Opportunities as chances: maximising the probability that everybody succeeds

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Abstract

We model opportunities in society as 'chances of success', that is as they are commonly described by practitioners. We show that a classical liberal principle of justice together with a limited principle of social rationality imply that the social objective should be to maximise the chance that everybody in society succeeds. Technically, this means using a 'Nash' welfare criterion. A particular consequence is that the failure of even only one individual must be considered maximally detrimental. We also study a refinement of this criterion and its extension to problems of intergenerational justice.

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"Until all people are happy, there is no individual happiness" Kenji Miyazawa (1896-1933)

1 Introduction

'Opportunities' are a central concept both in the public discourse and in economics. In this paper we propose a new approach to model this concept. We assume that each individual is regarded as a binary experiment with either 'success' or 'failure' as possible outcomes. Then, opportunities in society are expressed by the profile of 'chances of success' across individuals. By means of this simplification, we are able to offer several insights on the issue of the allocation of opportunities. For example, what is the social cost of one person in society not having any chance of success? Is it conceivable that such a sacrifice be justified by a sufficient increase in opportunities for the rest of society? Our theoretical framework offers insights to address this type of questions.

In our approach, we view people with more opportunities as people who face more favourable circumstances, and who, as a consequence, will tend to succeed more frequently. This meaning of the term 'opportunities' tallies with natural language,[1] and also with the interpretation of the social policy literature. Here, a single individual is a representative of a class or of a category. When in a social policy study it is claimed that some category of individuals has low opportunities, what is meant is that the probability of success (in some dimension) of an individual extracted at random from that category is lower than some benchmark. So it is quite common to read statements of this kind: "An adolescent of ethnic origin X and social background Y has half the average chances to be eventually admitted into a top university".[2] On this interpretation there is no mention of 'effort', 'responsibility' or 'talent'. Politicians frequently speak the language of chances.[3]

Academic economists too sometimes emphasise the 'favourable juncture of circumstances'

[1] Consider the Webster's definition of an opportunity: "a favourable juncture of circumstances". Similarly, in the Oxford Dictionary: "a good chance; a favourable occasion".

[2] The literature is too vast for a comprehensive set of references. See for example Mayer [34] and Duncan and Murnane [15], whose very titles both refer to children's 'life chances'.

[3] For example, Tony Blair: "If we are in politics for one thing it is to make sure that all children are given the best chance in life." (Labour Party conference speech, September 1999); J. F. Kennedy: "The Negro baby born in America today...has about one-half as much chance of completing a high school as a white baby born in the same place on the same day, one-third as much chance of completing college, one-third as much chance of becoming a professional man, twice as much chance of becoming unemployed, about one-seventh as much chance of earning $10,000 a year" (Civil Rights Address, June 1963). In a major independent report Frank Field [16] (a Labour MP) writes that "improving the life chances of under fives is the key to cutting social inequality".

aspect of opportunities. Notably, Deaton's [12] notion of 'escape' is not distant from our notion of 'success'. However, economists have more often adopted more sophisticated and indirect views of the notion of opportunity. Concepts such as talent and responsibility are placed at the forefront.[4] It is also usually taken as a given that opportunities, once properly formulated, should be equalised.

In this paper we try to take seriously the direct interpretation of practitioners. This approach, while lacking the sophistication of other approaches, has the advantage of interpreting opportunities in a way that is amenable to straightforward measurement. A target for social policy to equalise the proportion of students in top schools among the various ethnic groups, or the proportion in high-level jobs of students from different types of schools, is concrete and easy to understand and verify empirically, in a way in which, say, 'equalise capabilities across ethnic groups' is not.[5]

Our simplification pays off with some interesting insights in respect of a well-known difficulty with justifying egalitarian principles. Equalisation of any value measure across individuals can always be criticised (just like simple welfare egalitarianism) on the grounds that many individuals might have to face large aggregate losses for the sake of increasing only marginally the value for one individual. Our analysis, however, leads to a preference for some degree of equality that does not stem from the *nature* of the 'equalisandum' (opportunities as opposed to welfare), but rather from outside the stock of egalitarian principles, via a classical liberal 'Harm principle'.

This principle is liberal because it asserts a form of non-interference with individuals in society. The details are explained in section 3, but its core is the requirement that an individual who has suffered damage *without harming others* should not be interfered with.

By means of this and other properties we characterise some 'Nash-like' criteria: society should, broadly speaking, maximise the *product of opportunities*.[6] In the usual setting of social welfare, a drawback of the Nash product is that it raises a difficulty of interpretation about the object that is being maximised: what does a product of utilities mean? In contrast, classical criteria such as the Utilitarian and maximin ones are clearly interpretable as 'total'

[4] The literature here is vast too: an illustrative but far from comprehensive selection of contributions includes: Sen [42]; Fleurbaey [17, 18, 19]; Herrero [24]; Bossert and Fleurbaey [9]; Kranich [26]; Roemer [39, 40]; Laslier et al. [27]; Moreno-Ternero and Roemer [36]. This paper is closer in spirit to Bénabou and Ok [6] as they do not refer to responsibility. However, our focus is different in that we attempt to derive the desirability of equality from first principles.

[5] Our analysis here continues a research programme started in Mariotti and Veneziani [31], where we explored the notion of opportunities as chances in life and characterised a utilitarian ordering. Unlike in the latter contribution, however, we focus on ethical properties capturing a liberal perspective in social choice.

[6] It is standard in the social choice literature to call Nash orderings the binary relations that aggregate multiplicatively (see, e.g., Moulin [37], p.37 and Roemer [39], pp.61ff), and so we follow this convention.

utility or the utility of the worst off.[7] In our framework the Nash product, too, acquires a transparent meaning: to maximise the Nash product means to maximise the *probability that everybody in society succeeds* under the assumption that the individuals are independent experiments.[8]

An interesting feature of the Nash criterion in a context of opportunity profiles is a strongly egalitarian implication. In fact, it is sufficient for a profile to include one agent who fails with certainty for this profile to be at the bottom of the social ranking (no matter how many other individuals succeed). This answers the question of the opening paragraph.

Furthermore, we address a refinement issue. The straightforward application of the Nash criterion suffers from some lack of discriminatory power at the boundaries: we cannot distinguish situations where many individuals fail from situations in which only one of them fails (only weak, and not strong, Pareto is satisfied on the set of profiles in which some of the individuals fail). To address this issue, we also formulate a new variant of the Nash criterion, the *Two-Step Nash* criterion. This criterion refines the indifference classes and satisfies strong Pareto.

In the appendix, we extend our analysis to situations where the number of agents is infinite. This is relevant in intergenerational allocation problems. A concrete example of 'success' for a generation is the ability to enjoy a clean environment. At a much more abstract level, in an "Aristotelian" perspective, *self-realisation* - intended as developing human capacities - could be taken as the fundamental objective of mankind. In this interpretation, the probability of success of a generation is the probability that the generation will develop its inherently human capacities.[9] At the formal level, the main novelty is the introduction of the *Nash catching up* and the *Nash overtaking* criteria. This part of the analysis fits in a voluminous stream of recent work (including Alcantud [1], Asheim and Banerjee [3], Basu and Mitra [5], Bossert *et al* [10]. For a detailed survey, see Asheim [2]), and is necessarily more technical in nature.

[7] Provided of course that the appropriate assumptions on the comparability of the units and origin of the utility scale are made.

[8] Note that the independence assumption just concerns the *interpretation* of our model. It is not a formal assumption that underlies the formal results. We discuss independence in the conclusions.

[9] Indeed, our formal framework can be equivalently interpreted as focusing on the degree (in a scale from 0 to 1) of self-realisation (or success) of each generation or the proportion of individuals in a given generation who realise their potential (or are successful). More generally, although we believe that an interpretation of opportunities as chances does provide interesting new insights on a variety of social issues, from a strictly formal viewpoint, our analysis applies to any setting in which the measure of advantage lies in the $[0, 1]$ interval. So, for example, our notion of opportunities could be related to that of QALYs used in the health literature. We discuss this aspect in section 8. We thank an anonymous referee for this suggestion.

2 Opportunities in the box of life

There is a finite set of agents $\mathcal{N} = \{1, 2, ..., T\}$ in society. An *opportunity* for individual t in $\mathcal{N}$ is a number between 0 and 1. This number is interpreted as a 'chance of success' either in some given field or in life as a whole,[10] so that opportunities can be manipulated just as probabilities. We are interested in how opportunities should be allocated among the T individuals. The underlying idea is that some (limited) resources (possibly money) can be allocated so as to influence the distribution of opportunities.[11] An *opportunity profile* (or simply a *profile*) is a point in the '*box of life*' $B^T = [0,1]^T$. A profile $a = (a_1, a_2, ..., a_T) \in B^T$ lists the opportunities, or 'chances of success' of agents in $\mathcal{N}$ if a is chosen.

The points $\mathbf{0} = (0, 0, ..., 0) \in B^T$ and $\mathbf{1} = (1, 1, ..., 1) \in B^T$ can be thought of as *Hell* (no opportunities for anybody) and *Heaven* (full opportunities for everybody), respectively. We will also say that individual t *is in Hell* (resp., *Heaven*) at a if $a_t = 0$ (resp., $a_t = 1$). Let $B_+^T = \left\{a \in B^T | a \gg \mathbf{0}\right\}$ denote the interior of the box of life.[12]

A permutation π is a bijection of $\mathcal{N}$ onto itself. For any $a \in B^T$ and any permutation π, denote $\pi a = \left(a_{\pi(t)}\right)_{t \in \mathcal{N}}$, which we call a *permutation of* a. For all $a \in B^T$, let $\overline{a}$ be the permutation of a which ranks its elements in ascending order.

We aim to specify desirable properties for a *social opportunity relation* $\succcurlyeq^S$ on the box of life B^T.[13]

Two properties for $\succcurlyeq^S$ are the following, for all $a, b \in B^T$:

Strong Pareto: $a > b \Rightarrow a \succ^S b$.

Anonymity: $a = \pi b$ for some permutation $\pi \Rightarrow a \sim^S b$.

These properties are standard and will not be discussed further. We now define the two relations on the box of life that are the main object of this study.[14] For all $a, b \in B^T$, the

[10] Leading examples of 'success' that appear in the social policy literature are the following: no teenage childbearing; not dropping out of school; attainment of x years of formal education; attainment of fraction α of the average hourly wage, or yearly income; no male idleness (this is defined in Mayer [34] as the condition of a 24-year old not in school and not having done paid work during the previous year); no single motherhood. In a health context, success may be defined, for instance, by: surviving until age y; surviving a given operation; (for a group) mortality and morbidity below percentage β of a reference group's average; 'good health' (vs. death), as in the 'standard gamble' of the QALYs approach. In a social psychology context, success may be related to reported happiness being within a certain quantile of the population. And so on.

[11] See Mayer [34] for an interesting counterpoint to the effect of money on children's life chances.

[12] Vector notation: for all $a, b \in B^T$ we write $a \geq b$ to mean $a_t \geq b_t$, for all $t \in \mathcal{N}$; $a > b$ to mean $a \geq b$ and $a \neq b$; and $a \gg b$ to mean $a_t > b_t$, for all $t \in \mathcal{N}$.

[13] Given a binary relation $\succcurlyeq$ on a set X and $x, y \in X$, we write $x \succ y$ (the asymmetric factor) if and only if $x \succcurlyeq y$ and $y \not\succcurlyeq x$, and we write $x \sim y$ (the symmetric part) if and only if $x \succcurlyeq y$ and $y \succcurlyeq x$.

[14] We recall here some standard terminology. A binary relation $\succcurlyeq$ on a set X is said to be: *reflexive* if,

Nash social opportunity ordering $\succcurlyeq^N$ aggregates chances of success by multiplication:

$$a \succcurlyeq^N b \Leftrightarrow \prod_{t=1}^{T} a_t \geq \prod_{t=1}^{T} b_t.$$

Next, we introduce a new refinement of the Nash ordering on the boundary of the box of life, which we call the **Two-Step Nash social opportunity ordering** $\succcurlyeq^{2N}$. For all $a \in B^T$, let $P^a = \{t \in \mathcal{N} : a_t > 0\}$ and let $|P^a|$ denote the cardinality of P^a. Then for all $a, b \in B^T$:

$$\begin{aligned} a \quad \succ \quad ^{2N} & b \Leftrightarrow \text{either } |P^a| > |P^b|, \\ \text{or } |P^a| \quad = \quad & |P^b| \ \& \prod_{t \in P^a} a_t > \prod_{t \in P^b} b_t. \end{aligned}$$

Thus also:

$$a \sim^{2N} b \Leftrightarrow \left(|P^a| = |P^b|\right) \ \& \ \left(\prod_{t \in P^a} a_t = \prod_{t \in P^b} b_t\right),$$

which includes the case $|P^a| = |P^b| = 0$ and $a = b = \mathbf{0}$.[15] So, the Two-Step Nash ordering is equivalent to the standard Nash ordering when at least one profile is on the interior of the box of life, but unlike the standard Nash ordering it does not consider all profiles on the boundary indifferent to each other. If at least one of the two profiles has (at least) a zero component we count the positive entries. If they have the same number of positive entries, we apply Nash to them. If not, then the profile with the higher number of positive entries is preferred.

3 A Non-Interference Principle

Imagine that success is achieved by overcoming a series of 'hurdles'. For example, in order to be succesful in becoming a doctor, being a dustman's daughter combines hurdles that a doctor's son does not face (less favourable studying environment, lack of a high-level social network, possibly gender, and so on). A different example comes from Deaton's [12] idea of escape we mentioned earlier. It is instructive to cite the list of 'lucks' that Deaton deems crucial for his father's success (escape), because it is a concrete illustration of the "to success through hurdles" view that we propose here:

for any $x \in X$, $x \succcurlyeq x$; *complete* if, for any $x, y \in X$, $x \neq y$ implies $x \succcurlyeq y$ or $y \succcurlyeq x$; *transitive* if, for any $x, y, z \in X$, $x \succcurlyeq y \succcurlyeq z$ implies $x \succcurlyeq z$. $\succcurlyeq$ is a *quasi-ordering* if it is reflexive and transitive, while $\succcurlyeq$ is an *ordering* if it is a complete quasi-ordering.

[15] We use the convention that $\prod_{t \in P^a} a_t = \prod_{t \in P^b} b_t = 1$ when $P^a = P^b = \varnothing$.

"the luck not to be among those who died as children, the luck to be rescued from the pit by the war, the luck not to be on the wrong commando raid, the luck not to die from tuberculosis, and the luck to get a job in an easy labor market" ([12], Preface).

Observe that our view of success can encompass two distinct types of situations. In one, hurdles are 'events' that can happen to individuals (e.g. being drafted in the army). In the other, they are more like given individual characteristics (e.g. being a dustman's daughter).[16]

We consider hurdles such that the addition or removal of a hurdle has a multiplicative effect on the probability of success.[17] With this interpretation in mind, the next axiom imposes some minimal limits on the interference of society on an individual's opportunities. We assume that an individual has the right to prevent society from acting against her in all circumstances of reduction in her opportunities (due to an increase in the hurdles she faces), *whenever* the opportunities of no other individual are affected. By 'acting against her' we mean a switch against the individual in society's strict rankings of the chance profiles, with respect to the ranking of the original profiles (before the change in hurdles for the individual under consideration occurred). Crucially, the principle says nothing on society's possible actions aimed at increasing the individual's opportunities: an individual facing additional hurdles cannot demand (on the basis of our axiom) to be compensated by a switch of society's ranking in her favour. In this sense the principle we propose is libertarian rather than egalitarian.[18]

Probabilistic Harm Principle: Let $a, b, a', b' \in B^T$ be such that $a \succ^S b$ and, for some $t \in \mathcal{N}$ and for some $\rho \in (0, 1)$,

$$\begin{aligned} a'_t &= \rho \cdot a_t, \\ b'_t &= \rho \cdot b_t, \\ a'_j &= a_j, \text{ for all } j \neq t, \\ b'_j &= b_j, \text{ for all } j \neq t. \end{aligned}$$

[16]Obviously whether a given event is a hurdle depends on individual circumstances: for somebody who already has a job with good career prospects rather than a mining job in Scotland, being drafted in the army would be a hurdle, not an advantage.

[17]In other words, all hurdles whose effects are correlated are lumped together. For example, surviving tubercolosis and being drafted in the army can be considered non-correlated and thus separate hurdles, like being female and having a dustman father.

[18]In Mariotti and Veneziani [30], we explore a more radical formalisation of the principle, applied not to chances but to welfare levels, in which the 'no harm' conclusion follows even when the reduction in welfare is not proportional. This leads to the leximin principle. (See also Lombardi *et al* [28] for an extension.)

From a philosophical viewpoint, we interpret this principle as an incarnation of J.S. Mill's 'Harm Principle'. We dwell on philosophical issues in Mariotti and Veneziani [33].

Then $b' \not\succ^S a'$ whenever $a'_t > b'_t$.

In other words, when comparing two pairs of profiles interpreted as involving losses of opportunities for only individual t from an initial situation a, b to a final situation a', b' as described, there are three possibilities:

- Individual t is *compensated* for her loss (society abandons the strict preference for t's lower-chances profile).
- Individual t is *not harmed* further beyond the given opportunity damage (society prefers always the lower-chances or always the higher-chances profile for t).
- Individual t is *punished* (society switches preference from t's higher-chances profile to t's lower-chances profile).

What the Probabilistic Harm Principle does is to exclude the third possibility. Society's choice should not become less favourable to somebody *solely* because her position has worsened, without affecting others' opportunities.

Observe how in formulating this principle the *cause* of the reduction in opportunities for individual t (i.e. the specific hurdles that are raised) is completely ignored. It may have happened because of carelessness or because of sheer bad luck. All that matters is that *the other individuals are not affected* by individual t's change.

Note also the conclusion $b' \not\succ^S a'$ in the statement of the axiom. The veto power of the individual whose opportunities have decreased is limited, in that she cannot impose on society a ranking in complete agreement with her chances. This feature becomes especially relevant if we allow $\succcurlyeq^S$ to be incomplete (as in the impossibility results below), for in this case $b' \not\succ^S a'$ does not imply $a' \succcurlyeq^S b'$ and thus the requirement of the axiom becomes even weaker: the individual cannot prevent society from declaring the two alternatives either indifferent or noncomparable in the face of her strict preference.

At the formal level, observe that we allow for the possibility that $b_t = b'_t = 0$. Below we also explore another liberal axiom in which we require $b_t > 0$. This is important from both the theoretical and the analytical viewpoint. Theoretically, the question is whether the principle should be restricted to situations where a damage occurs in the strict sense, i.e. where opportunities strictly decrease. This may seem reasonable, but maybe it is not. If $b_t = 0$, - so that an agent would be in Hell both before and after the harm, should society choose against her, - changing social preferences to $b' \succ^S a'$ might be regarded as a very heavy punishment indeed on the logic of the Probabilistic Harm Principle.

Because the Probabilistic Harm Principle protects individuals from interference after a *reduction* in their opportunities, it may be tempting to conclude that it incorporates some form of minimal inequality aversion, or a moderate form of priority for the worse-off. At a closer look, however, the strongly individualistic and non-aggregative nature of the Principle suggests that this interpretation is misleading: the Probabilistic Harm Principle focuses on changes in the situation of a single agent while keeping everyone else indifferent, and ignores all information about both individual opportunity levels and differences in opportunity levels between individuals. Indeed, there exist several non-egalitarian or even anti-egalitarian social opportunity orderings that satisfy the Principle:

- *Universal indifference*: in this ordering all states in the box of life are indifferent. It satisfies Anonymity and (vacuously) the Probabilistic Harm Principle while being definitionally insensitive to any equity considerations.

- *Sufficientarianism*: let $\alpha \in B$ denote an (ethically determined) threshold denoting a sufficient, or satisfactory chance of success in life, and for all $a \in B^T$, let $P^a(\alpha) = \{i \in \mathcal{N} : a_i \geq \alpha\}$ denote the set of individuals who have a 'sufficient' chance of success at profile a. Then, for all $a, b \in B^T : a \succcurlyeq^\alpha b \Leftrightarrow |P^a(\alpha)| \geq \left|P^b(\alpha)\right|$. The sufficientarian ordering $\succcurlyeq^\alpha$ satisfies Anonymity and the Probabilistic Harm Principle, but it incorporates no concern for equality.[19] Indeed, sufficientarianism has explicitly been proposed as an alternative to egalitarianism and embodies the intuition that "equality is not, as such, of particular moral importance" (Frankfurt [21], p.21).

- *Lexicographic dictatorships*: these satisfy Strong Pareto and the Probabilistic Harm Principle, but are in direct conflict with any egalitarian (or even fairness) concerns.[20]

Observe that none of these three example orderings has a multiplicative structure à la Nash. This structure will emerge in our characterisations through the joint action of the Probabilistic Harm Principle together with other axioms.

4 Impossibilities

When attempting to apply the Probabilistic Harm Principle - together with the other basic requirements of Anonymity and Strong Pareto - we are immediately confronted with a difficulty.

[19] It is easy to show, for example, that $\succcurlyeq^\alpha$ violates both the Pigou-Dalton condition and the so-called Hammond Equity axiom (Hammond [22]).

[20] Formally, lexicographic dictatorships are defined as follows. Let π^L be a given permutation of $\mathcal{N}$ and for all $a \in B^T$, let $a^L = \pi^L a$. Then, for all $a, b \in B^T$, $a \sim^L b \Leftrightarrow a^L = b^L$ and $a \succ^L b \Leftrightarrow$ either $a_1^L > b_1^L$, or there is $t' > 1 : a_t^L = b_t^L$, for all $t < t'$, and $a_{t'}^L > b_{t'}^L$. The *class* of lexicographic dictatorships is identified varying π^L.

Theorem 1. *There exists no transitive social opportunity relation $\succcurlyeq^S$ on B^T that satisfies* ***Anonymity****,* ***Strong Pareto****, and* ***Probabilistic Harm Principle****.*

Proof: By example. Consider the profiles

$$a = (a_1, 0, x, x, ..., x)\,,\ \ b = (0, b_2, x, x, ..., x)\,,$$

where $1 \geq a_1 > b_2 > 0$ and $x \in [0,1]$, so that $a, b \in B^T$. By **Anonymity** and **Strong Pareto**, together with transitivity, we have $a \succ^S b$.

Consider next the following profiles obtained from a, b:

$$a' = (a'_1, 0, x, x, ..., x)\,, b' = b = (0, b_2, x, x, ..., x)$$

where $a'_1 = \rho a_1$, $b'_1 = \rho b_1 = 0$, for some $\rho \in (0,1)$ such that $\rho a_1 < b_2$. Since $a', b' \in B^T$ and $\rho a_1 > \rho b_1$, then by **Probabilistic Harm Principle**, it follows that $b' \not\succ^S a'$. However, by **Anonymity** and **Strong Pareto**, together with transitivity, $b' \succ^S a'$, a contradiction. ■

Observe that this result holds for social opportunity relations which are possibly incomplete. And even transitivity can be dispensed with, provided that Anonymity and Strong Pareto are replaced by the following axiom.

Suppes-Sen Grading Principle: If $a > \pi b$ for some permutation π then $a \succ^S b$.

Corollary 1. *There exists no social opportunity relation $\succcurlyeq^S$ on B^T that satisfies* ***Suppes-Sen Grading Principle*** *and* ***Probabilistic Harm Principle****.*

Proof: Straightforward modification of the previous proof. ■

It is worth noting that previous impossibility results concerning the application of the Nash criterion in the context of welfare orderings focus on axioms of a different nature, emphasising the role of continuity and ratio-scale invariance (see, e.g., Tsui and Weymark's [43] Theorem 1). The Probabilistic Harm Principle is logically strictly weaker than ratio-scale invariance (beside being interpreted very differently), given the restrictions $a'_t > b'_t$ and especially $\rho \in (0,1)$. In addition to that, the consequent in the statement of the axiom only requires that society's strict preference should not be reversed (which in our case allows both for indifference and for noncomparability). A further difference concerns the fact that, as noted, we dispense with both the completeness and the transitivity of $\succcurlyeq^S$.[21]

The result originates in the structure of the space of alternatives and the properties of the boundary of the box of life, coupled with the fact that the Probabilistic Harm Principle

[21] An equivalent of Theorem 1 holds also for infinite societies using Finite Anonymity and the infinite version of the Probabilistic Harm Principle defined in the Appendix.

applies also to profiles on the boundary, and to boundary values $b_t = 0$. In this sense, while the impossibility is formally robust, in that it holds for several combinations of similar axioms (e.g. Strong Pareto in the statement could be weakened in some ways) we do not deem it as expressing any deep contradiction between normative principles.

In the sequel we explore two specific ways in which the principles can be essentially reconciled. The first strategy consists of weakening Strong Pareto. For all $a, b \in B^T$:[22]

Pareto: $a > b \Rightarrow a \succsim^S b$ and $a \gg b \Rightarrow a \succ^S b$.

Weakening Strong Pareto to Pareto makes room for the addition of another desirable principle of social rationality.

5 Social Rationality

The new type of property we examine concerns the 'rationality' of the social opportunity relation. Consider first an axiom analogous to the sure-thing type of principle underlying Harsanyi's [23] defense of Utilitarianism (in a welfare context):

Sure Thing: Let $a, b, a', b' \in B^T$. If $a \succsim^S b$ and $a' \succsim^S b'$, then

$$\forall \lambda \in (0,1) : \lambda a + (1-\lambda)\, a' \succsim^S \lambda b + (1-\lambda)\, b',$$

with $\lambda a + (1-\lambda)\, a' \succ^S \lambda b + (1-\lambda)\, b'$ if at least one of the two preferences in the premise is strict.

Sure Thing is a classic independence property, and it can be justified in a standard way as follows. Denote the compound profiles $a'' = \lambda a + (1-\lambda)\, a'$ and $b'' = \lambda b + (1-\lambda)\, b'$. The profile a'' can be thought of as being obtained by means of a two-stage lottery: first, an event E can occur with probability λ. Then, if E occurs the profile is a, and otherwise it is a'. And b'' can be described analogously, as a compound event conditional on the occurrence or not of E. Then, when choosing between a'' and b'', it seems natural to adhere to this decomposition: if E occurs, it would have been better to choose a'' since a is better than b; and if E does not occur it would also have been better to choose a'' since a' is better than b'. Therefore, a'' should be regarded as better than b'' before knowing whether E occurs or not.

We think that a property akin to Sure Thing should be imposed, but that, in its full force, it displays some ethically unattractive features. The following argument may be reminiscent

[22] This is property S1 in Diamond's [13] classic paper.

of the classic 'Diamond critique' of the similar property in Harsanyi's Utilitarianism[23] (note that a Utilitarian social opportunity ordering would satisfy Sure Thing). Consider:

$$a = a' = b' = (0, 1)\,,\ b = (1, 0)\,,\ \lambda = \frac{1}{2}.$$

Then if Anonymity applies we have

$$a \sim^S b' \sim^S a' \sim^S b,$$

and by Sure Thing

$$a'' = (0, 1) \sim^S \left(\frac{1}{2}, \frac{1}{2}\right) = b''.$$

But having one individual in Hell and the other in Heaven for sure can hardly be reasonably regarded as socially indifferent to both individuals being half way between Heaven and Hell in the box of life. As Diamond ([14], p.766) would put it, "[b''] seems strictly preferable to me since it gives 1 a fair share while [a''] does not".

The reason for this unacceptable situation is, obviously, that 'mixing' opportunities across different individuals may produce ethically relevant effects. The problem of properties like Sure Thing, both in a utility context and in the present one, is precisely the potentially beneficial effect of this sort of 'diagonal mixing' in the box of life.

However, the property is immune from this line of criticism when the allowable mixings are restricted to ones that are parallel to the edges of the box: namely, the compound lotteries only concern *a single individual*. This seems to capture a position *à la* Diamond: "I am willing to accept the sure-thing principle for individual choice but not for social choice" ([14], p.766).

The following weakening of Sure Thing is then responsive to the Diamond critique:

Individual Sure Thing: Let $a, b \in B^T$ be such that $a \succcurlyeq^S b$ and let $a', b' \in B^T$ be such that there exists $t \in \mathcal{N}$ such that $a'_j = a_j$ and $b'_j = b_j$, for all $j \neq t$, and $a' \succcurlyeq^S b'$. Then

$$\forall \lambda \in (0, 1) : \lambda a + (1 - \lambda)\, a' \succcurlyeq^S \lambda b + (1 - \lambda)\, b',$$

with $\lambda a + (1 - \lambda)\, a' \succ^S \lambda b + (1 - \lambda)\, b'$ if at least one of the two preferences in the premise is strict.

6 Nash Retrouvé

Before proving the main characterisation result of this section, we establish a preliminary result, which is of interest in its own right. The Lemma proves that any two profiles that

[23] See also Fleurbaey [20]. To avoid confusions, we point out that Harsanyi and Diamond were discussing the application of the independence axiom to the lottery of social outcomes, while here the focus is on the vector of lotteries faced by individuals.

imply Hell for at least one individual are socially indifferent (we address this feature in the next section).

Lemma 1 *Let the social opportunity ordering $\succsim^S$ on B^T satisfy* ***Anonymity, Pareto, Probabilistic Harm Principle****, and* ***Individual Sure Thing****. Then, for all $a, b \in B^T$: $[a_t = 0,\ b_{t'} = 0,\ some\ t, t' \in \mathcal{N}] \Rightarrow a \sim^S b$.*

Proof: Consider any $a, b \in B^T$ such that $a_t = 0$, $b_{t'} = 0$, some $t, t' \in \mathcal{N}$. Suppose, without loss of generality, that $T > z = |P^b| \geq |P^a|$, and denote $h = |P^b| - |P^a|$. We proceed by induction on h.

1. ($h = 0$) Let $T > |P^b| = |P^a| = z$. If $z = 0$, then the result follows from reflexivity. If $z > 0$ and there is a permutation π such that $a = \pi b$, the result follows from **Anonymity**. Therefore suppose that $z > 0$ and there is no permutation π such that $a = \pi b$. Suppose, by way of contradiction, that $a \nsim^S b$. By completeness, and without loss of generality, suppose that $a \succ^S b$. By **Anonymity** and transitivity, we can focus on the ranked profiles $\overline{a}, \overline{b}$ where by assumption:

$$\overline{a} = (0, 0, ..., 0, \overline{a}_l, ..., \overline{a}_T)\,,\ \overline{b} = \left(0, 0, ..., 0, \overline{b}_l, ..., \overline{b}_T\right)\,,$$

and $z = T - l + 1$. We need to consider two cases.

Case 1. Suppose that $\overline{a} > \overline{b}$ and consider any k, $l \leq k \leq T$, such that $\overline{a}_k > \overline{b}_k$. Consider a profile a^π which is a permutation π of $\overline{a}$ such that $a^\pi_1 = \overline{a}_k$, $a^\pi_k = \overline{a}_1 = 0$, and all other entries are the same. By **Anonymity** and transitivity, $a^\pi \succ^S \overline{b}$.

Then, consider the profiles $a', b' \in B^T$ obtained from $a^\pi, \overline{b}$ as follows: $a'_1 = \rho a^\pi_1 = \rho \overline{a}_k < \overline{b}_k$, $b'_1 = \rho \overline{b}_1 = 0$, for some $\rho \in (0, 1)$, and $a'_j = a^\pi_j$, $b'_j = \overline{b}_j$ all $j \neq 1$. Since $a'_1 > b'_1$, **Probabilistic Harm Principle** implies $a' \succsim^S b'$. Therefore by **Individual Sure Thing**, $a^\lambda = \lambda a^\pi + (1 - \lambda) a' \succ^S \lambda \overline{b} + (1 - \lambda) b' = \overline{b}$, for all $\lambda \in (0, 1)$. Since $a^\pi_1 = \overline{a}_k > \overline{b}_k$ and $a'_1 < \overline{b}_k$, it follows that there is $\lambda^* \in (0, 1)$ such that $a^{\lambda^*}_1 = \overline{b}_k$. Let π^{-1} be the inverse of permutation π and let $a^1 = \pi^{-1} a^{\lambda^*}$. By **Anonymity** and transitivity, $a^1 \succ^S \overline{b}$.

If for all $j \neq k, l \leq j \leq T$, $\overline{a}_j = \overline{b}_j$, then $a^1 = \overline{b}$ and we obtain a contradiction by reflexivity. If, instead, there is $k' \neq k$ such that $\overline{a}_{k'} > \overline{b}_{k'}$, then the same argument can be applied iteratively m times to all entries of $\overline{a}$ such that $\overline{a}_j > \overline{b}_j$ to obtain a profile $a^m \in B^T$ such that $a^m \succ^S \overline{b}$, but $a^m = \overline{b}$, yielding a contradiction by reflexivity.

Case 2. Suppose that there exists some k, $l \leq k \leq T$, such that $\overline{a}_k < \overline{b}_k$. Let $L\left(\overline{a}, \overline{b}\right) = \left\{j \in \mathcal{N} | \overline{a}_j < \overline{b}_j\right\}$. Then consider the profile $\overline{b}' \in B^T$ such that $\overline{b}'_j = \overline{b}_j$ for all $j \notin L\left(\overline{a}, \overline{b}\right)$ and $\overline{b}'_j = \overline{a}_j$, for all $j \in L\left(\overline{a}, \overline{b}\right)$. By **Pareto** together with transitivity, it follows that $\overline{a} \succ^S \overline{b}'$. Then the same argument as in case 1 can be applied to derive the desired contradiction.

2. (Inductive step) Suppose the result holds for $T - 1 > h - 1 \geq 0$. Consider $a, b \in B^T$ such that $|P^b| < T$ and $|P^b| - |P^a| = h > 0$. Suppose, by way of contradiction, that $a \nsim^S b$.

By completeness, suppose that $a \succ^S b$. By **Anonymity** and transitivity, we can focus on the ranked profiles $\overline{a}, \overline{b}$ where by construction:

$$\overline{a} = (0, 0, ..., 0, \overline{a}_l, ..., \overline{a}_T), \ \overline{b} = \left(0, 0, ..., 0, \overline{b}_{l-h}, ..., \overline{b}_T\right),$$

with $l > l - h > 1$. Then consider the profile $a' \in B^T$ which is obtained from $\overline{a}$ as follows: $a'_j = \overline{a}_j$, for all $j \neq l-1$, and $a'_{l-1} \in (0,1]$: $a' = \left(0, 0, ..., a'_{l-1}, \overline{a}_l, ..., \overline{a}_T\right)$. By construction $|P^{\overline{b}}| - |P^{a'}| = h - 1 \geq 0$ and thus by the induction hypothesis, $a' \sim^S \overline{b}$. Then, by **Individual Sure Thing**, it follows that $a^\lambda = \lambda \overline{a} + (1-\lambda) a' \succ^S \overline{b} = \lambda \overline{b} + (1-\lambda) \overline{b}$, for all $\lambda \in (0,1)$. However, since $|P^{\overline{b}}| - |P^{a^\lambda}| = h - 1$ it must be $a^\lambda \sim^S \overline{b}$ by the induction hypothesis, a contradiction.

A similar argument rules out the possibility that $b \succ^S a$. ■

Given Lemma 1, we can now show that an ordering in the box of life can be completely characterised by the four axioms discussed before.[24]

Theorem 2. *A social opportunity ordering $\succcurlyeq^S$ on B^T satisfies* ***Anonymity, Pareto, Probabilistic Harm Principle****, and* ***Individual Sure Thing*** *if and only if $\succcurlyeq^S$ is the Nash ordering $\succcurlyeq^N$.*

Proof: ($\Rightarrow$) It is immediate that the Nash ordering $\succcurlyeq^N$ satisfies all four axioms.

($\Leftarrow$) Suppose that the social opportunity ordering $\succcurlyeq^S$ on B^T satisfies **Anonymity**, **Pareto**, **Probabilistic Harm Principle**, and **Individual Sure Thing.** For any $a, b \in B^T$, we first prove that $a \sim^N b \Rightarrow a \sim^S b$ holds, and then invoke **Pareto** to prove that $a \succ^N b \Rightarrow a \succ^S b$ also holds.

1. Suppose that $a, b \in B^T$ are such that $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t$. If $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t = 1$, then $a \sim^S b$ follows from reflexivity. If $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t = 0$, then it follows from Lemma 1.

2. Therefore suppose that $1 > \prod_{t=1}^T a_t = \prod_{t=1}^T b_t > 0$. If there exists a permutation π such that $a = \pi b$, then the result follows from **Anonymity**. Therefore, suppose that there exists no permutation π such that $a = \pi b$, and suppose, by way of contradiction, that $a \nsim^S b$. By completeness, and without loss of generality, suppose that $a \succ^S b$.

Let $U(a,b) = \{t \in \mathcal{N} |\ a_t > b_t\}$ and $L(a,b) = \{t \in \mathcal{N} |\ a_t < b_t\}$. By construction, $U(a,b) \neq \varnothing$ and $L(a,b) \neq \varnothing$. Take any $k \in U(a,b)$ and $l \in L(a,b)$. Then, from a, b construct a^1, b^1 as follows: let $a^1_k = \delta a_k$, and $b^1_l = \delta b_l$, where $\delta = \max\left(\frac{b_k}{a_k}, \frac{a_l}{b_l}\right)$, and let $a^1_j = a_j$ all $j \neq k$, and $b^1_j = b_j$ all $j \neq l$. Note that $0 < \delta < 1$ by construction and so $a^1, b^1 \in B^T$. We prove that $a^1 \succ^S b^1$.

[24] The axioms in Theorem 2, and indeed in all characterisation results below, can be shown to be independent. The details are available from the authors upon request.

3. Consider a profile b^π which is a permutation π of b such that $b^\pi_k = b_l$. By **Anonymity** and transitivity, $a \succ^S b^\pi$. Then, consider profiles $a', b' \in B^T$ such that $a'_k = 0$, $b'_k = 0$, and $a'_j = a_j, b'_j = b^\pi_j$ for all $j \neq k$. By Lemma 1, $a' \sim^S b'$. Therefore by **Individual Sure Thing**, $a^\lambda = \lambda a + (1-\lambda)a' \succ^S b^\lambda = \lambda b^\pi + (1-\lambda)b'$, for all $\lambda \in (0,1)$. Note that for all $\lambda \in (0,1)$, $a^\lambda_k = \lambda a_k$, $b^\lambda_k = \lambda b^\pi_k = \lambda b_l$, and $a^\lambda_j = a_j$, $b^\lambda_j = b^\pi_j$, for all $j \neq k$. Hence, $a^1 \succ^S b^1$ follows from **Anonymity** and transitivity by setting $\lambda = \delta$.

4. By construction $1 > \prod_{t=1}^T a^1_t = \prod_{t=1}^T b^1_t > 0$ and

$$E(a,b) = \{t \in \mathcal{N} |\ a_t = b_t\} \subset E(a^1, b^1) = \left\{t \in \mathcal{N} |\ a^1_t = b^1_t\right\}$$

If $U = \{k\}$ and $L = \{l\}$, then $\frac{b_k}{a_k} = \frac{a_l}{b_l}$ and so $E(a^1, b^1) = \mathcal{N}$, yielding the desired contradiction by reflexivity. Otherwise, the previous argument can be iterated m times to obtain profiles $a^m, b^m \in B^T$ such that $a^m \succ^S b^m$, but $E(a^m, b^m) = \mathcal{N}$, which again yields a contradiction by reflexivity.

5. This proves that for all $a, b \in B^T$, $a \sim^N b$ implies $a \sim^S b$. The proof is completed in a routine way by invoking **Pareto** and transitivity. ■

The interpretation of the Nash social opportunity ordering is of interest. In the present framework, each individual is a binary experiment, with outcome either success or failure. Imagining that such experiments are independent, the requirement to maximise the Nash ordering means that chances in life should be allocated so as to *maximise the probability that everybody succeeds*. As a particular implication, the failure of even only one individual must be considered as maximally detrimental.

Contrast this attempt to maximise the probability of Heaven with a Utilitarian type of ordering, which would maximise the sum of probabilities. In the proposed interpretation, that would amount to *maximising the expected number of successes*. Clearly, such a method would be biased, compared to the one proposed, against a minority of individuals with very low probability of success.

It is also interesting to compare the use of the Nash ordering in the present framework to that in a standard utility framework. In the latter, there are two problems of interpretation.

Firstly, the meaning of a product of utilities is unclear (as noted, e.g., by Rubinstein [41][25]). In a welfare world the Utilitarian process of aggregation has a 'natural' meaning, which the Nash product lacks. But in a world of chances, a process of aggregation by product is equally natural.

[25] "The formula of the Nash bargaining solution lacks a clear meaning. What is the interpretation of the product of two von Neumann Morgenstern utility numbers?" (p. 82). The interpretation he goes on to propose is related to non-cooperative bargaining. Here we are rather interested in an interpretation of the Nash ordering as an ethical allocation method. A different interpretation in this vein is in Mariotti [29].

Secondly, the maximisation of the Nash product on the positive orthant requires the external specification of a 'welfare zero'. In a bargaining context, this is assumed to be the 'disagreement point'; but its determination in a general social choice context is unclear, and it must be based on some external argument. On the contrary, the structure of the box of life, with its internal zero, makes this problem vanish.

7 The Two-Step Nash Ordering

A drawback of the Nash ordering - a consequence of relaxing Strong Pareto - is that it yields some very large indifference classes by considering all points on the boundary of the box of life as equally good (or bad). This may be deemed undesirable from an ethical perspective, and it may be a disadvantage for practical applications. For, a profile where *all* agents (potentially a very large number of individuals) are in Hell can hardly be seen as indifferent to one in which only one of them suffers.

In this section, we explore another way out of the impossibility in which Strong Pareto is not abandoned. This requires some adjustments in the axiomatic system. We restrict the application of the Probabilistic Harm Principle to strictly positive probabilities (as we discussed in section 3, this may be a reasonable restriction). Moreover, the same liberal logic that underlies the Probabilistic Harm Principle can be argued to extend to the case of *improvements* in individual opportunities, without being restricted to harms. The new Probabilistic Non-Interference principle below incorporates this extension.[26]

Probabilistic Non-Interference: Let $a, b, a', b' \in B^T$ be such that $a \succ^S b$ and, for some $t \in \mathcal{N}$ and for some $\rho > 0$,

$$\begin{aligned} a'_t &= \rho \cdot a_t, \\ b'_t &= \rho \cdot b_t, \\ a'_j &= a_j, \text{ for all } j \neq t, \\ b'_j &= b_j, \text{ for all } j \neq t. \end{aligned}$$

Then $a' \succ^S b'$ whenever $b_t \neq 0$ and $a'_t > b'_t$.

We can now state the main characterisation of this section:

Theorem 3. *A social opportunity ordering $\succcurlyeq^S$ on B^T satisfies **Anonymity, Strong Pareto**, and **Probabilistic Non-Interference** if and only if $\succcurlyeq^S$ is the Two-Step Nash ordering $\succcurlyeq^{2N}$.*

[26] In Mariotti and Veneziani [32, 33], we discuss in detail the conceptual underpinnings and implications of this extended liberal principle.

Proof: ($\Rightarrow$) It is immediate that $\succsim^{2N}$ satisfies all three axioms.

($\Leftarrow$) Suppose that the social opportunity ordering $\succsim^S$ on B^T satisfies **Anonymity**, **Strong Pareto**, and **Probabilistic Non-Interference**. For any $a, b \in B^T$, we shall prove that (i) $a \sim^{2N} b \Rightarrow a \sim^S b$; and (ii) $a \succ^{2N} b \Rightarrow a \succ^S b$.

Claim (i). We need to consider two cases.

Case 1. Suppose that $a, b \in B^T_+$ are such that $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t$. If $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t = 1$, then the result follows from reflexivity. Therefore suppose that $1 > \prod_{t=1}^T a_t = \prod_{t=1}^T b_t > 0$.

If there exists a permutation π such that $a = \pi b$, then the result follows from **Anonymity**. Therefore, suppose that there exists no permutation π such that $a = \pi b$, and suppose, by way of contradiction that $a \nsim^S b$. By completeness, and without loss of generality, suppose that $a \succ^S b$.

Let $U(a, b) = \{t \in \mathcal{N} | \ a_t > b_t\}$ and $L(a, b) = \{t \in \mathcal{N} | \ a_t < b_t\}$. By construction, $U(a, b) \neq \varnothing$ and $L(a, b) \neq \varnothing$. Take any $k \in U(a, b)$ and $l \in L(a, b)$. Suppose that $a_k > b_l$. (An analogous argument applies in the other two subcases with $a_k \leq b_l$ and $a_l > b_k$, or with $b_l \geq a_k > b_k \geq a_l$.)

Consider a profile b^π which is a permutation π of b such that $b^\pi_k = b_l$. By **Anonymity** and transitivity, $a \succ^S b^\pi$. Then, from a, b^π construct a', b' as follows. Let $a'_k = \rho a_k$, $b'_k = \rho b^\pi_k = \rho b_l$, where $\rho = \frac{b_k}{a_k} > 0$; and $a'_j = a_j$, $b'_j = b^\pi_j$, for all $j \neq k$. Because $\rho = \frac{b_k}{a_k} < 1$, $a', b' \in B^T_+$ and $a'_k = b_k$. Further, by **Probabilistic Non-Interference**, $a'_k > b'_k$ implies $a' \succ^S b'$.

Let π^{-1} be the inverse of permutation π and let $b^1 = \pi^{-1} b'$. Let $a^1 \equiv a'$. By **Anonymity** and transitivity, $a^1 \succ^S b^1$, and by construction $\prod_{t=1}^T a^1_t = \prod_{t=1}^T b^1_t > 0$. Further, $E(a, b) = \{t \in \mathcal{N} | \ a_t = b_t\} \subset E(a^1, b^1) = \{t \in \mathcal{N} | \ a^1_t = b^1_t\}$. If $U = \{k\}$ and $L = \{l\}$, then $\frac{b_k}{a_k} = \frac{a_l}{b_l}$ and so $E(a^1, b^1) = \mathcal{N}$, yielding the desired contradiction by reflexivity. Otherwise, the previous argument can be iterated m times to obtain profiles $a^m, b^m \in B^T_+$ such that $a^m \succ^S b^m$, but $E(a^m, b^m) = \mathcal{N}$, which again yields a contradiction by reflexivity.

Case 2. Suppose that $|P^a| = |P^b| < T$ and $\prod_{t \in P^a} a_t = \prod_{t \in P^b} b_t$. If $|P^a| = |P^b| > 0$, then by **Anonymity** and transitivity, it is possible to apply the same reasoning as for case 1 to the strictly positive entries of $a, b \in B^T$ in order to obtain the desired result. If $|P^a| = |P^b| = 0$, then $a \sim^S b$ by reflexivity.

Claim (ii). We need to consider two cases.

Case 1. Suppose that $a, b \in B^T$ are such that $|P^a| = |P^b| \leq T$ and $\prod_{t \in P^a} a_t > \prod_{t \in P^b} b_t$. Then there exists $a' \in B^T$, where $a_k > a'_k > 0$, some $k \in P^a$, $a'_j = a_j$ for all $j \neq k$, and $\prod_{t \in P^a} a'_t = \prod_{t \in P^b} b_t$. By **Strong Pareto**, $a \succ^S a'$, and therefore $a \succ^S b$ follows from claim (i) and transitivity.

Case 2. Suppose that $a, b \in B^T$ are such that $|P^a| > |P^b|$. If $|P^b| = 0$, the result follows from **Strong Pareto**. Therefore, suppose that $|P^b| > 0$. By **Anonymity** and transitivity, we can focus on the ranked profiles $\overline{a}, \overline{b}$. Let $k = \min\{t \in \mathcal{N} : \overline{a}_t > 0\}$ and let $l = \min\{t \in \mathcal{N} : \overline{b}_t > 0\}$. Since $|P^a| > |P^b|$, then $k < l$ and $\overline{a}_k > \overline{b}_k = 0$. If $\overline{a}_j \geq \overline{b}_j$, for all j, $T \geq j \geq l$, then the result follows from **Strong Pareto**. Therefore suppose that $\overline{b}_h > \overline{a}_h$, some $h \geq l$, and, in contradiction with claim (ii), $\overline{b} \succ^S \overline{a}$. Then by **Anonymity** and transitivity, consider profile b^π which is a permutation π of $\overline{b}$ such that $b^\pi_k = \overline{b}_h$. Then, from $\overline{a}, b^\pi$ construct $a', b' \in B^T$ as follows: $a'_k = \rho \overline{a}_k$, $b'_k = \rho b^\pi_k = \rho \overline{b}_h$, for some $\rho \in (0,1)$ such that $b'_k = \rho \overline{b}_h < \overline{a}_h$, and $a'_j = \overline{a}_j$, $b'_j = b^\pi_j$ all $j \neq k$. Since $h \geq l > k$, then $b'_k > a'_k$, and given that $\overline{a}_k \neq 0$, by **Probabilistic Non-Interference**, it follows that $b' \succ^S a'$.

Consider the ranked profiles $\overline{a}', \overline{b}'$. Note that $k' = \min\{t \in \mathcal{N} : \overline{a}'_t > 0\} = k$. By **Anonymity** and transitivity, $\overline{b}' \succ^S \overline{a}'$. If $\overline{a}' > \overline{b}'$, then the desired contradiction follows from **Strong Pareto**. Otherwise the previous argument can be iterated (always using the k-th entry of the ranked profiles $\overline{a}$, $\overline{a}'$, and so on) until the desired contradiction ensues.

This proves that $|P^a| > |P^b|$ implies $a \succcurlyeq^S b$. Suppose, contrary to claim (ii), that $a \sim^S b$. Then, for a sufficiently small $\varepsilon > 0$, it is possible to construct a profile $a^\varepsilon \in B^T$ such that $a^\varepsilon_t = a_t - \varepsilon > 0$ for some $t \in \mathcal{N}$, $a^\varepsilon_j = a_j$ all $j \neq t$, and $|P^{a^\varepsilon}| = |P^a| > |P^b|$. By **Strong Pareto** and transitivity, $b \succ^S a^\varepsilon$ and the previous argument can be applied. ■

As the reader will have noticed, another major difference in the conditions of Theorems 2 and 3, beside those already discussed, is the absence of Individual Sure Thing in the latter. In fact, the two-step Nash ordering does not satisfy Individual Sure Thing, as the following example demonstrates:

Example 1: $a = (\frac{3}{10}, \frac{4}{10}) \sim^{2N} b = (\frac{2}{10}, \frac{6}{10})$ *and* $a' = (\frac{3}{10}, 0) \succ^{2N} b' = (\frac{2}{10}, 0)$. *However,*

$$\forall \lambda \in (0,1) : a^\lambda = \lambda a + (1-\lambda)\, a' \sim^{2N} b^\lambda = \lambda b + (1-\lambda)\, b'.$$

In fact, $\forall \lambda \in (0,1)$ $a^\lambda = (\frac{3}{10}, \lambda \frac{4}{10})$ *and* $b^\lambda = (\frac{2}{10}, \lambda \frac{6}{10})$, *and thus* $\prod_{t=1}^2 a^\lambda_t = \prod_{t=1}^2 b^\lambda_t$.

This example implies immediately, together with Theorem 3, that it is impossible to impose on a social opportunity ordering $\succcurlyeq^S$ the four properties of Anonymity, Strong Pareto, Probabilistic Non-Interference, and Individual Sure Thing.[27]

8 Relation with the SWO literature

The main goal of this paper is to study an operational version of opportunities and to illustrate a new interpretation of the Nash criterion in this context. However, we would like

[27] Indeed, it can be proved that the clash between axioms remains even if one drops transitivity, Probabilistic Non-Interference is restricted only to the Harm part, and Strong Pareto is weakened to monotonicity. We thank J.C. Rodriguez Alcantud for pointing out these possible weakenings.

to clarify the connection between our work and the literature on the Nash social welfare orderings (SWOs) for the interested reader.

A first observation to make is that the characterisation of Theorem 2 is quite separate from those of the SWO literature. In the latter, a key axiom is typically one of Scale Invariance, while Theorem 2 uses a combination of a liberal and a social rationality principle. These principles are both formally and conceptually distinct from Scale Invariance properties.

The older part of the SWO literature focuses on the strictly positive orthant only (Boadway and Bruce [7]; Moulin [37]. See also Bosi *et al* [8]) and as we have seen profiles with zero entries create special technical problems. While still using a different domain (that of the box of life) our setting is closer to two more recent contributions by Tsui and Weymark [43] and Naumova and Yanovskaya [38], who explore larger domains.

Beside the one already mentioned, a further main technical difference from these papers is that we focus on Anonymity and do not assume any continuity property, whereas continuity axioms are central in both papers. Consequently, the arguments involved are entirely different. Notably, we do not use any results from functional analysis, nor properties of social welfare functions, since we cannot assume that our social ordering is representable.

To be more specific, Tsui and Weymark ([43], Theorem 5, p.252) elegantly characterise, using techniques from functional analysis, 'Cobb-Douglas' SWOs (of which the Nash ordering is a special case) on $\mathbb{R}^T$ by a continuity axiom, Weak Pareto and Ratio Scale Invariance. Once transferred to the appropriate domain, our ranking can be seen as the anonymous case within this class (obtained via Anonymity instead of continuity). They do not characterise SWOs similar to our Two-Step Nash ordering. Naumova and Yanovskaya [38] provide a general analysis of SWOs on $\mathbb{R}^T$ that satisfy Ratio-Scale Invariance, and they do characterise some *lexicographic* social welfare functions. Essentially, as compared to [43], they weaken the continuity properties. For example, they focus on the requirement that continuity should hold within *orthants*, which are unbounded sets of vectors whose individual components have always the same sign, positive, negative or zero (therefore the vectors $(1, 0, 1)$, $(1, 1, 0)$, and $(0, 1, 1)$, for instance, belong to the box of life B^3 but to three different orthants in the sense of [38]). The lexicographic SWOs characterised there differ markedly from ours in that they require a linear ordering of the orthants and therefore vectors on the boundary of the box of life (e.g., $(1, 0, 1)$, $(1, 1, 0)$, and $(0, 1, 1)$ in B^3) will never be indifferent. Therefore, contrary to our analysis, Anonymity is violated.

Finally, we recall a somewhat related approach in the specific context of health outcomes, namely the use of 'QALYs' (Quality-Adjusted Life Year) in health economics. QALYs are numbers between zero and one that express the degree to which perfect health (with a value normalised to one, whereas death is assigned zero) is attained. There is little axiomatic work on QALYs, but a recent one that is especially relevant in our context is by Moreno-Ternero and Østerdal [35], who also propose a 'multiplicative' aggregation procedure. Aside from the broad conceptual analogy, our work and theirs are quite distinct: they exclude zero lifetimes while dealing with the zero boundary has been a main theme for us; and the primitives of their model are not profiles of QALYs but rather matrices made of duplets each characterizing one individual, where the duplet refers to quantity and quality of life, respectively.

9 Concluding Remarks

We have proposed formulating opportunities as chances of success, an interpretation close to the standard use of the term by practitioners. This interpretation is easily amenable to concrete measurement, suitable to the formulation of policy targets, and close to common usage in the public debate.

We have highlighted some interesting conflicts between principles and discussed how such conflicts can be overcome. We have shown that strong limits to inequality in the profile of opportunities are implied by a liberal principle of justice and a property of social rationality. Beside the inequality aversion (concavity) of the social criterion, even only one person failing with certainty brings down the value of *any* profile to the minimum possible.

The use of the Nash social opportunity ordering acquires a natural interpretation in this context as the probability that everybody succeeds. Although not purely egalitarian, this 'maximise the probability of Heaven' criterion is likely in practice to avoid major disparities in opportunities, as profiles involving very low opportunities for one individual will appear very low in the social ordering. And, in the two-step refinement we have proposed, Hell should also be a sparsely populated place: that is, in practice, societies in which opportunities are confined to a tiny elite should be frowned upon. These partially egalitarian conclusions look stronger when one considers that they are obtained without any reference to issues of 'talent' or 'responsibility': the conclusions are partial but unconditional.[28]

One feature of our analysis is that in the 'Maximise the probability of Heaven' interpretation of the Nash criterion we have treated individuals as *independent* experiments. Note first that this relates only to the interpretation and not to the results themselves: the Nash criterion continues to follow from the axioms even without assuming such independence.

[28] One aim of our approach is to simplify the issue of egalitarianism in a context of 'social risk' as much as possible, which is obtained by assuming that success is binary. If social risk were to be considered allowing individual outcomes to be measured along a utility scale, the definition of an appropriate concept of egalitarianism would raise many additional thorny issues. See Fleurbaey [20] for a recent insightful contribution.

Secondly, at least to some extent, independence can be guaranteed by defining the notion of success in such a way as to factor out the common variables affecting success across individuals. For example, the chances of attaining a high paying job for the dustman's daughter and for the doctor's son are both affected by the possibility of an economic recession, and must therefore be partially correlated. To obtain independence, one might define a high-paying job independently for each state of nature or as an average across states. Thirdly, it seems nevertheless of interest to consider a framework in which the *input* of the analysis is the probability distribution over all logically conceivable profiles of success and failure, so as to include explicitly the possible correlations, instead of social preferences over profiles of 'marginal' distributions. This would be appropriate in cases where the correlation device is a relevant variable under the control of the social decision maker - imagine for instance the decision whether two officials on a wartime mission should travel on the same plane or on separate planes (with each plane having a probability p of crashing). Correlations are at the core of Fleurbaey's [20] study of risky social situations, which characterises a (mild) form of ex-post egalitarianism, allowing individual outcomes to be measured along a utility scale, for a fixed and strictly positive vector of probabilities on a given set of states of the world. An interesting development of our proposal would be to study the issue of correlations in our framework, with variable probabilities and a restricted range of outcomes. This is left for future research.

10 Appendix: Intergenerational justice and the Nash criterion

The focus on joint probability of success seems, at the conceptual level, as attractive an opportunity criterion when the agents are infinite in number as when there is only a finite number of them. And yet, a large set of infinite streams of probabilities yield a zero probability of joint success, making the criterion vacuous for practical purposes.

We propose two solutions to this dilemma. They consist of adapting two well-known methods for comparing infinite streams of utilities: namely, the *overtaking* and the *catching-up* criteria. In order to obtain the desired extensions of the social opportunity relations, we add properties that permit a link with the infinite case to (analogs of) the characterising axioms of the finite case. In this way, we obtain an overtaking version of the Nash criterion and a catching-up version of the Two-Step Nash criterion.

Almost without exception all uses of the Nash criterion we are aware of apply to a finite number of agents, and therefore our proposals may be of interest in their own right.[29]

[29] The only partial exception we are aware of is Cato [11], which however only considers the Nash overtaking

The previous notation is extended in a straightforward way to societies with an infinite set of agents $\mathbb{N} = \{1, 2, ...\}$, with the following specific additions. Let $B^\infty = [0,1]^\infty$ denote the set of countably infinite streams of probabilities of success for agents in $\mathbb{N}$. A profile is now denoted ${}_1a = (a_1, a_2, ...) \in B^\infty$, where a_t is the probability of success of generation $t \in \mathbb{N}$. For $T \in \mathbb{N}$, ${}_1a_T = (a_1, ..., a_T)$ denotes the T-*head* of ${}_1a$ and ${}_{T+1}a = (a_{T+1}, a_{T+2}, ...)$ denotes its T-*tail*, so that ${}_1a = ({}_1a_T, {}_{T+1}a)$. For any $x \in B$, ${}_1\mathbf{x} = (x, x, ...) \in B^\infty$ denotes the stream of constant probabilities equal to x. For all ${}_1a \in B^\infty$ and $T \in \mathbb{N}$, let $P^{{}_1a_T} = \{t \in \{1, ..., T\} : a_t > 0\}$.

A *permutation* π is now a bijective mapping of $\mathbb{N}$ onto itself. A permutation π of $\mathbb{N}$ is finite if there is $T \in \mathbb{N}$ such that $\pi(t) = t$, for all $t > T$, and Π is the set of all finite permutations of $\mathbb{N}$. For any ${}_1a \in B^\infty$ and any π, let $\pi({}_1a) = \left(a_{\pi(t)}\right)_{t\in\mathbb{N}}$ be a permutation of ${}_1a$. Finally, a relation $\succcurlyeq'$ on B^∞ is an *extension* of $\succcurlyeq$ if $\sim \subseteq \sim'$ and $\succ \subseteq \succ'$.

We can now consider the first infinite horizon version of the Nash criterion.

The Nash overtaking criterion: For all ${}_1a, {}_1b \in B^\infty$, ${}_1a \sim^{N^*} {}_1b \Leftrightarrow \exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T}$: $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t$; and ${}_1a \succ^{N^*} {}_1b \Leftrightarrow \exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T} : \prod_{t=1}^T a_t > \prod_{t=1}^T b_t$.

The characterisation results below are based on the following axioms which are analogous to those used in the finite context.

Finite Anonymity: For all ${}_1a \in B^\infty$ and all $\pi \in \Pi$, $\pi({}_1a) \sim^S {}_1a$.

Weak Dominance: For all ${}_1a, {}_1b \in B^\infty$ and all $x, y \in B$: ${}_1a > {}_1b \Rightarrow {}_1a \succcurlyeq^S {}_1b$, and $x < y \Rightarrow {}_1\mathbf{y} \succ^S (x, {}_2\mathbf{y})$.

Probabilistic Harm Principle*: Let ${}_1a, {}_1b \in B^\infty$ be such that ${}_1a = ({}_1a_T, {}_{T+1}b)$ for some $T \in \mathbb{N}$, and ${}_1a \succ^S {}_1b$; and let ${}_1a', {}_1b' \in B^\infty$ be such that for some $t \in \mathbb{N}$, and some $\rho \in (0,1)$,

$$\begin{aligned} a'_t &= \rho \cdot a_t, \\ b'_t &= \rho \cdot b_t, \\ a'_j &= a_j \text{ , for all } j \neq t, \\ b'_j &= b_j \text{ , for all } j \neq t. \end{aligned}$$

Then ${}_1b' \not\succ^S {}_1a'$ whenever $a'_t > b'_t$.

Individual Sure Thing*: Let ${}_1a, {}_1b \in B^\infty$ be such that ${}_1a = ({}_1a_T, {}_{T+1}b)$ for some $T \in \mathbb{N}$, and ${}_1a \succcurlyeq^S {}_1b$ and let ${}_1a', {}_1b' \in B^\infty$ be such that for some $t \leq T$, $a'_j = a_j$ and $b'_j = b_j$, for all $j \neq t$, and ${}_1a' \succcurlyeq^S {}_1b'$. Then

criterion on the strictly positive orthant.

$$\forall \lambda \in (0,1) : \lambda \ {}_1a + (1-\lambda) \ {}_1a' \succcurlyeq^S \lambda \ {}_1b + (1-\lambda) \ {}_1b',$$

with $\lambda \ {}_1a + (1-\lambda) \ {}_1a' \succ^S \lambda \ {}_1b + (1-\lambda) \ {}_1b'$ if at least one of the two preferences in the premise is strict.

Like in the finite case, Strong Pareto must be weakened to avoid impossibilities: Weak Dominance is one such weakening. Similar weakenings have been used in the literature (see e.g. Basu and Mitra [4], who use a slightly stronger version of the property we propose. For a discussion, see Asheim [2]).

In addition to the above axioms, a weak consistency requirement is imposed.

Weak Consistency: For all ${}_1a, {}_1b \in B^\infty$: (i) $\exists \tilde{T} \in \mathbb{N} : ({}_1a_T, {}_{T+1}\mathbf{1}) \succ^S ({}_1b_T, {}_{T+1}\mathbf{1})$ $\forall T \geq \tilde{T} \Rightarrow {}_1a \succ^S {}_1b$; (ii) $\exists \tilde{T} \in \mathbb{N} : ({}_1a_T, {}_{T+1}\mathbf{1}) \sim^S ({}_1b_T, {}_{T+1}\mathbf{1})$ $\forall T \geq \tilde{T} \Rightarrow {}_1a \sim^S {}_1b$.

Weak Consistency provides a link to the finite setting by transforming the comparison of two infinite profiles into an infinite number of comparisons of profiles each containing a finite number of generations. Axioms similar to Weak Consistency are common in the literature (see, e.g., Basu and Mitra [5], Asheim [2], Asheim and Banerjee [3]).[30]

Finally, the next axiom requires that $\succcurlyeq^S$ be complete at least when comparing elements of B^∞ with the same tail. This requirement is weak and it seems uncontroversial, for it is desirable to be able to rank as many profiles as possible.[31]

Minimal Completeness: For all ${}_1a, {}_1b \in B^\infty$, ${}_1a \neq {}_1b$: ${}_{T+1}a = {}_{T+1}b$ for some $T \in \mathbb{N} \Rightarrow$ ${}_1a \succcurlyeq^S {}_1b$ or ${}_1b \succcurlyeq^S {}_1a$.

Before proving our main characterisation result, we state the following Lemma which extends to B^∞ the equivalent result obtained in the finite context.[32]

Lemma 2 *Let the social opportunity quasi-ordering $\succcurlyeq^S$ on B^∞ satisfy* ***Finite Anonymity, Weak Dominance, Probabilistic Harm Principle*****, **Individual Sure Thing****, and* ***Minimal Completeness.*** *Then: for all ${}_1a, {}_1b \in B^\infty$ such that ${}_{T+1}a = {}_{T+1}b$ for some $T \in \mathbb{N}$, $[a_t = 0, b_{t'} = 0, \text{ some } t, t' \in \{1, ..., T\}] \Rightarrow {}_1a \sim^S {}_1b$.*

[30] Under Strong Pareto normally one needs only part (i) of Weak Consistency (or of a similar axiom), see e.g. Asheim and Banerjee [3], in particular Proposition 2. Here we only assume Weak Dominance and therefore the results in [3] do not hold. We thank Geir Asheim for alerting us to this issue.

[31] Lombardi *et al* [28] use Minimal Completeness to characterise the infinite leximin and maximin social welfare relations. Also, we note that all binary relations considered in this section are incomplete. However, by Szpilrajn's Theorem, the set of ordering extensions we characterise is obviously not empty.

[32] The proof of Lemma 2 is a straightforward modification of the proof of Lemma 1 and therefore is omitted. Details are available from the authors upon request.

Next, we derive a useful implication of **Weak Dominance** and **Individual Sure Thing***.

Lemma 3 *Let the social opportunity quasi-ordering $\succcurlyeq^S$ on B^∞ satisfy* ***Weak Dominance*** *and* ***Individual Sure Thing****. Then: for all ${}_1a$, ${}_1b \in B^\infty$ such that ${}_{T+1}a = {}_{T+1}b = {}_{T+1}\mathbf{1}$ for some $T \in \mathbb{N}$, ${}_1a_T \gg {}_1b_T \Rightarrow {}_1a \succ^S {}_1b$.*

Proof: We proceed by induction on T.

1. ($T = 1$) Take any ${}_1a$, ${}_1b \in B^\infty$ such that $a_1 > b_1$ and ${}_2a = {}_2b = {}_2\mathbf{1}$. By **Weak Dominance**, ${}_1\mathbf{1} \succ^S (b_1, {}_2\mathbf{1})$, and so if $a_1 = 1$, the result immediately follows. Suppose $a_1 < 1$. By reflexivity, ${}_1b \sim^S {}_1b$. Hence, by **Individual Sure Thing*** it follows that for all $\lambda \in (0,1) : \lambda_1\mathbf{1} + (1-\lambda)\,{}_1b \succ^S \lambda_1 b + (1-\lambda)\,{}_1b = {}_1b$, and noting that $1 > a_1 > b_1$, we obtain ${}_1a \succ^S {}_1b$.

2. (Inductive step.) Suppose that the result holds for $T-1 \geq 1$. Consider any ${}_1a, {}_1b \in B^\infty$ such that ${}_{T+1}a = {}_{T+1}b = {}_{T+1}\mathbf{1}$ for some $T > 1$, and ${}_1a_T \gg {}_1b_T$. By the induction hypothesis, $({}_1a_{T-1}, {}_T\mathbf{1}) \succ^S ({}_1b_{T-1}, {}_T\mathbf{1})$. By **Weak Dominance**, $({}_1b_{T-1}, {}_T\mathbf{1}) \succcurlyeq^S ({}_1b_{T-1}, b_T, {}_{T+1}\mathbf{1})$ and therefore by transitivity, $({}_1a_{T-1}, {}_T\mathbf{1}) \succ^S ({}_1b_{T-1}, b_T, {}_{T+1}\mathbf{1})$. If $a_T = 1$, the desired result follows. Therefore suppose that $a_T < 1$. By **Weak Dominance**, $({}_1a_{T-1}, b_T, {}_{T+1}\mathbf{1}) \succcurlyeq^S ({}_1b_{T-1}, b_T, {}_{T+1}\mathbf{1})$. But then, by **Individual Sure Thing*** it follows that for all $\lambda \in (0,1)$: $\lambda\,({}_1a_{T-1}, {}_T\mathbf{1}) + (1-\lambda)\,({}_1a_{T-1}, b_T, {}_{T+1}\mathbf{1}) \succ^S \lambda\,({}_1b_{T-1}, b_T, {}_{T+1}\mathbf{1}) + (1-\lambda)\,({}_1b_{T-1}, b_T, {}_{T+1}\mathbf{1}) = ({}_1b_T, {}_{T+1}\mathbf{1})$, and noting that $1 > a_T > b_T$, we obtain ${}_1a \succ^S {}_1b$. ■

Given Lemmas 2 and 3, the next Theorem proves that the above axioms jointly characterise the Nash overtaking quasi-ordering.

Theorem 4. *(NASH OVERTAKING) A social opportunity quasi-ordering $\succcurlyeq^S$ on B^∞ satisfies* ***Finite Anonymity****,* ***Weak Dominance****,* ***Probabilistic Harm Principle*****,* ***Individual Sure Thing*****,* ***Weak Consistency****, and* ***Minimal Completeness*** *if and only if $\succcurlyeq^S$ is an extension of $\succcurlyeq^{N^*}$.*

Proof: ($\Rightarrow$) Let $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$. It is immediate that $\succcurlyeq^S$ meets **Finite Anonymity** and **Weak Dominance**. By observing that $\succcurlyeq^{N^*}$ is complete for comparisons between profiles with the same tail, it is also easy to see that $\succcurlyeq^S$ satisfies **Weak Consistency** and **Minimal Completeness**. We need to show that $\succcurlyeq^S$ meets **Probabilistic Harm Principle*** and **Individual Sure Thing***.

To prove that $\succcurlyeq^S$ satisfies **Probabilistic Harm Principle***, take any ${}_1a, {}_1b \in B^\infty$ such that ${}_1a = ({}_1a_{\widehat{T}}, {}_{\widehat{T}+1}b)$ for some $\widehat{T} \in \mathbb{N}$, and ${}_1a \succ^S {}_1b$. Since $\succcurlyeq^{N^*}$ is complete for comparisons between profiles with the same tail, it follows that ${}_1a \succ^{N^*} {}_1b$. Then, let ${}_1a', {}_1b' \in B^\infty$ be such that for some $t' \in \mathbb{N}$, and some $\rho \in (0,1)$, $a'_{t'} = \rho a_{t'}$, $b'_{t'} = \rho b_{t'}$, $a'_j = a_j$, all $j \neq t'$, and $b'_j = b_j$, all $j \neq t'$. We need to prove that ${}_1b' \not\succcurlyeq^S {}_1a'$ whenever $a'_{t'} > b'_{t'}$.

By definition, ${}_1a \succ^{N^*} {}_1b$ implies that $\exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T}$: $\prod_{t=1}^T a_t > \prod_{t=1}^T b_t$. Consider any $T' \geq \max\left\{t', \tilde{T}\right\}$. Then note that $\forall T \geq T'$, $\prod_{t=1}^T a_t > \prod_{t=1}^T b_t$ implies $\prod_{t=1}^T a'_t = \rho \prod_{t=1}^T a_t > \prod_{t=1}^T b'_t = \rho \prod_{t=1}^T b_t$, for all $\rho \in (0,1)$. Hence ${}_1a' \succ^{N^*} {}_1b'$, and since $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$, it follows that ${}_1b' \not\succcurlyeq^S {}_1a'$.

To prove that $\succcurlyeq^S$ satisfies **Individual Sure Thing***, take any ${}_1a, {}_1b \in B^\infty$ such that ${}_1a = ({}_1a_{\widehat{T}}, {}_{\widehat{T}+1}b)$ for some $\widehat{T} \in \mathbb{N}$, and ${}_1a \succcurlyeq^S {}_1b$, and let ${}_1a', {}_1b' \in B^\infty$ be such that for some $t' \leq \widehat{T}$, $a'_j = a_j$ and $b'_j = b_j$, all $j \neq t'$, and ${}_1a' \succcurlyeq^S {}_1b'$. We show that

$$\forall \lambda \in (0,1): \ {}_1a^\lambda = \lambda {}_1a + (1-\lambda) \ {}_1a' \succcurlyeq^S \ {}_1b^\lambda = \lambda {}_1b + (1-\lambda) \ {}_1b',$$

with ${}_1a^\lambda \succ^S {}_1b^\lambda$ if at least one of the two preferences in the premise is strict.

First of all, since $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$ and $\succcurlyeq^{N^*}$ is complete for comparisons between profiles with the same tail, it follows that ${}_1a \succcurlyeq^{N^*} {}_1b$ and ${}_1a' \succcurlyeq^{N^*} {}_1b'$.

Next, note that by construction, $a_j^\lambda = a_j = a'_j$ and $b_j^\lambda = b_j = b'_j$, all $j \neq t'$, $a_{t'}^\lambda = \lambda a_{t'} + (1-\lambda) a'_{t'}$, and $b_{t'}^\lambda = \lambda b_{t'} + (1-\lambda) b'_{t'}$, $t' \leq \widehat{T}$. Therefore for all $T \geq \widehat{T}$, $\prod_{t=1}^T a_t^\lambda = (\lambda a_{t'} + (1-\lambda) a'_{t'}) \prod_{t \neq t'} a_t = \lambda \prod_{t=1}^T a_t + (1-\lambda) \prod_{t=1}^T a'_t$. A similar argument shows that $\prod_{t=1}^T b_t^\lambda = \lambda \prod_{t=1}^T b_t + (1-\lambda) \prod_{t=1}^T b'_t$.

Suppose that ${}_{\widehat{T}+1}a = {}_{\widehat{T}+1}b \gg {}_{\widehat{T}+1}\mathbf{0}$. By definition, and noting that ${}_{\widehat{T}+1}a = {}_{\widehat{T}+1}b$, ${}_1a \succcurlyeq^{N^*} {}_1b$ implies that either $\forall T \geq \widehat{T}$: $\prod_{t=1}^T a_t > \prod_{t=1}^T b_t$, or $\forall T \geq \widehat{T}$: $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t$. And a similar argument holds for ${}_1a' \succcurlyeq^{N^*} {}_1b'$. Therefore if $\forall T \geq \widehat{T}$: $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t$ and $\prod_{t=1}^T a'_t = \prod_{t=1}^T b'_t$, it follows that $\forall T \geq \widehat{T}$: $\prod_{t=1}^T a_t^\lambda = \prod_{t=1}^T b_t^\lambda$. Instead, if either $\forall T \geq \widehat{T}$: $\prod_{t=1}^T a_t > \prod_{t=1}^T b_t$, or $\forall T \geq \widehat{T}$: $\prod_{t=1}^T a'_t > \prod_{t=1}^T b'_t$, it follows that $\forall T \geq \widehat{T}$: $\prod_{t=1}^T a_t^\lambda > \prod_{t=1}^T b_t^\lambda$. In the former case, ${}_1a^\lambda \sim^{N^*} {}_1b^\lambda$, whereas in the latter case ${}_1a^\lambda \succ^{N^*} {}_1b^\lambda$. Since $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$, the desired result follows.

Suppose that $a_{\widetilde{T}} = b_{\widetilde{T}} = 0$ for some $\widetilde{T} > \widehat{T}$. Then $\forall T \geq \widetilde{T}$: $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t = \prod_{t=1}^T a'_t = \prod_{t=1}^T b'_t = 0$, and so $\forall T \geq \widetilde{T}$: $\prod_{t=1}^T a_t^\lambda = \prod_{t=1}^T b_t^\lambda = 0$. This implies ${}_1a^\lambda \sim^{N^*} {}_1b^\lambda$ and the desired result again follows from $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$.

($\Leftarrow$) Suppose that $\succcurlyeq^S$ on B^∞ satisfies **Finite Anonymity**, **Weak Dominance**, **Probabilistic Harm Principle***, **Individual Sure Thing***, **Weak Consistency**, and **Minimal Completeness**. We show that $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$, that is, for all ${}_1a$, ${}_1b \in B^\infty$,

$$ {}_1a \sim^{N^*} {}_1b \Rightarrow {}_1a \sim^S {}_1b, \tag{1}$$

and

$$ {}_1a \succ^{N^*} {}_1b \Rightarrow {}_1a \succ^S {}_1b. \tag{2}$$

Consider (1). Take any ${}_1a,\ {}_1b \in B^\infty$ such that $\exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T} : \prod_{t=1}^{T} a_t = \prod_{t=1}^{T} b_t$. We consider two cases.

Case 1. ${}_1a \gg {}_1\mathbf{0}$ and ${}_1b \gg {}_1\mathbf{0}$. If $\exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T} : \prod_{t=1}^{T} a_t = \prod_{t=1}^{T} b_t$, then ${}_{\tilde{T}+1}a = {}_{\tilde{T}+1}b$. Suppose, in contradiction, that ${}_1a \nsim^S {}_1b$. By **Minimal Completeness**, and without loss of generality, suppose that ${}_1a \succ^S {}_1b$. Fix $T' \geq \tilde{T}$. With an argument analogous to the finite case, we can use **Weak Dominance**, **Probabilistic Harm Principle***, **Individual Sure Thing***, **Finite Anonymity**, **Minimal Completeness**, and transitivity iteratively to derive profiles ${}_1a^m, {}_1b^m \in B^\infty$ such that ${}_1a^m = ({}_1a^m_{T'}, {}_{T'+1}a) \succ^S {}_1b^m = ({}_1b^m_{T'}, {}_{T'+1}b)$, but ${}_1a^m_{T'} = {}_1b^m_{T'}$ which contradicts reflexivity.

Case 2. $a_{T'} = 0$ for some $T' \in \mathbb{N}$ and $b_{T''} = 0$ for some $T'' \in \mathbb{N}$. Take any $T \geq \max\{T', T''\}$ and consider the profiles $({}_1a_T, {}_{T+1}\mathbf{1}), ({}_1b_T, {}_{T+1}\mathbf{1}) \in B^\infty$. By Lemma 2, $({}_1a_T, {}_{T+1}\mathbf{1}) \sim^S ({}_1b_T, {}_{T+1}\mathbf{1})$. Because this is true for all $T \geq \max\{T', T''\}$, **Weak Consistency** implies ${}_1a \sim^S {}_1b$.

Consider (2). Take any ${}_1a,\ {}_1b \in B^\infty$ such that $\exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T} : \prod_{t=1}^{T} a_t > \prod_{t=1}^{T} b_t$. Take any $T \geq \tilde{T}$ and consider the profiles $({}_1a_T, {}_{T+1}\mathbf{1}), ({}_1b_T, {}_{T+1}\mathbf{1}) \in B^\infty$. Let ${}_1x \equiv ({}_1b_T, {}_{T+1}\mathbf{1})$ and ${}_1y \equiv ({}_1a_T, {}_{T+1}\mathbf{1})$.

Note that $\prod_{t=1}^{T} a_t > \prod_{t=1}^{T} b_t$ implies that ${}_1a_T \gg {}_1\mathbf{0}_T$. Hence there is a sufficiently small $\varepsilon \in B_+^T$ such that ${}_1a_T^\varepsilon = (a_1 - \varepsilon_1, a_2 - \varepsilon_2, ..., a_T - \varepsilon_T) \in B^T$ and $\prod_{t=1}^{T} (a_t - \varepsilon_t) = \prod_{t=1}^{T} b_t$. By (1), it follows that ${}_1y^\epsilon \equiv ({}_1a_T^\varepsilon, {}_{T+1}\mathbf{1}) \sim^S {}_1x$. By Lemma 3, ${}_1y \succ^S {}_1y^\epsilon$ and therefore ${}_1y \succ^S {}_1x$ by transitivity.

Therefore $({}_1a_T, {}_{T+1}\mathbf{1}) \succ^S ({}_1b_T, {}_{T+1}\mathbf{1})$ and since the argument holds for any $T \geq \tilde{T}$, it follows from **Weak Consistency** that ${}_1a \succ^S {}_1b$. ∎

Next, we provide an extension of the Two-Step Nash criterion to the infinite context in the framework of Bossert *et al* [10]. As announced, the characterisation is based on infinite versions of the axioms used in Section 7. In addition to Finite Anonymity, we consider

Strong Pareto: For all ${}_1a,\ {}_1b \in B^\infty$, ${}_1a > {}_1b \Rightarrow {}_1a \succ^S {}_1b$.

Probabilistic Non-Interference*: Let ${}_1a, {}_1b \in B^\infty$ be such that ${}_1a = ({}_1a_T, {}_{T+1}b)$ for some $T \in \mathbb{N}$, and ${}_1a \succ^S {}_1b$; and let ${}_1a', {}_1b' \in B^\infty$ be such that for some $t \in \mathbb{N}$ and some $\rho > 0$,

$$\begin{aligned} a'_t &= \rho \cdot a_t, \\ b'_t &= \rho \cdot b_t, \\ a'_j &= a_j \text{ , for all } j \neq t, \\ b'_j &= b_j \text{ , for all } j \neq t. \end{aligned}$$

Then ${}_1a' \succ^S {}_1b'$ whenever $b_t \neq 0$ and $a'_t > b'_t$.

For each $T \in \mathbb{N}$, let the Two-Step Nash ordering on B^T be denoted as $\succsim_F^{2N}$. In analogy with Bossert *et al* [10], the Two-Step Nash social opportunity relation on B^∞ can be formulated as follows. Define $\succsim_T^{2N} \subseteq B^\infty \times B^\infty$ by letting, for all ${}_1a, {}_1b \in B^\infty$,

$$ {}_1a \succsim_T^{2N} {}_1b \Leftrightarrow {}_1a_T \succsim_F^{2N} {}_1b_T \text{ and } {}_{T+1}a \geq {}_{T+1}b. \tag{3}$$

The relation $\succsim_T^{2N}$ is reflexive and transitive for all $T \in \mathbb{N}$. Then the Two-Step Nash social opportunity relation is $\succsim^{2N^*} = \bigcup_{T \in \mathbb{N}} \succsim_T^{2N}$.

Theorem 5. *(**NASH CATCHING-UP**) A social opportunity ordering $\succsim^S$ on B^∞ satisfies **Finite Anonymity**, **Strong Pareto**, and **Probabilistic Non-Interference*** if and only if $\succsim^S$ is an ordering extension of $\succsim^{2N^*}$.*

Proof: (The proof adapts the one given for the leximin catching up by Bossert *et al* [10]. We report it for clarity.)

($\Rightarrow$) We first prove that the relations $\succsim_T^{2N}$ and $\succ_T^{2N}$ are nested. That is, for all $T \in \mathbb{N}$

$$\succsim_T^{2N} \subseteq \succsim_{T+1}^{2N}, \tag{4}$$

and

$$\succ_T^{2N} \subseteq \succ_{T+1}^{2N}. \tag{5}$$

To prove (4), suppose that ${}_1a \succsim_T^{2N} {}_1b$. By definition, ${}_1a \succsim_T^{2N} {}_1b \Leftrightarrow {}_1a_T \succsim_F^{2N} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$. Then, either ${}_1a_T \succ_F^{2N} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$ or ${}_1a_T \sim_F^{2N} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$. In either case, it is immediate to prove that ${}_1a_{T+1} \succsim_{T+1}^{2N} {}_1b_{T+1}$ and ${}_{T+2}a \geq {}_{T+2}b$, and so ${}_1a \succsim_{T+1}^{2N} {}_1b$.

To prove (5), suppose that ${}_1a \succ_T^{2N} {}_1b$. By definition at least one of the following statements is true:

$$ {}_1a_T \succ_F^{2N} {}_1b_T \text{ and } {}_{T+1}a \geq {}_{T+1}b \tag{6}$$

$$ {}_1a_T \succsim_F^{2N} {}_1b_T \text{ and } {}_{T+1}a > {}_{T+1}b. \tag{7}$$

If (6) holds, then it is immediate to prove that ${}_1a_{T+1} \succ_F^{2N} {}_1b_{T+1}$ and ${}_{T+2}a \geq {}_{T+2}b$ and so ${}_1a \succ_{T+1}^{2N} {}_1b$.

So, suppose (7) holds but (6) does not. If $a_{T+1} = b_{T+1}$, then ${}_1a_T \succsim_F^{2N} {}_1b_T$ and ${}_{T+1}a > {}_{T+1}b$ implies ${}_1a_{T+1} \succsim_F^{2N} {}_1b_{T+1}$ and ${}_{T+2}a > {}_{T+2}b$. If $a_{T+1} > b_{T+1}$, then ${}_1a_T \succsim_F^{2N} {}_1b_T$ and ${}_{T+1}a > {}_{T+1}b$ implies ${}_1a_{T+1} \succ_F^{2N} {}_1b_{T+1}$ and ${}_{T+2}a \geq {}_{T+2}b$. In either case ${}_1a \succ_{T+1}^{2N} {}_1b$.

In summary, we have proved that $\succsim_T^{2N} \subseteq \succsim_{T+1}^{2N}$ and $\succ_T^{2N} \subseteq \succ_{T+1}^{2N}$.

Then, using the same arguments as in Bossert et al. ([10], Theorem 1, p.584) it can be shown that $\succsim^{2N^*}$ is reflexive and transitive, and that it satisfies the following property ([10], p.586, equation (14)):

$$\forall\ {}_1a, {}_1b \in B^{\infty} : \exists T \in \mathbb{N} \text{ such that } {}_1a \succ_T^{2N}\ {}_1b \Leftrightarrow\ {}_1a \succ^{2N^*}\ {}_1b. \tag{8}$$

In order to complete the proof of necessity, we need to prove that any ordering extension $\succsim^S$ of $\succsim^{2N^*}$ satisfies the properties in the statement.

It it is immediate that $\succsim^S$ meets **Strong Pareto** and **Finite Anonymity**. To prove that **Probabilistic Non-Interference*** is satisfied, let ${}_1a, {}_1b \in B^{\infty}$ be such that ${}_1a = ({}_1a_T, {}_{T+1}b)$ for some $T \in \mathbb{N}$ and ${}_1a \succ^S {}_1b$. Suppose that ${}_1a', {}_1b' \in B^{\infty}$ are such that for some $t' \in \mathbb{N}$, and some $\rho > 0$, $a'_{t'} = \rho a_{t'}$, $b'_{t'} = \rho b_{t'}$, and $a'_j = a_j$, $b'_j = b_j$, all $j \neq t'$. We prove that if $\succsim^{2N^*} \subseteq \succsim^S$ then ${}_1a' \succ^S {}_1b'$ whenever $b_{t'} \neq 0$ and $a'_{t'} > b'_{t'}$.

Since $\succsim^{2N^*}$ is complete for comparisons between profiles with the same tail, it follows that ${}_1a \succ^{2N^*} {}_1b$. Therefore by property (8), there exists $T' \in \mathbb{N}$ such that ${}_1a \succ_{T'}^{2N} {}_1b$. Without loss of generality, let $T' = T$. Then ${}_1a \succ_T^{2N} {}_1b$ implies ${}_1a_T \succ_F^{2N} {}_1b_T$ and ${}_{T+1}a = {}_{T+1}b$. If $|P^{{}_1a_T}| > |P^{{}_1b_T}|$, then $|P^{{}_1a'_T}| = |P^{{}_1a_T}| > |P^{{}_1b'_T}| = |P^{{}_1b_T}|$. If $|P^{{}_1a_T}| = |P^{{}_1b_T}| \leq T$ and $\prod_{t \in P^{{}_1a_T}} a_t > \prod_{t \in P^{{}_1b_T}} b_t$, then $|P^{{}_1a'_T}| = |P^{{}_1a_T}| = |P^{{}_1b'_T}| = |P^{{}_1b_T}|$, $\prod_{t \in P^{{}_1a'_T}} a'_t = \rho \prod_{t \in P^{{}_1a_T}} a_t > \prod_{t \in P^{{}_1b'_T}} b'_t = \rho \prod_{t \in P^{{}_1b_T}} b_t$. In both cases, ${}_1a'_T \succ_F^{2N} {}_1b'_T$ and ${}_{T+1}a' = {}_{T+1}b'$, so that ${}_1a' \succ_T^{2N} {}_1b'$ and therefore by property (8), ${}_1a' \succ^{2N^*} {}_1b'$. The result follows noting that $\succsim^{2N^*} \subseteq \succsim^S$.

($\Leftarrow$) Suppose that $\succsim^S$ is an ordering on B^{∞} that satisfies **Finite Anonymity**, **Strong Pareto**, and **Probabilistic Non-Interference***. Fix $T \in \mathbb{N}$ and ${}_1c \in B^{\infty}$, and define the relation $\succsim_{{}_1c}^T \subseteq B^T \times B^T$ as follows. For any ${}_1a$, ${}_1b \in B^{\infty}$,

$${}_1a_T \succsim_{{}_1c}^T\ {}_1b_T \Leftrightarrow ({}_1a_T, {}_{T+1}c) \succsim^S ({}_1b_T, {}_{T+1}c)\,.$$

$\succsim_{{}_1c}^T$ is an ordering because $\succsim^S$ is. Moreover, for any ${}_1a$, ${}_1b \in B^{\infty}$,

$${}_1a_T \succ_{{}_1c}^T\ {}_1b_T \Leftrightarrow ({}_1a_T, {}_{T+1}c) \succ^S ({}_1b_T, {}_{T+1}c)\,.$$

The three axioms imply that $\succsim_{{}_1c}^T$ must satisfy the T-person versions of the axioms. Hence, using the characterisation of the T-person Two-Step Nash social opportunity ordering in Theorem 3, it follows that

$$\succsim_{{}_1c}^T = \succsim_F^{2N}\ .$$

Because T and ${}_1c$ were chosen arbitrarily, the latter statement is true for all $T \in \mathbb{N}$ and for any ${}_1c \in B^{\infty}$.

To prove that $\succsim^S$ is an ordering extension of $\succsim^{2N^*}$, we first establish that $\succsim^{2N^*} \subseteq \succsim^S$. Suppose that ${}_1a$, ${}_1b \in B^{\infty}$ are such that ${}_1a \succsim^{2N^*} {}_1b$. By the definition of $\succsim^{2N^*}$, there exists a T such that ${}_1a \succsim_T^{2N} {}_1b$, that is, ${}_1a_T \succsim_F^{2N} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$. Then, since $\succsim_{{}_1c}^T = \succsim_F^{2N}$, it follows that ${}_1a_T \succsim_{{}_1c}^T {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$, for all ${}_1c \in B^\infty$. Choosing ${}_1c = {}_1b$ and using the definition of $\succsim_{{}_1c}^T$, it follows that $({}_1a_T, {}_{T+1}b) \succsim^S ({}_1b_T, {}_{T+1}b)$. Because ${}_{T+1}a \geq {}_{T+1}b$, either reflexivity or **Strong Pareto**, together with transitivity imply $({}_1a_T, {}_{T+1}a) \succsim^S ({}_1b_T, {}_{T+1}b)$.

The proof is completed by showing that $\succ^{2N^*} \subseteq \succ^S$. Suppose that ${}_1a, {}_1b \in B^\infty$ are such that ${}_1a \succ^{2N^*} {}_1b$. By (8), there exists $T \in \mathbb{N}$ such that ${}_1a \succ_T^{2N} {}_1b$. Hence, at least one of (6) or (7) is true.

If (6) holds, since $\succsim_{{}_1c}^T = \succsim_F^{2N}$, it follows that ${}_1a_T \succ_{{}_1c}^T {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$, for all ${}_1c \in B^\infty$. Choosing ${}_1c = {}_1b$ and using the definition of $\succ_{{}_1c}^T$, it follows that $({}_1a_T, {}_{T+1}b) \succ^S ({}_1b_T, {}_{T+1}b)$. Then using either reflexivity or **Strong Pareto**, together with transitivity as in the proof of $\succsim^{2N^*} \subseteq \succsim^S$, we obtain $({}_1a_T, {}_{T+1}a) \succ^S ({}_1b_T, {}_{T+1}b)$.

If (7) holds, since $\succsim_{{}_1c}^T = \succsim_F^{2N}$, it follows that ${}_1a_T \succsim_{{}_1c}^T {}_1b_T$ and ${}_{T+1}a > {}_{T+1}b$, for all ${}_1c \in B^\infty$. Choosing ${}_1c = {}_1b$ and using the definition of $\succsim_{{}_1c}^T$, it follows that $({}_1a_T, {}_{T+1}b) \succsim^S ({}_1b_T, {}_{T+1}b)$. Then by **Strong Pareto** and transitivity, it follows that $({}_1a_T, {}_{T+1}a) \succ^S ({}_1b_T, {}_{T+1}b)$.

Therefore $\succ^{2N^*} \subseteq \succ^S$, which concludes the proof. ■

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