below. First, assuming $\Sigma(a) = (I', \phi)$, the judgement $\Sigma(a) : I$ holds iff $I = I'$ and $\Sigma(a).f$ stands for $\phi(f).$ Next we define typing rules for values in store contexts:

$v \in \text{Val}_{\text{void}} \implies \Sigma(v) : I \lor v = \text{null}1$

and write $\Sigma \vdash v : \theta \implies (I \vdash v : \theta \lor (I' \vdash v : I' \land I' \leq \theta)).$

We let $\text{Sto}$ be the set of all stores. We write $\text{dom}(\Sigma(a))$ for the set of all $f$ such that $\Sigma(a).f$ is defined. We let $\text{Sto}_0$ contain all stores $\Sigma$ such that:

$\forall a \in \text{dom}(\Sigma)., f \in \text{dom}(\Sigma(a)). \Sigma(a).f \in \{\ast, 0, \text{null}1\}$

and we call such a $\Sigma$ a default store.

Given arenas $A$ and $B$, plays in $AB$ will consist of sequences of moves (with store) which will be either moves from $M_A \cup M_B$, or moves representing method calls and returns. Formally, we define:

$M_{AB} = M_A \cup M_B \cup \text{Calls} \cup \text{Retns}$

where we set $	ext{Calls} = \{\text{call } a.m(\vec{a}) \mid a \in A \land \vec{a} \in \text{Val}^*\}$ and

$\text{Retns} = \{\text{ret } a.m(v) \mid a \in A \land v \in \text{Val}\}.$

**Definition 5.** A *legal sequence* in $AB$ is a sequence of moves from $M_{AB}$ that adheres to the following grammar (Well-Bracketing), where $m_A$ and $m_B$ range over $M_A$ and $M_B$ respectively.

$L_{AB} ::= \epsilon \mid m_A X \mid m_A Y m_B X$

$X ::= Y \mid Y \text{'(call } a.m(\vec{a})) X$

$Y ::= \epsilon \mid YY \mid (\text{call } a.m(\vec{a})) Y (\text{ret } a.m(v))$

We write $L_{AB}$ for the set of legal sequences in $AB$. In the last clause above, we say that $\text{call } a.m(\vec{a})$ *justifies* $\text{ret } a.m(v)$.

To each $s \in L_{AB}$ we assign a *polarity* function $p$ from move occurrences in $s$ to the set $\mathbb{P} = \{O, P\}$. Polarieties represent the two players in our game reading of programs: $O$ is the Opponent and $P$ is the Proponent in the game. The latter corresponds to the modelled program, while the former models the possible computational environments surrounding the program. Polarieties are complemented via $\overline{O} = \{P\}$ and $\overline{P} = \{O\}$. In addition, the polarity function must satisfy the condition:

- For all $m_X \in M_X (X = A, B)$ occurring in $s$ we have $p(m_X) = O$ and $p(m_B) = P$ (O-starting).

We write $L_{AB}$ for the set of legal sequences in $AB$. In the last clause above, we say that $\text{call } a.m(\vec{a})$ *justifies* $\text{ret } a.m(v)$.

A move-with-store in $AB$ is a pair $m^\Sigma$ with $\Sigma \in \text{Sto}$ and $m \in M_{AB}$. For each sequence $s$ of moves-with-store we define the set of available names of $s$ by:

$Av(s(\emptyset)) = \emptyset, Av(s(m^\Sigma) \cup \nu(m))$

where, for each $X \subseteq A$, we let $S^*(X) = \bigcup \Sigma^*(X)$, with

$\Sigma^0(X) = X, \Sigma^{i+1}(X) = \nu(\Sigma^i(X)).$

That is, a name is available in $s$ just if it appears inside a move in $s$, or it can be reached from an available name through some store in $s$. We write $s$ for the underlying sequence of moves of $s$ (i.e. $\pi_1(s)$), and let $\subseteq$ denote the prefix relation between sequences. If $s'm^\Sigma \subseteq s$ and $a \in \nu(m^\Sigma) \lor \nu(s')$ then we say $a$ is introduced by $m^\Sigma'$ in $s$. In such a case, we define the owner of the name $a$ in $s$, written $o(a)$, to be $p(m)$ (where $p$ is the polarity associated with $s$). For each polarity $X \in \{O, P\}$ we let

$X(s) = \{a \in \nu(s) \mid o(a) = X\}$

be the set of names in $s$ owned by $X$.

**Definition 6.** A *play* in $AB$ is a sequence of moves-with-store $s$ such that $s$ is a legal sequence and, moreover, for all $s'm^\Sigma' \subseteq s$:

- It holds that $\text{dom}(\Sigma) = Av(s'm^\Sigma')$. (Fragility)
- If $a \in \text{dom}(\Sigma)$ with $\Sigma(a) : I$ then:
  - if $m \in M_X$, for $X \in \{A, B\}$, then $I \leq \xi_X(m_a, a)$;
  - for all $n^\Sigma$ in $s'$, if $a \in \text{dom}(T)$ then $T(a) : I$;
  - if $\Delta(I).m = \emptyset \rightarrow \emptyset$ then:
    - if $m = \text{call } a.m(\vec{a})$ then $\Sigma \vdash \vec{a} : \emptyset$ for some $\emptyset \leq \emptyset$,
    - if $m = \text{ret } a.m(v)$ then $\Sigma \vdash v : \emptyset$ for some $\emptyset \leq \emptyset$.

We write $P_{AB}$ for the set of plays in $AB$.1

---

1By abuse of notation, we frequently write instead “$a$ is introduced by $m$ in $s$”. Recall also that $\nu(s)$ collects all names appearing in $s$; in particular, $\nu(m^\Sigma_1, \ldots, m^\Sigma_i) = \nu(m_1) \cup \nu(\Sigma_2) \cup \cdots \cup \nu(m_i) \cup \nu(\Sigma_i).$
Note above that, because of well-bracketing and alternation, if \( m = \text{ret} \cdot a \cdot m(v) \) then well-calling implies \( o(a) = p(m) \). Thus, the frugality condition stipulates that names cannot appear in a play in unreachable parts of a store (cf. [17]). Moreover, well-calling ensures that the typing information in stores is consistent and adheres to the constraints imposed by \( \Delta \) and the underlying arenas. Finally, well-calling implements the specification that each player need only call the other player’s methods. This is because calls to each player’s own methods cannot in general be observed and so should not be accounted for in plays.

Given arenas \( A, B, C \), next we define interaction sequences, which show how plays from \( AB \) and \( BC \) can interact to produce a play in \( AC \). The sequences will rely on moves with stores, where the moves come from the set:

\[
M_{ABC} = M_A \cup M_B \cup M_C \cup \text{Calls} \cup \text{Retns}.
\]

The moves will be assigned polarities from the set:

\[
\text{Pol}_2 = \{ O_L, P_L, O_R, P_R, O_{LR}, P_{LR} \}.
\]

The index \( L \) stands for “left”, while \( R \) means “right”. The indices indicate which part of the interaction (\( A, B \) or \( C \)) a move comes from, and what polarity it has therein. We also consider an auxiliary notion of pseudo-polarities:

\[
OO = \{ O_L, O_R \}, \quad PO = \{ P_L, P_L, O_L, P_R, O_R \}, \quad OP = \{ P_R, P_R, O_L, P_R, O_R \}.
\]

Each polarity has an opposite pseudo-polarity determined by:

\[
\overline{O}_T = \overline{O}_T \cdot P_{LR} = PO, \quad \overline{P}_R = \overline{P}_R \cdot \overline{O}_R = PO, \quad \overline{P}_L = \overline{P}_L \cdot \overline{O}_R = OO.
\]

Finally, each \( X \in \{ AB, BC, AC \} \) has a designated set of polarities given by:

\[
\begin{align*}
p(AB) &= \{ O_L, P_L, O_L, P_R, O_R \}, \\
p(BC) &= \{ O_R, P_R, O_L, P_R, O_R, P_R \}, \\
p(AC) &= \{ O_L, P_L, O_L, P_R, O_R \}.
\end{align*}
\]

Note the slight abuse of notation with \( p \), as it is also used for move polarities.

Suppose \( X \in \{ AB, BC, AC \} \). Consider a sequence \( s \) of moves-with-store from \( ABC \) (i.e. a sequence with elements \( m^X \) with \( m \in M_{ABC} \)) along with an assignment \( p \) of polarities from \( \text{Pol}_2 \) to moves of \( s \). Let \( s \mid X \) be the subsequence of \( s \) containing those moves-with-store \( m^X \) of \( s \) for which \( p(m) \in p(X) \). Additionally, we define \( s \mid \overline{X} \) to be \( \overline{\gamma}(s \mid X) \), where the function \( \gamma \) acts on moves-with-store by restricting the domains of stores to available names:

\[
\gamma(\epsilon) = \epsilon, \quad \gamma(s \cdot m^X) = \gamma(s) \cdot m^X \cdot \Sigma(\epsilon (s \cdot m^X)).
\]

Definition 7. An interaction sequence in \( ABC \) is a sequence \( s \) of moves-with-store in \( ABC \) satisfying the following conditions:

- For each \( s' \cdot m^X \subseteq s \), \( \text{dom}(\Sigma) = \Sigma(\epsilon (s \cdot m^X)) \). (Fragility)
- If \( s' \cdot m^X \subseteq s \) and \( a \in \text{dom}(\Sigma) \) with \( \Sigma(a) : T \) then:
  - if \( m \in M_X \), for \( X \in \{ A, B, C \} \), then \( \Sigma \preceq \chi_X(m, a) \);
  - for all \( n \in T \) in \( s' \), if \( a \in \text{dom}(T) \) then \( T(a) \subseteq T \);
  - if \( \Delta(T), m = \theta \Rightarrow \theta \) then:
    - if \( m = \text{call} \cdot a \cdot m(v) \) then \( \Sigma \top \vdash \top : \theta' \) for some \( \theta' \leq \theta \);
    - if \( m = \text{ret} \cdot a \cdot m(v) \) then \( \Sigma \top \vdash \top : \theta' \) for some \( \theta' \leq \theta \).
- There is a polarity function \( p \) from move occurrences in \( s \) to \( \text{Pol}_2 \) such that:
  - For all \( m \in M_X \) (\( X = A, B, C \)) occurring in \( s \) we have \( p(m) \) = \( O_L, P_L, O_R, P_R \) and \( p(m) \) = \( P_R \);
  - If \( m_n \) are consecutive moves in \( s \) then \( p(n) \) = \( p(m) \).
- \( (\text{Alternation}) \)

\[ \]
leads to the following definition of projections of polarities,
$$\pi_{AB}(X_L) = X \quad \pi_{AB}(X_L Y_R) = X \quad \pi_{AB}(Y_R) = \text{undef.}$$
$$\pi_{BC}(X_L) = \text{undef.} \quad \pi_{BC}(X_L Y_R) = Y \quad \pi_{BC}(Y_R) = Y$$
$$\pi_{AC}(X_L Y_R) = \text{undef.} \quad \pi_{AC}(Y_R) = Y$$
where $X, Y \in \{O, P\}$. We can now show the following.

**Lemma 2.** Let $s \in \text{Int}(ABC)$. Then, for each $X \in \{AB, BC, AC\}$ and each $m^{\Sigma}$ in $s$, if $p(m) \notin p(X)$ then $\pi_X(p(m)) = p_X(m)$, where $p_X$ is the polarity of $s \upharpoonright X$.

**Proof.** We show this for $X = AB$, the other cases are proven similarly, by induction on $|s| \geq 0$; the base case is trivial. For the inductive case, if $m$ is the first move in $s$ with polarity in $p(AB)$ then, by projecting, $m \in M_A$ and therefore $p(m) = O_L$ and $p(AB)(m) = O$, as required. Otherwise, let $n$ be the last move in $s$ with polarity in $p(AB)$ before $m$. By IH, $p(AB)(n) = \pi_{AB}(p(n))$. Now, by projecting, $p(AB)(m) = \pi_{AB}(p(n))$ and observe that, for all $X \in p(n)$, $\pi_{AB}(X) = \pi_{AB}(p(n))$, so in particular $\pi_{AB}(p(m)) = \pi_{AB}(p(n)) = p(AB)(n) = p(AB)(m)$.

The following lemma formulates a taxonomy on names appearing in interaction sequences.

**Lemma 3.** Let $s \in \text{Int}(ABC)$. Then,
1. $\nu(s) = O(s \upharpoonright AC) \uplus P(s \upharpoonright AB) \uplus P(s \upharpoonright BC)$;
2. if $s = tm^{\Sigma}$ and:
   - $p(m) \in OO$ and $s \upharpoonright AC = t'm^{\Sigma'}$,
   - or $p(m) \in PO$ and $s \upharpoonright AB = t'm^{\Sigma'}$,
   - or $p(m) \in OP$ and $s \upharpoonright BC = t'm^{\Sigma'}$,
   then $\nu(t) \cap \nu(m^{\Sigma'}) \subseteq \nu(t)$ and, in particular, if $m$ introduces a name in $t'm^{\Sigma'}$ then $m$ introduces a name in $s$.

**Proof.** For 1, by definition of interactions we have that these sets are disjoint. It therefore suffices to show the left-to-right inclusion. Suppose that $a \in \nu(s)$ is introduced in some $m^{\Sigma}$ in $s$, with $p(m) \in PO$, and let $s \upharpoonright AC = AB \cdots m^{\Sigma} \cdots$. If $a \in \nu(m^{\Sigma'})$ then $a \in P(s \upharpoonright AB)$, as required. Otherwise, by Laird’s last set of conditions, $a$ is copied from the move preceding $m^{\Sigma}$ in $s$, a contradiction to its being introduced at $m^{\Sigma}$. Similarly if $p(m) \in OP$. Finally, if $p(m) \in OO$ then we work similarly, considering $O(s \upharpoonright AC)$.

For 2, we show the first case, and the other cases are similar. It suffices to show that $(\nu(m^{\Sigma'}) \setminus \nu(t')) \cap \nu(t) = \emptyset$. Suppose $a \in \nu(m^{\Sigma'}) \setminus \nu(t')$, therefore $a \in O(s \upharpoonright AC)$. But then we cannot have $a \in \nu(t')$ as the latter, by item 1, would imply $a \in P(s \upharpoonright AB) \uplus P(s \upharpoonright BC)$.

**Proposition 4.** For all $s \in \text{Int}(ABC)$, the projections $s \upharpoonright AB$, $s \upharpoonright BC$ and $s \upharpoonright AC$ are plays in $AB$, $BC$ and $AC$ respectively.

**Proof.** By frugality of $s$ and application of $\gamma$, all projections satisfy frugality. Moreover, well-classing is preserved by projections. For well-calling, let $m = \text{call} \cdot m(s)$ be a move in $s$ and let $n^T$ be the move introducing $a$ in $s$. Suppose $p(m) \in P(AB)$ and let us assume $p_{AB}(m) = O$. We need to show that $\sigma_{AB}(m) = P$. By $p_{AB}(m) = O$ we obtain that $p(m) \in \{O_L, O_P, P_o\}$ and, by well-calling of $s$, we have that $a(\overline{p}) = O$. Thus, $p(n) \in PO$ and, by Lemma 3, $a$ introduces in $s \upharpoonright AB$ and therefore $\sigma_{AB}(m) = P$. Hence, by the other hand, $p(AB)(m) = P$ then we obtain $p(n) \in OO \cup OP$ and therefore by Lemma 3, $a \in P(s \upharpoonright BC) \cup O(s \upharpoonright AC)$. Thus, by the same lemma, $a \notin P(s \upharpoonright AB)$ and hence $\sigma_{AB}(a) = O$. The cases for the other projections are shown similarly.

In our setting programs will be represented by strategies between arenas. We shall introduce them next after a few auxiliary definitions. Intuitively, strategies capture the observable computational patterns produced by a program.

Let us define the following notion of subtyping between stores.

For $\Sigma, \Sigma' \in \text{Sto}$, $\Sigma \succeq \Sigma'$ holds if, for all names $a$,
$$\Sigma'(a) : T' \Rightarrow \Sigma(a) \subseteq T' \forall \nu \in \text{dom}(\Sigma(a)), \Sigma(a).f = \Sigma'(a).f$$

In particular, if $a$ is in the domain of $\Sigma'$, $\Sigma$ may contain more information about $a$ because of assigning to $a$ a larger interface. Accordingly, for plays $s, s' \in PA_B$, we say that $s$ is an $O$-extension of $s'$ if $s$ and $s'$ agree on their underlying sequences, while their stores may differ due to subtyping related to $O$-names. Where such subtyping leads to $s$ having stores with more fields than those in $s'$, $P$ is assumed to copy the values of those fields. Formally, $s \preceq_O s'$ is defined by the rules:

$$s \preceq_O s' \quad \Sigma \preceq \Sigma' \quad \Sigma \cap P(sm^{\Sigma}) \subseteq \Sigma' \quad p(m) = O^\prime$$

$$\text{sn}^T \preceq s' \quad \Sigma \preceq \Sigma' \quad \Sigma \cap P(sm^{\Sigma}) \subseteq \Sigma' \quad p(m) = P^\prime$$

where $\Sigma$ extends $\Sigma'$ by $T$ if:

- for all $a \in \text{dom}(\Sigma) \setminus \text{dom}(\Sigma')$, $\Sigma(a) = T(a)$;
- for all $a \in \text{dom}(\Sigma(a)) \setminus \text{dom}(\Sigma'(a))$, $\Sigma(a).f = T(a).f$.

The utility of $O$-extension is to express semantically the fact that the environment of a program may use up-casting to inject in its objects additional fields (and methods) not accessible to the program.

**Definition 8.** A strategy $\sigma$ in $AB$ is a non-empty set of even-length plays from $PA_B$ satisfying the conditions:
- if $sm^{\Sigma} \cdot sn^T \notin \sigma$ then $s \in \sigma$. (Even-prefix closure)
- if $sm^{\Sigma}, sn^T \in \sigma$ then $sm^{\Sigma} \sim sn^T$. (Determinacy)
- if $s \in \sigma$ and $s \parallel t$ then $t \in \sigma$. (Equivalence)
- if $s, t \in \sigma$ and $s \preceq t$ then $t \in \sigma$. (O-extension)

We write $\sigma : A \rightarrow B$ when $\sigma$ is a strategy in $AB$. If $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$, we define their composition $\sigma; \tau$ by:

$$\sigma; \tau = \{s \upharpoonright AC \mid s \in \sigma(\tau)\}$$

where $\sigma(\tau) = \{s \in \text{Int}(ABC) \mid s \upharpoonright AB \in \sigma \land s \upharpoonright BC \in \tau\}$.

In definitions of strategies we may often leave the presence of the empty sequence implicit, as the latter is a member of every strategy. For example, for each arena $A$, we define the strategy:

$$\text{id}_A : A \rightarrow A = \{m^\Sigma_A m^\Sigma_A \in PA_A\}$$

The next series of lemmata allow us to show that strategy composition is well defined.

**Lemma 5.** If $sn^T \cdot sm^{\Sigma} \in \sigma(\tau)$ with $p(m) \notin OO$ then $sn^T \sim \text{sn}^T$. Hence, if $s_1 m^\Sigma_1 \sim s_2 m^\Sigma_2$ then $s_1 m^\Sigma_1 \sim s_2 m^\Sigma_2$.

**Proof.** For the latter part, if $s_1 = \pi_1 \cdot s_2$ then, since $\pi \cdot (s_2 m^\Sigma_2) \in \sigma(\tau)$, by former part of the claim we have $s_1 m^\Sigma_1 \sim \pi \cdot (s_2 m^\Sigma_2)$ so $s_1 m^\Sigma_1 \sim s_2 m^\Sigma_2$.

Now, for the former part, suppose WLOG that $p(m) \in PO$. Then, by the interaction diagram, we also have $p(n) \in PO$. As $sm^\Sigma \cdot sn^T \in \sigma$, $\sigma \cap P = \sigma$, by determinacy of $\sigma$ we get $s m^\Sigma \sim s' m^\Sigma$.

2 Recall that, for any nominal set $X$ and $x, y \in X$, we write $x \sim y$ just if there is a permutation $\pi$ such that $x = \pi \cdot y$. 

with $s' m^{\Sigma} = s m^{\Sigma} \upharpoonright \gamma AB$ and $s' n^T = s n^T \upharpoonright \gamma AB$. We therefore have $(s', m^{\Sigma}) \sim (s', n^T)$ and, trivially, $(s', s') \sim (s', s')$. Moreover, by Lemma 3, $\nu(m^{\Sigma}) \cap \nu(s) \subseteq \nu(s')$ and $\nu(n^T) \cap \nu(s) \subseteq \nu(s')$ hence, by Strong Support Lemma [26], $s m^{\Sigma} \sim s n^T$. By Laird’s last set of conditions, the remaining values of $\Sigma, T$ are determined by the last store in $s$, hence $s m^{\Sigma} \sim s n^T$.

\section*{Lemma 6.} If $s_1, s_2 \in \sigma \| \tau$ end in moves with polarities in $p(AC)$ and $s_1 \upharpoonright AC = s_2 \upharpoonright AC$ then $s_1 \sim s_2$.

\begin{proof} By induction on $[s_1, AC] > 0$. The base case is encompassed in $s_1 = s_2'. m^{\Sigma}$ with $p(m) \in O \cup O, i = 1, 2$, where note that by IH $m$ will have the same polarity in $s_1, s_2$. Then, by IH we get $s_1 = s_2' \sim s_2$ for some $\pi$. Let $s_1'. m^{\Sigma} = s_2 \upharpoonright AC$, for $i = 1, 2$, so in particular $s_1' = s_2' = s_1' s_2' = (s_1', s_2')$. Moreover, by hypothesis, we trivially have $(m^{\Sigma}, s_1') \sim (m^{\Sigma}, s_2')$ and hence, by Lemma 3 and Strong Support Lemma [26], we obtain $s_1'm^{\Sigma} \sim s_2'm^{\Sigma}$ which implies $s_1 \sim s_2$ by Laird’s conditions. Suppose now $s_1 = s_2'. m^{\Sigma}, i = 1, 2$ with $p(m) \in P(AC) \cup O$ and the last move in $s_1$ being the last move in $s_2'm^{\Sigma}$ having polarity in $p(AC)$. By IH, $s_1' \sim s_2'$. Then, by consecutive applications of Lemma 5, we obtain $s_1 \sim s_2$. \end{proof}

\section*{Proposition 7.} If $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$ then $\sigma; \tau : A \rightarrow C$.

\begin{proof} We show that $\sigma; \tau$ is a strategy. Even-prefix closure and equivariance are clear. Moreover, since each $s \in \sigma[\tau]$ has even-length projections in $AB$ and $BC$, we can show that its projection in $AC$ is even-length too. For $O$-extension, if $s \in \sigma[\tau]$ and $t < s$ with $s = u \upharpoonright \gamma AC$ and $u \in \sigma[\tau]$, then we can construct a $v \upharpoonright AC$ such that $t = v \upharpoonright AC$ and $v < u$, where $< u$ is defined for interaction sequences in an analogous way as for plays (with condition $p(m) = O$ replaced by $p(m) \in O \cup O$, and $p(m) = P$ by $p(m) \in PO \cup OP$). Moreover, $v \upharpoonright AB \leq u \upharpoonright \gamma AC$ and $v \upharpoonright BC \leq u \upharpoonright \gamma BC$, so $t \in \sigma[\tau]$. Finally, for determinacy, let $s m^{\Sigma}, s n^T \in \sigma[\tau]$ be due to $s_1 s_1 m^{\Sigma} s_2 s_2 n^T \in \sigma[\tau]$ respectively, where $s_1, s_2$ both end in the last move of $s$. By Lemma 6, we have $s_1 \sim s_2$ and thus, by consecutive applications of Lemma 5, we get $s_1 s_1 m^{\Sigma} \sim s_2 s_2 n^T$, so $s m^{\Sigma} \sim s n^T$. \end{proof}

The above result shows that strategies are closed under composition. We can prove that composition is associative and, consequently, obtain a category of games.

\section*{Proposition 8.} For all $\rho : A \rightarrow B, \sigma : B \rightarrow C$ and $\tau : C \rightarrow D$, $(\rho; \sigma; \tau) : \tau = \rho (\sigma; \tau)$.

\begin{definition} Given a class table $\Delta$, we define the category $G_\Delta$ having arenas as objects and strategies as morphisms. Identity morphisms are given by $id_{\Delta}$ for each arena $A$.

Note that neutrality of identity strategies easily follows from the definitions and, hence, $G_\Delta$ is well defined. In the sequel, when $\Delta$ can be inferred from the context, we shall write $G_\Delta$ simply as $G$. As a final note, for class tables $\Delta \subseteq \Delta'$, we can define a functor $\Delta/\Delta' : G_{\Delta} \rightarrow G_{\Delta'}$ which acts as the identity map on arenas, and sends each $\sigma : A \rightarrow B$ of $G_{\Delta}$ to:

$$(\Delta/\Delta')(\sigma) = \{s \in P_{AB}^\Delta \mid \exists t \in \sigma, s \leq t \}$$

where $P_{AB}^\Delta$ refers to plays in $G_{\Delta}$. In the other direction, we can define a strategy transformation:

$$(\Delta' \backslash \Delta')(\sigma) = \sigma \cap P_{AB}^{\Delta'}$$

which satisfies $\Delta' \backslash (\Delta \backslash (\Delta')(\sigma)) = \sigma$. \end{definition}

\section*{4. Soundness} Here we introduce constructions that will allow us to build a model of IMJ. We begin by defining a special class of strategies. A strategy $\sigma : A \rightarrow B$ is called \textbf{evaluated} if there is a function $f_\sigma : M_A \rightarrow M_B$ such that:

$$\sigma = \{m^\Delta m_B \in P_{AB} \mid m_B = f_\sigma(m_A)\}.$$ 

Note that equivariance of $\sigma$ implies that, for all $m_A \in M_A$ and permutations $\pi$, it holds that $\pi \cdot f_\sigma(m_A) = f_\sigma(\pi \cdot m_A)$. Thus, in particular, $\nu(f_\sigma(m_A)) = \nu(v(m_A))$.

Recall that, for arenas $A$ and $B$, we can construct a product arena $A \times B$. We can also define projection strategies:

$$\pi_1 : A \times B \rightarrow A = \{(m_A, m_B) \in P_{AB_A} \}$$

and, analogously, $\pi_2 : A \times B \rightarrow B$. Note that the projections are evaluated. Moreover, for each object $A$,

$$!_A = \{m_A^\Sigma \in \{m_A \in P_{A^1}\}$$

is the unique evaluated strategy of type $A \rightarrow 1$.

Given strategies $\sigma : A \rightarrow B$ and $\tau : \tau : A \rightarrow C$, with $\tau$ evaluated, we define:

$$\langle \sigma, \tau \rangle : A \rightarrow B \times C = \{m^\Sigma \in \{((m_A, m_B), m_\Sigma) \mid m_A^\Sigma \in \sigma \} \}$$

where we write $m[n^\tau]$ for the sequence obtained from $s$ by replacing any occurrences of $m_\tau$ in it by $m^\tau$ (note that there can be at most one occurrence of $m_\tau$ in $s$).

The above structure yields products for evaluated strategies.

\section*{Lemma 9.} Evaluated strategies form a wide subcategory of $\mathcal{G}$ which has finite products, given by the above constructions. Moreover, for all $\sigma : A \rightarrow B$ and $\tau : A \rightarrow C$ with $\tau$ evaluated, $\langle \sigma, \tau \rangle : A \rightarrow B \times C = \{m^\Sigma \in \{((m_A, m_B), m_\Sigma) \mid m_A^\Sigma \in \sigma \} \}$ where $\pi_1$ is the isomorphism $\langle \pi_2, \pi_1 \rangle$. The above represents a notion of \textbf{left-pairing} of $\sigma$ and $\tau$, where the effects of $\sigma$ precede those of $\tau$. We can also define a \textbf{left-tensor} between strategies:

$$\sigma \times \tau = A \times B \langle \sigma_1, \sigma_2 \rangle A' \times B \langle \tau_1, \tau_2 \rangle A' \times B'$$

for any $\sigma : A \rightarrow A'$ and $\tau : B \rightarrow B'$. \end{lemma}

\section*{Proposition 10.} Let $\tau : A' \rightarrow A, \sigma : A \rightarrow B_1, \tau : A \rightarrow B_2, \sigma_1 : B_1 \times B_2 \rightarrow C_1$ and $\sigma_2 : B_2 \rightarrow C_2$, with $\tau$ and $\tau$ evaluated. Then $\langle \sigma_1, \sigma_2 \rangle, \langle \sigma_1, \tau_2 \rangle, \tau_1 : \tau : \tau_1 : \tau : \tau_2$.

\begin{proof} The result follows from the simpler statements:

$$\langle \sigma, \id \rangle = \langle \tau, \sigma \rangle, \langle \sigma, \id \rangle = \langle \sigma, \id \rangle, \langle \sigma', \tau_2 \rangle = \langle \sigma, \id \rangle, \langle \sigma', \id \rangle$$

for all appropriately typed $\sigma, \sigma', \tau$, with $\tau$ evaluated, and Lemma 9. \end{proof}\end{document}
Lemma 11. For all strategies $\sigma' : A' \to A$ and $\sigma, \tau : A \to B$,
- if $\sigma' \triangleright \sigma$, then $\langle \sigma', \tau \rangle = \tau$;
- if $\sigma'$ is equivalent then $\langle \sigma, \tau \rangle = \sigma$;
- if $\sigma'$ is evaluated then $\langle 1 \, \triangleright \, \sigma' \rangle ; \sigma, \tau = \langle \sigma', \sigma, \tau \rangle$.

Method definitions in IMJ induce a form of exponentiation:
$$\bigwedge_{n=1}^N (\Delta \cup \delta | \theta) \mapsto M : \Theta$$
the modelling of which requires some extra semantic machinery. Traditionally, in call-by-value game models, exponentiation leads to 'effectless' strategies, corresponding to higher-order value terms. In our case, higher-order values are methods, manifesting themselves via the objects they may inhabit. Hence, exponentiation necessarily passes through generation of fresh object names containing these values. These considerations give rise to two classes of strategies introduced below.

We say that an even-length play $s \in P_{AB}$ is total if it is either empty or $s = m_{AB}^s \times m_{AB}^T s'$ and:
- $T \in \text{Sto}_0$ and $\nu(m_B) \cap \nu(S) \subseteq \nu(m_A)$,
- if $s' = m_{AB}^s \times m_{AB}^T T$ and $a \in \text{dom}(S) \cap \nu(\gamma(m_{AB}^s \times m_{AB}^T s'))$, for $\Sigma_0 \in \text{Sto}_0$ such that $\nu(m_{AB}^s \times m_{AB}^T s') \in P_{AB}$, then $\phi \notin \nu(n)$ and $T'(a) = \Sigma'(a)$.

We write $P_{AB}^s$ for the set of total plays in $A B$. Thus, in total plays, the initial move $m_A$ is immediately followed by a move $m_B$, and the initial store $S$ is invisible to $P$ in the sense that $P$ cannot use its names nor their values. A strategy $\phi : A \to B$ is called single-threaded if it consists of total plays and satisfies the conditions:
- for all $m_A^s \in P_{AB}$ there is $m_A^s \times m_B^T s \in \phi$;
- if $m_A^s \times m_B^T s \in \phi$, then $m_A^s \times m_B^T T$ and $a \in \text{dom}(S)$ such that $\gamma(m_A^s \times m_B^T s') \in P_{AB}$, then $\phi \notin \nu(n)$ and $T'(a) = \Sigma'(a)$.

Thus, single-threaded strategies reply to every initial move $m_A^s$ with a move $m_B^T$ which depends only on $m_A^s$ (i.e. $P$ does not read before playing). Moreover, $m_A^s$ does not change the values of $S$ ($P$ does not write) and may introduce some fresh objects, albeit with default values. Finally, plays of single-threaded strategies consist of just one thread, where a thread is a total play in which there can be at most one call to names introduced by its second move.

Conversely, given a total play starting with $m_A^s \times m_B^T s'$ we can extract its threads by tracing back for each move in $s$ the method call of the object $a \in \nu(T)$ it is related to. Formally, for each total play $s = m_A^s \times m_B^T s'$ with $|s'| > 0$, the thread move of $s$, written $\text{thr}(s)$, is given by induction:
- $\text{thr}(s \times m_B^{s''}) = \text{thr}(s')$, if $p(m) = P$;
- $\text{thr}(s \times a \cdot m(\overline{v})^{s''}) = a \cdot m(\overline{v})^{s''}$, if $a \in \nu(T)$;
- $\text{thr}(s \times n^{s''}) = \text{thr}(s \times m_B^{s''})$, if $a \in \nu(T)$ and $n$ introduces $a$.

Then, the current thread $s$ is the sequence of $s$ containing only moves with the same thread move as $s$, that is, if $\text{thr}(s) = m_B^{s''}$ and $s = m_A^s \times m_B^T s'$ then
$$|s| = m_A^s \times m_B^T \langle s' \mid m_B^{s''} \rangle$$

Note that the use of the term "thread" here is internal to game semantics parlance and in particular should not be confused with Java threads.

where the restriction retains only those moves $n^{s''}$ of $s'$ such that $\text{thr}(n^{s''}) = m_B^{s''}$. We extend this to the case of $|s| = 2$ by setting $[s] = s$. Finally, we call a total play $s \in P_{AB}$ thread-independent if for all $s' \times m_B^{s''} \subseteq \text{eoeos} s$ with $|s'| > 2$:
- if $\gamma([s'^{-}m_B^{s''}]) = s'^{-}m_B^{s''}$ then $\nu(S) \cap \nu(s') \subseteq \nu(s')$;
- if $s'$ ends in some $n^{s''}$ and $a \in \text{dom}(S) \cap \nu(\gamma([s'^{-}m_B^{s''}]))$ then $\Sigma'(a) = T'(a)$.

We write $P_{AB}^s$ for the set of thread-independent plays in $A B$.

We can now define strategies which occur as interleavings of single-threaded ones. Let $\phi : A \to B$ be a single-threaded strategy. We define:
$$\phi^1 = \{ s \in P_{AB}^s \mid \forall s' \subseteq \text{eoeos} s, \gamma([s']) \in \phi \}.$$

Lemma 12. $\phi^1$ is a strategy, for each single-threaded $\phi$.

Proof. Equivariance. Even-prefix closure and O-extension follow from the corresponding conditions on $\phi$. For determinacy, if $s m_{AB}^s \times m_B^{s''} \in \phi^1$ with $|s| > 0$ then, using determinacy of $\phi$ and the fact that $P$-moves do not change the current thread, nor do they modify or use names from other threads, we can show that $s m_{AB}^s \sim s n^{s''}$.

We say that a strategy $\sigma$ is thread-independent if $\sigma = \sigma^1$ for some single-threaded strategy $\sigma$. Thus, thread-independent strategies do not depend on initial stores and behave in each of their threads in an independent manner. Note in particular that evaluated strategies are thread-independent (and single-threaded).

Lemma 13. Let $\sigma : A \to B$ and $\tau : A \to C$ be strategies with $\tau$ thread-independent. Then, $(\sigma, \tau) ; \pi_1 = \sigma$ and $\langle \sigma, \tau \rangle = A \times B \subseteq P_{A \times B} \subseteq B \times C$.

Proof. The former claim is straightforward. For the latter, we observe that the initial effects of $\sigma$ and $\tau$ commute: on initial move $m_{AB}^s \times m_B^T$, $\tau$ does not read the store updates that $\sigma$ includes in its response $m_{BC}^s$, while $\sigma$ cannot access the names created by $\tau$ in its second move $m_{BC}^s$.

It is worth noting that the above lemma does not suffice for obtaining categorical products. Allowing thread-independent strategies to create fresh names in their second move breaks universality of pairings. Considering, for example, the strategy:
$$\sigma : 1 \to I \times I = \{ \langle a, a \rangle \mid a \in P_{I \times I} \mid \Sigma \in \text{Sto}_0 \}$$
we can see that $\sigma \neq \langle \sigma, \pi_1, \pi_2, \pi_2 \rangle$, as the right-hand-side contains plays of the form $\langle a, (a, b) \rangle$ with $a \neq b$.

We can now define an appropriate notion of exponential for our games. Let us assume a translation assigning an arena $[\theta]$ to each type sequence $\theta$. Moreover, let $I$ be an interface such that
$$\Delta(I) \mid \text{Meas} = \{ m_0 : \theta_1 \to \theta_1, \ldots, m_n : \theta_n \to \theta_n \}$$
where $\theta_1 = \theta_{i_1}, \ldots, \theta_{i_m}$, for each $i$. For any arena $A$, given single-threaded strategies $\phi_1, \ldots, \phi_n : A \to I$ such that, for each $i$, if $m_{AB}^s \times m_B^{s''} \in \phi_i$ then
$$\phi_i \notin \nu(S) \cap \nu(T(a)) : A \times (a \cdot m(\overline{v})) \subseteq \text{eoeos} m_{AB}$$
we can group them into one single-threaded strategy:
$$\langle \phi_1, \ldots, \phi_n \rangle : A \to \bigcup_{i=1}^n \phi_i.$$
Note that the $a$ above is fresh for each $m_{AB}^s$ (i.e. $a \notin \nu(m_{AB}^s)$).
Let now \( \sigma_1, \ldots, \sigma_n \) be strategies with \( \sigma_i : A \times [\vec{\theta}_i] \to [\theta_i] \). For each \( i \), we define the single-threaded strategy \( \Lambda(\sigma_i) : A \to I : \\
\Lambda(\sigma_i) = \{(m, \vec{\theta}) \in P_{\Lambda_i}^T | \gamma((m, \vec{\theta})^{\sigma_i}) \in \sigma_i \} \\
\cup \{(m, \vec{\theta}) \in P_{\Lambda_i}^T | \text{call a.m.(v)}^{\sigma_i} \text{ ret a.m.(v)}^{T} \text{ s' \in P}_{\Lambda_i}^T | \\
\gamma((m, \vec{\theta})^{\sigma_i} s^{T} s') \in \sigma_i \} \cup \{(m, \vec{\theta}) \in P_{\Lambda_i}^T \}
\)
where \( a \notin \nu(\Sigma, \bar{v}, t, s, \Sigma', T') \) and \( T(a) : I \). By definition, \( \Lambda(\sigma_i) \) is single-threaded. Therefore, setting 
\( 
\Lambda(\sigma_1, \ldots, \sigma_n) = (\langle \Lambda(\sigma_1), \ldots, \Lambda(\sigma_n) \rangle)^T : A \to I,
\)
we obtain a thread-independent strategy implementing a simultaneous currying of \( \sigma_1, \ldots, \sigma_n \). In particular, given translations \([M_i]\) for each method in a method-set implementation \( M \), we can construct: 
\( [M] : \Gamma \to I = \Lambda([M_1], \ldots, [M_n]). \)

Finally, we define evaluation strategies \( ev_m : I \times \vec{\theta} \to [\theta] \) by (taking even-length prefixes of):
\( ev_m = \{ (a, \vec{v})^\Sigma \text{ call a.m.(v)}^{\sigma} \text{ ret a.m.(v)}^{T} \text{ s' \in P}_{A_i} | (\Sigma(a) \leq I) \}
\)
where \( A_i = (I \times \vec{\theta})[\vec{\theta}]. \) We can now show the following natural mapping from groups of strategies in \( A \times [\vec{\theta}] \to [\theta] \) to thread-independent ones in \( A \to I \).

**Lemma 14.** Let \( \sigma_1, \ldots, \sigma_n \) be as above, and let \( \tau : A' \to A \) be evaluated. Then,
\( \bullet \Lambda(\sigma_1, \ldots, \sigma_n) \times \text{id} : \Gamma \to \theta \)
\( \bullet \Lambda((\tau \times \text{id}) \sigma_1, \ldots, (\tau \times \text{id}) \sigma_n). \)

Apart from dealing with exponentials, in order to complete our translation we need also to address the appearance of \( x : I \) in the rule.
\( \Gamma, x : I, \Delta \vdash M : \Theta \\
\Gamma, \Delta \vdash \text{new}(x : I, M) : I \\
\Delta(\text{Methods}=\Theta), \Delta' \)
Recall that 
\( [M] : [\Gamma] \times \vec{\theta} \to I \)

(1) is obtained using exponentiation. Thus, the second move of \([M]\) will appear in the right-hand-side \( I \) above and will be a fresh name \( b \) which will serve as a handle to the methods of \( M \); in order to invoke \( m : \Sigma, M \) on input \( \bar{v} \), the Opponent would have to call \( b.m(\bar{v}). \) The remaining challenge is to merge the two occurrences of \( I \) in (1). We achieve this as follows. Let us assume a well-formed extension \( \Delta' \) of \( \Delta: \)
\( \Delta' = (\Gamma' : (f' : I)), \Delta \)
that is, \( \Delta' \) contains a single field \( f' \) of type \( I \). We next define the strategy \( \kappa_{\Sigma} : I \to \Gamma' \times \vec{\theta} \) of \( G_{\Delta}: \)
\( \kappa_{\Sigma} = \{ (a, \alpha)^{\Sigma_0} \text{ call a.m.(v)}^{\Sigma_0} \text{ ret a.m.(v)}^{T} \text{ s' \in P}_{\Lambda_i} | (\Sigma_0(a) \leq I) \}
\)
where \( m \in \text{dom}(\Delta(\Sigma)), \) \( b = \Sigma_0(a), f' \), and \( \Sigma_0 \in \text{ST}(b, a) \) is such that \( \Sigma_0(a) : I \) and \( \Sigma_0(a') : I'. \) We let \( \text{new}(x : I, M) \) be the strategy:
\( \Gamma' \mid \text{id}, \Sigma(a) : I \to \Gamma \times \vec{\theta} \mid \text{id} \times [M][\pi_2], \Gamma \times \vec{\theta} \times \vec{\theta} \) \( \text{asn}_{\Sigma} \times \text{id} \mid \text{id} \times [M][\pi_2], \Gamma \times \vec{\theta} \times \vec{\theta} \)
and \( \text{asn}_{\Sigma} \times \text{id} \) is the assignment strategy:
\( \text{asn}_{\Sigma} : I \times \vec{\theta} \to 1 = \{(a, \alpha)^{\Sigma_0} \text{ s' \in P}_{\Lambda_i} \} \)

\( \text{Note:} \) \( x \) may appear free in \( M_i; \) it stands for the keyword this of Java.

\( \text{Here we omit wrapping } [M] \text{ inside } \Delta/\Delta', \text{ as well as wrapping the whole } \text{new}(x : I, M) \text{ in } \Delta'/\Delta, \text{ for conciseness.} \)
Figure 4. The semantic translation of IMJ.

commute (we write A for \(\Gamma \times \overrightarrow{\mathcal{T}}\)).

\[
\begin{align*}
\Gamma \vdash \text{let } x = M' \text{ in } \theta : [\Gamma \times \overrightarrow{\mathcal{T}}] &\rightarrow [\Gamma \times \overrightarrow{\mathcal{T}}] \times \overrightarrow{\mathcal{T}}. \\
\Gamma \vdash \text{let } x = M' \text{ in } \theta : [\Gamma \times \overrightarrow{\mathcal{T}}] &\rightarrow [\Gamma \times \overrightarrow{\mathcal{T}}] \times [\Gamma \times \overrightarrow{\mathcal{T}}]. \\
\end{align*}
\]

where \(\text{null} : 1 \rightarrow \{\ast \text{null}\}\).

\[
\begin{align*}
\Gamma \vdash \text{let } x = M' \text{ in } \theta : [\Gamma \times \overrightarrow{\mathcal{T}}] &\rightarrow [\Gamma \times \overrightarrow{\mathcal{T}}] \times [\Gamma \times \overrightarrow{\mathcal{T}}]. \\
\end{align*}
\]

that the copy of \(M_1\) differs from the original just in the handle name (the one returned in the codomain of \([M_1]\)), but the latter is hidden via composition with \(\mathsf{ev}_m\). The latter diagram stipulates that if we create \(\bar{a}\) with methods \(\bar{M}\), then calling \(a_i\) on \(m\) is the same as calling \(M_i\) on \(m\). The latter holds because of the way that \(\mathsf{as}_{\ell}\) manipulates calls inside the interaction, by delegating calls to methods of \(a_i\) to \(M_i\).

**Proposition 16** (Computational Soundness). *For all \(\vdash M : \text{void}, if \(M \nvdash \) then \([M] = \{\ast\} \text{ (i.e. } [M] = [\text{skip}])\).*

**Proof.** This directly follows from Correctness.

**Proposition 17** (Computational Adequacy). *For all \(\vdash M : \text{void}, if \([M] = \{\ast\} \text{ then } M \nvdash \).*

**Proof.** Suppose, for the sake of contradiction, that \([M] = \{\ast\}\) and \(M \nvdash \). We notice that, by definition of the translation for blocking constructs (castings and conditionals may block) and due to Correctness, if \(M \nvdash \) were due to some reduction step being blocked then the semantics would also block. Thus, \(M \nvdash \) must be due to divergence. Now, the reduction relation restricted to all rules but METHODCL is strongly normalising, as each transition decreases the size of the term. Hence, if \(M\) diverges then it must involve infinitely many METHODCL reductions and our argument below shows that the latter would imply \([M] = \{\epsilon\}\). For any term \(\Gamma \vdash N : \theta\) and \(a \in A \setminus \text{dom}(\Gamma)\), construct \(\Gamma_a \vdash N_a\), where \(\Gamma_a = \Gamma \uplus \{a : \text{Var}_j\}\), by recursively replacing each subterm of \(N\) of the shape \(N'.m(N)\) with \(a.f := (a.f + 1); N'.m(N)\).
$\text{Var}_f$ is an interface with a sole field $f : \text{int}$. Observe that each $s \in [\Gamma]_{+}^P$ induces some $s' \in [\Gamma_a]_{+}^P$ such that $a$ appears in $s'$ only in stores (and in a single place in the initial move) and $O$ never changes the value of $a$, while $P$ never decreases the value of $a$. If we write $[\Gamma_a]_{+}^P$ for the subset of $[\Gamma_a]_{+}$ containing precisely these plays. Then, take $M_0$ to be the term $\text{let } x = \text{new}(x : \text{Var}_f) \text{ in } (M_0 \langle x/a \rangle ; x.f)$, where $x$ a fresh variable. Because $\star \in [\Gamma_0]$, we get $s \in [\Gamma_0]$ for some $j \in \mathbb{Z}$. Consider now the infinite reduction sequence of $(\theta, M)$. It must have infinitely many $\text{MethodCL}$ steps, so suppose $(\theta, M) \rightarrow (\theta, M') \rightarrow \cdots$ $(S, M')$ contains $j + 1$ such steps. Then, we obtain $(\theta, M) \rightarrow (\theta, M') \rightarrow \cdots$ $(S_a, (M')_a ; a.f)$, with $S_a(a.f) = j + 1$. By Correctness, we have that $s \in \text{Var}_f$. Hence, it follows that $j \in [\Gamma_a]_{+}^P$ and $j \geq 0$. Since in $[\Gamma_a]_{+}^P$, the value of $a$ cannot decrease, and its initial value is $j + 1$ (as stipulated by $S_a$), we reach a contradiction.

5. Full Abstraction

Recall that, given plays $s, s'$, we call $s$ an $O$-extension of $s'$ (written $s \leq_O s'$) if $s$, $s'$ are identical except the type information regarding $O$-names present in stores: the types of $O$-names in $s$ may be subtypes of those in $s'$. We shall write $s \leq s'$ for the dual notion involving $P$-names, i.e., $s \leq s'$ if $s$, $s'$ are the same, but the types of $P$-names in $s'$ may be subtypes of those in $s$. Then, given $X \in \{O, P\}$ and fixed $A, B$, let us define $cl_X (s) = \{s' \in P_X \mid s' \leq s \} \cap cl_X (s)$. We write $P_{A \rightarrow B}$ for $P_{A \rightarrow B}$. A play will be called complete if it is of the form $m_1 \cdot Y m_2 \cdot Y$.

Next we establish a definability result stating that any complete play (together with other plays implied by $O$-closures) originates from a term.

**Lemma 18 (Definability).** Let $s \in P_{\Delta' \rightarrow \theta}$ be a complete play. There exists $\Delta' \supseteq \Delta$ and $\Delta' [\Gamma] \equiv M : \theta$ such that $[\Delta' [\Gamma] \equiv M : \theta] = \text{cl}_O (s)$.

**Proof.** The argument proceeds by induction on $|s|$. For $s = \epsilon$, any divergent term suffices. For example, one can take $\Delta' = \Delta \cup \{\text{Div} \rightarrow (m : \text{void} \rightarrow \text{void})\}$, and pre-compose any term of type $\theta$ with new($x : \text{Div} ; m : \lambda (\cdot) (\cdot) (\cdot)$).

Suppose $s \neq \epsilon$. Then the second move can be a question or an answer. We first show how to reduce the former case to the latter, so that only the latter needs to be attacked directly.

Suppose $s = q^S a \cdot \text{call}\ m(u)^{\Sigma_1} \cdot s_1 \cdot \text{ret}\ m(v)^{\Sigma_2} \cdot s_2 \cdot \Sigma_3 \cdot s_3$,

where $o : \mathcal{T}' \rightarrow (\Delta(\mathcal{T}')) (m) : \mathcal{T}_l \rightarrow \mathcal{T}_r$. Consider $\Delta' = \Delta \cup \{\mathcal{T}' \rightarrow (\mathcal{T}_l, m : \mathcal{T}_r) \rightarrow \theta\}$ and the following play from $P_{\Delta' \rightarrow [\Sigma_1]^{\mathcal{T}_l \rightarrow \mathcal{T}_r}}$:

$s' = q^S o \cdot p^S s' \cdot s_1 \cdot \text{call}\ p.m(u)^{\Sigma_2} \cdot s_2 \cdot \text{ret}\ p.m(v)^{\Sigma_3} s_3$, where $p \notin \{o, s\} \cdot \Sigma_1 = \Sigma \cup \Sigma_2 \cdot \Sigma = \{p \rightarrow \Sigma_1 (f', f' \rightarrow u)\}$ and $s_j$ is the same as $s_j$ except that each store is extended by $\Sigma$. If $\Delta' [\Gamma] \equiv M : \mathcal{T}$ satisfies the Lemma for $s$ then, for $s$, one can take $\Delta' = \Delta' \cup \{\mathcal{T}_l \rightarrow (\mathcal{T}_l, m : \mathcal{T}_r) \rightarrow \theta\}$ and the following play in $P_{\Delta' \rightarrow [\Sigma_1]^{\mathcal{T}_l \rightarrow \mathcal{T}_r}}$:

Then to the reduction given above we can now assume that $s \in P_{\Delta' \rightarrow \theta}$ is non-empty and

$s = q^S \Sigma_1 \cdot m_0^{\Sigma_1} \cdot m_1^{\Sigma_1} \cdot \cdots m_{2k}^\Sigma$, where $m_0$ is an answer. We are going to enrich $s$ in two ways so that it is easier to decompose. Ultimately, the decomposition of $s$ will be based on the observation that the $m_{1}^{\Sigma_1} \cdots m_{2k}^{\Sigma_1}$ segment can be viewed as an interleaving of threads, each of which is started by a move of the form call $p$ for some P-name $p$. A thread consists of the starting move and is generated according to the following two rules: $m_0$ belongs to the thread of $m_{2i+1}$ and every answer-move belongs to the same thread as the corresponding question-move.

- The first transformation of $s$ brings forward the point of P-name creation to the second move. In this way, threads will never create objects and, consequently, it will be possible to compose without facing the problem of object fusion.

Suppose $P(s) = \Sigma_0$ and $p_i : \Sigma_0$. Let $\Delta' = \Delta \cup \{\mathcal{T}_p \rightarrow f, f_i : \mathcal{T}_p\}$. Consider $s' = (n, q)^{\Sigma_0} \cdot m_0^{\Sigma_0} \cdot m_1^{\Sigma_0} \cdots m_{2k}^{\Sigma_0}$, where $\Sigma_0 = \Sigma_0 \cup \{n \rightarrow \mathcal{T}_p, (\mathcal{T}_p, p_i) \equiv p_i \rightarrow \mathcal{T}_p(p_i, \text{null}) | \mathcal{T}_p(p_i) \text{ undefined}, p_i \in P(s)\}$.

- The second transformation consists in storing the unfolding play in a global variable. It should be clear that the recursive structure of types along with the ability to store names is sufficient to store plays in objects. Let $\mathcal{T}_{play}$ be a signature that makes this possible. This will be used to enforce the intended interleaving of threads after their composition (in the style of Innocent Factorization [5]). Let $\Delta = \Delta' \cup \{\text{History} \rightarrow \text{play} : I_{play}\}$ and $\Delta'' = \{x_b : \text{History} \rightarrow \theta\}$.

Suppose $\Delta'' \equiv (\text{History}, \text{play} \rightarrow \text{null})$,

$\Sigma_{2i+1} \equiv \{h \rightarrow (\text{History}, \text{play} \rightarrow \text{null})\}$.

Observe that $s'' \in P_{\Delta'' \rightarrow \theta}$.

Now we shall decompose $m_1^{\Sigma_1} \cdots m_{2k}^{\Sigma_1}$ into threads. Recall that each of them is a subsequence of $s''$ of the form

\[ call\ p.m(u)^{\Sigma_0} \rightarrow t \cdot \text{ret}\ p.m(v)^{\Sigma_r}, \]

where the segment $t$ contains moves of the form call $o$ or ret $o$ for some $o \in O(s)$. We would now like to invoke the IH for each thread but, since a thread is not a play, we do so for the closely related play $(h, n, q, u)^{\Sigma_0} \cdot t' \cdot \text{v} \cdot \Sigma_r$. Let us call the resultant term $M_{s' \rightarrow \Sigma_r}$. Next we combine terms related to the same $p : I_{play}$ into an object definition by

\[ M_p = \text{new}(x : I_{play} ; m : \lambda u.\text{case}(u, \Sigma_0)(M_{(p,m,u,\Sigma_0)})) \]

The case statement, which can be implemented in IMJ using nested if’s, is needed to recognize instances of $\Sigma_0$ and $\Sigma_r$ that really occur in stores related to $p$. In such cases the corresponding term $M_{(p,m,u,\Sigma_0)}$ will be run. Otherwise, the statement leads to divergence.

The term $M$ for $s$ can now be obtained by taking

\[ \text{let } x_n = \text{new}(x : I_{play}) \text{ in } \text{let } x_h = \text{new}(x : \text{History}) \text{ in } \text{let } x_p = M_p \text{ in } \text{assert}(q^{\Sigma_r}) ; \text{let } f_i = x_p ; \text{make}(\Sigma'_0) ; \text{play}(m_0) \]

where $M_p$ represents a series of bindings (one for each P-name $p_i \in P(s)$, $\text{assert}(h, n, q, u)^{\Sigma_0}$) is a conditional that converges if and only if the initial values of free $\Gamma$ identifiers as well as values accessible through them are consistent with $q$ and $\Sigma_r$ respectively, $\text{make}(\Sigma'_0)$ is a sequence of assignments that set values to those specified in $\Sigma'_0$ (up-casts need to be performed to ensure typability) and $\text{play}(m_0)$ is stop, i, null or, if $m_0$ is a name,
it is a term of the form \((\theta)y_1f\), where \(y_1\) is \(x_n\) or \((x : I_1) \in \Gamma\) such that \(y_1f\) gives an access path to \(m_0\) in \(\Sigma_0\).

We conclude with full abstraction results both in inequational and equational forms. For technical convenience, we shall use a modified (but equivalent) definition of contextual approximation.

**Lemma 19.** Let \(\Gamma = \{x_1 : I_1, \ldots, x_k : I_k\}\), \(\Delta[\Gamma] \vdash M : \theta (i = 1, 2)\), and \(\Delta' = \Delta \cup \{\text{Wrap}_{P_1, P_2} : f \mapsto (f(x_1, \ldots, x_k) \Rightarrow \theta)\}\). Then \(\Delta[\Gamma] \vDash M_1 \bowtie M_2\) if and only if, for all \(\Delta'' \supseteq \Delta'\) and \(\Delta''' : z : \text{Wrap}_{P_1, P_2} \vdash \text{test} : \text{void},\) if \(\text{Test}[M_1] \Downarrow\) then \(\text{Test}[M_2] \Downarrow\), where \(\text{Test}[\cdot] \equiv \text{let } z = \text{new}(x : \text{Wrap}_{P_1, P_2}; f : \lambda x[\Delta''](-))\) in \(\text{test}\).

**Proof.** The Lemma holds because, on the one hand, it relies on contexts of a specific shape and, on the other hand, any closing context \(\text{Test}[\cdot]\) for \(M\) can be presented in the above form with \(\text{Test}[\cdot] \equiv C[z, f(x_1, \ldots, x_k)]\).

Given a term \(\Delta[\Gamma] \vdash M : \theta\), let us write \(\Delta[\Gamma] \vdash M : \theta\) for the set of complete plays from \(\{\Delta[\Gamma] \vdash M : \theta\}\). In what follows, we shall often omit \(\Delta[\Gamma] \vdash \theta\) for brevity.

**Theorem 20 (Equational full abstraction).** Given \(\Delta[\Gamma] \vdash M : \theta (i = 1, 2)\), we have \(\Delta[\Gamma] \vDash M_1 \bowtie M_2\) if and only if \(\text{cl}_P([\Delta[\Gamma] \vdash M_1 : \theta]) \subseteq \text{cl}_P([\Delta[\Gamma] \vdash M_2 : \theta])\).

**Proof.** The proof uses the following play transformation. Given \(t = q^*s_1\alpha S_2s_2 \in P_{\Delta[\alpha]}\), we define \(t \in P_{\Delta\vdash \text{Wrap} \vdash \text{void}}\) as

\[
n_{\Sigma} \triangleright n. f (q)^{\Sigma_n} \triangleright \Sigma_n s_1 \triangleright \Sigma_n \triangleright \Sigma_n \triangleright \Sigma_n,
\]

where \(\Delta\), \(\text{Wrap}_{P_1, P_2}\) are the same as in the above Lemma, \(\Sigma_n = (\{n \rightarrow (\text{Wrap}_{P_1, P_2}, \emptyset)\}, \Sigma_n \triangleright \Sigma_n\) stands for \(s\) in which every store was augmented by \(\Sigma_n\) and \(\Sigma_n\) is the store of the last move in \(t\). Intuitively, \(t\) is the play that \(\text{Test}[\cdot]\) needs to provide for a terminating interaction with \(\theta\).

**Example 3.** Let us revisit Example 2. We have \(\text{cl}_P(\sigma_\emptyset) = \sigma_\emptyset\) and \(\text{cl}_P(\sigma_2) = \sigma_2 \cup \{s, n_{\emptyset}(\text{empty}, \emptyset)\}\), i.e. \(\text{cl}_P(\sigma_\emptyset) \subseteq \text{cl}_P(\sigma_2)\). Thus, it follows from Theorem 20 that \(\Delta[\emptyset] \vdash M_1 \bowtie M_2\) if and only if \([\Delta[\emptyset] \vdash M_1 : \emptyset] \subseteq [\Delta[\emptyset] \vdash M_2 : \emptyset]\).

**Theorem 21 (Equational full abstraction).** Given \(\Delta[\Gamma] \vdash M_i : \theta (i = 1, 2)\), \(\Delta[\Gamma] \vdash M_1 \bowtie M_2\) if and only if \([\Delta[\Gamma] \vdash M_1 : \theta] \subseteq [\Delta[\Gamma] \vdash M_2 : \theta]\).

**References**