Option pricing with generalized continuous time random walk models

Chao Li

School of Mathematical Sciences

Queen Mary University of London

Submitted in partial fulfilment of the requirements

of the Degree of Doctor of Philosophy

May 2016

I, Chao Li, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material.

I accept that the College has the right to use plagiarism detection software to check the electronic version of the thesis.

I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university.

Signature:

Date:

Abstract

The pricing of options is one of the key problems in mathematical finance. In recent years, pricing models that are based on the continuous time random walk (CTRW), an anomalous diffusive random walk model widely used in physics, have been introduced. In this thesis, we investigate the pricing of European call options with CTRW and generalized CTRW models within the Black-Scholes framework. Here, the non-Markovian character of the underlying pricing model is manifest in Black-Scholes PDEs with fractional time derivatives containing memory terms. The inclusion of non-zero interest rates leads to a distinction between different types of "forward" and "backward" options, which are easily mapped onto each other in the standard Markovian framework, but exhibit significant differences in the non-Markovian case. The backward-type options require us in particular to include the multi-point statistics of the non-Markovian pricing model. Using a representation of the CTRW in terms of a subordination (time change) of a normal diffusive process with an inverse Lévy-stable process, analytical results can be obtained. The extension of the formalism to arbitrary waiting time distributions and general payoff functions is discussed. The pricing of path-dependent Asian options leads to further distinctions between different variants of the subordination. We obtain analytical results that relate the option price to the solution of generalized Feynman-Kac equations containing non-local time derivatives such as the fractional substantial derivative. Results for Lévy-stable and tempered Lévy-stable subordinators, power options, arithmetic and geometric Asian options are presented.

Contents

Acknowledgement **9**

1 Introduction **10**

2 Fundamentals of stochastic processes **16**
2.1 Brownian motion in physics 16
2.1.1 General diffusion processes 22
2.2 Anomalous diffusion 30
2.2.1 The continuous time random walk 30
2.3 Brownian motion in finance 33
2.3.1 Geometric Brownian motion 34
2.3.2 Black-Scholes option pricing theory 38

3 Generalized CTRW models **45**
3.1 Introduction to fractional calculus 45
3.1.1 Special functions: Gamma and Mittag-Leffler functions 45
3.1.2 The Riemann-Liouville fractional integral 47
3.1.3 The Riemann-Liouville fractional operator 49
3.2 CTRW with x–dependent drift and diffusion 50
3.2.1 Lévy processes 51
3.2.2 The inverse one-sided Lévy stable process $S(t)$ 59
3.2.3 The two-point PDF of the process $S(t)$ 62
3.2.4 The fractional Fokker-Planck equation for the PDF of $Y(t)$ . . . 65
3.3 Subordinated process with general waiting times 67
3.3.1 Single point PDF and fractional Fokker-Planck equation 67
3.3.2 The two-point PDF of the process $S(t)$ 69
3.4 Asset pricing models beyond geometric Brownian motion 71

3.4.1 Subdiffusive geometric Brownian motion 71
3.4.2 Subordinated geometric Brownian motion 75
3.5 Evidence for CTRW-type pricing models from financial data 78
3.6 Numerical simulation of sample path 83

4 European call option pricing formula 87
4.1 *Type A* option cost in subdiffusive regime 91
4.1.1 Subdiffusive *type A* option cost 1 92
4.1.2 Subdiffusive *type A* option cost 2 96
4.2 *Type B* option cost in subdiffusive regime 103
4.2.1 Subdiffusive *type B* option cost 1 103
4.2.2 Subdiffusive *type B* option cost 2 104
4.3 Summary of chapter 110

5 Subdiffusive European call option pricing formula with subordinated GBM 111
5.1 Subdiffusive *type A* option cost 1 111
5.2 Subdiffusive *type A* option cost 2 115
5.3 Subdiffusive *type B* option cost 1 117
5.4 Subdiffusive *type B* option cost 2 118
5.5 Summary of chapter 124

6 General subdiffusive call option formula with arbitrary payoffs function 127
6.1 General subdiffusive call option formula with an arbitrary payoff function 128
6.1.1 Subdiffusive formula type 1 128
6.1.2 Subdiffusive formula type 2 129
6.2 Subdiffusive power option formula 130
6.2.1 Normal power option formula 131
6.2.2 Subdiffusive power option 1 132
6.2.3 Subdiffusive power option 2 134
6.2.4 Different cases of waiting times 135
6.3 Summary of chapter 140

7 Path dependent call options 141
7.1 The conventional Asian call option 141
7.2 Subdiffusive versions of the Asian call option 143

7.3 Summary of chapter 149

8 Concluding remarks and outlook **150**

Bibliography **153**

List of Abbreviations **166**

List of Figures

1.1 Subdiffusive data in financial market 13

2.1 Clarkia pulchella pollen image 17
2.2 Schematic representation of a random walk in two dimension. 18
2.3 Overview of Perrins work on Brownian motion 21
2.4 Erroneous behaviour of Brownian motion 22
2.5 Sample paths of Brownian motion 25
2.6 Schematic representation of a continuous random walk in two dimension 31
2.7 The data of the price in the financial market 35

3.1 Sample realizations of the GBM and subdiffusive GBM 72
3.2 The density function of subdiffusive GBM 74
3.3 Sample realizations of GBM and subordinated GBM 75
3.4 The density function of subordinated GBM for different α 77
3.5 The density function of subordinated GBM for different ζ 77
3.6 The cumulative distribution of $N(t)$ 79
3.7 Survival probability for BUND futures 81
3.8 Survival probability for the General-Electric shares 81
3.9 Average survival probability function for Irish stock market data 82
3.10 Mean and second moment of subdiffusive GBM 84
3.11 Mean and second moment of subordinated GBM for different α 85
3.12 Mean and second moment of subordinated GBM for different ζ 86

4.1 Illustration of Markovian and non-Markovian processes 88
4.2 Subdiffusive *type A* option cost 99
4.3 Subdiffusive *type A* option cost $C_2^{(\mathrm{A})}(x,t)$ 103
4.4 Subdiffusive *type B* option cost 108

4.5 Subdiffusive *type B* option cost $C_2^{(\mathrm{B})}(x,T,t)$ 109

5.1 Analytic expression of subdiffusive *type A* option cost 121
5.2 The comparison of subdiffusive *type A* option cost 122
5.3 Analytic expression of subdiffusive *type B* option cost 123
5.4 The comparison of subdiffusive *type B* option cost 125
5.5 Analytic expression of subdiffusive *type B* option cost $C_2^{(\mathrm{B})}(x,T,t)$ . . . 126
5.6 The comparison of subdiffusive *type B* option cost $C_2^{(\mathrm{B})}(x,T,t)$ 126

6.1 Subdiffusive power option cost with α-stable waiting times 136
6.2 Subdiffusive power option cost with tempered stable waiting times . . . 137
6.3 The comparison of subdiffusive power option cost 139

Acknowledgement

I would like to thank

- My supervisor Dr. Adrian Baule. I have been lucky to have a supervisor with high standard toward research, during my study who is helping me to be familiar with the research life, reading my manuscript and for always taking his time to discuss with me.
- My supervisor Dr. Wolfram Just, for providing me with useful advices and for efforts which have substantially improved my research life, for the suggestions of structuring the manuscript and for always paying his patience to listen to me. His helpful guidance and encouragement has been of great value.
- Dr Rainer Klages, the previous director of postgraduate research studies in the School of Mathematical Sciences at Queen Mary, for spending his time listening to my "funny" problems.
- Prof M Farber, the director of postgraduate research studies in the School of Mathematical Sciences at Queen Mary, for facilitating my research life.
- Prof Christian Beck and Prof T Prellberg, the assessors of my research progress, for their scrutinity, criticism, and valuable suggestions which help me to make further probing.
- All current and previous members in the School of Mathematical Sciences at Queen Mary for giving me a happy, helpful, friendly and stimulating environment.
- My family and friends for the support and encouragement while I pursue the study abroad.
- The Chinese Scholarship Council (CSC) for providing the funding which allows me to undertake this research.

Chapter 1

Introduction

The fundamental purpose of any financial theory is the investigation of the behaviour of economic agents in allocating and deploying their resources, both over time and across places, in an uncertain condition [107]. The elements of time and uncertainty play an important role in influencing these financial behaviours. Due to the inherent complexity of the interactions between such agents, a sophisticated mathematical framework is required to quantify the effect of their interactions on the observed behaviour of markets, thus leading to the field of mathematical finance. The origin of modern mathematical finance could be traced back to Louis Bachelier's magnificent dissertation, which marks the birth of option pricing theory, using a stochastic process in continuous time. In analysing the problem of option pricing, Bachelier derives a mathematical model now known as the Wiener process or Brownian motion. However, Bachelier's work has remained unknown in finance for a long time. Indeed, during most of this period, the stochastic description of financial markets has been studied and discussed but mathematical models seem to have had little influence on practice.

Later, a variant of Brownian motion, known as geometric (economical) Brownian motion (GBM), has been put forward by Osborne [121] and Samuelson [143,144], eventually becoming an important model in finance. The GBM provides a reasonable pricing model, since its predicted share price values remain always positive avoiding the drawback of Bachelier's original model, as Samuelson asserts. Undoubtedly, the most vital development imposing impact on practice is the Black-Scholes (BS) theory for option pricing based on GBM which brings the field to closure on the subject [106]. For the contribution of the Black-Scholes model to the theory of option pricing, Robert Merton and

Myron Scholes were awarded the Alfred Nobel Memorial Prize in Economical Sciences in 1997 [16]. It was a great pity that Fischer Black could not receive the award since he died in 1995. Virtually almost at that time, the Chicago Board Options Exchange (CBOE) started trading the first options on individual stocks in the United States [80]. Moreover, Texas Instruments even introduced a hand-held calculator to specially produce BS option prices. Such a complete and rapid acceptance of financial theory into financial practice has been unprecedented. Then the BS approach has been extended to a wide variety of options such as caps, floors, collars, collateralized mortage obligations, knockout options, swaptions, lookback options, barrier options and so on [34]. Nowadays, with the creation of many kinds of new mathematical models and options, the options market has become one of the most attractive areas in the financial markets.

Even though the BS model achieved a great success, however, empirical studies on financial time series indicate that a simple Markovian process, namely geometric Brownian motion serving as the underlying asset pricing model in the BS theory, can not capture the complex behaviour of asset prices [87]. Indeed, there are some common properties across different markets and time periods, known as stylized empirical facts [31], including absence of autocorrelations, heavy tails, gain/loss asymmetry, aggregational gaussianity, intermittency, volatility clustering, conditional heavy tails, slow decay of autocorrelation in absolute returns, leverage effect, volume/volatility correlation, asymmetry in time scales, and so on. Since the accurate modelling of financial time series can greatly affect the evaluation of the option price, these empirical studies indicate that the empirically observed differences between actual asset prices and existing models should be taken into account and hence result into departing from the BS model in finance.

In 1963, Benoit Mandelbrot proposed one of the earliest alternatives to the GBM underlying the BS model, known as Lévy stable processes [97]. In his paper "The Variation of Certain Speculative Prices", Mandelbrot stated that "the empirical distributions of price changes are usually too 'peaked' to be relative to samples from Gaussian populations." He also claimed that "The tails of the distributions of price changes are in fact so extraordinarily long that the sample second moments typically vary in an erratic fashion." Although Mandelbrot's initial study examined the price changes of cotton and wool prices, the Lévy stable process was also assumed to be a possible model for the distribution of stock price returns. That assumption was confirmed by Fama when he examined the random nature of stock prices in 1965 [38]. In his research, Fama claimed that Lévy stable processes seem to fit the data better than GBM [95].

Later in 1995, from the study of a particular economic index (the Standard Poor's 500), a truncated Lévy stable distribution was proposed by Mantegna and Stanley to describe the price changes. More precisely, their work indicated that the central part could be predicted very well by a Lévy stable process but for the tails there is an exponential fall off [99]. Mantegna and Stanley claimed that truncated Lévy distributions could be used to model a broad spectrum of phenomena ranging from turbulence to financial markets. Then Koponen was inspired to derive an analytical form for the characteristic function of a truncated Lévy distribution with an exponential cutoff in the tails [81]. Some empirical studies support the claim that the truncated Lévy distribution is a simple and effective model of financial data [59, 100].

Overall, these alternative models include fractional Brownian motion, generalized hyperbolic models, models based on Lévy processes, stochastic volatility and GARCH models, constant elasticity of variance (CEV) model, jump-diffusion models, a numerical procedure called "implied binomial trees", time changed processes, affine stochastic-volatility and affine jump-diffusion models [83]. Of course, the corresponding effect on option pricing has been studied with the advent of these models at the same time. For instance, the evaluation of options based on the truncated Lévy stable process has been studied in [19, 77, 103] and based on fractional Brownian motion in [28, 29, 36, 163].

Recently, financial data has been found to exhibit constant values or very small fluctuations during some long time periods [48, 65, 66]. This kind of behaviour is characteristic of subdiffusive phenomena in physics, which arise due to trapping events when the particle gets immobilized. Fig 1.1 indicates the subdiffusive characteristics in a financial time series. Since the continuous time random walk model (CTRW) introduced in the physics literature by Montroll and Weiss [117] is widely used to study subdiffusive dynamics [13, 18, 58, 76, 108], some effort has been made to solve the problem of option pricing with a subdiffusive CTRW model.

In 2003, Stanislavsky put forward a Black-Scholes model under subordination, which introduces long-term memory effects in the classical BS model [155]. In 2008, Montero proposes a CTRW model for option pricing [114, 115]. Then Magdziarz gives an explicit expression for the BS formula in the subdiffusive regime and introduces later more general time-changed BS models under subordination [88, 90]. Then Orzeł and Weron solve the problem of calibrating the parameters of the subdiffusive Black-Scholes model to real data [120]. Formulas for European put and call option prices are presented for the subdiffusive Bachelier model [91].

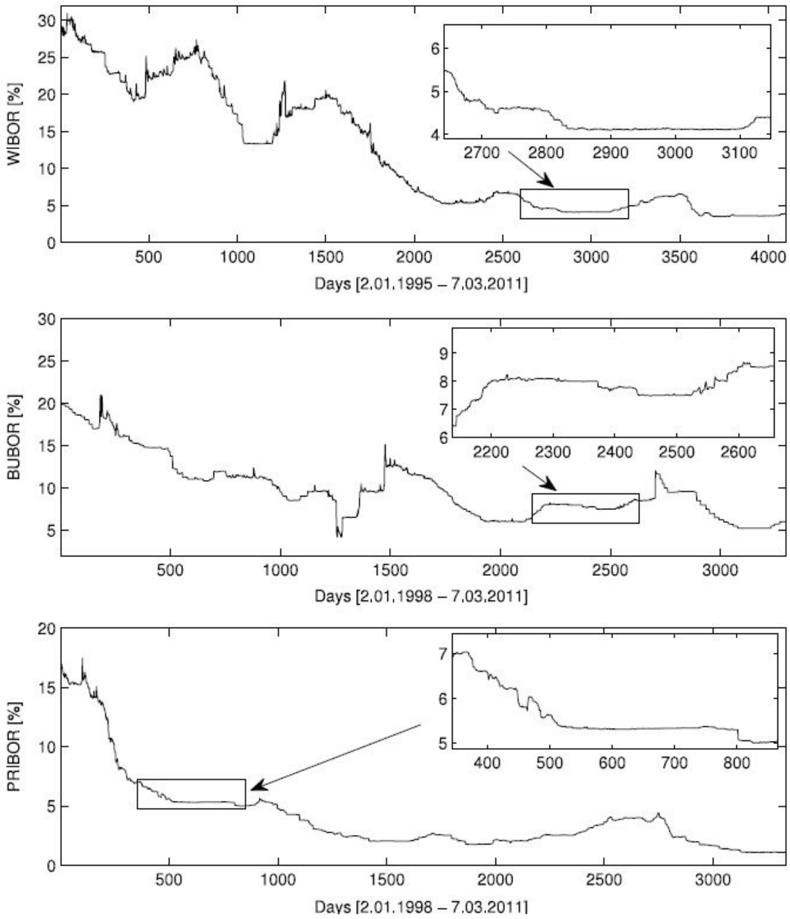

Figure 1.1: Subdiffusive data in financial market: the examined datasets of the 1-monthly Warsaw Interbank Offered Rate (top panel), Budapest Interbank Offered Rate (middle panel) and Prague Interbank Offered Rate (bottom panel) rates. The data and the figure is obtained in Ref. [66].

Hence, one could expect that such sophisticated mathematical models are the key to provide pricing formulas that match the behaviour of real data. In spite of several works in the literature probing the subdiffusive option pricing problem, as far as we know, no work has addressed the option price dependence on past time points. Although this is not a problem in the classical Black-Scholes theory which depends on a Markovian process, it is of great importance for the option pricing based on a non-Markovian model such as the CTRW. More importantly, it seems that all current subdiffusive option pricing formulas are discussed case by case and there is no formula which could unify these results together. Furthermore, previous works have focused on subdiffusive vanilla options .There is no discussion of exotic options in the subdiffusive regime. The reason might be that it is not easy to find analytical solutions for such subdiffusive option prices, especially for subdiffusive path-dependent options (such as Asian call options). Aside from these issues, much of the literature on option pricing has successfully applied Fourier analysis to determine option prices [9, 11, 25, 27, 56, 151], but no one used it to study subdiffusive options. Inspired by tackling these problems, we start to investigate them in our work.

In this thesis, we investigate the option pricing problem beyond the BS formula with an anomalous asset pricing model based on the CTRW. In particular, we are mainly concerned with the following topics: (1) Establishing different types of subdiffusive European call option formulas and deriving the corresponding partial differential equations (PDEs) that take into account memory effects. This elucidates the effect of multiple time points on the option price; (2) Generalizing these results to arbitrary waiting time distributions and payoff functions; (3) Deriving the PDEs for path dependent options in the presence of a subdiffusive pricing model.

The rest of this thesis consists of seven chapters. Chapters 2–3 present the necessary background material which is frequently used in the subsequent chapters. Even though some of them are well discussed in the literature, we give the main derivations for the purpose of making the content self-contained. The remaining five chapters are our own work.

In Chapter 4, we discuss the possible subdiffusive European call option pricing formulas with a CTRW. In particular, we propose two types of subdiffusive options: A "forward" (*type A*) and "backward" (*type B*) type option with non-zero interest rates based on a formulation of the subdiffusive pricing model in terms of a subordination. The standard BS formula is recovered in a well defined limit. We show that these two types of subdiffusive formulas could also be derived from corresponding fractional partial differential equations generalizing the celebrated BS PDE.

In Chapter 5, we investigate the subdiffusive European call option pricing formulas with general waiting times which is essentially a generalization of the results presented in Chapter 4.

In Chapter 6, we provide an option pricing formula based on a general payoff function in the anomalous regime, which could be used to derive the corresponding published subdiffusive option formulas in Refs. [88, 90, 91, 120, 155]. As the model used in this option pricing problem contains more parameters, it could be easily adapted to many different scenarios. As an application of our general formula, we discuss the special case of the anomalous power option formula. The fractional equations which could be used to describe this kind of new subdiffusive power option formula are derived. A comparison between classical and anomalous power options is made.

In Chapter 7, we provide a discussion of path dependent call options with general waiting times. In particular, we take Asian call options as an example and derive three types of formulas in the anomalous regime which provide us with more choices for practical applications.

In Chapter 8, some concluding remarks are made on the previous chapters. Finally, we discuss some interesting future topics.

Chapter 2

Fundamentals of stochastic processes

The main goal of this chapter is to review some fundamental material regarding stochastic processes. In particular we present Brownian motion both from physical and economical sides, which results into the classical approaches of studying general diffusion. In fact, stochastic expression of diffusion processes combined with classical hypotheses in Economics led to the further development of the theory of option pricing. The discovery of anomalous diffusion is believed to be useful to characterise new phenomenon in economics. Finally, the presentation of the Black-Scholes theory, on the one hand, provides us with a mathematical approaches in option's pricing. On the other hand, it greatly facilitates to understand future chapters.

2.1 Brownian motion in physics

Since Brownian motion plays an important role in the development of mathematical models and option pricing, it will serve as a starting point of our discussion.

"The story of Brownian motion is one of confused experiment, heated philosophy, belated theory, and, finally, precise and decisive measurement" [54]. As early as in 1785 Jan Ingenhousz found the irregular motion of coal dust particles on the surface of alcohol, but this kind of observed phenomenon took the name Brownian motion because of another fundamental pioneering work. In 1827, Scottish botanist Robert

Brown observed that when suspended in water, small pollen grain of the plant Clarkia pulchella were found to be in a very animated and irregular state of motion under his one lens microscope [20]. The image of Clarkia pulchella grains under a microscope is illustrated in Fig. 2.1. At first he thought this motion was a manifestation of life, but after systematically investigation he concluded that this kind of phenomenon existed apparently in any suspension of fine living or non-living particles.

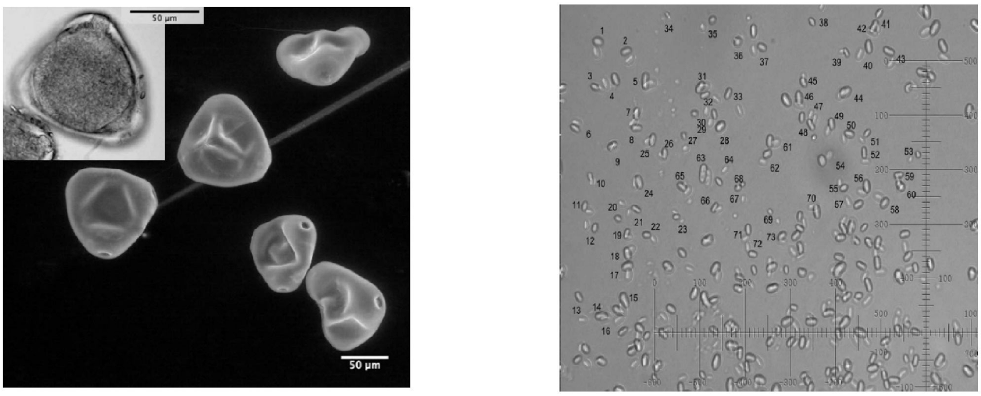

Figure 2.1: Left panel: Clarkia pulchella pollen imaged by an electron microscope. Right panel: Clarkia pulchella pollen with the ruler scale of 2 μm per unit. Both of the figures are adapted from Ref. [122].

At about the same time, in 1822, Joseph Fourier proposed the heat conduction equation, on the basis of which Adolf Fick presented the diffusion equation in 1855 [40]. However, a satisfactorily mathematical explanation of Brownian motion did not come until 1905, when Albert Einstein published the paper under the title “Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen” [35], which meant that “On the motion, required by the molecular-kinetic theory of heat, of particles suspended in fluids at rest” [75]. Einstein came up with two major points for the problem of Brownian motion as follows [49]

- The motion of the pollen grain is caused by the frequent force driven by the incessantly moving molecules of liquid in which it is suspended.
- The effect on the pollen grain from the motion of these molecules can only be described probabilistically by frequent statistically independent impacts due to the complexity of the motion of these molecules.

It seemed that a statistical explanation of these these fluctuations was inevitable. Maxwell and Boltzmann had previously used statistics for their famous gas theories, but only described possible states and the likelihood of their achievement. Rayleigh was actually the first one who considered a statistical description in this context [133], but Einstein was the first to establish the link between the erratic Brownian motion of individual particles and the thermodynamic laws of diffusion already known since the mid of the 19th century. Einstein's theory was based on the case of the free particle, that is, a particle on which only the forces due to the molecules of the surrounding medium are acting. His reasoning can be briefly summarized as follows [49, 70, 119]. Let us start with a discrete

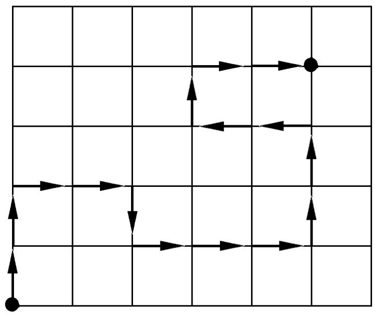

Figure 2.2: Schematic representation of a random walk in two dimensions. The walker jumps to a randomly chosen site with a fixed step length Δx.

time random walk in one dimension (1D) and assume that the walker's initial position is at the origin at time 0. The extension of these results to the n-dimensional case is straightforward. The walker jumps at each time step $t = 0, \Delta t, 2\Delta t, \cdots, n\Delta t, \cdots$ random with a constant step width $\Delta X_i = \Delta x$. Each jump is independent of the previous one. After N steps, the position $X(N)$ of the walker is

$$X(N) = \sum_{i=1}^{N} \Delta X_i \tag{2.1}$$

A discrete random walk with initial position $X(0) = 0$ at time $t = 0$ is illustrated in Fig. 2.2 in a two-dimensional lattice. Supposing that the motion of the free particle is on a straight line and $P_j(t_n)$ is the probability density that a Brownian particle is at position j at time $t_n = n\Delta t$, then such a process can be characterised by the master equation

$$P_j(t_n + \Delta t) = \frac{1}{2} P_{j-1}(t_n) + \frac{1}{2} P_{j+1}(t_n) \tag{2.2}$$

since the process is local in both space and time. Taylor expansions assuming small Δt and Δx indicate

$$P_j(t + \Delta t) = P_j(t) + \Delta t \frac{\partial P_j}{\partial t} + \mathcal{O}(\Delta t^2) \tag{2.3}$$

and

$$P_{j\pm 1}(t) = P(x,t) \pm \Delta x \frac{\partial P}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 P}{\partial x^2} + \mathcal{O}(\Delta x^3)\,. \tag{2.4}$$

Substituting Eqs. (2.3) and (2.4) in Eq. (2.2) and taking the continuum limit $\Delta t \to 0$ and $\Delta x \to 0$ yields the diffusion equation [108]

$$\frac{\partial}{\partial t} P(x,t) = D \frac{\partial^2}{\partial x^2} P(x,t) \tag{2.5}$$

where D is the continuum limit of

$$D = \lim_{\Delta x \to 0, \Delta t \to 0} \frac{(\Delta x)^2}{2\Delta t}\,. \tag{2.6}$$

D is called the coefficient of diffusion. Note that $P(x,t)$ is now a probability density function (PDF) normalized to one

$$\int_{-\infty}^{\infty} dx\, P(x,t) = 1\,. \tag{2.7}$$

Here the Dirac delta function $\delta(x)$ is introduced as

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \tag{2.8}$$

satisfying the identity

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1\,, \tag{2.9}$$

and the property

$$\int_{-\infty}^{\infty} dx\, f(x) \delta(x - x_0) = f(x_0) \tag{2.10}$$

for any real or complex valued continuous function $f(x)$. If the particle is at position x_0 at time t_0 so that $P(x, t_0 | x_0, t_0) = \delta(x - x_0)$, then by Fourier transform method, the solution to Eq. (2.5) could be found as follows [136]

$$P(x,t|0,0) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \tag{2.11}$$

and $P(x,t|0,0)$ is called the propagator. The average and the second moment can be derived as follows

$$\begin{aligned} \langle X(t)\rangle &= \int_{-\infty}^{\infty} dx\, x P(x,t|0,0) = 0 \\ \left\langle X^2(t)\right\rangle &= \int_{-\infty}^{\infty} dx\, x^2 P(x,t|0,0) = 2Dt\,. \end{aligned} \tag{2.12}$$

In fact, the diffusion equation obtained by Einstein is a special case of the Fokker-Planck equation (also known as Kolmogorov's equation) which could describe a large class of stochastic processes in which the system exhibits a continuous sample path. In this case, that implies that the particle's position, if thought of as a solution of the diffusion equation, in which time is continuous (not discrete, as assumed by Einstein), can be written as $X(t)$, where $X(t)$ is a continuous function of time but a random function. This leads us to consider the possibility of describing the dynamics of the system in a direct way by a stochastic differential equation for the path. Indeed, this was initiated by Langevin's treatment of Brownian motion with the assumption of an external erratic force [85] which was a kind of Itô's stochastic differential equation [62]. He derived the equation of motion for a Brownian particle as follows [49]

$$m\frac{d^2X(t)}{dt^2} = -6\pi\eta a\frac{dX(t)}{dt} + \xi(t) \tag{2.13}$$

where m is the mass of the particle, η is the viscosity of the fluid, a is the diameter of the spherical particle, and $\xi(t)$ is a random force, which we further specify later. Starting with this equation, one could find that

$$\left\langle X^2(t)\right\rangle = [kT/(3\pi\eta a)]t \tag{2.14}$$

This corresponds to Eq. (2.12) as derived by Einstein, provided we identify

$$D = kT/(6\pi\eta a), \tag{2.15}$$

where T is absolute temperature and k_{B} is Boltzmann's constant (or, equivalently, Avogadro-Loschmidt number N), which nowadays is called Stokes-Einstein relation. Al-

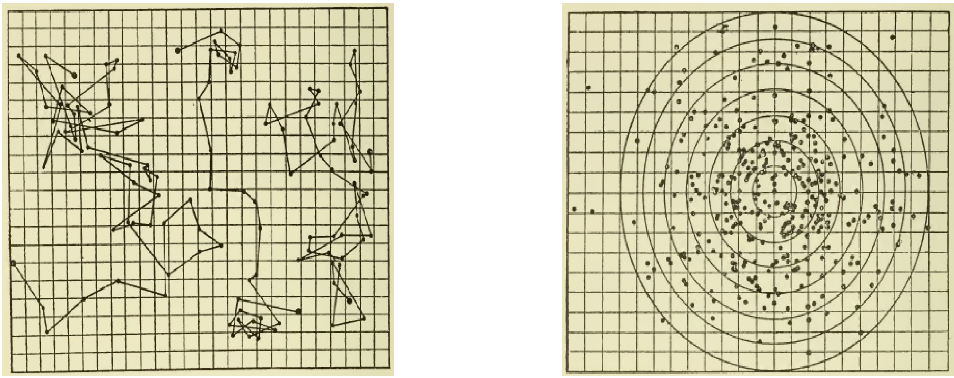

Figure 2.3: Overview of Perrins work on Brownian motion. Left panel: three sample trajectories of individual mastic granules obtained by tracing the segments at 30 seconds intervals. Right panel: distribution of 365 observations relating to granules of mastic. The mastic, used in the preparation of varnish, is obtained by making incisions in the bark of the Pistacia lentisciis (Chios Island). Both of these figures are adapted from Ref. [125]

though the notion of Boltzmanns constant k was not yet fully established at the time, this relation established a link between the macroscopic kinetic coefficients and the microscopic molecular world [45]. Einstein's work inspired Jean Perrin and his students [123–125] to perform a series of experiments to determine the value of Avogadro-Loschmidt number which won him the Nobel Prize in 1926. Some results of Perrin were displayed in Fig. 2.3. In Fig. 2.4, the data obtained by Kappler [72] with high-accuracy set-up using an optical detection method was showed and from his data he also got the Avogadro-Loschmidt number. Einsteins predictions could be elegantly verified for the Avogadro-Loschmidt number N in the range $(6.4 \div 6.9) \times 10^{23}/[mol]$ [52]. As a physical subject, Brownian motion was investigated extensively both from theory and experiment by Fokker [44], Planck [127], Smoluchowski, Klein, Kramers, Ornstein, Uhlenbeck, Chandrasekhar, Montroll and others. On the other hand, besides Albert Einstein, Thorvald Nicolai Thiele and Louis Bachelier were earliest ones who attempted to model Brownian motion mathematically [68]. However, it was Norbert Wiener who first demonstrated the construction of Brownian motion in a rigorous mathematical way [4] and showed that its trajectory was continuous everywhere but nowhere differentiable with self-similar in law which meant if one zooms in or zooms out on a Brownian motion it was still a Brownian motion. This kind of observation was related to the self-affine nature of the diffusion process. Due to his contribution, the Brownian motion sometimes was also known as Wiener process. Further important mathematical contributions were made by

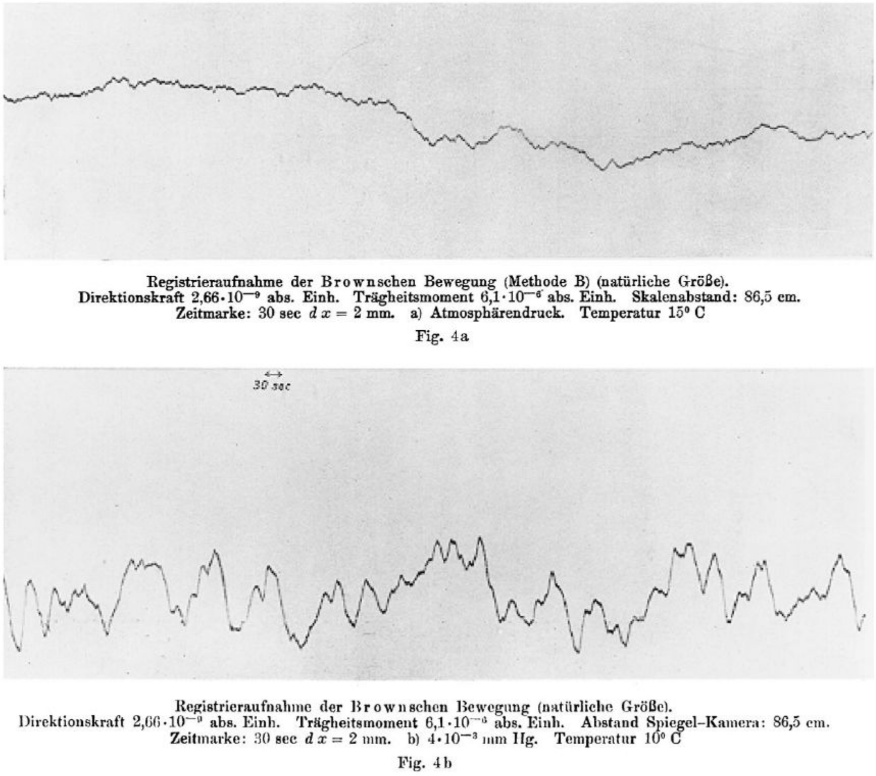

Figure 2.4: Erroneous behaviour of Brownian motion observed in a highprecision measurement obtained by Kappler in 1931 [72]. Both of these figures are adapted from Ref. [108].

Joseph Doob, Mark Kac, William Feller, and others.

2.1.1 General diffusion processes

Inspired by Brownian motion in physics, Brownian motion is defined mathematically as follows

Definition 1 *(Brownian motion) A stochastic process $\{W(t), t \geq 0\}$, also called a diffusion in physics as it can be used to model diffusions, is said to be a standard Brownian motion process(or Wiener process) if [111, 140]*

1. $W(0) = 0$.
2. $\{W(t), t \geq 0\}$ *has independent increments, in that for all* $t_1 < t_2 < \cdots < t_n$, $W(t_n) - W(t_{n-1}), W(t_{n-1}) - W(t_{n-2}), \ldots, W(t_2) - W(t_1), W(t_1)$ *are independent.*

3. $\{W(t),\, t \geq 0\}$ *has stationary increments, in that the distribution of* $W(t+s)-W(t)$ *does not depend on* t.

4. *For every* $t > 0$, $W(t)$ *is normally distributed with mean* 0 *and variance* t.

An equivalent definition could be the solution to a stochastic differential equation (SDE) as follows

$$\dot{W}(t) = \frac{dW(t)}{dt} = \xi(t) \tag{2.16}$$

with the initial condition $W(0) = 0$, where $\xi(t)$ is the random force of Eq. (2.13). From Def. 1, we thus see that $\xi(t)$ is a Gaussian random variable with zero mean

$$\langle \xi(t) \rangle = 0\,, \tag{2.17}$$

and δ correlation

$$\big\langle \xi(t')\xi(t'') \big\rangle = \delta(t' - t'')\,. \tag{2.18}$$

It is clear that integrating both sides of Eq. (2.16) could result into

$$W(t) = W(0) + \int_0^t \xi(t')dt'\,. \tag{2.19}$$

Thus, one has that

$$\begin{aligned} \langle W(t) \rangle &= \left\langle W(0) + \int_0^t \xi(t')dt' \right\rangle \\ &= \int_0^t \big\langle \xi(t') \big\rangle\, dt' \\ &= 0\,, \end{aligned} \tag{2.20}$$

$$\begin{aligned}
\left\langle (W(t)-W(0))^2 \right\rangle &= \left\langle \left(\int_0^t \xi(t')dt' \right)^2 \right\rangle \\
&= \left\langle \int_0^t \xi(t')dt' \int_0^t \xi(t'')dt'' \right\rangle \\
&= \int_0^t dt' \int_0^t \left\langle \xi(t')\xi(t'') \right\rangle dt'' \\
&= \int_0^t dt' \int_0^t \delta(t'-t'')dt'' \\
&= t\,,
\end{aligned}$$

and

$$\begin{aligned}
\left\langle (W(t+\tau)-W(t))^2 \right\rangle &= \left\langle \left(W(0) + \int_0^{t+\tau} \xi(t')dt' - W(0) - \int_0^t \xi(t')dt' \right)^2 \right\rangle \\
&= \left\langle \left(\int_0^t \xi(t')dt' + \int_t^{t+\tau} \xi(t')dt' - \int_0^t \xi(t')dt' \right)^2 \right\rangle \\
&= \int_t^{t+\tau} dt' \int_t^{t+\tau} \left\langle \xi(t')\xi(t'') \right\rangle dt'' \\
&= \tau\,.
\end{aligned} \tag{2.21}$$

In particular, for small τ, it follows

$$\left\langle (dW(t))^2 \right\rangle = \left\langle (W(t+dt)-W(t))^2 \right\rangle = dt\,. \tag{2.22}$$

Since $W(t)$ is actually not differentiable, $\dot{W}(t) = \xi(t)$ does not really exist. However, the notation $\xi(t)$ is preferred by physicists whereas $dW(t)$ is usually used in mathematics. Fig. 2.5 illustrates five sample paths of Brownian motion. In general the spectral density of a process $X(t)$ is defined by the Fourier transform of the correlation function as

$$S(v) = \int_{-\infty}^{\infty} e^{ivt} \left\langle X(t)X(0) \right\rangle dt\,. \tag{2.23}$$

Thus for the noise $\xi(t)$, it follows that

$$\begin{aligned}
S(v) &= \int_{-\infty}^{\infty} e^{ivt} \left\langle \xi(t)\xi(0) \right\rangle dt \\
&= \int_{-\infty}^{\infty} e^{ivt}\delta(t)dt = 1\,,
\end{aligned} \tag{2.24}$$

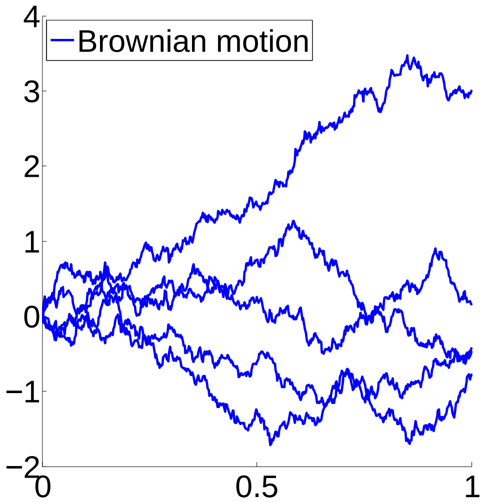

Figure 2.5: Sample paths of Brownian motion

which does not depend on v. That is , all frequencies contribute equally in the correlation function, which means by analogy, all colours contribute equally to make white light. Therefore, $\xi(t)$ is called Gaussian white noise. If the spectral density depends on v, the noise is termed coloured noise.

The methods of Einstein and Langevin represent the two main approaches in the theory of stochastic processes, which will be used to investigate general diffusion process. A general diffusion process $X(t)$ could be defined by a SDE

$$\dot{X}(t) = \mu(X(t)) + \sigma(X(t))\xi(t), \tag{2.25}$$

where $\dot{X}(t) = dX(t)/dt$, the functions $\mu(X)$ is continuously differentiable, $\sigma(X)$ is twice continuously differentiable [37], and $\xi(t)$ is Gaussian white noise with the properties Eqs. (2.17)–(2.18) as before. The initial condition is assumed to be $X(0) = x_0$. In order to specify the multiplicative term $\sigma(X(t))\xi(t)$, it is necessary to consider a discretized version of Eq. (2.25) by introducing a time step Δt:

$$X_{n+1} - X_n = \mu(X_n)\Delta t + \sigma(X_n)\xi_n \tag{2.26}$$

where $X_n = X(n\Delta t)$ and the increment ξ_n of the white noise $\xi(t)$ is defined as

$$\xi_n = \int_{t_n}^{t_n+\Delta t} \xi(t')dt' \tag{2.27}$$

where $t_n = n\Delta t$.

It should be remarked that the definition of the discretization of term $\sigma(X(t))\xi(t)$ in Eq. (2.26) corresponds to Itô's stochastic integral and forms the basis of Itô's stochastic calculus. Different definitions of the discretization could result in different stochastic integrals, such as Stratonovich's definition.

As $\xi(t)$ is a Gaussian random variable, it follows that ξ_n are Gaussian with the average and variance as follows

$$\begin{aligned} \langle \xi_n \rangle &= \left\langle \int_{t_n}^{t_n+\Delta t} \xi(t')dt' \right\rangle \\ &= 0\,, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} \mathrm{Var}(\xi_n) &= \left\langle \xi_n^2 \right\rangle - \langle \xi_n \rangle^2 \\ &= \left\langle \xi_n^2 \right\rangle \\ &= \int_{t_n}^{t_n+\Delta t} dt' \int_{t_n}^{t_n+\Delta t} \left\langle \xi(t')\xi(t'') \right\rangle dt'' \\ &= \int_{t_n}^{t_n+\Delta t} dt' \int_{t_n}^{t_n+\Delta t} \delta(t'-t'')dt'' \\ &= \Delta t\,. \end{aligned} \tag{2.29}$$

Because of the statistical independence of increments over different non-overlapping time periods, one could immediately obtain that

$$\langle \xi_i \xi_j \rangle = \delta_{ij}\Delta t \tag{2.30}$$

where δ_{ij} denotes the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j\,. \end{cases} \tag{2.31}$$

It is known that for a Gaussian random variable Y with average μ and variance σ^2, Y could be written as $Y = \mu + \sigma W$, where W here indicates a Gaussian variable with zero average and variance 1. As $\langle \xi_n \rangle = 0$ and $\mathrm{Var}(\xi_n) = \Delta t$, it is clear that

$$\xi_n = \sqrt{\Delta t}\zeta_n \tag{2.32}$$

where ζ_n is a Gaussian random variable with zero average and variance 1. Thus Eq. (2.26) could be expressed as

$$X_{n+1} - X_n = \mu(X_n)\Delta t + \sigma(X_n)\sqrt{\Delta t}\zeta_n\,. \tag{2.33}$$

As the diffusion Eq. (2.25) is a stochastic equation, which means that its solution are random trajectories that are different in each realization, it is convenient to know the probability that the trajectory reaches a certain position at a given time. If $P(x, t|x_0, 0)$ denotes the probability density function of the stochastic process $X(t)$ determined by Eq. (2.25) with initial condition $P(x, 0|x_0, 0) = \delta(x - x_0)$, it could be derived from a PDE like in the Einstein approach. This PDE is known as the Fokker-Planck equation (or the Kolmogorov's equation) which will greatly help to understand the dynamics of the process $X(t)$. In order to derive the corresponding Fokker-Planck equation for the SDE (2.25), we start with the discussion of Itô's formula [37].

Let us define the increment of the process $X(t)$ as

$$\Delta X = X(t + \Delta t) - X(t)\,, \tag{2.34}$$

then we could obtain the Taylor expansion of a function $u(X(t), t)$ up to quadratic order in ΔX and linear in Δt as

$$\begin{aligned} u(X(t+\Delta t), t+\Delta t) &= u(X(t), t) + \tfrac{\partial u(X(t),t)}{\partial x}\Delta X + \tfrac{\partial u(X(t),t)}{\partial t}\Delta t \\ &\quad + \tfrac{1}{2}\tfrac{\partial^2 u(X(t),t)}{\partial x^2}\Delta X^2 + \tfrac{\partial^2 u(X(t),t)}{\partial x \partial t}\Delta t \Delta X\,. \end{aligned} \tag{2.35}$$

With the help of Eq. (2.33), ΔX could be written as

$$\begin{aligned} \Delta X &= X(t + \Delta t) - X(t) \\ &= X_{n+1} - X_n \\ &= \mu(X_n)\Delta t + \sigma(X_n)\sqrt{\Delta t}\zeta_n\,. \end{aligned} \tag{2.36}$$

Therefore, one obtains that

$$\begin{aligned}(\Delta X)^2 &= (\mu(X_n)\Delta t)^2 + 2(\mu(X_n)\Delta t)(\sigma(X_n)\sqrt{\Delta t}\zeta_n) + (\sigma(X_n)\sqrt{\Delta t}\zeta_n)^2 \\ &\approx \sigma(X_n)^2\Delta t + \mathcal{O}(\Delta t^{3/2})\,. \end{aligned} \tag{2.37}$$

In the last step, ζ_n^2 is approximated by its expected value $\langle \zeta_n^2 \rangle = 1$. Substituting Eq. (2.37) into Eq. (2.35) and keeping only terms linear in Δt and ΔX leads to

$$\begin{aligned}u(X(t+\Delta t), t+\Delta t) = u(X(t),t) &+ \frac{\partial u(X(t),t)}{\partial x}\Delta X + \frac{\partial u(X(t),t)}{\partial t}\Delta t \\ &+ \frac{\sigma(X(t))^2}{2}\frac{\partial^2 u(X(t),t)}{\partial x^2}\Delta t, . \end{aligned} \tag{2.38}$$

Dividing by Δt and taking the limit $\Delta t \to 0$, one derives that

$$\begin{aligned}\frac{d}{dt}u(X(t),t) &= \frac{\partial u(X(t),t)}{\partial t} + \frac{\partial u(X(t),t)}{\partial x}\dot{X}(t) + \frac{\sigma(X(t))^2}{2}\frac{\partial^2 u(X(t),t)}{\partial x^2} \\ &= \frac{\partial u(X(t),t)}{\partial t} + \frac{\partial u(X(t),t)}{\partial x}(\mu(X(t)) + \sigma(X(t))\xi(t)) \\ &+ \frac{\sigma(X(t))^2}{2}\frac{\partial^2 u(X(t),t)}{\partial x^2}\,, \end{aligned} \tag{2.39}$$

which is usually known as Itô's formula. Comparing this formula with the usual chain rule, one finds that an additional term appears which is caused by the stochastic term. Now supposing that $u(x)$ is an arbitrary smooth function, based on Itô's formula, it follows

$$\frac{d}{dt}u(X(t)) = u'(X(t))(\mu(X(t)) + \sigma(X(t))\xi(t)) + \frac{\sigma(X(t))^2}{2}u''(X(t)) \tag{2.40}$$

where $u'(x) = du(x)/dx$ and $u''(x) = d^2u(x)/dx^2$. Taking the expectation for both sides of Eq. (2.40) with the condition $X(0) = x_0$, it yields

$$\begin{aligned}\left\langle \frac{d}{dt}u(X(t)) \right\rangle &= \langle u'(X(t))(\mu(X(t)) + \sigma(X(t))\xi(t)) \rangle \\ &+ \left\langle \frac{\sigma(X(t))^2}{2}u''(X(t)) \right\rangle . \end{aligned} \tag{2.41}$$

With the density function $P(x,t|x_0,0)$ and the definition of the average value, this equation could be expressed explicitly as

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\, u(x)P(x,t|x_0,0)\\&=\int_{-\infty}^{\infty} dx\, P(x,t|x_0,0)\left(u'(x)\mu(x)+\frac{\sigma(x)^2}{2}u''(x)\right)\,.\end{aligned} \tag{2.42}$$

The term $\langle u'(X(t))\sigma(X(t))\xi(t)\rangle$ disappears since we have in discretized form

$$\left\langle u'(X(t))\sigma(X(t))\xi(t)\right\rangle \approx \left\langle u'(X_n)\sigma(X_n)\xi_n\right\rangle = \left\langle u'(X_n)\right\rangle \langle\sigma(X_n)\rangle \langle\xi_n\rangle = 0\,, \tag{2.43}$$

which is due to the fact that in our Itô's discretization X_n depends only on all ξ_i with $i \leq n-1$. As a consequence X_n and ξ_n are statistically independent. As a result, the average in Eq. (2.43) could be factorized and the zero comes from the property $\langle\xi_n\rangle = 0$. By imposing some natural decay assumptions on $P(x,t|x_0,0)$: $P(x,t|x_0,0) \to 0$ and $\partial P(x,t|x_0,0)/\partial x \to 0$ as $x \to \pm\infty$, and performing integration by parts, the right hand side of Eq. (2.42) becomes

$$\begin{aligned}&\int_{-\infty}^{\infty} dx\, P(x,t|x_0,0)\left(u'(x)\mu(x)+\frac{\sigma(x)^2}{2}u''(x)\right)\\&=\int_{-\infty}^{\infty} dx\, u(x)\left(-\frac{\partial}{\partial x}(P(x,t|x_0,0)\mu(x))+\frac{1}{2}\frac{\partial^2}{\partial x^2}(P(x,t|x_0,0)\sigma^2(x))\right)\,.\end{aligned} \tag{2.44}$$

On the other hand, the left side of Eq. (2.42) could be written as

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\, u(x)P(x,t|x_0,0) = \int_{-\infty}^{\infty} dx\, u(x)\frac{\partial}{\partial t}P(x,t|x_0,0)\,. \tag{2.45}$$

Thus we obtain that

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, u(x)\frac{\partial P(x,t|x_0,0)}{\partial t} = \int_{-\infty}^{\infty} dx\, u(x)\bigg(&-\frac{\partial}{\partial x}(P(x,t|x_0,0)\mu(x))\\&+\frac{1}{2}\frac{\partial^2}{\partial x^2}(P(x,t|x_0,0)\sigma^2(x))\bigg)\,.\end{aligned} \tag{2.46}$$

Since the above integral holds for every smooth function u, this indicates that

$$\frac{\partial}{\partial t}P(x,t|x_0,0) = -\frac{\partial}{\partial x}(P(x,t|x_0,0)\mu(x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(P(x,t|x_0,0)\sigma^2(x)) \tag{2.47}$$

with initial condition $P(x,0|x_0,0) = \delta(x-x_0)$ which indicates that the process starts at the fixed position x_0 at time 0. Eq. (2.47) is exactly what we are looking for and called Fokker-Planck equation [136].

2.2 Anomalous diffusion

Despite the success of models dependent on Brownian motion and diffusion processes, over the last two decades it seems that many dynamical systems in a wide variety of fields, ranging from biology to physics, can not properly be described within this framework [18,109,152]. Deviating from the well known Ficks law of purely thermalized systems [75, 108], anomalous diffusion, known since Richardson's study in turbulent diffusion in 1926 [134], is typically defined in terms of the the mean square displacement (MSD)

$$\left\langle x^2(t)\right\rangle \propto Kt^{\alpha}, \quad \alpha \in (0,\infty)\,. \tag{2.48}$$

Different from normal diffusion with the linear time dependence of the MSD $\left\langle x^2(t)\right\rangle \propto Kt$ for $\alpha = 1$. Based on the the value of the anomalous diffusion index α, for $0 < \alpha < 1$, the process is called subdiffusion whereas for $\alpha > 1$ it is superdiffusion [108]. Various generalizations of diffusion processes have been proposed to account for such anomalous diffusion, such as fractional Brownian motion, CTRW models, generalised master equations and so on. The approach to anomalous kinetics which we are going to present is given in terms of CTRWs.

2.2.1 The continuous time random walk

The CTRW, which was first introduced by Montroll and Weiss [117], became one of the most widely discussed methods for investigating anomalous diffusion. The CTRW has been successfully applied to model anomalous diffusion in various fields [13,18,58,76,108]. These applications include transport in amorphous materials [150], random networks [14] , earthquake [55], and so on. In particular, the CTRW formalism has also been extended to study phenomena occurring in financial markets [101,102]. In what follows, we present the essential ideas underlying a CTRW .

Different form the discrete time random walk formulation, the CTRW assumes that the waiting times between two successive jumps and the length of a given jump can be

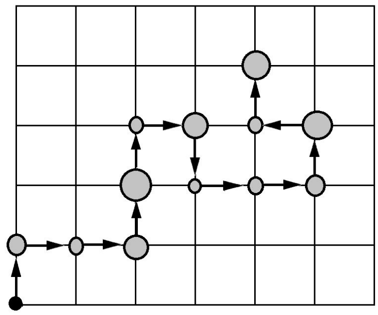

Figure 2.6: Schematic representation of a CTRW in two dimension. The waiting times are symbolised by the waiting circle and the diameter of the each circles is proportional to the waiting times spent on a fixed position until the next jump happens. The jump lengths are assumed to keep constant here.

drawn from a joint PDF $\rho(\xi,\eta)$ with the normalization condition $\int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta\, \rho(\xi,\eta) = 1$. By this assumption, the PDF of the waiting times

$$\psi(\eta) = \int_{-\infty}^{\infty} d\xi\, \rho(\xi,\eta) \tag{2.49}$$

as well as the PDF of the jump lengths

$$\vartheta(\xi) = \int_{0}^{\infty} d\eta\, \rho(\xi,\eta) \tag{2.50}$$

can be derived. The probability of the waiting times in the interval $(\eta, \eta + d\eta)$ could be calculated by $\psi(\eta)d\eta$ and the probability of the jump length in the interval $(\xi, \xi + d\xi)$ is from $\vartheta(\xi)d\xi$. If the waiting times and the jump lengths are independent random variables, the joint probability density $\rho(\xi,\eta)$ can be factorized in terms of the marginal probability densities for jump lengths $\vartheta(\xi)$ and waiting times $\psi(\eta)$ as $\rho(\xi,\eta) = \psi(\eta)\vartheta(\xi)$, which is known as a decoupled CTRW. For the coupled case, one finds that $\rho(\xi,\eta) = p(\xi|\eta)\psi(\eta)$ or $\rho(\xi,\eta) = p(\eta|\xi)\vartheta(\xi)$. The correlations between jumps and waiting times depend on the physical context [132]. In the present discussion, we consider only the decoupled case. An illustration of the CTRW with initial position $X(t_0) = 0$ at time $t_0 = 0$ is given by Fig. 2.6 for a two-dimensional lattice.

Although the CTRW was originally introduced as a natural generalization of a random walk on a lattice, a convenient stochastic representation of these processes can be given in terms of coupled Langevin equations [41–43]

$$\dot{X}(s) = \xi(s), \tag{2.51a}$$

$$\dot{T}(s) = \eta(s), \tag{2.51b}$$

where $\xi(s)$ is a white Gaussian noise with properties $\langle \xi(s) \rangle = 0$ and $\langle \xi(s_2)\xi(s_1) \rangle = \delta(s_2 - s_1)$, and $\eta(s)$ is a one-sided Lévy process of order α with $0 < \alpha < 1$. The two processes $\xi(s)$ and $\eta(s)$ are assumed to be statistically independent. The CTRW could then be derived as a time-changed (or subordinated) process

$$Y(t) = X(S(t)) \tag{2.52}$$

where the process $S(t)$, the inverse of $T(s)$, can be defined as a collection of first passage times

$$S(t) = \inf\{s > 0 : T(s) > t\}\,. \tag{2.53}$$

In this formulation, the CTRW as a subordinated normal diffusive processes can be regarded as the continuum limit of the original renewal picture of Montroll and Weiss [117]. In the CTRW, the number of steps N made by the walker in a time interval $(0, t)$ is a random variable. Starting on the origin at time 0, a random walker stays fixed to its position until time t_1, and then it makes a random jump to ξ_1. The walker is keeping at the same place ξ_1 until time $t_2 > t_1$ when it jumps randomly to a new position $\xi_1 + \xi_2$. The process is then renewed. If $Y(t)$ denotes the position of a random walker at time t, $\xi_i = Y(t_i) - Y(t_{i-1})$ denotes a random jump occurring at a random time t_i and $\eta_i = t_i - t_{i-1}$ is the waiting time between two successive jumps, then the position $Y(t)$ of a CTRW could be characterised by two sets of random variables $\{(\xi_i, \eta_i)\}_{i,\cdots,N(t)}$ [23, 50, 145, 146, 159]

$$Y(t) = \sum_{i=1}^{N(t)} \xi_i\,, \tag{2.54}$$

where $t_0 = 0$, $Y(0) = 0$ and $N(t)$ represents the number of jumps occurred up to time t. Here we suppose that the waiting times $\{\eta_1, \eta_2, \cdots\}$ and the jumps $\{\xi_1, \xi_2, \cdots\}$ are independent identically distributed (*i.i.d.*) random variables and that each η_i is independent of ξ_i.

On the other hand, direct integration of Eqs. (2.51a)–(2.51b) yields

$$\begin{aligned} Y(t) &= X(S(t)) \\ &= \int_0^{S(t)} \dot{X}(\tau) d\tau \\ &= \int_0^{S(t)} \xi(\tau) d\tau \,. \end{aligned} \tag{2.55}$$

Comparing with Eq. (2.54), we see that Fogedby's approach [41] leads to the resulting trajectory of the random walk in the continuum limit by parametrizing both the path of the walker $X(\cdot)$ and the time elapsed $T(\cdot)$ with an arbitrary continuous arc length s. The stochastic process $S(\cdot)$, the inverse of $T(\cdot)$, measures the arc length as a function of the physical time. The continuum limit of the random variable $N(t)$ thus is represented by $S(t)$ that counts the number of steps in the renewal picture [23]. We will continue our discussion of CTRW further in next chapter.

2.3 Brownian motion in finance

As we have seen, the physical term Brownian motion describes the erratic motion of small particles suspended in a liquid due to the random bombardment by surrounding liquid molecules. A similar random phenomena is observed in the erratic fluctuations in the price of certain financial assets, in which case the "microscopic" fluctuations are brought by a vast amount of individual financial transactions happening during the stock exchange. For all of these phenomena a statistically identifiable collective behaviour arises because of the large number of individual random events happening independently of each other. Thus it was not surprising to see that a probabilistic analysis equivalent to Einsteins Brownian analysis, had actually already been applied to a range of the kind of financial transactions on the Paris stock market by a French doctoral student named Louis Bachelier, who was recognized nowadays as the founder of the modern mathematical finance [6, 7, 156]. In fact, 5 years earlier than Albert Einstein, dating back to 1900, Bachelier first derived the Brownian motion mathematically to study the pricing of shares and European options. He introduced the idea of the relative value of a share as [49]

$$X(t) - X_0 \tag{2.56}$$

which means the difference between its value $X(t)$ at time t and the the known value X_0 at time 0. He then deduced that $X(t)$ follows a process known as Brownian motion today. Furthermore, by dividing time into discrete intervals and considering discrete jumps in the share prices, he arrived finally at the heat equation (2.5). Despite the fundamental importance of Bachelier's process as Brownian motion, his work was ignored and forgotten until it was rediscovered by Jimmie Savage in 1955, who reminded Paul Samuelson [15]. Samuelson [98, 143, 144, 158] pointed out the deficiency of Bachelier's method for taking negative value for prices of shares, and further put forward a correct quantity, known as the return on the share price, given by

$$\frac{X(t)}{X_0}\,. \tag{2.57}$$

The return is the fractional gain or loss in the share price, which results into a formulation in which

$$\ln\left(\frac{X(t)}{X_0}\right) \tag{2.58}$$

is regarded as the quantity that undergoes Brownian motion. It is evident that this formulation sets up the natural range $(0, \infty)$ of the price. The improvement over Bachelier's result is so successful that it is the preferred model for share prices to this day. Samuelson termed the new process GBM which will be presented in the following section.

2.3.1 Geometric Brownian motion

Since GBM was accepted as a reasonable price model, it was used to simulate assets prices in real life. Figs. 2.7 demonstrate the similarity of geometric Brownian motion sample paths and real asset prices.

Before we define geometric Brownian motion mathematically, we first introduce Brownian motion with drift.

Definition 2 *(Brownian motion with drift) A stochastic process $\{B(t),\, t \geq 0\}$ is said to be a Brownian motion process with drift μ and variance σ^2 if [140]*

1. *$B(0) = 0$.*
2. *$\{B(t),\, t \geq 0\}$ has independent and stationary increments.*
3. *$B(t)$ is normally distributed with mean μt and variance $\sigma^2 t$.*

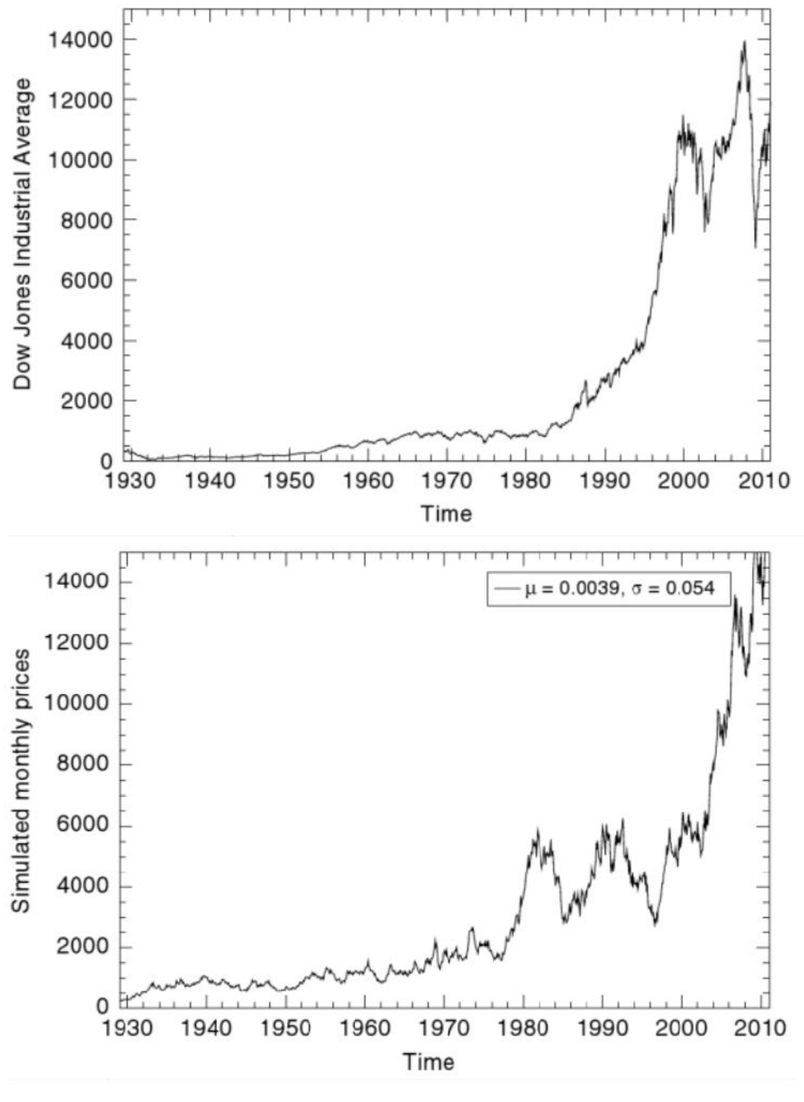

Figure 2.7: Top panel: The monthly data of the Dow Jones Industrial Average over the period from October 1928 to August 2011. Bottom panel: The price simulation by geometric Brownian motion in Eq. (2.64). Both of these figures are adapted from Ref. [135].

A Brownian motion with drift could also be defined as the solution to a SDE as follows

$$dB(t) = \mu dt + \sigma dW(t) \tag{2.59}$$

where $\{W(t),\, t \geq 0\}$ is standard Brownian motion and the initial condition is that $B(0) = 0$. This is how Louis Bachelier specified stock prices in his PhD dissertation. An equivalent version of this equation can be written as

$$\dot{B}(t) = \mu + \sigma\xi(t) \tag{2.60}$$

with the same initial condition $B(0) = 0$, where $\xi(t)$ is the Gaussian white noise as before. We shall, unless otherwise stated, use this kind of notation for stochastic differential equations throughout this thesis.

According to the Fokker-Planck equation (2.47), it is rather straightforward to find the corresponding Fokker-Planck equation for Brownian motion with drift as

$$\frac{\partial P(x,t|x_0,0)}{\partial t} = -\mu\frac{\partial P(x,t|x_0,0)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 P(x,t|x_0,0)}{\partial x^2} \tag{2.61}$$

where $P(x,t|x_0,0)$ is the density function of Brownian motion with drift and the initial condition is that $P(x,0|x_0,0) = \delta(x - x_0)$.

Definition 3 *(Geometric Brownian motion) If $\{B(t),\, t \geq 0\}$ is a Brownian motion process with drift μ and variance σ^2, then GBM process $\{X(t),\, t \geq 0\}$ with drift parameter μ and variance parameter σ^2 is defined by [139]*

$$X(t) = e^{B(t)} = e^{\sigma W(t)+\mu t}\,. \tag{2.62}$$

If $X(0) = x$, then GBM could be written as

$$X(t) = xe^{B(t)} \tag{2.63}$$

where $B(t)$ is Brownian motion with drift and $\{W(t),\, t \geq 0\}$ is standard Brownian motion.

It is not hard to deduce that the process $\ln X(t)$ is normally distributed with mean $\ln x + \mu t$ and variance $\sigma^2 t$. A Geometric Brownian Motion $\{X(t), t \geq 0\}$ could also be defined as the solution of an SDE of the type of Eq. (2.25) with a linear drift $\mu(x) = \mu\, x$ and a linear x-dependent diffusion coefficient $\sigma(x) = \sigma\, x$

$$\dot{X}(t) = \mu X(t) + \sigma X(t)\xi(t) \tag{2.64}$$

with initial value $X(0) = x_0$. Here we interpret the multiplicative term $\sigma X(t)\xi(t)$ in

Itô's sense.

Now by using Itô's formula, we can also find the solution to the geometric Brownian motion described by Eq. (2.64). Let us introduce a new process $Z(t)$ as

$$Z(t) = \ln X(t)\,. \tag{2.65}$$

Applying Itô's formula to the function $u(x) = \ln x$, we obtain

$$\begin{aligned}\frac{d}{dt}Z(t) &= u'(X(t))\left(\mu(X(t)) + \sigma(X(t))\xi(t)) + \frac{\sigma(X(t))^2}{2}u''(X(t)\right)\\ &= \frac{1}{x}\left(\mu x + \sigma x\xi(t)\right) - \frac{1}{x^2}\frac{\sigma^2 x^2}{2}\\ &= \left(\mu + \sigma\xi(t)\right) - \frac{\sigma^2}{2}\\ &= \left(\mu - \frac{\sigma^2}{2}\right) + \sigma\xi(t)\,.\end{aligned} \tag{2.66}$$

This equation could now be directly integrated, so we obtain

$$Z(t) = Z(0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\int_0^t \xi(\tau)d\tau\,, \tag{2.67}$$

which means that

$$\ln X(t) = \ln X(0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\int_0^t \xi(\tau)d\tau\,, \tag{2.68}$$

and hence we derive

$$X(t) = x_0 e^{(\mu-\sigma^2/2)t+\sigma\int_0^t \xi(\tau)d\tau}\,. \tag{2.69}$$

By the relation $dW(t) = \xi(t)dt$, it could also be written as

$$X(t) = x_0 e^{(\mu-\sigma^2/2)t+\sigma W(t)}\,. \tag{2.70}$$

With Eq. (2.47) the Fokker-Planck equation for geometric Brownian motion follows immediately as

$$\frac{\partial P(x,t|x_0,0)}{\partial t} = -\mu\frac{\partial(P(x,t|x_0,0)x)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2(P(x,t|x_0,0)x^2)}{\partial x^2} \tag{2.71}$$

where $P(x,t|x_0,0)$ is the density function of geometric Brownian motion and the initial condition is that $P(x,0|x_0,0) = \delta(x-x_0)$.

Using a Fourier transform, it is straightforward to solve Eq. (2.71):

$$P(x,t|x_0,0) = \frac{1}{x\sqrt{2\pi\sigma^2 t}} \exp(-\frac{(\ln x - \ln x_0 - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t}) \,. \tag{2.72}$$

With the help of Mathematica software, applying the Laplace transform with respect to t, we could find that the Laplace transform of the probability density function $P(x,t|x_0,0)$ of GBM given in Eq. (2.72), which will be used later, $\tilde{P}(x,\lambda|x_0,0)$ takes the form as follows

$$\begin{aligned}
\tilde{P}(x,\lambda|x_0,0) &= \frac{1}{x\sigma\sqrt{2\lambda+\frac{\hat{\mu}^2}{\sigma^2}}} \exp\left(\frac{(\ln x - \ln x_0)\left(\hat{\mu} - \sigma\sqrt{2\lambda+\frac{\hat{\mu}^2}{\sigma^2}}\right)}{\sigma^2}\right), && x > x_0 \\
\tilde{P}(x,\lambda|x_0,0) &= \frac{1}{x\sigma\sqrt{2\lambda+\frac{\hat{\mu}^2}{\sigma^2}}}\,, && x = x_0 \\
\tilde{P}(x,\lambda|x_0,0) &= \frac{1}{x\sigma\sqrt{2\lambda+\frac{\hat{\mu}^2}{\sigma^2}}} \exp\left(\frac{(\ln x - \ln x_0)\left(\hat{\mu} + \sigma\sqrt{2\lambda+\frac{\hat{\mu}^2}{\sigma^2}}\right)}{\sigma^2}\right), && x < x_0
\end{aligned} \tag{2.73}$$

where $\hat{\mu} = \mu - \frac{1}{2}\sigma^2$. These result will become useful later in Ch. 4.

2.3.2 Black-Scholes option pricing theory

Although a description of market processes in terms of stochastic processes was put forward, it was not clear how it could be applied to investment decisions. The key breakthrough came with the advent of the BS option pricing formula, for which Robert Merton and Myron Scholes were awarded the Alfred Nobel Memorial Prize in Economic Sciences in 1997 [16]. Then the Black-Scholes approach has been extended to a wide variety of exotic options such as caps, floors, collars, collateralized mortage obligations, knockout options, swaptions, lookback options, barrier options and so on [34].

An option is a financial contract which gives the holder the right to buy or sell an asset with certain conditions within a specified period of time. A call option gives the holder the right to purchase shares of a stock at a specified price (*strike price*), on or before a specific date (*expiration time*). For instance, a call option on IBM stock provides its holder the right to buy the IBM shares that underlie the option at the exercise price. An European option means that it can be exercised only on the expiration time.

A key challenge in mathematical finance is to determine the fair price of a financial contract. The central concept underlying theories of asset pricing is the condition of *no arbitrage* : The price should be such that it is not possible to make a profit by a self-financing strategy without any probability of an intermediate loss. In other words, there is no "free lunch". Mathematically, this statement has been made precise as the *Fundamental Theorem of Asset Pricing:* A market model defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with asset prices $X(t)$ is arbitrage-free if and only if there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted asset prices are martingales with respect to $\mathbb{Q}$ [32]. The probabilities $\mathbb{Q}$ are also called *risk-neutral* probabilities. A risk-neutral asset pricing model thus requires that

$$\left\langle e^{-rt} X(t) \right\rangle^{\mathbb{Q}} = X(0) \tag{2.74}$$

during a given time interval $[0, t]$, where r represents a nominal interest rate [60] and $\langle ... \rangle^{\mathbb{Q}}$ denotes an expected value with respect to the risk-neutral probabilities. Eq. (2.74) indicates that the expected value of the asset price at time t is just that of a risk-free investment under continuous compounding. So we see that it is not possible to make a risk-free profit by either (i) borrowing money from a bank account and investing into the share or (ii) shortselling the share and investing the money into a bank account.

As an example, we can consider GBM. Recall that equivalent probability measures are those that define the same set of possible scenarios, i.e., let A denote a set of possible events then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if [153]

$$\mathbb{P}(A) = 1 \qquad \Longleftrightarrow \qquad \mathbb{Q}(A) = 1 \tag{2.75}$$

Equivalent probability measures can thus always be generated by reweighting the original probability measure with a process $Z(t)$ that satisfies $\langle Z(t) \rangle^{\mathbb{P}} = 1$, since in this case $\mathbb{Q}(A) = \langle Z(t) 1_A \rangle^{\mathbb{P}} = 1$ if $\mathbb{P}(A) = 1$. If we now choose for $Z(t)$ the process

$$Z(t) = \exp\left\{ -\frac{1}{2}\theta^2 t - \theta\, W(t) \right\}, \qquad \theta = \frac{\mu - r}{\sigma} \tag{2.76}$$

then the GBM $X(t)$ of Eq. (2.70) for a general drift parameter μ and volatility σ can be likewise expressed as

$$X(t) = x_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) t + \sigma \tilde{W}(t)\right\}, \qquad \tilde{W}(t) = W(t) + \theta\, t, \tag{2.77}$$

where it can be shown that $\tilde{W}(t)$ is a Brownian motion under the measure $\mathbb{Q}$ [153]. Under the risk-neutral measure, the process $X(t)$ thus satisfies the SDE

$$\dot{X}(t) = rX(t) + \sigma X(t)\tilde{\xi}(t) \tag{2.78}$$

with initial value $X(0) = x_0$. As before we interpret the multiplicative term $\sigma X(t)\xi(t)$ in Itô's sense and $\tilde{\xi}(t)$ is Gaussian white noise under the risk-neutral measure. We see that the expected value under the measure $\mathbb{Q}$ now satisfies the property Eq. (2.74), since

$$\begin{aligned} \langle X(t)\rangle^{\mathbb{Q}} &= \langle Z(t)X(t)\rangle^{\mathbb{P}} \\ &= x_0 \left\langle \exp\left\{-\frac{1}{2}\theta^2 t - \theta\, W(t) + (r - \sigma^2/2)t + \sigma\, \tilde{W}(t)\right\}\right\rangle^{\mathbb{P}} \\ &= x_0 e^{rt} \left\langle \exp\left\{-\frac{1}{2}\theta^2 t + \theta\sigma\, t - \sigma^2 t/2 - (\theta - \sigma)W(t)\right\}\right\rangle^{\mathbb{P}} \\ &= x_0 e^{rt}. \end{aligned} \tag{2.79}$$

Having obtained the risk-neutral probabilities of asset prices, we can express the price of a financial contract under the condition that no arbitrage is possible as

$$C(x,t) = \left\langle e^{-r(T-t)} Q(X(T))\right\rangle^{\mathbb{Q}}_{X(t)=x} \tag{2.80}$$

where $Q(X(T))$ is the general payoff function at the expiration time T and $\langle \ldots \rangle^{\mathbb{Q}}_{X(t)=x}$ denotes the expected value under the risk-neutral measure conditional on $X(t) = x$. The payoff denotes the value of the financial contract at T, e.g., the value of the option. Eq. (2.80) essentially means that the fair price of the contract (option) at a time $t < T$ is the expected value of the contract at the expiration time under the risk-neutral measure discounted to the time t.

Another important concept is *market completeness* : A market is said to be complete if any financial contract can be replicated by a self financing strategy (perfect hedge). The *Second Fundamental theorem of Asset Pricing* then states that a market is complete if and only if there is a unique risk-neutral measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. For the risk-neutral GBM discussed above one can indeed show that this measure is unique and thus market models based on it are complete [153].

The classical Black-Scholes theory derives a closed pricing form of the European call option. The standard Black Scholes formula for the option price, first put forward by F. Black and M. Scholes [17, 49], and by Merton in a different way [105], is based on some essential assumptions [17], namely

1. The short-term interest rate is known and is constant through time
2. The stock price follows the geometric Brownian motion.
3. The stock pays no dividends or other distributions.
4. The option is "European", that is, it can only be exercised at the expiration time.
5. There are no transaction costs in buying or selling the stock or the option.
6. It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate
7. There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date

The payoff of a plain European call option is given by

$$(X(T)-K)^+ = \begin{cases} X(T)-K, & if\ X(T)\geq K \\ 0, & if\ X(T)<K \end{cases} \tag{2.81}$$

where $K \geq 0$ the strike price. With the above assumptions, the European call option price at time $t < T$ under the no arbitrage condition in the Black-Scholes (BS) framework is

$$C_{\mathrm{BS}}(x,t) = \left\langle e^{-r(T-t)}(X(T)-K)^+ \right\rangle^{\mathbb{Q}}_{X(t)=x}. \tag{2.82}$$

The expectation value, Eq. (2.82), can be evaluated in analytical form leading to the classical Black-Scholes formula [139]

$$C_{\mathrm{BS}}(x,t) = x\Phi(\omega) - Ke^{-r(T-t)}\Phi(\omega - \sigma\sqrt{(T-t)}), \tag{2.83}$$

where ω is given by

$$\omega = \frac{r(T-t) + \sigma^2(T-t)/2 - \ln(K/x)}{\sqrt{\sigma^2(T-t)}} \tag{2.84}$$

and Φ denotes the cumulative distribution function of a standard normal random variable $\Phi(x) = \int_{-\infty}^{x} \exp(-u^2/2)du/\sqrt{2\pi}$.

The classical Black-Scholes formula Eq. (2.83) can also be found as a solution of a PDE. In order to derive this equation we require the Feynman-Kac formula, which means that we will find a PDE for the general discounted final time payoff

$$u(x,t) = \left\langle e^{-\int_t^T r(X(\tau))d\tau} Q(X(T)) \right\rangle_{X(t)=x} \tag{2.85}$$

where $X(\tau)$ is defined in Eq. (2.25), $r(x)$ is some specified function and $Q(x)$ is the payoff at maturity time $T > t$ with the condition $X(t) = x$. Now let

$$\begin{aligned} z_1(s) &= e^{-\int_t^s r(X(\tau))d\tau} \\ z_2(s) &= u(X(s), s) \end{aligned} \tag{2.86}$$

where $u(x,t)$ solves the equation

$$\frac{\partial u(x,t)}{\partial t} + \mu(x)\frac{\partial u(x,t)}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2 u(x,t)}{\partial x^2} - r(x)u(x,t) = 0\,, \tag{2.87}$$

with final time condition $u(x,T) = Q(x)$. Taking the derivative of $z_1(s)z_2(s)$ with respect to s one obtains

$$\frac{d}{ds}\left(z_1(s)z_2(s)\right) = \dot{z}_1(s)z_2(s) + z_1(s)\dot{z}_2(s)\,. \tag{2.88}$$

Here there is no additional term compared to the derivative of normal functions as no correlations exist between functions $z_1(s)$ and $z_2(s)$. Although the function $z_1(s)$ contains the underlying stochastic process, it is actually an integral of $X(\tau)$ which leads to the loss of higher order terms during the calculation of $\frac{d}{ds}\left(z_1(s)z_2(s)\right)$. By Itô's formula, it follows that

$$
\begin{aligned}
\dot{z_1}(s) &= -r(X(s))z_1(s) \\
\dot{z_2}(s) &= \left(\frac{\partial u(X(s),s)}{\partial s} + \frac{\partial u(X(s),s)}{\partial x}(\mu(X(s)) + \sigma(X(s))\xi(s))\right. \\
&\quad \left. + \frac{\sigma(X(s))^2}{2}\frac{\partial^2 u(X(s),s)}{\partial x^2}\right) . \qquad (2.89)
\end{aligned}
$$

Then $\frac{d}{ds}\left(z_1(s)z_2(s)\right)$ in Eq. (2.88) becomes

$$
\begin{aligned}
\frac{d}{ds}\left(z_1(s)z_2(s)\right) &= \left(-r(X(s))z_1(s)\right)z_2(s) + z_1(s)\left(\tfrac{\partial u(X(s),s)}{\partial s}\right. \\
&\quad + \tfrac{\partial u(X(s),s)}{\partial x}(\mu(X(s)) + \sigma(X(s))\xi(s)) \\
&\quad \left. + \tfrac{\sigma(X(s))^2}{2}\tfrac{\partial^2 u(X(s),s)}{\partial x^2}\right) . \qquad (2.90)
\end{aligned}
$$

Taking the conditional expected value on both sides of Eq. (2.90) and noticing that $u(x,s)$ is the solution of Eq. (2.87), we obtain

$$
\begin{aligned}
\frac{d}{ds}\left\langle e^{-\int_t^s r(X(\tau))d\tau}u(X(s),s)\right\rangle_{X(t)=x} &= \left\langle \frac{d}{ds}\left(z_1(s)z_2(s)\right)\right\rangle_{X(t)=x} \\
&= \left\langle z_1(s)\frac{\partial u(X(s),s)}{\partial x}\sigma(X(s))\right\rangle_{X(t)=x} \langle \xi(s)\rangle \\
&= 0 \, . \qquad (2.91)
\end{aligned}
$$

Here the right hand side can be factorized again because there is no correlation among these terms. Therefore one derives that $\langle z_1(s)z_2(s)\rangle_{X(t)=x}$ is constant for all $s \geq t$ and thus

$$
\langle z_1(T)z_2(T)\rangle_{X(t)=x} = z_1(t)z_2(t) = u(x,t), \qquad (2.92)
$$

where $u(x,t)$ satisfies Eq. (2.87). Eq. (2.87) is known as the Feynman-Kac formula [79].

Let $r(x) = r$ and $Q(X(T)) = (X(T) - K)^+$ in Eq. (2.85), and we consider the expected value of GBM under the risk-neutral measure $\langle ... \rangle^{\mathbb{Q}}$. We will find that the European call option $C_{\mathrm{BS}}(x,t)$ in Eq. (2.82), according to the Feynman-Kac formula, is the solution of the equation

$$
\left(\frac{\partial}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} - r + rx\frac{\partial}{\partial x}\right)C_{\mathrm{BS}}(x,t) = 0 \qquad (2.93)
$$

with the initial and boundary conditions

$$C_{\rm BS}(x,T) = \max\{x-K,0\}, \qquad x \geq 0 \tag{2.94a}$$
$$C_{\rm BS}(0,t) = 0, \qquad t \leq T \tag{2.94b}$$
$$C_{\rm BS}(x,t) \to x, \qquad x \to \infty \tag{2.94c}$$

which is known as the Black-Scholes PDE.

If we let $t=0$ and set $T=t$ in the expectation value Eq. (2.82), then we could get the expression

$$C^{(\rm A)}_{\rm BS}(x,t) = \left\langle e^{-rt}(X(t)-K)^{+}\right\rangle^{\mathbb{Q}}_{X(0)=x} \tag{2.95}$$

which can be likewise evaluated by the Black-Scholes formula Eq. (2.83). Moreover, $C^{(\rm A)}_{\rm BS}(x,t)$ satisfies the PDE

$$\frac{\partial}{\partial t}C^{(\rm A)}_{\rm BS}(x,t) = \left(\frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} - r + rx\frac{\partial}{\partial x}\right)C^{(\rm A)}_{\rm BS}(x,t) \tag{2.96}$$

with the initial and boundary conditions

$$C^{(\rm A)}_{\rm BS}(x,0) = \max\{x-K,0\}, \qquad x \geq 0 \tag{2.97a}$$
$$C^{(\rm A)}_{\rm BS}(0,t) = 0, \qquad t \geq 0 \tag{2.97b}$$
$$C^{(\rm A)}_{\rm BS}(x,t) \to x, \qquad x \to \infty \tag{2.97c}$$

Eq. (2.96) follows directly from the observation that the BS formula Eq. (2.83) implies

$$\frac{\partial}{\partial t}C_{\rm BS} = -\frac{\partial}{\partial T}C_{\rm BS} \tag{2.98}$$

for $C_{\rm BS}$ given by the expected value Eq. (2.82). In the remainder, we will generally distinguish between options defined by expected values as in Eq. (2.95) (we shall call it *type A* or forward option in our discussion) and in Eq. (2.80) (we shall call it *type B* or backward option). Even though there is no essential difference in the standard BS theory in view of Eq. (2.98), taking into account non-Markovian effects in the underlying asset pricing model requires us to distinguish the two. The *type B* option then exhibits an additional dependency on the initial time 0 and price $X(0)$ (see Ch. 4).

Chapter 3

Generalized CTRW models

3.1 Introduction to fractional calculus

Since fractional operators are used in our later work, it is necessary to recall some results on the fractional calculus firstly. The theory of derivatives of non integer order goes back to Leibniz in 1695. Regarding the notation $\frac{d^n y}{dt^n}$ for the nth derivative of a function y with respect to t, L'Hospital wrote in a letter to Leibniz: "What if $n = 1/2$?" In a response, Leibniz said, "This is an apparent paradox from which, one day, useful consequences will be drawn" [33]. After more than three century's effort made by the scientists, different possible ways are proposed to extend the ordinary calculus to define fractional derivatives, but we will focus on the Riemann-Liouville definition.

3.1.1 Special functions: Gamma and Mittag-Leffler functions

One of the basic functions of the fractional calculus is Euler's gamma function $\Gamma(z)$, which extends the factorial $n!$ and allows n to take non integer and even complex values.

The Gamma function $\Gamma(z)$ is defined by the integral [130]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \,, \tag{3.1}$$

which converges in the right half of the complex plane when $Re(z) > 0$. Indeed, it

follows

$$\begin{aligned}\Gamma(x+iy) &= \int_0^\infty e^{-t}t^{x-1+iy} \\ &= \int_0^\infty e^{-t}t^{x-1}e^{iy\ln(t)} \\ &= \int_0^\infty e^{-t}t^{x-1}[\cos(y\ln(t)) + i\sin(y\ln(t))]dt \end{aligned} \tag{3.2}$$

The expression in the square brackets above is bounded for all t, convergence at infinity is provided by e^{-t}, and for the convergence at $t = 0$, we require $x = Re(z) > 1$.

Some properties of the Gamma function are listed as follows [130]

- One of the basic properties of the Gamma function is

$$\Gamma(z+1) = z\Gamma(z)\,, \tag{3.3}$$

which could be easily proved by integrating by parts

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t}t^{z}dt \\ &= [-e^{-t}t^{z}]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}dt \\ &= z\Gamma(z)\,. \end{aligned} \tag{3.4}$$

Obviously, $\Gamma(1) = 1$ and by the relation above we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!\,. \tag{3.5}$$

While the exponential function e^z, plays a very important role in the theory of integer order differential equations, its one parameter generalization defined as [130]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}\,, \tag{3.6}$$

was introduced by Mittag-Leffler [112, 113] and investigated also by Wiman [162]. For special values of α, the following special cases of the Mittag-Leffler function can be obtained

- for $\alpha = 0$, $E_0(z) = \frac{1}{1-z}, \quad |z| < 1$

- for $\alpha = 1$, $E_1(z) = e^z$
- for $\alpha = 2$, $E_2(z) = \cosh(\sqrt{z})$, where $\cosh(x)$ is defined as

$$\cosh(x) = \frac{e^x + e^{-x}}{2}\,. \tag{3.7}$$

Now let us introduce one special case of Mittag-Leffler function as

$$E_\alpha(at^\alpha) = \sum_{k=0}^{\infty} \frac{(at^\alpha)^k}{\Gamma(\alpha k + 1)} \tag{3.8}$$

and we are interested in its Laplace transform as we will need it in our later discussion. As we know that

$$\mathcal{L}\{t^{\alpha-1}\} = \int_0^\infty e^{-\lambda t} t^{\alpha-1} dt \tag{3.9}$$

by letting $u = \lambda t$ we could get

$$\mathcal{L}\{t^{\alpha-1}\} = \frac{1}{\lambda^\alpha} \int_0^\infty e^{-u} u^{\alpha-1} du = \frac{1}{\lambda^\alpha}\Gamma(\alpha)\,. \tag{3.10}$$

As one could see that

$$E_\alpha(at^\alpha) = \frac{1}{\Gamma(1)} + \frac{at^\alpha}{\Gamma(\alpha+1)} + \frac{(at^\alpha)^2}{\Gamma(2\alpha+1)} + \frac{(at^\alpha)^3}{\Gamma(3\alpha+1)} + \cdots \tag{3.11}$$

Applying Laplace transform to each term, as well as $\mathcal{L}\{1\} = 1/\lambda$ and the result from Eq. (3.10), it follows that

$$\begin{aligned} \mathcal{L}\{E_\alpha(at^\alpha)\} &= \frac{1}{\lambda} + \frac{a}{\lambda^{\alpha+1}} + \frac{(a)^2}{\lambda^{2\alpha+1}} + \frac{(a)^3}{\lambda^{3\alpha+1}} + \cdots \\ &= \frac{1}{\lambda}\left(1 + \frac{a}{\lambda^\alpha} + \frac{(a)^2}{\lambda^{2\alpha}} + \frac{(a)^3}{\lambda^{3\alpha}} + \cdots\right) \\ &= \frac{\lambda^{\alpha-1}}{\lambda^\alpha - a}\,, \quad \text{if } \mathrm{Re}(\lambda) > |\mathrm{a}|^{1/\alpha}\,. \end{aligned} \tag{3.12}$$

3.1.2 The Riemann-Liouville fractional integral

Let us now turn to the theory of derivatives of arbitrary order, known as fractional derivatives which have generalized the notions of integer-order differentiation. Throughout our work, the Riemann-Liouville fractional integral [130] is defined through

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{3.13}$$

where $\alpha > 0$ and the subscripts a and t denote the two limits related to the operation of fractional differentiation. The Riemann-Liouville fractional integral will be used to construct a fractional derivative or the derivative of arbitrary real order as

$$ {}_aD_t^{\beta} f(t) = \frac{d^n}{dt^n} {}_aD_t^{\beta-n} f(t) \tag{3.14}$$

with $\beta > 0$ and $n-1 \leq \beta < n$. The Riemann-Liouville type derivative is not the only way to define the derivative of arbitrary real order, and there are other possible definitions, such as the Caputo derivative, where the order of differentiation and integration is changed compared with Eq. (3.14). But we will only consider Riemann-Liouville type derivative in our discussion. It could be observed that the definition of the fractional differentiation is non local due to the presence of the integral.

The Laplace convolution of two function $f(t)$ and $g(t)$ is defined as

$$\begin{aligned} f(t) * g(t) &= \int_0^\infty f(t-\tau) g(\tau) d\tau \\ &= \int_0^\infty f(\tau) g(t-\tau) d\tau \end{aligned} \tag{3.15}$$

with the assumption that both function are equal to zero for $t < 0$. The Laplace transform of the convolution is equal to the product of the Laplace transforms of the functions

$$\mathcal{L}\{f(t) * g(t)\} = F(\lambda) G(\lambda) \tag{3.16}$$

under the assumption that both $\mathcal{L}\{f(t)\} = F(\lambda)$ and $\mathcal{L}\{g(t)\} = G(\lambda)$ exist. As a consequence, we observe that the Riemann-Liouville integral with $a = 0$ can be written as

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t)\,. \tag{3.17}$$

Therefore, the Laplace transform is a useful tool in solving fractional order differential equations. Since the Laplace transform of the function $t^{\alpha-1}$ is given in Eq. (3.10) as [130], using the formula for the Laplace transform of the convolution, the Laplace transform of the Riemann-Liouville fractional integral with $a = 0$ is found as

$$\mathcal{L}\{{}_0D_t^{-\alpha} f(t)\} = \lambda^{-\alpha} F(\lambda) \tag{3.18}$$

Another useful formula is the Laplace transform of the derivative of an integer order n of the function $f(t)$

$$\mathcal{L}\{f^{(n)}(t)\} = \lambda^n F(\lambda) - \sum_{k=0}^{n-1} \lambda^{n-k-1} f^{(k)}(0) = \lambda^n F(\lambda) - \sum_{k=0}^{n-1} \lambda^k f^{(n-k-1)}(0) \tag{3.19}$$

which could be obtained from the definition by integrating by parts under the assumption that the corresponding integrals exist.

3.1.3 The Riemann-Liouville fractional operator

The Riemann-Liouville fractional operator, ${}_0D_t^{1-\alpha}$, is defined through [130]

$${}_0D_t^{1-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{3.20}$$

where $0 < \alpha < 1$. Finally, the use of the formula for the Laplace transform leads to [130]

$$\mathcal{L}\{{}_0D_t^{1-\alpha} f(t)\} = \lambda^{1-\alpha} \tilde{f}(\lambda) \tag{3.21}$$

where $\tilde{f}(\lambda)$ is the Laplace transform of $f(t)$.

When $\alpha = 0$, the Riemann-Liouville derivative becomes a normal derivative.

As an example, we consider the fractional Riemann-Liouville derivative ${}_0D_t^{1-\alpha}$ of the power function given by

$${}_0D_t^{1-\alpha} t^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(\mu+\alpha)} t^{\mu+\alpha-1} . \tag{3.22}$$

When $\mu = 0$, the equation above becomes

$${}_0D_t^{1-\alpha} 1 = \frac{\Gamma(1)}{\Gamma(\alpha)} t^{\alpha-1} , \tag{3.23}$$

which indicates that the fractional derivative of a constant does not vanish. Indeed, this is a difference between the fractional derivative and a standard derivative. But if we let $\alpha \to 0$ first, we will find that the result is zero, so the limits $\alpha \to 0$ and $\mu \to 0$ can not be interchanged.

3.2 CTRW with x–dependent drift and diffusion

The stochastic representation of a force-free CTRW in terms of coupled Langevin equations has already been given in Eqs. (2.51a)–(2.51b). In the following, we consider the generalization to arbitrary drift and diffusion terms in the sense of the general SDE Eq. (2.25). We thus introduce a generalized CTRW model as

$$\dot{X}(s) = \mu(X(s)) + \sigma(X(s))\xi(s) \tag{3.24a}$$

$$\dot{T}(s) = \eta(s). \tag{3.24b}$$

Here, $\mu(x)$ and $\sigma(x)$ satisfy the same conditions as in Eq. (2.25) and $\xi(s)$ is white Gaussian noise with properties $\langle \xi(s) \rangle = 0$ and $\langle \xi(s_2)\xi(s_1) \rangle = \delta(s_2 - s_1)$. The term $\sigma(X(s))\xi(s)$ is still defined in Itô sense so that Eq. (3.24a) defines a normal diffusive process $X(s)$ in the operational time s. The process $T(s)$ is a one-sided Lévy process assumed to be statistically independent from $X(s)$. The generalized CTRW is again defined as the subordinated process $Y(t) = X(S(t))$, where $S(t)$ is the inverse of the Lévy subordinator, see Eq. (2.53). Note that the description of $T(s)$ in terms of an equation of motion drive by the associated noise $\eta(s)$ is not necessary in principle. However, the introduction of the Lévy noise is useful when discussing functionals of $Y(t)$ [22].

To understand the generalized CTRW process, it is important to highlight the non-Markovian nature of the process due to possibly long waiting times in $T(s)$. Therefore, single-time or conditional PDFs alone are not sufficient to characterize the process. The one- and two-point PDFs of $Y(t)$ follow in a straightforward way [12]

$$
\begin{aligned}
f_Y(x,t) &= \langle \delta(x - Y(t)) \rangle \\
&= \langle \delta(x - X(S(t))) \rangle \\
&= \left\langle \int_0^\infty \delta(x - X(s))\delta(s - S(t))ds \right\rangle \\
&= \int_0^\infty \langle \delta(x - X(s)) \rangle \langle \delta(s - S(t)) \rangle \, ds \\
&= \int_0^\infty f_X(x,s)h(s,t)ds
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
&f_Y(x_2,t_2,x_1,t_1) \\
&= \langle \delta(x_2 - Y(t_2))\delta(x_1 - Y(t_1)) \rangle \\
&= \langle \delta(x_2 - X(S(t_2)))\delta(x_1 - X(S(t_1))) \rangle \\
&= \left\langle \int_0^\infty ds_2 \int_0^\infty \delta(x_2 - X(s_2))\delta(s_2 - S(t_2))\delta(x_1 - X(s_1))\delta(s_1 - S(t_1))ds_1 \right\rangle \\
&= \int_0^\infty ds_2 \int_0^\infty \langle \delta(x_2 - X(s_2))\delta(x_1 - X(s_1)) \rangle \langle \delta(s_2 - S(t_2))\delta(s_1 - S(t_1)) \rangle \, ds_1 \\
&= \int_0^\infty ds_2 \int_0^\infty f_X(x_2,s_2,x_1,s_1)h(s_2,t_2,s_1,s_1)ds_1
\end{aligned}
\tag{3.26}
$$

which can be extended to n-point by analogy. Here, $f_X(x,s)$ and $f_X(x_2,s_2,x_1,s_1)$ are the one- and two-point PDFS of $X(s)$ defined in Eq. (3.24a), respectively, and $h(s,t)$ and $h(s_2,t_2,s_1,s_1)$ are the one- and two-point PDFs of $S(t)$. Before we discuss the properties of $S(t)$ and its associated PDFs in more detail, we provide a basic introduction into Lévy processes.

3.2.1 Lévy processes

Lévy processes are named after the French mathematician Paul Lévy whose work plays an instrumental role in bringing together an understanding and characterization of processes with stationary independent increments. Generally speaking, a Lévy process is a continuous time stochastic process with independent and stationary increments. Its strict definition is given as [32, 84]

Definition 4 *(Lévy process) A stochastic process* $\{X(t),\, t \geq 0\}$ *defined on a probability* *space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be a Lévy process if*

1. *The paths of* $X(t)$ *are right continuous with left limits* $\mathbb{P}$*-almost surely.*
2. $X(0) = 0$.
3. *For* $0 \leq t_1 \leq t_2$, $X(t_2) - X(t_1)$ *is equal in distribution to* $X(t_2 - t_1)$.
4. *For* $0 \leq t_1 \leq t_2$, $X(t_2) - X(t_1)$ *is independent of* $\{X(u), u \leq t_1\}$.

Comparing with the definition of Brownian motion in Definition 1, one could find that Brownian motion falls indeed into the class of Lévy processes. However, it contains many more processes.

As the notion of an infinitely divisible distribution has an intimate relationship with Lévy process, it is necessary to spend a little time on infinite divisibility.

Definition 5 *(Infinitely divisible distribution) A real valued random variable* X *has an infinitely divisible distribution if for each positive integer* n*, there exist a sequence of independent identically distributed random variables* $X_1, \cdots, X_n$ *such that the equality*

$$X_1 + \cdots + X_n = X \tag{3.27}$$

holds in distribution.

Usually the infinitely divisible distribution is characterised by its characteristic exponent Ψ which is known as the Lévy-Khintchine formula.

Theorem 1 *(Lévy-Khintchine formula). [84] A real valued random variable* X *that is infinitely divisible has a characteristic exponent* Ψ *for every real number* v

$$\left\langle e^{iXv} \right\rangle = \int_{-\infty}^{\infty} e^{ixv} f(x) dx = e^{-\Psi(v)} \tag{3.28}$$

where $f(x)$ *is the PDF of* X *and*

$$\Psi(v) = iav + \frac{1}{2}\sigma^2 v^2 + \int_{-\infty}^{\infty} \left(1 - e^{ivx} + ivx\mathbf{1}_{|x|<1}\right) \Pi(dx) \, . \tag{3.29}$$

Here, $\Pi(dx)$ *is a so called Lévy measure satisfying* $\int_{-\infty}^{\infty} \max(1, x^2)\Pi(dx) < \infty$*, and* $\sigma \geq 0$ *and* a *are real valued numbers.*

According to the definition of a Lévy process it could be found that for any $t > 0$, $X(t)$ is a random variable with the property of infinite divisibility. This could be derived from the fact that for any positive integer n

$$X(t) = X(t/n) + (X(2t/n) - X(t/n)) + (X(3t/n) - X(2t/n)) \\ + \cdots + (X((n-1)t) + X((n-2)t/n)) + (X(t) - X((n-1)t/n)) \tag{3.30}$$

as well as the fact that X has stationary independent increments. Now we can define for any real number v and $t > 0$

$$\Psi_t(v) = -\ln \left\langle e^{iX(t)v} \right\rangle . \tag{3.31}$$

Then using Eq. (3.30) as well as stationary increments of X, it follows that for any two positive integers m, n

$$X(m) = X(1) + (X(2) - X(1)) + (X(3) - X(2)) + \cdots + (X(m-1) \\ - X(m-2)) + (X(m) - X(m-1)) = mX(1)\,, \tag{3.32}$$

and

$$X(m) = X(m/n) + (X(2m/n) - X(m/n)) + (X(3m/n) - X(2m/n)) + \cdots \\ + (X((n-1)m) + X((n-2)m/n)) + (X(m) - X((n-1)m/n)) \\ = nX(m/n)\,, \tag{3.33}$$

which immediately result into

$$\Psi_m(v) = -\ln \left\langle e^{iX(m)v} \right\rangle = -\ln \left\langle e^{imX(1)v} \right\rangle = -m \ln \left\langle e^{iX(1)v} \right\rangle , \tag{3.34}$$

and

$$\Psi_{m/n}(v) = -\ln \left\langle e^{iX(m)v} \right\rangle = -\ln \left\langle e^{inX(m/n)v} \right\rangle = -n \ln \left\langle e^{iX(m/n)v} \right\rangle . \tag{3.35}$$

Here $\langle e^{imX(1)v} \rangle$ and $\langle e^{inX(m/n)v} \rangle$ can be factorized because of the independent increments of X. Hence

$$m\Psi_1(v) = \Psi_m(v) = n\Psi_{m/n}(v) \tag{3.36}$$

which indicates that for any rational $t = m/n > 0$

$$\Psi_t(v) = t\Psi_1(v)\,. \tag{3.37}$$

The same property also holds for irrational t. Hence any Lévy process has the property

$$\left\langle e^{iX(t)v} \right\rangle = e^{-t\Psi(v)} \tag{3.38}$$

where $\Psi(v) = \Psi_1(v)$ represents the characteristic exponent of X_1, which has an infinite divisible distribution. Next we will discuss special cases.

- Taking Brownian motion with PDF given in Eq. (2.11), one could obtain that

$$\int_{-\infty}^{\infty} e^{ixv} f(x,t)dx = e^{-Dtv^2}\,. \tag{3.39}$$

  It is immediately derived that the characteristic exponent $\Psi(v) = Dv^2$.

- For Brownian motion with drift defined by Eq. (2.60), its PDF could be derived as

$$f(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}\,, \tag{3.40}$$

  Hence, its characteristic function can be found as

$$\int_{-\infty}^{\infty} e^{ixv} f(x,t)dx = e^{i\mu tv - \frac{1}{2}\sigma^2 tv^2} \tag{3.41}$$

  with characteristic exponent $\Psi(v) = -i\mu v + \frac{1}{2}\sigma^2 v^2$.

- A compound Poisson process with intensity $\lambda > 0$ and jump size distribution F is a stochastic process $X(t)$ defined as

$$X(t) = \sum_{i=1}^{N(t)} \xi_i \tag{3.42}$$

  where jump sizes ξ_i are i.i.d with distribution F and $N(t)$ is a Poisson process with intensity λ, independent from ξ_i. Recall that a random viable X is said to be a Poisson random variable with some parameter $\lambda > 0$ if $P\{X = k\} = e^{-\lambda}\lambda^k/k!$

for $k = 0, 1, 2, \cdots$. Then it could be found that

$$\sum_{k\geq 0} e^{ivk} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k\geq 0} \frac{e^{ivk}\lambda^k}{k!} = e^{-\lambda} e^{\lambda e^{iv}} = e^{-\lambda(1-e^{iv})}, \tag{3.43}$$

from which we see that in the Lévy-Khintchine formula $a = \sigma = 0$ and $\Pi = \lambda\delta(x-1)$.

A Poisson process $\{N(t), t \geq 0\}$ is a Lévy process which is Poisson distributed with parameter λt at time $t > 0$. By Eq. (3.43) it follows that

$$\left\langle e^{ivN(t)} \right\rangle = e^{-\lambda t(1-e^{iv})} \tag{3.44}$$

and hence the characteristic exponent of the Poisson process $N(t)$ is $\Psi(v) = \lambda(1 - e^{iv})$. For the compound Poisson process, one could see that

$$\left\langle e^{ivX(t)} \right\rangle = \left\langle \left\langle e^{iv \sum_{i=1}^{n} \xi_i} | N(t) = n \right\rangle \right\rangle . \tag{3.45}$$

As we could derive that

$$\begin{aligned}
\left\langle \left\langle e^{iv \sum_{i=1}^{n} \xi_i} | N(t) = n \right\rangle \right\rangle &= \sum_{n\geq 0} \left\langle e^{iv \sum_{i=1}^{n} \xi_i} | N(t) = n \right\rangle P(N(t) = n) \\
&= \sum_{n\geq 0} \left\langle e^{iv\xi_1} \right\rangle^n \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= e^{-\lambda t} \sum_{n\geq 0} \left\langle e^{iv\xi_1} \right\rangle^n \frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t} \exp\left(\lambda t \left\langle e^{iv\xi_1} \right\rangle\right) \\
&= \exp\left(-\lambda t \left(1 - \left\langle e^{iv\xi_1} \right\rangle\right)\right) \\
&= \exp\left(-\lambda t \int_{-\infty}^{\infty} \left(1 - e^{ivx}\right) F(x) dx\right) .
\end{aligned} \tag{3.46}$$

Hence by introducing $\Pi(dx) = \lambda F(x)dx$, in the Lévy-Khintchine formula for a compound Poisson process, its characteristic exponent has the following form

$$\Phi(\lambda) = \int_0^{\infty} \left(1 - e^{-\lambda x}\right) \Pi(dx) . \tag{3.47}$$

Comparing with Eq. (3.29), we see that a Lévy process can be intuitively interpreted as a stochastic process containing continuous fluctuations in the form of a Brownian motion with drift and, in addition, jumps occurring at Poissonian time points with a certain jump PDF Π. However, the mathematical framework also allows for non-normalizable functions Π. An important example are stable distributions, which are infinitely divisible distributions defined as: [39, 84]

Definition 6 *(Stable distribution) A random variable, X, is said to have a stable distribution if the distributional equality*

$$X_1 + \cdots + X_n = a_n X + b_n \tag{3.48}$$

holds for all $n \geq 1$, where $X_1, \ldots, X_n$ are independent copies of X, $a_n > 0$ and b_n.

By subtracting b_n/n from each term on the left hand side of Eq. (3.48), one could see that this definition indicates that any stable random variable is infinitely divisible. It has been found that necessarily $a_n = n^{1/\alpha}$ for $\alpha \in (0, 2]$ by Feller [39]. If a stable distribution observes Eq. (3.48) but with $b_n = 0$, it becomes one smaller class known as the α-Stable distribution observing

$$X_1 + \cdots + X_n = n^{1/\alpha} X\,. \tag{3.49}$$

When $\alpha = 2$, it corresponds to zero mean Gaussian random variables. Stable random variables with $\alpha \in (0, 1) \cup (1, 2)$ observing the relation Eq. (3.48) have characteristic exponents of the form

$$\psi(v) = c|v|^\alpha \left(1 - i\beta \tan\frac{\pi\alpha}{2}\mathrm{sgn}v\right) + iv\eta \tag{3.50}$$

where η is real number, $\beta \in [-1, 1]$, and $c > 0$. Stable random variables with $\alpha = 1$ observing the relation Eq. (3.48) have characteristic exponents of the form

$$\psi(v) = c|v| \left(1 + i\beta\frac{2}{\pi}\ln|v|\mathrm{sgn}v\right) + iv\eta \tag{3.51}$$

where η is real number, $\beta \in [-1, 1]$, and $c > 0$. The connection with the Lévy-Khintchine formula is established if we note that these characteristic exponents arise from the power-law jump amplitudes $\sigma = 0$ and

$$\Pi(dx) = \begin{cases} c_1 x^{-1-\alpha} dx, & if\ x \in (0, \infty) \\ c_2 |x|^{-1-\alpha} dx, & if\ x \in (-\infty, 0) \end{cases} \tag{3.52}$$

where $c = c_1 + c_2$, $c_1, c_2 \geq 0$ and $\beta = (c_1 - c_2)/(c_1 + c_2)$ if $\alpha \in (0,1) \cup (1,2)$ and $c_1 = c_2$ if $\alpha = 1$.

Now let us introduce increasing Lévy processes, which are also known as subordinators because such processes can be used as time changes for other process. Subordinators are very important ingredients for forming subordinated models in finance [26]. For the convenience of our discussion of subordinators, we present the Laplace exponent Φ.

Theorem 2 *(Lévy-Khintchine formula). If Φ is the Laplace exponent of a subordinator $X(t)$, then there exist a unique pair (k, d) of nonnegative real number with $\int_0^\infty (1 \wedge x^2)\Pi(dx) < \infty$, such that for every $\lambda \geq 0$*

$$\left\langle e^{-\lambda X(t)} \right\rangle = \int_0^\infty e^{-\lambda x} f(x,t) dx = e^{-t\Phi(\lambda)} \tag{3.53}$$

with the PDF $f(x,t)$ of $X(t)$ and

$$\Phi(\lambda) = k + d\lambda + \int_0^\infty \left(1 - e^{-\lambda x}\right) \Pi(dx) . \tag{3.54}$$

In the special case when $\int_0^\infty \Pi(dx) < \infty$, $X(t)$ is of finite activity which could be written as a compound Poisson process. In cases where this does not hold, $X(t)$ is an infinite activity process as it has an infinite number of very small jumps in any finite time interval which include the one sided Lévy-stable process and the tempered Lévy-stable process.

The one side Lévy stable distribution L_α with $0 < \alpha < 1$ can be represented by $k = d = 0$ and

$$\Pi(dx) = \begin{cases} c x^{-1-\alpha} dx, & \text{if } x \in (0, \infty) \\ 0, & \text{if } x \in (-\infty, 0) \end{cases} \tag{3.55}$$

The associated Laplace exponent Φ is thus

$$\begin{aligned}\Phi(\lambda) &= \lim_{\epsilon\to 0}\int_\epsilon^\infty \left(1-e^{-\lambda x}\right) c x^{-1-\alpha} dx \\ &= \lim_{\epsilon\to 0} c\left[-\frac{x^{-\alpha}}{\alpha}(1-e^{-\lambda x})\Big|_\epsilon^\infty + \frac{\lambda}{\alpha}\int_\epsilon^\infty x^{-\alpha} e^{-\lambda x} dx\right] \\ &= \frac{c\lambda}{\alpha}\int_0^\infty x^{-\alpha} e^{-\lambda x} dx. \end{aligned} \tag{3.56}$$

By the variable transform $y = \lambda x$, it leads to

$$\begin{aligned}\int_0^\infty \left(1-e^{-\lambda x}\right) c x^{-1-\alpha} dx &= \frac{c\lambda^\alpha}{\alpha}\int_0^\infty y^{-\alpha} e^{-y} dy \\ &= \frac{c\lambda^\alpha}{\alpha}\Gamma(1-\alpha) \end{aligned} \tag{3.57}$$

In particular, by choosing

$$c = \frac{\alpha}{\Gamma(1-\alpha)}\,, \tag{3.58}$$

we see that $L_\alpha(x)$ with $0 < \alpha < 1$ has a simple Laplace exponent given by

$$\Phi(\lambda) = \lambda^\alpha. \tag{3.59}$$

Hence, $L_\alpha(x)$ could be characterized in terms of Laplace transform [10, 82]

$$\int_0^\infty e^{-\lambda x} L_\alpha(x) dx = e^{-\lambda^\alpha}\,. \tag{3.60}$$

We see that $L_\alpha(x)$ is associated with power-law jump amplitudes in the Lévy-Khintchine formula. In a physics context, it is sometimes desirable to truncate power-laws at large scales in order to obtain finite moments. A mathematically convenient way to introduce such a truncation is by exponential tempering. A tempered distribution has the Laplace exponent

$$\begin{aligned}\Phi(\lambda,\zeta) &= \int_0^\infty \left(1-e^{-\lambda x}\right) e^{-\zeta x}\Pi(dx) \\ &= \int_0^\infty \left(1-e^{-(\lambda+\zeta)x}\right)\Pi(dx) - \int_0^\infty \left(1-e^{-\zeta x}\right)\Pi(dx) \\ &= \Phi(\lambda+\zeta) - \Phi(\zeta)\,. \end{aligned} \tag{3.61}$$

In particular, the tempered Lévy-stable distribution has the Laplace exponent

$$\Phi(\lambda, \zeta) = (\lambda + \zeta)^\alpha - \zeta^\alpha \,. \tag{3.62}$$

Note that in the case, the jump amplitudes Π are still non-normalizable, i.e., the associated process is of infinite activity. After the simple calculation, one could find that

$$\begin{aligned}\Phi(\lambda, \zeta) &= (\lambda + \zeta)^\alpha - \zeta^\alpha \\ &= \lambda^\alpha \left(1 + \frac{\zeta}{\lambda}\right)^\alpha - \zeta^\alpha \\ &= \lambda^\alpha \left(1 + \frac{\zeta}{\lambda}\alpha + \cdots\right) - \zeta^\alpha \\ &\approx \lambda^\alpha, \quad \lambda \longrightarrow \infty \end{aligned} \tag{3.63}$$

and

$$\begin{aligned}\Phi(\lambda, \zeta) &= (\lambda + \zeta)^\alpha - \zeta^\alpha \\ &= \zeta^\alpha \left(1 + \frac{\lambda}{\zeta}\right)^\alpha - \zeta^\alpha \\ &= \zeta^\alpha \left(1 + \frac{\lambda}{\zeta}\alpha + \cdots\right) - \zeta^\alpha \\ &\approx \alpha\zeta^{\alpha-1}\lambda, \quad \lambda \longrightarrow 0 \,, \end{aligned} \tag{3.64}$$

which indicate that for small times (large λ) Eq. (3.62) recovers a Levy-stable process, while for large times (small λ) it recovers a normal one with exponential waiting times. This is the whole point of the tempering. The pure subdiffusive CTRW is recovered when $\zeta = 0$, whereas the normal diffusion is obtained when $\zeta \longrightarrow \infty$. Therefore the tempered Lévy-stable case exhibits crossover scaling between subdiffusive and normal diffusive regime.

3.2.2 The inverse one-sided Lévy stable process $S(t)$

The introduction of the intermediate process $S(t)$ in Eq. (2.51a–2.51b), greatly affects the Markovian process $X(s)$ defined in Eq. (3.24a). Unlike the process $X(s)$, the new subordinated process $Y(t)$ exhibits non Markovian characteristics when the process $T(s)$ is of infinite activity, e.g., in the Lévy-stable or tempered Lévy-stable case. We will first focus on the case of $T(s)$ given as a one-sided Lévy-stable process with parameter $0 < \alpha < 1$ and characteristic function [41]

$$\langle e^{-\lambda T(s)} \rangle = e^{-\lambda^\alpha s} \,. \tag{3.65}$$

The inverse process $S(t)$ can be defined as

$$S(t) = \inf\{s > 0 : T(s) > t\} \,, \tag{3.66}$$

i.e., as a collection of first passage times. By the monotonicity of the process $T(s)$ and $S(t)$

$$s_2 > s_1 \Rightarrow T(s_2) > T(s_1) \,, \tag{3.67}$$

one could get the relationship [12]

$$\Theta(s - S(t)) = 1 - \Theta(t - T(s)) \,. \tag{3.68}$$

where $\Theta(x)$ is the Heaviside step function

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0 \\ 1/2, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases} \tag{3.69}$$

Eq. (3.68) allows us to derive an evolution equation for the single time PDF of the the process $S(t)$ as follows. Taking derivative with respect to s in both side of Eq. (3.68), one could obtain

$$\delta(s - S(t)) = \frac{\partial}{\partial s}\Theta(s - S(t)) = -\frac{\partial}{\partial s}\Theta(t - T(s)) \tag{3.70}$$

Let $h(s,t)$ denote the PDF of the process $S(t)$, then by taking the average in Eq. (3.70), we get

$$h(s,t) = \langle \delta(s - S(t)) \rangle = -\frac{\partial}{\partial s} \langle \Theta(t - T(s)) \rangle \tag{3.71}$$

As by the definition of the process we have $S(0) = 0$, see [12], the density function obeys the initial condition $h(s,0) = \delta(s)$ and thus can be viewed as a special case of a conditional probability.

If we apply the Laplace transform to $h(s,t)$, then Eq. (3.71) results in

$$\begin{aligned}
\tilde{h}(s,\lambda) &= \int_0^{+\infty} dt e^{-\lambda t} h(s,t) \\
&= -\frac{\partial}{\partial s}\int_0^{+\infty} dt e^{-\lambda t}\left\langle \Theta(t-T(s))\right\rangle \\
&= -\frac{\partial}{\partial s}\left\langle \int_0^{+\infty} dt e^{-\lambda t}\Theta(t-T(s))\right\rangle \\
&= -\frac{\partial}{\partial s}\left\langle \frac{1}{\lambda}\int_0^{+\infty} dt e^{-\lambda t}\delta(t-T(s))\right\rangle \\
&= -\frac{\partial}{\partial s}\left\langle \frac{e^{-\lambda T(s)}}{\lambda}\right\rangle \\
&= \lambda^{\alpha-1}e^{-\lambda^{\alpha}s}.
\end{aligned} \tag{3.72}$$

when using the moment generating function Eq. (3.65). From the computation above, we know that

$$\int_0^{+\infty} e^{-\lambda t}\langle\Theta(t-T(s))\rangle dt = \frac{1}{\lambda}\langle e^{-\lambda T(s)}\rangle\,. \tag{3.73}$$

Clearly the derivative of the Laplace transform obeys

$$-\frac{\partial}{\partial s}\tilde{h}(s,\lambda) = \lambda^{\alpha}\tilde{h}(s,\lambda)\,. \tag{3.74}$$

After performing the inverse Laplace transform, a fractional equation for $h(s,t)$ is obtained as [12]

$$\frac{\partial}{\partial t}h(s,t) = -{}_0D_t^{1-\alpha}\frac{\partial}{\partial s}h(s,t) \tag{3.75}$$

where the operator ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville fractional differential operator defined in Eq. (3.20). The fractional evolution equation Eq. (3.75) will play an important role in our discussion later. On the other hand, according to Eq. (3.71), one can obtain

$$h(s,t) = -\frac{\partial}{\partial s}\int_0^t \left\langle \delta(t'-T(s))\right\rangle dt' \tag{3.76}$$

Denoting by $p(t,s)=\langle\delta(t-T(s))\rangle$ the PDF of the one sided Lévy-stable process $T(s)$, it could derived that

$$h(s,t) = -\frac{\partial}{\partial s}\int_0^t p(t',s)dt' = -\frac{\partial}{\partial s}\int_0^t \frac{1}{s^{1/\alpha}} L_\alpha\left(\frac{t'}{s^{1/\alpha}}\right) dt' \tag{3.77}$$

Here we use the fact that

$$\langle e^{-\lambda T(s)}\rangle = \int_0^\infty e^{-\lambda t} p(t,s)dt = e^{-\lambda^\alpha s} = e^{-(\lambda s^{1/\alpha})^\alpha}\,, \tag{3.78}$$

as well as Eq. (3.60), so we could derive that

$$p(t,s) = \frac{1}{s^{1/\alpha}} L_\alpha\left(\frac{t}{s^{1/\alpha}}\right)\,. \tag{3.79}$$

Then by the variable transformation $x = t'/s^{1/\alpha}$, Eq. (3.77) results into [10]

$$h(s,t) = -\frac{\partial}{\partial s}\int_0^{\frac{t}{s^{1/\alpha}}} L_\alpha(x)dx = \frac{t}{\alpha s^{1+1/\alpha}} L_\alpha(\frac{t}{s^{1/\alpha}}) \tag{3.80}$$

Alternatively $h(s,t)$ can be found in the form of a series as [12]

$$h(s,t) = \sum_{n=0}^{\infty} \frac{(-st^\alpha)^n}{\Gamma(1+n\alpha)}. \tag{3.81}$$

3.2.3 The two-point PDF of the process $S(t)$

Due to the jumps in $T(s)$ representing large waiting times, the inverse process $S(t)$ is non-Markovian. As such, only specifying the one-point PDF is not sufficient to characterize the process. The complete multi-point structure has been characterized in [12], which is briefly summarized here.

If we denote by $h(s_2,t_2,s_1,t_1)$ the two point PDF of $S(t)$, then it could be expressed as

$$h(s_2,t_2,s_1,t_1) = \langle \delta(s_2 - S(t_2))\delta(s_1 - S(t_1))\rangle = \frac{\partial}{\partial s_2}\frac{\partial}{\partial s_1}\langle \Theta(s_2 - S(t_2))\Theta(s_1 - S(t_1))\rangle \tag{3.82}$$

As the result of the monotonicity of $T(s)$ and $S(t)$, see Eq. (3.68), it could be found that

$$\begin{aligned}\langle\Theta(s_2-S(t_2))\Theta(s_1-S(t_1))\rangle &= \langle(1-\Theta(t_2-T(s_2)))\,(1-\Theta(t_1-T(s_1)))\rangle \\ &= \langle 1-\Theta(t_2-T(s_2))-\Theta(t_1-T(s_1)) \\ &\quad +\Theta(t_2-T(s_2))\Theta(t_1-T(s_1))\rangle \end{aligned} \tag{3.83}$$

If we take derivatives with respect to s_2 and s_1 on both sides of Eq. (3.83) as well as use Eq. (3.82), we find

$$h(s_2,t_2,s_1,t_1)=\frac{\partial}{\partial s_2}\frac{\partial}{\partial s_1}\langle\Theta(t_2-T(s_2))\Theta(t_1-T(s_1))\rangle \tag{3.84}$$

Applying the two-time Laplace transform to $h(s_2,t_2,s_1,t_1)$

$$\begin{aligned}\tilde{h}(s_2,\lambda_2,s_1,\lambda_1) &= \int_0^{+\infty}\int_0^{+\infty} dt_2dt_1e^{-\lambda_2t_2}e^{-\lambda_1t_1}h(s_2,t_2,s_1,t_1) \\ &= \frac{\partial}{\partial s_2}\frac{\partial}{\partial s_1}\int_0^{+\infty}\int_0^{+\infty} dt_2dt_1e^{-\lambda_2t_2}e^{-\lambda_1t_1}\langle\Theta(t_2-T(s_2))\Theta(t_1-T(s_1))\rangle \\ &= \frac{\partial}{\partial s_2}\frac{\partial}{\partial s_1}\left\langle\frac{e^{-\lambda_2T(s_2)}e^{-\lambda_1T(s_1)}}{\lambda_2\lambda_1}\right\rangle \end{aligned} \tag{3.85}$$

By the independence of the increments of $T(s)$, and for the two cases $s_2>s_1$ and $s_1>s_2$, we can calculate the expected value as follows

$$\begin{aligned}\left\langle e^{-\lambda_2T(s_2)}e^{-\lambda_1T(s_1)}\right\rangle &= \int_0^{\infty}dt_2\int_0^{\infty}dt_1\,e^{\lambda_2t_2-\lambda_1t_1}p(t_2,s_2,t_1,s_1) \\ &= \left\langle e^{-\lambda_2\int_0^{s_2}\eta(s)ds}e^{-\lambda_1\int_0^{s_1}\eta(s)ds}\right\rangle \\ &= \Theta(s_2-s_1)\left\langle e^{-\lambda_2\int_{s_1}^{s_2}\eta(s)ds-(\lambda_1+\lambda_2)\int_0^{s_1}\eta(s)ds}\right\rangle \\ &\quad+\Theta(s_1-s_2)\left\langle e^{-\lambda_1\int_{s_2}^{s_1}\eta(s)ds-(\lambda_1+\lambda_2)\int_0^{s_2}\eta(s)ds}\right\rangle \\ &= \Theta(s_2-s_1)\left\langle e^{-\lambda_2\int_{s_1}^{s_2}\eta(s)ds}\right\rangle\left\langle e^{-(\lambda_1+\lambda_2)\int_0^{s_1}\eta(s)ds}\right\rangle \\ &\quad+\Theta(s_1-s_2)\left\langle e^{-\lambda_1\int_{s_2}^{s_1}\eta(s)ds}\right\rangle\left\langle e^{-(\lambda_1+\lambda_2)\int_0^{s_2}\eta(s)ds}\right\rangle \\ &= \Theta(s_2-s_1)e^{-(s_2-s_1)\lambda_2^{\alpha}}e^{-s_1(\lambda_1+\lambda_2)^{\alpha}} \\ &\quad+\Theta(s_1-s_2)e^{-(s_1-s_2)\lambda_1^{\alpha}}e^{-s_2(\lambda_1+\lambda_2)^{\alpha}} \end{aligned} \tag{3.86}$$

where $p(t_2,s_2,t_1,s_1)$ is two-point PDF of $T(s)$. By performing the inverse Laplace transform, Eq. (3.86) results into

$$p(t_2, s_2, t_1, s_1) = \Theta(s_2 - s_1)\frac{1}{s_1^{1/\alpha}} L_\alpha\left(\frac{t_1}{s_1^{1/\alpha}}\right)\frac{1}{(s_2 - s_1)^{1/\alpha}} L_\alpha\left(\frac{t_2 - t_1}{(s_2 - s_1)^{1/\alpha}}\right) + \quad (3.87)$$

$$\Theta(s_1 - s_2)\frac{1}{s_2^{1/\alpha}} L_\alpha\left(\frac{t_2}{s_2^{1/\alpha}}\right)\frac{1}{(s_1 - s_2)^{1/\alpha}} L_\alpha\left(\frac{t_1 - t_2}{(s_1 - s_2)^{1/\alpha}}\right), \quad (3.88)$$

which indicate that $p(t_2, s_2; t_1, s_1)$ factorizes and is thus a Markovian process. In particular, when $s_1 < s_2$, the conditional PDF $p(t_2, s_2|t_1, s_1)$ is

$$p(t_2, s_2|t_1, s_1) = \frac{1}{(s_2 - s_1)^{1/\alpha}} L_\alpha\left(\frac{t_2 - t_1}{(s_2 - s_1)^{1/\alpha}}\right). \quad (3.89)$$

Coming back to the process $S(t)$, we can evaluate Eq. (3.85) by performing the derivatives with respect to s_1 and s_2 in Eq. (3.86). We obtain

$$\begin{aligned}\tilde{h}(s_2, \lambda_2, s_1, \lambda_1) = {} & \delta(s_2 - s_1)\frac{\lambda_1^\alpha - (\lambda_1 + \lambda_2)^\alpha + \lambda_2^\alpha}{\lambda_2\lambda_1} e^{-s_1(\lambda_1+\lambda_2)^\alpha} \\ & + \Theta(s_2 - s_1)\frac{\lambda_2^\alpha[(\lambda_1 + \lambda_2)^\alpha - \lambda_2^\alpha]}{\lambda_2\lambda_1} e^{-s_1(\lambda_1+\lambda_2)^\alpha} e^{-(s_2-s_1)\lambda_2^\alpha} \\ & + \Theta(s_1 - s_2)\frac{\lambda_1^\alpha[(\lambda_1 + \lambda_2)^\alpha - \lambda_1^\alpha]}{\lambda_2\lambda_1} e^{-s_2(\lambda_1+\lambda_2)^\alpha} e^{-(s_1-s_2)\lambda_1^\alpha}, \end{aligned} \quad (3.90)$$

Obviously, Eq. (3.90) is the analytical expression for the Laplace transform of the two-point PDF. Unfortunately, an exact result for the Laplace inversion is not known and thus $h(s_2, t_2; s_1, t_1)$ needs to be evaluated numerically. The extension from two to n points can be performed in complete analogy [12].

It is also possible to derive a fractional evolution equation for the two-point PDF. Starting from the Eq. (3.90), it is evident that

$$\left(\frac{\partial}{\partial s_2} + \frac{\partial}{\partial s_1}\right)\tilde{h}(s_2, \lambda_2, s_1, \lambda_1) = -(\lambda_1 + \lambda_2)^\alpha \tilde{h}(s_2, \lambda_2, s_1, \lambda_1) \quad (3.91)$$

After performing the inverse Laplace transform in Eq. (3.91), we derive the FDE for $h(s_2, t_2, s_1, t_1)$ as

$$\left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_1}\right) h(s_2, t_2, s_1, t_1) = -\left(\frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_1}\right)^{1-\alpha}\left(\frac{\partial}{\partial s_2} + \frac{\partial}{\partial s_1}\right) h(s_2, t_2, s_1, t_1) \quad (3.92)$$

where the fractional operator $\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)^{1-\alpha}$ of two times is defined as

$$\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)^{1-\alpha} g(t_2,t_1) := \left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)^{-\alpha} g(t_2,t_1) \tag{3.93}$$

and the fractional operator$\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)^{-\alpha}$ in Laplace space is given as

$$\mathcal{L}\left\{\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)^{-\alpha} g(t_2,t_1)\right\} = \frac{1}{(\lambda_2+\lambda_1)^{\alpha}}\,. \tag{3.94}$$

Noting that $\mathcal{L}\{\rho(t_2)\delta(t_2-t_1)\} = \tilde{\rho}(\lambda_1+\lambda_2)$ for an arbitrary one-parameter function g, one could obtain

$$\begin{aligned}&\int_0^\infty dt_2 \int_0^\infty dt_1\, e^{-\lambda_2 t_2-\lambda_1 t_1}\frac{1}{\Gamma(\alpha)}t_1^{\alpha-1}\delta(t_2-t_1)\\ &\quad= \int_0^\infty dt_1\, e^{-(\lambda_2+\lambda_1)t_1}\frac{1}{\Gamma(\alpha)}t_1^{\alpha-1} = \frac{1}{(\lambda_2+\lambda_1)^{\alpha}}\,.\end{aligned} \tag{3.95}$$

as it has been shown that $\mathcal{L}\left\{t^{\alpha-1}\right\} = \Gamma(\alpha)/\lambda^{\alpha}$ holds. Hence, we see that the fractional operator in Eq. (3.93) can also be expressed as

$$\begin{aligned}\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)^{-\alpha} g(t_2,t_1) = \frac{1}{\Gamma(\alpha)}\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)\int_0^{\min(t_2,t_1)} \tau^{\alpha-1}\\ \times\, g(t_2-\tau,t_1-\tau)d\tau\,.\end{aligned} \tag{3.96}$$

3.2.4 The fractional Fokker-Planck equation for the PDF of $Y(t)$

The stochastic differential equation in Eq. (3.24a) describes the normal Markovian process $X(s)$ and its PDF is given as the solution of the associated Fokker-Planck Equation [136] as discussed in Sec. (2.1.1)

$$\frac{\partial}{\partial t} f_X(x,s) = L_{\mathrm{FP}} f_X(x,s)\,, \tag{3.97}$$

where

$$L_{\mathrm{FP}} = -\frac{\partial}{\partial x}\mu(x,s) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma(x,s)^2\,, \tag{3.98}$$

and $f_X(x,s)$ is the single-point PDF of the process $X(s)$.

As usual the initial condition $f_X(x,0)$ determines the distribution of the initial conditions $X(0)$. We now aim at deriving a similar equation for the density function $f_Y(x,t)$ of the process $Y(t)$. If we take the derivative with respect to t, we get

$$\begin{aligned}\frac{\partial}{\partial t}f_Y(x,t) &= \int_0^\infty ds \frac{\partial}{\partial t}h(s,t)f_X(x,s)\\ &= -{}_0D_t^{1-\alpha}\left(h(s,t)f_X(x,s)|_0^\infty - \int_0^\infty ds\, h(s,t)\frac{\partial}{\partial s}f_X(x,s)\right)\\ &= {}_0D_t^{1-\alpha}\left(\int_0^\infty ds\, h(s,t)\frac{\partial}{\partial s}f_X(x,s)\right) \end{aligned} \tag{3.99}$$

Here we have used the fact that $h(s,t)$ satisfies the Eq. (3.75) and we have taken into account the boundary condition $h(0,t) = h(\infty,t) = 0$ into account, which follows, e.g., from the Laplace transform Eq. (3.72). Since $f_X(x,s)$ obeys the usual Fokker-Planck equation (3.97) we find that Eq. (3.99) becomes

$$\begin{aligned}\frac{\partial}{\partial t}f_Y(x,t) &= {}_0D_t^{1-\alpha}\left(\int_0^\infty ds\, h(s,t)L_{\mathrm{FP}}f_X(x,s)\right)\\ &= {}_0D_t^{1-\alpha}\left(L_{\mathrm{FP}}\int_0^\infty ds\, h(s,t)f_X(x,s)\right)\\ &= {}_0D_t^{1-\alpha}L_{\mathrm{FP}}f_Y(x,t)\end{aligned} \tag{3.100}$$

Hence we end up with the fractional Fokker-Planck equation of the PDF of $Y(t)$ as follows

$$\frac{\partial}{\partial t}f_Y(x,t) = {}_0D_t^{1-\alpha}\left(-\frac{\partial}{\partial x}\mu(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma(x,t)^2\right)f_Y(x,t). \tag{3.101}$$

It is of course rather straightforward as well to calculate moments of the process $Y(t)$. For instance the definition for the expectation value yields

$$\langle Y(t)\rangle = \langle X(S(t))\rangle = \int_0^\infty ds\, \langle X(s)\rangle\, h(s,t) \tag{3.102}$$

Also, we could calculate the second moment $\langle Y^2(t)\rangle$ by

$$\left\langle Y^2(t)\right\rangle = \langle X(S(t))\rangle = \int_0^\infty ds\, \left\langle X^2(s)\right\rangle h(s,t)\,. \tag{3.103}$$

In particular, letting $\mu(x) = 0$ and $\sigma(x) = \text{const}$ in Eq. (3.24a), one could immediately find that

$\left\langle X^2(s)\right\rangle = \sigma^2 s$ and obtain

$$\left\langle Y^2(t)\right\rangle = \sigma^2 \int_0^\infty ds\, sh(s,t)\,. \tag{3.104}$$

By applying Laplace transform, one could find that $\mathcal{L}\{\left\langle Y^2(t)\right\rangle\} = \sigma^2/\lambda^{\alpha+1}$. Now performing inverse Laplace transform, one obtains a subdiffusive mean-square displacement

$$\left\langle Y^2(t)\right\rangle = \frac{\sigma^2}{\Gamma(\alpha+1)} t^\alpha\,. \tag{3.105}$$

3.3 Subordinated process with general waiting times

In the general case, the waiting time process $T(s)$ is characterized by the characteristic function

$$\left\langle e^{-\lambda T(s)}\right\rangle = e^{-\Phi(\lambda)s} \tag{3.106}$$

where $\Phi(\lambda)$ is the Laplace exponent as before, with the representation Eq. (3.54). As the result of many possible choices for the jump amplitudes $\Pi(dx)$ and thus $\Phi(\lambda)$, a lot of different waiting time statistics could be studied. For $\Phi(\lambda) = \lambda^\alpha$ with $0 < \alpha < 1$, we recover the CTRW [41,46,160]. If $\Phi(\lambda) = \lambda$, this means $T(s) = s$ and the subordination simply replaces s with t, such that $Y(t)$ describes a normal Brownian diffusion [22].

Following similar steps as above, it is then straightforward to obtain the generalizations of the n-point PDFs and fractional Fokker-Planck equation for a Laplace exponent $\Phi(\lambda)$.

3.3.1 Single point PDF and fractional Fokker-Planck equation

Suppose that $h(s,t)$ denotes the probability density function of the process $S(t)$. By Eqs. (3.67)–(3.70), we likewise have the relation

$$h(s,t) = \langle \delta(s - S(t))\rangle = -\frac{\partial}{\partial s}\langle \Theta(t - T(s))\rangle \tag{3.107}$$

As in Laplace space we know that $\int_0^{+\infty} e^{-\lambda t} \langle \Theta(t - T(s)) \rangle \, dt = \left\langle \frac{e^{-\lambda T(s)}}{\lambda} \right\rangle$, then Eq. (3.107) results in

$$\tilde{h}(s, \lambda) = -\frac{\partial}{\partial s} \left\langle \frac{e^{-\lambda T(s)}}{\lambda} \right\rangle = \frac{\Phi(\lambda)}{\lambda} e^{-\Phi(\lambda) s} \tag{3.108}$$

where the property in Eq. (3.106) is used. Taking the derivative of $\tilde{h}(s, \lambda)$ with respect to s, it is clear that

$$-\frac{\partial}{\partial s} \tilde{h}(s, \lambda) = \Phi(\lambda) \tilde{h}(s, \lambda) \, . \tag{3.109}$$

Multiplying $\lambda / \Phi(\lambda)$ to both sides of the equation above, it could become

$$-\frac{\lambda}{\Phi(\lambda)} \frac{\partial}{\partial s} \tilde{h}(s, \lambda) = \lambda \tilde{h}(s, \lambda) \, . \tag{3.110}$$

After performing the inverse Laplace transform, a fractional equation for $h(s, t)$ is derived as

$$\frac{\partial}{\partial t} h(s, t) = -F_t \left(\frac{\partial}{\partial s} h(s, t) \right) \tag{3.111}$$

where the operator F_t is defined as

$$F_t \left(g(t) \right) = \frac{\partial}{\partial t} \int_0^t K(t - \tau) g(\tau) d\tau \tag{3.112}$$

with the memory kernel $K(t)$

$$\tilde{K}(\lambda) = \int_0^\infty e^{-\lambda t} K(t) = \frac{1}{\Phi(\lambda)}. \tag{3.113}$$

We see that when $\Phi(\lambda) = \lambda^\alpha$, the operator F_t becomes the Riemann-Liouville fractional differential operator ${}_0D_t^{1-\alpha}$ as defined in Eq. (3.20).

After the fractional equation governing the dynamics of $h(s, t)$ is obtained, we could make a step further to derive the fractional Fokker-Planck type equation for the density function of the subordinated process with general waiting times $Y(t) = X(S(t))$. Also for general $\Phi(\lambda)$ Eq. (3.25) holds, i.e., $f_Y(x, t) = \int_0^\infty ds f_X(x, s) h(s, t)$. Taking the derivative of $f_Y(x,t)$ with respect to t, we will get

$$\begin{aligned}\frac{\partial}{\partial t}f_Y(x,t) &= \int_0^\infty ds \frac{\partial}{\partial t}h(s,t)f_X(x,s) \\ &= -F_t\left(h(s,t)f_X(x,s)|_0^\infty - \int_0^\infty ds\, h(s,t)\frac{\partial}{\partial s}f_X(x,s)\right) \\ &= F_t\left(\int_0^\infty ds\, h(s,t)\frac{\partial}{\partial s}f_X(x,s)\right) . \end{aligned} \tag{3.114}$$

Here, we use the fact that $h(s,t)$ satisfies Eq. (3.111) and the boundary conditions $h(0,t) = h(\infty,t) = 0$. As $f_X(x,s)$ satisfies the standard Fokker-Planck equation (3.97), we find that Eq. (3.114) could be converted into

$$\begin{aligned}\frac{\partial}{\partial t}f(x,t) &= F_t\left(\int_0^\infty ds\, h(s,t)L_{\text{FP}}f_X(x,s)\right) \\ &= F_t\left(L_{\text{FP}}\int_0^\infty ds\, h(s,t)f_X(x,s)\right) \\ &= F_t\left(-\frac{\partial}{\partial x}\mu(x,t)f_X(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma(x,t)^2 f_X(x,t)\right) . \end{aligned} \tag{3.115}$$

Clearly, when $K(t) = 1$, Eq. (3.115) becomes the normal Fokker-Planck equation (3.97). It is evident that Eqs. (3.102) and (3.103) still hold for calculating the moments of the process $Y(t)$ here.

3.3.2 The two-point PDF of the process $S(t)$

Once again let us denote the two time PDF of the process $S(t)$ as $h(s_2,t_2,s_1,t_1)$. We know that in Laplace space, $\tilde{h}(s_2,\lambda_2,s_1,\lambda_1)$ can be expressed as (see Eq. (3.85))

$$\tilde{h}(s_2,\lambda_2,s_1,\lambda_1) = \frac{1}{\lambda_2\lambda_1}\frac{\partial}{\partial s_2}\frac{\partial}{\partial s_1}\left\langle e^{-\lambda_2 T(s_2)}e^{-\lambda_1 T(s_1)}\right\rangle \tag{3.116}$$

For the two cases $s_2 > s_1$ and $s_1 > s_2$, we can calculate as in Eq. (3.86)

$$\begin{aligned}\left\langle e^{-\lambda_2 T(s_2)}e^{-\lambda_1 T(s_1)}\right\rangle = \Theta(s_2 - s_1)e^{-s_1\Phi(\lambda_1+\lambda_2)}e^{-(s_2-s_1)\Phi(\lambda_2)} + \\ \Theta(s_1 - s_2)e^{-s_2\Phi(\lambda_1+\lambda_2)}e^{-(s_1-s_2)\Phi(\lambda_1)} \end{aligned} \tag{3.117}$$

because of the independence of the increments of $T(s)$ as well as the moment generating function Eq. (3.106). Taking the derivative with respect to s_2 and s_1 in the equation above, Eq. (3.116) results in [22]

$$\tilde{h}(s_2,\lambda_2,s_1,\lambda_1) = \delta(s_2-s_1)\frac{\Phi(\lambda_1)-\Phi(\lambda_1+\lambda_2)+\Phi(\lambda_2)}{\lambda_2\lambda_1}e^{-s_1\Phi(\lambda_1+\lambda_2)}+ \\ \Theta(s_2-s_1)\frac{\Phi(\lambda_2)[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)]}{\lambda_2\lambda_1}e^{-s_1\Phi(\lambda_1+\lambda_2)}e^{-(s_2-s_1)\Phi(\lambda_2)}+ \\ \Theta(s_1-s_2)\frac{\Phi(\lambda_1)[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_1)]}{\lambda_2\lambda_1}e^{-s_2\Phi(\lambda_1+\lambda_2)}e^{-(s_1-s_2)\Phi(\lambda_1)} \tag{3.118}$$

Clearly $\tilde{h}(s_2,\lambda_2,s_1,\lambda_1)$ satisfies

$$\left(\frac{\partial}{\partial s_2}+\frac{\partial}{\partial s_1}\right)\tilde{h}(s_2,\lambda_2,s_1,\lambda_1) = -\Phi(\lambda_1+\lambda_2)\tilde{h}(s_2,\lambda_2,s_1,\lambda_1)\,. \tag{3.119}$$

By the inverse Laplace transform, we could derive a fractional equation for the density function $h(s_2,t_2,s_1,t_1)$

$$\left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)h(s_2,t_2,s_1,t_1) = -F_{t_2+t_1}\left(\frac{\partial}{\partial s_2}+\frac{\partial}{\partial s_1}\right)h(s_2,t_2,s_1,t_1) \tag{3.120}$$

where the fractional operator $F_{t_2+t_1}$ of two times is defined as

$$F_{t_2+t_1}g(t_2,t_1) = \left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)\int_0^{t_2}\int_0^{t_1}K_2(\tau_2,\tau_1)g(t_2-\tau_2,t_1-\tau_1)d\tau_2 d\tau_1 \tag{3.121}$$

The kernel $K_2(t_2,t_1)$ is formally given in Laplace space as

$$\tilde{K}_2(t_2,t_1) := \frac{1}{\Phi(\lambda_2+\lambda_1)}. \tag{3.122}$$

Like Eq. (3.96), the fractional operator in Eq. (3.121) can also be expressed as

$$F_{t_2+t_1}g(t_2,t_1) = \left(\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_1}\right)\int_0^{\min(t_2,t_1)}K(\tau)g(t_2-\tau,t_1-\tau)d\tau, \tag{3.123}$$

where the kernel K is given by Eq. (3.122).

The fractional time derivative in Eq. (3.120) reveals the non-Markovian characteristics of the processes $S(t)$ and $Y(t)$. With similar steps, the equations governing the probability density function of the process $S(t)$ for the n times times can be derived, but we omit details here. Readers could refer to Ref. [22] for more details.

3.4 Asset pricing models beyond geometric Brownian motion

The classical Black Scholes theory is based on geometric Brownian motion. Here, by subordination, we will extend the analysis to two larger classes of processes. For the convenience of the reader we will give a short review of the definition and some basic properties of such non-Markovian processes.

3.4.1 Subdiffusive geometric Brownian motion

The first model, which we will investigate has been introduced in Ref. [88] as subdiffusive geometric Brownian motion. In a nutshell, subdiffusive geometric Brownian motion is given by $Y(t) = X(s(t))$, where $X(s)$ is a normal geometric Brownian motion $T(s)$ a one-sided Lévy-stable process with parameter $0 < \alpha < 1$. It can be represented by the coupled Langevin equations (cf. Eq. (2.51))

$$\dot{X}(s) = \mu X(s) + \sigma X(s)\xi(s) \tag{3.124a}$$

$$\dot{T}(s) = \eta(s), \qquad T(0) = 0 \tag{3.124b}$$

where again μ is the drift parameter , σ is the volatility and $\xi(s)$ a white Gaussian noise. The two processes $\xi(s)$ and $\eta(s)$ are assumed to be statistically independent. The process defined by Eq. (3.124) is a natural extension of the standard risk-neutral geometric Brownian motion incorporating waiting times with a power-law distribution as in the CTRW. As a hands on illustration Figure 3.1 shows numerical realisations of the paths of the processes $X(s)$, $Y(t) = X(S(t))$, and $S(t)$, respectively. In what follows we will summarise in more detail the required analytical properties of subdiffusive geometric Brownian motion.

The Fokker-Planck Equation (2.71) greatly helps us to understand the normal geometric Brownian motion as it governs the corresponding probability density function $f_X(x,t)$. As usual the initial condition $f_X(x,0)$ determines the distribution of the initial conditions $X(0)$. We now aim at deriving a similar equation for the density function $f_Y(y,t)$ of the process Eq. (3.124) assuming that the density $h(s,t)$ of the stochastic transformation is given, see Eq. (3.75). Obviously the subdiffusive GBM is a special case of the process in Eq. (3.24) when $\mu(X(s)) = \mu X(s)$ and $\sigma(X(s)) = \sigma X(s)$ with $\mu,\ \sigma = \text{const}$ in Eq. (3.24a). Hence using Eq. (3.101), we end up with the fractional

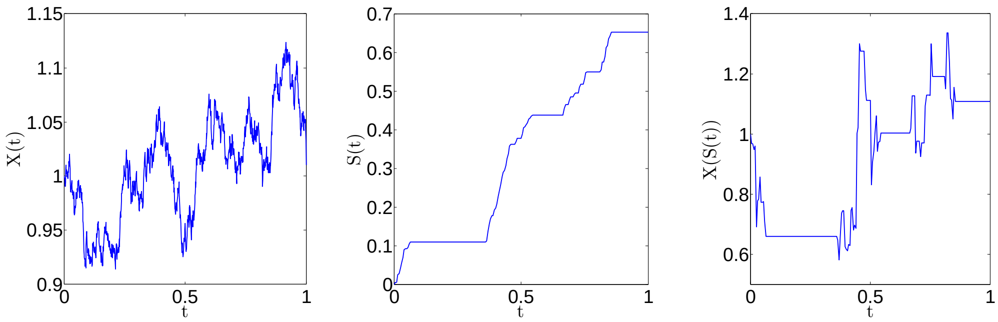

Figure 3.1: Sample realizations of the Geometric Brownian motion $X(t)$, subdiffusive Geometric Brownian motion $Y(t) = X(S(t))$, and the inverse process $S(t)$, according to Eq. (3.124) with parameters $\sigma = 1$, $X(0) = 1$, $\mu = 0.5$ and $\alpha = 0.7$, as obtained by the algorithms [78, 93] (see also Sec. 3.6). The constant intervals of $X(S(t))$ show the heavy-tailed waiting times. It is obvious that the subdiffusive Geometric Brownian motion is quite different from the Geometric Brownian motion due to the inverse process $S(t)$. The most evident phenomenon is the appearance of the flat path sections during some time periods.

Fokker-Planck equation of the PDF of the subdiffusive GBM as follows

$$\frac{\partial}{\partial t} f_Y(x,t) = {}_0D_t^{1-\alpha} \left((-\mu \frac{\partial}{\partial x} x + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} x^2) f_Y(x,t) \right) \tag{3.125}$$

which is the same as that found by Magdziarz [88].

In fact we could solve this equation to get the probability density function $f_Y(x,t)$ of subdiffusive GBM by Laplace transform, but considering the redundancy of the procedure, we resort to another method to derive its solution. According to Eq. (3.25), we know that the solution of Eq. (3.125) can be expressed as

$$f_Y(x,t) = \int_0^\infty ds f_X(x,s) h(s,t) \tag{3.126}$$

where $f_X(x,s)$ is the probability density function of GBM $X(s)$ in determined in Eq. (3.124a), and $h(s,t)$ is the density function of $S(t)$ given by Eq. (3.71).

In Laplace space, we could obtain that

$$\begin{aligned}\tilde{f}_Y(x,\lambda) &= \int_0^\infty dt\, e^{-\lambda t} f_Y(x,t) \\ &= \int_0^\infty ds\, f_X(x,s)\tilde{h}(s,\lambda) \\ &= \lambda^{\alpha-1}\int_0^\infty ds f_X(x,s)e^{-\lambda^\alpha s} \\ &= \lambda^{\alpha-1}\tilde{f}_X(x,\lambda^\alpha)\end{aligned} \tag{3.127}$$

where $\tilde{f}_X(x,\lambda)$ is the Laplace transform of the probability density function of GBM given in Eq. (2.73). Using Eq. (3.127), then the exact expression of $f_Y(x,t)$ in Laplace space is derived as

$$\begin{aligned}\tilde{f}_Y(x,\lambda) &= \frac{\lambda^{\alpha-1}}{x\sigma\sqrt{2\lambda^\alpha+\frac{\hat{\mu}^2}{\sigma^2}}}\exp\left(\frac{(\ln x-\ln x_0)\left(\hat{\mu}-\sigma\sqrt{2\lambda^\alpha+\frac{\hat{\mu}^2}{\sigma^2}}\right)}{\sigma^2}\right), & x>x_0 \\ &= \frac{\lambda^{\alpha-1}}{x\sigma\sqrt{2\Phi(\lambda)+\frac{\hat{\mu}^2}{\sigma^2}}}, & x=x_0 \\ &= \frac{\lambda^{\alpha-1}}{x\sigma\sqrt{2\lambda^\alpha+\frac{\hat{\mu}^2}{\sigma^2}}}\exp\left(\frac{(\ln x-\ln x_0)\left(\hat{\mu}+\sigma\sqrt{2\lambda^\alpha+\frac{\hat{\mu}^2}{\sigma^2}}\right)}{\sigma^2}\right), & x<x_0\end{aligned} \tag{3.128}$$

By performing the inverse Laplace transform to $\tilde{f}_Y(x,\lambda)$ in Eq. (3.128), the exact solution $f_Y(x,t)$ of Eq. (3.125) could be obtained numerically. Fig. 3.2 shows the change of the density $f_Y(x,t)$ with t and x. It is of course rather straightforward as well to calculate moments of the process $Y(t)$. Again calculations considerably simplify if the Laplace transform

$$\langle\tilde{Y}(\lambda)\rangle = \int_0^\infty \exp(-\lambda t)\langle Y(t)\rangle dt \tag{3.129}$$

is used. Then Eq. (3.129) results in

$$\left\langle\tilde{Y}(\lambda)\right\rangle = \int_0^\infty dt\, e^{-\lambda t}\left\langle X(S(t))\right\rangle = \int_0^\infty ds\, \tilde{h}(s,\lambda)\left\langle X(s)\right\rangle . \tag{3.130}$$

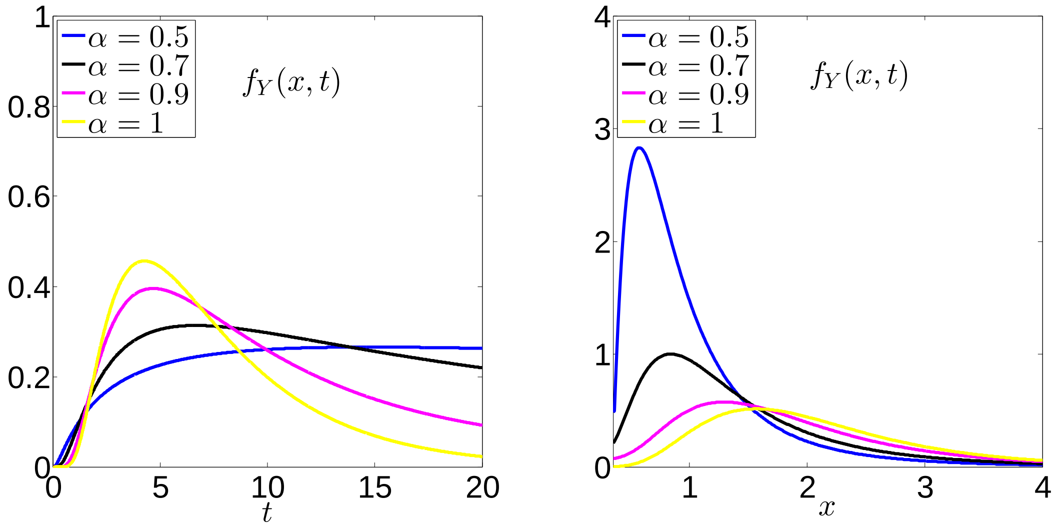

Figure 3.2: The density function of subdiffusive geometric Brownian motion [See Eq. (3.128)] with respect to t with parameters $\sigma = 0.2$, $x_0 = 1$, $x = 2$ and $\mu = 0.15$ (left panel) and to x with parameters $\sigma = 0.2$, $x_0 = 1$, $t = 5$ and $\mu = 0.15$ (right pane).

It is pretty straightforward to derive $\langle X(s)\rangle = x_0 e^{\mu s}$ directly from Eq. (3.124a) where we assume the initial condition $X(0) = x_0$. With the help of Eq. (3.72) we can then compute the integral to result in $\langle \tilde{Y}(\lambda)\rangle = x_0\lambda^{\alpha-1}/(\lambda^\alpha - \mu)$. Performing the inverse Laplace transformation, with the help of the one-parameter Mittag-Leffler function (see Eq. (3.6)) we obtain

$$\langle Y(t)\rangle = x_0 E_\alpha(\mu t^\alpha)\,. \tag{3.131}$$

With the same procedure as above, we can calculate the second moment $\langle Y^2(t)\rangle$ as well if we take $\left\langle X^2(s)\right\rangle = x_0^2 e^{2\mu s+\sigma^2 s}$ into account

$$\left\langle Y^2(t)\right\rangle = x_0^2 E_\alpha((2\mu+\sigma^2)t^\alpha)\,. \tag{3.132}$$

Obviously, by Eq. (3.132), one could find that the subdiffusive GBM does not actually represent a subdiffusive process, but for the convenience of our later discussion we still stick to this name as this price model has already been termed as subdiffusive GBM in previous literature [88].

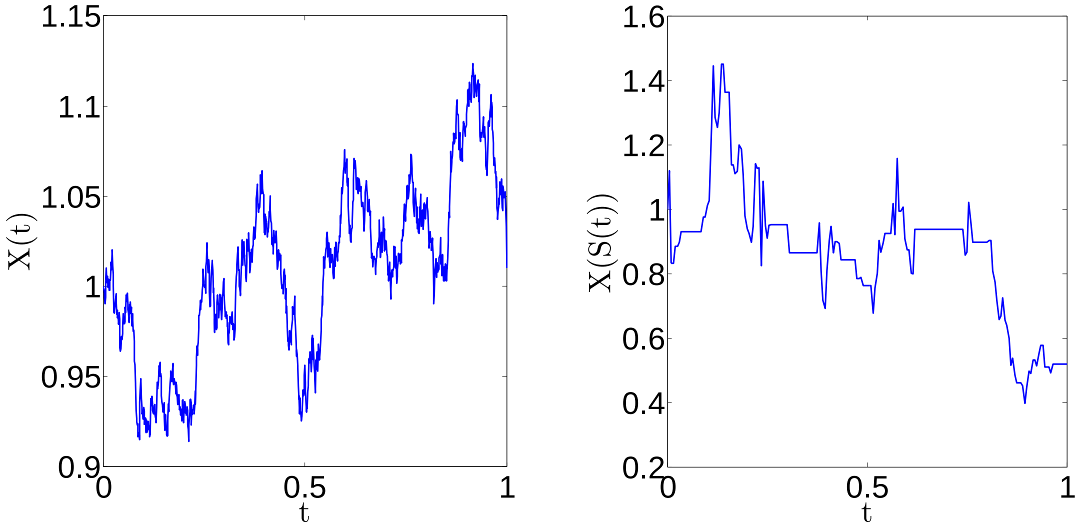

Figure 3.3: Sample realizations of the standard Geometric Brownian motion $X(t)$ (left panel) with Euler method [57], and the subordinated Geometric Brownian motion $Y(t) = X(S(t))$ with $\Phi(\lambda) = (\lambda + \zeta)^\alpha - \zeta^\alpha$ (right panel), according to Eqs. (3.124) and (3.106) with parameters $\sigma = 1$, $X(0) = 1$, $r = 0.5$, $\zeta = 0.001$ and $\alpha = 0.7$, as obtained by the algorithms [8, 78, 90] (see Sec. 3.6). It is obvious that the subordinated Geometric Brownian motion is quite different from the standard Geometric Brownian motion due to the process $S(t)$. The constant intervals of $X(S(t))$ show the effect of the heavy-tailed waiting times, which is typical characteristic for subdiffusion.

3.4.2 Subordinated geometric Brownian motion

We now consider the coupled Langevin equations (3.124a)–(3.124b), but generalize the waiting time process as in Eq. (3.106), i.e., we consider a Laplace exponent $\Phi(\lambda)$. With Eq. (3.115) we could derive the fractional Fokker-Planck equation for the PDF of the subordinated GBM immediately as follows

$$\frac{\partial}{\partial t} f_Y(x,t) = F_t \left((-r \frac{\partial}{\partial x} x + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} x^2) f_Y(x,t) \right) \tag{3.133}$$

In order to obtain the PDF $f_Y(x,t)$ of the subordinated GBM, we have to solve this fractional equation in principle. However, as in the CTRW case, we know already that the solution is given by the integral transformation Eq. (3.25).

By analogy with Eq. (3.127) it is clear that in Laplace space

$$
\begin{aligned}
\tilde{f}_Y(x,\lambda) &= \int_0^\infty dt\, e^{-\lambda t} f_Y(x,t) \\
&= \int_0^\infty ds\, f_X(x,s)\tilde{h}(s,\lambda) \\
&= \int_0^\infty ds\, f_X(x,s)\frac{\Phi(\lambda)}{\lambda}e^{-\Phi(\lambda)s} \\
&= \frac{\Phi(\lambda)}{\lambda}\tilde{f}_X(x,\Phi(\lambda))
\end{aligned} \tag{3.134}
$$

where $\tilde{f}_X(x,\lambda)$ is the Laplace transform of the probability density function $f_X(x,s)$ of GBM given in Eq. (2.73). By Eq. (2.73), the exact expression of $f_Y(x,t)$ in Laplace space is derived as

$$
\begin{aligned}
&\tilde{f}_Y(x,\lambda) \\
&= \frac{\Phi(\lambda)}{\lambda}\frac{1}{x\sigma\sqrt{2\Phi(\lambda)+\frac{\hat{\mu}^2}{\sigma^2}}}\exp\left(\frac{(\ln x-\ln x_0)\left(\hat{\mu}-\sigma\sqrt{2\Phi(\lambda)+\frac{\hat{\mu}^2}{\sigma^2}}\right)}{\sigma^2}\right), \qquad x>x_0 \\
&= \frac{\Phi(\lambda)}{\lambda}\frac{1}{x\sigma\sqrt{2\Phi(\lambda)+\frac{\hat{\mu}^2}{\sigma^2}}}, \qquad x=x_0 \\
&= \frac{\Phi(\lambda)}{\lambda}\frac{1}{x\sigma\sqrt{2\Phi(\lambda)+\frac{\hat{\mu}^2}{\sigma^2}}}\exp\left(\frac{(\ln x-\ln x_0)\left(\hat{\mu}+\sigma\sqrt{2\Phi(\lambda)+\frac{\hat{\mu}^2}{\sigma^2}}\right)}{\sigma^2}\right), \qquad x<x_0
\end{aligned} \tag{3.135}
$$

By performing the inverse Laplace transform of $\tilde{f}_Y(x,\lambda)$ in Eq. (3.135), the exact solution $f_Y(x,t)$ of Eq. (3.133) could be obtained for general $\Phi(\lambda)$. This allows us to investigate in particular the effect of the exponential tempering on the waiting times by considering the tempered Lévy-stable Laplace exponent, Eq. (3.62). Fig. 3.4 indicates the changes of the density function of subordinated geometric Brownian motion $f_Y(x,t)$ according t and x for different α at fixed μ. Fig. 3.5 shows the changes of the density function $f_Y(x,t)$ of the subordinated geometric Brownian motion according t and x based on different value of μ.

With Eqs. (3.102)–(3.103) and (3.108), the first moment of subordinated geometric

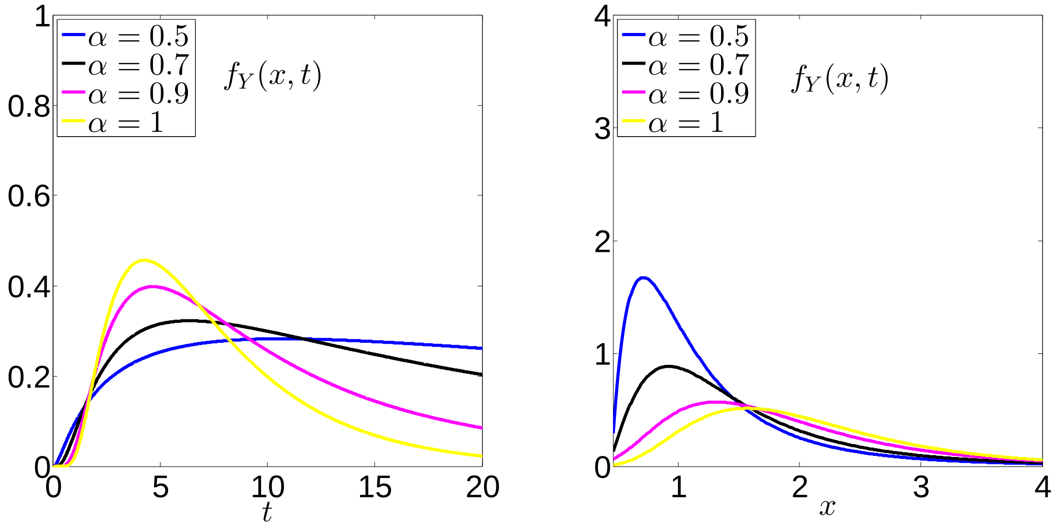

Figure 3.4: The density function $f(x,t)$ of subordinated geometric Brownian motion with $\Phi(\lambda) = (\lambda + \zeta)^{\alpha} - \zeta^{\alpha}$ with respect to t with parameters $\sigma = 0.2$, $x_0 = 1$, $x = 2$, $\zeta = 0.005$ and $r = 0.15$ (left panel) and to x with parameters $\sigma = 0.2$, $x_0 = 1$, $\zeta = 0.005$, $t = 5$ and $r = 0.15$ (right panel).

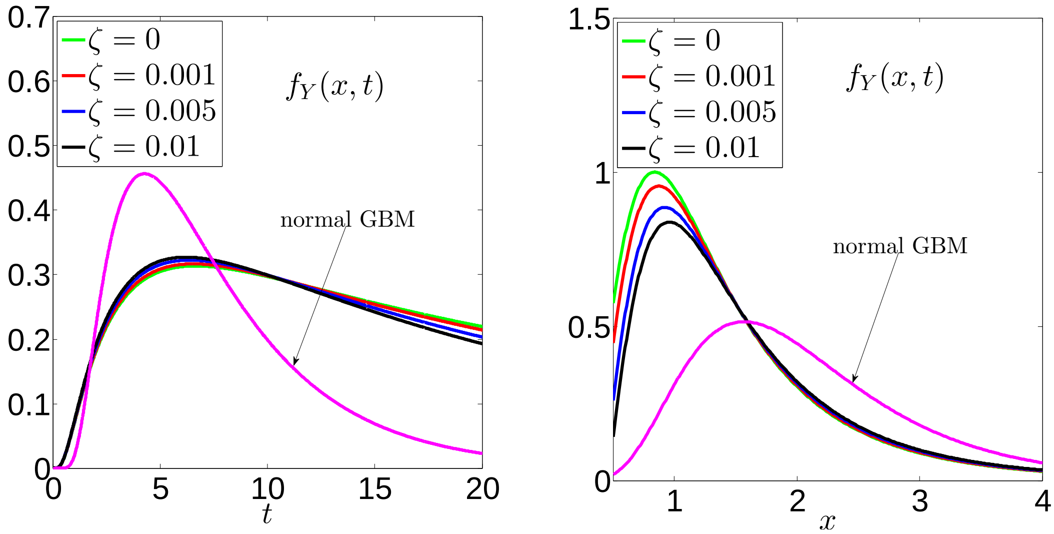

Figure 3.5: The density function of subordinated geometric Brownian motion with $\Phi(\lambda) = (\lambda + \zeta)^{\alpha} - \zeta^{\alpha}$ with respect to t with parameters $\sigma = 0.2$, $x_0 = 1$, $x = 2$ $\alpha = 0.7$ and $r = 0.15$ and to x with parameters $\sigma = 0.2$, $x_0 = 1$, $\alpha = 0.7$, $t = 5$ and $r = 0.15$.

Brownian motion in Laplace space could be derived as follows

$$
\begin{aligned}
\langle \tilde{Y}(\lambda) \rangle &= \int_0^\infty \exp(-\lambda t) \langle Y(t) \rangle dt \\
&= \int_0^\infty \exp(-\lambda t) \, \langle X(S(t)) \rangle \, dt \\
&= \int_0^\infty \tilde{h}(s, \lambda) \, \langle X(s) \rangle \, ds \\
&= \int_0^\infty x_0 \exp(rs) \frac{\Phi(\lambda)}{\lambda} \exp(-\Phi(\lambda) s) ds \\
&= x_0 \frac{\Phi(\lambda)}{\lambda} \int_0^\infty \exp(-[\Phi(\lambda) - r] s) \\
&= \frac{x_0 \Phi(\lambda)}{\lambda(\Phi(\lambda) - r)} \,.
\end{aligned} \tag{3.136}
$$

With the similar steps, the second moment of subordinated geometric Brownian motion in Laplace space is found as

$$
\begin{aligned}
\left\langle \tilde{Y}^2(\lambda) \right\rangle &= \int_0^\infty dt \, e^{-\lambda t} \left\langle X^2(S(t)) \right\rangle \\
&= \int_0^\infty ds \, \tilde{h}(s, \lambda) \left\langle X^2(s) \right\rangle \\
&= \int_0^\infty ds \, \frac{\Phi(\lambda)}{\lambda} e^{-\Phi(\lambda) s} x_0^2 \, e^{2rs + \sigma^2 s} \\
&= x_0^2 \frac{\Phi(\lambda)}{\lambda} \int_0^\infty ds \, e^{-(\Phi(\lambda) - 2r - \sigma^2) s} \\
&= x_0^2 \frac{\Phi(\lambda)}{\lambda} \frac{1}{\Phi(\lambda) - 2r - \sigma^2}
\end{aligned} \tag{3.137}
$$

as we know that for normal geometric Brownian motion given by Eq. (3.124a), $\left\langle X^2(s) \right\rangle = x_0^2 e^{2\mu s + \sigma^2 s}$ and $\langle X(s) \rangle = x_0 e^{\mu s}$.

3.5 Evidence for CTRW-type pricing models from financial data

Stochastic models expressed in terms of a subordination have been proposed early on for the modelling of asset prices in financial markets [30, 96, 131]. Approaches based on a CTRW description as outlined in Sec. 2.2.1, focusing on waiting times between price changes that do not follow an exponential distribution, have become popular more recently at the turn of the millennium. In a financial interpretation of a CTRW the particle jumps will be represented by the log-returns $\xi_i = \ln X(t_{i+1}) - \ln X(t_i)$ and the waiting times by the delay $\eta_i = t_{i+1} - t_i$ between transactions, where $N(t)$ transactions take place in a given time interval $[0, t]$, see Eq. (2.54). In Ref. [129], using this approach, 1000 US stocks have been analyzed in a two-year period 1994-95. The cumulative distribution of $N(t)$ has indeed been shows to follow a power-law $\Pr(N(t) > x) \sim x^{-\beta}$ with a mean-value $\beta = 3.4 \pm 0.05$, see Fig. 3.6. From an investigation of the correlation function of $N(t)$ the existence of long-range correlations in time has also been demonstrated.

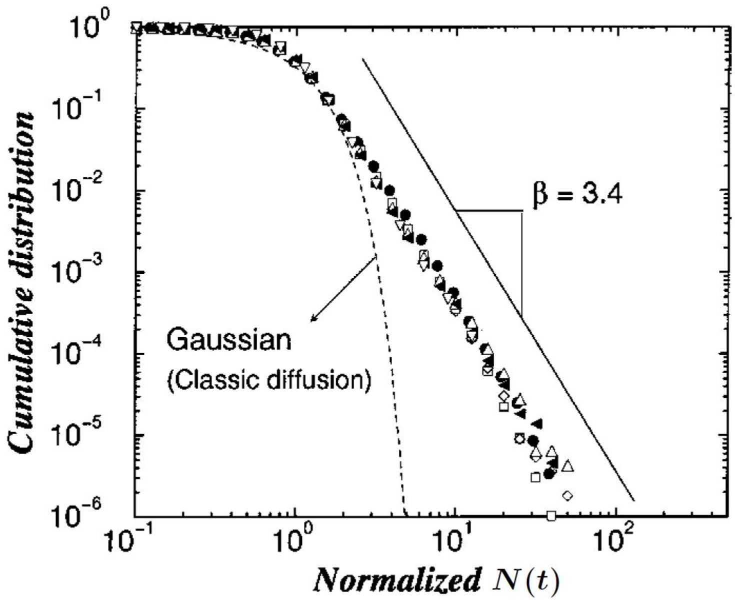

Figure 3.6: The cumulative distribution of $N(t)$. This figure is adapted from Ref. [129].

A substantial amount of work on waiting times in financial data has been performed by Scalas *et al.* [51, 94, 132, 145–149]. Assuming that ξ_i and η_i are independent and identically distributed random variables, one can consider the survival probability as [51]

$$\Omega(\eta) = 1 - \int_0^{\eta} \psi(\eta')d\eta' = \int_{\eta}^{\infty} \psi(\eta')d\eta', \qquad \psi(\eta) = -\frac{d}{dt}\Omega(\eta) \tag{3.138}$$

where $\psi(\eta)$ is the PDF of the waiting times. The integral $\int_0^{\eta} \psi(\eta')d\eta'$ gives the probability that the price changes at some instant in the interval $[0, \eta)$. Thus $\Omega(\eta)$ is the probability that the price does not change during a time interval of duration η after a jump [51]. In particular, for Markovian process, one could derive that

$$\psi(t) = \frac{1}{T} e^{-\eta/T}\,, \eta \geq 0\,, \tag{3.139}$$

where T is the average waiting time and consequently

$$\Omega(\eta) = e^{-\eta/T}\,. \tag{3.140}$$

On the other hand, a CTRW with power-law distributed waiting times corresponds to a survival probability [148]

$$\Omega(t) = E_\alpha(-t^\alpha) \tag{3.141}$$

given as a Mittag-Leffler function generalizing the simple exponential decay of the Markovian case.

In Ref. [94], the anomalous non-exponential behaviour of the survival probability has indeed been observed for BUND future prices. Using a two-parameter fit with the function

$$\Omega(\eta) = E_\alpha(-(\gamma\eta)^\alpha)\,, \tag{3.142}$$

where γ is a time-scale factor depending on the time unit excellent agreement with the empirical data can be observed, see Fig. 3.7 . In Ref. [132] the survival probability obtained from high-frequency data of General-Electric shares has been shown to follow a stretched exponential $\exp(-(\eta/\eta_0)^\alpha/\Gamma(1+\alpha))$, see Fig. 3.8. Since the Mittag-Leffler asymptotically converges to a stretched exponential for small times, this study also provides evidence for power-law waiting times in financial data. The same phenomenon is also found by the empirical analysis of 30 New York Stock Exchange (NYSE) stocks [147]. In Ref. [149] the authors argue that the waiting times between consecutive trades are non-exponentially distributed after carefully examining nearly 800,000 orders and 540,000 trades of Glaxo Smith Kline and Vodafone stocks.

The anomalous non-exponential behaviour of the waiting time distribution manifest in the survival probability has been corroborated by the market analysis of other groups as well. A study of two completely different financial markets, namely the Irish stock market during the 19th century over the period 1850 to 1854 and the Japanese yen currency fluctuations during the latter part of the 20th century (1989–1992) have been performed in Ref. [141]. Both of the data sets confirm power law tails in the survival

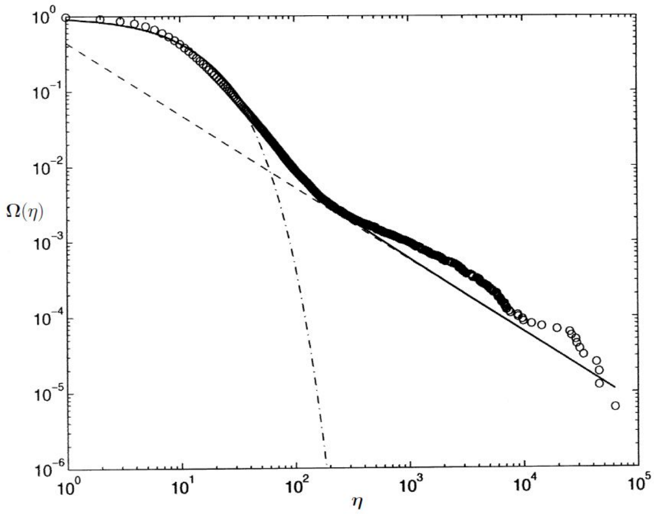

Figure 3.7: Survival probability for BUND futures with delivery date:June 1997. The line (——) indicates the Mittag-Leffler function with parameters $\alpha = 0.96, \gamma = 1/12$. The figure is adapted from Ref. [94].

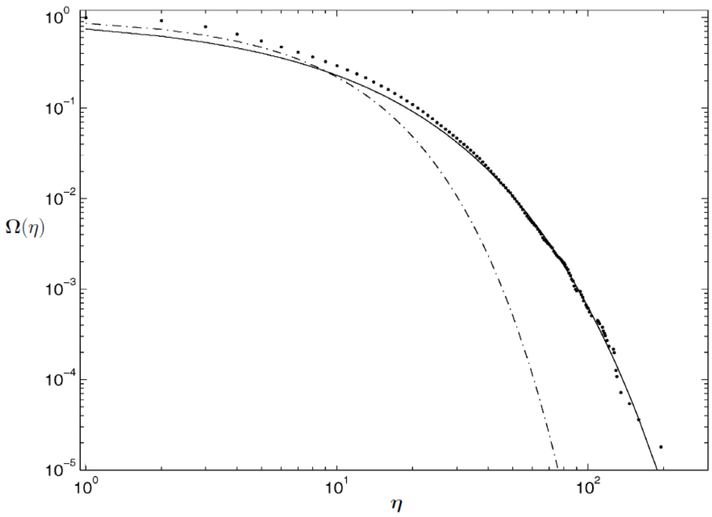

Figure 3.8: Survival probability for the high-frequency data of General-Electric shares. The solid line (——) indicates the stretched exponential with parameters $\alpha = 0.7, \eta_0 = 6.6$. The figure is adapted from Ref. [132].

probability. However, only the Irish stock market data follows also the Mittag-Leffler decay over a considerable range, see Fig. 3.9 A decay following a stretched exponential has been observed for bond futures in the Korean Futures Exchange market [73].

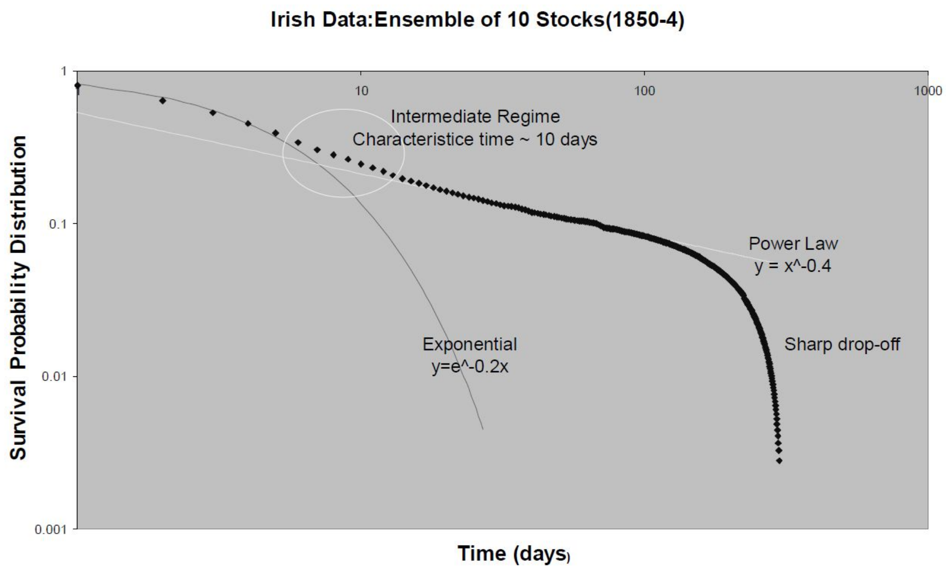

Figure 3.9: Average survival probability function for Irish stock market data between 1850 and 1854. Fit parameters for Mittag-Leffler function with parameters $\alpha = 0.4, \gamma = 0.025$. The figure is adapted from Ref. [141].

A power-law behaviour of the waiting times between successive price changes has been directly observed in Ref. [86] for the Korean stock market index KOSPI. The quantitative investigation of the calm–time intervals of price changes for 800 companies listed in the Tokyo Stock Exchange also support that the interval distribution obeys a power law decay [71] .

After analysing the sequence of time intervals between consecutive stock trades of thirty companies representing eight sectors of the US economy over a period of four years, the authors in Ref. [63] point out that their results " support the hypothesis that the dynamics of transaction times may play a role in the process of price formation, and may have implications for financial modelling based on continuous time random walks and subordinated-processes."

Recently, there has been an increasing interest on the CTRW formalism which are used to describe the price processes [48, 65, 66]. All these studies establish that the CTRW with either power-law distributed waiting times or waiting times following a more complicated distribution is a useful model to explain the statistical properties of financial data. The investigation of other problems based on CTRW have been put forward, such as mean exit times of asset prices [116] and option pricing [91]. In Ch. 4 the problem of pricing options with CTRW-type asset pricing models is discussed in detail.

3.6 Numerical simulation of sample path

Finally, let us introduce a algorithm for numerical simulation sample path of subordinated GBM $Y(t) = X(S(t))$ given by Eq. (3.124). The algorithm for subordinated GBM has been given by authors [78,93]. Here we present a summary. With Euler scheme, we can obtain the discrete analogy of Eq. (3.124) as

$$X_k = X_{k-1} + X_{k-1}(r\Delta s + \sigma\sqrt{\Delta s}\zeta_k) \tag{3.143a}$$

$$T_k = T_{k-1} + \eta_k \tag{3.143b}$$

where ζ_k is a standard Gaussian random variable with zero average and variance 1. The variable η_k a random variable, which can be derived as

$$\eta_k = (\Delta s)^{1/\alpha} \frac{\sin[\alpha(V_k + \frac{\pi}{2})]}{[\cos(V_k)]^{1/\alpha}} \left(\frac{\cos[V_k - \alpha(V_k + \frac{\pi}{2})]}{W_k} \right)^{(1-\alpha)/\alpha} \tag{3.144}$$

where V_k is uniformly distributed on $[-\pi/2, \pi/2]$ and W_k has exponential distribution with mean 1, which could be generated as follows

$$\begin{aligned} V_k &= \pi(\zeta_1 - \frac{1}{2}) \\ W_k &= -\ln(1 - \zeta_2)\,. \end{aligned} \tag{3.145}$$

To get the numerical simulation of trajectories $X(S(t))$ at discrete times $t_j = k\Delta t$, $j = 0, \cdots, N$, the following algorithm can be applied

- Initialization of $X(0) = 1$ and $T(0) = 0$, set $s = 0$.
- For each j, increase s by Δs (we choose $\Delta s < \Delta t$), and increase X_k and T_k by Eq. (3.143a)–(3.143b) while $T_k < t_j$.
- Set $X(S(t_j)) = X_k$.

As far as for subordinated GBM, the procedure is quite similar. The different is that here the random variable η_k in Eq. (3.143b) is required to generated as follows [8,78,90]

- Generate exponential random variable W with man $1/\lambda$.
- Generate a random variable η by Eq. (3.144).
- Let $\eta_k = \eta$ when $W > \eta$, otherwise go to step 1.

We use the simulation method above to study the mean and the second moment of subdiffusive and subordinated GBM, respectively. It is obviously that the theoretical results agree well with the simulation results. Fig. 3.10 shows the mean and the second moment of subdiffusive geometric Brownian motion for different α, respectively. Fig. 3.11

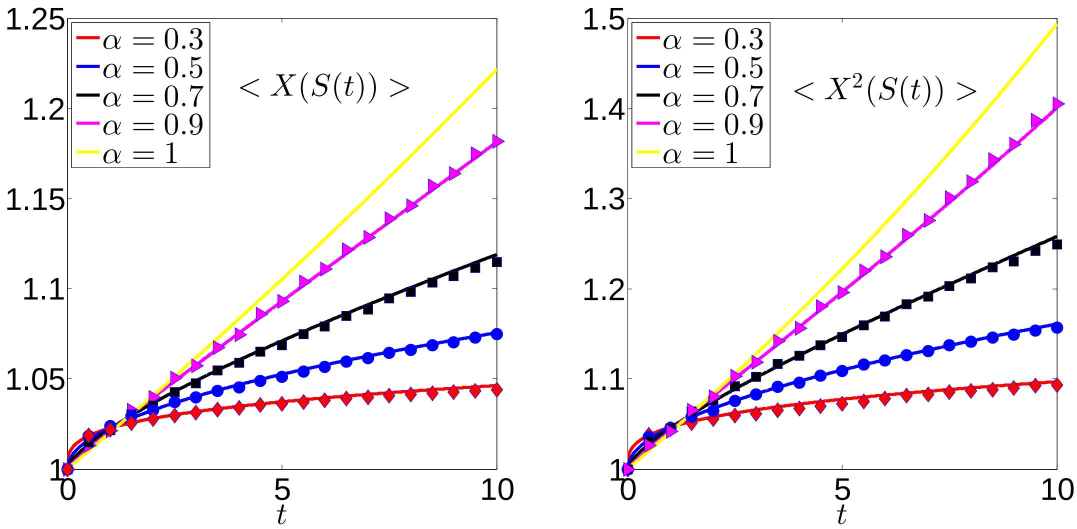

Figure 3.10: Mean (left panel) [see Eq. (3.131)] and second moment (right panel) [see Eq. (3.132)] of subdiffusive geometric Brownian motion with parameters $\sigma = 0.01$, $x_0 = 1$ and $\mu = 0.02$. Ensembles of 1000 trajectories of $X(S(t))$ are simulated with the algorithms [78,93]. Lines correspond to the analytic expressions of Eqs. (3.131)–(3.132) and the simulation results (markers) agree well with the exact expressions.

shows mean and the second moment of the subordinated geometric Brown motion for different α. Fig. 3.12 clearly indicates mean and the second moment of the subordinated geometric Brown motion with $\Phi(\lambda) = (\lambda + \zeta)^\alpha - \zeta^\alpha$ for different ζ.

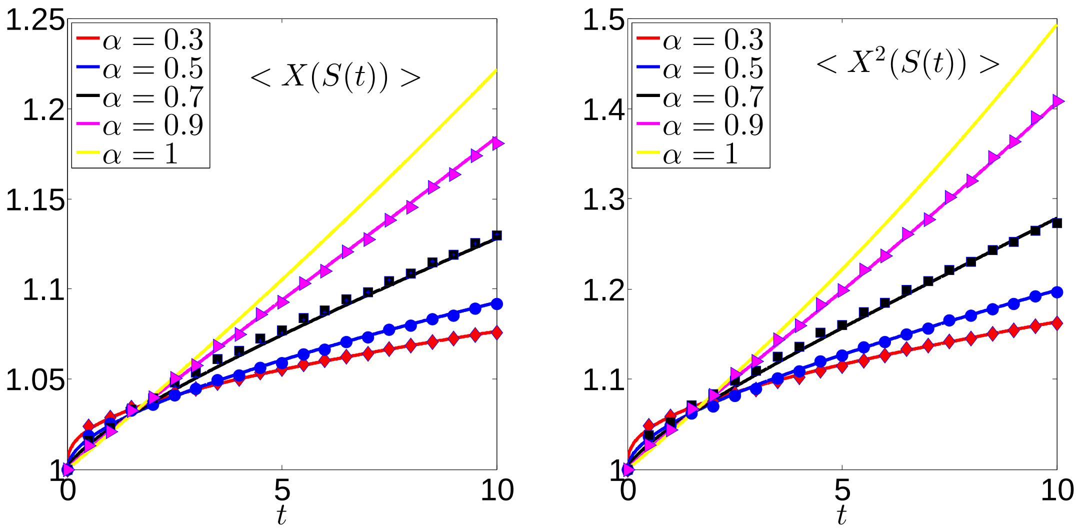

Figure 3.11: Sample mean [see Eq. (3.136)](left panel) and second moment [see Eq. (3.137)] (right panel) of subordinated geometric Brownian motion with $\Phi(\lambda) = (\lambda + \zeta)^\alpha - \zeta^\alpha$ as well as the parameters $\sigma = 0.01$, $x_0 = 1$, $\mu = 0.02$ and $\zeta = 0.005$. Ensembles of 1000 trajectories of $X(S(t))$ are simulated with the algorithms [8, 78, 90]. Lines correspond to the analytic expressions of Eqs. (3.136)–(3.137) and the simulation results (markers) agree well with the exact expressions.

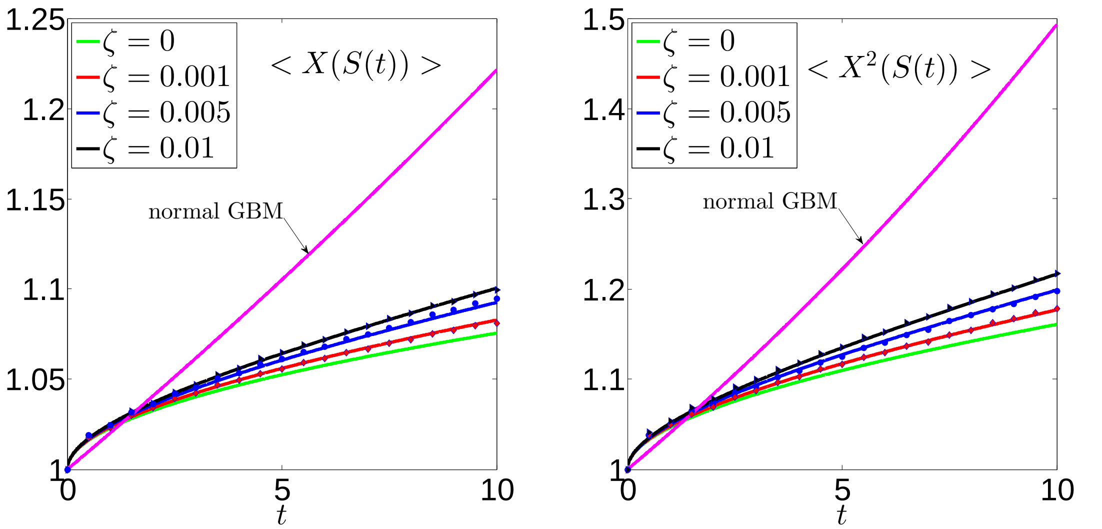

Figure 3.12: Sample mean [see Eq. (3.136)] (left panel) and second moment [see Eq. (3.137)] (right panel) of subordinated geometric Brownian motion with $\Phi(\lambda) = (\lambda + \zeta)^\alpha - \zeta^\alpha$ as well as the parameters $\sigma = 0.01$, $x_0 = 1$, $\mu = 0.02$ and $\alpha = 0.5$. Ensembles of 1000 trajectories of $X(S(t))$ are simulated with the algorithms [8, 78, 90]. Lines correspond to the analytic expressions of Eqs. (3.136)–(3.137) and the simulation results (markers) agree well with the exact expressions.

Chapter 4

European call option pricing formula

The key issue in relating option pricing and financial market is appropriate construction of price model. The lack of prediction of price in the future makes it impossible to form a fair price for an option. As a result we cannot impose any efficient analysis on option pricing and assets' price. Moreover, there is no way to consider how far the price of an option can go in next months, since we only have the history of the assets price and no mathematical analysis can be performed to exploit the future price. To circumvent this problem, an effective modelling solution is necessary and required. In the next chapter a more general model for the assets' price will be presented, describing the prices as a subordinated process with general waiting times. Before going through this process however, let us start with European call option pricing with generalised CTRW model.

Supposing that the assets price model follows subdiffusive GBM Eq. (3.124), which is no longer a Markovian processes, new characteristics for this model may arise. Fig. 4.1 gives illustrations of Markovian and non-Markovian processes, respectively. For Markovian processes, the conditional probability $p(x, t|x_0, t_0)$ at time t is entirely determined by the initial position x_0 at time t_0 shown by the left one of Fig 4.1. However, for non-Markovian processes, conditional probabilities have a more intricate structure. For instance, the conditional probability $p(x, t_2|x_1, t_1, x_0, t_0)$ at time t_2 now explicitly depends on the entire history. As a result of non-Markovian properties, the subdiffusive processes $Y(t)$ is much more complex than the simple assets processes $X(t)$, which gives new feature to our new assets price model.

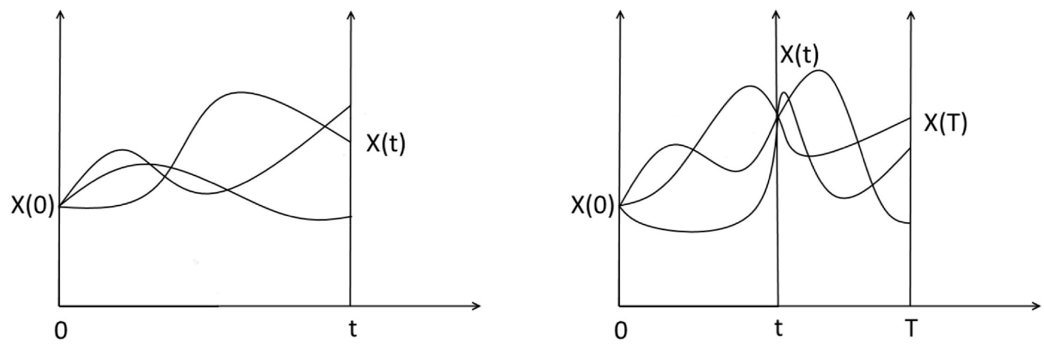

Figure 4.1: Illustration of the difference between Markovian and non-Markovian processes. The conditional probability density function $p(x, t|x_0, t_0)$ of a Markovian processes at time t only depends on the initial position x_0 at time t_0. However, for non-Markovian processes, the conditional probability $p(x, t_2|x_1, t_1, x_0, t_0)$ at time t_2 not only depends on the initial position x_1 at time t_1 but also on the starting position x_0 at time t_0

In the classical Black Scholes setup the expiration date t together with the initial data x_0 at $t = 0$ determines the cost of the option. The picture is entirely unchanged if given data x_0 at $t = 0$ we start the trading at $t_1 = t$ with data x_1 and expiration time T. Because of the assumed Markovian property of the asset price the information x_0 at $t = 0$ drops from the expression and we can still apply the Black Scholes theory with expiration time $T - t$.

In the case considered here, i.e., assuming a non Markovian asset price the situation is fundamentally different, and both types of options will differ. We call an option to be of *type A* if, along the lines of the traditional Black Scholes theory, given initial data x_0 at time $t = 0$ the expiration date is given by t. However, if we start trading at t with initial data x_1 and expiration time T then the additional knowledge of the asset at $t = 0$ with value x_0 can make a difference. We will call the corresponding option an option of *type B*.

An additional layer of complexity is added by the way how to take the interest rate, i.e., the discounting into account. We can set the discounting and the trading dates according to the subordinated time, i.e., at $S(0) = 0$ and $S(t)$ or at the real time $t = 0$ and t. We will call these different types of options cost 1 and cost 2.

As discussed in Ch. 2 an important requirement for option pricing is that it should not include arbitrage chances, or equivalently that a risk-neutral measure can be found.

In order to find a risk-neutral measure we need to show that there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that Eq. (2.74) holds. For a complete model, this measure also needs to be unique. For the pricing model considered here, namely subdiffusive geometric Brownian motion $Y(t)$ given by Eq. (3.124), one needs to distinguish the cases of $r = 0$ and $\neq 0$.

The case $r = 0$ has been discussed in detail in Ref. [89]. In this case, a risk-neutral measure is given by replacing $Z(t)$ of Eq. (2.76) by

$$H(t) = Z(S(t)) = \exp\left\{-\frac{1}{2}\theta^2 S(t) - \theta W(S(t))\right\}, \tag{4.1}$$

with $\theta = \mu/\sigma$. It is straightforward to show that then

$$\begin{aligned}\langle Y(t)\rangle^{\mathbb{Q}} &= \langle Y(t)H(t)\rangle^{\mathbb{P}} \\ &= \langle X(S(t))Z(S(t))\rangle^{\mathbb{P}} \\ &= \int_0^\infty \mathrm{d}s\, h(s,t)\,\langle X(s)Z(s)\rangle^{\mathbb{P}} \\ &= x_0 \int_0^\infty \mathrm{d}s\, h(s,t) \\ &= x_0. \end{aligned}\tag{4.2}$$

Therefore $Y(t)$ is a martingale under $\mathbb{Q}$, i.e., it satisfies Eq. (2.74) for $r = 0$. The key is to recognize that $\langle X(s)Z(s)\rangle^{\mathbb{P}}$ with Z given by Eq. (2.76) is just GBM under the risk-neutral measure satisfying Eq. (2.79).

Moreover, it has been shown in Ref. [88] that this risk-neutral measure is not unique, indicating the incompleteness of the market according to the second fundamental theorem of asset pricing [32]. Thus it is not possible to find a self-financing strategy. Because of the incompleteness of the market, different probability measures will result into different prices. But the probability measure $\mathbb{Q}$ defined in Eq. (4.1) has its own advantage. It is clear that in the Brownian limit, where $S(t) \to t$, $\mathbb{Q}$ becomes the probability measure of the classical Black-Scholes model, which is arbitrage free and complete. Therefore it can be used to compare the obtained prices of of the subdiffusive and classical models.

However, for $r \neq 0$, the situation is more complicated. In Ref. [120] it has been suggested that by using Eq. (4.1) with $\theta = (r + \mu)/\sigma$ as an equivalent measure, $Y(t)$ will satisfy Eq. (2.74) even for $r \neq 1$. Assuming that there might have been a typo in Ref. [120] and instead θ should be chosen as $\theta = (\mu - r)/\sigma$ as in Eq. (2.76), we obtain for the expected value under $\mathbb{Q}$

$$\begin{aligned}\langle Y(t)\rangle^{\mathbb{Q}} &= \int_0^\infty \mathrm{d}s\, h(s,t)\, \langle X(s)Z(s)\rangle^{\mathbb{P}} \\ &= x_0 \int_0^\infty \mathrm{d}s\, e^{rs}\, h(s,t) \\ &= x_0 E_\alpha(rt^\alpha) \end{aligned} \tag{4.3}$$

using the result Eq. (2.79). Clearly, multiplying both sides by e^{-rt} will not result in a martingale and Eq. (2.74) is violated. It is still an open question whether an equivalent measure can be found for the $r \neq 0$ case such that Eq. (2.74) holds. At this point it is difficult to see how the properties $\langle Z(t)\rangle^{\mathbb{P}} = 1$ and Eq. (2.74) can be simultaneously satisfied.

On the other hand, we see that $Y(t)$ under the equivalent measure Eq. (4.1) and $\theta = (\mu - r)/\sigma$ satisfies the property

$$\begin{aligned}\left\langle e^{-rS(t)} Y(t)\right\rangle^{\mathbb{Q}} &= \left\langle e^{-r\,S(t)} X(S(t)) Z(S(t))\right\rangle^{\mathbb{P}} \\ &= \int_0^\infty \mathrm{d}s\, h(s,t) e^{-r\,s}\, \langle X(s)Z(s)\rangle^{\mathbb{P}} \\ &= x_0 \int_0^\infty \mathrm{d}s\, h(s,t) \\ &= x_0. \end{aligned} \tag{4.4}$$

Therefore a modified no arbitrage statement holds: Rather than discounting in the physical time t, the asset price needs to be discounted with respect to the auxiliary time $S(t)$. As for the case $r = 0$ we do not expect the resulting market model to be complete although this has not been proven.

For the rest of the thesis, we will investigate the subdiffusive option pricing for these two different types of discounting. For the type A option we have the two versions with option prices being determined by

$$C_1^{(A)}(x,t) = \left\langle e^{-rS(t)} (X(S(t)) - K)^+ \right\rangle^{\mathbb{Q}}_{X(0)=x} \tag{4.5}$$

and

$$C_2^{(A)}(x,t) = \langle e^{-rt} (X(S(t)) - K)^+ \rangle^{\mathbb{Q}}_{X(0)=x} \tag{4.6}$$

respectively, where $\langle \cdots \rangle^{\mathbb{Q}}$ denotes the conditional expectation values with respect to the risk-neutral measure $\mathbb{Q}$ of Eq. (4.1) with $\theta = (\mu - r)/\sigma$. The same considerations can be applied for type B options which take the additional information from the memory into account. A type B cost 1 option is thus determined by

$$C_1^{(B)}(x, T, t) = \left\langle e^{-r(S(T)-S(t))} \left(X(S(T)) - K\right)^+ \right\rangle_{X(S(t))=x}^{\mathbb{Q}} \tag{4.7}$$

and the price for the cost 2 option reads

$$C_2^{(B)}(x, T, t) = e^{-r(T-t)} \left\langle (X(S(T)) - K)^+ \right\rangle_{X(S(t))=x, X(0)=x_0}^{\mathbb{Q}} \tag{4.8}$$

The last expression clearly displays a dependence on the additional information available at time $t = 0$. This additional constraint drops from the type B cost 1 option as the time points and the discounting is based on the subordinate time and the process $X(s)$, i.e., the asset price on the time scale $S(t)$, is still Markovian.

Crucially, all four option prices reduce to the ones of the standard Black-Scholes theory when the subdiffusive GBM reduces to conventional GBM for $S(t) \to t$. The precise way how this limit is achieved depends on the model for $S(t)$ expressed by the Laplace exponent ϕ. In the conventional power-law case $\phi(\lambda) = \lambda^\alpha$, this limit is simply $\alpha \to 1$. Even though none of the option prices Eqs. (4.5)–(4.8) satisfies a no arbitrage condition in the traditional sense, we still expect to be able to obtain useful information due to the correspondence with the standard Black-Scholes option prices in the appropriate limit. In fact, pricing models violating the no arbitrage condition Eq. (2.74), such as fractional Brownian motion, have been widely discussed in the mathematical finance literature. For convenience we will drop in the remainder of this thesis the superscript $\mathbb{Q}$. All expected values are implicitly assumed with respect to the measure $\mathbb{Q}$ of Eq. (4.1) with $\theta = (\mu - r)/\sigma$.

This chapter is organized as follows. We begin by showing the difference of two kinds of subdiffusive *type A* options with generalised CTRW model. Subsequently, we will study subdiffusive *type B* options. Finally, the summary and conclusion are given.

4.1 *Type A* option cost in subdiffusive regime

First we will investigate the subdiffusive cases for *type A* option cost in the following sections. The two different versions of costing will be discussed separately.

4.1.1 Subdiffusive *type A* option cost 1

Let us assume that the asset price follows the subdiffusive geometric Brownian motion. If we assume that the discounting takes place on the subordinated timescale the price of the option is given by Eq. (4.5) and can be thus written as

$$\begin{aligned} C_1^{(\mathrm{A})}(x,t) &= \left\langle e^{-rS(t)}(X(S(t))-K)^+\right\rangle_{X(0)=x} \\ &= \left\langle \int_0^\infty dse^{-rs}(X(s)-K)^+\delta(s-S(t))\right\rangle \\ &= \int_0^\infty ds\left\langle e^{-rs}(X(s)-K)^+\right\rangle h(s,t) \\ &= \int_0^\infty dsC_{\mathrm{BS}}(x,s)h(s,t) \end{aligned} \tag{4.9}$$

where $C_{\mathrm{BS}}(x,s)$ denotes the classical Black Scholes expression, Eq. (2.95) and the density of the inverse one-sided Lévy stable process of order α, $h(s,t)$, is determined by Eq. (3.71). The other parameters have their usual meaning, i.e., r is the risk free rate, t is the exercise date, and K the strike price. Such an expression can be fairly easily dealt with if we apply the Laplace transform to Eq. (4.9)

$$\begin{aligned} \tilde{C}_1^{(\mathrm{A})}(x,\lambda) &= \int_0^\infty dsC_{\mathrm{BS}}(x,s)\tilde{h}(s,\lambda) \\ &= \int_0^\infty dsC_{\mathrm{BS}}(x,s)\lambda^{\alpha-1}e^{-\lambda^\alpha s} \\ &= \lambda^{\alpha-1}\int_0^\infty dsC_{\mathrm{BS}}(x,s)e^{-\lambda^\alpha s} \\ &= \lambda^{\alpha-1}\tilde{C}_{\mathrm{BS}}(x,\lambda^\alpha) \end{aligned} \tag{4.10}$$

The Laplace transform $\tilde{C}_{\mathrm{BS}}(x,\lambda)$ of the Black-Scholes formula can be for instance found in [110] as follows,

$$\tilde{C}_{\mathrm{BS}}(x,\lambda) = \left(\frac{m_2(\lambda+r)}{\lambda+r} + \frac{1-m_2(\lambda+r)}{\lambda}\right)\frac{K^{1-m_1(\lambda+r)}}{(m_1(\lambda+r)-m_2(\lambda+r))} \times x^{m_1(\lambda+r)}, \qquad x \le K \tag{4.11a}$$

$$\tilde{C}_{\mathrm{BS}}(x,\lambda) = \left(\frac{m_1(\lambda+r)}{\lambda+r} + \frac{1-m_1(\lambda+r)}{\lambda}\right)\frac{K^{1-m_2(\lambda)}}{(m_1(\lambda+r)-m_2(\lambda+r))} \times x^{m_2(\lambda+r)} + \frac{x}{\lambda} - \frac{K}{r+\lambda}, \qquad x \ge K \tag{4.11b}$$

where we have introduced the abbreviations

$$m_{1/2}(\Lambda) = \frac{-(r-\sigma^2/2) \pm \sqrt{(r-\sigma^2/2)^2 + 2\sigma^2\Lambda}}{\sigma^2}\,. \tag{4.12}$$

Replacing λ with λ^α in Eq. (4.11) and substituting it into Eq. (4.10), yields the final result. It is however difficult to perform the inverse Laplace transform by analytical methods and to obtain an explicit analytic formula for the option price in the time domain. Alternatively we can use the Talbot method [2, 3, 157] to compute the option value in the time domain, by numerical inversion of the Laplace transform with the help of the Mathematica software package [1].

When $r = 0$, our subdiffusive option cost in Eq. (4.9) becomes Black-Scholes formula in subdiffusive regime [88].

While Eq. (4.10) gives the solution in the Laplace space and we will used this result to illustrate as well the use of fractional BS equations for the computation of the option price. For that purpose let us first derive the equation of motion for the quantity Eq. (4.5). By differentiating $C_1^{(\mathrm{A})}(x,t)$ in Eq. (4.9) with respect to t, we get

$$\frac{\partial}{\partial t}C_1^{(\mathrm{A})}(x,t) = \int_0^\infty ds C_{\mathrm{BS}}(x,s)\frac{\partial}{\partial t}h(s,t)\,. \tag{4.13}$$

Using Eq. (3.75) this can be rewritten as

$$
\begin{aligned}
\frac{\partial}{\partial t}C_1^{(\mathrm{A})}(x,t) &= -{}_0D_t^{1-\alpha}\left(\int_0^\infty dsC_{\mathrm{BS}}(x,s)\frac{\partial}{\partial s}h(s,t)\right) \\
&= -{}_0D_t^{1-\alpha}\left(h(s,t)C_{\mathrm{BS}}(x,s)|_0^\infty - \int_0^\infty ds\frac{\partial}{\partial s}C_{\mathrm{BS}}(x,s)h(s,t)\right) \\
&= -{}_0D_t^{1-\alpha}\left(-\int_0^\infty ds(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2}C_{\mathrm{BS}}(x,s) - rC_{\mathrm{BS}}(x,s)\right. \\
&\quad \left. +rx\frac{\partial}{\partial x}C_{\mathrm{BS}}(x,s))h(s,t)\right) \\
&= -{}_0D_t^{1-\alpha}\left(-\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} + r - rx\frac{\partial}{\partial x}\right)C_1^{(\mathrm{A})}(x,t) \\
&= {}_0D_t^{1-\alpha}\left(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} - r + rx\frac{\partial}{\partial x}\right)C_1^{(\mathrm{A})}(x,t) \qquad (4.14)
\end{aligned}
$$

where we have again used the appropriate boundary conditions for $h(s,t)$ at $t=0$ and $t=\infty$. The resulting fractional BS equation has to be supplemented with the initial and boundary conditions

$$C_1^{(\mathrm{A})}(x,0) = \max((x-K),0), \qquad x \geq 0 \qquad (4.15\mathrm{a})$$

$$C_1^{(\mathrm{A})}(0,t) = 0, \qquad t \geq 0 \qquad (4.15\mathrm{b})$$

$$C_1^{(\mathrm{A})}(x,t) \to x, \qquad x \to \infty \qquad (4.15\mathrm{c})$$

which follow immediately from Eq. (4.5) if we use $S(0)=0$ and the positivity of the process defined by Eq. (3.66). Of course in the special case $\alpha=1$, the fractional equation (4.14) becomes the normal Black-Scholes equation Eq. (2.96). The result computed previously in Eq. (4.10) can be obtained as well from the fractional BS equation if we follow the similar procedure described in [110]. Applying the Laplace transform to Eq. (4.14) we obtain an ordinary differential equation (ODE)

$$\frac{\sigma^2x^2}{2}\tilde{C}_1''^{(\mathrm{A})}(x,\lambda) + rx\tilde{C}_1'^{(\mathrm{A})}(x,\lambda) - (\lambda^\alpha + r)\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = -\frac{C_1^{(\mathrm{A})}(x,0)}{\lambda^{1-\alpha}}. \qquad (4.16)$$

The inhomogeneous part is given by the initial condition, Eq. (4.15), and we will discuss the two cases $x \leq K$ and $x > K$ separately. For $x \leq K$ the inhomogeneous part vanishes and the general solution of Eq. (4.16) is given by the homogeneous solution

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = Ax^{m_1(\lambda^\alpha+r)} + Bx^{m_2(\lambda^\alpha+r)} \qquad (4.17)$$

where we have used the abbreviations introduced in Eq. (4.12). Obviously we have $m_1 \geqslant 0 \geqslant m_2$. In order to ensure for a nonsingular solution in the limit $x \to 0$ we need to require that $B = 0$. Hence we are left with

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = A x^{m_1(\lambda^\alpha + r)}, \quad x \leqslant K\,. \tag{4.18}$$

In the case $x \geqslant K$ the inhomogeneous part of Eq. (4.16) is given by $C_1^{(\mathrm{A})}(x,0) = \max((x-K),0) = x-K$ and with a suitable particular solution of the nonhomogeneous equation the general solution then reads

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = A x^{m_1(\lambda^\alpha + r)} + B x^{m_2(\lambda^\alpha + r)} + \frac{x}{\lambda} - \frac{K}{\lambda^{1-\alpha}(\lambda^\alpha + r)}, \quad x \geq K\,. \tag{4.19}$$

Here boundedness of $\tilde{C}_1^{\prime(\mathrm{A})}(x,\lambda)$ requires that $A = 0$ and we are left with

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = B x^{m_2(\lambda^\alpha + r)} + \frac{x}{\lambda} - \frac{K}{\lambda^{1-\alpha}(\lambda^\alpha + r)}, \quad x \geqslant K\,. \tag{4.20}$$

When $x = K$, the option function given by Eqs. (4.18) and (4.20) is required to be continuous and differentiable. The corresponding matching conditions result in

$$B = \left(\frac{m_1(\lambda^\alpha + r)}{\lambda^{1-\alpha}(\lambda^\alpha + r)} + \frac{1 - m_1(\lambda^\alpha + r)}{\lambda}\right) \frac{K^{1-m_2(\lambda^\alpha + r)}}{(m_1(\lambda^\alpha + r) - m_2(\lambda^\alpha + r))} \tag{4.21a}$$

$$A = \left(\frac{m_2(\lambda^\alpha + r)}{\lambda^{1-\alpha}(\lambda^\alpha + r)} + \frac{1 - m_2(\lambda^\alpha + r)}{\lambda}\right) \frac{K^{1-m_1(\lambda^\alpha + r)}}{(m_1(\lambda^\alpha + r) - m_2(\lambda^\alpha + r))} \tag{4.21b}$$

and we finally obtain

$$\begin{aligned} \tilde{C}_1^{(\mathrm{A})}(x,\lambda) &= \left(\frac{m_2(\lambda^\alpha + r)}{\lambda^{1-\alpha}(\lambda^\alpha + r)} + \frac{1 - m_2(\lambda^\alpha + r)}{\lambda}\right) \frac{K^{1-m_1(\lambda^\alpha + r)}}{(m_1(\lambda^\alpha + r) - m_2(\lambda^\alpha + r))} \\ &\quad \times x^{m_1(\lambda^\alpha + r)}, \quad x \leq K, \end{aligned} \tag{4.22a}$$

$$\begin{aligned} \tilde{C}_1^{(\mathrm{A})}(x,\lambda) &= \left(\frac{m_1(\lambda^\alpha + r)}{\lambda^{1-\alpha}(\lambda^\alpha + r)} + \frac{1 - m_1(\lambda^\alpha + r)}{\lambda}\right) \frac{K^{1-m_2(\lambda^\alpha + r)}}{(m_1(\lambda^\alpha + r) - m_2(\lambda^\alpha + r))} \\ &\quad \times x^{m_2(\lambda^\alpha + r)} + \frac{x}{\lambda} - \frac{K}{\lambda^{1-\alpha}(\lambda^\alpha + r)}, \quad x \geq K. \end{aligned} \tag{4.22b}$$

It is obvious that the $\tilde{C}_1^{(\mathrm{A})}(x,\lambda)$ in Eq. (4.22) is the same as the result obtained in Eq. (4.10).

4.1.2 Subdiffusive *type A* option cost 2

We now focus on the evaluation of the option price for a subdiffusive *type A* option cost with discounting on the physical time scale. The corresponding expression, Eq. (4.6), differs from the previous case and that becomes evident if we express the conditional expectation value in terms of the classical Black Scholes formula, Eq. (2.95)

$$\begin{aligned} C_2^{(\mathrm{A})}(x,t) &= \left\langle e^{-rt}(X(S(t))-K)^+\right\rangle_{X(0)=x} \\ &= e^{-rt}\left\langle \int_0^\infty ds (X(s)-K)^+ \delta(s-S(t))\right\rangle \\ &= e^{-rt}\int_0^\infty ds \left\langle (X(s)-K)^+\right\rangle h(s,t) \\ &= e^{-rt}\int_0^\infty ds e^{rs} C_{\mathrm{BS}}(x,s) h(s,t)\,. \end{aligned} \tag{4.23}$$

We can use such an expression to derive the corresponding equation of motion by taking the derivative with respect to t

$$\begin{aligned} \frac{\partial}{\partial t} C_2^{(\mathrm{A})}(x,t) = &-re^{-rt}\int_0^\infty ds e^{rs} C_{\mathrm{BS}}(x,s) h(s,t) \\ &+ e^{-rt}\int_0^\infty ds e^{rs} C_{\mathrm{BS}}(x,s) \frac{\partial}{\partial t} h(s,t)\,. \end{aligned} \tag{4.24}$$

With the help of Eq. (4.23), Eq. (3.75) and Eq. (2.96), the equation above becomes

$$
\begin{aligned}
\frac{\partial}{\partial t} C_2^{(\mathrm{A})}(x,t) =& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}{}_0D_t^{1-\alpha}\left(\int_0^\infty ds e^{rs} C_{\mathrm{BS}}(x,s)\frac{\partial}{\partial s}h(s,t)\right)\\
=& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}{}_0D_t^{1-\alpha}\left(h(s,t)C_{\mathrm{BS}}(x,s)e^{rs}|_0^\infty\right.\\
&\left. - \int_0^\infty ds \frac{\partial}{\partial s}(e^{rs} \times C_{\mathrm{BS}}(x,s))h(s,t)\right)\\
=& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}{}_0D_t^{1-\alpha}\left(-r\int_0^\infty ds e^{rs} C_{\mathrm{BS}}(x,s)h(s,t)\right.\\
&\left. - \int_0^\infty ds e^{rs} \times \frac{\partial}{\partial s} C_{\mathrm{BS}}(x,s)h(s,t)\right)\\
=& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}{}_0D_t^{1-\alpha}\left(-r\int_0^\infty ds e^{rs} C_{\mathrm{BS}}(x,s)h(s,t)\right.\\
&- \int_0^\infty ds e^{rs}\left(\frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2}C_{\mathrm{BS}}(x,s) - rC_{\mathrm{BS}}(x,s)\right.\\
&\left.\left. + rx\frac{\partial}{\partial x}C_{\mathrm{BS}}(x,s)\right)h(s,t)\right)\\
=& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}{}_0D_t^{1-\alpha}\left(e^{rt}(-\frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} - rx\frac{\partial}{\partial x})C_2^{(\mathrm{A})}(x,t)\right)
\end{aligned}
\tag{4.25}
$$

and we finally arrive at a modified fractional BS equation

$$
\frac{\partial}{\partial t} C_2^{(\mathrm{A})}(x,t) = -rC_2^{(\mathrm{A})}(x,t) + e^{-rt}\ {}_0D_t^{1-\alpha}\left(e^{rt}(\frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x})C_2^{(\mathrm{A})}(x,t)\right) . \tag{4.26}
$$

As in the previous case the initial and boundary conditions are given by

$$
C_2^{(\mathrm{A})}(x,0) = \max((x-K),0), \qquad x \geq 0 \tag{4.27a}
$$

$$
C_2^{(\mathrm{A})}(0,t) = 0, \qquad t \geq 0 \tag{4.27b}
$$

$$
C_2^{(\mathrm{A})}(x,t) \to xe^{-rt}E_\alpha(rt^\alpha), \qquad x \to \infty \tag{4.27c}
$$

Eqs. (4.14) and (4.26) differ in the way the discounting is embedded in the option pricing. If discounting takes place on the subordinated timescale then a plain fractional BS equation governs the dynamics, while the discounting at the real timescale adds an additional complexity to the problem, turning the equation of motion in a true non autonomous system.

With the same method mentioned as above, we can easily compute the solution as

follows

$$\tilde{C}_2^{(\mathrm{A})}(x,\lambda) = \left(\frac{m_2((\lambda+r)^\alpha)}{\lambda+r} + \frac{1-m_2((\lambda+r)^\alpha)}{((\lambda+r)^\alpha - r)(\lambda+r)^{1-\alpha}}\right) \times \frac{K^{1-m_1((\lambda+r)^\alpha)}}{(m_1((\lambda+r)^\alpha) - m_2((\lambda+r)^\alpha))} x^{m_1((\lambda+r)^\alpha)}, \quad x \le K \tag{4.28a}$$

$$\tilde{C}_2^{(\mathrm{A})}(x,\lambda) = \left(\frac{m_1((\lambda+r)^\alpha)}{\lambda+r} + \frac{1-m_1((\lambda+r)^\alpha)}{((\lambda+r)^\alpha - r)(\lambda+r)^{1-\alpha}}\right) \times \frac{K^{1-m_2((\lambda+r)^\alpha)}}{m_1((\lambda+r)^\alpha) - m_2((\lambda+r)^\alpha} x^{m_2((\lambda+r)^\alpha)} + \frac{x}{((\lambda+r)^\alpha - r)(\lambda+r)^{1-\alpha}} - \frac{K}{\lambda+r}, \quad x \ge K\,. \tag{4.28b}$$

Of course we can as well apply the Laplace transform directly to Eq. (4.23), to derive the same result, namely

$$\begin{aligned}
\tilde{C}_2^{(\mathrm{A})}(x,\lambda) &= \int_0^\infty ds e^{rs} C_{\mathrm{BS}}(x,s) h(s,\lambda+r) \\
&= \int_0^\infty ds e^{rs} C_{\mathrm{BS}}(x,s) (\lambda+r)^{\alpha-1} e^{-((\lambda+r)^\alpha - r)s} \\
&= (\lambda+r)^{\alpha-1} \int_0^\infty ds C_{\mathrm{BS}}(x,s) e^{-((\lambda+r)^\alpha - r)s} \\
&= (\lambda+r)^{\alpha-1} \tilde{C}_{\mathrm{BS}}(x, (\lambda+r)^\alpha - r)
\end{aligned} \tag{4.29}$$

Replacing λ with $(\lambda+r)^\alpha - r$ in Eq. (4.11) and substituting it into Eq. (4.29), we recover Eq. (4.28).

The value of option prices $C_1^{(\mathrm{A})}(x,t)$ and $C_2^{(\mathrm{A})}(x,t)$ are expected to be quite close while the small interest rate is taken as it is evident that these two formulas are the same when interest $r = 0$. As for the large interest rate, the situation will become different. The formulas $C_1^{(\mathrm{A})}(x,t)$ and $C_2^{(\mathrm{A})}(x,t)$ will give different value. In order to confirm whether this is right or not, we plot $C_1^{(\mathrm{A})}(x,t)$ and $C_2^{(\mathrm{A})}(x,t)$ with interest rate $r = 0.02$ and $r = 0.5$, respectively. Fig. 4.2 shows a comparison of the time dependence of the option prices $C_1^{(\mathrm{A})}(x,t)$ and $C_2^{(\mathrm{A})}(x,t)$ for parameter values $K = 2$, $x = 1$, $\sigma = 1$, and two different values for the interest rate $r = 0.02$ and $r = 0.5$. It turns out that when $r = 0.02$ which is quite close to 0, the value of $C_1^{(\mathrm{A})}(x,t)$ and $C_2^{(\mathrm{A})}(x,t)$ are nearly the same whereas when $r = 0.5$ the value is completely different. To some extent, these result confirms our expectation.

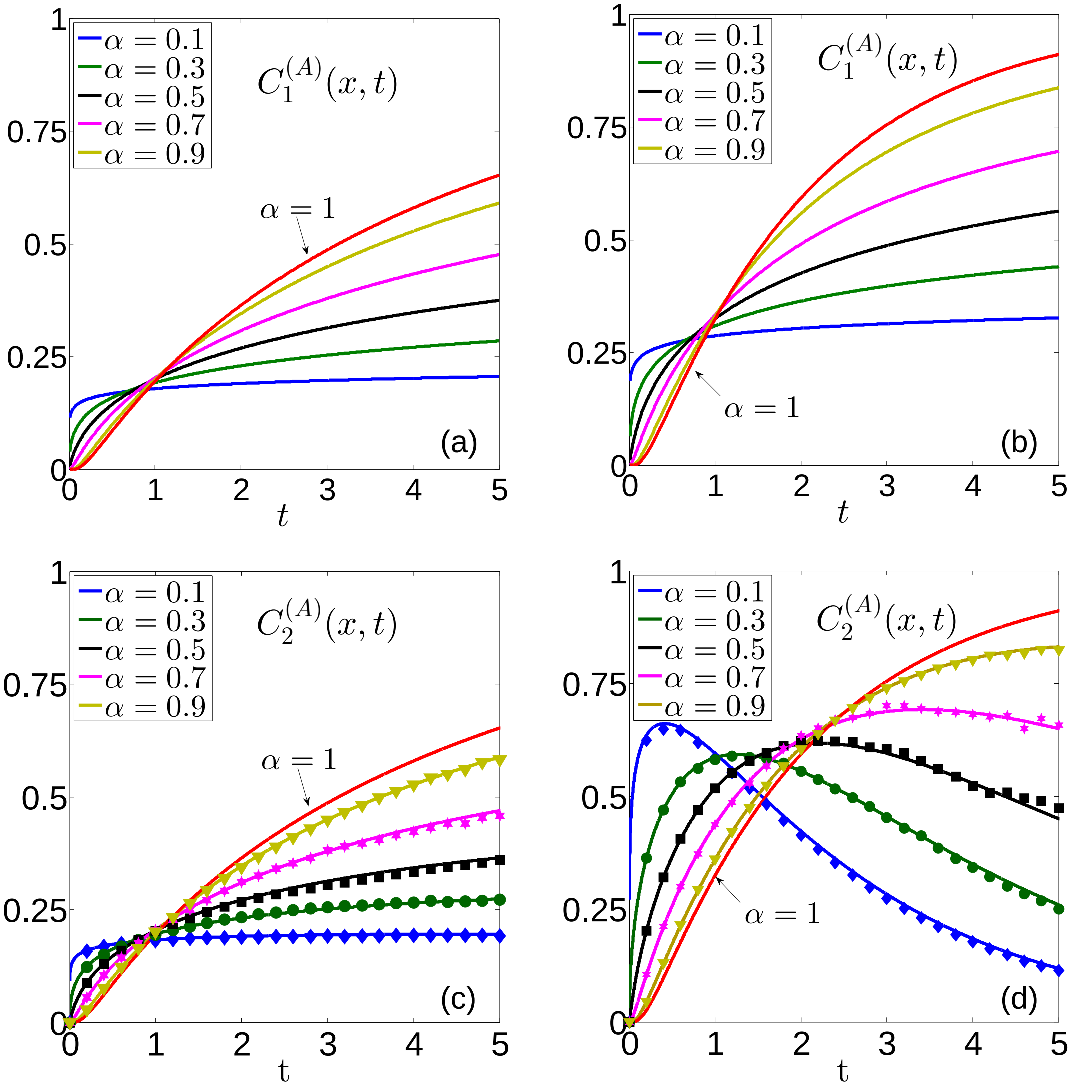

Figure 4.2: Analytic expression of subdiffusive *type A* option cost formula $C_1^{(A)}(x,t)$ [see Eq. (4.22a)] and $C_2^{(A)}(x,t)$ [see Eq. (4.28a)] with parameter values $K = 2$, $x = 1$, $\sigma = 1$ for different interest rates: (a)(c) $r = 0.02$ and (b)(d) $r = 0.5$. We use simulation to confirm the result of $C_2^{(A)}(x,t)$. Ensembles of 10^5 trajectories of $X(S(t))$ are simulated with the algorithms [78,93]. Lines correspond to the numerical inversion of the Laplace transform of Eq. (4.28a) with implementation the Talbot method [2,3,157]. The simulation results (markers) agree well with the exact expressions.

We focus in particular on the impact of the subdiffusive behaviour, controlled by the parameter α. Fig. 4.2 (a) and 4.2 (b) display the option $C_1^{(\mathrm{A})}(x,t)$ with discounting on the subordinated time scale. It is clearly visible that the classical Black-Scholes European call option formula overvalues the option when the asset prices follows a subdiffusive dynamics where our new expression would provide a more reasonable pricing model. Of course the classical Black-Scholes behaviour is recovered in the limiting case $\alpha \to 1$. Furthermore the results indicate that the option price is increasing with increasing values of α for large times, while that behaviour is reversed on short time scales. Fig. 4.2 (c) and 4.2 (d) shows that the subdiffusive formula $C_2^{(\mathrm{A})}(x,t)$ which takes the discounting on the real time scale into account exhibits a similar characteristics as the $C_1^{(\mathrm{A})}(x,t)$ for small interest rates. Both of the subdiffusive formulas $C_1^{(\mathrm{A})}(x,t)$ and $C_2^{(\mathrm{A})}(x,t)$ give a qualitatively similar result for $r = 0.02$. It is remarkable that the shape of $C_2^{(\mathrm{A})}(x,t)$ changes considerably when larger interest rates r are considered. There seems to be a complete reverse of the α dependence of the option price on longer time scales, but to some extent such a behaviour could as well be an artifact of the pricing model.

Evaluation of the subdiffusive *type A* option cost 2 by Fourier method

As Fourier analysis has been successfully used to evaluate the option pricing in many literature [25,61,74,151,164], we would like to use this method to evaluate our subdiffusive *type A* option cost 2 determined by Eq. (4.6) in this part. If we let

$$\begin{aligned} Z(t) &= \ln X(S(t)) \qquad , \\ k &= \ln K \qquad , \end{aligned} \tag{4.30}$$

where $X(S(t))$ is the subdiffusive GBM defined by Eq. (3.124), then by definition, $C_2^{(\mathrm{A})}(x,t)$ in Eq. (4.6) could be expressed as

$$\begin{aligned} C_2^{(\mathrm{A})}(x,t) &= \left\langle e^{-rt}(X(S(t)) - K)^+ \right\rangle_{X(0)=x} \\ &= e^{-rt} \int_k^\infty dz (e^z - e^k) f_t(z) \,, \end{aligned} \tag{4.31}$$

where $f_t(z)$ is density function of $Z(t)$.

By modifying the option cost $C_2^{(\mathrm{A})}(x,t)$, we obtain a square integrable function $c_t(k)$

defined as

$$c_t(k) = e^{bk} C_2^{(\mathrm{A})}(x,t), \tag{4.32}$$

where $b > 0$ helps to ensure the integrability of the modified option value $c_t(k)$. Appropriate choice of parameter b has been discussed in [25].

Now applying the Fourier transform to $c_t(k)$ with respect to k , we could find that

$$\begin{aligned}
\hat{c}_t(v) &= \int_{-\infty}^{\infty} e^{ivk} c_t(k) dk \\
&= \int_{-\infty}^{\infty} e^{ivk} e^{bk} C_2^{(\mathrm{A})}(x,t) dk \\
&= \int_{-\infty}^{\infty} e^{ivk} e^{bk} \left(\int_k^{\infty} e^{-rt} (e^z - e^k) f_t(z) dz \right) dk \\
&= e^{-rt} \int_{-\infty}^{\infty} dz f_t(z) \int_{-\infty}^{z} dk e^{k(iv+b)} (e^z - e^k) \\
&= e^{-rt} \int_{-\infty}^{\infty} dz f_t(z) \left[e^z \frac{e^{k(iv+b)}}{iv+b} \Big|_{-\infty}^{z} - \frac{e^{k(iv+b+1)}}{iv+b+1} \Big|_{-\infty}^{z} \right] \\
&= e^{-rt} \int_{-\infty}^{\infty} dz f_t(z) \left[\frac{e^{z(iv+b+1)}}{iv+b} - \frac{e^{z(iv+b+1)}}{iv+b+1} \right] \\
&= e^{-rt} \int_{-\infty}^{\infty} dz f_t(z) \frac{e^{z(iv+b+1)}}{b^2+b-v^2+i(2b+1)v} \\
&= \frac{e^{-rt} \left\langle e^{iZ(t)[v-(b+1)i]} \right\rangle}{b^2+b-v^2+i(2b+1)v} \\
&= \frac{e^{-rt} \varphi_t(v-(b+1)i)}{b^2+b-v^2+i(2b+1)v}
\end{aligned} \tag{4.33}$$

where $\varphi_t(u)$ is the characteristic function of $Z(t)$, which could be derived as

$$\begin{aligned}
\varphi_t(u) &= \left\langle e^{iuZ(t)} \right\rangle \\
&= \left\langle e^{iu \ln X(S(t))} \right\rangle \\
&= \int_0^{\infty} ds \left\langle e^{iu \ln X(s)} \right\rangle h(s,t) \\
&= \int_0^{\infty} ds e^{iu(\ln x_0 + \hat{\mu} s) - \frac{\sigma^2 u^2}{2} s} h(s,t) \\
&= e^{iu \ln(x_0)} \int_0^{\infty} ds e^{sg(u)} h(s,t)
\end{aligned} \tag{4.34}$$

where $g(u) = iu\hat{\mu} - \frac{1}{2}\sigma^2 u^2$ and $\hat{\mu} = (\mu - \frac{1}{2}\sigma^2)$. Here we use the fact that $\ln X(s)$ represents a normal process with mean $\ln x_0 + \hat{\mu}s$ and variance $\sigma^2 s$.

With the help of Eq. (3.72), we could obtain $\varphi_t(u)$ in Laplace space as

$$\begin{aligned}
\hat{\varphi}_\lambda(u) &= \int_0^\infty dt e^{-\lambda t}\varphi_t(u) \\
&= e^{iu\ln(x_0)} \int_0^\infty ds e^{sg(u)} h(s,\lambda) \\
&= e^{iu\ln(x_0)} \int_0^\infty ds e^{sg(u)} \lambda^{\alpha-1} e^{-\lambda^\alpha s} \\
&= \lambda^{\alpha-1} \int_0^\infty ds e^{-(\lambda^\alpha - g(u))s} \\
&= \frac{\lambda^{\alpha-1}}{\lambda^\alpha - g(u)} \,.
\end{aligned} \tag{4.35}$$

After performing the inverse Laplace transform, via one parameter Mittag-Leffler function, $\varphi_t(u)$ could be expressed as

$$\varphi_t(u) = e^{iu\ln(x_0)} E_\alpha\left(g(u)t^\alpha\right) \,. \tag{4.36}$$

Therefore, $\hat{c}_t(v)$ in Eq. (4.33) could be written as

$$\hat{c}_t(v) = \frac{e^{-rt} e^{i(v-(b+1)i)\ln(x_0)} E_\alpha\left(g(v-(b+1)i)t^\alpha\right)}{b^2 + b - v^2 + i(2b+1)v} \tag{4.37}$$

The closed form of the $\hat{c}_t(v)$ in Eq. (4.37) facilitates us to get the value of $c_t(v)$. Finally by performing the inverse Fourier transform, we could get the value of $c_t(k)$ as

$$c_t(k) = \frac{e^{-bk}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_t(v) dv \tag{4.38}$$

Thereafter from the Eq. (4.32) it is evident that

$$\begin{aligned}
C_2^{(\mathrm{A})}(x,t) &= e^{-bk} c_t(k) \\
&= \frac{e^{-bk}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \hat{c}_t(v) dv \\
&= \frac{e^{-bk}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \frac{e^{-rt} e^{i(v-(b+1)i)\ln(x_0)} E_\alpha\left(g(v-(b+1)i)t^\alpha\right)}{b^2 + b - v^2 + i(2b+1)v} dv
\end{aligned} \tag{4.39}$$

However, it seems unlikely to get the closed form for the option value $C_2^{(\mathrm{A})}(x,t)$, so

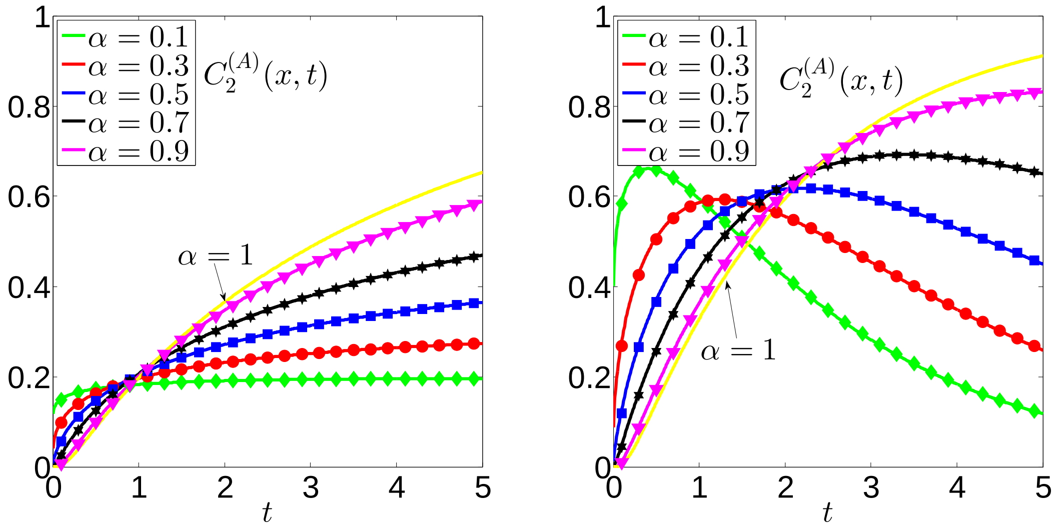

Figure 4.3: Analytic expression of subdiffusive *type A* option cost formula $C_2^{(\mathrm{A})}(x,t)$ [see Eq. (4.28a)] with parameter values $K = 2$, $x = 1$, $\sigma = 1$ and $b = 0.1$ for different interest rates: $r = 0.02$ (left panel) and $r = 0.5$ (right panel). Lines correspond to the numerical inversion of the Laplace transform of Eq. (4.28a) and points to be numerical Fourier transform evaluation of Eq. (4.39).

we use Mathematica software to get its numerical value. Figure 4.3 shows that the expressions Eq. (4.28a) and Eq. (4.39) give the same results, as expected.

4.2 *Type B* option cost in subdiffusive regime

We are now going to analyse *type B* options which take in addition information from the history of the price evolution into account. For such models the non Markovian nature of the asset price has the potential to turn out to be crucial.

4.2.1 Subdiffusive *type B* option cost 1

Let us begin with by considering the costing on the subordinated time scale. In this case the effective dynamics is still markovian and the historical information drops from the corresponding conditional expectation vale, Eq. (4.7). Following the previous reasoning and using the two time density function (cf. Eq. (3.82)) the option price can be again expressed in terms of the classical Black Scholes expression (2.95)

$$\begin{aligned}
C_1^{(\mathrm{B})}(x,T,t) &= \left\langle e^{-r(S(T)-S(t))}\left(X(S(T))-K\right)^+\right\rangle_{X(S(t))=x} \\
&= \left\langle \int_0^\infty ds_2 \int_0^\infty ds_1 e^{-r(s_2-s_1)}(X(s_2)-K)^+ \right. \\
&\quad \left. \times\, \delta(s_2-S(T))\delta(s_1-S(t)) \right\rangle \\
&= \int_0^\infty ds \left\langle e^{-r(s_2-s_1)}(X(s_2)-K)^+ \right\rangle h(s_2,T,s_1,t) \\
&= \int_0^\infty ds_1 \int_0^\infty ds_2 C_{\mathrm{BS}}(x,s_2-s_1)h(s_2,T,s_1,t) \\
&= \int_0^\infty ds_1 \int_0^\infty d\tau C_{\mathrm{BS}}(x,\tau)h(s_1+\tau,T,s_1,t)\,. \qquad (4.40)
\end{aligned}$$

Applying the two-time Laplace transform with respect to T and t we end up with

$$\begin{aligned}
\tilde{C}_1^{(\mathrm{B})}(x,\lambda_2,\lambda_1) &= \int_0^\infty ds_1 \int_0^\infty d\tau C_{\mathrm{BS}}(x,\tau)\tilde{h}(s_1+\tau,\lambda_2,s_1,\lambda_1) \\
&= \int_0^\infty ds_1 \int_0^\infty d\tau C_{\mathrm{BS}}(x,\tau)\frac{\left((\lambda_1+\lambda_2)^\alpha-\lambda_2^\alpha\right)\lambda_2^\alpha}{\lambda_1\lambda_2} \\
&\quad \times e^{-s_1((\lambda_1+\lambda_2)^\alpha-\lambda_2^\alpha)}e^{-(s_1+\tau)\lambda_2^\alpha} \\
&= \frac{\left((\lambda_1+\lambda_2)^\alpha-\lambda_2^\alpha\right)\lambda_2^\alpha}{\lambda_1\lambda_2(\lambda_1+\lambda_2)^\alpha}\int_0^\infty d\tau C_{\mathrm{BS}}(x,\tau)e^{-\tau\lambda_2^\alpha} \\
&= \frac{\left((\lambda_1+\lambda_2)^\alpha-\lambda_2^\alpha\right)\lambda_2^\alpha}{\lambda_1\lambda_2(\lambda_1+\lambda_2)^\alpha}\tilde{C}_{\mathrm{BS}}(x,\lambda_2^\alpha) \qquad (4.41)
\end{aligned}$$

and using Eq. (4.11) we can easily write down the expression in closed analytical form.

4.2.2 Subdiffusive *type B* option cost 2

If we perform the costing according to the physical timescale then the corresponding conditional expectation value, Eq. (4.8), depends explicitly on the historical information. Hence at this stage the non markovian character of the underlying asset price will turn out to become crucial. As usual, we denote by by $P_Y(y,T|x,t,x_0,0)$ the conditional probability of subdiffusive GBM [53], i.e., the conditional probability of the asset price. Similarly, as before, joint probabilities are denoted by $f(y,T;x,t;x_0,0)$ and $f(x,t;x_0,0)$

with $x_0 > 0$. Then Eq. (4.8) can be simply rewritten as

$$
\begin{aligned}
C_2^{(\mathrm{B})}(x,T,t) &= e^{-r(T-t)} \left\langle (X(S(T)) - K)^+ \right\rangle_{X(S(t))=x, X(0)=x_0} \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty} (y-K)^+ P_Y(y,T|x,t,x_0,0)dy \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty} (y-K)^+ \frac{f(y,T;x,t;x_0,0)}{f(x,t;x_0,0)} dy \\
&= e^{-r(T-t)} \int_{K}^{\infty} (y-K) \frac{f(y,T;x,t;x_0,0)}{f(x,t;x_0,0)} dy \\
&= \frac{e^{-r(T-t)}}{f(x,t;x_0,0)} \int_{K}^{\infty} (y-K) f(y,T;x,t;x_0,0) dy \,.
\end{aligned} \tag{4.42}
$$

The joint probabilities of the non Markovian process, Eq. (3.124), can now again be expressed by properties of the process X and one and two time distribution functions. Let us denote by $P_X(x,s|x_0,0)$ the conditional probability of GBM, as obtained in Eq. (2.72), and by $\tilde{P}_X(x,\lambda|x_0,0)$ its Laplace transform given in Eq. 2.73. The joint probabilities of the process Y can now be written in terms of the conditional probabilities of the Markovian dynamics and the one and two point distribution functions of the subordination (see Eq. (3.26))

$$
\begin{aligned}
f(y,T;x,t;x_0,0) = &\int_0^{\infty} ds_1 \int_{s_1}^{\infty} ds_2 P_X(y,s_2|x,s_1) P_X(x,s_1|x_0,0) \\
&\times P_X(x_0) h(s_2,T,s_1,t)
\end{aligned} \tag{4.43}
$$

and

$$
f(x,t;x_0,0) = \int_0^{\infty} ds P_X(x,s|x_0,0) P_X(x_0) h(s,t) \tag{4.44}
$$

with $P_X(x)$ denoting the stationary distribution of GBM.

With the help of Eq. (4.43) and Eq. (4.44), we find that $C_2^{(\mathrm{B})}(x,T,t)$ in Eq. (4.42) can be written as

$$
C_2^{(\mathrm{B})}(x,T,t) = \frac{e^{-r(T-t)}}{\Pi_1(x,t)} \Pi_2(x,T,t) \tag{4.45}
$$

if we introduce the abbreviations

$$\Pi_1(x,t) = \int_0^\infty ds P_X(x,s|x_0,0)h(s,t) \tag{4.46a}$$

$$\Pi_2(x,T,t) = \int_0^\infty ds_1 \int_{s_1}^\infty ds_2 P_X(x,s_1|x_0,0)h(s_2,T,s_1,t) \times \int_K^\infty (y-K)P_X(y,s_2|x,s_1)dy\,. \tag{4.46b}$$

With the substitution $\tau = s_2 - s_1$, $\Pi_2(x,T,t)$ in Eq. (4.46b) simplifies to

$$\Pi_2(x,T,t) = \int_0^\infty ds_1 \int_0^\infty d\tau P_X(x,s_1|x_0,0)h(s_1+\tau,T,s_1,t)e^{r\tau}C_{\mathrm{BS}}(x,\tau)\,. \tag{4.47}$$

The above equation holds as we know that $e^{-r(s_2-s_1)}\int_K^\infty (y-K)P_X(y,s_2|x,s_1)dy$ is actually the expression of $e^{-r(s_2-s_1)}\langle (X(s_2)-K)^+\rangle_{X(s_1)=x}$ which is equivalent to $C_{\mathrm{BS}}(x,\tau)$ in Eq. (2.95) with expiration time $\tau = s_2 - s_1$ and initial price $X(0) = x$.

Applying the two time Laplace transform to $\Pi_2(x,T,t)$ in Eq. (4.47) and with the help of Eq. (3.90) , we get

$$\begin{aligned}
\tilde{\Pi}_2(x,\lambda_2,\lambda_1) &= \int_0^\infty \int_0^\infty e^{-\lambda_2 T} e^{-\lambda_t t}\Pi_2(x,T,t)dTdt \\
&= \int_0^\infty ds_1 \int_0^\infty d\tau P_X(x,s_1|x_0,0)\tilde{h}(s_1+\tau,\lambda_2,s_1,\lambda_1)e^{r\tau}C_{\mathrm{BS}}(x,\tau) \\
&= \int_0^\infty ds_1 \int_0^\infty d\tau P_X(x,s_1|x_0,0)\frac{((\lambda_1+\lambda_2)^\alpha - \lambda_2^\alpha)\,\lambda_2^\alpha}{\lambda_1\lambda_2} e^{-s_1((\lambda_1+\lambda_2)^\alpha - \lambda_2^\alpha)} \times \\
&\quad e^{-(s_1+\tau)\lambda_2^\alpha} e^{r\tau} C_{\mathrm{BS}}(x,\tau) \\
&= \frac{((\lambda_1+\lambda_2)^\alpha - \lambda_2^\alpha)\,\lambda_2^\alpha}{\lambda_1\lambda_2} \int_0^\infty ds_1 P_X(x,s_1|x_0,0)e^{-((\lambda_1+\lambda_2)^\alpha)s_1} \times \\
&\quad \int_0^\infty d\tau e^{-\tau(\lambda_2^\alpha - r)} C_{\mathrm{BS}}(x,\tau) \\
&= \frac{((\lambda_1+\lambda_2)^\alpha - \lambda_2^\alpha)\,\lambda_2^\alpha}{\lambda_1\lambda_2} \tilde{P}_X(x,(\lambda_1+\lambda_2)^\alpha|x_0,0)\tilde{C}_{\mathrm{BS}}(x,\lambda_2^\alpha - r)\,.
\end{aligned} \tag{4.48}$$

Similarly the Laplace transform of $\Pi_1(x,t)$ in Eq. (4.46a) is found as follows

$$\begin{aligned}\tilde{\Pi}_1(x,\lambda_1) &= \int_0^\infty e^{-\lambda_1 t}\Pi_1(x,t)dt \\ &= \int_0^\infty ds P_X(x,s|x_0,0)\tilde{h}(s,\lambda_1) \\ &= \lambda_1^{\alpha-1}\int_0^\infty ds P_X(x,s|x_0,0)e^{-\lambda_1^\alpha s} \\ &= \lambda_1^{\alpha-1}\tilde{P}_X(x,\lambda_1^\alpha|x_0,0) \end{aligned} \tag{4.49}$$

as the expression of $\tilde{h}(s,\lambda_1)$ has been found in Eq. (3.72).

Let us recall that the Black-Scholes formula and the conditional probability of GBM in Laplace space $\tilde{C}_{\mathrm{BS}}(x,\lambda)$ and $\tilde{P}_X(x,\lambda|x_0,0)$ have been found in Eq. (4.11) and Eq. (2.73), respectively. Therefore, the exact analytic expressions of $\Pi_2(x,T,t)$ in Eq. (4.48) and $\Pi_1(x,t)$ in Eq. (4.49) in Laplace space could be derived straightforwardly. Moreover, their closed forms could be derived if the inverse Laplace transform is applied and the expression of $C_2^{(\mathrm{B})}(x,T,t)$ in Eq. (4.45) could then be easily obtained. However, considering the difficulty and complexity of this job, we resort to the numerical methods. We get $\Pi_2(x,T,t)$ numerically by using a two dimensional version of the algorithm proposed in [161] and $\Pi_1(x,t)$ by the Talbot method. As well as the help of Eq. (4.45), finally we could evaluate $C_2^{(\mathrm{B})}(x,T,t)$ numerically.

It is well known that in the frame of the classical Black-Scholes theory, the option value can be evaluated by $C_{\mathrm{BS}}(x,t)$ in Eq. (2.95) with the initial price x and expiration time $T-t$ without considering the concrete time t as long as we know the asset price x at the current time t and the expiration time T. Fig. 4.4 shows that how *type B* option $C_1^{(\mathrm{B})}(x,T,t)$ and $C_2^{(\mathrm{B})}(x,T,t)$ change with respect to the remaining time to expiration $T-t$ based on $r=0.02$ and $r=0.5$ respectively with parameters $K=2, \sigma=1, x_0=2, x=1$. It is clearly shown that all the *type B* option takes the same value as the corresponding *type A* option when $\alpha\to 1$, which is exactly the classical Black-Scholes case. However, this fact does not hold any more for subdiffusive option with $\alpha\neq 1$ as we could easily find the *type B* option and the corresponding *type A* option give different values, which might be greatly due to the non-Markovian properties of the new asset model $X(S(t))$. From the definition in Eq. (4.40) and Eq. (4.42), it is evident that the *type B* option $C_1^{(\mathrm{B})}(x,T,t)$ and $C_2^{(\mathrm{B})}(x,T,t)$ will give the the same when interest rate $r=0$. Thus when $r=0.02$, pretty close to 0, from Fig. 4.4 (a) and 4.4 (c), it is observable that these two formulas give the similar value, but when $r=0.5$, a larger

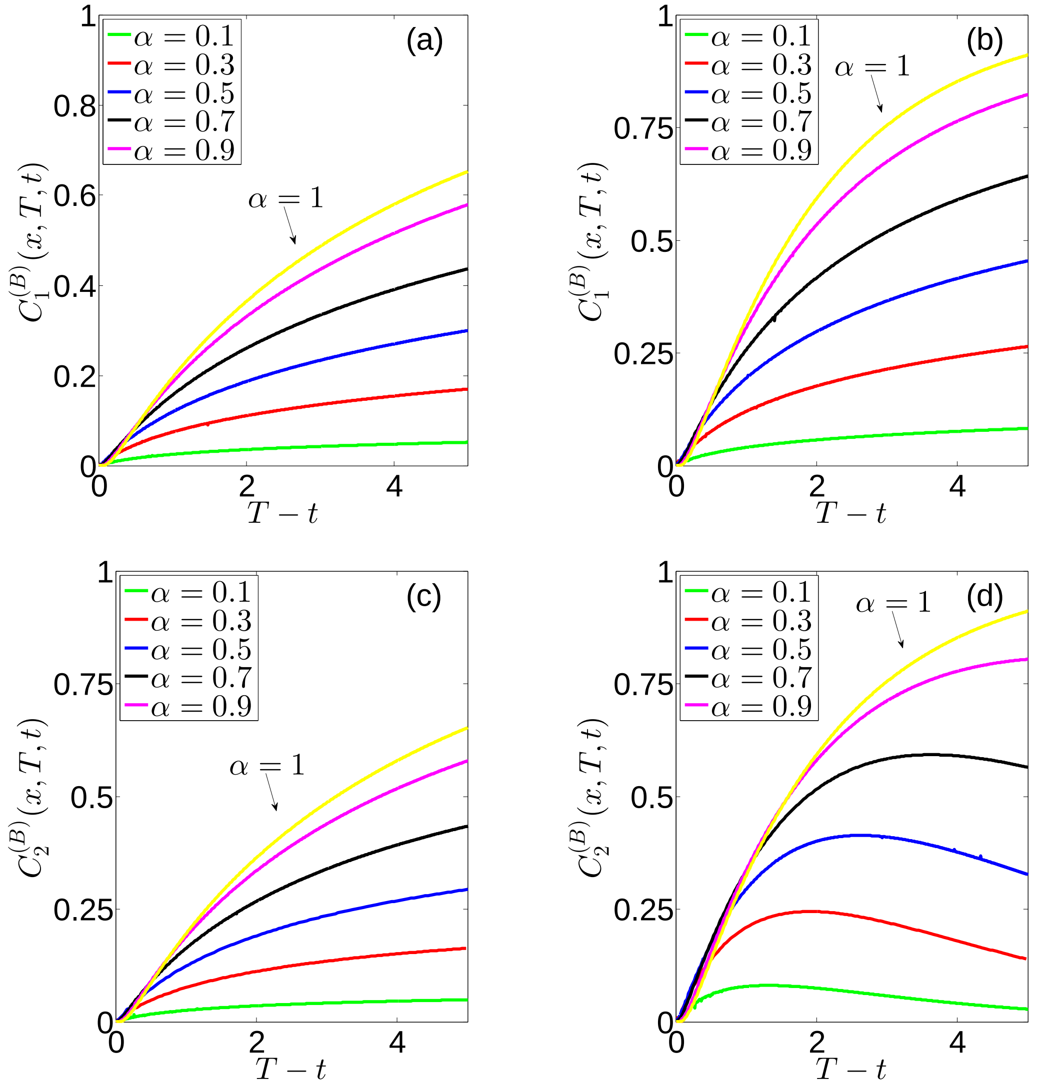

Figure 4.4: Analytic expression of subdiffusive *type B* option cost formula $C_1^{(\mathrm{B})}(x,T,t)$ [see Eq. (4.41)] and $C_2^{(\mathrm{B})}(x,T,t)$ [see Eq. (4.45)] with parameter values $K = 2, x = 1, x_0 = 2, \sigma = 1, t = 0.3$ for different interest rates: (a)(c) $r = 0.02$ and (b)(d) $r = 0.5$.

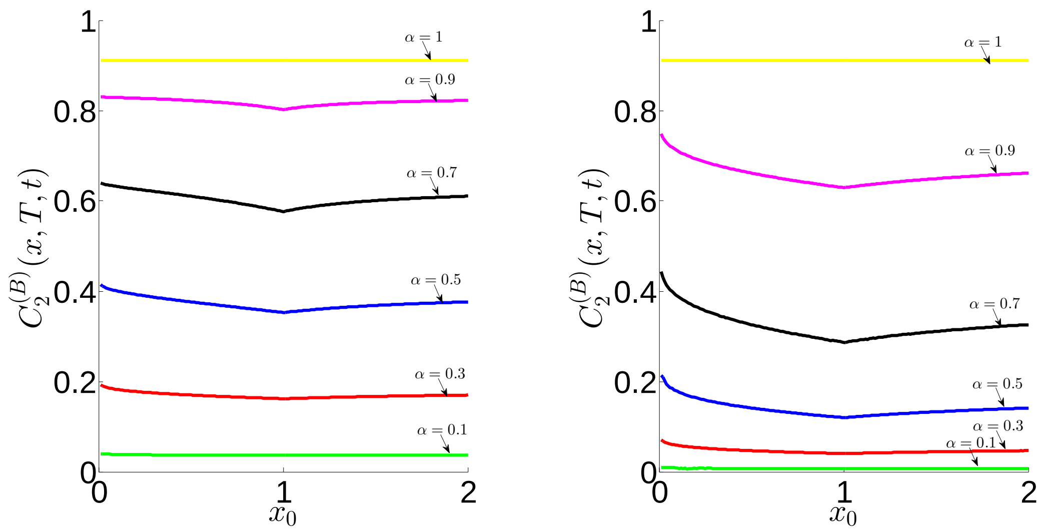

Figure 4.5: Analytic expression of subdiffusive *type B* option cost formula $C_2^{(\mathrm{B})}(x,T,t)$ [see Eq. (4.45)] with parameter values $K=2,\sigma=1,x=1,r=0.5$ for different initial and expiration times: (a) $t=0.1,T=5.1$ and (b) $t=5,T=10$.

interest rate, it is not difficult to found from Fig. 4.4 (b) and 4.4 (d) that different values are reached by these two formulas.

Meanwhile, we could find that the classical Black-Scholes European call option formula usually gives higher value than the subdiffusive option formulas whereas they could take a low value, which can also be found for the *type A* option. It indicates that the subdiffusive formula provides more reasonable price. All of the Fig. 4.4 have the same trend that when $\alpha \to 1$, the subdiffusive option prices tend to the classical value. It could be observed that the smaller the value of indicator α takes, the lower the value of the subdiffusive option takes for large time $T-t$. From Fig. 4.4 (c) and 4.4 (d), we could observe that the subdiffusive formula $C_2^{(\mathrm{B})}(x,T,t)$ also exhibit the similar characteristics as the $C_1^{(\mathrm{B})}(x,T,t)$. The shape of $C_2^{(\mathrm{B})}(x,T,t)$ changes a lot when different interest rate r is taken whereas $C_1^{(\mathrm{B})}(x,T,t)$ seems little change for different interest rate for large time scale, which has already found for *type A* option. We could conclude that the subdiffusive formula $C_2^{(\mathrm{B})}(x,T,t)$ is more sensitive to interest than the subdiffusive formulas $C_1^{(\mathrm{B})}(x,T,t)$ for the same expiration time $T-t$, which is quite similar to the *type A* option. However, compared with corresponding *type A* option, the *type B* option usually give a different value except the case when $\alpha=1$ for the same time to expiration. This phenomenon is most evident for $C_2^{(\mathrm{A})}(x,t)$ and $C_2^{(\mathrm{B})}(x,T,t)$. It is also interesting to consider how the subdiffusive *type B* option cost $C_2^{(\mathrm{B})}(x,T,t)$ changes with different value of starting price x_0 as our asset price follows the non-Markovian processes. Fig. 4.5 shows how $C_2^{(\mathrm{B})}(x,T,t)$ changes with respect to x_0. From the figure, it's easy to find that the subdiffusive option value $C_2^{(\mathrm{B})}(x,T,t)$ is higher than $x_0 = x$ when $x_0 < x$ as well as $x_0 > x$. It indicates that the larger value α takes, the higher value the subdiffusive option reach. It is obvious that when $\alpha \to 1$ the starting price x_0 has no impact on $C_2^{(\mathrm{B})}(x,T,t)$ any more as the classic Black-Scholes case appear again. However, $C_2^{(\mathrm{B})}(x,T,t)$ is always affected by the value of x_0 for small and large time t, which is quite different from the normal idea.

4.3 Summary of chapter

In this chapter, we put forward different subdiffusive *type A* and B option costs. For the subdiffusive *type A* option costs, they are found to be as the solutions to their corresponding fractional differential equations. Based on the starting time $t = 0$ and $t \neq 0$, we differentiate the *type A* and *type B* option. Comparison between different subdiffusive *type A* and B option costs are made. Due to the new subdiffusive model for the price process which exhibit non Markovian properties, great differences appear between subdiffusive option cost and classical option cost. We find that subdiffusive option formulas provide lower value than classical option which is more acceptable when the prices processes show the characteristics of subdiffusive dynamics. However, the subdiffusive model cannot guarantee the risk-neutral property, which is pretty important in the real market. It needs to be studied further. Our discussion of the subdiffusive option formulas would help to study subdiffusive phenomenon in other fields.

Chapter 5

Subdiffusive European call option pricing formula with subordinated GBM

In this chapter, we continue our study of European call options but with subordinated GBM. Our goal is to extend the pricing model from subdiffusive GBM to more complex subordinated GBM. We will show that the two categories namely, *type A* and *type B* can be generalised to cases with more general waiting times. Consequently, the results of the previous chapter can be recovered in special cases. The structure of this chapter is as follows. In Sec. 5.1, subdiffusive *type A* option cost 1 is investigated. In Sec. 5.2, the other subdiffusive *type A* option is studied. In Sec. 5.3, subdiffusive *type B* option cost 1 is discussed. In Sec. 5.4, the second subdiffusive *type B* option is presented. Finally, the summary is made in Sec. 5.5.

5.1 Subdiffusive *type A* option cost 1

Instead of subdiffusive GBM used in the previous chapter, we shall assume that the asset price follows subordinated GBM in the following sections. By analogy, we could still get four different types of option costs as described in Chap. 4. In this section we will first take a look at the first one of the *type A* option costs which is described by Eq. (4.5). With the same procedure used in the previous chapter, the subdiffusive *type*

A option cost 1 could be expressed as

$$\begin{aligned} C_1^{(\mathrm{A})}(x,t) &= \left\langle e^{-rS(t)}(X(S(t))-K)^+ \right\rangle_{X(0)=x} \\ &= \int_0^\infty ds\, C_{BS}(x,s) h(s,t) \end{aligned} \tag{5.1}$$

where here $h(s,t)$ is the PDF of the process $S(t)$ characterized by a Laplace exponent Eq. (3.106). The other parameters have the same meaning, i.e., $C_{BS}(x,s)$ denotes the classical Black Scholes expression given by Eq. (2.95), r is the interest rate, t is the expiration time, and K is the strike price. Applying the Laplace transform to Eq. (5.1) with respect to t leads to

$$\begin{aligned} \tilde{C}_1^{(\mathrm{A})}(x,\lambda) &= \int_0^\infty ds\, C_{BS}(x,s)\tilde{h}(s,\lambda) \\ &= \int_0^\infty ds\, C_{BS}(x,s) \frac{\Phi(\lambda)}{\lambda} e^{-\Phi(\lambda)s} \\ &= \frac{\Phi(\lambda)}{\lambda} \int_0^\infty ds\, C_{BS}(x,s) e^{-\Phi(\lambda)s} \\ &= \frac{\Phi(\lambda)}{\lambda} \tilde{C}_{BS}(x,\Phi(\lambda)) \,, \end{aligned} \tag{5.2}$$

which could be easily evaluated by replacing λ with $\Phi(\lambda)$ in Eq. (4.11) and substituting it into Eq. (5.2). Numerical inversion of the Laplace transform with the help of the Mathematica software package [1] is used here again to investigate its behaviour.

When $\Phi(\lambda) = \lambda^\alpha$, our subdiffusive option cost in Eq. (5.2) becomes the subdiffusive *type* A option cost 1 in Eq. (4.10).

We are also interested in obtaining a fractional differential equation for the quantity $C_1^{(\mathrm{A})}(x,t)$ as in the previous chapter. Differentiating $C_1^{(\mathrm{A})}(x,t)$ in Eq. (5.1) with respect to t results into

$$\frac{\partial}{\partial t} C_1^{(\mathrm{A})}(x,t) = \int_0^\infty ds\, C_{BS}(x,s) \frac{\partial}{\partial t} h(s,t) \,. \tag{5.3}$$

As $h(s,t)$ satisfies Eq. (3.111), it follows that

$$\begin{aligned}
\frac{\partial}{\partial t}C_1^{(\mathrm{A})}(x,t) &= -F_t\left(\int_0^\infty ds\, C_{BS}(x,s)\frac{\partial}{\partial s}h(s,t)\right) \\
&= -F_t\left(h(s,t)C_{BS}(x,s)|_0^\infty - \int_0^\infty ds\,\frac{\partial}{\partial s}C_{BS}(x,s)h(s,t)\right) \\
&= -F_t\left(-\int_0^\infty ds\,(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2}C_{BS}(x,s) - rC_{BS}(x,s)\right. \\
&\quad \left. +rx\frac{\partial}{\partial x}C_{BS}(x,s))h(s,t)\right) \\
&= -F_t\left(-\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} + r - rx\frac{\partial}{\partial x}\right)C_1^{(\mathrm{A})}(x,t) \\
&= F_t\left(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} - r + rx\frac{\partial}{\partial x}\right)C_1^{(\mathrm{A})}(x,t)
\end{aligned} \tag{5.4}$$

where again we have used the appropriate boundary conditions for $h(s,t)$ at $t=0$ and $t=\infty$. It is obvious that the same initial and boundary conditions for the fractional BS equation is derived as

$$C_1^{(\mathrm{A})}(x,0) = \max((x-K),0), \qquad x \geq 0 \tag{5.5a}$$

$$C_1^{(\mathrm{A})}(0,t) = 0, \qquad t \geq 0 \tag{5.5b}$$

$$C_1^{(\mathrm{A})}(x,t) \to x, \qquad x \to \infty \tag{5.5c}$$

One could find that for the special case $\Phi(\lambda) = \lambda^\alpha$, the fractional equation (5.4) becomes the subdiffusive BS Eq. (4.14). The solutions to Eq. (5.4) can be derived explicitly by the procedure described in Sec. 4.1.1. Performing the Laplace transform, we can obtain an ODE

$$\frac{\sigma^2x^2}{2}\tilde{C}_1''^{(\mathrm{A})}(x,\lambda) + rx\tilde{C}_1'^{(\mathrm{A})}(x,\lambda) - (\Phi(\lambda)+r)\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = -\frac{\Phi(\lambda)}{\lambda}C_1^{(\mathrm{A})}(x,0)\,. \tag{5.6}$$

The inhomogeneous part is given by the initial condition, Eq. (5.5), and two cases $x \leq K$ and $x > K$ will be discussed separately.

For $x \leq K$ the inhomogeneous part vanishes and Eq. (5.6) becomes

$$\frac{\sigma^2x^2}{2}\tilde{C}_1''^{(\mathrm{A})}(x,\lambda) + rx\tilde{C}_1'^{(\mathrm{A})}(x,\lambda) - (\Phi(\lambda)+r)\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = 0\,. \tag{5.7}$$

The general solution of Eq. (5.7) is given by

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = Ax^{m_1(\Phi(\lambda)+r)} + Bx^{m_2(\Phi(\lambda)+r)} \tag{5.8}$$

where we have used the abbreviations introduced in Eq. (4.12). Obviously the relations $m_1 \geq 0 \geq m_2$ still hold and $B = 0$ is required to ensure for a nonsingular solution in the limit $x \to 0$ here. Hence the expression becomes

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = Ax^{m_1(\Phi(\lambda)+r)}, \quad x \leqslant K\,. \tag{5.9}$$

For the case $x \geq K$ the inhomogeneous part of Eq. (5.6) is given by $C_1^{(\mathrm{A})}(x,0) = \max((x-K),0) = x-K$ and with a suitable particular solution of the nonhomogeneous equation the general solution then reads

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = Ax^{m_1(\Phi(\lambda)+r)} + Bx^{m_2(\Phi(\lambda)+r)} + \frac{x}{\lambda} - \frac{\Phi(\lambda)}{\lambda}\frac{K}{\Phi(\lambda)+r}, \quad x \geq K\,. \tag{5.10}$$

Here boundedness of $\tilde{C}_1^{\prime(\mathrm{A})}(x,\lambda)$ implies that $A = 0$ and it follows that

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = Bx^{m_2(\Phi(\lambda)+r)} + \frac{x}{\lambda} - \frac{\Phi(\lambda)}{\lambda}\frac{K}{\Phi(\lambda)+r}, \quad x \geq K\,. \tag{5.11}$$

When $x = K$, the option function given by Eqs.(5.9) and (5.11) is required to be continuous and differentiable. The corresponding matching conditions result in

$$\begin{aligned} B = &\left(\frac{\Phi(\lambda)}{\lambda}\frac{m_1(\Phi(\lambda)+r)}{\Phi(\lambda)+r} + \frac{1-m_1(\Phi(\lambda)+r)}{\lambda}\right) \\ &\times \frac{K^{1-m_2(\Phi(\lambda)+r)}}{(m_1(\Phi(\lambda)+r) - m_2(\Phi(\lambda)+r))}\,, \end{aligned} \tag{5.12a}$$

$$\begin{aligned} A = &\left(\frac{\Phi(\lambda)}{\lambda}\frac{m_2(\Phi(\lambda)+r)}{\Phi(\lambda)+r} + \frac{1-m_2(\Phi(\lambda)+r)}{\lambda}\right) \\ &\times \frac{K^{1-m_1(\Phi(\lambda)+r)}}{(m_1(\Phi(\lambda)+r) - m_2(\Phi(\lambda)+r))}\,. \end{aligned} \tag{5.12b}$$

and hence the solution to Eq. (5.6) is derived as

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = \left(\frac{\Phi(\lambda)}{\lambda}\frac{m_2(\Phi(\lambda)+r)}{\Phi(\lambda)+r} + \frac{1-m_2(\Phi(\lambda)+r)}{\lambda}\right) \times \frac{K^{1-m_1(\Phi(\lambda)+r)}}{(m_1(\Phi(\lambda)+r)-m_2(\Phi(\lambda)+r))} \times x^{m_1(\Phi(\lambda)+r)}, \qquad x \leq K, \tag{5.13a}$$

$$\tilde{C}_1^{(\mathrm{A})}(x,\lambda) = \left(\frac{\Phi(\lambda)}{\lambda}\frac{m_1(\Phi(\lambda)+r)}{\Phi(\lambda)+r} + \frac{1-m_1(\Phi(\lambda)+r)}{\lambda}\right) \times \frac{K^{1-m_2(\Phi(\lambda)+r)}}{(m_1(\Phi(\lambda)+r)-m_2(\Phi(\lambda)+r))} \times x^{m_2(\Phi(\lambda)+r)} + \frac{x}{\lambda} - \frac{\Phi(\lambda)}{\lambda}\frac{K}{\Phi(\lambda)+r}, \qquad x \geq K. \tag{5.13b}$$

One can check that the $\tilde{C}_1^{(\mathrm{A})}(x,\lambda)$ in Eq. (5.13) is the same as the result obtained in Eq. (5.2).

5.2 Subdiffusive *type A* option cost 2

In this section, we will consider the evaluation of the subdiffusive *type A* option cost 2 with discounting on the physical time scale. The corresponding expression, Eq. (4.6), can be also expressed in terms of the classical BS Eq. (2.95) as

$$\begin{aligned} C_2^{(\mathrm{A})}(x,t) &= \langle e^{-rt}(X(S(t))-K)^+\rangle_{X(0)=x} \\ &= e^{-rt}\int_0^\infty ds\, e^{rs} C_{BS}(x,s)h(s,t)\,. \end{aligned} \tag{5.14}$$

Once again by taking the derivative with respect to t on both sides of Eq. (5.14), we obtain

$$\begin{aligned} \frac{\partial}{\partial t}C_2^{(\mathrm{A})}(x,t) &= -re^{-rt}\int_0^\infty ds\, e^{rs} C_{BS}(x,s)h(s,t) \\ &\quad + e^{-rt}\int_0^\infty ds\, e^{rs} C_{BS}(x,s)\frac{\partial}{\partial t}h(s,t)\,. \end{aligned} \tag{5.15}$$

Considering Eq. (5.14), Eq. (3.111) and Eq. (2.96), the equation above becomes

$$\begin{aligned}\frac{\partial}{\partial t}C_2^{(\mathrm{A})}(x,t) =& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}F_t\left(\int_0^\infty ds\, e^{rs}C_{BS}(x,s)\frac{\partial}{\partial s}h(s,t)\right)\\ =& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}F_t\,(h(s,t)C_{BS}(x,s)e^{rs}|_0^\infty\\ & -\int_0^\infty ds\,\frac{\partial}{\partial s}(e^{rs}\times C_{BS}(x,s))h(s,t))\\ =& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}F_t\left(-r\int_0^\infty ds\, e^{rs}C_{BS}(x,s)h(s,t)\right.\\ &\left. -\int_0^\infty ds\, e^{rs}\times\frac{\partial}{\partial s}C_{BS}(x,s)h(s,t)\right)\\ =& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}F_t\left(-r\int_0^\infty ds\, e^{rs}C_{BS}(x,s)h(s,t)\right.\\ & -\int_0^\infty ds\, e^{rs}\left(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2}C_{BS}(x,s) - rC_{BS}(x,s)\right.\\ &\left.\left.+rx\frac{\partial}{\partial x}C_{BS}(x,s)\right)h(s,t)\right)\\ =& -rC_2^{(\mathrm{A})}(x,t) - e^{-rt}F_t\left(e^{rt}(-\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} - rx\frac{\partial}{\partial x})C_2^{(\mathrm{A})}(x,t)\right)\end{aligned} \tag{5.16}$$

and another modified fractional BS equation is obtained

$$\frac{\partial}{\partial t}C_2^{(\mathrm{A})}(x,t) = -rC_2^{(\mathrm{A})}(x,t) + e^{-rt}\,F_t\left(e^{rt}(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x})C_2^{(\mathrm{A})}(x,t)\right). \tag{5.17}$$

The initial and boundary conditions are given by

$$C_2^{(\mathrm{A})}(x,0) = \max((x-K),0), \qquad x \geq 0 \tag{5.18a}$$

$$C_2^{(\mathrm{A})}(0,t) = 0, \qquad t \geq 0 \tag{5.18b}$$

$$C_2^{(\mathrm{A})}(x,t) \to xe^{-rt}\mathcal{L}^{-1}\left\{\frac{\Phi(\lambda)}{\lambda(\Phi(\lambda)-r)}\right\}, \qquad x \to \infty \tag{5.18c}$$

With the same method mentioned as above, we can easily compute the solution as

follows

$$\tilde{C}_2^{(\mathrm{A})}(x,\lambda) = \left(\frac{m_2(\Phi(\lambda+r))}{\lambda+r} + \frac{\Phi(\lambda+r)}{\lambda+r}\frac{1-m_2(\Phi(\lambda+r))}{\Phi(\lambda+r)-r}\right) \times \frac{K^{1-m_1(\Phi(\lambda+r))}}{(m_1(\Phi(\lambda+r))-m_2(\Phi(\lambda+r)))} x^{m_1(\Phi(\lambda+r))}, \qquad x \le K\,, \tag{5.19a}$$

$$\tilde{C}_2^{(\mathrm{A})}(x,\lambda) = \left(\frac{m_1(\Phi(\lambda+r))}{\lambda+r} + \frac{\Phi(\lambda+r)}{\lambda+r}\frac{1-m_1(\Phi(\lambda+r))}{\Phi(\lambda+r)-r}\right) \times \frac{K^{1-m_2(\Phi(\lambda+r))}}{m_1(\Phi(\lambda+r))-m_2(\Phi(\lambda+r)} \times x^{m_2(\Phi(\lambda+r))} + \frac{\Phi(\lambda+r)}{\lambda+r}\frac{x}{\Phi(\lambda+r)-r} - \frac{K}{\lambda+r}, \qquad x \ge K\,. \tag{5.19b}$$

On the other hand, the Laplace transform can be used directly to Eq. (5.14) to derive the same result, namely

$$\begin{aligned}\tilde{C}_2^{(\mathrm{A})}(x,\lambda) &= \int_0^\infty ds\, e^{rs} C_{BS}(x,s)\tilde{h}(s,\lambda+r) \\ &= \int_0^\infty ds\, e^{rs} C_{BS}(x,s)\frac{\Phi(\lambda+r)}{\lambda+r}e^{-(\Phi(\lambda+r)-r)s} \\ &= \frac{\Phi(\lambda+r)}{\lambda+r}\int_0^\infty ds\, C_{BS}(x,s)e^{-(\Phi(\lambda+r)-r)s} \\ &= \frac{\Phi(\lambda+r)}{\lambda+r}\tilde{C}_{BS}(x,\Phi(\lambda+r)-r)\end{aligned} \tag{5.20}$$

Replacing λ with $\Phi(\lambda+r)-r$ in Eq. (4.11) and substituting it into Eq. (5.20), the solutions given by Eq. (5.19) are recovered.

5.3 Subdiffusive *type B* option cost 1

In this section, the subdiffusive *type B* option cost 1 with the subordinated GBM will be presented. Following the previous procedure and using the two time density function, the corresponding conditional expectation value, Eq. (4.7) can be again expressed in terms of the classical Black Scholes expression (2.82)

$$\begin{aligned}
C_1^{(\mathrm{B})}(x,T,t) &= \left\langle e^{-r(S(T)-S(t))}\left(X(S(T))-K\right)^+\right\rangle_{X(S(t))=x} \\
&= \left\langle \int_0^\infty ds_2 \int_0^\infty ds_1 e^{-r(s_2-s_1)}(X(s_2)-K)^+\delta(s_2-S(T))\delta(s_1-S(t))\right\rangle \\
&= \int_0^\infty ds\,\left\langle e^{-r(s_2-s_1)}(X(s_2)-K)^+\right\rangle h(s_2,T,s_1,t) \\
&= \int_0^\infty ds_1 \int_0^\infty ds_2 C_{BS}(x,s_2-s_1)h(s_2,T,s_1,t) \\
&= \int_0^\infty ds_1 \int_0^\infty d\tau C_{BS}(x,\tau)h(s_1+\tau,T,s_1,t)\,.
\end{aligned} \tag{5.21}$$

With the two-time Laplace transform with respect to T and t we end up with

$$\begin{aligned}
\tilde{C}_1^{(\mathrm{B})}(x,\lambda_2,\lambda_1) &= \int_0^\infty ds_1 \int_0^\infty d\tau C_{BS}(x,\tau)\tilde{h}(s_1+\tau,\lambda_2,s_1,\lambda_1) \\
&= \int_0^\infty ds_1 \int_0^\infty d\tau C_{BS}(x,\tau)\frac{\left(\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)\right)\Phi(\lambda_2)}{\lambda_1\lambda_2} \\
&\quad \times e^{-s_1(\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2))}e^{-(s_1+\tau)\Phi(\lambda_2)} \\
&= \frac{\left(\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)\right)\Phi(\lambda_2)}{\lambda_1\lambda_2\Phi(\lambda_1+\lambda_2)}\int_0^\infty d\tau C_{BS}(x,\tau)e^{-\tau\Phi(\lambda_2)} \\
&= \frac{\left(\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)\right)\Phi(\lambda_2)}{\lambda_1\lambda_2\Phi(\lambda_1+\lambda_2)}\tilde{C}_{BS}(x,\Phi(\lambda_2))\,,
\end{aligned} \tag{5.22}$$

which can be explicitly expressed if results given in Eq. (4.11) are used.

5.4 Subdiffusive *type B* option cost 2

In this section, we take the subdiffusive type option cost 2 into consideration. As in the previous chapter, we denote by $P_Y(y,T|x,t,x_0,0)$ the conditional probability of subordinated GBM, i.e., the conditional probability of the asset price. Similarly, as before, joint probabilities are denoted by $f(y,T;x,t;x_0,0)$ and $f(x,t;x_0,0)$ with $x_0>0$. Then Eq. (4.8) can be expressed as

$$\begin{aligned}
C_2^{(\mathrm{B})}(x,T,t) &= e^{-r(T-t)}\left\langle (X(S(T))-K)^+\right\rangle_{X(S(t))=x,X(0)=x_0} \\
&= \frac{e^{-r(T-t)}}{f(x,t;x_0,0)}\int_K^\infty (y-K)f(y,T;x,t;x_0,0)dy\,.
\end{aligned} \tag{5.23}$$

Again the joint probabilities of the non-Markovian process can now be expressed by the properties of the process X and one and two time distribution functions. Let us denote the conditional probability of GBM by $P_X(x, s|x_0, 0)$ (see Eq. 2.72) and its Laplace transform by $\tilde{P}_X(x, \lambda|x_0, 0)$ (see Eq. (2.73)). The joint probabilities of the process Y can now be written in terms of the conditional probabilities of the Markovian dynamics and the one and two time distribution functions of the subordinator (see Eq. (3.26))

$$f(y, T; x, t; x_0, 0) = \int_0^\infty ds_1 \int_{s_1}^\infty ds_2 P_X(y, s_2|x, s_1) P_X(x, s_1|x_0, 0) \times P_X(x_0) h(s_2, T, s_1, t) \tag{5.24}$$

and

$$f(x, t; x_0, 0) = \int_0^\infty ds\, P_X(x, s|x_0, 0) P_X(x_0) h(s, t) \tag{5.25}$$

with $P_X(x)$ denoting the stationary distribution of GBM.

With the help of Eq. (5.24) and Eq. (5.25), we find that $C_2^{(\mathrm{B})}(x, T, t)$ in Eq. (5.23) can be written as

$$C_2^{(\mathrm{B})}(x, T, t) = \frac{e^{-r(T-t)}}{\Pi_1(x, t)} \Pi_2(x, T, t) \tag{5.26}$$

via the abbreviations given By Eq. (4.46).

By applying the two time and one time Laplace transform to $\Pi_2(x, T, t)$ and $\Pi_1(x, t)$

in Eq. (4.46), and with the help of Eq. (3.118), one can see that

$$\begin{aligned}
\tilde{\Pi}_2(x,\lambda_2,\lambda_1) &= \int_0^\infty \int_0^\infty e^{-\lambda_2 T} e^{-\lambda_t t} \Pi_2(x,T,t) dT dt \\
&= \int_0^\infty ds_1 \int_0^\infty d\tau P_X(x,s_1|x_0,0) \tilde{h}(s_1+\tau,\lambda_2,s_1,\lambda_1) e^{r\tau} C_{BS}(x,\tau) \\
&= \int_0^\infty ds_1 \int_0^\infty d\tau P_X(x,s_1|x_0,0) \frac{(\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2))\,\Phi(\lambda_2)}{\lambda_1\lambda_2} \\
&\qquad \times e^{-s_1(\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2))} e^{-(s_1+\tau)\Phi(\lambda_2)} e^{r\tau} C_{BS}(x,\tau) \\
&= \frac{(\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2))\,\Phi(\lambda_2)}{\lambda_1\lambda_2} \int_0^\infty ds_1 P_X(x,s_1|x_0,0) \\
&\qquad \times e^{-\Phi(\lambda_1+\lambda_2)s_1} \int_0^\infty d\tau e^{-\tau(\Phi(\lambda_2)-r)} C_{BS}(x,\tau) \\
&= \frac{(\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2))\,\Phi(\lambda_2)}{\lambda_1\lambda_2} \tilde{P}_X(x,\Phi(\lambda_1+\lambda_2)|x_0,0) \\
&\qquad \times \tilde{C}_{BS}(x,\Phi(\lambda_2)-r)\,.
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
\tilde{\Pi}_1(x,\lambda_1) &= \int_0^\infty e^{-\lambda_1 t} \Pi_1(x,t) dt \\
&= \int_0^\infty ds\, P_X(x,s|x_0,0) \tilde{h}(s,\lambda_1) \\
&= \frac{\Phi(\lambda_1)}{\lambda_1} \int_0^\infty ds\, P_X(x,s|x_0,0) e^{-\Phi(\lambda_1)s} \\
&= \frac{\Phi(\lambda_1)}{\lambda_1} \tilde{P}_X(x,\Phi(\lambda_1)|x_0,0)\,.
\end{aligned} \tag{5.28}$$

As the Black-Scholes formula and the conditional probability of GBM in Laplace space $\tilde{C}_{BS}(x,\lambda)$ and $\tilde{P}_X(x,\lambda|x_0,0)$ have been given in Eq. (4.11) and Eq. (2.73), respectively, the exact analytic expressions of $\Pi_2(x,T,t)$ in Eq. (5.27) and $\Pi_1(x,t)$ in Eq. (5.28) in Laplace space could be derived straightforwardly. Moreover, their closed forms could be derived if the inverse Laplace transform is applied and the expression for $C_2^{(\mathrm{B})}(x,T,t)$ in Eq. (5.26) could then be obtained. However, considering the difficulty and complexity of this job, we still resort to the numerical method. The quantity $\Pi_2(x,T,t)$ can be obtained numerically using a two dimensional version of the algorithm proposed in [161] and $\Pi_1(x,t)$ by the Talbot method [2,3,157]. Then by Eq. (5.26), finally we can evaluate $C_2^{(\mathrm{B})}(x,T,t)$ numerically.

Fig. 5.1 shows the subdiffusive *type A* option formula with subordinated GBM for tempered stable waiting times $\Phi(\lambda) = (\lambda + \zeta)^\alpha - \zeta^\alpha$ according to different values of the real time t with parameters $K = 2$, $x = 1$, $\sigma = 1$ and $\zeta = 0.001$ for the interest rates $r = 0.02$ and $r = 0.5$, respectively. It indicates that when $\alpha \to 1$, the subdiffusive

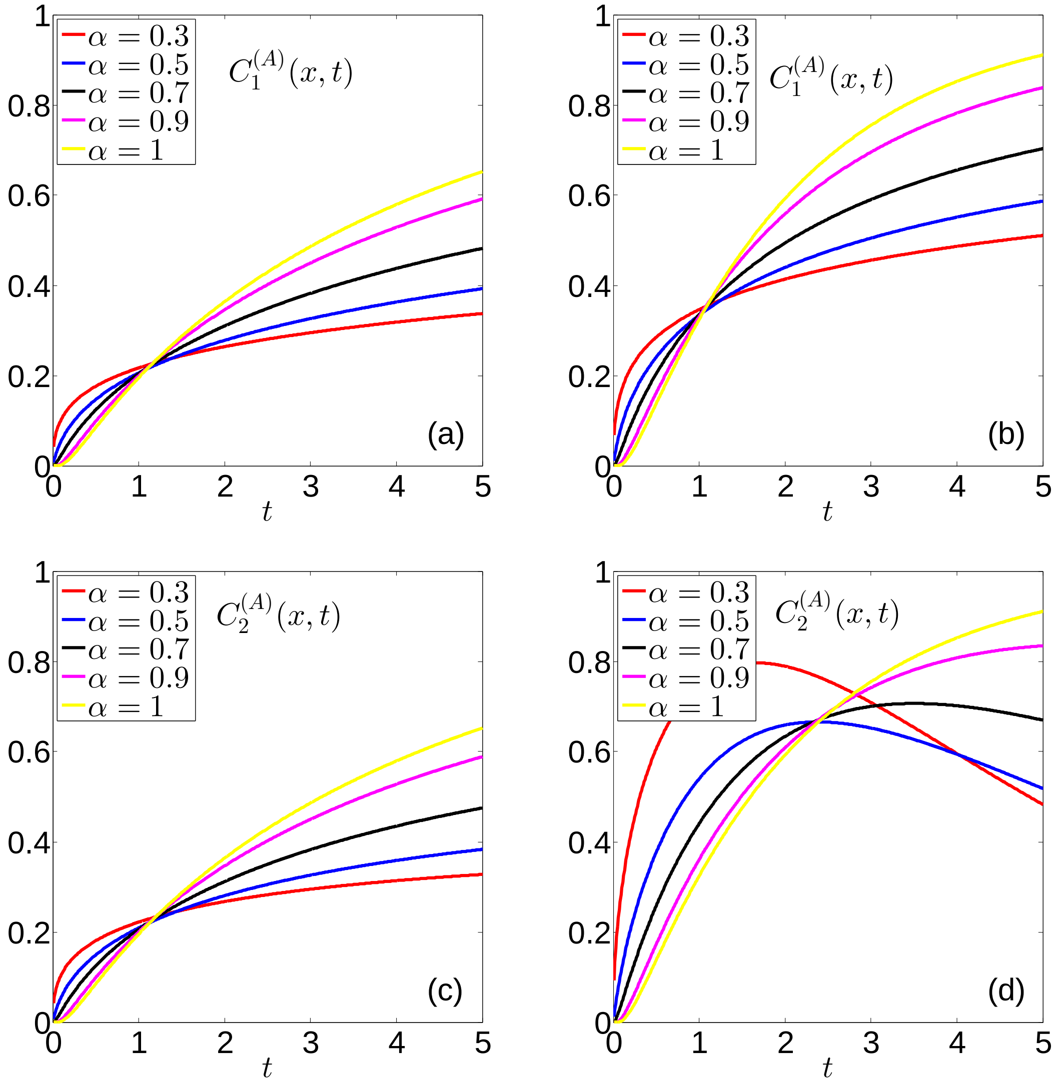

Figure 5.1: Analytic expression of subdiffusive *type A* option cost formula with subordinated GBM $C_1^{(\mathrm{A})}(x,t)$ [see Eq. (5.13a)] and $C_2^{(\mathrm{A})}(x,t)$ [see Eq. (5.19a)] for tempered stable waiting times $\Phi(\lambda) = (\lambda + \zeta)^\alpha - \zeta^\alpha$ with parameter values $K = 2$, $x = 1$, $\sigma = 1$ and $\zeta = 0.001$ for different interest rates: (a)(c) $r = 0.02$ and (b)(d) $r = 0.5$.

option values are approaching the standard Black-Scholes formula prices.

The comparison of the subdiffusive *type A* option formula with subordinated GBM for tempered stable waiting times $\Phi(\lambda) = (\lambda+\zeta)^\alpha - \zeta^\alpha$ is made in Fig. 5.2 for different values of the parameter ζ.

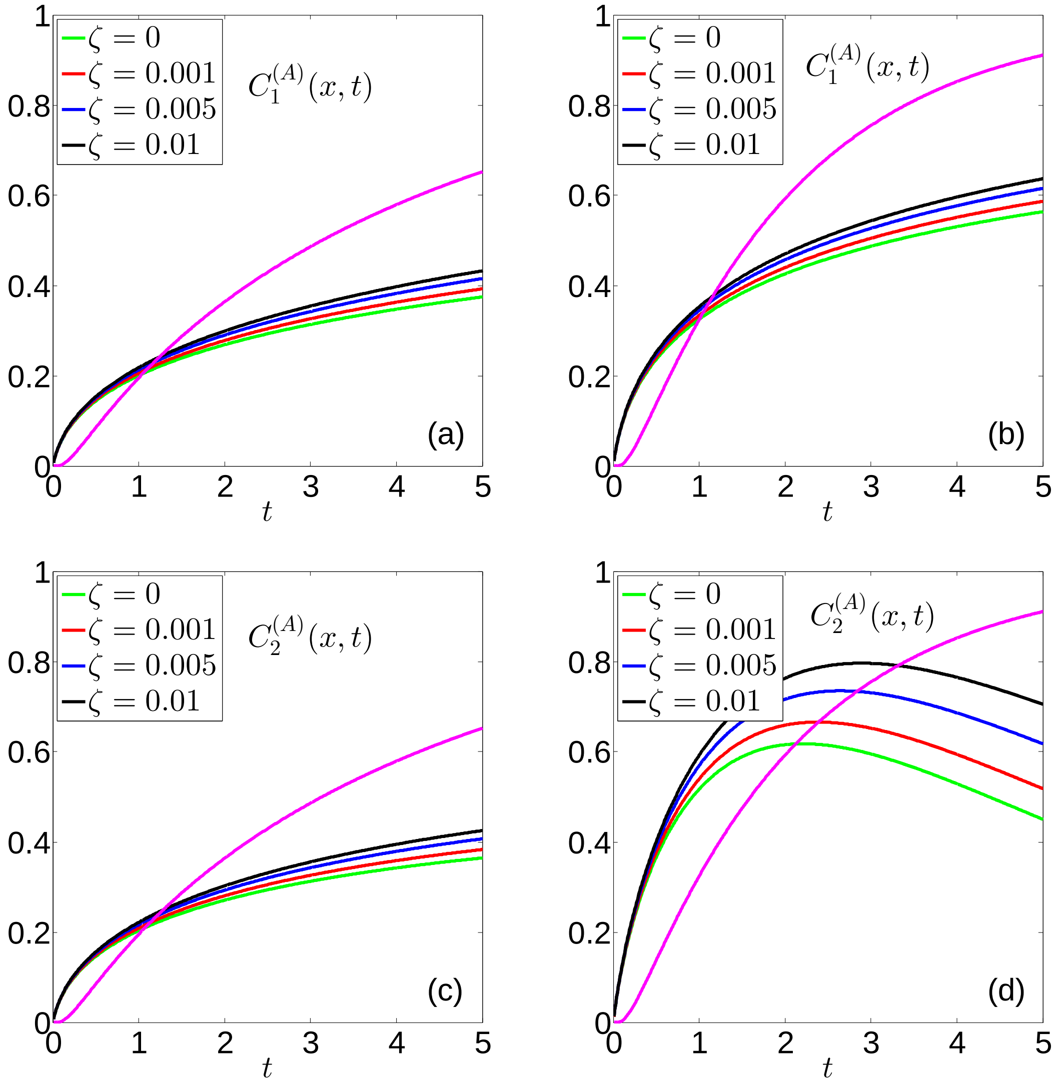

Figure 5.2: The comparison of subdiffusive *type A* option cost formula with subordinated GBM $C_1^{(\mathrm{A})}(x,t)$ [see Eq. (5.13a)] and $C_2^{(\mathrm{A})}(x,t)$ [see Eq. (5.19a)] for tempered stable waiting times $\Phi(\lambda) = (\lambda+\zeta)^\alpha - \zeta^\alpha$ with parameter values $K = 2$, $x = 1$, $\sigma = 1$ and $\alpha = 0.5$ for different interest rates: (a)(c) $r = 0.02$ and (b)(d) $r = 0.5$.

Fig. 5.3 shows the subdiffusive *type B* option formula with subordinated GBM for tempered stable waiting times $\Phi(\lambda) = (\lambda+\zeta)^\alpha - \zeta^\alpha$ for different values of the time difference $T-t$ with parameters $K = 2$, $x = 1$, $x_0 = 2$, $\sigma = 1$, $t = 0.3$ and $\zeta = 0.001$ for the interest rate $r = 0.02$ and $r = 0.5$, respectively. It also shows that when $\alpha \to 1$, the subdiffusive option values are approaching the standard Black-Scholes formula prices. The comparison of the subdiffusive *type B* option formula with subordinated GBM for

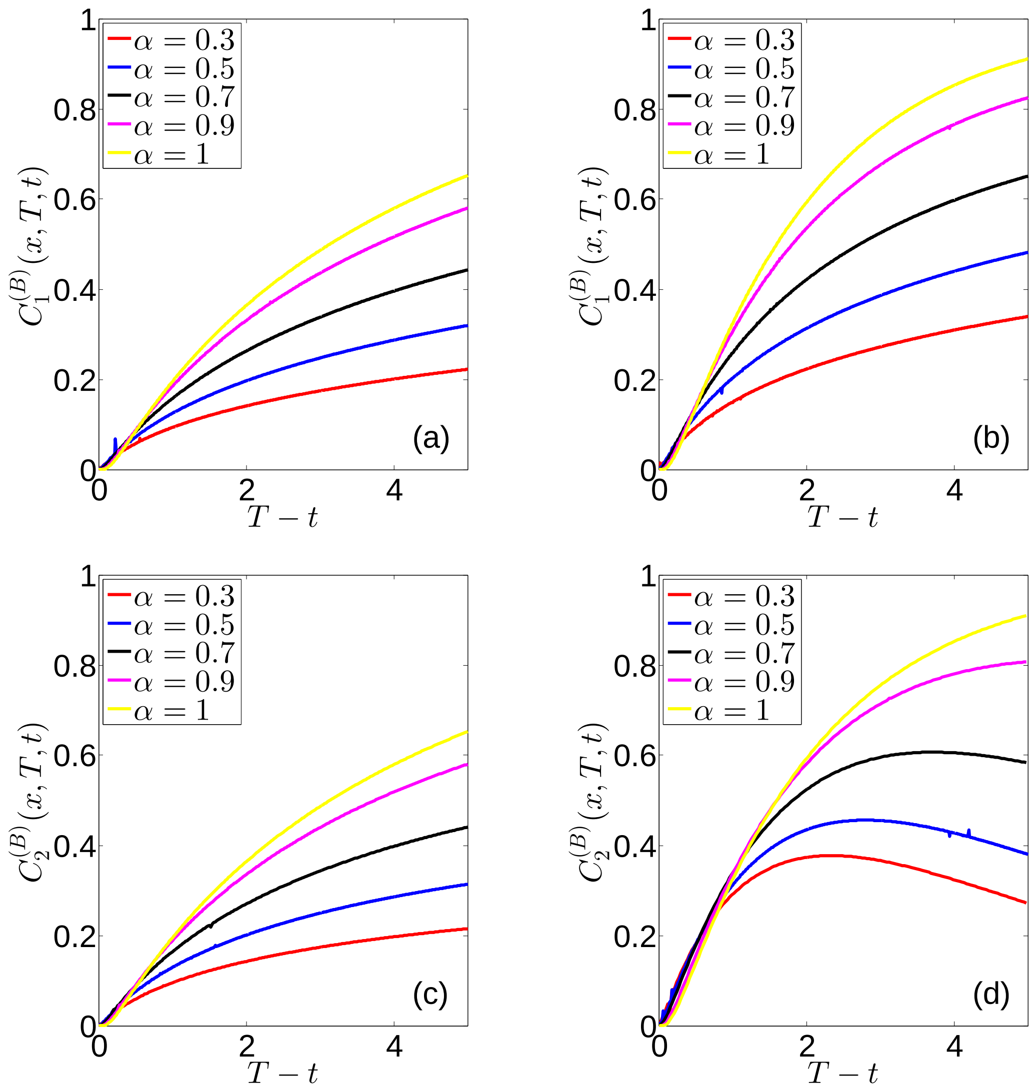

Figure 5.3: Analytic expression of subdiffusive *type B* option cost formula with subordinated GBM $C_1^{(\mathrm{B})}(x, T, t)$ [see Eq. (5.21)] and $C_2^{(\mathrm{B})}(x, T, t)$ [see Eq. (5.26)] for tempered stable waiting times $\Phi(\lambda) = (\lambda + \zeta)^\alpha - \zeta^\alpha$ with parameter values $K = 2$, $x = 1$, $x_0 = 2$, $\sigma = 1$, $\zeta = 0.001$, and $t = 0.3$ for different interest rates: (a)(c) $r = 0.02$ and (b)(d) $r = 0.5$.

tempered stable waiting times $\Phi(\lambda) = (\lambda + \zeta)^\alpha - \zeta^\alpha$ is made in Fig. 5.4 for different values of the parameter ζ.

As in Chapter 4 we are also interested in the impact of the starting price x_0 on the option values. Fig. 5.5 shows how $C_2^{(B)}(x,T,t)$ changes with respect to x_0. And the comparison of the subdiffusive *type B* option $C_2^{(B)}(x,T,t)$ with subordinated GBM is made in Fig. 5.6

5.5 Summary of chapter

In this chapter, we have examined the pricing of subdiffusive European call options based on a subordinated GBM which includes the case of subdiffusive GBM. By introducing general waiting times manifest in a Laplace exponent ϕ, we generalize the results obtained in the previous chapter. As in Chapter 4, we also derive two types of subdiffusive *type A* and *type B* option costs with non-zero interest rate, respectively. We show that each subdiffusive call option pricing formula could also be derived from corresponding fractional differential equations. Finally, we show the behaviour of the subdiffusive *type A* and *type B* option pricing formulas for different values of the parameter.

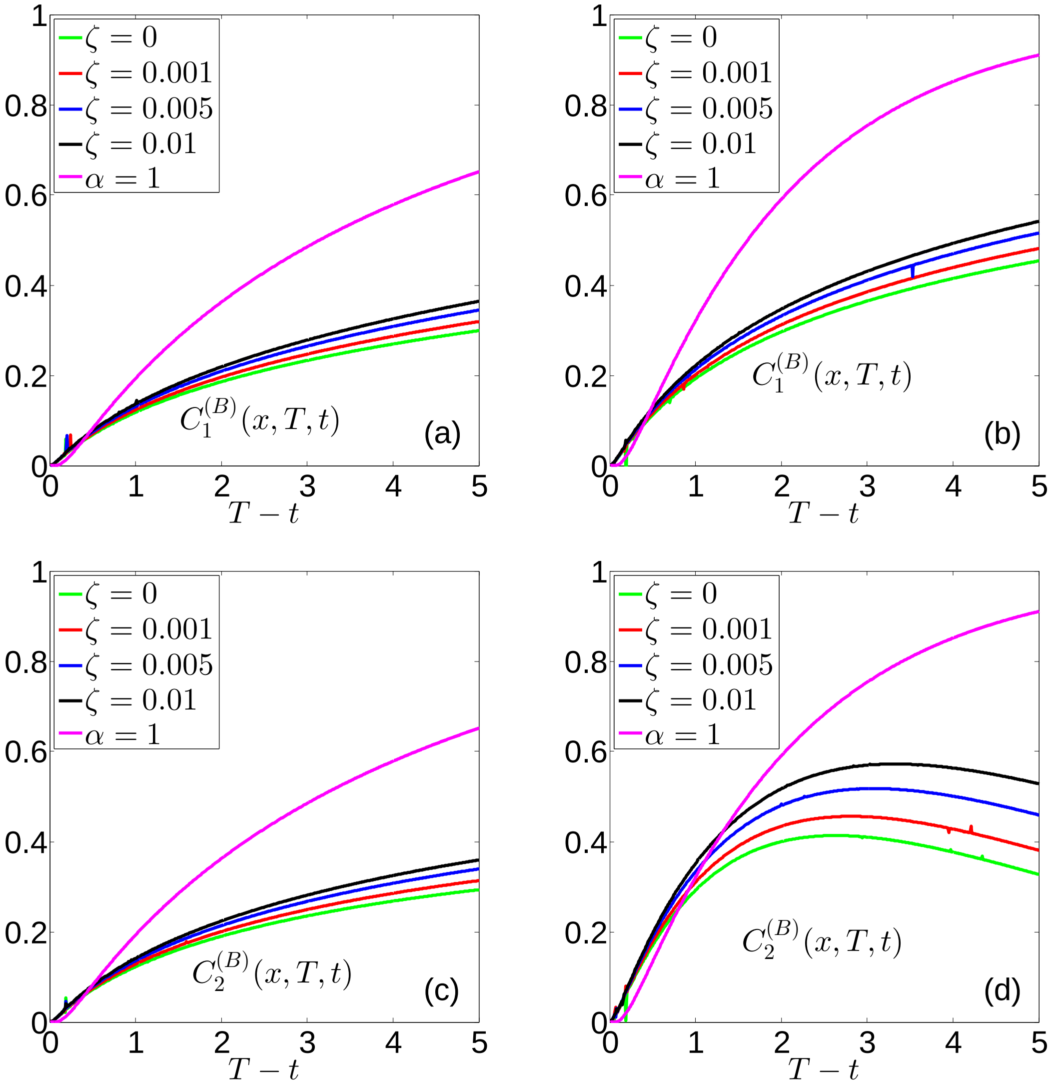

Figure 5.4: The comparison of subdiffusive *type B* option cost formula with subordinated GBM $C_1^{(\mathrm{B})}(x,T,t)$ [see Eq. (5.21)] and $C_2^{(\mathrm{B})}(x,T,t)$ [see Eq. (5.26)] for tempered stable waiting times $\Phi(\lambda) = (\lambda+\zeta)^\alpha - \zeta^\alpha$ with parameter values $K=2$, $x=1$, $x_0=2$, $\sigma=1$, $t=0.3$ and $\alpha=0.5$ for different interest rates: (a)(c) $r=0.02$ and (b)(d) $r=0.5$.

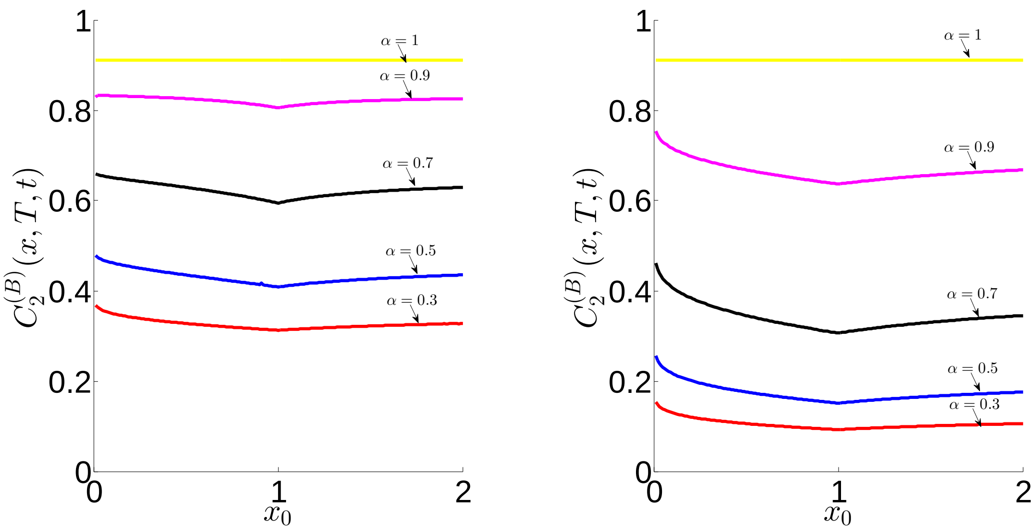

Figure 5.5: Analytic expression of subdiffusive *type B* option cost formula $C_2^{(\mathrm{B})}(x,T,t)$ [see Eq. (5.26)] for tempered stable waiting times $\Phi(\lambda) = (\lambda+\zeta)^\alpha - \zeta^\alpha$ with parameter values $K = 2, \sigma = 1, x = 1, r = 0.5, \zeta = 0.001$ for different initial and expiration times: (left panel) $t = 0.1, T = 5.1$ and (right panel) $t = 5, T = 10$.

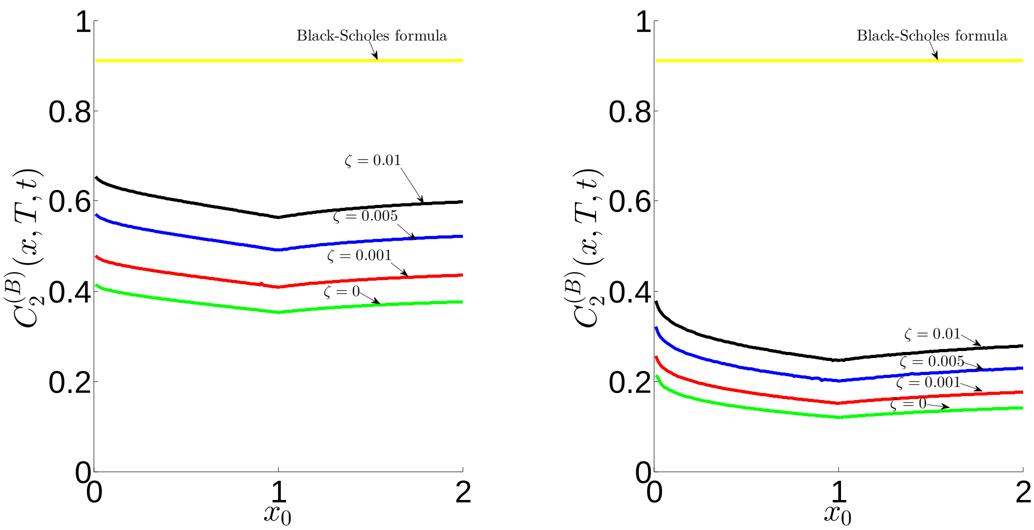

Figure 5.6: The comparison of subdiffusive *type B* option cost formula with subordinated GBM $C_2^{(\mathrm{B})}(x,T,t)$ [see Eq. (5.26)] for tempered stable waiting times $\Phi(\lambda) = (\lambda+\zeta)^\alpha - \zeta^\alpha$ with parameter values $K = 2$, $x = 1$, $\sigma = 1$, $r = 0.5$, $t = 0.1$, $T = 5.1$ and $\alpha = 0.5$ for different initial and expiration times: (left panel) $t = 0.1, T = 5.1$ and (right panel) $t = 5, T = 10$.

Chapter 6

General subdiffusive call option formula with arbitrary payoffs function

In this chapter, we will study general option pricing with subordinated processes defined by Eq. (3.24) with a Laplace exponent given in Eq. (3.106). By general option pricing, we mean that the payoff function is arbitrary. We should emphasize that general option pricing with subordinated processes can be used to address several interesting option pricing problems if certain assumptions are made. We will demonstrate that a general option pricing formula for a subordinated pricing model can be expressed in terms of its normal version and the corresponding density of the subordinator. For simplicity, we will examine the power option with a subordinated GBM. In particular, we will consider stable and tempered stable waiting times. This option is, perhaps, the simplest option we could examine that extends the normal European call option case. Thus, we hope that these results demonstrate the practical use of our approach. This chapter is organized as follows. In Sec. 6.1, the general subdiffusive call option pricing formulas will be put forward for arbitrary payoff functions. In Sec. 6.2, we give an application of our general formula by considering the special case of the subdiffusive power option. We also derive the fractional equations that can be likewise used to obtain the option cost and derive its solution. Finally, the comparison between the classical and subdiffusive power options are made.

6.1 General subdiffusive call option formula with an arbitrary payoff function

In this section, the general subdiffusive call option formula with an arbitrary payoff function will be presented. Moreover, we assume that the initial time of the option is the current time $t = 0$. Recall that the normal option with an arbitrary payoff function is defined as [139]

$$C(x,t) = e^{-rt} \left\langle (g(X(t)) - K)^+ \right\rangle_{X(0)=x} \tag{6.1}$$

where r is the interest rate, g is an appropriate arbitrary specified function, $(g(X(t)) - K)^+$ is $\max\{0, g(X(t)) - K\}$, t is exercise time, K is strike price and $X(t)$ is the stochastic process of the asset price. It is implicitly assumed that the expected value is with respect to a suitable risk-neutral measure. Supposing that the underlying assets price follows the subordinated process $Y(t) = X(S(t))$ defined by Eq. (3.24) with Laplace exponent given in Eq. (3.106), interesting properties may be found as the pricing model is not a normal Markovian process any more.

According to the different ways of discounting either with respect to the subordinator or the physical time, there will be two versions of subdiffusive option costs corresponding to the normal option with arbitrary payoffs defined in Eq. (6.1). Next we will investigate these expressions in detail.

6.1.1 Subdiffusive formula type 1

The first type of subdiffusive call option cost assumes that the discounting takes place with respect to the subordinator $S(t)$

$$C_1(x,t) = \left\langle e^{-rS(t)} \left(g(X(S(t))) - K\right)^+ \right\rangle_{X(0)=x} \tag{6.2}$$

where as usual r is the interest rate, g is an appropriate arbitrary specified function, $(g(X(S(t)) - K)^+$ is $\max\{0, g(X(S(t)) - K\}$, t is exercise time, K is strike price and $X(S(t))$ is the assets' price.

In fact, the price of option $C_1(x,t)$ given by Eq. (6.2) could be expressed in the form of the normal option pricing cost with an arbitrary payoff, Eq. (6.1), as follows

$$\begin{aligned} C_1(x,t) &= \left\langle e^{-rS(t)}(g\left(X(S(t))\right) - K)^+ \right\rangle \\ &= \left\langle \int_0^\infty ds\, e^{-rs}(g\left(X(s)\right) - K)^+ \delta(s - S(t)) \right\rangle \\ &= \int_0^\infty ds\, e^{-rs} \left\langle g\left(X(s)\right) - K)^+ \right\rangle h(s,t) \\ &= \int_0^\infty ds\, C(x,s) h(s,t) \end{aligned} \tag{6.3}$$

where $C(x,s)$ represents the normal option pricing cost defined by Eq. (6.1) and $h(s,t)$ is the density function of the process $S(t)$ defined in Eq. (3.107). If we take the Laplace transform to $C_1(x,t)$ with respect to t in Eq. (6.3), by Eq. (3.108) it is evident that

$$\begin{aligned} \tilde{C}_1(x,\lambda) &= \int_0^\infty dt e^{-\lambda t} \int_0^\infty ds\, C(x,s) h(s,t) \\ &= \int_0^\infty ds\, C(x,s) \tilde{h}(s,\lambda) \\ &= \int_0^\infty ds\, C(x,s) \frac{\Phi(\lambda)}{\lambda} e^{-\Phi(\lambda)s} \\ &= \frac{\Phi(\lambda)}{\lambda} \int_0^\infty ds\, C(x,s) e^{-\Phi(\lambda)s} \\ &= \frac{\Phi(\lambda)}{\lambda} \tilde{C}(x,\Phi(\lambda)) \end{aligned} \tag{6.4}$$

where $\tilde{C}(x,\Phi(\lambda)) = \int_0^\infty ds C(x,s) e^{-\Phi(\lambda)s}$, which would be obtained by applying the Laplace transform to $C(x,s)$ in Eq. (6.1). Then performing the inverse Laplace transform to $\tilde{C}_1(x,\lambda)$ in Eq. (6.4), the expression for the subdiffusive option $C_1(x,t)$ determined by Eq. (6.2) in the time domain could be derived explicitly.

6.1.2 Subdiffusive formula type 2

Another type of subdiffusive call option cost is based on the assumption that the discounting is affected by the real time

$$C_2(x,t) = \left\langle e^{-rt} \left(g(Y(t)) - K\right)^+ \right\rangle_{X(0)=x} \tag{6.5}$$

With the same procedure as above, the average value determined by Eq. (6.5) could be derived as

$$\begin{aligned} C_2(x,t) &= e^{-rt}\left\langle \left(g\left(X(S(t))\right)-K\right)^+\right\rangle \\ &= e^{-rt}\int_0^\infty ds\, e^{rs}C(x,s)h(s,t)\,. \end{aligned} \tag{6.6}$$

where $C(x, s)$ represents the normal option pricing cost with an arbitrary payoff, Eq. (6.1), and $h(s, t)$ is the density function of the process $S(t)$.

If we apply the Laplace transform directly to this formula in Eq. (6.6), we can derive that

$$\begin{aligned} \tilde{C}_2(x,\lambda) &= \int_0^\infty dt e^{-\lambda t}e^{-rt}\int_0^\infty ds\, e^{rs}C(x,s)h(s,t) \\ &= \int_0^\infty ds\, e^{rs}C(x,s)\tilde{h}(s,\lambda+r) \\ &= \int_0^\infty ds\, e^{rs}C(x,s)\frac{\Phi(\lambda+r)}{\lambda+r}e^{-\Phi(\lambda+r)s} \\ &= \frac{\Phi(\lambda+r)}{\lambda+r}\int_0^\infty ds\, C(x,s)e^{-(\Phi(\lambda+r)-r)s} \\ &= \frac{\Phi(\lambda+r)}{\lambda+r}\tilde{C}(x,\Phi(\lambda+r)-r)\,. \end{aligned} \tag{6.7}$$

Here we use the fact that the density function $h(s, t)$ of the process $S(t)$ in Laplace space satisfies Eq. (3.108). $\tilde{C}(x, \Phi(\lambda + r) - r)$ can be obtained from the calculation $\int_0^\infty ds\, C(x, s)e^{-(\Phi(\lambda+r)-r)s}$, which is the Laplace transform of the normal option with an arbitrary payoff in Eq. (6.1). As we have found the exact expression for the subdiffusive option $\tilde{C}_2(x, \lambda)$ (see Eq. (6.7)) in Laplace space, its closed form could be obtained if the inverse Laplace transform is performed to $\tilde{C}_2(x, \lambda)$. Note that when $S(t) = t$, which means that only the real time takes effect in the option pricing, both the subdiffusive call option costs of Eq. (6.2) and Eq. (6.5) become the normal call option cost $C(x, t)$ of Eq. (6.1).

6.2 Subdiffusive power option formula

As the general subdiffusive call option formulas for an arbitrary payoff have been presented, it will provide us with a useful tool to analyse a variety of option types in the subdiffusive regime conveniently and efficiently. In order to show the efficiency of the subdiffusive option formulas we have derived, we will apply it to the case of the power option to obtain the corresponding formulas and investigate their behaviour. For the convenience of the following discussion, we will take a look at the normal power option first.

6.2.1 Normal power option formula

In the option price cost defined by Eq. (6.1), if we let $g(X(t) = X(t)^\beta$, and $X(t)$ is GBM under the risk-neutral measure we get the power option $C_\beta(x,t)$ with power parameter β as [139]

$$C_\beta(x,t) = e^{-rt}\left\langle (X(t)^\beta - K)^+ \right\rangle_{X(0)=x} \tag{6.8}$$

where $(X(t)^\beta - K)^+$ is $\max\{0, X(t)^\beta - K\}$, t is exercise time, K is strike price. Furthermore, the option given by Eq. (6.8) can be evaluated by [139]

$$C_\beta(x,t) = e^{(\beta-1)(r+\frac{\beta\sigma^2}{2})t} C_{\mathrm{BS}}(x^\beta, t, K, \beta\sigma, r_\beta) \tag{6.9}$$

where $C_{\mathrm{BS}}(x,t,K,\sigma,r)$ is the Black-Scholes formula (see Eq. (2.83)).

Moreover, the power option $C_\beta(x,t)$ with power parameter β can be described by the equation

$$\frac{\partial}{\partial t}C_\beta(x,t) = \left(\frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} - r + \left(r + (1-\beta)(1-\frac{\sigma^2}{2})\right) x\frac{\partial}{\partial x}\right) C_\beta(x,t) \tag{6.10}$$

with the initial and boundary conditions

$$C_\beta(x,0) = \max((x^\beta - K), 0), \qquad x \geq 0 \tag{6.11a}$$

$$C_\beta(0,t) = 0, \qquad t \geq 0 \tag{6.11b}$$

$$C_\beta(x,t) \to x^\beta, \qquad x \to \infty\,. \tag{6.11c}$$

To derive the equation above, we use the fact that the standard Black-Scholes price $C_{\mathrm{BS}}(x,t,K,\sigma,r)$ is found as a solution of the BS Eq. (2.96).

6.2.2 Subdiffusive power option 1

By introducing a subordinated process defined by Eq. (3.24) with Laplace exponent given in Eq. (3.106) as a pricing model, we will derive the subdiffusive formulas corresponding to the normal power option in Eq. (6.8). From the previous discussion, we know that there will be two versions in the subdiffusive regime. According to the formula derived in Eq. (6.3), the first type of the subdifffusive power option formula can be written as

$$C_1(x,t) = \int_0^\infty ds\, C_\beta(x,s)h(s,t) \tag{6.12}$$

where $C_\beta(x,s)$ is the normal power option defined by Eq. (6.8) and $h(s,t)$ is the density of the process $S(t)$. In particular, if we choose $\beta = 1$, the subdiffusive option pricing in Eq. (6.12) becomes Black-Scholes formula time changed by an inverse subordinators [90].

In fact, the quantity in Eq. (6.12) can likewise be characterized by a fractional differential equation which we will derive in the following part. Resorting to Eq. (3.111), taking the derivative of $C_1(x,t)$ with respect to t, it is evident that

$$\frac{\partial}{\partial t}C_1(x,t) = \int_0^\infty ds\, C_\beta(x,s)\frac{\partial}{\partial t}h(s,t) \tag{6.13}$$

Using Eq. (3.111), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}C_1(x,t) &= -F_t\left(\int_0^\infty ds\, C_\beta(x,s)\frac{\partial}{\partial s}h(s,t)\right) \\
&= -F_t\left(h(s,t)C_\beta(x,s)|_0^\infty - \int_0^\infty ds\, \frac{\partial}{\partial s}C_\beta(x,s)h(s,t)\right) \\
&= -F_t\left(-\int_0^\infty ds\, \left(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} - r + \left(r + (1-\beta)(1-\frac{\sigma^2}{2})\right)x\frac{\partial}{\partial x}\right)C_\beta(x,s)h(s,t)\right) \\
&= F_t\left(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} - r + \left(r + (1-\beta)(1-\frac{\sigma^2}{2})\right)x\frac{\partial}{\partial x}\right)C_1(x,t)\,.
\end{aligned} \tag{6.14}$$

Here of course the appropriate boundary conditions are required for $h(s,t)$ at $t = 0$ and $t = \infty$.

Finally we obtain the fractional equation for the diffusive power option formula as

$$\frac{\partial}{\partial t}C_1(x,t) = F_t\left(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} - r + \left(r + (1-\beta)(1-\frac{\sigma^2}{2})\right)x\frac{\partial}{\partial x}\right)C_1(x,t) \tag{6.15}$$

with the initial and boundary conditions

$$C_1(x,0) = \max((x^\beta - K),0), \qquad x \geq 0 \tag{6.16a}$$
$$C_1(0,t) = 0, \qquad t \geq 0 \tag{6.16b}$$
$$C_1(x,t) \to x^\beta, \qquad x \to \infty\,, \tag{6.16c}$$

which follow instantly from the definition in Eq. (6.1). It is obvious that the Black-Scholes Eq. (2.96) is recovered when $\beta = 1$. So far we have succeeded in deriving the fractional differential equation for $C_1(x,t)$ and it is straightforward to solve it in Laplace space which is common for solving such fractional equations, but we omit this method here for simplicity. Alternatively, we can apply the Laplace transform directly to $C_1(x,t)$ in Eq. (6.12). After a simple calculation, it is easy to see that $C_1(x,t)$ in Laplace space can be written as

$$\begin{aligned}\tilde{C}_1(x,\lambda) &= \frac{\Phi(\lambda)}{\lambda}\tilde{C}_\beta(x,\Phi(\lambda)) \\ &= \frac{\Phi(\lambda)}{\lambda}\tilde{C}_{\mathrm{BS}}(x^\beta,\Phi(\lambda)-(\beta-1)(r+\frac{\beta\sigma^2}{2}),K,\beta\sigma,r_\beta)\,.\end{aligned} \tag{6.17}$$

Here we use the relation between the normal power option $\tilde{C}_\beta(x,\lambda)$ and Black-Scholes formula $\tilde{C}_{\mathrm{BS}}(x,\lambda,K,\sigma,r)$ in Laplace space as follows

$$\begin{aligned}\tilde{C}_\beta(x,\lambda) &= \int_0^\infty dt e^{-\lambda t} C_\beta(x,t) \\ &= \int_0^\infty dt e^{-\lambda t} e^{(\beta-1)(r+\frac{\beta\sigma^2}{2})t} C_{\mathrm{BS}}(x^\beta,t,K,\beta\sigma,r_\beta) \\ &= \tilde{C}_{\mathrm{BS}}(x^\beta,\lambda-(\beta-1)(r+\frac{\beta\sigma^2}{2}),K,\beta\sigma,r_\beta)\,.\end{aligned} \tag{6.18}$$

Since in Laplace space $\tilde{C}_{\mathrm{BS}}(x,\lambda,K,\sigma,r)$ is given by Eq.(4.11), the closed analytic form of the subdiffusive power option $C_1(x,t)$ in Eq. (6.12) could be derived by performing the inverse Laplace transform of the expression in Eq. (6.17).

6.2.3 Subdiffusive power option 2

Let us now turn to the other type of the subdiffusive power option, which is obtained from Eq. (6.6) as

$$C_2(x,t) = e^{-rt} \int_0^\infty ds\, e^{rs} C_\beta(x,s) h(s,t) \tag{6.19}$$

We are interested in the fractional equation which could characterise the evolution of $C_2(x,t)$ above. If we take the derivative of $C_2(x,t)$ with respect to t, we find that

$$\frac{\partial}{\partial t} C_2(x,t) = -re^{-rt} \int_0^\infty ds\, e^{rs} C_\beta(x,s) h(s,t) + e^{-rt} \int_0^\infty ds\, e^{rs} C_\beta(x,s) \frac{\partial}{\partial t} h(s,t)\,. \tag{6.20}$$

By Eq. (3.111) and Eq. (6.19), the equation above can be converted into

$$\begin{aligned}
\frac{\partial}{\partial t} C_2(x,t) &= -rC_2(x,t) - e^{-rt} F_t \left(\int_0^\infty ds\, e^{rs} C_\beta(x,s) \frac{\partial}{\partial s} h(s,t) \right) \\
&= -rC_2(x,t) - e^{-rt} F_t \left(h(s,t) C_\beta(x,s) e^{rs}|_0^\infty - \int_0^\infty ds\, \frac{\partial}{\partial s} (e^{rs} C_\beta(x,s)) h(s,t) \right) \\
&= -rC_2(x,t) - e^{-rt} F_t \left(-r \int_0^\infty ds\, e^{rs} C_\beta(x,s) h(s,t) \right. \\
&\quad \left. - \int_0^\infty ds\, e^{rs} \frac{\partial}{\partial s} C_\beta(x,s) h(s,t) \right)
\end{aligned} \tag{6.21}$$

with the assumption that the appropriate boundary conditions are satisfied by the density function $h(s,t)$ both at time $t = 0$ and $t = \infty$.

As the normal power option satisfies Eq. (6.10), substituting it into the equation above, we can derive the fractional equation for $C_2(x,t)$ given by Eq. (6.19) as follows

$$\begin{aligned}\frac{\partial}{\partial t}C_2(x,t) &= -rC_2(x,t) + e^{-rt}F_t\left(r\int_0^\infty ds\, e^{rs}C_{\mathrm{BS}}(x,s)h(s,t)\right.\\&\quad + \int_0^\infty ds\, e^{rs}\left(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2} - r\right.\\&\quad \left.\left.+ \left(r + (1-\beta)(1-\frac{\sigma^2}{2})\right)x\frac{\partial}{\partial x}\right)C_\beta(x,s)h(s,t)\right)\\&= -rC_2(x,t) + e^{-rt}F_t\left(e^{rt}\left(\frac{\sigma^2x^2}{2}\frac{\partial^2}{\partial x^2}\right.\right.\\&\quad \left.\left.+ \left(r + (1-\beta)(1-\frac{\sigma^2}{2})\right)x\frac{\partial}{\partial x}\right)C_2(x,t)\right) . \qquad (6.22)\end{aligned}$$

The initial and boundary conditions for the equation above should satisfy

$$C_2(x,0) = \max((x^\beta - K),0), \qquad x \geq 0 \qquad (6.23a)$$

$$C_2(0,t) = 0, \qquad t \geq 0 \qquad (6.23b)$$

$$C_2(x,t) \to x^\beta e^{-rt}\mathcal{L}^{-1}\left\{\frac{\Phi(\lambda)}{\lambda(\Phi(\lambda)-r)}\right\}, \qquad x \to \infty . \qquad (6.23c)$$

The fractional equation (6.22) also becomes the standard Black-Scholes Equation 2.96 when we let $\beta = 1$. Using the formula which we have obtained for the general case given by Eq. (6.7), the solution to the fractional equation (6.22) in Laplace space could be derived as

$$\begin{aligned}\tilde{C}_2(x,\lambda) &= \frac{\Phi(\lambda+r)}{\lambda+r}\tilde{C}_\beta(x,\Phi(\lambda+r)-r)\\&= \frac{\Phi(\lambda+r)}{\lambda+r}\tilde{C}_{\mathrm{BS}}(x^\beta,\Phi(\lambda+r)-r-(\beta-1)(r+\frac{\beta\sigma^2}{2}),K,\beta\sigma,r_\beta) . \qquad (6.24)\end{aligned}$$

Now the closed exact form of the subdiffusive power option $C_2(x,t)$ in Eq. (6.19) can be obtained by performing the inverse Laplace transform to $\tilde{C}_2(x,\lambda)$ in Eq. (6.24).

6.2.4 Different cases of waiting times

α-stable waiting times

According to different expressions of the Laplace exponent in Eq. (3.106), we are able to model different statistics of the waiting times. As a special case, we will consider the

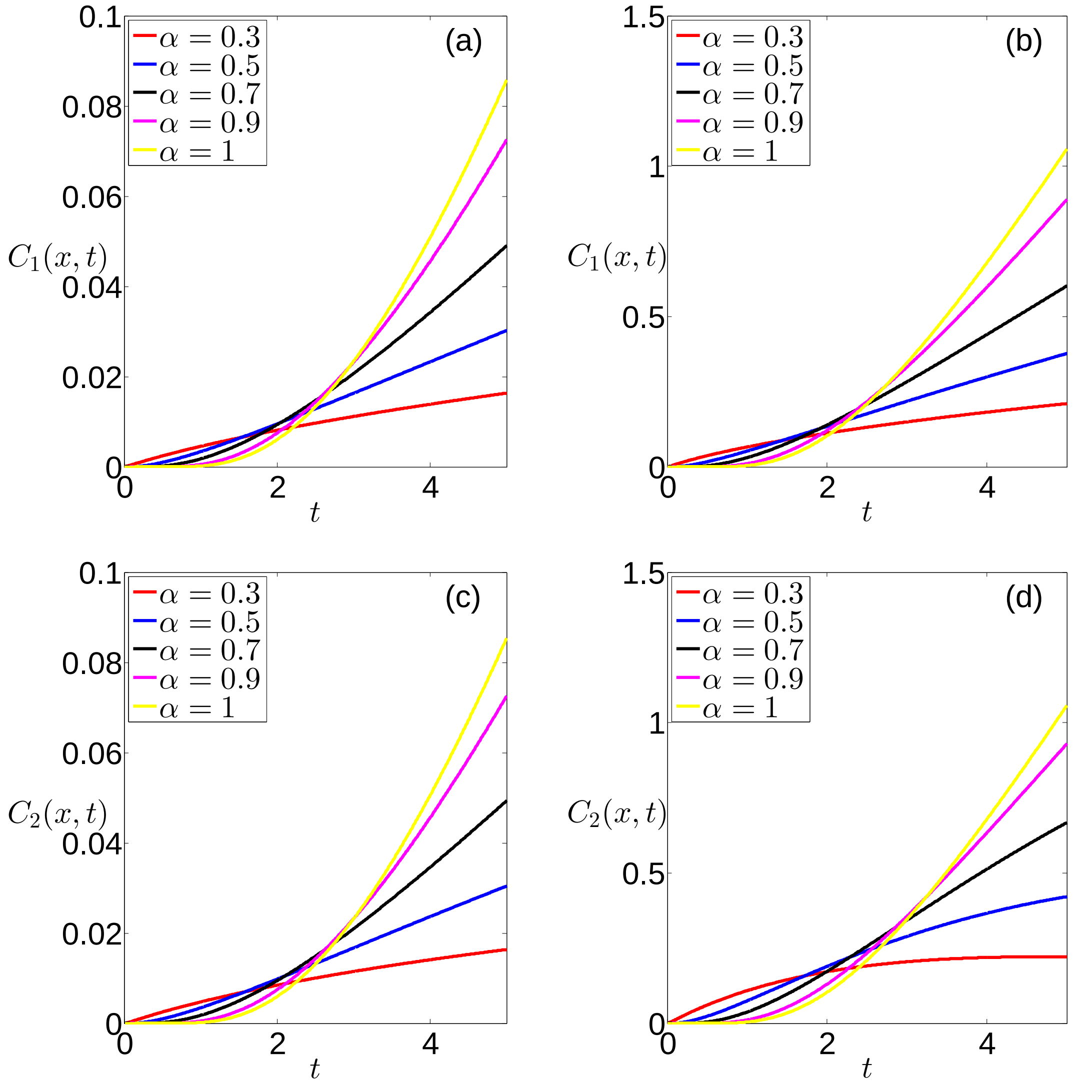

Figure 6.1: Analytic expression of subdiffusive power option cost formula with α-stable waiting times $C_1(x,t)$ [see Eqs. (6.18)] and $C_2(x,t)$ [see Eqs. (6.24)] with parameter values $K = 3$, $x = 1.1$, $\beta = 1.5$, $\sigma = 0.2$ for different interest rates: (a)(c) $r = 0.02$ and (b)(d) $r = 0.2$.

α-stable waiting times [23,64,104,142], which means the Laplace exponent in Eq. (3.106) reads $\Phi(\lambda) = \lambda^\alpha$. For this case, it actually describes the subdiffusive dynamics based on the continuous-time random walk (CTRW) [23].

In Fig. 6.1, the subdiffusive power option with α-stable waiting times is compared for different α values with parameters $K = 3$, $x = 1.1$, $\beta = 1.5$, $\sigma = 0.2$ for two different values for the interest rate $r = 0.02$ and $r = 0.2$. We see that when $\alpha \to 1$, the subdiffusive option value with stable waiting times is approaching the standard power option prices.

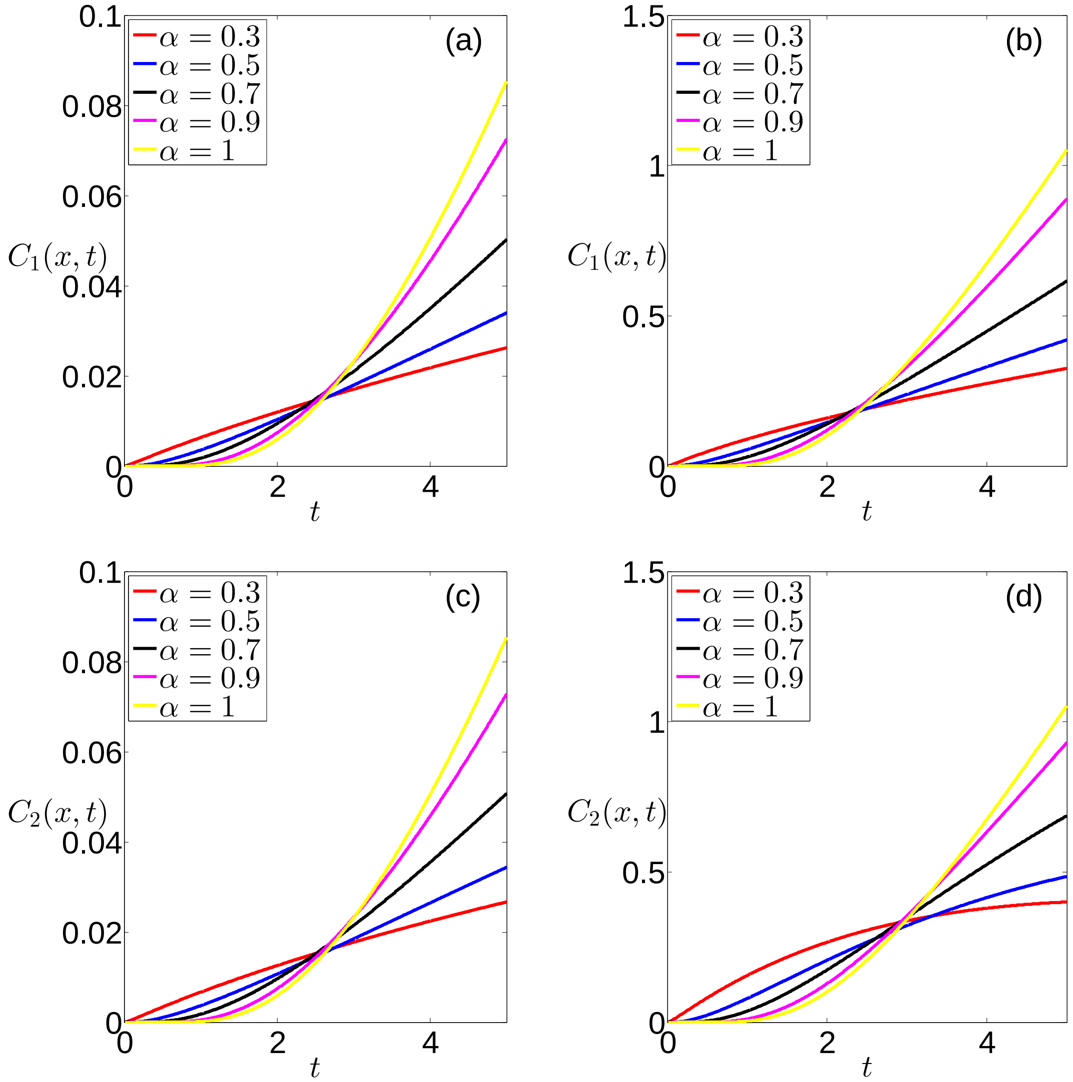

Figure 6.2: Analytic expression of subdiffusive power option cost formula with tempered stable waiting times $C_1(x,t)$ [see Eqs. (6.18)] and $C_2(x,t)$ [see Eqs. (6.24)] with parameter values $K = 3$, $x = 1.1$, $\beta = 1.5$, $\sigma = 0.2$ and $\zeta = 0.001$ for different interest rates: (a)(c) $r = 0.02$ and (b)(d) $r = 0.2$.

The tempered stable waiting times

We also consider tempered stable waiting times, which means that in Eq. (3.106) $\Phi(\lambda) = (\lambda+\zeta)^\alpha - \zeta^\alpha$ with $0 < \alpha < 1$ [8, 32, 67]. The tempered stable waiting times could resemble stable laws in many fields [138]. Particularly, the transition from the initial subdiffusive character of motion in short times to the standard diffusion in long times is observed [126,154], which is suitable to model many experiments results [21,69,118,128]. When $\zeta = 0$ the CTRW case is recovered while for $\zeta \to \infty$ the Brownian limit is obtained. As it is quite difficult to get closed forms for $C_1(x,t)$ and $C_2(x,t)$ by applying the inverse Laplace transform to the expressions derived in Eqs. (6.18)–(6.24) directly, we resort to the so-called Talbot method [2,3,157] to compute the option value in the time domain, and we use the numerical inverse Laplace transform of the Mathematica software [1].

Fig. 6.2 shows the changes of the subdiffusive power option with tempered stable waiting times according to different values of the real time t with parameters $K = 3$, $x = 1.1$, $\beta = 1.5$, $\sigma = 0.2, \zeta = 0.001$ for the interest rate $r = 0.02$ and $r = 0.2$, respectively. It also shows that when $\alpha \to 1$, the subdiffusive option values with stable waiting times are approaching the standard power option prices. The comparison of the subdiffusive power option cost formula 1 $C_1(x,t)$ with tempered stable waiting times and formula 2 $C_2(x,t)$ is made in Fig. 6.3 for different values of the tempering parameter ζ.

The difference between the values of subdiffusive power option costs with either stable waiting times or tempered waiting times $C_1(x,t)$ and $C_2(x,t)$ should be very tiny when the interest rate r takes a small value, which is evident as the formula $C_1(x,t)$ in Eq. (6.12) and $C_2(x,t)$ in Eq. (6.19) would be equal to each other when $r = 0$. However, when the interest rate increases, the prices are different, which could be observed both from Figs. 6.1 and 6.2. When $r = 0.02$ which is quite close to 0, the value of $C_1(x,t)$ and $C_2(x,t)$ are almost the same whereas when $r = 0.2$ the value is quite different. This behaviour is as expected.

We are interested in the impact of the subdiffusive behaviour, controlled by the parameter α. If we take a look at stable waiting times shown by Figs. 6.1, Fig. 6.1 (a) and 6.1 (b) describe values of the power option $C_1(x,t)$ with discounting on the subordinator time scale. It is clearly visible that the standard power call option formula overvalues the option when the asset prices follows a subdiffusive dynamics where the subdiffusive option formula would provide a more reasonable price. Of course the standard power option is recovered in the limiting case $\alpha \to 1$. Furthermore the figures indicate that for long times the larger the value of α takes, the higher the value that

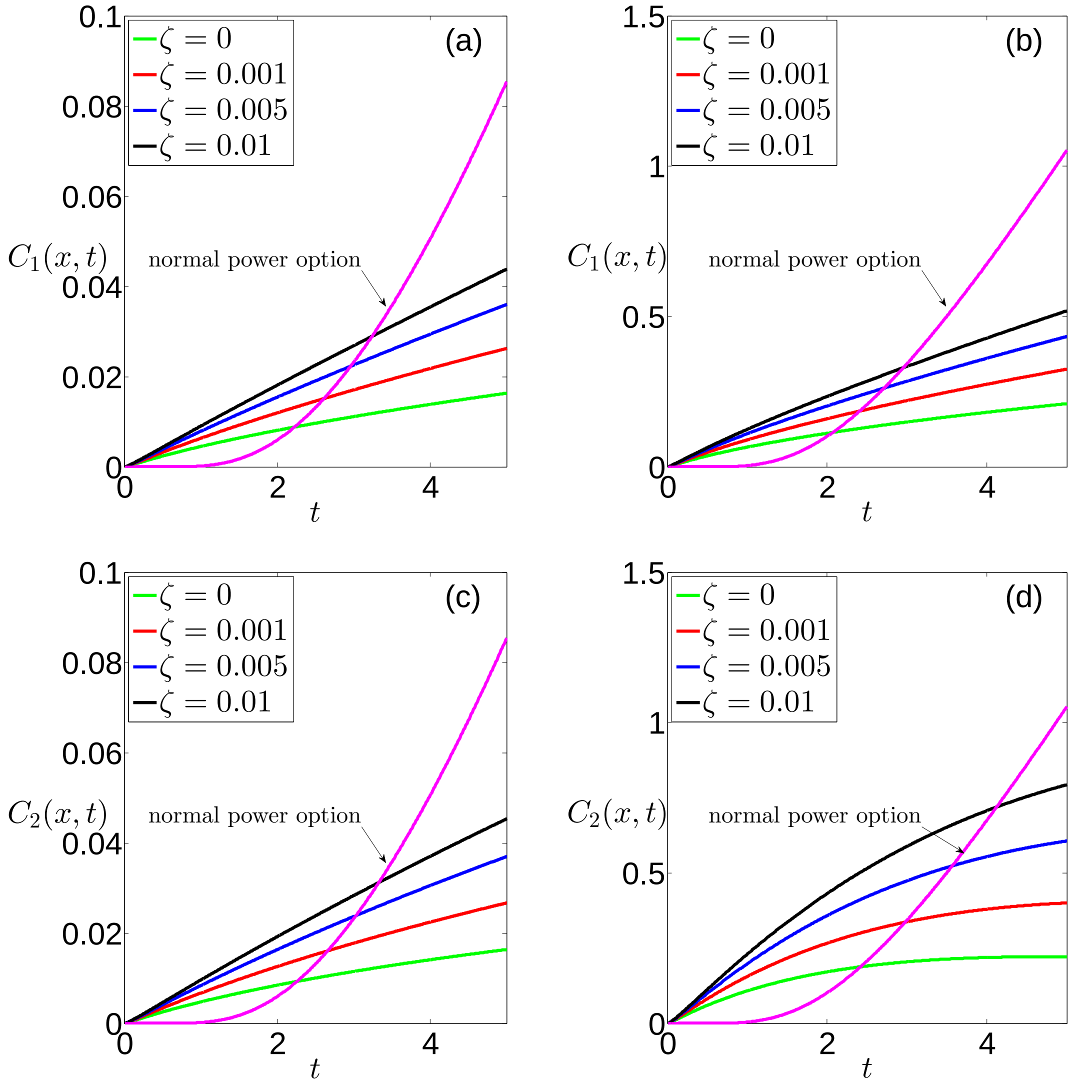

Figure 6.3: The comparison of subdiffusive power option cost formula with tempered stable waiting times $C_1(x,t)$ [see Eqs. (6.18)] and $C_2(x,t)$ [see Eqs. (6.24)] with parameter values $K = 3$, $x = 1.1$, $\beta = 1.5$, $\sigma = 0.2$ and $\alpha = 0.3$ for different interest rates: (a)(c) $r = 0.02$ and (b)(d) $r = 0.2$.

the subdiffusive option reaches. For short times, this behaviour is reversed. Fig. 6.1 (c) and 6.1 (d) show that the subdiffusive formula $C_2(x,t)$ which takes the discounting on the real time scale into consideration exhibits similar trends as $C_1(x,t)$. It is remarkable that both of the subdiffusive formulas $C_1(x,t)$ and $C_2(x,t)$ give a qualitatively similar result for $r = 0.02$. The same behaviour also takes place for the tempered waiting times which can be observed in Fig. 6.2.

In the case of the tempered stable waiting times, also the parameter ζ has an impact on the value of the option. Fig. 6.3 provides us with a clear view of the resulting behaviour. For non-zero ζ the subdiffusive option is intermediate between the subdiffusive power option with stable waiting times and the standard power option. When ζ takes a value near 0, the value of the subdiffusive power option with tempered waiting times are more close to the subdiffusive power option with stable waiting times. When the value of ζ increases, the value of the subdiffusive power option with tempered waiting times approaches a high level, which is still below the cost of the normal power option. This behaviour holds for both $C_1(x,t)$ and $C_2(x,t)$, which means that the parameters ζ and α both affect the value of the subdiffusive option pricing cost.

6.3 Summary of chapter

In this chapter, we have put forward a general subdiffusive call option pricing formula for arbitrary payoffs, which is assumed to be able to capture the constant periods in the asset price dynamics. Our formulas, depending on a subdiffusive pricing model with general waiting times, can be used to obtain the corresponding subdiffusive option formulas published in Refs. [88,90,91,120,155]. Since the payoff function is arbitrary, our results can be applied to a variety of different option pricing problems. As an example, we derived the subdiffusive power option formula. Once a specific payoff function is given, one could follow the standard procedure outlined here to derive its subdiffusive version. Our formulas will be particularly useful for pricing options in markets which exhibit subdiffusive dynamics.

Chapter 7

Path dependent call options

In this chapter, we will examine the exotic options with subordinated processes. Up till this point, we have investigated the vanilla options with subordinated process. Unfortunately, this perfection is depend only on the assets' price at the maturity time but ignores the price path leading to it. Rather than the vanilla options, we aim to investigate exotic options in the subdiffusive regime that depend not only on the asset price at the expiration time but also on the previous price history. In particular, we will examine Asian call option. In this case, the subordination can be formulated in three different ways leading to three different option formulas and PDEs. This chapter is arranged as follows. In Sec. 7.1, we present the conventional Asian call option and discuss a solution method based on a Laplace transformation of the strike price. In Sec. 7.2, the possible subdiffusive versions are introduced and solution methods discussed. Finally, we summarize this chapter in Sec. 7.3.

7.1 The conventional Asian call option

The no arbitrage price of a normal arithmetic Asian option $C(x,t)$ can be expressed as [5, 137]

$$C(x,t) = e^{-rt} \left\langle \left(\frac{1}{t} \int_0^t g(X(\tau))d\tau - K \right)^+ \right\rangle_{X(0)=x} \tag{7.1}$$

where g is an arbitrary function specifying the payoff, r is the interest rate, t is expiration time, and K is strike price. $X(t)$ denotes the price of the underlying asset which is expected to follow the risk-neutral Geometric Brownian motion (GBM) Eq. (2.78). We see that the share price at the expiration time as in the normal European call option is replaced by a functional that includes all share prices from time 0 (the initial time of the option) until the expiration time. In particular, when $g(x) = x$, Eq. (7.1) is referred to as arithmetic Asian call option and when $g(x) = \ln(x)$ as geometric Asian call option. Note that throughout this chapter the brackets $\langle ... \rangle$ always denote a conditional expected value $\langle ... \rangle_{X(0)=x}$.

An elegant way to translate Eq. (7.1) into more tractable expressions is to use a Laplace transform with respect to the strike price K as described in Ref. [47] for $g(x) = x$, but the approach can be applied for a general function g. Let us first review the major steps of this method. If the option could be expressed as follows

$$C(x, t, K) = e^{-rt} \left\langle \beta \left(g(X(t)) - K\right)^{+} \right\rangle \tag{7.2}$$

where $\beta \left(X(t) - K\right)^{+}$ is the payoff function for some constant β and function g. Then

$$\begin{aligned} C(x, t, K) &= e^{-rt} \left\langle \beta \left(g(X(t)) - K\right)^{+} \right\rangle \\ &= \beta e^{-rt} \int_0^\infty (g(z) - K)^{+} f(z|x) dx \end{aligned} \tag{7.3}$$

where $f(z|x)$ is the conditional PDF of $X(t)$. If we apply a Laplace transform with respect to K, we obtain further

$$\begin{aligned} C(x, t, v) &= \mathcal{L}\{C(x, t, K)\} \\ &= \int_0^\infty e^{-vK} C(x, t, K) dK \\ &= \beta e^{-rt} \int_0^\infty e^{-vK} \int_0^\infty (g(z) - K)^{+} f(z|x) dz\, dK \\ &= \beta e^{-rt} \int_0^\infty \left(\int_0^{g(z)} e^{-vK} (g(z) - K) dK \right) f(z|x) dz \\ &= \beta e^{-rt} \int_0^\infty \frac{e^{-vg(z)} + g(z)v - 1}{v^2} f(z|x) dz \\ &= \beta e^{-rt} \left(\frac{\left\langle e^{-vg(X(t))} \right\rangle}{v^2} + \frac{\langle g(X(t)) \rangle}{v} - \frac{1}{v^2} \right) \end{aligned} \tag{7.4}$$

Applying the inverse Laplace transform to it, we get

$$C(x,t,K) = \beta e^{-rt}\left(\mathcal{L}^{-1}\left\{\frac{\left\langle e^{-vg(X(t))}\right\rangle}{v^2}\right\} + \langle g(X(t))\rangle - K\right) \tag{7.5}$$

The option pricing could be obtained explicitly if we know the inversion of $\frac{\left\langle e^{-vg(X(t))}\right\rangle}{v^2}$ and the average value $\langle g(X(t))\rangle$.

Comparing to our option formula Eq. (7.1) we notice that the payoff is

$$\frac{1}{t}\left(\int_0^t g(X(\tau))d\tau - Kt\right)^+, \tag{7.6}$$

we see that $g(X(t))$ in Eq. (7.5) is replaced by $\int_0^t g(X(\tau))d\tau$ here. Furthermore, $\beta = 1/t$ and $K \to Kt$. Therefore, the Asian call option price is determined by

$$C(x,t,K) = \frac{1}{t}e^{-rt}\left(\mathcal{L}^{-1}\left\{\frac{\left\langle e^{-v\int_0^t g(X(\tau))d\tau}\right\rangle}{v^2}\right\} + \left\langle\int_0^t g(X(\tau))d\tau\right\rangle - Kt\right). \tag{7.7}$$

Apart from performing the inverse Laplace transform, the challenge is thus to evaluate the expected values

$$\left\langle\int_0^t g(X(\tau))d\tau\right\rangle = \int_0^t \langle g(X(\tau))\rangle\, d\tau, \tag{7.8}$$

and, more importantly,

$$u(x,t) = \left\langle e^{-v\int_0^t g(X(\tau))d\tau}\right\rangle \tag{7.9}$$

which satisfies the conventional Feynman-Kac (FK) equation (2.87). In order to extend Eq. (7.1) to a subdiffusive pricing model, we thus need to generalize the FK equation accordingly. We first discuss the three different ways of introducing the subordination in Eq. (7.1).

7.2 Subdiffusive versions of the Asian call option

By replacing the risk-neutral Geometric Brownian motion (GBM) Eq. (2.78) with the subordinated geometric Brownian motion for the asset price, we then obtain three different types of subdiffusive option formulas depending on the way the physical time is represented:

1. The simplest way to include the subordination is by replacing t with $S(t)$ throughout Eq. (7.1). This yields:

$$C_1(x,t) = \left\langle e^{-rS(t)} \left(\frac{1}{S(t)} \int_0^{S(t)} g(X(\tau))d\tau - K \right)^+ \right\rangle . \tag{7.10}$$

2. A variant of this version is obtained by replacing t only in the integral limit by $S(t)$, which keeps the integral over the asset price process simple

$$C_2(x,t) = \left\langle e^{-rt} \left(\frac{1}{t} \int_0^{S(t)} g(X(\tau))d\tau - K \right)^+ \right\rangle . \tag{7.11}$$

3. Finally, we obtain the third variant by replacing the asset price process $X(t)$ with $Y(t)$ while keeping the physical time everywhere as in Eq. (7.1), where $Y(t) = X(S(t))$ is as before the CTRW in physical time. This means the integral is now over the physical time rather than over the operational time

$$C_3(x,t) = \left\langle e^{-rt} \left(\frac{1}{t} \int_0^{t} g(X(S(\tau)))d\tau - K \right)^+ \right\rangle . \tag{7.12}$$

   As we will see below, this convention, which is in a way the most consistent extension of Eq. (7.1), requires us to introduce an entirely different fractional time operator.

The option price in version 1 is obtained in a straightforward way as an integral transformation. However, for versions 2 and 3, we need to apply results on generalized FK formulas from the literature. Note that the conditional expected value on $X(0) = x$ is applicable for all three versions since $X(S(0)) = X(0) = x$.

Formula 1

The option price Eq. (7.10) can be further manipulated as follows

$$\begin{aligned} C_1(x,t) &= \left\langle e^{-rS(t)} \left(\frac{1}{S(t)} \int_0^{S(t)} g(X(\tau))d\tau - K \right)^+ \right\rangle \\ &= \int_0^\infty ds \left\langle e^{-rs} \left(\frac{1}{s} \int_0^s g(X(\tau))d\tau - K \right)^+ \right\rangle h(s,t) \\ &= \int_0^\infty ds\, C(x,s) h(s,t) \end{aligned} \tag{7.13}$$

where $C(x,s)$ is the price of the normal Asian call option and $h(s,t)$ is the density function of the general inverse Lévy subordinator $S(t)$ Eq. (3.107). By applying the Laplace transform to Eq. (7.13), we obtain further

$$\begin{aligned} \tilde{C}_1(x,\lambda) &= \int_0^\infty dt e^{-\lambda t} \int_0^\infty ds\, C(x,s) h(s,t) \\ &= \int_0^\infty ds\, C(x,s) \tilde{h}(s,\lambda) \\ &= \int_0^\infty ds\, C(x,s) \frac{\Phi(\lambda)}{\lambda} e^{-\Phi(\lambda)s} \\ &= \frac{\Phi(\lambda)}{\lambda} \int_0^\infty ds\, C(x,s) e^{-\Phi(\lambda)s} \\ &= \frac{\Phi(\lambda)}{\lambda} \tilde{C}(x,\Phi(\lambda)) \end{aligned} \tag{7.14}$$

To derive the result above, Eq. (3.108) is used here. We conclude that if $C_1(x,t)$ is obtained in Laplace space, it is straightforward to evaluate the subdiffusive version 1 option price by applying the inverse Laplace transform to Eq. (7.14).

Formula 2

In the case of the second option formula Eq. (7.11), we can use the method outlined above by performing a Laplace transform with respect to the strike price K leading to Eq. (7.7). Noticing that the payoff is

$$\frac{1}{t} \left(\int_0^{S(t)} g(X(\tau))d\tau - Kt \right)^+ , \tag{7.15}$$

instead of Eq. (7.15), we see that we can use the solution Eq. (7.7) to evaluate Eq. (7.11) if we replace the two expected values in Eq. 7.7 by the expressions $\left\langle e^{-v\int_0^{S(t)} g(X(\tau))d\tau}\right\rangle$ and $\left\langle \int_0^{S(t)} g(X(\tau))d\tau\right\rangle$. The expected value

$$u(x,t) = \left\langle e^{-v\int_0^{S(t)} g(X(\tau))d\tau}\right\rangle \tag{7.16}$$

requires a generalization of the conventional FK equation (2.87), which, in fact, has been solved for a general Laplace exponent Φ of the subordinator $S(t)$ in Ref. [92]. The expected value Eq. (7.16) is the solution of the fractional equation

$$\frac{\partial u(x,t)}{\partial t} = F_t\left(\left(rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) u(x,t) - vg(x)u(x,t)\right) \tag{7.17}$$

with initial condition

$$u(x,0) = 1 \tag{7.18}$$

where the operator F_t is defined as in Eq. (3.112). Eq. (7.17) is obtained analogous to the fractional Fokker-Planck equation (3.115) by noting that

$$\begin{aligned} u(x,t) &= \left\langle e^{-v\int_0^{S(t)} g(X(\tau))d\tau}\right\rangle \\ &= \left\langle \int_0^\infty ds\, \delta(s-S(t)) e^{-v\int_0^{s} g(X(\tau))d\tau}\right\rangle \\ &= \int_0^\infty ds\, h(s,t) \left\langle e^{-v\int_0^{s} g(X(\tau))d\tau}\right\rangle \\ &= \int_0^\infty ds\, h(s,t) u_0(x,s), \end{aligned} \tag{7.19}$$

where $u_0(x,s)$ satisfies the conventional FK equation (2.87).

Therefore, one of the required terms in Eq. (7.7) could be calculated if we are able to calculate the integral in Eq. (7.19) from the conventional solution of the FK equation. Moreover, we have

$$\left\langle \int_0^{S(t)} g(X(\tau))d\tau \right\rangle = \int_0^\infty \left(\int_0^s \langle g(X(\tau))\rangle\, d\tau\right) h(s,t)ds. \tag{7.20}$$

For the special case of an arithmetic Asian call option with $g(x) = x$, we obtain further

$$\begin{aligned}\int_0^\infty \left(\int_0^s \langle g(X(\tau))\rangle\, d\tau\right) h(s,t)ds &= \int_0^\infty \left(\int_0^s x_0 e^{r\tau}\right) h(s,t)ds \\ &= x_0 \int_0^\infty \frac{e^{rs}-1}{r} h(s,t)ds \\ &= \frac{x_0}{r}\left(\int_0^\infty e^{rs} h(s,t)ds - 1\right) \end{aligned} \tag{7.21}$$

where $h(s,t)$ is the PDF of the process $S(t)$ in Eq. (3.107). Overall, we are thus able to evaluate the arithmetic Asian call option via Eq. (7.7) if we can provide Eq. (7.21) and the solution Eq. (7.19) of the generalized Feynman-Kac equation (7.17) for the case $g(x) = x$. As far as the case of an geometric Asian call option with $g(x) = \ln(x)$, it also can obtained with the similar steps unless we get

$$\begin{aligned}\int_0^\infty \left(\int_0^s \langle g(X(\tau))\rangle\, d\tau\right) h(s,t)ds &= \int_0^\infty \left(\int_0^s \left(\ln x_0 + \left(r - \frac{\sigma^2}{2}\right)\tau\right) d\tau\right) h(s,t)ds \\ &= \int_0^\infty \left(\ln x_0 s + \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right) s^2\right) h(s,t)ds \\ &= \ln x_0 \int_0^\infty s h(s,t)ds \\ &\quad + \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)\int_0^\infty s^2 h(s,t)ds\,. \end{aligned} \tag{7.22}$$

Formula 3

In order to evaluate the third version of a subdiffusive Asian call option, Eq. (7.12), we can proceed as for the second one, only that now the payoff function is

$$\frac{1}{t}\left(\int_0^t g(X(S(\tau)))d\tau - Kt\right)^+ . \tag{7.23}$$

Therefore, we need to calculate the two expected values $\left\langle e^{-v\int_0^t g(X(S(\tau)))d\tau}\right\rangle$ and $\left\langle \int_0^t g(X(S(\tau)))d\tau\right\rangle$ in order to apply Eq. (7.7). We thus seek the Feynman-Kac equation that provides as solution the expected value

$$u(x,t) = \left\langle e^{-v\int_0^t g(X(S(\tau)))d\tau}\right\rangle . \tag{7.24}$$

To our knowledge this equation has not been derived yet for the underlying general CTRW model Eq. (3.24) with x-dependent drift and diffusion and general waiting times. However, specific cases have been treated in the literature. In [24,160] the FK equation[1] has been derived for an x-dependent drift only and the special case of an inverse Lévy stable subordinator, i.e., the conventional CTRW case. In [22] the full model Eq. (3.24) has been treated for general waiting times, but only the forward FK equation has been derived formally. Combining these results from the literature it is straightforward to conjecture, even without derivation, that the FK equation sought in the present case should have the form

$$\frac{\partial u(x,t)}{\partial t} = \left(rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \right) \mathcal{D}_t u(x,t) - vg(x)u(x,t) \tag{7.25}$$

with initial condition

$$u(x,0) = 1. \tag{7.26}$$

In Eq. (7.25), the outstanding feature is the presence of a particular fractional time derivative, the so-called *fractional substantial derivative* $\mathcal{D}_t$, which has first been derived for the joint position-velocity PDF of anomalous random walkers [46]. For a general Laplace exponent, it is defined in Laplace space as [22]

$$\mathcal{L}\left\{\mathcal{D}_t u(x,t)\right\} = \frac{\lambda + vg(x)}{\Phi(\lambda + vg(x))}\tilde{u}(x,\lambda). \tag{7.27}$$

and can also be written explicitly in terms of an integral

$$\mathcal{D}_t u(x,t) = \left(\frac{\partial}{\partial t} + vg(x) \right) \int_0^t d\tau\, K(t-\tau) e^{-vg(x)(t-\tau)} u(x,t), \tag{7.28}$$

which corresponds formally to the inverse Laplace transform of the rhs in Eq. (7.27). The kernel K is again related to the Laplace exponent Φ via Eq. (3.113). Clearly, Eq. (7.25) is a complicated integro-differential equation. No non-trivial solution has been found yet, which is also due to the fact that the solution can not be expressed in terms of a simple integral transform as in the Formula 2, Eq. (7.19).

[1] More precisely, it is the backward FK equation, since the spacial derivatives act on the independent variable at the initial time.

The remaining expected value can be simplified as follows

$$\left\langle \int_0^t g(X(S(\tau)))d\tau \right\rangle = \left\langle \int_0^t d\tau \int_0^\infty ds\, g(X(s))\delta(s - S(\tau)) \right\rangle$$
$$= \int_0^t d\tau \int_0^\infty ds \, \langle g(X(s)) \rangle \, h(s,\tau) \tag{7.29}$$

For the arithmetic Asian call option with $g(x) = x$ we thus obtain

$$\int_0^t d\tau \int_0^\infty ds \, \langle g(X(s)) \rangle \, h(s,\tau) = x_0 \int_0^t d\tau \int_0^\infty ds\, e^{rs} h(s,\tau) \tag{7.30}$$

However, the main challenge remains to solve Eq. 7.25, which has not been possible so far, even for the simple arithmetic case $g(x) = x$. For the geometric Asian call option with $g(x) = \ln x$ we can derive

$$\int_0^t d\tau \int_0^\infty ds \, \langle g(X(s)) \rangle \, h(s,\tau) = \int_0^t d\tau \int_0^\infty ds \, \left(\ln x_0 + \left(r - \frac{\sigma^2}{2} \right) s \right) h(s,\tau) \,. \tag{7.31}$$

7.3 Summary of chapter

In this chapter, we went beyond the vanilla options and considered exotic options in the subdiffusive regime that are determined not only by the asset price at the expiration time but also by the previous price history. We discussed three different types of subdiffusive Asian call options with a general payoff function. In the first case, the subdiffusive option price is readily obtained if the result on the conventional Asian call option is known, since then the integral transformation with respect to the single-point PDF of the subordinator $S(t)$ can be applied. In the second case, an approach based on a Laplace transform with respect to the strike price needs to be applied, but the relevant quantities are again obtained from the conventional ones via the integral transform. In the third and most realistic case, such a simple transformation can not be used. Nevertheless, we were able to express the option price in terms of the solution of a generalized FK equation containing a substantial fractional derivative. Solutions to this equation are not known at this point in time, so a further study of this equation would be interesting future work.

Chapter 8

Concluding remarks and outlook

Conclusions of this thesis

In this thesis, we have discussed subdiffusive European call options with CTRW, the extension to general waiting times, subdiffusive call options with arbitrary payoff functions as well as general waiting times, and path dependent call options. In particular, we took subdiffusive geometric Brownian motion and subordinated geometric Brownian motion as the underlying asset price models which are related to subdiffusive phenomena. The main results of this thesis are listed as follows.

In Chapter 4, we use the subdiffusive GBM to analyse the subdiffusive European call option pricing formula with CTRW. Our study shows that there are two types of subdiffusive options, type A and type B option costs with non-zero interest rate based on the CTRW formalization of the subdiffusive pricing model. We indicated that these two types of subdiffusive formulas could be derived from corresponding fractional differential equations. During the investigation, the fractional Fokker-Planck equation governing the dynamics of the subdiffusive model is found again, which is different from the original derivation in the literature. Comparison between two types of subdiffusive option formulas are made and the effect of past time points studied.

In Chapter 5, by extending the subdiffusive GBM to the subordinated GBM, we investigate the subdiffusive European call option pricing formula with general waiting times which is actually an extension of what we discussed in chapter 4. We derive two types of subdiffusive type A and type B option costs with non-zero interest rate. We also find that these two types of subdiffusive formulas could also be derived from corresponding fractional differential equations.

In Chapter 6, we investigate a general option formula with arbitrary payoffs, whose underlying pricing model is assumed to be able to capture the anomalous characteristics of assets price. We find two types of subdiffusive formulas which give more choices to model markets with anomalous dynamics. By illustrating the anomalous power option formula, we present an application of our general formula. The fractional equations which would be used to describe this kind of new power option formula are derived. The comparison between the classical and anomalous power option are made.

In Chapter 7, we take subdiffusive exotic options into consideration which are determined not only by the asset price at the expiration time but also by the previous price history. Particularly, we present three different types of subdiffusive Asian call options. For the first case, the subdiffusive option price can be readily obtained based on the classical Asian call option and the integral transformation with respect to the single-point PDF of the subordinator $S(t)$. For the second case, an efficient approach based on a Laplace transform with respect to the strike price can be put into use, but the relevant quantities are again obtained from the conventional ones via the integral transform. For the last and most realistic case, such a simple transformation can not be used. However, we can express the option price in terms of the solution of a generalized FK equation containing a substantial fractional derivative. Solutions to this equation are not known yet, so further studies need to be completed to analyse this scenario.

Outlook

Starting with what we have derived so far, it is rather interesting to explore other option pricing formula in the subdiffusive regime. In what follows, some interesting problems are listed.

- It is rather interesting to get the real data in financial market exhibiting the anomalous dynamics which could help to adjust the parameters in the pricing model to better capture the real data.
- To improve the current model with more practical assumptions would be a good way to find new option pricing formulas. It would be sensible to introduce, for example, stochastic volatility, as the constant volatility assumed here is not an ideal way to analyse real data. For example, a new subdiffusive Heston model can

be formulated as follows

$$\dot{X}(s) = (r - \frac{\sigma^2}{2})x + \sqrt{v}x\xi_1(s)$$
$$\dot{v}(s) = k(\theta - v) + \sigma\sqrt{v}\xi_2(s)\,,$$
$$\dot{T}(s) = \eta(s)\,, \tag{8.1a}$$

where θ is the long-time mean of v, k is the rate of relaxation to this mean, σ is the variance noise. $\xi_1(s)$, $\xi_2(s)$ are standard white Gaussian noises. $\xi_1(s)$, $\xi_2(s)$ and $\eta(s)$ are also assumed to be independent noises such that X, v and T are statistically independent processes. The noise η is characterised by Eq.(3.106). By this improvement, a new versatile pricing model will be obtained, which would result in new option pricing cost formulas.

- The path dependent Asian options also deserve further investigation. It would be great progress in the field of anomalous dynamics to obtain relevant solutions for the generalized FK equation (2.87) with the fractional substantial derivative.
- Throughout this thesis, we have only considered the Gaussian white noise for the pricing model. These results are quite limited. As far as we know, more different noises can be taken into consideration which would result into new findings.

Bibliography

[1] A Mathematica implementation of this method is available at http://library.wolfram.com/infocenter/MathSource/5026.

[2] J. Abate and P. P. Valkó. Multi-precision Laplace transform inversion. *International Journal for Numerical Methods in Engineering*, 60:979–993, 2004.

[3] J. Abate and W. Whitt. A unified framework for numerically inverting Laplace transforms. *INFORMS Journal on Computing*, 18:408–421, 2006.

[4] E. Akyıldırım and H. M. Soner. A brief history of mathematics in finance. *Borsa Istanbul Review*, 14:57–63, 2014.

[5] B. Alziary, J. P. Décamps, and P. F. Koehl. A P.D.E. approach to Asian options: analytical and numerical evidence. *Journal of Banking & Finance*, 21:613–640, 1997.

[6] L. Bachelier. Théorie de la spéculation. *Annales scientifiques de l'École Normale Supérieure*, 17:21–86, 1900.

[7] L. Bachelier. *Louis Bachelier's Theory of Speculation: The Origins of Modern Finance*. Princeton University Press, 2011.

[8] B. Baeumer and M. M. Meerschaert. Tempered stable Lévy motion and transient super-diffusion. *Journal of Computational and Applied Mathematics*, 233:2438–2448, 2010.

[9] G. S. Bakshi and Z. Chen. An alternative valuation model for contingent claims. *Journal of Financial Economics*, 44:123–165, 1997.

[10] E. Barkai. Fractional Fokker-Planck equation, solution, and application. *Physical Review E*, 63:046118, 2001.

[11] D. S. Bates. Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *Review of Financial Studies*, 9:69–107, 1996.

[12] A. Baule and R. Friedrich. Joint probability distributions for a class of non-Markovian processes. *Physical Review E*, 71:026101, 2005.

[13] D. Ben-Avraham and S. Havlin. *Diffusion and Reactions in Fractals and Disordered Systems*. Cambridge University Press, 2000.

[14] B. Berkowitz and H. Scher. Anomalous transport in random fracture networks. *Physical Review Letters*, 79:4038–4041, 1997.

[15] P. L. Bernstein. *Capital Ideas: The Improbable Origins of Modern Wall Street*. Wiley, 2012.

[16] J. R. Birge and V. Linetsky. *Handbooks in Operations Research and Management Science: Financial Engineering*. Elsevier, 2007.

[17] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.

[18] J. P. Bouchaud and A. Georges. Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications. *Physics Reports*, 195:127–293, 1990.

[19] S. I. Boyarchenko and S. Z. Levendorskiy. Option pricing for truncated Lévy processes. *International Journal of Theoretical and Applied Finance*, 3:549–552, 2000.

[20] R. Brown. XXVII. A brief account of microscopical observations made in the months of June, July and August 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *Philosophical Magazine, or Annals of Chemistry, Mathematics, Astronomy, Natural History and General Science*, 4:161–173, 1828.

[21] A. C. Cadavid, J. K. Lawrence, and A. A. Ruzmaikin. Anomalous diffusion of solar magnetic elements. *Astrophysical Journal*, 521:844–850, 1999.

[22] A. Cairoli and A. Baule. Anomalous processes with general waiting times: functionals and multipoint structure. *Physical Review Letters*, 115:110601, 2015.

[23] A. Cairoli and A. Baule. Langevin formulation of a subdiffusive continuous-time random walk in physical time. *Physical Review E*, 92:012102, 2015.

[24] S. Carmi and E. Barkai. Fractional Feynman-Kac equation for weak ergodicity breaking. *Physical Review E*, 84:061104, 2011.

[25] P. Carr and D. B. Madan. Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2:61–73, 1999.

[26] P. Carr and L. Wu. Time-changed Lévy processes and option pricing. *Journal of Financial Economics*, 71:113–141, 2004.

[27] R. R. Chen and L. Scott. Pricing interest rate options in a two-factor cox–ingersoll–ross model of the term structure. *Review of Financial Studies*, 5:613–636, 1992.

[28] P. Cheridito. Mixed fractional Brownian motion. *Bernoulli*, 7:913–934, 2001.

[29] P. Cheridito. *Regularizing fractional Brownian motion with a view towards stock price modelling*. PhD thesis, Swiss Federal Instute of Technology Zurich, 2001.

[30] P. K. Clark. A subordinated stochastic process model with finite variance for speculative prices. *Econometrica*, 41:135–155, 1973.

[31] R. Cont. Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1:223–236, 2001.

[32] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman & Hall, 2004.

[33] S. Das. *Functional Fractional Calculus*. Springer, Berlin, 2011.

[34] D. Duffie. Black, Merton and Scholes-their central contributions to economics. *Scandinavian Journal of Economics*, 100:411–423, 1998.

[35] A. Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik*, 322:549–560, 1905.

[36] R. J. Elliott and L. Chan. Perpetual american options with fractional Brownian motion. *Quantitative Finance*, 4:123–128, 2004.

[37] L. Evans. *An Introduction to Stochastic Differential Equations*. American Mathematical Society, 2012.

[38] E. F. Fama. The behavior of stock-market prices. *Journal of Business*, 38:34–105, 1965.

[39] W. Feller. *An Introduction to Probability Theory, Vol. 1.* Wiley, New York, NY, third edition, 1968.

[40] A. Fick. Üeber diffusion. *Annalen der Physik*, 170:59–86, 1855.

[41] H. C. Fogedby. Langevin equations for continuous time Lévy flights. *Physical Review E*, 50:1657–1660, 1994.

[42] H. C. Fogedby. Lévy flights in random environments. *Physical Review Letters*, 73:2517–2520, 1994.

[43] H. C. Fogedby. Lévy flights in quenched random force fields. *Physical Review E*, 58:1690–1712, 1998.

[44] A. D. Fokker. Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld. *Annalen der Physik*, 348:810–820, 1914.

[45] E. Frey and K. Kroy. Brownian motion: a paradigm of soft matter and biological physics. *Annalen der Physik*, 14:20–50, 2005.

[46] R. Friedrich, F. Jenko, A. Baule, and S. Eule. Anomalous diffusion of inertial, weakly damped particles. *Physical Review Letters*, 96:230601, 2006.

[47] G. Fusai and A. Roncoroni. *Implementing Models in Quantitative Finance: Methods and Cases*. Springer, 2008.

[48] J. Gajda and A. Wyłomańska. Geometric Brownian motion with tempered stable waiting times. *Journal of Statistical Physics*, 148:296–305, 2012.

[49] C. Gardiner. *Stochastic Methods: A Handbook for the Natural and Social Sciences.* Springer Berlin Heidelberg, 2010.

[50] G. Germano, M. Politi, E. Scalas, and R. L. Schilling. Itô and Stratonovich integrals on compound renewal processes: the normal/Poisson case. *Communications in Nonlinear Science and Numerical Simulation*, 15:1583–1588, 2010.

[51] R. Gorenflo, F. Mainardi, E. Scalas, and M. Raberto. *Fractional Calculus and Continuous-Time Finance III : the Diffusion Limit*, pages 171–180. Birkhäuser Basel, 2001.

[52] P. Hänggi and F. Marchesoni. Introduction: 100 years of Brownian motion. *Chaos*, 15:026101, 2005.

[53] P. Hänggi and H. Thomas. Time evolution, correlations, and linear response of non-Markov processes. *Zeitschrift für Physik B Condensed Matter*, 26:85–92, 1977.

[54] M. D. Haw. Colloidal suspensions, Brownian motion, molecular reality: a short history. *Journal of Physics: Condensed Matter*, 14:7769–7779, 2002.

[55] A. Helmstetter and D. Sornette. Diffusion of epicenters of earthquake aftershocks, Omori's law, and generalized continuous-time random walk models. *Physical Review E*, 66:061104, 2002.

[56] S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

[57] D. J. Higham. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Review*, 43:525–546, 2001.

[58] F. Höfling and T. Franosch. Anomalous transport in the crowded world of biological cells. *Reports on Progress in Physics*, 76:046602, 2013.

[59] B. Holdom. From turbulence to financial time series. *Physica A*, 254:569–576, 1998.

[60] J. C. Hull. *Options, Futures, and Other Derivatives.* Pearson Education, 2006.

[61] T. R. Hurd and Z. Zhou. A Fourier transform method for spread option pricing. *SIAM Journal on Financial Mathematics*, 1:142–157, 2010.

[62] K. Itô. Differential equations determining Markov processes. *Zenkoku Shijo Sugaku Danwakai*, 244:1352–1400, 1942.

[63] P. C. Ivanov, A. Yuen, B. Podobnik, and Y. Lee. Common scaling patterns in intertrade times of us stocks. *Physical Review E*, 69:056107, 2004.

[64] F. Jacob. *Risk Estimation on High Frequency Financial Data: Empirical Analysis of the DAX 30.* Springer, 2015.

[65] J. Janczura, S. Orzeł, and A. Wyłomańska. Subordinated α-stable Ornstein–Uhlenbeck process as a tool for financial data description. *Physica A*, 390:4379–4387, 2011.

[66] J. Janczura and A. Wyłomańska. Subdynamics of financial data from fractional Fokker-Planck equation. *Acta Physica Polonica B*, 40:1341–1351, 2009.

[67] J. Janczura and A. Wyłomańska. Anomalous diffusion models: different types of subordinator distribution. *Acta Physica Polonica B*, 43:1000–1016, 2012.

[68] R. Jarrow and P. Protter. A short history of stochastic integration and mathematical finance: the early years, 1880-1970. *Lecture Notes-Monograph Series*, 45:75–91, 2004.

[69] J. H. Jeon, V. Tejedor, S. Burov, E. Barkai, C. Selhuber-Unkel, K. Berg-Sørensen, L. Oddershede, and R. Metzler. In vivo anomalous diffusion and weak ergodicity breaking of lipid granules. *Physical Review Letters*, 106:048103, 2011.

[70] M. Kac. Random walk and the theory of Brownian motion. *American Mathematical Monthly*, 54:369–391, 1947.

[71] T. Kaizoji and M. Kaizoji. Power law for the calm-time interval of price changes. *Physica A*, 336:563–570, 2004.

[72] E. Kappler. Versuche zur Messung der Avogadro-Loschmidtschen Zahl aus der Brownschen Bewegung einer Drehwaage. *Annalen der Physik*, 403:233–256, 1931.

[73] K. Kim and S. M. Yoon. Dynamical behavior of continuous tick data in futures exchange market. *Fractals*, 11:131–136, 2003.

[74] J. L. Kirkby. Efficient option pricing by frame duality with the fast Fourier transform. *SIAM Journal on Financial Mathematics*, 6:713–747, 2015.

[75] J. Klafter, M. F. Shlesinger, and G. Zumofen. Beyond Brownian motion. *Physics Today*, 49:33–39, 1996.

[76] R. Klages, G. Radons, and I. M. Sokolov. *Anomalous Transport: Foundations and Applications*. Wiley, 2008.

[77] H. Kleinert. Option pricing from path integral for non-gaussian fluctuations. natural martingale and application to truncated Lévy distributions. *Physica A*, 312:217–242, 2002.

[78] D. Kleinhans and R. Friedrich. Continuous-time random walks: simulation of continuous trajectories. *Physical Review E*, 76:061102, 2007.

[79] R. V. Kohn. PDE for finance notes, spring 2011-section 1. 2011.

[80] R. Kolb and J. Overdahl. *Financial Derivatives*. Wiley, 2003.

[81] I. Koponen. Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process. *Physical Review E*, 52:1197–1199, 1995.

[82] T. Koren, A. Chechkin, and J. Klafter. On the first passage time and leapover properties of Lévy motions. *Physica A*, 379:10–22, 2007.

[83] S. G. Kou. Jump-diffusion models for asset pricing in financial engineering. *Handbooks in Operations Research and Management Science*, 15:73–116, 2007.

[84] A. E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications.* Springer Berlin Heidelberg, 2006.

[85] P. Langevin. Sur la théorie du mouvement brownien. *CR Acad. Sci. Paris*, 146:530–533, 1908.

[86] J. W. Lee, K. E. Lee, and P. A. Rikvold. Waiting-time distribution for korean stock-market index kospi. *Journal of the Korean Physical Society*, 48:S123–S126, 2006.

[87] J. D. MacBeth and L. J. Merville. An empirical examination of the Black-Scholes call option pricing model. *Journal of Finance*, 34:1173–1186, 1979.

[88] M. Magdziarz. Black-Scholes formula in subdiffusive regime. *Journal of Statistical Physics*, 136:553–564, 2009.

[89] M. Magdziarz. Path properties of subdiffusion a martingale approach. *Stochastic Models*, 26:256–271, 2010.

[90] M. Magdziarz and J. Gajda. Anomalous dynamics of Black-Scholes model time-changed by inverse subordinators. *Acta Physica Polonica B*, 43:1093–1110, 2012.

[91] M. Magdziarz, S. Orzeł, and A. Weron. Option pricing in subdiffusive Bachelier model. *Journal of Statistical Physics*, 145:187–203, 2011.

[92] M. Magdziarz and R. Schilling. Asymptotic properties of Brownian motion delayed by inverse subordinators. *Proceedings of the American Mathematical Society*, 143:4485–4501, 2015.

[93] M. Magdziarz and A. Weron. Competition between subdiffusion and Lévy flights: a Monte Carlo approach. *Physical Review E*, 75:056702, 2007.

[94] F. Mainardi, M. Raberto, R. Gorenflo, and E. Scalas. Fractional calculus and continuous-time finance II: the waiting-time distribution. *Physica A*, 287:468–481, 2000.

[95] S. W. Malone. Alternative price processes for Black-Scholes: Empirical evidence and theory. 2002.

[96] B. Mandelbrot and H. M. Taylor. On the distribution of stock price differences. *Operations Research*, 15:1057–1062, 1967.

[97] B. B. Mandelbrot. The variation of certain speculative prices. *Journal of Business*, 36:394–419, 1963.

[98] B. B. Mandelbrot and R. L. Hudson. *The Misbehaviour of Markets: A Fractal View of Risk, Ruin and Reward*. Profile Books, 2004.

[99] R. N. Mantegna and H. E. Stanley. Scaling behaviour in the dynamics of an economic index. *Nature*, 376:46–49, 1995.

[100] R. N. Mantegna and H. E. Stanley. *Introduction to Econophysics: Correlations and Complexity in Finance*. Cambridge University Press, 1999.

[101] J. Masoliver, M. Montero, J. Perelló, and G. H. Weiss. The continuous time random walk formalism in financial markets. *Journal of Economic Behavior & Organization*, 61:577–598, 2006.

[102] J. Masoliver, M. Montero, and G. H. Weiss. Continuous-time random-walk model for financial distributions. *Physical Review E*, 67:021112, 2003.

[103] A. Matacz. Financial modeling and option theory with the truncated Lévy process. *International Journal of Theoretical and Applied Finance*, 3:143–160, 2000.

[104] M. M. Meerschaert and P. Straka. Inverse stable subordinators. *Mathematical Modelling of Natural Phenomena*, 8:1–16, 2013.

[105] R. C. Merton. Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.

[106] R. C. Merton. *Continuous-Time Finance*. Wiley, 1992.

[107] R. C. Merton, R. V. Simons, and A. D. Wilkie. Influence of mathematical models in finance on practice: past, present and future. *Philosophical Transactions of the Royal Society London A*, 347:451–463, 1994.

[108] R. Metzler and J. Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339:1–77, 2000.

[109] R. Metzler and J. Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *Journal of Physics A: Mathematical and General*, 37:R161–R208, 2004.

[110] T. Michael and S. Ravi. Black-Scholes option pricing formula.

[111] S. Mitra. A review of volatility and option pricing. *International Journal of Financial Markets and Derivatives*, 2:149–179, 2011.

[112] G. M. Mittag-Leffler. Sur la nouvelle fonction $e_\alpha(x)$. *CR Acad. Sci. Paris*, 137:554–558, 1903.

[113] G. M. Mittag-Leffler. Sur la representation analytique dune branche uniforme dune fonction monogene. *Acta Mathematica*, 29:101–181, 1905.

[114] M. Montero. Perpetual American options within CTRWs. *Physica A*, 387:3936–3941, 2008.

[115] M. Montero. Renewal equations for option pricing. *European Physical Journal B*, 65:295–306, 2008.

[116] M. Montero, J. Perelló, J. Masoliver, F. Lillo, S. Miccichè, and R. N. Mantegna. Scaling and data collapse for the mean exit time of asset prices. *Physical Review E*, 72:056101, 2005.

[117] E. W. Montroll and G. H. Weiss. Random walks on lattices. ii. *Journal of Mathematical Physics*, 6:167–181, 1965.

[118] K. Murase, T. Fujiwara, Y. Umemura, K. Suzuki, R. Iino, H. Yamashita, M. Saito, H. Murakoshi, K. Ritchie, and A. Kusumi. Ultrafine membrane compartments for molecular diffusion as revealed by single molecule techniques. *Biophysical Journal*, 86:4075–4093, 2004.

[119] E. Nelson. *Dynamical theories of Brownian motion.* Princeton University Press,, 1967.

[120] S. Orzeł and A. Weron. Calibration of the subdiffusive Black-Scholes model. *Acta Physica Polonica B*, 41:1151–1159, 2010.

[121] M. F. M. Osborne. Brownian motion in the stock market. *Operations Research*, 7:145–173, 1959.

[122] P. Pearle, B. Collett, K. Bart, D. Bilderback, D. Newman, and S. Samuels. What Brown saw and you can too. *American Journal of Physics*, 78:1278–1289, 2010.

[123] J. Perrin. Lagitation moléculaire et le mouvement brownien. *Comptes rendus*, 146:967–970, 1908.

[124] J. Perrin. Mouvement brownien et réalité moléculaire. *Annales de Chimie et de Physique*, 18:5–104, 1909.

[125] J. Perrin. *Brownian Movement and Molecular Reality.* Taylor and Francis, 1910.

[126] A. Piryatinska, A. I. Saichev, and W. A. Woyczynski. Models of anomalous diffusion: the subdiffusive case. *Physica A*, 349:375–420, 2005.

[127] M. Planck. An essay on statistical dynamics and its amplification in the quantum theory. *Sitz. Ber. Preuss. Akad. Wiss*, 325:324–341, 1917.

[128] M. Platani, I. Goldberg, A. I. Lamond, and J. R. Swedlow. Cajal Body dynamics and association with chromatin are ATP-dependent. *Nature Cell Biology*, 4:502–508, 2002.

[129] V. Plerou, P. Gopikrishnan, L. A. N. Amaral, X. Gabaix, and H. E. Stanley. Economic fluctuations and anomalous diffusion. *Physical Review E*, 62:R3023, 2000.

[130] I. Podlubny. *Fractional Differential Equations.* Academic Press, 1999.

[131] S. J. Press. A compound events model for security prices. *Journal of Business*, 40:317–335, 1967.

[132] M. Raberto, E. Scalas, and F. Mainardi. Waiting-times and returns in high-frequency financial data: an empirical study. *Physica A*, 314:749–755, 2002.

[133] L. Rayleigh. Steady motion in a corner of a viscous fluid. *Scientific Papers*, 6:8–21, 1920.

[134] L. F. Richardson. Atmospheric diffusion shown on a distance-neighbour graph. *Proceedings of the Royal Society of London.*, 110:709, 1926.

[135] P. Richmond, J. Mimkes, and S. Hutzler. *Econophysics and Physical Economics.* Oxford University Press, 2013.

[136] H. Risken. *Fokker-Planck Equation*. Springer, Berlin, 1998.

[137] L. C. G. Rogers and Z. Shi. The value of an Asian option. *Journal of Applied Probability*, 32:1077–1088, 1995.

[138] J. Rosiński. Tempering stable processes. *Stochastic Processes and their Applications*, 117:677–707, 2007.

[139] S. M. Ross. *An Elementary Introduction to Mathematical Finance*. Cambridge University Press, 2011.

[140] S. M. Ross. *Introduction to Probability Models*. Academic Press, 2014.

[141] L. Sabatelli, S. Keating, J. Dudley, and P. Richmond. Waiting time distributions in financial markets. *European Physical Journal B*, 27:273–275, 2002.

[142] G. Samoradnitsky and M. S. Taqqu. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Taylor & Francis, 1994.

[143] P. A. Samuelson. Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review*, 6:41–49, 1965.

[144] P. A. Samuelson. Mathematics of speculative price. *SIAM Review*, 15:1–42, 1973.

[145] E. Scalas. The application of continuous-time random walks in finance and economics. *Physica A*, 362:225–239, 2006.

[146] E. Scalas. Five years of continuous-time random walks in econophysics. In A. Namatame, T. Kaizouji, and Y. Aruka, editors, *The Complex Networks of Economic Interactions: Essays in Agent-Based Economics and Econophysics*, pages 3–16. Springer, Berlin, 2006.

[147] E. Scalas, R. Gorenflo, H. Luckock, F. Mainardi, M. Mantelli, and M. Raberto. Anomalous waiting times in high-frequency financial data. *Quantitative Finance*, 4:695–702, 2004.

[148] E. Scalas, R. Gorenflo, and F. Mainardi. Fractional calculus and continuous-time finance. *Physica A*, 284:376–384, 2000.

[149] E. Scalas, T. Kaizoji, M. Kirchler, J. Huber, and A. Tedeschi. Waiting times between orders and trades in double-auction markets. *Physica A*, 366:463–471, 2006.

[150] H. Scher and E. W. Montroll. Anomalous transit-time dispersion in amorphous solids. *Physical Review B*, 12:2455–2477, 1975.

[151] L. O. Scott. Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: applications of Fourier inversion methods. *Mathematical Finance*, 7:413–426, 1997.

[152] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter. Strange kinetics. *Nature*, 363:31–37, 1993.

[153] S. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer, 2004.

[154] A. Stanislavsky, K. Weron, and A. Weron. Diffusion and relaxation controlled by tempered α-stable processes. *Physical Review E*, 78:051106, 2008.

[155] A. A. Stanislavsky. Black–Scholes model under subordination. *Physica A*, 318:469–474, 2003.

[156] E. J. Sullivan and T. M. Weithers. Louis Bachelier: The father of modern option pricing theory. *Journal of Economic Education*, 22:165–171, 1991.

[157] A. Talbot. The accurate numerical inversion of Laplace transforms. *IMA Journal of Applied Mathematics*, 23:97–120, 1979.

[158] J. L. Teall. *Financial Market Analytics*. Quorum Books, 1999.

[159] M. Teuerle, A. Wyłomańska, and G. Sikora. Modeling anomalous diffusion by a subordinated fractional Lévy-stable process. *Journal of Statistical Mechanics: Theory and Experiment*, 2013:P05016, 2013.

[160] L. Turgeman, S. Carmi, and E. Barkai. Fractional feynman-kac equation for non-brownian functionals. *Physical Review Letters*, 103:190201, 2009.

[161] P. P. Valkó and J. Abate. Numerical inversion of 2-D Laplace transforms applied to fractional diffusion equations. *Applied Numerical Mathematics*, 53:73–88, 2005.

[162] A. Wiman. Über den Fundamentalsatz in der Teorie der Funktionen $E_\alpha(x)$. *Acta Mathematica*, 29:191–201, 1905.

[163] W. L. Xiao, W. G. Zhang, X. L. Zhang, and Y. L. Wang. Pricing currency options in a fractional Brownian motion with jumps. *Economic Modelling*, 27:935–942, 2010.

[164] O. Zhylyevskyy. A fast Fourier transform technique for pricing American options under stochastic volatility. *Review of Derivatives Research*, 13:1–24, 2010.

List of Abbreviations

PDF	probability density function
SDE	stochastic differential equation
GBM	geometric Brownian motion
BS	Black-Scholes
PDE	partial differential equation
ODE	ordinary differential equation
CTRW	continuous time random walk