Conserved Quantities in Lemaître-Tolman-Bondi Cosmology

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(Dated: October 20, 2014)

We study linear perturbations to a Lemaître-Tolman-Bondi (LTB) background spacetime. Studying the transformation behaviour of the perturbations under gauge transformations, we construct gauge invariant quantities. We show, using the perturbed energy conservation equation, that there are conserved quantities in LTB, in particular a spatial metric trace perturbation, $\zeta_{\rm SMTP}$, which is conserved on all scales. We then briefly extend our discussion to the Lemaître spacetime, and construct gauge-invariant perturbations in this extension of LTB spacetime.

I. INTRODUCTION

Conserved quantities are useful tools with a wide range of applications in cosmology. In particular, they allow us to relate early and late times in a cosmological model, without explicitly having to solve the evolution equations, either exactly or taking advantage of some limiting behaviour. These quantities have been studied extensively within the context of cosmological perturbation theory, and usually applied to a Friedmann-Robertson-Walker(FRW) background spacetime.

Using metric based cosmological perturbation theory [1, 2], we can readily construct gauge-invariant quantities which are also conserved, that is constant in time (see e.g. Ref. [3] for early work on this topic). In a FRW background spacetime, ζ, the curvature perturbation on uniform density hypersurfaces, is conserved on large scales for adiabatic fluids. To show that ζ is conserved and under what conditions, we only need the conservation of energy [4]. This was first shown to work for fluids at linear order, but it holds also at second order in the perturbations, and in the fully non-linear case, usually referred to as the δN formalism [4–6][1].

Instead of, or in addition to, cosmological perturbation theory, we can also use other approximation schemes to deal with the non-linearity of the Einstein equations. In particular gradient expansion schemes have proven to be useful in the context of conserved quantities, again with the main focus on FRW spacetimes [6–9]. But conserved quantities have also been studied for spacetimes other than FRW, such as braneworld models (see e.g. Ref. [10], and anisotropic spacetime (e.g. Ref. [11]).

The Lemaître-Tolman-Bondi (LTB) spacetime [12] is a more general solution to Einstein's field equations than the Friedmann-Robertson-Walker (FRW) model. While LTB is invariant under rotations, FRW is rotation and translations invariant, and hence has homogeneous and isotropic, maximally symmetric spatial sections [13].

Recent research into LTB cosmology has been motivated by seeking an alternative explanation for the late time accelerated expansion of the universe, as indicated by e.g. SNIa observations [14]. Inhomogeneous cosmologies, including LTB, have been suggested as such an alternative explanation of these observations (see e.g. Refs. [15, 16]). Other observations such as galaxy surveys, large scale structure surveys, the CMB and indeed any redshift dependent observations (see for example Refs. [17], [18], [19]) are usually interpreted assuming a flat FRW cosmology - isotropic and homogeneous on large scales. In order to test the validity of this assumption other, inhomogeneous, cosmologies such as LTB should also be considered. Consequently there is much active research into LTB and other inhomogeneous spherically symmetric cosmologies, both at background order and with perturbations (see e.g. Refs. [20–33] for theory and comparison with observation in general, see e.g. Refs. [34–39] for research relating to CMB and see e.g. Refs. [40–48] for research more specific to the kinetic Sunyaev-Zeldovich effect, see e.g. Refs. [49], [50], [51] for structure formation in LTB, including N-body simulations).

Gauge-invariant perturbations in general spherically symmetric spacetimes have been studied already in the 1970s by Gerlach and Sengupta [52, 53], using a 2+2 split on the background spacetime. Recent works studying perturbed LTB spacetimes performs a 1+1+2 split (see e.g. Refs. [54], [55], [56]). These splits allow for a decomposition of the tensorial quantities on the submanifolds into axial and polar scalars and vectors, similar to the scalar-vector-tensor decomposition in FRW [1, 2]. In this work we perform a 1+3 split of spacetime, without further decomposing the spatial submanifold. This prevents us from decomposing tensorial quantities on the spatial submanifold further into

[1] How the gauge invariant curvature perturbation ζ is constructed is briefly discussed in Subsection III D 1.

axial and polar scalars and vectors, but provides us with much simpler expressions, well suited for the construction of conserved quantities.

We study systematically how to construct gauge-invariant quantities in perturbed LTB spacetimes. While the existence of conserved quantities in perturbed spacetimes in general is well known, in this paper a conserved gauge invariant quantity, $\zeta_{\rm SMTP}$ is constructed for the first time in LTB (and Lemaître) spacetimes. To this end we derive the transformation rules for matter and metric variables under small coordinate- or gauge-transformations. We use these derived gauge-invariant quantities. We also derive the perturbed energy density evolution equation, which allows us to derive a very simple evolution equation for the spatial metric perturbation on uniform density and comoving hypersurfaces, $\zeta_{\rm SMTP}$. The differences between $\zeta_{\rm SMTP}$ and ζ in FRW will be discussed in Subsection III D 2. The evolution equation for $\zeta_{\rm SMTP}$ and the conditions under which this variable is conserved will be discussed in Subsection III E.

The paper is structured as follows. In the next section we present the governing equations in covariant form. In Section III we review standard Lemaître-Tolman-Bondi cosmology at the background level. We then extend the standard results by adding perturbations to the Lemaître-Tolman-Bondi background. We also look at the transformation behaviour of matter and metric variables and construct gauge invariant quantities, including a small review of the procedures applied to FRW. Finally in this section we look at the evolution of the spatial metric trace perturbation, $\zeta_{\rm SMTP}$. In Section IV we briefly extend our work to the Lemaître spacetime in order to allow non-zero background pressure. Finally in Section V we discuss the implications of our work for studies of Lemaître-Tolman-Bondi spacetime, and the evolution of structure therein.

II. GOVERNING EQUATIONS

In this section we present the governing equations in covariant form, that is we do not assume a particular background spacetime, and do not split quantities into background and perturbations. However, we assume Einstein's General Relativity (here and throughout this paper). We also define some covariant quantities that will be useful in later sections.

Einstein's field equations are given by

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ the energy-momentum tensor, and G is Newton's constant. The energy-momentum tensor for a perfect fluid is given by,

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P\, g^{\mu\nu}\,, \tag{2.2}$$

where $g^{\mu\nu}$ is the metric tensor, ρ is the energy density, P the pressure, and u^μ the 4-velocity of the fluid. The metric tensor is subject to the constraint,

$$g^{\mu\nu} g_{\nu\gamma} = \delta^\mu_\gamma, \tag{2.3}$$

where δ^μ_γ is the Kronecker delta. The 4-velocity is defined by,

$$u^\mu = \frac{dx^\mu}{d\tau}\,, \tag{2.4}$$

where τ is the proper time along the curves to which u^μ is tangent, related to the line element ds by

$$ds^2 = -d\tau^2\,. \tag{2.5}$$

The 4-velocity is subject to the constraint,

$$u^\mu u_\mu = -1\,. \tag{2.6}$$

We get energy-momentum conservation from the Einstein equations, Eq. (2.1), through the Bianchi identities,

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{2.7}$$

The metric tensor allows us to define a unit time-like vector field orthogonal to constant-time hypersurfaces,

$$n_\mu \propto \frac{\partial t}{\partial x^\mu}\,, \tag{2.8}$$

subject to the constraint

$$n^\mu n_\mu = -1 \,. \tag{2.9}$$

The covariant derivative of any 4-vector can be decomposed as (see for example [13], [57]),

$$\nabla_\mu n_\nu = -n_\mu u^\alpha \nabla_\alpha n_\nu + \frac{1}{3}\Theta_n \mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \,, \tag{2.10}$$

where we use the unit normal vector, n^μ, purely as an example, since Eq. (2.10) is true for any 4-vector e.g the 4-velocity, u^μ. Here Θ_n is the expansion factor, $\sigma_{\mu\nu}$ the shear tensor, $\omega_{\mu\nu}$ the vorticity tensor, and $\mathcal{P}_{\mu\nu}$ is the spatial projection tensor.

The expansion factor defined with respect to the unit normal vector is,

$$\Theta_n = \nabla_\mu n^\mu \,, \tag{2.11}$$

the shear, $\sigma_{\mu\nu}$, is given by,

$$\sigma_{\mu\nu} = \frac{1}{2}\mathcal{P}^\alpha_\mu \mathcal{P}^\beta_\nu \left(\nabla_\beta n_\alpha + \nabla_\alpha n_\beta\right) - \frac{1}{3}\Theta_n \mathcal{P}_{\mu\nu} \,, \tag{2.12}$$

where the spatial projection tensor is defined as

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \,. \tag{2.13}$$

III. LEMAÎTRE-TOLMAN-BONDI SPACETIME

In this section we first briefly review standard Lemaître-Tolman-Bondi (LTB) cosmology at the background level. We then extend the standard results by adding perturbations to the LTB background. In order to remove any unwanted gauge modes, we study the transformation behaviour of the perturbations, which then allows us to construct gauge-invariant quantities, in particular the equivalent to the curvature perturbation. We show under which conditions this curvature perturbation is conserved.

Throughout this section we assume zero pressure in the background spacetime (see Section IV for the addition of non-zero background pressure). We do however allow for a pressure perturbation in the later subsections.

A. Background

The LTB metric can be written in various forms [12], [58], [54]. Here we shall use the following form of the metric [12], [13],

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + Y^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{3.1}$$

where X and Y are scale factors dependent upon both the radial spatial and time co-ordinates and are not independent, and the indices $0,1,2,3$ are t,r,θ,ϕ respectively. The scale factors are related by,

$$X = \frac{1}{W(r)}\frac{\partial Y}{\partial r} , \tag{3.2}$$

where $W(r)$ is an arbitrary function of r, following Bondi [12], arising from the Einstein field equations.

The 4-velocity in the background is given from its definition, Eq. (2.4), as

$$u^\mu = [1,0,0,0] \,, \tag{3.3}$$

since we assume we are comoving with respect to the background coordinates and hence, $dr = d\theta = d\phi = 0$, and therefore $d\tau^2 = dt^2$ (that is in the local rest frame).

From the definition of the energy-momentum tensor, Eq. (2.2), we immediately find that in the absence of pressure the only non-zero component is, $T^{00} = \rho$. For later convenience we define Hubble parameter equivalents for our two scale factors such that,

$$H_X = \frac{\dot{X}}{X} \,, \qquad H_Y = \frac{\dot{Y}}{Y} \,. \tag{3.4}$$

where the "dot" denotes the derivative with respect to coordinate time t.

The Einstein equations are, from Eq. (2.1), for the $0-0$ component,

$$\frac{1}{Y^2}+{H_Y}^2+2\frac{X'Y'}{X^3Y}+2H_XH_Y-\left(\frac{Y'}{XY}\right)^2-2\frac{Y''}{X^2Y}=8\pi G\rho\,, \tag{3.5}$$

where a prime denotes a derivative with respect to the radial coordinate r. For the $0-r$ component we find,

$$\frac{2}{Y}\left(Y'H_X-\dot{Y}'\right)=0\,, \tag{3.6}$$

for the $r-r$ component,

$$\left(\frac{Y'}{XY}\right)^2-\frac{1}{Y^2}-H_Y^2-2\frac{\ddot{Y}}{Y}=0\,, \tag{3.7}$$

and for $\theta-\theta$ and $\phi-\phi$ components we get,

$$\frac{Y''}{X^2Y}-\frac{X'Y'}{YX^2}-\frac{\ddot{Y}}{Y}-H_XH_Y-\frac{\ddot{X}}{X}=0\,. \tag{3.8}$$

The other components are identically zero. The energy conservation equation, obtained from Eq. (2.7), is

$$\dot{\rho}+\rho(H_X+2H_Y)=0\,. \tag{3.9}$$

B. Perturbations

In this section we add perturbations to the LTB background. Unlike recent works studying perturbed LTB models, e.g. Refs. [54], we do not decompose the perturbations into polar and axial scalars and vectors, and multi-poles, which considerably simplifies our governing equations.

We split quantities into a t and r dependent background part, and a perturbation depending on all four coordinates. For example we decompose the energy density ρ as follows,

$$\rho=\bar{\rho}(t,r)+\delta\rho(x^\mu)\,, \tag{3.10}$$

where here and in the following a "bar" denotes a background quantity, if there is a possibility for confusion.

We perturb the metric in a similar way as in the flat FRW case, the LTB metric being very similar to flat FRW in spherical polar coordinates, save for the two scale factors and the factor of r being absorbed into Y.

Hence we split the metric tensor as

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+\delta g_{\mu\nu}, \tag{3.11}$$

where $\bar{g}_{\mu\nu}$ is given by Eq. (3.1). For the perturbed part of the metric, $\delta g_{\mu\nu}$, we make the ansatz,

$$\delta g_{\mu\nu}=\begin{pmatrix} -2\Phi & XB_r & YB_\theta & Y\sin\theta B_\phi \\ XB_r & 2X^2C_{rr} & XYC_{r\theta} & XY\sin\theta C_{r\phi} \\ YB_\theta & XYC_{r\theta} & 2Y^2C_{\theta\theta} & Y^2\sin\theta C_{\theta\phi} \\ Y\sin\theta B_\phi & XY\sin\theta C_{r\phi} & Y^2\sin\theta C_{\theta\phi} & 2Y^2\sin^2\theta C_{\phi\phi}\end{pmatrix}\,. \tag{3.12}$$

Here Φ is the lapse function, and B_n, where $n=r,\theta,\phi$, are the shift functions for each spatial coordinate. Similarly, C_{nm}, where $n,m=r,\theta,\phi$, are the spatial metric perturbations. As already pointed out, we do not decompose B_n and C_{nm} further into scalar and vector perturbations (see however Ref. [54]).

Using the perturbed metric we can construct the perturbed 4-velocities using the definition, Eq. (2.4). Proper time is to linear order in the perturbations given by,

$$d\tau=(1+\Phi)dt\,, \tag{3.13}$$

and defining the 3-velocity as $v^i=\frac{dx^i}{dt}$, from Eq. (2.4) we get the contravariant 4-velocity vector,

$$u^\mu=[(1-\Phi),v^r,v^\theta,v^\phi]. \tag{3.14}$$

By lowering the index using the perturbed metric we obtain the covariant form,

$$u_\mu = \left[-(1+\Phi), X\left(B_r + Xv^r\right), Y\left(B_\theta + Yv^\theta\right), Y\sin(\theta)\left(B_\phi + Y\sin(\theta)v^\phi\right)\right]. \tag{3.15}$$

Conservation of the energy-momentum tensor, Eq. (2.7), allows us together with its definition, Eq. (2.2), to derive the perturbed energy conservation equation,

$$\delta\dot{\rho} + (\delta\rho + \delta P)\left(H_X + 2H_Y\right) + \bar{\rho}' v^r + \bar{\rho}\left(\dot{C}_{rr} + \dot{C}_{\theta\theta} + \dot{C}_{\phi\phi} + v^{r\prime} + \partial_\theta v^\theta + \partial_\phi v^\phi + \left[\frac{X'}{X} + 2\frac{Y'}{Y}\right] v^r + \cot\theta v^\theta\right) = 0, \tag{3.16}$$

where we used Eq. (3.10), and the LTB background requires $\bar{P} = 0$. The perturbed momentum conservation equations are

$$\dot{\bar{\rho}} v^r + \bar{\rho}(\dot{v}^r + \frac{\dot{B}_r}{X} + \frac{B_r}{X} H_X + (3H_X + 2H_Y)v^r) + \frac{1}{X^2}\delta P' = 0\,, \tag{3.17}$$

$$\dot{\bar{\rho}} v^\theta + \bar{\rho}(\dot{v}^\theta + \frac{\dot{B}_\theta}{Y} + \frac{B_\theta}{Y} H_Y + (H_X + 4H_Y)v^\theta) + \frac{1}{Y^2}\partial_\theta \delta P = 0\,, \tag{3.18}$$

$$\dot{\bar{\rho}} v^\phi + \bar{\rho}\left(\dot{v}^\phi + \frac{\dot{B}_\phi}{Y\sin\theta} + \frac{B_\phi H_Y}{Y\sin\theta} + (H_X + 4H_Y)v^\phi\right) + \frac{1}{Y^2\sin^2\theta}\partial_\phi \delta P = 0\,, \tag{3.19}$$

which we do not use in this work.

C. Gauge Transformation

In order to construct gauge-invariant perturbations, we have to study the transformation behaviour of our matter and metric variables. Here and in the following, we use the Bardeen approach to cosmological perturbation theory [1, 2]. For a general discussion and references to primary literature, we refer the reader to Ref. [59] and references therein.

Using the active point of view, linear order perturbations of a tensorial quantity $\mathbf{T}$ transform as

$$\delta\tilde{\mathbf{T}} = \delta\mathbf{T} + \pounds_{\delta x^\mu}\bar{\mathbf{T}}\,, \tag{3.20}$$

where the tilde denotes quantities evaluated in the "new" coordinate system.The old and the new coordinate systems are related by

$$\tilde{x}^\mu = x^\mu + \delta x^\mu\,, \tag{3.21}$$

where $\delta x^\mu = [\delta t, \delta x^i]$ is the gauge generator. The Lie derivative is denoted by $\pounds_{\delta x^\mu}$.

1. *Metric and Matter Quantities*

From Eq. (3.20) and Eq. (3.10) we find that the density perturbation transforms simply as,

$$\delta\tilde{\rho} = \delta\rho + \dot{\bar{\rho}}\delta t + \bar{\rho}'\delta r\,, \tag{3.22}$$

since the background energy density depends on t and r. The perturbed spatial part of the 4-velocities, defined in Eq. (3.14) transform as,

$$\tilde{v}^i = v^i - \dot{\delta x^i}\,, \tag{3.23}$$

where $i = r, \theta, \phi$.

The perturbed metric transforms, using Eq. (3.20), as

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \delta x^\gamma \partial_\gamma \bar{g}_{\mu\nu} + \bar{g}_{\gamma\nu}\partial_\mu \delta x^\gamma + \bar{g}_{\mu\gamma}\partial_\nu \delta x^\gamma. \tag{3.24}$$

From the $0-0$ component of Eq. (3.24) we find that the lapse function transforms as

$$\tilde{\Phi} = \Phi - \delta\dot{t}\,. \tag{3.25}$$

For the perturbations on the spatial trace part of the metric we find for the r coordinate from Eq. (3.24),

$$\tilde{C}_{rr} = C_{rr} + \delta t\frac{\dot{X}}{X} + \delta r\frac{X'}{X} + \delta r'\,, \tag{3.26}$$

for the θ coordinate,

$$\tilde{C}_{\theta\theta} = C_{\theta\theta} + \delta t\frac{\dot{Y}}{Y} + \delta r\frac{Y'}{Y} + \partial_\theta\delta\theta\,, \tag{3.27}$$

and for the ϕ coordinate,

$$\tilde{C}_{\phi\phi} = C_{\phi\phi} + \delta t\frac{\dot{Y}}{Y} + \delta r\frac{Y'}{Y} + \delta\theta\cot\theta + \partial_\phi\delta\phi\,. \tag{3.28}$$

For later convenience we define a spatial metric perturbation, ψ, as,

$$3\psi = \frac{1}{2}\delta g^k_k = C_{rr} + C_{\theta\theta} + C_{\phi\phi}\,, \tag{3.29}$$

that is the trace of the spatial metric, in analogy with the curvature perturbation $\psi_{\rm FRW}$ in perturbed FRW spacetimes (see Section III D 1 below). The relation between ψ here and the curvature perturbation in perturbed FRW can be most easily seen from the perturbed expansion scalar, given in Eq. (3.37) below, which is very similar to its FRW counterpart (see e.g. Ref. [59], Eq. (3.19)). The relation is not obvious from calculating the spatial Ricci scalar for the perturbed LTB spacetime, as can be seen from Eq. (A10), given in the appendix.

From the above ψ transforms as

$$3\tilde{\psi} = 3\psi + \left[\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right]\delta t + \left[\frac{X'}{X} + 2\frac{Y'}{Y}\right]\delta r + \partial_i\delta x^i + \delta\theta\cot\theta\,, \tag{3.30}$$

where $i = r, \theta, \phi$. In addition, from Eq. (3.24) the off diagonal spatial metric perturbations transform as,

$$\begin{aligned}
\tilde{C}_{r\theta} &= C_{r\theta} + \frac{Y}{X}\delta\theta' + \frac{X}{Y}\partial_\theta\delta r\,, && (3.31)\\
\tilde{C}_{r\phi} &= C_{r\phi} + \frac{Y\sin\theta}{X}\delta\phi' + \frac{X}{Y\sin\theta}\partial_\phi\delta r\,, && (3.32)\\
\tilde{C}_{\theta\phi} &= C_{\theta\phi} + \frac{\sin\theta}{X}\partial_\theta\delta\phi + \frac{1}{\sin\theta}\partial_\phi\delta\theta\,. && (3.33)
\end{aligned}$$

The mixed temporal-spatial perturbations of the metric, that is the shift vector, from Eq. (3.24) transform as

$$\begin{aligned}
\tilde{B}_r &= B_r + X\dot{\delta r} - \frac{\delta t'}{X}\,, && (3.34)\\
\tilde{B}_\theta &= B_\theta + Y\dot{\delta\theta} - \frac{\partial_\theta\delta t}{Y}\,, && (3.35)\\
\tilde{B}_\phi &= B_\phi + Y(\sin\theta)\dot{\delta\phi} - \frac{\partial_\phi\delta t}{Y(\sin\theta)}\,. && (3.36)
\end{aligned}$$

2. *Geometric Quantities*

The expansion scalar, defined in Eq. (2.11) with u^μ in place of n^μ, calculated from the 4-velocity, given in Eq. (3.14), is,

$$\Theta = (H_X + 2H_Y) + 3\dot{\psi} + \partial_i v^i - (H_X + 2H_Y)\,\Phi + \left(\frac{X'}{X} + 2\frac{Y'}{Y}\right)v^r + (\cot\theta)\,v^\theta\,, \tag{3.37}$$

where $i = r, \theta, \phi$. Alternatively, the expansion factor defined with respect to the unit normal vector field defined in Eq. (2.11), is given by,

$$\Theta_n = (H_X + 2H_Y) + 3\dot{\psi} - (H_X + 2H_Y)\,\Phi - \frac{B_r'}{X} - \frac{\partial_\theta B_\theta}{Y} - \frac{\partial_\phi B_\phi}{Y\sin\theta} - \frac{2B_r Y'}{XY} - \frac{B_\theta \cot\theta}{Y}\,. \tag{3.38}$$

In order to have the possibility to define later hypersurfaces of uniform expansion, on which the perturbed expansion is zero, we have to find the transformation behaviour of the expansion scalar. We find, that e.g. Θ_n transforms as,

$$\begin{aligned}\tilde{\Theta}_n &= \Theta_n + \left[\dot{H}_X + 2\dot{H}_Y\right]\delta t + \left[H_X + 2H_Y\right]\dot{\delta t} + \left(\frac{\dot{X}'}{X} - \frac{\dot{X}X'}{X^2} + 2\frac{\dot{Y}'}{Y} - 2\frac{\dot{Y}Y'}{Y^2}\right)\delta r \\ &+ \left[\frac{1}{X^2}\partial_{rr} + \frac{1}{Y^2}\partial_{\theta\theta} + \frac{1}{Y^2\sin^2\theta}\partial_{\phi\phi}\right]\delta t + \frac{2Y'}{YX^2}\delta t' + \frac{\cot\theta}{Y^2}\partial_\theta \delta t\,.\end{aligned} \tag{3.39}$$

We immediately see that the transformation behaviour of Θ_n is rather complicated, and we therefore do not use it to specify a gauge.

D. Gauge invariant quantities

We can now use the results from the previous section, to construct gauge-invariant quantities. Luckily, we can use the results derived for the FRW background spacetime as guidance. We follow in particular Ref. [4], which showed that the evolution equation for the curvature perturbation on uniform density hypersurfaces can be derived solely from the energy conservation equations (on large scales).

1. FRW spacetime

We will first consider the construction of gauge-invariant quantities in perturbed FRW spacetime, which is the homogeneous limit of LTB. The perturbed FRW metric is [60]

$$ds^2 = -(1+2\phi)dt^2 + 2aB_{,i}dtdx^i + a^2\left[(1-2\psi_{\rm FRW})\delta_{ij} + 2E_{,ij}\right]dx^i dx^j\,, \tag{3.40}$$

where we have performed a scalar-vector-tensor decomposition, and kept only the scalar part. Eq. (3.20) and Eq. (3.21) then give [59]

$$\begin{aligned}\widetilde{\psi_{\rm FRW}} &= \psi_{\rm FRW} + \frac{\dot{a}}{a}\delta t\,, & (3.41)\\ \widetilde{\delta\rho_{\rm FRW}} &= \delta\rho_{\rm FRW} + \dot{\bar{\rho}}\delta t\,, & (3.42)\\ \tilde{E} &= E + \delta x\,. & (3.43)\end{aligned}$$

where $a = a(t)$ is the scale factor and $\bar{\rho} = \bar{\rho}(t)$ is the background energy density. We can now choose a gauge condition, to get rid of the gauge artefacts, here δt. To this end, the uniform density gauge can then be specified by the choice $\widetilde{\delta\rho_{\rm FRW}} \equiv 0$, which implies

$$\delta t = -\frac{\delta\rho_{\rm FRW}}{\dot{\bar{\rho}}}\,. \tag{3.44}$$

Combining Eq. (3.41) and Eq. (3.44), we are then led to define

$$-\zeta \equiv \psi_{\rm FRW} + \frac{\dot{a}/a}{\dot{\bar{\rho}}}\delta\rho_{\rm FRW}\,, \tag{3.45}$$

which is gauge-invariant under Eq. (3.20), as can be seen by direct calculation.

2. LTB spacetime

We can now proceed to construct gauge-invariant quantities in the perturbed LTB model, taking the FRW case as guidance. From the transformation equation of the perturbed spatial metric trace, ψ, we see that here we have to substitute for δt and δx^i, that is we have to choose temporal *and* spatial hypersurfaces.

From the density perturbation transformation, Eq. (3.22), choosing uniform density hypersurfaces, $\delta\tilde{\rho} = 0$, to fix the temporal gauge, we get

$$\delta t\Big|_{\delta\tilde{\rho}=0} = -\frac{1}{\dot{\bar{\rho}}}\left[\delta\rho + \bar{\rho}'\delta r\right] . \tag{3.46}$$

Substituting this into Eq. (3.30), the transformation of the metric trace, we get

$$-\zeta_{\rm SMTP} = \psi - \frac{1}{3}\left[\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right]\left(\frac{\delta\rho + \bar{\rho}'\delta r}{\dot{\bar{\rho}}}\right) + \frac{1}{3}\left\{\left[\frac{X'}{X} + 2\frac{Y'}{Y}\right]\delta r + \partial_i\delta x^i + \delta\theta\cot\theta\right\} , \tag{3.47}$$

where we chose the sign convention and notation to coincide with the FRW case. We can now choose comoving hypersurfaces to fix the remaining spatial gauge freedom. This gives for the spatial gauge generators from the transformation of the 3-velocity perturbation, Eq. (3.23),

$$\delta x^i = \int v^i dt\,. \tag{3.48}$$

Substituting the above equations into Eq. (3.47) we finally get the gauge-invariant spatial metric trace perturbation on comoving, uniform density hypersurfaces,

$$-\zeta_{\rm SMTP} = \psi + \frac{\delta\rho}{3\bar{\rho}} + \frac{1}{3}\left\{\left(\frac{X'}{X} + 2\frac{Y'}{Y} + \frac{\bar{\rho}'}{\bar{\rho}}\right)\int v^r dt + \partial_r\int v^r dt + \partial_\theta\int v^\theta dt + \partial_\phi\int v^\phi dt + \cot\theta\int v^\theta dt\right\} , \tag{3.49}$$

i.e. $\zeta_{\rm SMTP} = -\frac{1}{6}\widetilde{\delta g^k_k}|_{\widetilde{\delta\rho}=0,v=0}$. We can check by direct calculation, i.e. by substituting Eq. (3.30), Eq. (3.22), and Eq. (3.23) into Eq. (3.49), that $\zeta_{\rm SMTP}$ is gauge invariant.

Instead of using $\delta\rho$ to specify our temporal gauge, we can just as easily use the spatial metric trace perturbation, that is define hypersurfaces where $\widetilde{\psi} \equiv 0$. This gives for δt

$$\delta t = -\frac{1}{H_X + 2H_Y}\left[3\psi + \left(\frac{X'}{X} + 2\frac{Y'}{Y}\right)\delta r + \partial_i\delta x^i + \delta\theta\cot\theta\right] . \tag{3.50}$$

This allows us to construct another gauge invariant quantity, the density perturbation on uniform spatial metric trace perturbation hypersurfaces, using Eq. (3.22), as

$$\delta\tilde{\rho}\Big|_{\psi=0} = \delta\rho + \bar{\rho}\left\{3\psi + \left(\frac{X'}{X} + 2\frac{Y'}{Y} + \frac{\bar{\rho}'}{\bar{\rho}}\right)\int v^r dt + \partial_r\int v^r dt + \partial_\theta\int v^\theta dt + \partial_\phi\int v^\phi dt + \cot\theta\int v^\theta dt\right\} , \tag{3.51}$$

where the spatial gauge generators were eliminated by selecting the comoving gauge Eq. (3.48) again. The density perturbation defined in Eq. (3.51) can be written in terms of $\zeta_{\rm SMTP}$, defined in Eq. (3.49), simply as

$$\delta\tilde{\rho}\Big|_{\psi=0} = -3\bar{\rho}\zeta_{\rm SMTP} . \tag{3.52}$$

This expression allows us to relate the density perturbation at different times to the spatial metric trace perturbation, which, as we shall see in Section III E, is conserved or constant in time on all scales for barotropic fluids.

Alternatively, in both cases above, Eq. (3.49) and Eq. (3.51), we could have used the shift functions instead of the 3-velocities to define the spatial gauge, in analogy with the Newtonian or longitudinal gauge condition in perturbed FRW. In this case the spatial gauge generators are

$$\delta r = -\int dt\left[\frac{\partial_r}{X^2}\left(\frac{\delta\rho}{\dot{\bar{\rho}}} + \frac{B_r}{X}\right)\right] - \int dt\left[\frac{\partial_r}{X^2}\left(\frac{\delta r\bar{\rho}'}{\dot{\bar{\rho}}}\right)\right] , \tag{3.53}$$

$$\delta\theta = -\int dt\left[\frac{\partial_\theta}{Y^2}\left(\frac{\delta\rho}{\dot{\bar{\rho}}} + \frac{B_\theta}{Y}\right)\right] - \int dt\left[\frac{\partial_\theta}{Y^2}\left(\frac{\delta r\bar{\rho}'}{\dot{\bar{\rho}}}\right)\right] , \tag{3.54}$$

$$\delta\phi = -\int dt\left[\frac{\partial_\phi}{Y^2\sin^2\theta}\left(\frac{\delta\rho}{\dot{\bar{\rho}}} + \frac{B_\phi}{Y\sin\theta}\right)\right] - \int dt\left[\frac{\partial_\phi}{Y^2\sin^2\theta}\left(\frac{\delta r\bar{\rho}'}{\dot{\bar{\rho}}}\right)\right] . \tag{3.55}$$

Since the expressions are considerably longer than Eq. (3.48) above, we did not pursue this choice of spatial gauge any further.

Another alternative would be to choose a more geometric definition of the longitudinal or Newtonian gauge, namely use a zero shear condition to fix temporal and spatial gauge, again in analogy with FRW, i.e.,

$$\widetilde{\delta\sigma_{ij}} = 0\,. \tag{3.56}$$

However, again we find that this leads to much more complicated gauge conditions (since we do not decompose into axial and polar scalar and vector parts), and we here do not pursue this further. See however appendix A 2 for the components of the shear tensor.

For the spatial metric trace perturbation expressed in terms of the perturbations of the Gerlach and Sengupta formalism see appendix B.

E. Evolution of $\zeta_{\rm SMTP}$

Before we derive the evolution equation for spatial metric trace perturbation $\zeta_{\rm SMTP}$, we briefly discuss the decomposition of the pressure perturbation in the LTB setting. We assume that the pressure $P = P(\rho, S)$, where ρ is the density and S the entropy of the system. We can then expand the pressure as

$$\delta P = c_{\rm s}^2 \delta\rho + \delta P_{\rm nad}\,, \tag{3.57}$$

where $\delta P_{\rm nad}$ is the entropy or non-adiabatic pressure perturbation, and the adiabatic sound speed is define as

$$c_{\rm s}^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_S\,, \tag{3.58}$$

for a pedagogical introduction to this topic see e.g. Ref. [61]. Since in LTB background quantities are t and r dependent, therefore allowing for now $P \equiv P(t,r)$, we find that

$$c_{\rm s}^2 = \frac{\dot{\bar{P}} + \bar{P}' v^r}{\dot{\bar{\rho}} + \bar{\rho}' v^r}\,. \tag{3.59}$$

However, since in LTB $\bar{P} = 0$, we have that on uniform density hypersurfaces $\delta P = \delta P_{\rm nad}$.

The evolution equation for spatial metric trace perturbation on uniform density and comoving hypersurfaces, $\zeta_{\rm SMTP}$, using the time derivative of Eq. (3.49), Eq. (3.16) and background conservation equation, Eq. (3.9), is

$$\dot{\zeta}_{\rm SMTP} = \frac{H_X + 2H_Y}{3\bar{\rho}} \delta P_{\rm nad}\,. \tag{3.60}$$

This result is valid on all scales. We see that $\zeta_{\rm SMTP}$ is conserved for $\delta P_{\rm nad} = 0$, e.g. for barotropic fluids. While this result is similar to the FRW case [4], we do not have to assume the large scale limit here, which is a striking contrast to be discussed in Section V.

F. Spatial Metric Trace Perturbation in FRW

In this subsection we will now compare the behaviour of the $\zeta_{\rm SMTP}$ variable that we defined in LTB with the spatial metric trace perturbation on comoving constant density hypersurfaces in FRW spacetime, including background pressure. From Eq. (3.40), the trace of the perturbed part of the spatial metric can be seen to be given in FRW by

$$\frac{1}{2}\delta g^k_{k\,\rm FRW} = -3\psi_{\rm FRW} + \nabla^2 E\,. \tag{3.61}$$

This quantity can be seen to transform under Eq. (3.20) as

$$\widetilde{\frac{1}{2}\delta g^k_k} = \frac{1}{2}\delta g^k_k - 3H\delta t + \nabla^2 \delta x\,. \tag{3.62}$$

The 3-velocity transformation has the same form as in LTB, and is given by Eq. (3.23). Additionally, the density perturbation evolves as

$$\delta\dot{\rho} + 3H\left(\delta\rho + \delta P\right) - 3\left(\bar{\rho} + \bar{P}\right)\dot{\psi}_{\rm FRW} + \left(\bar{\rho} + \bar{P}\right)\frac{\nabla^2}{a^2}\left(av + a^2\dot{E}\right) = 0\,. \tag{3.63}$$

Taking the time derivative of Eq. (3.61) and substituting into Eq. (3.63) we then find that the spatial metric trace perturbation on comoving constant density hypersurfaces evolves as

$$\dot{\zeta}_{\rm SMTP} = -\frac{1}{6}\dot{\widetilde{\delta g^k_k}}|_{\widetilde{\delta\rho}=0,v=0} = \frac{H}{(\bar{\rho}+\bar{P})}\delta P_{\rm nad}\,. \tag{3.64}$$

This equation is again valid on all scales, and can again be seen to demonstrate that the spatial metric trace perturbation on comoving constant density hypersurfaces is conserved for barotropic fluids. It should be noted that in order to relate this spatial metric trace perturbation on comoving constant density hypersurfaces in FRW to observables such as the density perturbation both the density perturbation and 3-velocity need to be specified on flat hypersurfaces. It should also be noted that this quantity is not the same as the curvature perturbation, ζ, from the standard FRW literature. Both Eq. (3.64) and Eq. (3.60) differ from the result for the Lemaître spacetime, as shall be seen in Section IV below.

IV. THE LEMAÎTRE SPACETIME

Although the main focus of this paper is on LTB cosmology, we here briefly also discuss perturbations around a Lemaître background spacetime. The Lemaître spacetime is a generalisation of LTB, allowing for non-zero pressure in the background [62]. Although no exact solution are known in this case, we nevertheless think it is interesting to extend the discussion of the previous sections to this spacetime.

The Lemaître background metric is given by

$$ds^2 = -f^2dt^2 + X^2(r,t)dr^2 + Y^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,, \tag{4.1}$$

where f is an additional factor, $f \equiv f(t,r)$. The background four velocity, from Eq. (2.4), is,

$$u^\mu = \left[\frac{1}{f}, 0, 0, 0\right], \tag{4.2}$$

and energy-momentum tensor, from Eq. (2.2), becomes,

$$T^{\mu\nu} = \begin{pmatrix} \frac{\rho}{f^2} & 0 & 0 & 0 \\ 0 & \frac{P}{X^2} & 0 & 0 \\ 0 & 0 & \frac{P}{Y^2} & 0 \\ 0 & 0 & 0 & \frac{P}{Y^2\sin^2\theta} \end{pmatrix}. \tag{4.3}$$

Energy conservation is similar to LTB but with an additional pressure term,

$$\dot{\rho} + (\rho + P)(H_X + 2H_Y) = 0\,. \tag{4.4}$$

If we now perturb the metric in a similar way to LTB, Eq. (3.12), we get,

$$\delta g_{\mu\nu} = \begin{pmatrix} -2f^2\Phi & fXB_r & fYB_\theta & fY\sin\theta B_\phi \\ fXB_r & 2X^2C_{rr} & XYC_{r\theta} & XY\sin\theta C_{r\phi} \\ fYB_\theta & XYC_{r\theta} & 2Y^2C_{\theta\theta} & Y^2\sin\theta C_{\theta\phi} \\ fY\sin\theta B_\phi & XY\sin\theta C_{r\phi} & Y^2\sin\theta C_{\theta\phi} & 2Y^2\sin^2\theta C_{\phi\phi} \end{pmatrix}. \tag{4.5}$$

The perturbed 4-velocity, from Eq. (2.4), is,

$$u^\mu = \frac{1}{f}\left[(1-\Phi), v^r, v^\theta, v^\phi\right]\,, \tag{4.6}$$

As in the LTB case, we can now study how the perturbations in this case change under the transformation Eq. (3.21). The perturbed energy density $\delta\rho$, and the 3-velocities, v^i, transform as in the LTB background Eq. (3.22) and Eq. (3.23). The perturbed metric components transform as

$$\tilde{\Phi} = \Phi - \frac{\dot{f}}{f}\delta t - \frac{f'}{f}\delta r + \dot{\delta t}\,, \tag{4.7}$$

and

$$\begin{aligned} \tilde{B}_r &= B_r + \frac{X}{f}\dot{\delta r} - \frac{f\delta t'}{X}\,, &(4.8)\\ \tilde{B}_\theta &= B_\theta + \frac{Y}{f}\dot{\delta\theta} - \frac{f\partial_\theta\delta t}{Y}\,, &(4.9)\\ \tilde{B}_\phi &= B_\phi + \frac{Y(\sin\theta)}{f}\dot{\delta\phi} - \frac{f\partial_\phi\delta t}{Y(\sin\theta)}\,. &(4.10)\end{aligned}$$

The transformation behaviour of the perturbed metric components C_{ij}, and hence ψ, are unchanged from the LTB case, see Eq. (3.26) - Eq. (3.28) and Eq. (3.30) - Eq. (3.33).

The perturbed energy conservation equation is,

$$\begin{aligned} \delta\dot{\rho} \;&+\; (\delta\rho+\delta P)\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right) + \left(\bar{\rho}'+\bar{P}'\right)v^r + \frac{fB_r}{X}\bar{P}' + \left(\partial_\theta\frac{B_\theta}{Y}+\partial_\phi\frac{B_\phi}{Y\sin\theta}\right)f\bar{P}\\ &+\; (\bar{\rho}+\bar{P})\left(\dot{\psi}+{v^r}'+\partial_\theta v^\theta+\partial_\phi v^\phi+\left[\frac{f'}{f}+\frac{X'}{X}+2\frac{Y'}{Y}\right]v^r+\frac{B_rf'}{X}+\cot\theta v^\theta\right)=0\,. \end{aligned} \tag{4.11}$$

As in the previous section, we can now construct gauge-invariant quantities. We choose hypersurfaces of vanishing perturbed energy density to define the temporal gauge, that is,

$$\delta t = \frac{\delta\rho}{\dot{\bar{\rho}}} + \frac{\bar{\rho}'}{\dot{\bar{\rho}}}\delta r\,, \tag{4.12}$$

and choose again co-moving gauge, where $v^i=0$, to get for the spatial coordinate shifts

$$\delta x^i = \int v^i dt\,. \tag{4.13}$$

Then using the transformation for perturbed metric trace ψ, given above in Eq. (3.29), we can construct the gauge-invariant spatial metric trace perturbation on uniform density and comoving hypersurfaces,

$$-\zeta_{\rm SMTP} = \psi + \frac{\delta\rho}{3(\bar{\rho}+\bar{P})} + \frac{1}{3}\left\{\left(\frac{X'}{X}+2\frac{Y'}{Y}+\frac{\bar{\rho}'}{\bar{\rho}+\bar{P}}\right)\int v^r dt + \partial_r\int v^r dt + \partial_\theta\int v^\theta dt + \partial_\phi\int v^\phi dt + \cot\theta\int v^\theta dt\right\}\,. \tag{4.14}$$

The evolution equation for $\zeta_{\rm SMTP}$ is then found from Eq. (4.11), using the decomposition of the pressure perturbation, Eq. (3.57), and the definition of the adiabatic sound speed, Eq. (3.59), as

$$\begin{aligned} -\dot{\zeta}_{\rm SMTP} \;&=\; \frac{\dot{\bar{\rho}}}{\left(\bar{\rho}+\bar{P}\right)^2}\delta P_{\rm nad} - \frac{\bar{P}'}{\left(\bar{\rho}+\bar{P}\right)}v^r - \frac{fB_r}{X\left(\bar{\rho}+\bar{P}\right)}\bar{P}' - \left(\partial_\theta\frac{B_\theta}{Y}+\partial_\phi\frac{B_\phi}{Y\sin\theta}\right)\frac{f\bar{P}}{\left(\bar{\rho}+\bar{P}\right)}\\ &+\; \left[\partial_t\left(\frac{X'}{X}+\frac{Y'}{Y}+\frac{\bar{\rho}'}{\bar{\rho}+\bar{P}}\right)\right]\int v^r dt - \frac{f'}{f}v^r + \frac{B_rf'}{X}\,. \end{aligned} \tag{4.15}$$

By transforming the coordinates to Cartesian using the chain rule and taking the spatial derivatives to be negligible on large scales, Eq. (4.15), reduces to,

$$\dot{\zeta}_{\rm SMTP} \;=\; \frac{H_X+2H_Y}{3(\bar{\rho}+\bar{P})}\,\delta P_{\rm nad}\,. \tag{4.16}$$

This can be seen to be similar to that for LTB, Eq. (3.60), but as with the standard ζ in FRW and unlike $\zeta_{\rm SMTP}$ in both LTB and FRW is only valid at large scales.

V. DISCUSSION AND CONCLUSION

In this paper we have constructed gauge-invariant quantities in perturbed LTB spacetime. In particular we have constructed the gauge-invariant spatial metric trace perturbation on comoving, uniform density hypersurfaces, $\zeta_{\rm SMTP}$. We derived the evolution equation for $\zeta_{\rm SMTP}$ and found that it is conserved on all scales for barotropic fluids (when $\delta P_{\rm nad} = 0$). We found this result for the evolution equation for $\zeta_{\rm SMTP}$ also holds for FRW. This is in contrast to the standard FRW result, where the similar gauge-invariant quantity, ζ, is only conserved on large scales. It was also found that the evolution equation for $\zeta_{\rm SMTP}$, in Lemaître spacetime which would be conserved in the case of barotropic fluids is only found in the large scale limit, as with the result for the standard ζ in FRW.

Deriving these results in LTB is more involved than in the FRW case, because the background is t and r dependent, whereas the FRW background is homogeneous and isotropic, and hence only t dependent. Additional complications often arise in LTB because it suggests a 1+1+2 decomposition, and not "simply" a 1+3 one, as in FRW. This makes a multi-pole decomposition much more complicated, and hence we did not use it here to construct conserved quantities.

The main application of the results derived in this paper lies in providing conserved quantities in LTB and Lemaître spacetimes. Conserved quantities have proved to be very useful in the FRW case, and they can be used to directly relate observable quantities, such as the density perturbation, between the beginning of the epoch modelled by the LTB spacetime and late times, without having to solve the field equations. Furthermore, these conserved quantities can be used as a consistency check for numerical simulations, as for example those described in Ref.[56]. By solving for the density contrast numerically and comparing this result to the one obtained by using $\zeta_{\rm SMTP}$, the accuracy of the code can easily be checked.

Acknowledgments

The authors are grateful to Tim Clifton and David Mulryne for useful and insightful discussions. AL is funded by a STFC studentship, and KAM is supported, in part, by STFC grant ST/J001546/1. The computer algebra package Cadabra[63] was used in the derivation of some of the equations.

Appendix A: Additional material for LTB

In this section of the appendix we present some material that is not essential to follow the main body of this work. However, since it might be useful and save time in reproducing or extending some or all of the calculations, we reproduce it here.

1. Contravariant LTB Metric Perturbations

Using the constraint Eq. (2.3) we get the contravariant perturbed metric components,

$$\delta g^{\mu\nu} = \begin{pmatrix} 2\Phi & \frac{B_r}{X} & \frac{B_\theta}{Y} & \frac{B_\phi}{Y\sin\theta} \\ \frac{B_r}{X} & -\frac{2C_{rr}}{X^2} & -\frac{C_{r\theta}}{XY} & -\frac{C_{r\phi}}{XY\sin\theta} \\ \frac{B_\theta}{Y} & -\frac{C_{r\theta}}{XY} & -\frac{2C_{\theta\theta}}{Y^2} & -\frac{C_{\theta\phi}}{Y^2\sin\theta} \\ \frac{B_\phi}{Y\sin\theta} & -\frac{C_{r\phi}}{XY\sin\theta} & -\frac{C_{\theta\phi}}{Y^2\sin\theta} & -\frac{2C_{\phi\phi}}{Y^2\sin^2\theta} \end{pmatrix}. \tag{A1}$$

2. LTB Shear

The $t-t$ component of the shear is zero. To linear order we find that the $r-r$ component is,

$$\sigma_{rr} = -\frac{1}{3}X^2\Big(\dot{\psi} - 2(1-\Phi)(H_X - H_Y) - 4C_{rr}(H_X - H_Y) - 2\frac{B_r}{XY}Y' - \frac{B_\theta\cot\theta}{Y} \tag{A2}$$
$$+ \frac{2}{X}B_r' - \frac{1}{Y}\partial_\theta B_\theta - \frac{1}{Y\sin\theta}\partial_\phi B_\phi - 3\dot{C_{rr}}\Big),$$

the $\theta-\theta$ component,

$$\sigma_{\theta\theta} = -\frac{1}{3}Y^2\Big(\dot{\psi}+(1-\Phi)(H_X-H_Y)+2C_{\theta\theta}(H_X-H_Y)+\frac{B_rY'}{XY}-\frac{B_\theta}{Y}\cot\theta - \frac{B_r'}{X}+\frac{2}{Y}\partial_\theta B_\theta-\frac{\partial_\phi B_\phi}{Y\sin\theta}-3\dot{C}_{\theta\theta}\Big), \tag{A3}$$

and the $\phi-\phi$ component,

$$\sigma_{\phi\phi} = -\frac{1}{3}Y^2\sin^2\theta\Big(\dot{\psi}+(1-\Phi)(H_X-H_Y)+2C_{\phi\phi}(H_X-H_Y)+\frac{B_rY'}{XY}+2\frac{B_\theta}{Y}\cot\theta - \frac{B_r'}{X}-\frac{\partial_\theta B_\theta}{Y}-\frac{\partial_\phi B_\phi}{Y\sin\theta}-3\dot{C}_{\phi\phi}\Big). \tag{A4}$$

We also need the off-diagonal components. For the mixed temporal-spatial components we get,

$$\sigma_{tr}=\frac{2B_rX}{3}(H_X-H_Y),\qquad \sigma_{t\theta}=-\frac{B_\theta Y}{3}(H_X-H_Y),\qquad \sigma_{t\phi}=-\frac{B_\phi Y\sin\theta}{3}(H_X-H_Y). \tag{A5}$$

For the mixed spatial components we get,

$$\sigma_{r\theta} = \frac{1}{3}C_{r\theta}XY(H_X-H_Y)+\dot{C_{r\theta}}XY-\frac{1}{2}YB_\theta'+\frac{1}{2}B_\theta Y'-\frac{1}{2}X\partial_\theta B_r. \tag{A6}$$

$$\sigma_{\theta\phi} = -\frac{1}{3}C_{\theta\phi}Y^2\sin\theta(H_X-H_Y)+\frac{1}{2}\dot{C_{\theta\phi}}Y^2\sin\theta-\frac{1}{2}Y\sin\theta\partial_\theta B_\phi+\frac{1}{2}Y\cos\theta B_\phi. \tag{A7}$$

$$\sigma_{r\phi} = \frac{1}{6}C_{r\phi}XY\sin\theta(H_X-H_Y)+\frac{1}{2}\dot{C_{r\phi}}XY\sin\theta-\frac{1}{2}Y\sin\theta B_\phi'+\frac{1}{2}\sin\theta B_\phi Y'. \tag{A8}$$

3. The LTB Ricci 3-scalar

The Ricci scalar on the spatial 3-hypersurfaces is given, in the background, as,

$$\bar{R}^{(3)} = \frac{4X'Y'}{X^3Y}-\frac{2Y'^2}{X^2Y^2}-\frac{4Y''}{X^2Y}+\frac{2}{Y^2}, \tag{A9}$$

and the perturbed Ricci scalar is given by,

$$\begin{aligned}\delta R^{(3)} =& \frac{4C_{rr}}{X^2Y}\left(\frac{Y'^2}{Y}+2Y''-\frac{2X'Y'}{X}\right)-\frac{2}{Y^2}(2C_{\theta\theta})-\frac{2}{X^2}(C_{\theta\theta}''+C_{rr}'')+\frac{2C_{r\theta}\cot\theta}{X^2Y}(X'-Y')\\ &+\frac{2X'(\partial_\theta C_{r\phi})}{X^2Y}+\frac{2X'C_{\theta\theta}'}{X^3}+\frac{4Y'C_{rr}'}{X^2Y}+\frac{2X'C_{\phi\phi}'}{X^3}+\frac{2\cot\theta(\partial_\theta C_{\theta\theta})}{Y^2}\\ &-\frac{6Y'(C_{\theta\theta}'+C_{\phi\phi}')}{X^2Y}-\frac{4\cot\theta(\partial_\theta C_{\phi\phi})}{Y^2}-\frac{2(\partial_{\theta\theta}C_{\phi\phi})}{Y^2}+\frac{2(\partial_\theta C_{r\theta}')}{XY}+\frac{2(\partial_\phi C_{r\phi}')}{XY\sin\theta}+\frac{2(\partial_{\theta\phi}C_{\theta\phi})}{Y^2\sin\theta}.\end{aligned} \tag{A10}$$

Appendix B: The Spatial Metric Trace Perturbation in 2+2 Spherical Harmonic Formalism

1. Background

The background LTB metric in the Clarkson, Clifton and February formalism is [54]

$$ds^2=-dt^2+\frac{a_\parallel^2(t,r)}{(1-\kappa r^2)}dr^2+a_\perp^2(t,r)r^2d\Omega^2. \tag{B1}$$

This is the same metric in the same coordinates as that used in this paper Eq. (3.1). This allows us to compare directly the perturbed metric components once the relations between the background functions are known. In the rest of this section, where a symbol is used in the Clarkson, Clifton and February formalism which has a different meaning to the same symbol in this paper we have made it calligraphic, except for "v" which is made "v". Also, a radial derivative is later defined which differs slightly from that used in earlier sections of this paper. To distinguish this alternative radial derivative we use a dagger in place of the prime used in Ref. [54]. From Eq. (B1) and Eq. (3.1) we get,

$$X = \frac{a_{\|}}{\sqrt{(1-\kappa r^2)}}\,, \qquad Y = a_{\perp} r\,, \qquad H_{\|} \equiv \frac{\dot{a_{\|}}}{a_{\|}} = H_X\,, \qquad H_{\perp} \equiv \frac{\dot{a_{\perp}}}{a_{\perp}} = H_Y\,, \tag{B2}$$

where $\kappa \equiv \kappa(r)$. The radial derivative defined in Ref. [54] for an arbitrary function, F, is

$$F^{\dagger} = \frac{\sqrt{(1-\kappa r^2)}}{a_{\|}} F' = \frac{F'}{X}\,, \tag{B3}$$

where the time derivative of the above radial derivative behaves as

$$(\dot{F})^{\dagger} - (\dot{F^{\dagger}}) = H_{\|} F^{\dagger} = H_X \frac{F'}{X}\,, \tag{B4}$$

2. Perturbations

The perturbed portion of the metric for axial perturbations is given as

$$\delta g_{\mu\nu} \equiv \begin{pmatrix} 0 & h_A^{\rm axial}\bar{\mathcal{Y}}_a \\ h_A^{\rm axial}\bar{\mathcal{Y}}_a & h\,\bar{\mathcal{Y}}_{ab} \end{pmatrix} \tag{B5}$$

and for the polar perturbations as

$$\delta g_{\mu\nu} \equiv \begin{pmatrix} h_{AB}\mathcal{Y} & h_A^{\rm polar}\mathcal{Y}_a \\ h_A^{\rm polar}\mathcal{Y}_a & a_{\perp}^2 r^2 (K\mathcal{Y}\gamma_{ab} + G\mathcal{Y}_{:ab}) \end{pmatrix} \tag{B6}$$

In the above equations $\mathcal{Y} \equiv \mathcal{Y}^{(lm)}$ and are the various spherical harmonic functions for scalar vector and tensor perturbations (see Ref. [54]). The index A runs over t and r, while a runs over θ and ϕ. The colon represents the covariant derivative with respect to the metric on the unit sphere. $h_A^{\rm axial}, h, h_{AB}, h_A^{\rm polar}, K, G$ are the perturbation variables and are functions of x^A. By direct comparison between the perturbed metrics in both formalisms we find,

$$\psi = \frac{1}{3}(C_{rr} + C_{\theta\theta} + C_{\phi\phi}) = \frac{1}{6}\left(\frac{h_{rr}\mathcal{Y}}{X^2} + \frac{h\,\bar{\mathcal{Y}}_{\theta\theta}}{Y^2} + \frac{h\,\bar{\mathcal{Y}}_{\phi\phi}}{Y^2\sin^2\theta} + 2K\mathcal{Y} + G\mathcal{Y}_{:\theta\theta} + \frac{G\mathcal{Y}_{:\phi\phi}}{\sin^2\theta}\right), \tag{B7}$$

where we have used Bondi's scale factors, X and Y, for brevity. The covariant form of the axial perturbed 4-velocities is

$$\delta u_{\mu} = (0, \bar{\rm v}\,\bar{\mathcal{Y}}_a) \tag{B8}$$

and the scalar perturbed 4-velocities are

$$\delta u_{\mu} = \left[\left(\tilde{w}\hat{n}_A + \frac{1}{2}h_{AB}\hat{u}^B\right)\mathcal{Y}, \tilde{\rm v}\,\mathcal{Y}_a\right], \tag{B9}$$

where $\bar{\rm v}, \tilde{w}, \tilde{\rm v}$ are all functions of x^A, and $\hat{n}_A$ is the unit spacelike radial vector and $\hat{u}^A$ is the unit timelike vector. The contravariant form of the perturbed metric for axial perturbations is

$$\delta g^{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & \frac{1}{Y^2}h_t^{\rm axial}\bar{\mathcal{Y}}_\theta & \frac{1}{Y^2\sin^2\theta}h_t^{\rm axial}\bar{\mathcal{Y}}_\phi \\ 0 & 0 & -\frac{1}{X^2Y^2}h_r^{\rm axial}\bar{\mathcal{Y}}_\theta & -\frac{1}{X^2Y^2\sin^2\theta}h_r^{\rm axial}\bar{\mathcal{Y}}_\phi \\ \frac{1}{Y^2}h_t^{\rm axial}\bar{\mathcal{Y}}_\theta & -\frac{1}{X^2Y^2}h_r^{\rm axial}\bar{\mathcal{Y}}_\theta & -\frac{1}{Y^4}h\,\bar{\mathcal{Y}}_{\theta\theta} & -\frac{1}{Y^4\sin^2\theta}h\,\bar{\mathcal{Y}}_{\theta\phi} \\ \frac{1}{Y^2\sin^2\theta}h_t^{\rm axial}\bar{\mathcal{Y}}_\phi & -\frac{1}{X^2Y^2\sin^2\theta}h_r^{\rm axial}\bar{\mathcal{Y}}_\phi & -\frac{1}{Y^4\sin^2\theta}h\,\bar{\mathcal{Y}}_{\theta\phi} & -\frac{1}{Y^4\sin^4\theta}h\,\bar{\mathcal{Y}}_{\phi\phi} \end{pmatrix} \tag{B10}$$

and for the polar perturbations is

$$\delta g^{\mu\nu} \equiv \begin{pmatrix} -h_{tt}\mathcal{Y} & \frac{1}{X^2}h_{tr}\mathcal{Y} & \frac{1}{Y^2}h_t^{\rm polar}\mathcal{Y}_\theta & \frac{1}{Y^2\sin^2\theta}h_t^{\rm polar}\mathcal{Y}_\phi \\ \frac{1}{X^2}h_{tr}\mathcal{Y} & -\frac{1}{X^4}h_{rr}\mathcal{Y} & -\frac{1}{X^2Y^2}h_r^{\rm polar}\mathcal{Y}_\theta & -\frac{1}{X^2Y^2\sin^2\theta}h_r^{\rm polar}\mathcal{Y}_\phi \\ \frac{1}{Y^2}h_t^{\rm polar}\mathcal{Y}_\theta & -\frac{1}{X^2Y^2}h_r^{\rm polar}\mathcal{Y}_\theta & -\frac{1}{Y^2}(K\mathcal{Y}+G\mathcal{Y}_{:\theta\theta}) & -\frac{1}{Y^2\sin^2\theta}G\mathcal{Y}_{:\theta\phi} \\ \frac{1}{Y^2\sin^2\theta}h_t^{\rm polar}\mathcal{Y}_\phi & -\frac{1}{X^2Y^2\sin^2\theta}h_r^{\rm polar}\mathcal{Y}_\phi & -\frac{1}{Y^2\sin^2\theta}G\mathcal{Y}_{:\theta\phi} & -\frac{1}{Y^2\sin^4\theta}(K\mathcal{Y}\sin^2\theta+G\mathcal{Y}_{:\phi\phi}) \end{pmatrix} \tag{B11}$$

where we have once again used Bondi's scale factors, X and Y, for brevity. The perturbed 4-velocity in contravariant form is

$$\begin{aligned} u^\mu \;=\; \Bigg[\quad & 1+\frac{1}{2}h_{tt}\mathcal{Y}, \qquad\qquad -\frac{\mathcal{Y}}{X^2}\left(\frac{1}{2}h_{tr}+X\tilde{w}\right), \qquad\qquad\qquad\qquad (\mathrm{B12}) \\ & \frac{1}{Y^2}\left(\bar{\mathrm{v}}\,\bar{\mathcal{Y}}_\theta+\tilde{\mathrm{v}}\,\mathcal{Y}_\theta-h_t^{\rm axial}\bar{\mathcal{Y}}_\theta-h_t^{\rm polar}\mathcal{Y}_\theta\right), \qquad \frac{1}{Y^2\sin^2\theta}\left(\bar{\mathrm{v}}\,\bar{\mathcal{Y}}_\phi+\tilde{\mathrm{v}}\,\mathcal{Y}_\phi-h_t^{\rm axial}\bar{\mathcal{Y}}_\phi-h_t^{\rm polar}\mathcal{Y}_\phi\right)\Bigg], \end{aligned}$$

where the last three terms correspond directly with v^r, v^θ, v^ϕ respectively in the formalism of this paper. Substituting Eq. (B12), Eq. (B7), Eq. (B3) and Eq. (B2) into Eq. (3.49) we get

$$\begin{aligned} -\zeta_{\rm SMTP} \;=\; & \frac{1}{6}\left(\frac{(1-\kappa r^2)}{a^2_\parallel}h_{rr}\mathcal{Y}+\frac{h\,\bar{\mathcal{Y}}_{\theta\theta}}{a_\perp{}^2r^2}+\frac{h\,\bar{\mathcal{Y}}_{\phi\phi}}{a_\perp{}^2r^2\sin^2\theta}+2K\mathcal{Y}+G\mathcal{Y}_{:\theta\theta}+\frac{G\mathcal{Y}_{:\phi\phi}}{\sin^2\theta}\right)+\frac{\delta\rho}{3\bar{\rho}} \qquad (\mathrm{B13}) \\ +\; & \frac{1}{3}\Bigg\{\partial_\theta\int\frac{1}{a_\perp{}^2r^2}\left(\bar{\mathrm{v}}\,\bar{\mathcal{Y}}_\theta+\tilde{\mathrm{v}}\,\mathcal{Y}_\theta-h_t^{\rm axial}\bar{\mathcal{Y}}_\theta-h_t^{\rm polar}\mathcal{Y}_\theta\right)dt \\ +\; & \partial_\phi\int\frac{1}{a_\perp{}^2r^2\sin^2\theta}\left(\bar{\mathrm{v}}\,\bar{\mathcal{Y}}_\phi+\tilde{\mathrm{v}}\,\mathcal{Y}_\phi-h_t^{\rm axial}\bar{\mathcal{Y}}_\phi-h_t^{\rm polar}\mathcal{Y}_\phi\right)dt \\ +\; & \cot\theta\int\frac{1}{a_\perp{}^2r^2}\left(\bar{\mathrm{v}}\,\bar{\mathcal{Y}}_\theta+\tilde{\mathrm{v}}\,\mathcal{Y}_\theta-h_t^{\rm axial}\bar{\mathcal{Y}}_\theta-h_t^{\rm polar}\mathcal{Y}_\theta\right)dt-\partial_r\int\frac{\mathcal{Y}(1-\kappa r^2)}{a_\parallel{}^2}\left(\frac{1}{2}h_{tr}+\frac{a_\parallel}{\sqrt{(1-\kappa r^2)}}\tilde{w}\right)dt \\ -\; & \left(\left(\frac{a_\parallel}{\sqrt{(1-\kappa r^2)}}\right)^\dagger+2\frac{(a_\perp r)^\dagger a_\parallel}{a_\perp r\sqrt{(1-\kappa r^2)}}+\frac{\bar{\rho}^\dagger a_\parallel}{\bar{\rho}\sqrt{(1-\kappa r^2)}}\right)\int\frac{(1-\kappa r^2)}{a^2_\parallel}\mathcal{Y}\left(\frac{1}{2}h_{tr}+\frac{a_\parallel}{\sqrt{(1-\kappa r^2)}}\tilde{w}\right)dt\Bigg\}, \end{aligned}$$

which is our gauge invariant quantity, conserved on all scales with only adiabatic pressure perturbations, but expressed in terms of the perturbation functions used in [54]. As mentioned in III D 2, this relates directly to the density perturbation on constant curvature hypersurfaces through Eq. (3.52)

$$\delta\tilde{\rho}\Big|_{\psi=0} = -3\bar{\rho}\zeta_{\rm SMTP}\,.$$

Eq. (B13) is clearly more complicated than Eq. (3.49).

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