RELATIVITY AND THE METROLOGY OF TIME

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The motivation for this work is two-fold: the application of general relativity to the metrology of time on one hand (part II), and the use of the methods and technology of time metrology for tests of relativity on the other (part III).

In Part II a detailed theory for the treatment of the metrology of time in a relativistic context is developed. It provides mathematical expressions for application to the syntonisation and synchronisation of clocks and the realisation of the time coordinates of space-time reference systems. The theoretical expressions are developed to accuracies exceeding those of previous publications in order to accommodate any development in clock and time-transfer technology that can be expected in the near future.

Part III presents two original experiments which test the theory of special relativity using state-of-the-art time metrology. The first experiment uses data from clock comparisons between ground clocks and clocks on board the Global Positioning System (GPS) satellites to test the second postulate of special relativity (the universality of the speed of light). The experiment is sensitive to a possible anisotropy of the one-way speed of light in any spatial direction, and on a non-laboratory scale (baselines $\geq 20000$ km) and provides the most stringent limits for the anisotropy published up to date. The second is a proposal for a test of special relativity using a spacecraft that carries an onboard atomic clock and uses a two way time transfer system. The potential accuracy of such a test is evaluated for the ESA/RSA ExTRAS (Experiment on Timing Ranging and Atmospheric Sounding) experiment which was planned for launch in 1997 but is now "on hold".
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<th>Full Form</th>
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<tr>
<td>BIPM</td>
<td>Bureau International des Poids et Mesures</td>
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<td>BRS</td>
<td>Barycentric Reference System</td>
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<tr>
<td>CCDS</td>
<td>Comité Consultatif pour la Définition de la Seconde</td>
</tr>
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<td>CCIR</td>
<td>(now ITU) International Telecommunication Union</td>
</tr>
<tr>
<td>CGPM</td>
<td>Conférence Générale des Poids et Mesures</td>
</tr>
<tr>
<td>CIPM</td>
<td>Comité International des Poids et Mesures</td>
</tr>
<tr>
<td>CODE</td>
<td>Center for Orbit Determination in Europe</td>
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<tr>
<td>EAL</td>
<td>Echelle Atomique Libre (free atomic time scale)</td>
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<tr>
<td>ESA</td>
<td>European Space Agency</td>
</tr>
<tr>
<td>ET</td>
<td>Ephemeris Time</td>
</tr>
<tr>
<td>ExTRAS</td>
<td>Experiment on Timing Ranging and Atmospheric Sounding</td>
</tr>
<tr>
<td>FTZ</td>
<td>Forschungs- und Technologie Zentrum (Germany)</td>
</tr>
<tr>
<td>GLONASS</td>
<td>GLObal NAvigation Satellite System</td>
</tr>
<tr>
<td>GP-A</td>
<td>Gravity Probe A (Vessot et al. 1980)</td>
</tr>
<tr>
<td>GPS</td>
<td>Global Positioning System</td>
</tr>
<tr>
<td>GRS</td>
<td>Geocentric Reference System</td>
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<tr>
<td>GWEP</td>
<td>Gravitational Weak Equivalence Principle</td>
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<tr>
<td>IAU</td>
<td>International Astronomical Union</td>
</tr>
<tr>
<td>IERS</td>
<td>International Earth Rotation Service</td>
</tr>
<tr>
<td>IGS</td>
<td>International GPS Service for Geodynamics</td>
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<tr>
<td>ITRF</td>
<td>IERS Terrestrial Reference Frame</td>
</tr>
<tr>
<td>JPL</td>
<td>Jet Propulsion Laboratory (USA)</td>
</tr>
<tr>
<td>LASSO</td>
<td>Laser Synchronisation from Stationary Orbit</td>
</tr>
<tr>
<td>LITS</td>
<td>Linear Ion Trap Standard</td>
</tr>
<tr>
<td>LPTF</td>
<td>Laboratoire Primaire du Temps et des Fréquences (France)</td>
</tr>
<tr>
<td>MJD</td>
<td>Modified Julian Date</td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Agency (USA)</td>
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<tr>
<td>NIST</td>
<td>National Institute of Standards and Technology (USA)</td>
</tr>
<tr>
<td>OCA</td>
<td>Observatoire de la Côte d'Azur (France)</td>
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<tr>
<td>PRARE</td>
<td>Precise Range And Range Rate Equipment</td>
</tr>
<tr>
<td>PRN</td>
<td>Pseudo Random Noise code (unique identification number of GPS satellites)</td>
</tr>
<tr>
<td>Acronym</td>
<td>Description</td>
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<tr>
<td>---------</td>
<td>-------------</td>
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<tr>
<td>PTB</td>
<td>Physikalisch Technische Bundesanstalt (Germany)</td>
</tr>
<tr>
<td>RSA</td>
<td>Russian Space Agency</td>
</tr>
<tr>
<td>SA</td>
<td>Selective Availability</td>
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<tr>
<td>SEP</td>
<td>Strong Equivalence Principle</td>
</tr>
<tr>
<td>SI</td>
<td>International System of units</td>
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<tr>
<td>STANAG</td>
<td>NATO STANdardisation AGreement</td>
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<tr>
<td>T2L2</td>
<td>Time Transfer by Laser Light</td>
</tr>
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<td>TAI</td>
<td>International Atomic Time</td>
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<tr>
<td>TCB</td>
<td>Barycentric Coordinate Time</td>
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<tr>
<td>TCG</td>
<td>Geocentric Coordinate Time</td>
</tr>
<tr>
<td>TT</td>
<td>Terrestrial Time</td>
</tr>
<tr>
<td>TWSTT</td>
<td>Two Way Satellite Time Transfer</td>
</tr>
<tr>
<td>UT1</td>
<td>Universal Time</td>
</tr>
<tr>
<td>UTC</td>
<td>Coordinated Universal Time</td>
</tr>
<tr>
<td>WGS</td>
<td>World Geodetic System</td>
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</table>
Notation

The Einstein summation convention over repeated indices is used:

\[ g_{ab} T^{ab} = \sum_a \sum_b g_{ab} T^{ab}. \]

Latin indices, \( i, j, k \ldots \) take the values 1, 2, or 3.

Greek indices, \( \alpha, \beta, \gamma \ldots \) take the values 0, 1, 2, or 3.

3-vectors, \( \nu' \), are denoted by \( \bar{\nu} \), with their magnitude, \( (\nu' \nu')^{1/2} \), denoted by \( \nu \).

Differentiation is denoted by

\[ \frac{\partial}{\partial x^\alpha} \equiv \left( \begin{array}{c} \alpha \end{array} \right). \]

The Kronecker symbol, \( \delta_j^i \), is

\[ \delta_j^i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}. \]

The metric tensor of flat space-time (special relativity), \( \eta_{\alpha\beta} \), is

\[ \eta_{00} = -1, \ \eta_{\alpha0} = \eta_{0\alpha} = 0 \]

\[ \eta_{ij} = \delta_j^i \]

The metric tensor of curved space-time is denoted by \( g_{\alpha\beta} \), with \( g_{\alpha\beta} = g_{\beta\alpha} \).

Contravariant tensors are denoted by \( A^\alpha \) with the covariant components obtained using

\[ A_\alpha = g_{\alpha\beta} A^\beta. \]
The motivation for this work is two-fold: the application of general relativity to the metrology of time on one hand (part II), and the use of the methods and technology of time metrology for tests of relativity on the other (part III). These two fields of study are, of course, closely related. With decreasing observational uncertainties applied science (time metrology and related fields in this case) requires a theoretical basis which allows the treatment of the practical problems at hand with the required accuracy. Conversely fundamental research (tests of relativity in this case) is unthinkable without the technology and know-how developed in applied fields. Studying both fields in parallel provides a "global" view which may lead to new possibilities and ideas. This thesis presents the results of such a study.

In part I of the thesis the basic theory, concepts, notation and vocabulary used throughout the work are introduced. Of particular interest are the relativistic definitions of the desynchronisation (time difference) and desyntonisation (relative rate) of clocks or coordinate time scales (sections 1.2.2 and 1.2.3) and the introduction of a notation that explicitly distinguishes the unit of proper time from the scale units of coordinate time scales (section 1.2.1). This notation is based on the distinction between two kinds of quantities (proper time and coordinate time) the theoretical basis of which is discussed in more detail in the appendix, using the concepts and principles of quantity calculus.

In Part II a detailed theory for the treatment of the metrology of time in a relativistic context is developed. It provides mathematical expressions for application to the syntonisation and synchronisation of clocks and the realisation of the time coordinates of space-time reference systems. The theoretical expressions are developed to accuracies exceeding those of previous publications in order to accommodate any development in clock and time-transfer technology that can be expected in the near future.

Section II.1 presents a relativistic theory for the syntonisation of clocks in the vicinity of the Earth (within a geocentric sphere of 300000 km radius), including all terms larger than one part in $10^{18}$. This theory is based on recent work by Wolf & Petit (1995), Petit & Wolf (1996). The space-time metric for the geocentric reference system, including terms of order $c^{-4}$, has been derived in Brumberg & Kopejkin (1988), Kopejkin (1988) and Damour, Soffel & Xu
The relationship between Geocentric Coordinate Time (TCG) and the proper time of a clock in the vicinity of the Earth is given by Brumberg & Kopejkin (1990), Klioner (1992) and Brumberg et al. (1993). In these papers tidal terms of order $10^{-17}$, including the response of an elastic Earth are given at an accuracy sufficient for the purposes of this thesis. However, effects of oceanic tides and of non-tidal origin are not mentioned, although they can reach the same order of magnitude (section II.1.1.1). Furthermore, these papers are incomplete as, on one hand, they specify tidal terms of order $10^{-17}$, while, on the other hand, the expressions given for the geopotential cannot be used for syntonisation at accuracies better than $10^{-14}$. In sections (II.1.1.1; II.1.1.2) the methods that can be used to obtain the value of the geopotential with sufficient accuracy are detailed. Using such methods the uncertainty of syntonization is of order $10^{-17}$ for clocks on the Earth's surface and $10^{-18}$ for clocks on board terrestrial satellites.

A relativistic theory for the synchronization of remote clocks in the vicinity of the Earth is presented in section II.2. Recent theoretical studies in this field claim an accuracy of 0.1 nanoseconds (Klioner 1992), and in some cases (Allan & Ashby 1986, CCIR 1990, CCDS 1980) the provided formulae are expressed in terms of path-integrals making them more difficult to use than explicit expressions. The theory presented here (based on recent work by Petit & Wolf (1994)) gives explicit expressions for the synchronization of two remote clocks including all terms that, in the vicinity of the Earth (within a geocentric sphere of 200000 km radius), are greater than one picosecond.

Part III presents two original experiments which test the theory of special relativity using state-of-the-art time metrology. The first experiment (Wolf & Petit 1996) uses data from clock comparisons between ground clocks and clocks on board the Global Positioning System (GPS) satellites to test the second postulate of special relativity (the universality of the speed of light). The experiment is sensitive to a possible anisotropy of the one-way speed of light in any spatial direction, and on a non-laboratory scale (baselines $\geq 20000$ km) and provides the most stringent limits for the anisotropy published up to date. The second (Wolf 1995) is a proposal for a test of special relativity using a spacecraft that carries an onboard atomic clock and uses a two way time transfer system. The potential accuracy of such a test is evaluated for the ESA/RSA ExTRAS (Experiment on Timing Ranging and Atmospheric Sounding) experiment which was planned for launch in 1997 but is now "on hold". Both experiments use systems that are intended primarily for use in other fields of science (metrology, navigation, geodesy, atmospheric studies) with no need for additional equipment specific to the tests of
special relativity. As a result they can be considered essentially low-cost experiments which is generally an important factor for research in fundamental science. The GPS experiment, for example, can be carried out by virtually anyone with a minimum of financial investment as all the necessary data is freely available on the internet via anonymous ftp.
I. INTRODUCTION AND CONCEPTS

In common practice relativistic effects in time metrology are taken into account by applying small relativistic corrections to results obtained on the basis of classical Newtonian theory. Such a procedure may lead to the omission or duplication of some corrections, whilst the basic principles are ignored. Furthermore Newtonian definitions, in particular of space-time reference systems, can be ambiguous in a relativistic context and have to be reconsidered leading to new, relativistic definitions like, for example, the 1980 definition of International Atomic Time (TAI) (BIPM 1980, see section I.3.3.2) and the 1991 resolution of the International Astronomical Union (IAU) (IAU 1991). Therefore, a global treatment of the metrology of time as a whole within the theoretical framework of general relativity, like the one presented here, is preferable.

In this first part of the thesis the basic theory, concepts, notation and vocabulary used throughout the work are introduced. Of particular interest are the relativistic definitions of the desynchronisation (time difference) and desyntonisation (relative rate) of clocks or coordinate time scales (sections I.2.2 and I.2.3) and the introduction of a notation that explicitly distinguishes the unit of proper time from the scale units of coordinate time scales (section I.2.1). The theoretical basis of this notation is discussed in more detail in the appendix, using the concepts and principles of quantity calculus.
I.1 General Relativity: Proper and Coordinate Quantities

Einstein's general theory of relativity is one of the cornerstones of modern physics, and as such is the subject of a large number of textbooks. A general introduction to the theory is out of the scope of this thesis, instead only the parts relevant to the issues treated in subsequent chapters will be expounded. For a more detailed description the reader is referred to e.g. Schutz (1985), Misner, Thorne and Wheeler (1973), Møller (1972), Wald (1984), Einstein (1956). A more specific treatment, related to Parts I. and II. of my thesis, is available in Brumberg (1991a) or Soffel (1989a). In Will (1993) the theory is examined in the light of experimental tests (related to Part III. of the thesis).

I.1.1. Proper and Coordinate Quantities

*It might appear possible to overcome all the difficulties attending the definition of “time” by substituting “the position of the small hand of my watch” for “time”. And in fact such a definition is satisfactory when we are concerned with defining a time exclusively for the place where the watch is located.*

Albert Einstein (1905) p. 39

*...it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning.*

Albert Einstein (in Schilpp 1949, pp. 65-67)

For the treatment of metrology within the framework of general relativity it turns out to be useful to distinguish between two kinds of quantities, proper and coordinate quantities, defined as:
(i) **Proper quantities** are the direct results of observation without involving any information that is dependent on conventions (such as, for example, the choice of a space-time reference frame or a convention of synchronisation (c.f. section I.2.2)). For the metrology of time the most fundamental such quantity is the time measured by a particular clock (the output of that clock), which will be referred to as proper time. This also includes quantities that are not the real, physical results of observations, but correspond, in principle, to proper quantities (e.g. the proper time of a clock placed at the geocentre or infinitely far from the solar system), as these are also independent of any conventions.

(ii) **Coordinate quantities** are dependent on conventional choices, e.g. of a space-time coordinate system, a convention of synchronisation etc... Examples in the context of the metrology of time are the coordinate time difference between two events (the difference between the time coordinates of these events) or the rate of a clock with respect to the coordinate time of some space-time reference system, which are both dependent on the chosen reference system.

Due to the curvature of space-time the scale units of space-time coordinate systems have, in general, no globally constant relation to proper quantities. In the framework of Newtonian mechanics (using Euclidean geometry) it is always possible to define coordinates in such a way that their scale units are equal to proper quantities everywhere, and it is therefore not necessary to explicitly distinguish between them. This is impossible in general relativity where the relation between proper quantities and coordinate scale units is dependent on the position in space-time of the measuring observer. For time metrology this implies, for example, that the relation between a coordinate time interval and a measured proper time interval is dependent on the position of the measuring clock, hence distinguishing explicitly between these quantities, in my view, significantly enhances the understanding of the issues at hand.

**I.1.2. The Equivalence Principle**

The Equivalence Principle is the postulate that forms the foundation of general relativity. Its roots go back to Newton and even Galilei, as it is based on the observation that in
a gravitational field, roughly stated, all bodies fall with the same acceleration regardless of their mass or internal structure. Using Newtonian theory this can be expressed as

\[ m_i = m_g \]  

(I.1.1)

where \( m_i \) and \( m_g \) are the inertial and gravitational mass of a body appearing in Newton's second law \( (F = m_i \, a) \) and the law of gravitation \( (F = GM \, m_g \, r^2) \) respectively. In a more rigorous formulation it can be stated as what Will (1993) termed the Gravitational Weak Equivalence Principle (GWEP):

*If an uncharged body is placed at an initial event in space-time and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition.*

Where "uncharged" means electrically neutral.

Based on this principle Einstein conjectured that therefore all physical experiments lead to the same results when performed in a laboratory that is freely falling in a gravitational field or in one that is not subject to a gravitational field (as long as the laboratory is sufficiently small to be able to neglect the effects of the inhomogeneity of the field). This is expressed by Will (1993) as the Strong Equivalence Principle (SEP) stating that:

(i) GWEP is valid, (ii) the outcome of any local test experiment is independent of the velocity of the (freely falling) apparatus, and (iii) the outcome of any local test experiment is independent of where and when in the universe it is performed.

where a "local test experiment" is an experiment performed in a laboratory that is sufficiently small so that effects due to inhomogeneities in the external field are negligible compared to the measurement uncertainties.

When dealing with realistic situations where laboratories are not in free fall and of a non-negligible size, SEP is only valid after corrections (as predicted by general relativity) have been made. For the metrology of time this is particularly important when comparing measurements performed at different locations, e.g. when comparing the frequencies of two clocks that are subject to different gravitational fields.
It is a direct consequence of the above principle, that the proper time measured by a clock depends only on its path in space-time, and not on the nature of the clock (as long as it is insensitive to accelerations). This prediction has been verified experimentally by comparing the frequencies of a Mg and a Cs clock in the varying gravitational field of the sun (Godone et al. 1995). The results confirmed the predictions setting an upper limit on the relative frequency variation of $7 \times 10^{-4}$ of the frequency shift caused by the variation in the external gravitational potential.

I.1.3. The Metric Equation

A system of coordinates in general relativity is defined by its metric tensor $g_{\alpha\beta}(x^\lambda)$ ($x^\lambda$ denote the four space-time coordinates with $x^0=ct$ where $t$ is the coordinate time) which is coordinate dependent and needs to be known for the whole region of space-time within which the coordinate system is used. The metric tensor provides the local relationship between proper and coordinate quantities by the basic equation

$$ds^2 = -c^2 d\tau^2 = g_{\alpha\beta}(x^\lambda) dx^\alpha dx^\beta$$

(I.1.2)

where $ds$ is the relativistic line element, $c$ is the velocity of light in vacuum ($c = 299792458$ m/s), and $\tau$ denotes proper time with $d\tau$ being the increment of proper time measured between two events that lie on the world-line of the measuring clock (i.e. that take place in its immediate vicinity). The $dx^\lambda$ are increments in coordinates between the two events and the Einstein summation convention over repeated indices is adopted.

If the two events have a finite separation the proper time interval measured by the clock is the integral of $d\tau$ along the path in space-time of the clock. As $g_{\alpha\beta}(x^\lambda)$ is position dependent the integral will, in general, differ for different paths. This means that the proper time interval measured between the same two events is a function of the trajectory of the clock during the measurement.
In general relativity gravitation is understood as a curvature of space-time and can be described using a curvilinear coordinate system \((x^\alpha)\) and its associated metric tensor \(g_{\alpha\beta}(x^\alpha)\), with \(g_{\alpha\beta}(x^\alpha) = g_{\beta\alpha}(x^\alpha)\), and the usual transformation law,

\[
G_{\alpha\beta}(x^\alpha) = \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta} g_{\mu\nu}(X^\lambda) \tag{I.1.3}
\]

for a coordinate transformation \(x^\alpha \rightarrow x'^\alpha\).

According to the equivalence principle one can always define a laboratory (of infinitely small size, and in free fall) in which all laws of physics are equivalent to those observed in a laboratory that is not subject to a gravitational field i.e. in which the theory of special relativity is valid. This can be expressed geometrically by stating that it is always possible to define a local coordinate system \((X^\mu)\) so that \(g_{\alpha\beta}(x^\alpha)\) transforms to \(\eta_{\alpha\beta}\) the metric of special relativity and (I.1.2) is then written as

\[
ds^2 = -c^2 d\tau^2 = \eta_{\alpha\beta} dX^\alpha dX^\beta \tag{I.1.4}
\]

where \(\eta_{00} = -1, \eta_{11} = \eta_{22} = \eta_{33} = 1\) and all other components = 0.

In the local coordinate system \((X^\mu)\) the geometry is euclidean, and \(ds\) is measured along the straight line (in the euclidean sense) joining the two points (events). Therefore the proper time interval measured between two infinitesimally close events is unique, as opposed to the case of two events with a finite separation where the measured proper time interval is a function of the trajectory of the clock.

In practice the criteria for distinguishing between the infinitesimal case where special relativity is valid and the more general case of finite separations are the uncertainties of the considered measurements. If these are smaller or of the same order as the corrections predicted by general relativity the space-time domain in question cannot be considered infinitesimally small. For example, the frequency comparison between two clocks in the same terrestrial laboratory but at altitudes that differ by 1 m can be treated using only special relativity as long
as the measurement uncertainties in relative frequency \( \Delta f/f \) are much greater than \( 1 \times 10^{-16} \) (the gravitational red shift predicted by general relativity).

### 1.1.4. Einstein's Equations, the Weak Field Approximation

The components of the metric tensor can be obtained from a particular solution of Einstein's equations

\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = - \kappa T_{\alpha\beta}
\]  

(I.1.5)

neglecting the term containing the cosmological constant which is only important for cosmological problems. Equations (I.1.5) contain the Ricci tensor \( R_{\alpha\beta} \), scalar curvature \( R \), the energy-momentum tensor \( T_{\alpha\beta} \) and a constant \( \kappa \) (\( \kappa = 8\pi G/c^2 \) where \( G \) is the Newtonian gravitational constant) which is determined by passing to the limiting case of Newtonian theory. The Ricci tensor and scalar curvature are functions of \( g_{\alpha\beta} \), whereas the energy-momentum tensor is a function of the mass density and velocity of matter (for details see any relativity textbook e.g. Brumberg (1991a) chapter 1). So (I.1.5) gives the basic relationship between the curvature of space-time (characterised by \( g_{\alpha\beta} \)) and the distribution of mass-energy.

Einstein's equations form a set of ten equations containing fourteen unknown functions: ten components of the metric tensor \( g_{\alpha\beta} \), three components of the velocity of matter \( v_i \), and the mass density \( \rho \). For a solution in a particular system of coordinates it is necessary to add four equations called coordinate conditions. The choice of coordinate conditions is arbitrary, but will in general affect the values obtained for the components of the metric tensor \( g_{\alpha\beta} \).

For any problem at hand some particular coordinate conditions might be preferable for reasons of mathematical simplicity, or because some characteristic of the coordinates is required (e.g. spatial isotropy). For the problems of celestial mechanics harmonic coordinate conditions
where \( g \) is the determinant of \( g_{\alpha \beta} \), often simplify the mathematical treatment.

For celestial mechanics within the solar system the solution of Einstein's equations generally used is the "weak field" or "post Newtonian" approximation, valid for \( \varepsilon^2 \approx \frac{U}{c^2} \approx \frac{v^2}{c^2} \ll 1 \), where \( U \) is the total Newtonian gravitational potential and \( v \) a typical relative velocity of celestial bodies. Within the solar system \( \varepsilon^2 < 10^{-5} \). In this approximation the components of the metric tensor are expressed by a power series based on small corrections to the Minkowski metric of special relativity in terms of \( \varepsilon^n \),

\[
\begin{align*}
g_{00} &= -1 + \frac{h_{00}^{(2)}}{c^2} + \frac{h_{00}^{(4)}}{c^4} + O(c^{-6}) \\
g_{0i} &= \frac{h_{0i}^{(2)}}{c^2} + O(c^{-4}) \\
g_{ij} &= \delta_{ij} + \frac{h_{ij}^{(2)}}{c^2} + \frac{h_{ij}^{(4)}}{c^4} + O(c^{-6}),
\end{align*}
\]

where the \( h_{\alpha \beta}^{(n)}/c^n \) are of order \( \varepsilon^n \).

Substituting (I.1.7) into the Einstein equations (I.1.5) and applying the harmonic coordinate conditions gives, after some calculation (see e.g. Brumberg 1991a, p.48-51 for details), the components of the metric tensor. For the purposes of this thesis, as shown in subsequent chapters (c.f. sections II.1, II.2), the order \( \varepsilon^2 \)

\[
h_{00}^{(2)} = h_{ij}^{(2)} = 2U
\]

with the algebraic sign of \( U \) taken as positive, is sufficient.

It should be noted that the choice of coordinate conditions is not relevant at the order \( \varepsilon^2 \). Terms that are specific to some particular coordinate conditions only appear at the order \( \varepsilon^4 \),
and can therefore be neglected. Nonetheless, it is important to realise that general relativity presents additional degrees of freedom with respect to Newtonian theory, and that a system of space-time coordinates in general relativity, is specified by the choice of not only an origin and a time dependent rotation of the spatial axes, but also by the choice of a set of coordinate conditions.

1.5. Space-Time Coordinate Systems within the Solar System

In principle one is free to use any set of coordinates for the description of space-time. However, it turns out that defining several overlapping systems of coordinates, each particularly suited to a restricted region, can significantly simplify the treatment of practical problems and the relationship between coordinates and proper quantities (IAU 1991, Damour 1989, Soffel 1989b). Such definitions provide several time coordinates for use in particular regions of space-time with the relation between them given by relativistic coordinate transformations.

At its 1991 General Assembly, the International Astronomical Union (IAU) explicitly adopted the general theory of relativity as the theoretical framework for the definition and realisation of space-time coordinate systems (IAU 1991). In particular a Barycentric Reference System (BRS) and a Geocentric Reference System (GRS) were defined. The corresponding time coordinates were termed Barycentric Coordinate Time (TCB) and Geocentric Coordinate Time (TCG). Additionally another geocentric time coordinate, Terrestrial Time (TT) was defined, as the ideal form of the International Atomic Time (TAI). These definitions are detailed in section 1.3.
I.2 Time and Frequency: Definitions, Conventions, Notation

“So you are saying that human agreement decides what is true and what is false?” - It is what human beings say that is true and false; and they agree in the language they use. That is not agreement in opinions but in form of life.
Ludwig Wittgenstein (1953) §241.

When placing the metrology of time into a relativistic framework certain ambiguities and new concepts, which were not present in the Newtonian case, may lead to new definitions and conventions. Examples are the 1980 definition of International Atomic Time (BIPM 1980) and the 1991 resolution of the International Astronomical Union (IAU 1991). But in spite of the already existing agreements, it seems to me that there is still potential for confusion and misunderstanding. For example, there exists no conventional notation for distinguishing coordinate scale units from the unit of proper time, currently both are denoted by the same symbol "s". It might also be useful to distinguish between the scale units of different coordinate times like TCB and TCG. Furthermore there exists no rigorous, conventional definition of the frequency difference between two distant clocks (section I.2.3), and current practice of assuming that it is the frequency difference measured "if the two clocks were co-located" only makes sense if this difference is not a function of time.

In this section a system of definitions and notations is introduced which will be used throughout the thesis in the hope of reducing the danger of confusion and misunderstanding. In my opinion such a system, if used extensively, could facilitate communication and understanding among scientists, especially in the future when, with decreasing uncertainties, the issues treated here will become more and more important.

The explicit distinction between the unit of proper time and the time coordinate scale units introduced here (section I.2.1 and appendix) is different from the Newtonian case where there was only one temporal quantity (absolute time) and therefore a single unit was in principle sufficient. In the appendix this "new" situation is clarified using the concepts and
principles of quantity calculus pioneered by Maxwell and developed by Wallot in the 1920s (see De Boer 1994/95 for a comprehensive introduction). Although not indispensable for the practical application of the results presented in this thesis, the investigation into quantity calculus presented in the appendix is, in my view, helpful for a better understanding of the "co-existence" of the different quantities and units.

L2.1 Concepts of Time Metrology in a Relativistic Context

A clock A delivers a series of physical electric pulses separated from one another by a duration of 1 second as realised by the clock, often designated as "series of 1 pps". Each pulse is an event with an associated number, the reading $h_A(e)$ of the clock for that particular event. The origin of $h_A$ is arbitrary, but it is incremented by 1 second at each new pulse. The readings of A provide, with a stated uncertainty, a realisation of proper time along its world line $\tau_A$. By convention (IAU 1991) the unit of proper time is the SI second defined as (BIPM 1991) "...the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.\"., and denoted by the symbol "s".

An event can be characterised by its space-time coordinates $x^\alpha$ in some space-time coordinate system $RF$. The time axis of such a coordinate system is referred to as the coordinate time scale of that system with its associated scale unit denoted by the symbol "$s_{RF}$". In the general case coordinate time will be denoted by $u$, with particular time coordinates denoted by their acronyms (e.g. Geocentric Coordinate Time TCG, Terrestrial Time TT, Barycentric Coordinate Time TCB). Note that International Atomic Time, TAI, is defined as a realised coordinate time (see section 1.3.3.2) of a geocentric reference frame, its ideal form being TT.

A clock that realises some coordinate time scale $u$ provides events (pulses) $e$ whose readings $h_A(e)$ are identical to the corresponding time coordinates $u(e)$. In general such a clock will have to be steered, in order to compensate for a phase offset and a frequency shift due to the gravitational field at its location, its velocity in $RF$, environmental effects etc..
The relative rate of two clocks A and B that are collocated and at rest with respect to each other is defined as,

\[ R_{AB}(u) = \frac{dh_A(u)}{dh_B(u)} \] (I.2.1)

where \( R_{AB}(u) \) is the rate of clock A with respect to clock B at an event with time coordinate \( u \) that is collocated with the clocks, and \( dh_A \) and \( dh_B \) are increments of the clock readings of A and B between two infinitesimally close events at \( u \) that lie on the world-line of the clocks.

If clock B is ideal the rate

\[ R_{AB}(u) = \frac{dh_A(u)}{d\tau_A(u)} = \phi_A(u) \] (I.2.2)

is the normalised frequency of clock A at coordinate time \( u \). The relative rate of two clocks \( R_{AB}(u) \) and consequently the normalised frequency \( \phi_A(u) \) also are dimensionless numbers. They are proper quantities in the sense of the definition given in section I.1.1.

In general the frequency \( f_P(u) \) of a periodic phenomenon P is the inverse of the duration (in SI seconds) of its period. The unit used is the Hz (1 Hz = 1 s\(^{-1}\)). The normalised frequency is then obtained by

\[ \phi_P(u) = \frac{f_P(u)}{f_{P_0}} \] (I.2.3)

where \( f_{P_0} \) is the nominal frequency of P.
L2.2 Synchronisation

It is well known that in relativity the notion of simultaneity is not defined a priori so that a conventional choice of a definition has to be made. This choice will then lead to a corresponding definition of clock synchronisation as synchronised clocks must simultaneously produce the same readings. The definition adopted by convention (IAU 1991) is that of coordinate simultaneity and corresponding coordinate synchronisation, as given, for example, by Klioner (1992):

"Two events fixed in some reference system by the values of their coordinates \((u_1, x_1, y_1, z_1)\) and \((u_2, x_2, y_2, z_2)\) are considered to be simultaneous with respect to this reference system, if the values of time coordinate corresponding to them are equal: \(u_1 = u_2\). In the following this definition of simultaneity (and corresponding definition of synchronisation) we shall call coordinate simultaneity (and coordinate synchronisation)."

Clearly, synchronisation by this definition is entirely dependent on the chosen space-time coordinate system.

The advantage of this convention is that it allows transitivity between synchronised clocks i.e. when clocks A and B as well as B and C are synchronised, then A and C are also synchronised, which is not the case when using the so called Einstein synchronism (Einstein 1905) in a general coordinate system for which \(g_{\mu\nu} \neq 0\).

L2.2.1 Time Comparisons between two Clocks

Having adopted a convention of synchronisation, it is of interest for the metrology of time to define a quantity that expresses the amount of desynchronisation (by the above definition). Two alternatives can be considered:

(a) the desynchronisation of two clocks A and B is defined as the difference of their coordinate simultaneous readings at a coordinate time \(u\),
where \( x_{AB}(u) \) denotes the desynchronisation. It can be expressed in "s" but is a coordinate quantity as it is dependent on the convention of synchronisation and the choice of a reference system. It should not be misunderstood as a measured proper time interval.

(b) the desynchronisation of two clocks A and B is defined as the coordinate time interval

\[
\Delta u_{AB}(h) = u_A(h) - u_B(h)
\]

where \( u_i(h) \) is the time coordinate of the event on clock \( i \) with reading \( h \). In this case the desynchronisation is a coordinate time interval between two events, expressed in "sRF".

Definition (b) has the advantage of being closer to experimental practice, where a counter is started by the pulse coming from one clock and stopped by the pulse with the same reading coming from the other one. However, (a) can be easier extended to the definition of the desynchronisation of two theoretical time scales (e.g. TCB-TCG) and has the additional advantage of being already in use. It will also be consistently used in this thesis.

### 1.2.2.2 Time Comparisons between a Clock and a Coordinate Time

The desynchronisation between a clock and a coordinate time can now be defined as the desynchronisation of two clocks where one of the clocks is realising the coordinate time scale in question. For example the desynchronisation between a clock A and TT in a frame of synchronisation RF is expressed in the form,

\[
x_{ATT}(u) = h_A(u) - h_{TT}(u)
\]

where \( h_{TT}(u) \) is the reading of the clock that realises TT at coordinate time \( u \) in RF, and \( x_{ATT}(u) \) can be expressed in "s".
However, the above definition is ambiguous as, in principle, there exist an infinite number of clocks that are realising TT (one for each spatial point of the reference frame). When comparing any two of these clocks to A, the resulting values of $x_{\text{ATT}}(u)$ will in general differ.

It is therefore necessary to specify, by convention, the spatial position of the clock that is realising the coordinate time. The most practical choice is the location of clock A, then the definition reads:

*The time difference between a clock and a coordinate time is defined as the difference between two colocated clocks where one of the clocks is realising the coordinate time in question.*

Note that the frame of synchronisation RF may be the same as the one whose coordinate time is realised, so in the example used in equation (1.2.6) the coordinate time $u$ is in fact TT. In this case the ambiguity due to the location of the clock that realises the coordinate time does not arise, as all such clocks are (by definition) synchronised in the frame whose coordinate time they realise.

### 1.2.2.3 Time Comparison between two Coordinate Times

Knowing the time coordinate of an event in some coordinate system, one might wish to obtain, by means of a transformation, the time coordinate of the same event in another coordinate system. In its general form (within the weak field approximation) the relation between the coordinate times of two space-time coordinate systems A and B can be written as,

$$\frac{d_{u_A}}{d_{u_B}} = [1 + q(x^4)]$$  \hspace{1cm} (1.2.7)

where $d_{u_A}$ (expressed in "s_A") is the increment of the time coordinate of frame A between two infinitesimally close events, $d_{u_B}$ (expressed in "s_B") is the increment of the time coordinate between the same two events in frame B and $[1 + q(x^4)]$ is close to unity and expressed in "s_A/s_B". Knowing the definitions of the spatio-temporal origins of the two frames, an explicit
form of the transformation, giving the desynchronisation of the time scales, can be obtained by integrating (1.2.7).

L2.3 Syntonisation

From the definition given in section 1.2.1 it is clear that the relative rate of two clocks A and B, $R_{AB}(u)$, is a measurable quantity. It is the result of a direct measurement, presupposing that both clocks be located at the same site, with no relative velocity between them. For many applications in time metrology, it is desirable to determine the rate of a clock with respect to some coordinate time, or the relative rate of two distant clocks, so unambiguous definitions of these quantities are required.

I2.3.1 The Rate of a Clock with Respect to Coordinate Time

The rate of a clock A (assumed ideal) with respect to a coordinate time can be determined theoretically (see section I.1.1) from a knowledge of the coordinate position and velocity of A and the metric equation of the reference frame, in the form,

$$\frac{d\tau_A}{du}(u) = [1 + q(x')]$$  \hspace{1cm} (I.2.8)

where $d\tau_A$ is the increment of proper time (the clock reading) between two infinitesimally close events, $du$ is the increment of coordinate time between the same two events and $[1 + q(x')]$ is close to unity and expressed in “s/sRF” with $|q(x')| < 10^{-9}$ in the vicinity of the Earth for a geocentric reference frame.

The rate of a clock with respect to a coordinate time is a coordinate quantity that can be determined from theory.
I.2.3.2 The Relative Rate of Two Distant Clocks

The relative rate of two distant ideal clocks in a frame RF can be defined as,

\[
\left[ \frac{d\tau_A}{d\tau_B} (u) \right]_{RF} = \frac{d\tau_A}{du} (u) \frac{du}{d\tau_B} (u) \quad (I.2.9)
\]

where \( \frac{du}{d\tau_B} (u) \) is the inverse of the rate of B with respect to the coordinate time, determined at \( u \).

In general the \( d\tau/du \) vary with time hence the moment at which they are evaluated has to be fixed by convention. According to (I.2.9) they have to be determined at the same moment of coordinate time \( u \) and hence the convention of coordinate simultaneity is implicitly applied. So the quantity \( \left[ \frac{d\tau_A}{d\tau_B} (u) \right]_{RF} \) is a coordinate quantity dependent on the reference frame used. It can be determined by theory or by repeated time comparisons (time transfers) between A and B, using the convention of coordinate simultaneity in RF.

In the case where A and B are real clocks their relative rate when separated

\[
\left[ \frac{dh_A}{dh_B} (u) \right]_{RF}
\]

which is a coordinate quantity should not be confused with \( R_{AB}(u) \) as defined in equation (I.2.1) which is a proper quantity presupposing that the two clocks are colocated. However, for many practical applications the approximation

\[
\left[ \frac{dh_A}{dh_B} (u_0) \frac{d\tau_B}{d\tau_A} (u_0) \right]_{RF} = \left[ \frac{dh_A}{du} (u_0) \frac{d\tau_B}{dh_B} (u_0) \right]_{RF} \approx R_{AB}(u_1) \quad (I.2.10)
\]

can be used. In (I.2.10) \( \left[ \frac{dh_A}{dh_B} (u_0) \right]_{RF} \) is determined by repeated time transfers, \( \left[ \frac{d\tau_B}{d\tau_A} (u_0) \right]_{RF} \) is obtained from theory, \( R_{AB}(u_1) \) is the rate measured locally at coordinate time \( u_1 \) when the two clocks have been brought in colocation, and the approximation holds if the differences \((dh/d\tau)(u_1) - (dh/d\tau)(u_0)\) are negligible or can be estimated with sufficient accuracy.
The difference of normalised frequencies of two distant clocks can also be defined using the convention of coordinate simultaneity. One obtains a coordinate quantity which can be related to the proper difference of normalised frequencies (measured when the two clocks are colocated) by the approximation

$$\left[ \varphi_A(u_0) - \varphi_B(u_0) \right]_{\text{RF}} = \frac{dh_A}{d\tau_A}(u_0) - \frac{dh_B}{d\tau_B}(u_0) \approx \varphi_A(u_1) - \varphi_B(u_1)$$

(I.2.11)

where the two clocks are colocated at coordinate time $u_1$ and the approximation is valid under the same conditions as (I.2.10).

Consider an observer O who measures locally the relative rate of two signals emitted by A and B and received at O, each signal locked to the frequency of the emitting clock. The obtained relative rate is a proper quantity and can differ significantly from $\left[ \frac{dh_A}{dh_B}(u) \right]_{\text{RF}}$. To relate the two quantities the effects of light propagation have to be corrected for (see section I.1.3).

I.2.3.3 The Relative Rate of Two Coordinate Times

The general form of expressions for the relative rate of two coordinate times is given in equation (I.2.7). Such expressions are obtained directly from the definitions of the reference frames and the appropriate metric equations. For barycentric and geocentric coordinate time scales such expressions together with limits for their evaluation can be found in section II.3.3 and II.3.4.
I.3 Definitions of Barycentric and Geocentric Space-Time Reference Systems

In general relativity a space-time coordinate system is defined by a time dependent spatial origin, a time dependent rotation of the spatial axes, a temporal origin, and a set of coordinate conditions. It is described by its metric tensor \( g_{\alpha\beta}(x^5) \). The coordinate systems used are defined by convention, the last such definitions given by the International Astronomical Union (IAU) in its resolution A4 (IAU 1991). It was recognised, that defining several overlapping systems of coordinates, each particularly suited to a restricted region, can significantly simplify the treatment of practical problems and the relationship between coordinates and proper quantities. Accordingly definitions were given for coordinate systems centred at the barycenter of any ensemble of masses.

The metric tensors were defined including only those terms required at the present level of observational accuracy (up to order \( \varepsilon^2 \)), with the remark that higher order terms may be added as deemed necessary by users. Terms that are specific to a particular set of coordinate conditions are of higher order, therefore the IAU did not specify any such conditions. The influence of masses which are external to the ensemble to which the coordinate system pertains should be characterised by tidal terms which vanish at the space origin.

It was stated explicitly that all time coordinates be derived from a time scale realised by atomic clocks operating on the Earth, and that the units of proper time and proper length be the second and the meter of the International System of Units (SI) (BIPM 1991).

The solar system barycentric coordinate system (BRS) and the geocentric coordinate system (GRS) are of particular interest for a large number of applications. Their explicit definitions are given in the following subsections.

Space-time coordinate systems are realised by determining as accurately as possible the coordinates of reference events. For spatial coordinates this implies that the time-dependent spatial position of reference points (on the Earth or astronomical) be given, whilst coordinate times are realised by determining the time coordinates of the events (pulses) of particular
clocks which observations are referenced to. The realisation of coordinate times is treated in more detail in section II.3.

I.3.1 The Barycentric Coordinate System

The spatial origin of the barycentric reference system (BRS) is defined to coincide with the centre of mass of the solar system, with the spatial coordinate grid showing no rotation with respect to a set of distant extragalactic objects (IAU 1991). It is assumed that the average rotation of a large number of extragalactic objects can be considered to represent the rotation of the universe which is assumed to be zero.

The origin of its time coordinate (Barycentric Coordinate Time, TCB) was fixed with respect to a reference event located at the geocentre and dated in TAI (International Atomic Time): the reading of TCB should be 1977 January 1.0 h_{TCB} 0 min_{TCB} 32,184 s_{TCB} exactly on 1977 January 1.0 h_{TAI} 0 min_{TAI} 0 s_{TAI} exactly at the geocentre. The origin of TCB has been arbitrarily set so that it coincides with Geocentric Coordinate Time (TCG, c.f. section I.3.2) and Terrestrial Time (TT, c.f. section I.3.3.1) on 1977 January 1.0 h_{TAI} 0 min_{TAI} 0 s_{TAI} at the geocentre.

The metric equation in barycentric coordinates (x^0=c_{TCB}, x^i) is obtained directly from the weak-field approximative solution of the Einstein equations (see section I.1.4). It can be expressed as

\[ ds^2 = -c^2dt^2 \]
\[ = -(1-\frac{U}{c^2})c^2dTCB^2 + \left( 1 + \frac{U}{c^2} \right) (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \]  

(1.3.1)

where \( U \) is the total Newtonian gravitational potential with its algebraic sign taken as positive.

One can see that far from the masses of the solar system, and neglecting the effect of external masses, the metric tensor reduces to \( \eta_{\alpha\beta} \), that of special relativity. In this region the
proper time interval measured by an ideal clock that is stationary \((dx' = 0)\) is identical to the corresponding coordinate time interval.

I.3.2 The Geocentric Coordinate System

The spatial origin of the geocentric reference frame (GRS) is defined to coincide with the centre of mass of the Earth, with the spatial axes showing no rotation with respect to a set of distant extragalactic objects and therefore with respect to the BRS space coordinates. This condition specifies a so-called kinematically non-rotating geocentric frame, as opposed to a dynamically non-rotating one which is characterised by the vanishing coriolis and centrifugal forces (realised for example by the angular momentum vectors of torque free gyroscopes). The two differ by the geodesic precession, resulting from the motion of the Earth-Moon system about the sun, which manifests itself as a term of order \(\varepsilon^3\) in the \(\tilde{g}_{0i}\) term of the geocentric metric. This term is negligible for the purposes of my thesis (see section II.1.1), hence the distinction between kinematically and dynamically non-rotating frames is not significant at the required level of accuracy.

The origin of Geocentric Coordinate Time (TCG) is identical to that of TCB (see section I.3.1) i.e. the reading of TCG should be 1977 January 1. 0 h\(_{TCG}\) 0 min\(_{TCG}\) 32,184 s\(_{TCG}\) exactly on 1977 January 1. 0 h\(_{TAI}\) 0 min\(_{TAI}\) 0 s\(_{TAI}\) exactly at the geocentre.

One might think that the transformation from BRS to GRS could take the simple form

\[
\bar{w} = \bar{x} - \bar{x}_E(TCB)
\]

\[
TCG = TCB
\]  

(I.3.2)

where GRS coordinates are represented by \(w^o_c(TCG, w')\) and the subscript \(E\) refers to the centre of mass of the Earth.

However, it turns out that the geocentric coordinates obtained by using a transformation of the kind (I.3.2) are inconvenient when treating problems in the vicinity of the
Earth. They introduce unnecessarily large and complex relativistic terms when used, for example, for the description of the motion of Earth satellites. These terms subsequently disappear when the coordinate quantities are reduced to proper quantities which are compared to observations, leaving only relativistic effects which are much smaller than the coordinate ones. Consequently it would be desirable to find "good" geocentric coordinates in which these spuriously large terms do not arise, which would significantly facilitate the treatment of the problem at hand and simplify the relation between coordinate and proper quantities.

According to the principle of equivalence there must exist a freely falling local (infinitesimal) coordinate system in which physics can be described by special relativity. For the real case of a geocentric coordinate system this implies that a frame exists in which external bodies (all celestial bodies except of the Earth) manifest themselves only by tidal terms which vanish at the geocentre and are due to the inhomogeneities of the external field. This requirement (also expressed by the IAU, 1991) in fact specifies the "good" geocentric coordinates mentioned earlier, where the description of problems in the vicinity of the Earth take their simplest form.

One method for obtaining a "good" geocentric coordinate system proceeds through the following steps (see Brumberg 1991a, 1991b, or, for more detail, Kopejkin 1988):

(i) A general form of the geocentric metric is written in the form of the expansion (I.1.7), where, for purposes of this thesis, only terms up to order $\varepsilon^2$ are relevant. In the coefficients $h_{00}^{(2)}$ and $h_{0}^{(2)}$ proper Earth terms and terms due to external masses (in the form of tidal terms) are separated.
(ii) The proper Earth terms are determined directly from the solution of Einstein's equations for the one body problem (e.g. Brumberg 1991a).
(iii) The two coordinate systems (BRS and GRS) are related by a generalisation of the Lorentz transformations of special relativity expanded in terms of $\varepsilon^\alpha$.
(iv) Finally the coefficients of the external terms in the GRS metric as well as the coefficients of the generalised Lorentz transformations are determined by matching the GRS metric to the BRS one using relation (I.1.3).

For the $h_{00}^{(2)}$ term one obtains (Brumberg 1991b, Brumberg et al. 1992)
\[ H_{00}^{(3)} = 2\left[ U_E(\bar{w}) + Q_k w^k + U(\bar{x}_E + \bar{\nu}) - U(\bar{x}_E) - U_{\nu_k}(\bar{x}_E)w^k \right], \]  

where \( U_E(\bar{w}) \) and \( U(\bar{x}) \) are the Newtonian gravitational potentials of the Earth and of external masses respectively, and \( Q_k \) is the correction for the non-geodesic barycentric motion of the Earth.

The second term in (I.3.3), arising from the interaction of the Earth's quadrupole moments and the external masses (given explicitly e.g. in Brumberg 1991a) gives rise to a correction which is negligible for the problems addressed here (see section II.1.1) and will henceforth be omitted.

Then the GRS metric up to order \( \varepsilon^2 \) is

\[
\begin{align*}
ds^2 &= -c^2 d\tau^2 \\
&= \left\{1 - \frac{2}{c^2} \left[ U_E(\bar{w}) + U(\bar{x}_E + \bar{\nu}) - U(\bar{x}_E) - U_{\nu_k}(\bar{x}_E)w^k \right]\right\} c^2 dTCG^2 \\
&\quad + \left\{1 + \frac{2}{c^2} \left[ U_E(\bar{w}) + U(\bar{x}_E + \bar{\nu}) - U(\bar{x}_E) - U_{\nu_k}(\bar{x}_E)w^k \right]\right\} \delta_{ij} dw^i dw^j
\end{align*}
\]  

(I.3.4)

A different, in some respects more rigorous, approach to determining a "good" geocentric coordinate system has been developed by Damour, Soffel & Xu (1991). At the order \( \varepsilon^2 \) the results of the two methods are equivalent, hence the latter will not be further discussed.

### I.3.3 Other Geocentric Coordinate Time Scales

International Atomic Time (TAI) has been defined (see section I.3.3.2) as a realised geocentric coordinate time scale. As the errors in the realisation of TAI are not always negligible the IAU (1991) found it necessary to define an ideal form of TAI, designated
Terrestrial Time (TT) which differs from TCG as defined in the previous section by a constant rate.

Universal Time (UT1) is a realised dynamical time scale derived from the observation of the Earth's rotation: it is proportional to the angle of rotation of the Earth on its axis. The coefficient of proportionality is chosen so that 24 hours of UT1 are close to the mean duration of the day. It is combined with TAI to obtain Coordinated Universal Time (UTC) the basis for the distribution of time around the world. In the following explicit definitions of TT, TAI and UTC are given.

L3.3.1 Terrestrial Time (TT)

Terrestrial Time is defined (IAU 1991) as a geocentric coordinate time differing from TCG by a constant rate, the scale unit of TT being chosen so that it agrees with the unit of proper time (the SI second) on the rotating geoid.

The origin of TT is identical to that of TCG and TCB, i.e. the reading of TT should be 1977 January 1.0 h TT 0 min TT 32,184 s TT exactly on 1977 January 1.0 h TAI 0 min TAI 0 s TAI exactly at the geocentre. TT is an ideal form of TAI apart from the 32,184sTT offset which was introduced to ensure an approximate continuity with the previously used Ephemeris Time (ET).

L3.3.2 International Atomic Time (TAI)

The definition of TAI (a realisation of TT) was approved by the Comité International des Poids et Mesures (CIPM) in 1970, and recognised by the Conférence Générale des Poids et Mesures (CGPM) in 1971. It reads as follows:

International Atomic time (TAI) is the time reference coordinate established by the Bureau International de l'Heure on the basis of the readings of atomic clocks operating in various establishments in accordance with the definition of the second, the unit of time of the
International System of Units. [In 1988, the responsibility for TAI was transferred to the time section of the Bureau International des Poids et Mesures, BIPM].

In order to suit a relativistic context the definition was completed by the Comité Consultatif pour la Définition de la Seconde (CCDS) in its 9th session held in 1980: TAI is a coordinate time scale defined in a geocentric reference frame with the SI second as realised on the rotating geoid as the scale unit.

The origin of TAI is not well defined. It has been agreed officially that TAI and Universal Time (UT1) should coincide on 1958 January 1, but this definition is subject to uncertainties in different local realisations of UT1 at the time, and to subsequent re-evaluations of UT1 (Guinot 1995).

I.3.3.3 Coordinated Universal Time (UTC)

Coordinated Universal Time (UTC) was defined in 1972. It is a combination of TAI and UT1 (Universal Time, based on the Earth's rotation) defined by

\[
UTC(u) - TAI(u) = n \text{s}_{TAI} \quad (n \text{ integer})
\]

\[
|UTC(u) - UT1(u)| < 0.9 \text{s}_{TAI}
\]  

(1.3.5)

for an event with time coordinate \(u\).

UTC and TAI differ by an integer number of \(s_{TAI}\) (so called leap-seconds), equal to 30 from 1996 January 1. 0 h_{TAI} 0 min_{TAI} 0 s_{TAI}. The International Earth Rotation Service (IERS) which is responsible for the publication of UT1, decides on the adjustment of leap-seconds by reference to the predicted divergence between TAI and UT1. They are introduced at the end of a month, normally at the end of June or December.

By definition UTC has the same metrological qualities as TAI, which is a coordinate time based on atomic clocks. In addition it follows the rotation of the Earth to within 1 s_{TAI}. It is the general basis for the distribution of time around the world. Local times are derived from UTC by a shift of, in general, an integer number of hours (which can change from winter time
to summer time), decided by administrations of individual countries or regional groups. All
time signals, at whatever level, including signals distributed by TV, radio, or speaking clocks,
are synchronised to these local times, and thus to UTC.

I.3.4 The Geocentric Rotating Coordinate System

For most terrestrial applications a geocentric frame, rotating with the Earth, is used. Such
frames have the advantage that the spatial coordinates of points fixed on the Earth's
surface vary in time by only small amounts which are often negligible.

A geocentric rotating coordinate system (GRSR) can be defined by applying a rotation
of angular velocity $\omega_e$ (the angular velocity of the Earth) to GRS. The time coordinate of
GRSR is the same as for GRS, i.e. TCG.

Several realisations of GRSR are available, the most generally used ones being the
World Geodetic System, WGS84, and the IERS (International Earth Rotation Service)
Terrestrial Reference Frame, e.g. ITRF94.
I.4 The Measurement of Proper Time

If language is to be a means of communication there must be agreement not only in definitions but also (queer as this may sound) in judgements. This seems to abolish logic, but does not do so.—It is one thing to describe methods of measurement, and another to obtain and state results of measurement. But what we call “measuring” is partly determined by a certain constancy in results of measurement.


Agreement on "....a certain constancy in results of measurement" leads us to the hypothesis that there is an underlying unique physical quantity which is being measured, that the outcome of our measurements is not coincidental but a, more or less exact, representation of this quantity. If there is to be communication concerning this quantity one has to agree on a definition for it, and describe the methods of measurement to be used. Defining the ideal quantity that is measured by clocks, defining its unit, describing the methods of measurement and their evaluation, and performing such measurements are some of the major tasks of the metrology of time.

I.4.1 Proper Time and its Unit

In the theoretical framework of general relativity the ideal quantity that time measurements are based on is proper time along a particular world-line $\tau_A$ (c.f. section I.1.1). Expressed geometrically, an interval of proper time $\Delta\tau_A$ along a world-line $A$ is the integral (c.f. equation I.1.2)

$$\Delta\tau_A = \frac{1}{c} \int_A \sqrt{-g} \, ds = \frac{1}{c} \int_A \sqrt{-g_{\alpha\beta}} \, dx^\alpha dx^\beta$$

(I.4.1)
where the integral is taken along the time-like \((ds^2 < 0)\) path \(A\). In more physical terms, \(\Delta\tau_A\) is the quantity measured by an ideal clock that moves along the path \(A\). By "ideal clock" is meant a device that perfectly realises the conventionally defined unit of proper time.

By convention (IAU 1991) the unit of proper time is the second of the international system of units (SI), defined as follows (BIPM 1991):

_The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom._

It is denoted by the symbol "s".

4.2 Evaluation of Real Clock Performance

The performance a real clock, or an ensemble of clocks, is usually described and evaluated using the notions of accuracy, stability and reliability. These key concepts of the metrology of time are introduced in the following sub-sections. They will be applied not only to the measurement of proper time, but also to realisations of coordinate times and are therefore discussed in both contexts.

4.2.1 Accuracy

The accuracy of a clock or a time scale is its ability to realise a scale interval as close as possible to its definition. For a clock this is the ability to produce the SI second as defined above. For a realisation of a coordinate time scale it is the ability to maintain a scale unit close to its definition. In general the _normalised frequency deviation_ of a clock or a time scale is determined and stated together with the uncertainty of the determination. This uncertainty is usually referred to as the "accuracy" of the clock.
The normalised frequency deviation of a clock \( \Delta \), denoted \( y_\Delta(u) \), is defined as

\[
y_\Delta(u) = \phi_\Delta(u) - 1
\]  

(I.4.2)

with \( \phi_\Delta(u) \) defined in section (I.2.1 (equation I.2.2)). It can be evaluated by comparison to another clock which is known to be more accurate. If this is not possible \( y_\Delta(u) \) is determined by estimating the influence of all known physical effects that may modify the output frequency with respect to the definition. An uncertainty budget is then constituted giving an estimate of the normalised frequency deviation and its uncertainty.

The normalised frequency deviation of a realised time scale is the deviation from its definition or its ideal form. For TAI

\[
y_{\text{TAI}}(u) = \frac{d\text{TAI}}{dT}(u) - 1 .
\]  

(I.4.3)

It is determined by comparing the scale unit of the realised time scale to a clock, taking into account the appropriate relativistic terms. For the above example

\[
y_{\text{TAI}}(u) = \frac{d\text{TAI}}{dh_\Delta}(u) \frac{d\tau_\Delta}{dT}(u) \phi_\Delta(u) - 1
\]  

(I.4.4)

where \( d\text{TAI}/dh_\Delta(u) \) is the result of the comparison, \( d\tau_\Delta/dTT(u) \) is determined theoretically (c.f. section I.2.3.1 and II.1.1) and \( \phi_\Delta(u) \) is obtained from an evaluation of the clock. The uncertainty of \( y_{\text{TAI}}(u) \) is given by the sum of the uncertainties in the comparison method and the determinations of \( \phi_\Delta(u) \) and of the relativistic part \( d\tau_\Delta/dTT(u) \) (c.f. section II.1.1).

I.4.2.2 Stability

The stability of a clock or time scale can be defined as its ability to maintain a constant scale interval, even if it differs from the ideal one. A measure of stability thus consists of the estimation of the dispersion of the normalised frequency deviation values \( y(u) \) with time. Some
statistical tools have been developed to estimate stability. They are efficient for the characterisation of the usual types of random noise which affect clock signals. The most common such tool is the two-sample, or Allan variance $\sigma_y^2(T)$ which depends upon the sampling or integration time $T$.

The Allan Variance

The recommended [Recommendation 538-2, CCIR 1992, 1990] measure for stability is the two-sample standard deviation $\sigma_y(T)$, which is the square root of the Allan variance $\sigma_y^2(T)$ defined as:

$$
\sigma_y^2(T) = \frac{1}{2T^2} \left\langle \left( x_{AB}(u+2T) - 2x_{AB}(u+T) - x_{AB}(u) \right)^2 \right\rangle
$$

(I.4.5)

where $\langle \ldots \rangle$ denotes an infinite time average, and $x_{AB}$ is the desynchronisation of two clocks or time scales (c.f. section I.2.2 and II.2). A more detailed descriptions of the Allan, and related variances, and of their application in practice (for finite time series) can be found in (Allan 1987).

Note that $\sigma_y(T)$ is a dimensionless number assuming that the integration time $T$ is measured using the same units as for $x_{AB}$.

It is essential to note that $\sigma_y(T)$ as defined in (I.4.5) represents the stability of time differences between A and B (i.e. their coupled stability) rather than the stability of an individual clock or time scale with respect to the ideal (its intrinsic stability). If one of the two clocks is known to be much more stable than the other the instability may be ascribed entirely to the less stable one. When this is not the case the intrinsic stabilities can be estimated if 3 or more independent clocks or time scales and comparisons between them are available using the so called N-cornered hat technique.
The N-Cornered Hat Technique

Suppose that $N$ independent clocks or time scales $H_i$, and $K$ series of time difference measurements between each pair of the clocks, $x_i(u)$, are available ($K = \sum_{k=1}^{N-1} k$ and $ij = 1,2,\ldots,N$ with $i \neq j$). Denoting the coupled Allan variance of clocks $H_i$ and $H_j$ by $\sigma_{x(ij)}^2(T)$ and the intrinsic variance of $H_i$ by $\sigma_{x(i)}^2(T)$ a set of $K$ equations

$$\sigma_{x(ij)}^2(T) = \sigma_{x(i)}^2(T) + \sigma_{x(j)}^2(T)$$

(I.4.6)

can be obtained (Barnes 1982, Allan 1987). For $N \geq 3$ the equations are deterministic and can be solved, obtaining the intrinsic variances $\sigma_{x(i)}^2(T)$ from the $\sigma_{x(ij)}^2(T)$ which were obtained from the measurements of the $x_i(u)$.

It must be stressed that this method only applies to clocks or time scales which are known to be independent. If the independence is not verified, variances and covariances should be handled together for a complete analysis (Premoli & Tavella 1993, Tavella & Premoli 1994).

I.4.2.3 Reliability

Reliability of a clock or an ensemble of clocks or of a realisation of a coordinate time scale is simply the requirement of continued operation. Ensuring reliability in general requires redundancy and, eventually, national and international collaboration between laboratories maintaining atomic clocks.

The simplest solution to the failure of a clock and the resulting loss of reliability is to replace it by another one. This presupposes that several clocks are operated and intercompared continuously one of them, the so called "Master Clock", providing the time scale. If this clock should fail, it can be replaced a posteriori by any other clock in the ensemble using the last available comparison data.
More often, reliability is ensured by computing an ensemble time (e.g. TAI). These times rarely have a physical representation, instead they are provided in deferred time in the form of time differences between each individual clock and the calculated ensemble time. In the computation it is necessary to minimise the perturbations that result as clocks enter and leave the ensemble, hence a large number of participating clocks not only ensures reliability but also helps to reduce the negative effects of entering or leaving clocks.

I.4.3 Current and Expected Clock Performance

In this section the performance of some of the best clocks and clock types is examined. All clocks, with the exception of Pulsar Time, are based on atomic transitions, with Caesium and Hydrogen being the most widely used ones. The presently achieved and published accuracies and stabilities as well as expected developments in the near future are discussed.

I.4.3.1 Atomic Clocks

Currently the clocks used for the most demanding applications are all based on atomic transitions. The types and designs vary, ranging from the widely used caesium-beam clocks and hydrogen masers to more "exotic" prototypes like caesium-fountains and linear ion trap standards (LITS). Concerning the availability of these clocks one can distinguish between commercially available atomic clocks and laboratory prototypes or primary standards which exist in small numbers in a few time laboratories. Commercially available clocks (for high performance usually Cs-beam clocks or H-masers) are generally optimised for stability and reliability with less importance accorded to their accuracy. Laboratory standards, on the other hand, are usually subject to a detailed evaluation of their accuracy including a complete uncertainty budget, but often (with a few exceptions) do not operate continuously.

International Atomic Time (TAI) is obtained by a combination of a large number (currently \( \approx 230 \)) of commercial clocks and laboratory standards (see section II.3). In a first step the data from the participating clocks is combined providing a free atomic time scale
(Echelle Atomique Libre, EAL). The scale unit of EAL is then compared to a few selected primary standards in order to evaluate its normalised frequency deviation $\gamma_{EAL}(u)$ (using equation 1.4.4). This deviation is then compensated by frequency steering, the resulting time scale being TAI. The steering corrections are kept smaller than the instability of EAL in order to try and avoid degradation of the short and medium term ($\approx$ 30-60 days) stability of TAI. The long term (> 1 year) stability of TAI is equal to the accuracy of the primary standards used for the steering, whereas the long term stability of EAL may be degraded by a frequency drift (i.e. a variation of $\gamma_{EAL}(u)$ in the long term).

Until recently the best accuracies were achieved by laboratory Cs-beam standards, in particular those of the PTB in Germany and the NIST in the U.S.A. The NIST-7 standard has an estimated uncertainty of $1,0 \times 10^{-14}$ while the two standards at the PTB (CS2 and CS3) reach accuracies of $1,5 \times 10^{-14}$ and $1,4 \times 10^{-14}$ respectively. In October 1995 a preliminary evaluation of a Cs-fountain standard operating at the LPTF (LPTF-FO1) in France was published estimating its uncertainty to be $2-3 \times 10^{-15}$ (Clairon et al. 1995). Commercially available clocks can reach accuracies of $1 \times 10^{-12}$ for H-masers and below $1 \times 10^{-12}$ for Cs-beam clocks (Cutler 1993). The accuracy of TAI is that of the standards used for the steering (currently NIST-7, CS2, CS3 and LPTF-FO1) combined with the uncertainty due to the transfer of this accuracy to TAI (see section II.3.2). For the future, laser cooling of commercial Cs-clocks is expected to yield improvement of their accuracies by a factor of 3-5 (Cutler 1993) whilst further progress in Cs-fountain standards should substantially improve their accuracy.

Presently hydrogen masers reach frequency stabilities of $1 \times 10^{-15}$ or slightly below (square root of the Allan variance) for averaging times of $\approx 10^4$ s (Vessot et al. 1992; Busca 1993; Cutler 1993). For longer averaging times the best stabilities are displayed by caesium clocks: commercial Cs-beam clocks are stable to less than one part in $10^{14}$ (Cutler 1993) whereas the laboratory Cs-beam standard CS2 displays a stability of $\approx 5 \times 10^{-15}$ for averaging times > 40 days (Thomas & Azoubib 1996). The coupled stability of the Cs-fountain at the LPTF compared to a H-maser reaches $2 \times 10^{-15}$ for an integration time of $10^4$ s but is limited by the H-maser for longer integration times (Clairon et al. 1995). The Linear Ion Trap Standards (LITS) at JPL (Tjoelker et al. 1996) show a coupled stability (LITS3 vs LITS4) in the upper $10^{-16}$ region for integration times of $\approx 10^5$ s.

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The stability of the free atomic time scale EAL has been evaluated using the N-cornered hat technique at $3.1 \times 10^{-15}$ for integration times around 40 days (Thomas & Azoubib 1996). In the future improvements of the Cs-fountain and the linear ion trap (LITS) may yield stabilities of a few parts in $10^{17}$ and $10^{18}$ respectively (Rolston & Phillips 1991; Itano 1991; Maleki 1993). Clock stabilities (including EAL) are summarised in figure I.1.
Fig. 1. Allan standard deviation (estimate of relative stability) for EAL and various clocks (from C. Thomas, 1996).

1. EAL
2. Commercial caesium clock
3. PTB CS2 primary caesium clock
4. BNM-LPTF caesium fountain
5. Passive hydrogen maser
6. Active hydrogen maser
7. Linear ion trap (LITS)
### I.4.3.2 Pulsar Time

Pulsar Time (PT) designates a time scale based on the observation of one or an ensemble of several millisecond pulsars (Petit et al. 1992, Petit & Tavella 1996). The arrival times of the observed radiopulses originating from a millisecond pulsar are subject to a number of physical effects which have to be corrected for (e.g. the period derivative, the proper motion, the angular position of the pulsar, its distance from the solar system, the interstellar medium etc...). The parameters for these corrections are not known a priori and have to be determined by adjusting a model to the pulsar timing data (which is obtained using atomic time, AT, as a reference). Hence PT cannot be considered as a time scale that is independent of atomic time as realised by atomic clocks (Blandford et al. 1984, Guinot & Petit 1991). In particular, it is impossible to obtain the accuracy of pulsar time, as this is entirely subject to the adjustment. However, one can determine the coupled stability of the PT-AT residuals after adjustment of the model. Calculating an ensemble pulsar time PT\(_{\text{ens}}\), the PT\(_{\text{ens}}\)-AT residuals can be used to detect instabilities in atomic time for integration times \(T\) exceeding one year but not larger than half the total observation time of the pulsars (Petit & Tavella 1996).

The stabilities of TAI-PSR1937+21 and of TAI-PSR1855+09 are shown in figure I.2. The stability of TAI-PSR1937+21 seems to be limited at \(\approx 4 \times 10^{-14}\) whereas that of TAI-PSR1855+09 currently reaches \(\approx 2 \times 10^{-14}\) for integration times of a few years. The past and present long term stability of TAI is \(\approx 1 \times 10^{-14}\) (see section I.4.3.1 or Petit 1995) hence the instability of TAI-PSR1937+21 is probably due to effects in PSR1937+21 itself. On the other hand PSR1855+09 shows no such limit. Combining the data of the two pulsars one can draw a tentative conclusion indicating a long term (\(\approx 2\) years) instability in TAI of a few parts in \(10^{14}\) (Petit & Tavella 1996) which could be related to the fact that, in the computation of TAI, frequency steering corrections were applied between June 1989 and April 1992 amounting to \(6 \times 10^{-14}\) in relative frequency (BIPM 1994).

From the investigation of the contributing physical effects Petit (1995) estimates that a future ensemble pulsar time based on a number of selected millisecond pulsars should have an intrinsic stability close to \(2 \times 10^{-15}\) for integration times of a few years. However, to realise such an ensemble pulsar time and to take advantage of its long term stability a large number of
millisecond pulsars have to be observed over several years, therefore such a time scale will not be available in the near future.

So currently pulsars can provide time scales with stabilities of the same order as, or slightly better than that of atomic time for integration times of a few years, with an expected improvement when more millisecond pulsar data is available. However, atomic time is also expected to improve so it is not certain that it will then be possible to use $PT_{\text{ens}}$ for the detection of instabilities in AT. Nevertheless it could still be useful for the transfer of the accuracy (using $PT_{\text{ens}}$ as a "flywheel") of a future, improved atomic time to the present one and for the study of the pulsars themselves.

![Fig. L2: Allan deviation for PSR1937+21, PSR1855+09 and an ensemble pulsar time ($PT_{\text{ens}}$) vs. TAI (from Petit & Tavella 1996).](image)
II. APPLICATION OF GENERAL RELATIVITY TO THE METROLOGY OF TIME

Modern metrology of time is concerned with measurements of time and frequency, the comparison of such measurements (often over non-laboratory distances) and the realisation of the time coordinates of space-time reference systems. Generally speaking, a relativistic treatment of metrology becomes necessary when the measurement and comparison uncertainties are of the same order or smaller than the corrections due to relativistic effects. Presently the relative accuracy of the best atomic clocks reaches two parts in $10^{15}$ (Clairon et al. 1995) with best frequency stabilities of $\approx 1 \times 10^{-15}$ (Vessot et al. 1992, Busca 1993, Tjoelker et al. 1996). Time comparisons using satellite methods reach accuracies of a few ns (Lewandowski et al. 1993) with precisions <100 ps (Veillet et al. 1992, Veillet & Fridelance 1993). Relativistic effects can cause shifts in relative rate of $1,1 \times 10^{-13}$ per kilometre of altitude for clocks on the surface of the Earth, and necessitate corrections to time transfers which can reach 400 ns (Allan & Ashby 1985, Petit & Wolf 1994, Wolf & Petit 1995). This clearly indicates that modern metrology of time requires a relativistic treatment when working with clocks and comparison methods at the current uncertainty limit.

In Part II. of this thesis a detailed theory for the treatment of the metrology of time in a relativistic context is presented. It provides mathematical expressions for application to the syntonisation and synchronisation of clocks and the realisation of the time coordinates of space-time reference systems. The theoretical expressions are developed to accuracies exceeding those of previous publications in order to accommodate any development in clock and time-transfer technology that can be expected in the near future.

Section II.1 presents a relativistic theory for the syntonisation of clocks in the vicinity of the Earth (within a geocentric sphere of 300000 km radius), including all terms larger than one part in $10^{18}$. This theory is based on recent work by Wolf & Petit (1995), Petit & Wolf 1996. The space-time metric for the geocentric reference system, including terms of order $\varepsilon^4$ (terms in $c^{-4}$), has been derived in Brumberg & Kopejkin (1988), Kopejkin (1988) and Damour, Soffel & Xu (1991). The relationship between TCG and the proper time of a clock in the vicinity of the Earth is given by Brumberg & Kopejkin (1990), Klioner (1992) and
Brumberg et al. (1993). In these papers tidal terms of order $10^{17} \text{s/s}_{\text{TCG}}$, including the response of an elastic Earth are given at an accuracy sufficient for the purposes of this thesis. However, effects of oceanic tides and of non-tidal origin which can contribute up to a few $10^{17} \text{s/s}_{\text{TCG}}$, are not considered. Furthermore, these papers are incomplete as, on one hand, they specify tidal terms of order $10^{-17} \text{s/s}_{\text{TCG}}$, while, on the other hand, the expressions given for the geopotential cannot be used for syntonisation at accuracies better than $10^{-14} \text{s/s}_{\text{TCG}}$ (c.f. equation I.1.3). In sections (II.1.1.1, II.1.1.2) the methods that can be used to obtain the value of the geopotential with sufficient accuracy are detailed. Using such methods the uncertainty of syntonisation is of order $10^{17} \text{s/s}_{\text{TCG}}$ for clocks on the Earth’s surface and $10^{-18} \text{s/s}_{\text{TCG}}$ for clocks onboard terrestrial satellites.

A relativistic theory for the synchronisation of remote clocks in the vicinity of the Earth is presented in section II.2. Recent theoretical studies in this field claim an accuracy of 0.1 nanoseconds (Klioner 1992), and in some cases (Allan & Ashby 1986, CCIR 1990, CCDS 1980) the provided formulae are expressed in terms of path-integrals making them more difficult to use than explicit expressions. The theory presented here (based on recent work by Petit & Wolf (1994)) gives explicit expressions for the synchronisation of two remote clocks including all terms that in the vicinity of the Earth (within a geocentric sphere of 200000 km radius) are greater than one picosecond.

Finally (section II.3) describes the application of the synchronisation and syntonisation of remote clocks to the realisation of the coordinate time scales TCG, TT and TCB. The transformations relating these three coordinate time scales are given, together with their limitations mainly due to uncertainties in our knowledge of geophysical and astronomical constants.
II.1 Syntonisation in the Vicinity of the Earth

In general relativity the rate of a clock A with respect to coordinate time \( \frac{d\tau_A}{du}(u) \) is dependent on the space-time coordinate system used and its metric tensor (a direct result of equation (I.1.2)). In the approximation used here (c.f. equation (I.1.8)) this means that \( \frac{d\tau_A}{du}(u) \) is subject to the gravitational field the clock is submitted to and to the coordinate velocity of the clock. Consequently two clocks that move with a relative velocity, and are submitted to different gravitational fields will in general be observed to run at different rates, a prediction that has been verified experimentally with a relative uncertainty of \( 7 \times 10^{-4} \) using two hydrogen masers (Vessot 1980). Therefore, when comparing the rates of two distant clocks or using a clock for the realisation of a coordinate time this rate shift has to be taken into account at a level of accuracy that should be below the instability and inaccuracy of the clocks in question. A relativistic theory for the syntonisation of distant clocks should therefore provide a formalism for the calculation of the relativistic rate shift, and details for its application in practice including all terms that may be significant at the required level of accuracy.

The measured and expected accuracies and stabilities of clocks were discussed in section I.4.3 (see also fig. I.1). Stabilities are expected to attain the \( 10^{-17}, 10^{-18} \) range while accuracies should soon reach parts in \( 10^{16} \). In this chapter a relativistic theory for the syntonisation of clocks in the vicinity of the Earth (within a geocentric sphere of 300000 km radius) is presented, including all terms larger than one part in \( 10^{18} \). Outside this sphere the effect of the lunar quadrupole moment may exceed \( 1 \times 10^{-18} \) and should be included.

In section II.1.1 the syntonisation of an ideal clock with respect to TCG (the determination of \( \frac{d\tau_A}{dT_{CG}}(T_{CG}) \)) which allows the syntonisation of two distant ideal clocks within GRS (the determination of \( \left[ \frac{d\tau_A}{dT_{B}}(T_{CG}) \right]_{GRS} \) as defined in section I.2.3.1 will be considered. These quantities are of interest for the realisation of TCG using a real clock, and also when two real clocks are compared using repeated time transfers (providing
and one is interested in extracting \( R_{AB}(TCG) \) or \( \phi_\alpha(TCG) - \phi_\beta(TCG) \) using approximations (I.2.10) and (I.2.11) respectively. In section II.1.3 the case where two distant clocks are syntonised using an electromagnetic signal locked to A and received by B will be treated. When determining the relative rate of two distant clocks one might be interested in time varying effects only (i.e. effects that influence the observed frequency stability). These are investigated in section II.1.2. Tidal variations of the gravitational field as well as those of non-tidal origin (atmospheric pressure changes, movements of the Earth's crust, polar motion etc.) are considered.

Note that \( \frac{d\tau}{dT_{CG}} \) is expressed in "s/s\( T_{CG} \)" using different notations for the unit of proper time and the scale units of time coordinates (c.f. section I.2.1 and appendix).

On the surface of the Earth a relativistic rate shift of \( 1 \times 10^{-14} \) s/s\( T_{CG} \) corresponds to a change in altitude of \( \approx 1 \) cm. Therefore, effects on syntonisation due to the size of the clocks and the interaction region of the atoms may not be negligible when such accuracies are required and should be accounted for when using the theory provided here.

II.1.1 Syntonisation with respect to TCG

A general expression for the rate of an ideal clock with respect to coordinate time in the weak field approximation is obtained by substituting (I.1.7) into the metric equation (I.1.2) and solving for \( d\tau/du \) with \( u = x^0/c \),

\[
\frac{d\tau}{du} = 1 - \frac{\kappa_{\alpha\beta}^{(2)}}{2c^2} - \frac{v^2}{2c^2} - \frac{\kappa_{\theta\phi}^{(2)}}{2c^2} - \frac{\kappa_{\theta\phi}^{(0)}}{8c^4} - \frac{\kappa_{\phi\phi}^{(0)}}{2c^2} - \frac{\kappa_{\phi\phi}^{(2)}}{8c^4} - \frac{\kappa_{\phi\phi}^{(2)}}{4c^4} + O(c^{-6}),
\]

(II.1.1)

where \( v' = dx'/du \) is the coordinate velocity of the clock.
For a geocentric coordinate system with TCG as coordinate time and non-rotating spatial coordinates, the components of the space-time metric up to order \( h_{00}^{(3)} \) are given, for example, by Brumberg et al. (1992). The fourth order term \( h_{00}^{(4)} \) is derived in Kopejkin (1988). Substituting these results into (II.1.1) it can be seen that in the vicinity of the Earth all terms of order \( \varepsilon^4 \) (terms in \( c^4 \)) amount to some \( 10^{-19} \) s/s\(_{TCG} \) or less. In particular the \( h_{00}^{(4)}/2c^4 \) term and terms due to the geodesic precession (in \( h_{0i}^{(3)}/c^4 \)), which require the specification of coordinate conditions (harmonic, standard post-Newtonian etc...) and the state of rotation of the frame (kinematically or dynamically non-rotating), are below the \( 10^{-18} \) s/s\(_{TCG} \) limit. The choice of coordinate conditions and of the state of rotation (in the above sense) of the frame is therefore not significant for syntonisation at an accuracy of \( 10^{-18} \) s/s\(_{TCG} \).

For this reason only the \( h_{00}^{(2)} \) component of the metric tensor is required here. It is given in equation (I.3.3). The term due to the non-geodesic motion of the centre of the Earth, \( Q\omega^k \) (given explicitly e.g. in Brumberg 1991a), gives rise to a correction of less than a few \( 10^{-19} \) s/s\(_{TCG} \) in the vicinity of the Earth and can therefore be neglected for the purposes of this thesis.

Hence the rate of a clock with respect to coordinate time (TCG) in the vicinity of the Earth, including all terms larger than \( 1 \times 10^{-18} \) s/s\(_{TCG} \), is

\[
\frac{d\tau}{dTCG} = 1 - \frac{1}{c^2} \left[ U_E(\vec{w}) + \frac{v^2}{2} + \nabla(\vec{x}_E + \vec{w}) - \nabla(\vec{x}_E) - \nabla_{\vec{x}}(\vec{x}_E)\vec{w}^2 \right].
\]

Orders of magnitude of the individual terms in (II.1.2), and their calculation at the required accuracy, are considered in detail in the following sub-sections.

**II.1.1.1 Clocks on the Earth’s Surface**

The limiting factor for syntonisation with respect to coordinate time, of a clock on the surface of the Earth is the inaccuracy in the determination of the Earth's gravitational potential. Currently this uncertainty is \( \approx 1 \text{ m}^2/\text{s}^2 \) for the total (gravitational + centrifugal) potential on the geoid, \( W_0 \), (Bursa et al. 1992, Bursa 1993) which is equivalent to a 10 cm error in radial
distance. This corresponds to an uncertainty of $\approx 1 \times 10^{-17}$ $s/s_{TCO}$ in (II.1.2). In this section I therefore only consider effects whose influence on the terms in (II.1.2) is larger than this value. These are summarised in table II.1, together with orders of magnitude and uncertainties of the associated corrections.

<table>
<thead>
<tr>
<th>Effect</th>
<th>Order of magnitude</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth's grav. pot.</td>
<td>$7 \times 10^{-10}$</td>
<td>$10^{-17}$</td>
</tr>
<tr>
<td>Centrifugal pot. ($v^2/2c^2$)</td>
<td>$1 \times 10^{-12}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
<tr>
<td>Volcanic and coseismic</td>
<td>$&lt; 10^{-16}$</td>
<td>?</td>
</tr>
<tr>
<td>(highly localised)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>External masses (moon, sun)</td>
<td>$10^{-17}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
<tr>
<td>Solid Earth tides</td>
<td>$10^{-17}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
<tr>
<td>Ocean tides</td>
<td>$10^{-17}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
</tbody>
</table>

Table II.1: Effects on syntonization with respect to TCG of clocks on the Earth's surface; orders of magnitude and uncertainties of the corrections (in $s/s_{TCO}$).

The gravitational potential of the Earth, $U_E(\bar{w})$, can be expressed as a series expansion in spherical harmonics. However, owing to mass irregularities, such a series cannot be considered convergent at the surface of the Earth (Moritz 1961). Nonetheless, due to the predominantly ellipsoidal shape of the Earth, one can use the first two terms of this series expansion as a first approximation (Allan & Ashby 1986, CCIR 1990, Kliker 1992, Brumberg et al. 1993). Thus,

$$U_E(\bar{w}) = \frac{GM_E}{\bar{w}} + \frac{GM_Ea_1^2J_2}{2\bar{w}^3}(1-3\cos^2\theta) + \ldots$$

(II.1.3)

where $G$ is the Newtonian gravitational constant, $M_E$ is the mass of the Earth ($GM_E = 3.9860044 \times 10^{14}$ $\text{m}^3/\text{s}^2$), $a_1$ and $J_2$ are, respectively, the equatorial radius and the quadrupole moment coefficient of the Earth ($a_1 = 6378136.3$ $\text{m}$, $J_2 = 1.0826 \times 10^{-3}$) and $\theta$ is the geocentric colatitude of the point of interest.
Substituting (11.1.3) into the second term of (11.1.2) gives terms which can amount to \( \approx 7 \times 10^{-10} \) \( \text{s/s}_{\text{TGG}} \) and \( \approx 8 \times 10^{-13} \) \( \text{s/s}_{\text{TGG}} \) for points on the surface of the Earth.

Considering the third term in (11.1.2), one can see that with

\[
\nu = \omega_E w \sin \theta,
\]

for a clock fixed on the surface of the Earth (where \( \omega_E \) represents the angular velocity of rotation of the Earth, \( \omega_E = 7,292115 \times 10^{-5} \text{ rad/s} \)) this term is equivalent to the centrifugal potential divided by \( c^2 \). Its magnitude can reach \( 1.2 \times 10^{-12} \text{s/s}_{\text{TGG}} \).

The effect on this term of the movement of the pole (\( \Delta \theta \)) and variations in the length of day (\( \Delta \omega_E \)) are of order \( 10^{-18} \text{s/s}_{\text{TGG}} \) and smaller and are treated in more detail in section II.1.2.

The surface obtained when setting \( U_e(\vec{w}) = W_o - (\omega_E w \sin \theta)^2 / 2 \) in (11.1.3) differs from the ellipsoid of the Earth model by less than 10 m. Therefore, an estimate of the accuracy of (11.1.3) can be obtained by considering the maximal difference between the geoid and the reference ellipsoid. This can amount to \( \approx 100 \) m (Vanicek & Krakiwsky 1986), so expression (11.1.3) for the Earth's gravitational potential should not be used if accuracies better than \( 1 \times 10^{-14} \text{s/s}_{\text{TGG}} \) are required.

For improved accuracy the second and third term in (11.1.2) should not be computed separately using (11.1.3) and (11.1.4). Instead, their combined effect should be determined using

\[
U_e(\vec{w}) + \frac{(\omega_E w \sin \theta)^2}{2} = W_o - \int g \, dH = W_o - \bar{g} H
\]

where \( g \) is the Earth's gravitational + centrifugal acceleration, and \( H \) is the height above the geoid. A value of \( g \) averaged between 0 and \( H \), \( \bar{g} \), obtained from a gravimetric model, can be used instead of computing the integral when the required accuracy in (II.1.2) is of order \( 10^{-15} \text{s/s}_{\text{TGG}} \).

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Using a geodetic GPS (Global Positioning System) receiver and a geoid model the height above the geoid, $H$, can be obtained with an accuracy of the order of 10 m. This allows the determination of the total (gravitational and centrifugal) effect on the clock with an accuracy of $\approx 1 \times 10^{-15}$ s/$s_{TCG}$. Similar accuracy can also be obtained by using a topographic map for the determination of $H$.

When higher accuracy is required, precise levelling should be used. Levelling measurements are referred to a zero-level reference point which can be compared to mean sea level using a tidal gauge. This level differs from the geoid by what is known as Sea Surface Topology (SST) which can amount to $\pm 0.7$ m (Torge 1989). The SST can be determined with an accuracy of $\approx 0.1$ m (Torge 1989) using oceanographic methods and satellite altimetry which induces an uncertainty of $\approx 1 \times 10^{-17}$ s/$s_{TCG}$ in (II.1.2). The uncertainty in the potential on the geoid, $W_0$, which is of order 1 m$^2$/s$^2$ (Bursa et al. 1992; Bursa 1993), contributes another $1 \times 10^{-17}$ s/$s_{TCG}$. The sum of the gravitational and centrifugal potential differences between mean sea level and an arbitrary point far from the coast can be obtained by geometrical levelling with simultaneous gravimetric measurements. The accumulated uncertainty when using modern levelling techniques and gravimetry is below $(0.5 \sqrt{D}/$km$)$ mm (Kasser 1989), where $D$ is the distance between the reference point and the point of interest, and does therefore not exceed a few centimetres even over large distances. In many countries levelling networks have been established at accuracies of $\approx (2 \sqrt{D}/$km$)$ mm for primary points, the use of which would again induce errors at the centimetric level. Alternatively levelling can be achieved with accuracies of order 10 cm (for distances of $\approx 100$ km) using differential GPS (Milbert 1992).

Therefore the constant part of the total potential at any point on the Earth's surface can be determined with an ultimate uncertainty less than 2.5 m$^2$/s$^2$ using a tidal gauge and good geometrical levelling. The main contributions to this uncertainty are inaccuracies in the determination of $W_0$ and the SST. This limits the evaluation of (II.1.2) at the level of $(2-3) \times 10^{-17}$ s/$s_{TCG}$, which is the limit for syntonisation of clocks with respect to coordinate time (TCG) on the surface of the Earth.

Additionally account has to be taken of the time varying part of the potential on the surface of the Earth caused by the gravitational attraction of external masses (tidal effects) and changes in the Earth's own gravitational field (non-tidal effects).
A number of effects gives rise to relativistic rate shifts which are larger than $1 \times 10^{-18}$ $s/s_{TcG}$ and are of a periodic nature. For clock comparisons using time transfers with a synchronisation accuracy of one picosecond (c.f. section II.2), such terms are negligible if their period is sufficiently short to prevent their amplitude in the time domain from exceeding this limit. They might, however, be of interest when comparing two distant clocks using an electromagnetic signal (see section II.1.3) and are therefore included in this study.

At $10^{-17}$ $s/s_{TcG}$ accuracy non-tidal effects are highly localised and can be neglected in general. They relate mainly to movements of the Earth's crust caused by volcanic and coseismic processes. The resulting change in the second term of (II.1.2) can amount to $\approx 10^{-16}$ $s/s_{TcG}$ on time scales ranging from a few days to one year (Torge 1989). From gravimetric measurements Ervin & McGinnis (1986) have inferred local elevation changes in the Mississippi embayment of up to 15 cm, caused by the surface loading associated with changes in river stage. The resulting change in the second term of (II.1.2) is $\approx 1.5 \times 10^{-17}$ $s/s_{TcG}$. In section II.1.2 a more detailed treatment of non-tidal effects at the $10^{-18}$ $s/s_{TcG}$ accuracy level is presented.

The third, fourth and fifth terms of (II.1.2) represent the effect of external masses, mainly the Moon and Sun. The Newtonian potential of external bodies can be expressed in the spherical approximation by,

$$U(x) = \sum_{A \in E} \frac{G M_A}{r_A}, \quad \text{(II.1.6)}$$

where $r_A$ is the coordinate distance between the point of interest and the centre of mass of the body $A$, $M_A$ is the mass of body $A$, the summation is over all celestial bodies apart from the Earth, and multipole terms are neglected as their effect on the surface of the Earth does not exceed $10^{-18}$ $s/s_{TcG}$. Note that it is not essential whether $r_A$ is expressed in geocentric or barycentric coordinates, as the induced error is several orders of magnitude smaller than $10^{-18}$ $s/s_{TcG}$. 

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Substituting (II.1.6) into the third, fourth and fifth terms of (II.1.2), expanding the third term in a Taylor series and using Love numbers to characterise the response of the Earth to the tidal potential (the solid Earth tide) gives,

\[ U_T(\vec{w}) = (1 + k_2 - h_2) \sum_{E,A} \left[ \frac{GM_A}{2r_{EA}^2} \left( 3 \frac{r'_{EA}}{r_{EA}^3} \frac{r'_{EA}}{r_{EA}^2} - \delta_4 \right) w'w' + o\left( \frac{GM_A w^3}{r_{EA}^4} \right) \right], \]  

(II.1.7)

where \( r'_{EA} = r_E - r_{EA} \), \( r_{EA} = (r'_{EA}r'_{EA})^{1/2} \), and where \( k_2 \) and \( h_2 \) are the Love numbers. For most Earth models \( (1 + k_2 - h_2) = 0.69 \) (Farrell 1972). It should be checked that a part of the solid Earth tide has not been already included when calculating (II.1.5).

Evaluation of expression (II.1.7) for the moon and the sun gives a correction in (II.1.2) which is smaller than \( 4 \times 10^{-17} \) s/s_{TCG}. Contributions from other planets and higher order terms in (II.1.7) add corrections which are smaller than \( 1 \times 10^{-18} \) s/s_{TCG}. The effect of oceanic tides can amount to \( 9 \times 10^{-18} \) s/s_{TCG} for the M2 lunar tide in a few regions, and roughly twice this value for the total tide (Scherneck 1994). It is discussed in more detail in section II.1.2.

**II.1.1.2 Clocks Onboard Terrestrial Satellites**

The accuracy of syntonisation of satellite clocks with respect to TCG is limited by uncertainties in the geopotential model and the orbit determination. Solid Earth tides, ocean tides, polar motion and changes in atmospheric pressure may give rise to corrections of some \( 10^{-18} \) s/s_{TCG} for low flying satellites, but can be neglected at altitudes exceeding 4000 km. The tidal potentials of external masses become more important with increasing altitude and so require an exact expression, rather than a series expansion as in (II.1.7), for their evaluation. Table II.2. lists all relevant effects together with orders of magnitude and uncertainties of the associated corrections.
<table>
<thead>
<tr>
<th>Effect</th>
<th>Order of magnitude</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth's grav. pot.</td>
<td>$&lt; 6 \times 10^{-10}$</td>
<td>few $10^{-18}$ (GEM-T3)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$&lt; 10^{-18}$ at $h &gt; 4000$ km</td>
</tr>
<tr>
<td>2nd order Doppler ($\nu^2/2c^2$)</td>
<td>$&lt; 3 \times 10^{-10}$</td>
<td>few $10^{-18}$ (due to 5 cm orbit uncertainty)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$&lt; 10^{-18}$ at $h &gt; 10000$ km</td>
</tr>
<tr>
<td>External masses:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moon (at $h = 300000$ km)</td>
<td>$4 \times 10^{-13}$</td>
<td></td>
</tr>
<tr>
<td>Sun</td>
<td>$4 \times 10^{-14}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
<tr>
<td>Venus</td>
<td>$6 \times 10^{-18}$</td>
<td></td>
</tr>
<tr>
<td>Solid Earth tides</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ocean tides</td>
<td>$10^{-18}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
<tr>
<td>Polar motion</td>
<td>(at low altitudes)</td>
<td></td>
</tr>
<tr>
<td>Atmospheric pressure</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table II.2: Effects on syntonization with respect to TCG of clocks onboard terrestrial satellites; orders of magnitude and uncertainties of the corrections (in s/s$_{TCG}$) where $h$ represents the altitude of the satellite.

The rate of a clock with respect to TCG is given by equation (II.1.2). The geopotential in the second term can be expressed as a series expansion in spherical harmonics

$$U_E(\bar{w}) = \frac{GM_E}{w} \left[ 1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} P_{nm}(\cos \theta) \left( \frac{a_1}{w} \right)^n \left( C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda) \right) \right],$$

where $\theta$ and $\lambda$ are the geocentric colatitude and longitude of the satellite, $P_{nm}(\cos \theta)$ are associated Legendre polynomials, and the $C_{nm}$ and $S_{nm}$ are coefficients determined by fitting the data from satellite observations. For the latest model (GEM-T3) these coefficients are given with an accuracy of a few parts in $10^9$ up to degree and order 50 (Lerch et al. 1992, IERS 1992). For low altitudes ($< 4000$ km) this results in syntonisation uncertainties which exceed $1 \times 10^{-18}$ s/s$_{TCG}$, but decrease with increasing altitude. The effects of solid Earth, oceanic, and
pole tides can be included in the model as small variations of the coefficients. The effect of atmospheric pressure variations may amount to $(1-2) \times 10^{-18} \text{s/sTCG}$ for altitudes < 4000 km but corrections to an accuracy of $10^{-18} \text{s/sTCG}$ are possible (see section II.1.2).

The third, fourth and fifth terms in (II.1.2) characterise the effect of external masses on the rate of a clock relative to TCG. It is not practical to use the Taylor expansion of the third term as in the previous section, since at higher altitudes a large number of terms would be needed to achieve the required $10^{-18} \text{s/sTCG}$ accuracy. Instead, substituting (II.1.6) into (II.1.2) for the external potentials and differentiating the fifth term, one obtains

$$U(\mathbf{r}_E + \mathbf{w}) - U(\mathbf{r}_E) - U_k(\mathbf{r}_E)w^k = \sum_{A \neq E} GM_A \left[ \frac{1}{r_{PA}} - \frac{1}{r_{EA}} + \frac{r_{EA}^k r_{EA}^k}{r_{EA}^3} \right]$$

(II.1.9)

for the total tidal potential. Here $r_{PA}^k = r_k^F - r_A^k$ is the vector from the centre of mass of body A to the point of interest with magnitude $r_{PA} = (r_{PA}^k r_{PA}^k)^{1/2}$, the subscript E stands for the Earth, and the summation is carried out over all celestial bodies except the Earth. At the required accuracy either barycentric or geocentric coordinates may be used.

The maximal magnitude of these terms in (II.1.2) is $\approx 4 \times 10^{-13} \text{s/sTCG}$ for the moon, $\approx 4 \times 10^{-14} \text{s/sTCG}$ for the sun and $\approx 6 \times 10^{-18} \text{s/sTCG}$ for Venus, for satellites at high altitudes. The effects of other planets and asteroids are negligible. The constraints on the knowledge of the planetary ephemerides are $\approx 117$ m for the Earth-moon distance, $\approx 1200$ km for the Earth-sun distance and $\approx 10^6$ m for the distance to Venus, which present no difficulties for modern astrometry.

Syntonisation of satellite clocks with respect to TCG is limited mainly by orbitography errors. For syntonisation to $10^{-18} \text{s/sTCG}$ the required accuracies are of order 1 cm on position and $1 \times 10^{-5}$ m/s on velocity for a satellite at 1000 km altitude. For satellites in higher orbits the constraints are less severe, being about 40 cm and $3 \times 10^{-5}$ m/s for a geostationary one.

The data necessary for orbit determination can be obtained by satellite ranging from a number of ground stations on the Earth using timing measurements of electromagnetic signals. For this purpose the clocks at the different stations have to be synchronised and syntonised, so
the problems of satellite orbitography and time metrology are not entirely independent. However, the accuracy required for station clock synchronisation and syntonisation for the realisation of centimetric orbits is of order one microsecond and parts in $10^{11}$ respectively (as typical satellite velocities are $< 10$ km/s and signal transmission times $< 1$ s), values which can be achieved even with terrestrial methods. This is why the two problems can be separated.

At present satellite laser ranging produces measurements at a precision of a few millimetres, with an accuracy of roughly one centimetre (Degnan 1993), the limiting error source being uncertainty in the atmospheric propagation delay. The satellite orbit is determined from the ranging measurements using an orbital model, which introduces further inaccuracies. Differences between orbits obtained using different ranging techniques and models for the Topex/Poseidon mission (altitude 1200 km) are typically a few centimetres (Yunck 1994, Nouel 1994, Schutz 1994) (consistent with an uncertainty of $\approx 1 \times 10^{-3}$ m/s in velocity) which is an indication of the accuracy of the orbits obtained. Beutler et al. (1994) estimate the uncertainty of precise GPS ephemerides (altitude 20000 km) to be $\approx 15$ cm ($\approx 1 \times 10^{-3}$ m/s in velocity).

In conclusion, the syntonisation of satellite clocks with respect to TCG is at present limited by uncertainties in the geopotential model (GEM-T3) and the satellite orbits at an accuracy of a few $10^{-18}$ s/s$_{TCG}$ at low altitudes, with a decrease of this limit below $1 \times 10^{-18}$ s/s$_{TCG}$ with increasing altitude ($> 10000$ km), which is an order of magnitude better than the uncertainty for clocks on the Earth's surface. Therefore it seems likely that future time will be provided from space.

II.1.2 Small Time Varying Effects

Section II.1.1 treats the syntonisation of clocks with respect to coordinate time, TCG. It was found that for clocks on the Earth's surface the uncertainty of this syntonisation is limited at $10^{-17}$ s/s$_{TCG}$ by uncertainties in the geopotential. For this reason several time varying effects with amplitudes below this level (polar motion, atmospheric pressure etc...) were neglected. In this section I will consider the case where only the stability of the relative rate
between two clocks is of interest. Then only time varying effects need be considered which, as shown below, can be calculated to $10^{-18}$ s/s$_{TCG}$ accuracy even for clocks on the Earth's surface. These effects are summarised in Table II.3., together with orders of magnitude and uncertainties of the associated corrections.

<table>
<thead>
<tr>
<th>Effect</th>
<th>Order of magnitude</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volcanic and coseismic</td>
<td>$&lt; 10^{-16}$</td>
<td>?</td>
</tr>
<tr>
<td>(highly localised)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Geodynamic and man-made</td>
<td>$&lt; 10^{-16}$</td>
<td>?</td>
</tr>
<tr>
<td>(localised and long-term $&gt; 1$ year)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>External masses (moon, sun)</td>
<td>$10^{-17}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
<tr>
<td>Solid Earth tides</td>
<td>$10^{-17}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
<tr>
<td>Ocean tides</td>
<td>$10^{-17}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
<tr>
<td>Polar motion</td>
<td>$10^{-18}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
<tr>
<td>(long-term $\approx 430$ days)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Atmospheric pressure</td>
<td>$10^{-18}$</td>
<td>$&lt; 10^{-18}$</td>
</tr>
</tbody>
</table>

Table II.3: Time varying effects on the Earth's surface for the determination of the relative rate of two clocks; orders of magnitude and uncertainties of the corrections (in s/s$_{TCG}$).

When comparing two distant clocks using either repeated time transfers (equation (1.2.9)) or by transmission of an electromagnetic signal (equation (II.1.13)) the obtained results are dimensionless numbers which are dependent on the rate of the individual clocks with respect to coordinate time, $d\tau/dTCG$. Therefore the time varying effects investigated in this section are those affecting $d\tau/dTCG$ and given in "s/s$_{TCG}$".

The correction due to external masses (moon and sun) and the solid Earth tide can be calculated (see equation (II.1.7)) with an uncertainty of $\approx 4 \times 10^{-19}$ s/s$_{TCG}$ due to the uncertainty in the determination of the Love numbers (Farrell 1972).

The effect of the ocean tide, including the associated loading, can be obtained using global oceanic tide models (Schwiderski 1983) and the loading deformation coefficients of
Farrell (1972). The results of one such calculation (Scherneck 1994) describe the total effect of the oceanic tides on the potential at the surface of the Earth in a 10x10 grid covering most of the globe. For the M2 lunar tide, the correction can reach 9x10^{-18} s/s_{T2G} in a few regions with a maximum of twice this value for the total tide, the uncertainty of these values being ≈2x10^{-19} s/s_{T2G} (IERS 1992).

Atmospheric pressure variations may cause rate shifts of a few 10^{-18} s/s_{T2G}. The direct gravitational potential of the atmosphere can be calculated adapting a method employed by Merriam (1992), in order to obtain the effect on the gravitational potential rather than the gravitational acceleration. The contribution to the potential at some point P, due to a thin column of air of infinitesimal area at azimuth α and a geocentric angular distance γ from the point of interest can be calculated as a function of the pressure and temperature at its base. Using the ideal gas law and assuming hydrostatic equilibrium and an isothermal atmosphere gives for the potential of one column

\[ U(\gamma, \alpha) = \frac{G P_0 \alpha \sin \gamma \, dy \, d\alpha}{RT_0} \int_0^{z_{max}} \frac{e^{-\gamma/H}}{[\omega^2 + (\alpha + z)^2 - 2\omega(\alpha + z)\cos \gamma]^2} \, dz \]  

(II.1.10)

where \(P_0\) and \(T_0\) are the pressure and the temperature at the base of the column, \(R\) is the specific gas constant for dry air (\(R = 287.05\) J kg\(^{-1}\) K\(^{-1}\)), \(z_{max}\) is the height of the atmosphere (\(z_{max} \approx 50\) km), and \(H\) is the scale height of the atmosphere. Typically, \(H\) varies from about 8 km near the surface to about 7 km in the stratosphere (Merriam 1992). Summing these functions over the globe using surface pressure and temperature data and small increments \(d\alpha\) and \(dy\) gives the total gravitational potential of the atmosphere. The additional change caused by the associated atmospheric loading can be calculated using the above method and the surface load Love numbers \(k_s\) and \(h_s\) (Farrell 1972). Alternatively, a simple regression formula by Rabbel and Zschau (1985) can be used, giving the radial displacement of a point on the Earth's surface by

\[ \Delta w / \text{mm} = (-0.35p - 0.55\bar{p}) / \text{mbar} \]  

(II.1.11)

where \(p\) is the pressure variation at the point of interest and \(\bar{p}\) the average of the pressure variation in the surrounding area of 2000 km radius with the pressure values set equal to zero.
over ocean areas. Rabbel and Zschau (1985) estimate the uncertainty of this expression to be less than 1 mm. For pressure variations of 10 mbar on a global scale (corresponding to seasonal changes) the effect on the rate of a clock on the Earth’s surface can reach $2 \times 10^{-18}$ s/sTGG due to the direct potential (equation (II.1.10)) but the magnitude of the displacement effect remains below $10^{-18}$ s/sTGG. Local pressure changes ((anti)cyclones) can cause displacements of up to 2.5 cm, corresponding to a correction of $2.7 \times 10^{-18}$ s/sTGG but have a negligible direct potential. Finally the secondary potential due to the deformation of the Earth can be neglected as the appropriate surface load Love numbers $k'_n$ are at least a factor 4 smaller than the corresponding $h'_n$, which are used to calculate the displacement effect (Farrell 1972).

For a clock on the Earth’s surface the centrifugal potential is given by $(\omega^2 \sin \theta)^2/2$. Differentiating this expression with respect to $\omega$ and to $\theta$, dividing by $c^2$ and including Love numbers allows the calculation of the total correction due to polar motion $\Delta \theta$ and variations in the angular velocity of the Earth $\Delta \omega_E$,

$$(1 + k_2 - h_2) \frac{\Delta (\omega_E \sin \theta)^2}{2c^2}$$

$$= (1 + k_2 - h_2) \frac{1}{c^2} \left( \frac{1}{2} \omega_E^2 \sin^2 \theta \Delta \theta + \omega_E \sin^2 \theta \Delta \omega_E \right)$$

Because the spherical harmonic contribution of the centrifugal potential is restricted to degree two (Hinderer et al. 1982) the total effect can be obtained using the classical Love numbers $k_2$, $h_2$ with $(1 + k_2 - h_2) = 0.69$. Maximum values for $\Delta \theta$ and $\Delta \omega_E$ are $2.4 \times 10^{-6}$ rad and $7 \times 10^{-12}$ rad/s respectively (Torge 1989) which result in corrections of up to $2 \times 10^{-18}$ s/sTGG and $1.6 \times 10^{-19}$ s/sTGG for the first and second terms in (II.1.12).

Finally I mention long term effects of a geodynamic nature, and give some examples of highly localised effects of volcanic, coseismic and man made origin which might have to be taken into account at certain sites.

On tectonic plate boundaries, geodynamic effects may give rise to corrections of up to $10^{-16}$ s/sTGG over a period of several years. For example in northern Iceland elevation changes of 1 m (corresponding to a correction of the order $10^{-16}$ s/sTGG) were observed between 1975
and 1980 (Torge 1989). In other regions the magnitude of geodynamic effects may only marginally reach the $10^{-18}$ s/s$\text{TCG}$ level on time scales exceeding 1 year.

Volcanic and coseismic activities observed in Hawaii and California caused elevation changes of the order of 1 m (Torge 1989) over periods up to several months.

Mass displacements by human interference (e.g. exploitation of oil, gas, geothermal fields) may lead to local corrections of order $10^{-17}$ s/s$\text{TCG}$ per year (Torge 1989).

### II.1.3 Syntonisation using an Electro-Magnetic Signal

Suppose that two distant clocks are compared using an electromagnetic signal locked to A and transmitted from A to B where its relative rate with respect to B is measured. Denoting by $\frac{dh_B}{dt}$ the normalised frequency of the signal at location $i$ (the rate of the signal with respect to the ideal clock $i$ as measured locally) one obtains on reception

$$
\frac{dh_B}{dt} \left( t_e \right) \frac{d\tau^A}{dh^A}(t_e) = \frac{d}{dt} \left( \frac{d\tau^A}{dh^A}(t_e) \right) \left[ \frac{1 + \tilde{k} \cdot \vec{w}_B(t_e)}{1 + \tilde{k} \cdot \vec{w}_A(t_e)} \right] + \frac{d}{dt} \left[ \frac{2GM_e}{c^3} \left( \frac{\vec{w}_B(t_e) - \tilde{k} \cdot \vec{w}_A(t_e)}{\vec{w}_A(t_e) - \tilde{k} \cdot \vec{w}_A(t_e)} \right) \right]
$$

where a dot denotes differentiation with respect to TCG which in (II.1.13), (II.1.14) and (II.1.15) is replaced by "t", the subscripts e and r refer to the emission and reception of the signal, and $\tilde{k} = (\vec{w}_A(t_e) - \vec{w}_B(t_e)) / |\vec{w}_A(t_e) - \vec{w}_B(t_e)|$.

The result of a comparison of two distant clocks using an electro-magnetic signal, $\frac{dh_B}{dt} \left( t_e \right) \frac{d\tau^A}{dh^A}(t_e)$, which is a proper quantity measured locally should not be confused with $\left[ \frac{d\tau^A}{dr} \right]_{\text{TCG}}$ as defined in (I.2.9) which is a coordinate quantity. The two differ essentially by the Doppler terms of order $\varepsilon^1$ that can reach parts in $10^5$ and by terms of order $\varepsilon^3$ of
gravitational and Doppler origin inferior to a few parts in $10^{14}$. They can be related by substituting into (II.1.13) the approximation

$$\frac{d\tau_A}{dt}(t_s) \frac{dt}{d\tau_B}(t_s) \approx \left[ \frac{d\tau_A}{d\tau_B}(t_s) \right]_{GRS}$$

(II.1.14)

valid if the differences $d\tau/dt(t_s) - d\tau/dt(t_0)$ are negligible or can be estimated with sufficient accuracy.

For two real clocks A and B a measurement of $\frac{d\tau_B}{dt}(t_s) \frac{d\tau_A}{dt}(t_s)$, can be related to the relative rate $R_{AB}(t_e)$, measured when the two clocks are colocated at some later time $t_e$, using the approximation

$$\frac{d\sigma_B}{d\sigma_A}(t_s) \frac{d\tau_A}{d\tau_B}(t_s) \approx R_{AB}(t_e) \left( \frac{d\sigma_B}{d\sigma_A}(t_e) \frac{d\tau_A}{d\tau_B}(t_s) \right)$$

(II.1.15)

valid under the condition that the differences $d\sigma/d\tau(t_e) - d\sigma/d\tau(t_0)$ are negligible or can be estimated with sufficient accuracy.
II.2 Synchronisation in the Vicinity of the Earth

In section 1.2.2 a convention of simultaneity and synchronisation, so called coordinate synchronisation, was adopted. According to this convention simultaneity and synchronisation are entirely dependent on the choice of coordinate system. For clocks on the Earth’s surface or in its immediate vicinity the reference system chosen for synchronisation is GRS as defined in section 1.3.2. Because GRS and the rotating frame GRSR (see section 1.3.4) have the same time coordinate (TCG) synchronisation in GRS is equivalent to synchronisation in GRSR.

In practice the synchronisation of two clocks A and B requires the determination of their desynchronisation \(x_{AB}(u)\) (c.f. section 1.2.2) which is also dependent on the choice of frame of synchronisation. However, defining desynchronisation as in equation (1.2.4) to some extent loosens this dependence, because \(x_{AB}(u)\) is invariant under coordinate transformations, \(x^\lambda \rightarrow x'^\lambda\), where the time transformation is independent of space coordinates i.e. \(u = u(u)\). For geocentric frames this implies that \(x_{AB}(TCG) = x_{AB}(TT)\).

Coordinate synchronisation between two distant clocks can be achieved by transport of a third, mobile clock or by the exchange of an electromagnetic signal. Both methods are affected by relativistic effects which require corrections that are significant when compared to currently achieved synchronisation accuracies. Hence a relativistic theory for clock synchronisation is required which provides expressions for the calculation of the desynchronisation of clocks, including all terms that may exceed the uncertainties achieved in practice.

A relativistic treatment of synchronisation by clock transport at the nanosecond accuracy level can be found in Allan & Ashby (1986) or Klioner (1991). With the advent of widely used satellite time transfer systems (in particular the Global Positioning System GPS) which are comparatively easy to operate and more accurate than synchronisation by clock transport the latter method has become all but obsolete for general use and will therefore not be further discussed here.
Recently, the precision of clock synchronisation between remote clocks on the surface of the Earth using electromagnetic signals has reached the sub-nanosecond level (Hetzel & Soring 1993, Veillet et al. 1992, Veillet & Fridelance 1993) with further improvements expected in the near future. For these applications it seems sensible to develop a relativistic theory including all terms greater than one picosecond. Recent theoretical studies in this field claim an accuracy of 0.1 nanoseconds (Klioner 1992), and in some cases (Allan & Ashby 1986, CCIR 1990, CCDS 1980) the provided formulae are expressed in terms of path-integrals making them more difficult to use than explicit expressions.

The theory presented in this section is based on recent work by Petit & Wolf (1994). Explicit expressions for synchronisation in GRS of two clocks that have their positions given in the rotating system are provided including all terms that in the vicinity of the Earth (within a geocentric sphere of 200000 km radius) are greater than one picosecond. Outside this sphere terms due to the potential of the Moon may amount to more than 1 ps and need to be accounted for separately. First local time comparisons within a laboratory are treated briefly (section II.2.1). The main part of this chapter is then devoted to a relativistic theory for time comparisons via electromagnetic signals using the one way method (section II.2.2.1) or the special cases of two way satellite time transfer, TWSTT, (section II.2.2.2) and LASSO (LAser Synchronisation from Stationary Orbit, section II.2.2.3) time transfers via a geostationary satellite. Here a possible small residual velocity of the satellite (<1 m/s) results in further terms contributing some tens of picoseconds for TWSTT and LASSO time transfers. In section II.2.2.4 the limits of applicability of the obtained expressions to practical situations are examined and finally, the analytical formula obtained for TWSTT is applied to particular situations comparing the results to those obtained using a more exact numerical analysis (section II.2.2.5).

II.2.1 Local Time Comparisons

The measured quantity, when locally comparing two clocks, A and B, is the increment of the reading of a third clock (usually an interval counter) $\Delta t_c$ between the arrival of a second pulse from clock A and the arrival of a subsequent second pulse from clock B. Denoting the
reading of clock $i$ at emission of a second pulse by $n_t$ the desynchronisation can be expressed as

$$x_{AB}(TCB_B) = h_A(TCB_B) - h_B(TCB_B)$$

$$= (n_A + \Delta h_A) - (n_B + \Delta h_B)$$

where $TCB_B$ is the coordinate time of arrival of the pulse from B at the counter and

$$\Delta h_A = \Delta h_{AT} + \int_0^{\Delta h} R_{AC}(\tau)d\tau$$

$$\Delta h_B = \Delta h_{BT}$$

with $\Delta h_{IT}$ denoting the travel time of the pulse from clock $i$ to the counter (cable delays etc.) as measured by $i$, and $R_{AC}(\tau)$ is the rate of A with respect to the counter (c.f. section I.2.1).

For picosecond accuracy the integral in (II.2.2) can be replaced by $\Delta h_{C}$ when $R_{AC}(\tau)$ is smaller than $1 \times 10^{-12}$, or by $\Delta h_{C} R_{AC}(\tau)$ when the variation of $R_{AC}(\tau)$ with time does not exceed $1 \times 10^{-12}$ s$^{-1}$.

II.2.2 Synchronising Distant Clocks

The widespread use of highly accurate atomic clocks, operating on the surface of the Earth and onboard terrestrial satellites calls for increasingly accurate methods for their synchronisation. Currently the most common such method is satellite time transfer, in particular the use of the Global Positioning System (GPS). Alternatives are Two Way Satellite Time Transfer (TWSTT) and LAser Synchronisation from Stationary Orbit (LASSO) which use geostationary satellites, Precise RAinge and range Rate Experiment (PRAREtime), and GLONASS which is a Russian system similar to GPS. The GPS system consists of 24 operational satellites in six orbital planes each equipped with an atomic clock and a dual-frequency microwave emission system. The emitted time signals are received on the ground and compared to the local clock. This constitutes a so called one way system (there are no
return signals) whose accuracy is limited at a few nanoseconds by uncertainties in the atmospheric propagation delays and in the knowledge of the satellite and station coordinates. TWSTT, LASSO and PRAREtime are two way systems (signals are transmitted in both directions along almost identical paths) which are affected much less by atmospheric delay uncertainties and coordinate errors, and presently reach sub-nanosecond precisions. These systems, however, are more difficult to operate and call for more complex and expensive equipment. Additionally LASSO, which works at optical frequencies, is weather dependent and can therefore not be used for routine operation.

All the methods mentioned above work at accuracies which are several orders of magnitude smaller than the main relativistic corrections. In the following sub-sections a relativistic theory for satellite time transfer is developed including all terms larger than one picosecond, for the general case of a one-way system and the special cases of TWSTT and LASSO via geostationary satellites. In practice station and satellite coordinates are given in a rotating geocentric frame (e.g. WGS84 or ITRF), so the theory should provide expressions for the desynchronisation of two distant clocks given as a function of the clock positions and satellite ephemerides in GRSR.

Throughout section II.2 Geocentric Coordinate Time (TCG) will be denoted by "t" for reasons of formal simplicity.

**II.2.2.1 One Way Time Transfer**

A light signal is emitted by clock A at coordinate time $t_0$ when its reading is $h_A(t_0)$, and received by clock B at coordinate time $t$ and clock reading $h_B(t)$. The desynchronisation is then given by

$$x_{AB}(t) = h_A(t_0) + \int_{t_0}^{t} \frac{dh_A}{dt}(t)dt - h_B(t)$$

(II.2.3)

where $T_t$ is the total coordinate transmission time, $T_t = t - t_0$.

The calculation of the rate of A with respect to coordinate time $dh_A/dt$ is detailed in section II.1.1. In the vicinity of the Earth $T_t$ does not exceed 0.5 s$_{TCG}$, so for picosecond
accuracy the integral in (II.2.3) can be replaced by \( \frac{dh}{dt}(t_0) T_t \) provided that \( \frac{dh}{dt} \) does not change by more than \( 2 \times 10^{-12} \text{s/s}_{TCG} \) during transmission.

For synchronisation it is now necessary to determine \( T_t \) as a function of the positions in GRSR of A and B. Using the geocentric metric (I.3.4), setting \( ds^2 = 0 \) for a light signal and solving for \( dt \) provides an expression for the transmission coordinate time \( T_t \):

\[
T_t = \int_{\phi_a(t_0)}^{\phi_b(t_f)} \left( \frac{dl}{c + \frac{2U dl}{c^2}} \right) + O(c^{-4})
\]

where \( U \) is the total potential (Earth + tidal), \( dl \) is the increment of coordinate length along the transmission path and the integral is to be taken along the path of the light signal.

Terms of order \( \varepsilon^3 \) and higher in the metric result in additional terms in (II.2.4) which are much smaller than \( 1 \text{ps}_{TCG} \) (see section II.2.2.4 for more detail). In the non-rotating frame the path of the light signal can be approximated as a straight line. Deviations due to gravitational bending and atmospheric refraction result in corrections to the transmission time of less than \( 1 \text{ps}_{TCG} \) for all practical purposes considered here (see section II.2.2.4).

The transmission time in (II.2.4) can be separated into a "geometrical" (the first term) and a "gravitational" (second term) part

\[
T_t = T + T_g.
\]

Considering at first only the geometry, and bearing in mind that generally the known quantities are the positions in the rotating frame, GRSR, of A and B at the time of signal emission \( t_e \), the transmission time can be written as:

\[
T = \frac{1}{c} |\vec{w}_{RB}(t_e) - \vec{w}_{RA}(t_e)| + s
\]
where the subscript \( "R! \) refers to coordinates in GRS\(_R\) and \( s \) represents the coordinate time taken for the signal to travel the extra path due to the motion of \( B \) in the non-rotating frame during transmission (see Fig. II.1). This is usually referred to as the Sagnac correction, first discovered experimentally by Sagnac (1913).

![Diagram showing one way time transfer in the non-rotating frame](image)

**Fig. II.1:** One way time transfer in the non-rotating frame. The dashed line represents \( \vec{w}_{RB} - \vec{w}_{RA} \) while the continuous line shows the path taken by the signal in the non-rotating frame.

I define:

\[
\begin{align*}
\vec{R}_0 &= \vec{w}_{RB} - \vec{w}_{RA} \\
\vec{v}_B &= \vec{\alpha}_E \times \vec{w}_{RB} + \vec{v}_{RB} \\
\vec{a}_B &= \vec{\alpha}_E \times (\vec{\alpha}_E \times \vec{w}_{RB}) + \vec{\alpha}_E \times \vec{v}_{RB} + \vec{a}_{RB}
\end{align*}
\]  

(II.2.7)

with \( \vec{w}_{RI}, \vec{v}_{RI} \) and \( \vec{a}_{RI} \) being the position, velocity and acceleration of \( i \) in the rotating frame at coordinate time \( t_e \), and the two frames coinciding at \( t_e \). The path travelled by the signal \( \vec{R}(T) \) can be expressed as a series expansion in terms of \( T \) in the non-rotating frame:

\[
\vec{R}(T) = \vec{R}_0 + \vec{v}_B T + \frac{1}{2} \vec{a}_B T^2 + O(T^3),
\]  

(II.2.8)

and its magnitude is given by:
Expanding the square-root and writing $R(T) = cT$ one obtains

$$T = \frac{1}{c} \left[ R_0 + \frac{R_0 \cdot \bar{v}_B}{R_0} T + \frac{\bar{v}_B^2 + R_0 \cdot \bar{a}_B}{2R_0} T^2 + O(T^3) \right].$$

Starting with $T = R_0/c$ and iterating twice yields an expression for the transmission time in terms of the known quantities $R_0, v_B$ and $a_B$:

$$T = \frac{R_0}{c} + \frac{R_0 \cdot \bar{v}_B}{c^2} + \frac{\left( \bar{v}_B^2 + R_0 \cdot \bar{a}_B + \frac{\left( R_0 \cdot \bar{v}_B \right)^2}{R_0^2} \right) R_0}{2c^2} + O(c^{-4}).$$

Substituting from (II.2.7) for $R_0$ and $v_B$ in the second term of (II.2.11) shows that, when the receiving station is stationary in GRS$(v_{RB}=0)$, this term is equivalent to the generally used expression for the Sagnac correction:

$$\frac{R_0 \cdot \bar{v}_B}{c^2} = \frac{-\bar{w}_{RA} \cdot (\bar{a}_B \times \bar{w}_{RB})}{c^2} = \frac{2\omega_E A_E}{c^2},$$

where $A_E$ is the area of equatorial projection of the triangle whose vertices are the centre of the Earth and the positions of the clocks in the rotating frame. $A_E$ is positive for signal propagation in the eastward direction and negative otherwise. This term can amount to $\approx 200$ psTGC for a one-way transfer between a geostationary satellite and a station on the surface of the Earth.

The third term, not found in previous publications, is of the next higher order and can amount to $\approx 10$ psTGC for a geostationary satellite and a station on the surface of the Earth.
The gravitational term of (II.2.4) can be integrated along a straight line in the non-rotating frame. Approximating the total gravitational potential by

\[ U = \frac{GM_E}{w} \]  

(II.2.13)

and integrating the second term of (II.2.4) gives

\[ T_s = \frac{2GM_E}{c^3} \ln \left( \frac{w_B(t_e) + \vec{n} \cdot \vec{w}_B(t_e)}{w_A(t_e) + \vec{n} \cdot \vec{w}_A(t_e)} \right) \]  

(II.2.14)

where \( \vec{n} = \vec{R}_B / R_0 \) is the unit vector along the path of the signal.

This term can amount to about 200 ps\(_{TCG}\) for a one way time transfer in the vicinity of the Earth. Higher order terms due to a more realistic expression for the gravitational potential (including the Earth's quadrupole moment and tidal terms) amount to some 10\(^{-2}\) ps\(_{TCG}\) and can be neglected.

Replacing \( \vec{w}_B(t_e) \) by \( \vec{w}_B(t_e) \) in (II.2.14) induces an error in \( T_s \) of less than 1 ps\(_{TCG}\), hence the total transmission time \( T_t \) can be written as

\[ T_t = \frac{R_0}{c} + \delta \]

\[ = \frac{R_0}{c} + \frac{\vec{R}_B \cdot \vec{v}_B}{c^2} + \frac{\left( \vec{v}_B^2 + \vec{R}_B \cdot \vec{a}_B + \frac{(\vec{R}_0 \cdot \vec{v}_B)^2}{R_0^2} \right) R_0}{2c^3} \]

\[ + \frac{2GM_E}{c^3} \ln \left( \frac{w_{RB} + \vec{n} \cdot \vec{w}_{RB}}{w_{RA} + \vec{n} \cdot \vec{w}_{RA}} \right) + O(c^{-4}). \]  

(II.2.15)

The above expression provides the coordinate transmission time for a light signal travelling from A to B in the vicinity of the Earth (within a geocentric sphere of 200000 km radius) with the coordinates of the two stations given in an Earth fixed rotating frame (GRSR). All terms that are greater than one picosecond are included. Note however, that atmospheric
delays which can amount to several tens of nanoseconds are not considered and need to be taken into account separately (see also section II.2.2.4). Substituting (II.2.15) into (II.2.3) finally provides the desynchronisation of the two clocks.

For time transfer with a geostationary satellite the terms in \( e^3 \) can amount to around 10 ps \( r_{co} \) for the Sagnac correction and 80 ps \( r_{co} \) for the gravitational delay. At present, one way time transfers are not accurate enough to necessitate the consideration of these terms. However, with accuracy expected to increase in the near future, and in view of possible satellite to satellite transfers (which would eliminate uncertainties due to atmospheric delays) these terms might become significant.

**II.2.2.1 Two Way Satellite Time Transfer (TWSTT)**

Consider TWSTT between two stations C and D, fixed on the surface of the Earth, via a geostationary satellite S (as shown in Fig. II.2).

Two signals are transmitted in opposite directions leaving C and D at \( t_0 \) and \( t_0 + \Delta t \) respectively. They reach the satellite at \( t_1 \) and \( t_3 \) where they are immediately retransmitted, and arrive at the opposite stations at \( t_2 \) and \( t_4 \). The measured quantities are the increments of the
readings of the two clocks between emission and reception of the signals, $\Delta h_C$ and $\Delta h_B$. Using the rate of the clocks with respect to coordinate time, $dh/dt$ (see section II.1.1), they can be transformed to two coordinate time intervals $t_C$ and $t_D$ which can be expressed as (see Fig. II.2):

$$
t_C = t_4 - t_0
t_D = t_2 - t_0 - \Delta t.
$$

The desynchronisation of the two clocks is given by

$$
x_{CD}(t_0 + \Delta t) = h_C(t_0) + \int_0^{t_0} dh_C(t)dt - h_D(t_0 + \Delta t)
$$

where the first and third term are obtained directly from the clocks and the evaluation of the integral requires the calculation of $\Delta t$.

It is assumed that the clocks have been synchronised previously to within 0.1 sTCG, a typical station to satellite transmission time (which can be achieved without difficulty in practice), and that the satellite has a residual velocity $v$, smaller than 1 m/sTCG and a residual acceleration in the rotating frame of less than $10^{-5}$ m/s$^2$TCG. These values have been chosen as typical after consultation of the EUTELSAT satellite control centre.

The defining equations for the transmission coordinate times are:

$$
T_1 = t_1 - t_0
T_2 = t_2 - t_1
T_3 = t_3 - t_0 - \Delta t
T_4 = t_4 - t_3
$$

and solving for $\Delta t$ yields

$$
\Delta t = \frac{1}{2}(t_C - t_D) + \delta
$$

$$
\delta = \frac{1}{2}(T_1 + T_2 - T_3 - T_4).
$$
The correction \( \delta \) arises from the motion of the stations and the satellite in the non-rotating frame and the gravitational and atmospheric delays for the individual transmissions \( T_1 \) to \( T_4 \).

At the required accuracy the gravitational and atmospheric delays cancel in the differences \( T_1 - T_4 \) and \( T_2 - T_3 \) (see also section II.2.2.4). Therefore only the geometrical terms need to be considered. For the individual links these are given by (II.2.11) when substituting:

for \( T_1 \):
\[
\begin{align*}
\vec{R}_0 &= \vec{\omega}_{RS} - \vec{\omega}_{RC} \\
\vec{v}_b &= \vec{\omega}_E \times \vec{\omega}_{RS} + \vec{v}_r \\
\vec{a}_b &= \vec{\omega}_E \times (\vec{\omega}_E \times \vec{\omega}_{RS}) + \vec{\omega}_E \times \vec{v}_r + \frac{\text{d}\vec{v}_r}{\text{d}t}
\end{align*}
\]

for \( T_2 \):
\[
\begin{align*}
\vec{R}_0 &= -(\vec{\omega}_{RS} - \vec{\omega}_{RD} + \vec{v}_r T_1) \\
\vec{v}_b &= \vec{\omega}_E \times \vec{\omega}_{RD} \\
\vec{a}_b &= \vec{\omega}_E \times (\vec{\omega}_E \times \vec{\omega}_{RD})
\end{align*}
\]

for \( T_3 \):
\[
\begin{align*}
\vec{R}_0 &= \vec{\omega}_{RS} - \vec{\omega}_{RD} + \vec{v}_r \Delta t \\
\vec{v}_b &= \vec{\omega}_E \times (\vec{\omega}_{RS} + \vec{v}_r \Delta t) + \vec{v}_r \\
\vec{a}_b &= \vec{\omega}_E \times [\vec{\omega}_E \times (\vec{\omega}_{RS} + \vec{v}_r \Delta t)] + \vec{\omega}_E \times \vec{v}_r + \frac{\text{d}\vec{v}_r}{\text{d}t}
\end{align*}
\]

for \( T_4 \):
\[
\begin{align*}
\vec{R}_0 &= -(\vec{\omega}_{RS} - \vec{\omega}_{RC} + \vec{v}_r (\Delta t + T_3) \\
\vec{v}_b &= \vec{\omega}_E \times \vec{\omega}_{RC} \\
\vec{a}_b &= \vec{\omega}_E \times (\vec{\omega}_E \times \vec{\omega}_{RC})
\end{align*}
\]

All positions are given at \( t=t_0 \) when the two frames coincide.
The first term is equivalent to $2\omega_{E}A_{E}/c^{2}$ with $A_{E}$ now being the equatorial projection of the area of the quadrangle whose vertices are the centre of the Earth and the positions of the satellite and the stations in the rotating frame.

Note that there are no terms of order $c^{-1}$ and $c^{-3}$ corresponding to the first and the third term in (II.2.11) as these terms cancel in the differences $T_1 - T_4$ and $T_2 - T_3$. This is also the reason why atmospheric delays and station and satellite coordinates are less important than in the one way case.

The second term of (II.2.21) varies with $v_{t}$ and $\Delta t$, and can amount to several hundred picoseconds. If $\Delta t \approx 0$ it can amount to several tens of picoseconds, depending on the residual velocity which is in general not well known. However, one can compensate for it by intentionally introducing a desynchronisation in order to drive this term towards zero, which is the case when the two signals arrive at S at about the same time (ie. $t_1 \approx t_3$).
II.2.2.3 Laser Synchronisation from Stationary Orbit (LASSO)

In this method laser pulses emitted from the stations C and D at \( t_0 \) and \( t_0 + \Delta t \) respectively are reflected by the geostationary satellite and return to the stations (as shown in fig. II.3).

The satellite is equipped with a clock which measures the time elapsed between arrival of the signals. Hence three coordinate time intervals (after transformation of the measured time intervals using the rate of the clocks with respect to coordinate time, \( \frac{dh}{dt} \), see section II.1.1) are obtained which can be expressed as (see Fig. II.3):

\[
t_c = t_2 - t_0 \\
t_d = t_4 - t_0 - \Delta t \\
t_s = t_3 - t_1
\]  

(II.2.22)

The desynchronisation of the two clocks C and D is given by (II.2.17) again requiring the determination of \( \Delta t \). Similarly to TWSTT, the defining equations (II.2.18) for \( T_1 \) to \( T_4 \) yield:

\[
\Delta t = \frac{1}{2}(t_c - t_d) + t_s + \delta \\
\delta = \frac{1}{2}(T_1 - T_2 - T_3 + T_4).
\]
Again the gravitational and atmospheric delays cancel to the required accuracy in the differences $T_1 - T_2$ and $T_4 - T_3$.

In order to calculate the individual transmission coordinate times $T_1$ to $T_4$ substitute into (II.2.11),

for $T_1$:

\[
\begin{align*}
\vec{R}_0 &= \vec{w}_{RS} - \vec{w}_{RC} \\
\vec{v}_B &= \vec{a}_E \times \vec{w}_{RS} + \vec{v}_r \\
\vec{a}_B &= \vec{a}_E \times (\vec{a}_E \times \vec{w}_{RS}) + \vec{a}_E \times \vec{v}_r + \frac{d\vec{v}_r}{dt}
\end{align*}
\]  

(II.2.24a)

for $T_2$:

\[
\begin{align*}
\vec{R}_0 &= -(\vec{w}_{RS} - \vec{w}_{RC} + \vec{v}_r T_1) \\
\vec{v}_B &= \vec{a}_E \times \vec{w}_{RC} \\
\vec{a}_B &= \vec{a}_E \times (\vec{a}_E \times \vec{w}_{RC})
\end{align*}
\]  

(II.2.24b)

for $T_3$:

\[
\begin{align*}
\vec{R}_0 &= \vec{w}_{RS} - \vec{w}_{RD} + \vec{v}_r \Delta t \\
\vec{v}_B &= \vec{a}_E \times (\vec{w}_{RS} + \vec{v}_r \Delta t) + \vec{v}_r \\
\vec{a}_B &= \vec{a}_E \times \left[ (\vec{a}_E \times (\vec{a}_E \times \vec{w}_{RS} + \vec{v}_r \Delta t)) \right] + \vec{a}_E \times \vec{v}_r + \frac{d\vec{v}_r}{dt}
\end{align*}
\]  

(II.2.24c)

for $T_4$:

\[
\begin{align*}
\vec{R}_0 &= -(\vec{w}_{RS} - \vec{w}_{RD} + \vec{v}_r (\Delta t + T_1)) \\
\vec{v}_B &= \vec{a}_E \times \vec{w}_{RD} \\
\vec{a}_B &= \vec{a}_E \times (\vec{a}_E \times \vec{w}_{RD})
\end{align*}
\]  

(II.2.24d)

All positions are given at $t=t_0$ when the two frames coincide.

As for TWSTT $T_1$ and $T_3$ can be replaced by their first order approximations in (24b) and (24d) inducing an error of some $10^3 \text{ ps}_{\text{TCG}}$.

Substituting the expressions obtained for $T_1$ to $T_4$ into (II.2.23) and neglecting terms smaller than $10^2 \text{ ps}_{\text{TCG}}$ (see section II.2.2.4) yields the total correction $\delta$ for LASSO time transfers:
\[
\begin{align*}
\delta &= \frac{R_{\text{CD}} \cdot (\vec{\omega}_e \times \vec{w}_{RS})}{c^2} + \frac{\Delta t (\vec{\omega}_e \times \vec{v}_e) \cdot \vec{w}_{RD}}{c^2} + \mathcal{O}\left[\left(\frac{v}{c}\right)\left(\frac{v}{c}\right)\left(\frac{R_0}{c}\right)\right] \tag{II.2.25}
\end{align*}
\]

Note that (II.2.25) is written with the position of the satellite in the rotating frame at \(t_0\).

When considering the position at \(t_0 + \Delta t/2\) it can be rewritten in a more symmetric form:

\[
\begin{align*}
\delta &= \frac{R_{\text{CD}} \cdot (\vec{\omega}_e \times \vec{w}_{RS})}{c^2} + \frac{\Delta t (\vec{\omega}_e \times \vec{v}_e) \cdot \frac{1}{2} (\vec{w}_{RC} + \vec{w}_{RD})}{c^2} + \mathcal{O}\left[\left(\frac{v}{c}\right)\left(\frac{v}{c}\right)\left(\frac{R_0}{c}\right)\right] \tag{II.2.26}
\end{align*}
\]

As in (II.2.21) the first term is equivalent to \(2\omega_e A_e/c^2\).

There are again no terms in \(c^1\) and \(c^3\) corresponding to the first and the third term in (II.2.11). They cancel when the differences \(T_1 - T_2\) and \(T_4 - T_3\) are formed.

The second term varies with \(v_e\) and \(\Delta t\). This term is smaller than \(10^{-2}\) \(\text{ps}_{\text{TCC}}\) for \(v_e \approx 1\) \(\text{m/s}_{\text{TCC}}\) and \(\Delta t \approx 0.1\) \(\text{s}_{\text{TCC}}\), which is the case for TWSTT and hence it does not appear in (II.2.21). However, for LASSO \(\Delta t\) can amount to several minutes in practice (Veillet et al. 1992, Veillet & Fridelance 1993) and therefore the second term in (II.2.25) and (II.2.26) can contribute up to \(10\) \(\text{ps}_{\text{TCC}}\).

Note also that while the second term of (II.2.21) can be minimised by an appropriate choice of \(\Delta t\), this is not the case in (II.2.25) and (II.2.26).

The fact that the second terms in (II.2.25) and (II.2.26) are of higher order than the second term of (II.2.21) reflects the only indirect dependence of the correction \(\delta\) for LASSO on the velocity of the satellite. For TWSTT cancellation takes place when forming the differences \(T_1 - T_4\) and \(T_2 - T_3\). The magnitudes of these differences are dependent on the
movement of the satellite between \( t_1 \) and \( t_3 \). This is not the case for LASSO, where the differences \( T_1 - T_2 \) and \( T_4 - T_3 \) are independent of the motion of the satellite.

Time transfer techniques such as LASSO or TWSTT provide higher precision than one way techniques. Recently a two way time transfer between PTB (Braunschweig, Germany) and FTZ (Darmstadt, Germany) with a precision of less than 300 ps, and a LASSO time transfer between McDonald(USA) and OCA(France) at a level of precision better than 100 ps were carried out (Hetzel & Soring 1993, Veillet et al. 1992, Veillet & Fridelance 1993).

The main errors in computing the corrections \( \delta \) for these techniques (using (II.2.21) and (II.2.25)) are due to the uncertainties in the position and the residual velocity of the satellite. The uncertainty in the position leads to an error in the computation of \( 2\omega_E A_E / c^2 \) of the order of 10 ps\( _{TCG} \) for both techniques (see section II.2.2.4). The uncertainty in the residual velocity affects the two techniques differently. For LASSO the second term in (II.2.25) is typically of the order of 10 ps\( _{TCG} \), hence reducing the overall uncertainty for LASSO requires better knowledge of the satellite position as well as consideration of the additional term. For TWSTT, on the other hand, the second term in (II.2.21) can reach 80 ps\( _{TCG} \) (for \( \Delta t = 0 \)). Hence reducing this term by an appropriate choice of \( \Delta t \) will improve the overall accuracy of the time transfer even in the case where \( v_r \) is unknown.

In both techniques, the precision of experiments repeated over periods of several weeks could be affected by the variation of the residual velocity of the satellite, if the corresponding terms are not accounted for.

**II.2.2.4 Limits of Applicability in Practice**

*Neglected Terms*

Higher order terms in the metric (I.3.4) are of order \( U^2 / c^4 \) and \( G\omega_E I_E / \omega^2 c^3 \) respectively (where \( I_E \) is the angular momentum of the Earth). The gravitational time delay due to these terms is given by the integral along the path of the signal of terms of order \( U^2 / c^5 \) and \( G\omega_E I_E / \omega^2 c^4 \), the result of which is much less than one picosecond for the situations considered.
The next terms in equation (II.2.21) are of order \((v/c)(\nu/v)\Delta t\) and \((v/c)(\nu/v)T\) and contribute a correction of less than \(10^{-2}\ \text{ps}_{\text{TGG}}\) for a transfer with \(\Delta t \approx 0.1\ \text{s}_{\text{TGG}}\), via a geostationary satellite with a residual velocity of \(1\ \text{m/s}_{\text{TGG}}\).

In (II.2.25) missing terms are of order \((v/c)(\nu/v)(R_v/c)\) and smaller, and contribute less than \(10^{-2}\ \text{ps}_{\text{TGG}}\) for a residual velocity of \(1\ \text{m/s}_{\text{TGG}}\).

**Computation of the Corrections**

For picosecond accuracy, the correction \(\delta\) contains terms in \(c^2\) and in \(c^3\) in the case of one-way time transfers (II.2.15), and terms in \(c^2\) only in the case of TWSTT (II.2.21) and LASSO (II.2.25) transfers.

The term in \(c^2\) can amount to a few hundred nanoseconds, depending on the relative positions of the transmission and reception points. For example, between the Earth and a geostationary orbit, the maximum value is about \(200\ \text{ns}_{\text{TGG}}\) for the one way- and \(400\ \text{ns}_{\text{TGG}}\) for the two way case. In order to compute this term with picosecond accuracy, it is sufficient for all quantities in the term in \(c^2\) to be known with a relative uncertainty of one or two parts in \(10^6\). This requires coordinates known to within 6-12 m for the Earth stations, including uncertainties in the realisation of the reference frame which are below \(\approx 1\ \text{m}\) for e.g. WGS84 and ITRF. This is generally the case for time laboratories. The satellite position should be known to within some tens of metres, depending on its orbit, and this is generally not the case a priori for a satellite without geodesic objectives. In addition the velocity of the satellite should be known to the same relative uncertainty of one or two parts in \(10^6\), which is also not the case in general. Typically the position of a geostationary satellite is known to an accuracy of \(\approx 1\ \text{km}\) which results in an error in the computation of the \(c^2\) term of \(\approx 10\ \text{ps}_{\text{TGG}}\). Similar arguments can be made to set constraints in the case of higher orbits or satellite to satellite time transfers.

If we assume a perfect geostationary orbit, the term in \(c^2\) can be computed with picosecond accuracy using the formula \(2\omega_E A_E/c^2\). Indeed, to obtain the required accuracy, the vector \(\vec{\omega}_E\) can be taken colinear to the Z-axis and the true pole coordinates ignored. The
effect of this approximation can only marginally reach 1 ps_TCG for TWSTT via a geostationary satellite in very special cases. This can occur when the two stations have the same longitude and are close to the poles, with the value of the longitude depending on the position of the pole. The IAU recommended value of the mean angular velocity of the Earth \( \omega_E \) (7.292115 \times 10^{-5} \text{ rad/s}_TCG) is to be used, and the constraints on the positions deduced in the previous paragraph apply for the computation of \( A_E \).

In the real case of a non-perfect geostationary orbit, the constraint on the knowledge of the velocity of the satellite is transferred to the residual velocity \( v_x \). For the one way and TWSTT techniques, this constraint is about 1 cm/s_TCG for picosecond accuracy but in the two way technique it can be completely relaxed by an intentional desynchronisation of the emission of the signals at the two stations, as mentioned in section II.2.2.2. For LASSO, the constraint on \( v_x \) is about 10 cm/s_TCG if one wishes to use laser pulses from the two stations separated by \( \Delta t \) of several minutes. The constraint on \( v_x \) can be relaxed by severing that on \( \Delta t \).

With the term in \( c^3 \) for one way transfers amounting to some tens of picoseconds, the constraints to obtain picosecond accuracy are of a few percent on positions, velocities and accelerations, and do not pose any practical problem.

*Propagation through the Atmosphere*

When one of the stations is on the Earth, propagation through the atmosphere is one of the major problems for one way time transfer. It leads to delays that can reach several tens of nanoseconds and can certainly not be calibrated to picosecond accuracy. This problem is not considered in this study. However the effects cancel to the picosecond level in the TWSTT (provided the up and down frequencies are close enough) and LASSO techniques.

For Earth to satellite time transfer the deviation from a straight line of the trajectory of a signal in an inertial frame, due to atmospheric refraction, does not exceed \( 10^{-4} \text{ rad} \) and the resulting terms due to the additional path and area are less than one picosecond for satellite elevations greater than 10 degrees. However they can become significant for satellite to satellite transfer when the signal traverses the atmosphere, but in this case the limiting factor
would be the uncertainty in the propagation delay itself. And, of course, traversing the atmosphere for satellite to satellite transfer can be avoided in practice.

*Gravitational Bending*

Deviation from a straight line of the trajectory due to the gravitational field of the Earth is of the order of some $10^{-9}$ rad, hence its effect on the transmission time of a light signal in the vicinity of the Earth is below one picosecond for all possible paths.

### II.2.2.5 Numerical Application for TWSTT

Equation (II.2.21) has been applied to two way time transfers within Europe and across intercontinental distances. The results have been compared with those obtained using

$$
\delta = \frac{2\omega_n A_1}{c^2}
$$

and from a more exact numerical method.

The numerical method is based on the derivation by Klioner (1992): in this $T_2$, $T_3$ and $T_4$ are calculated as functions of $t_c$, $t_D$, $t_S$ and $\Delta t$, based on the velocity and acceleration of the satellite and the stations in the inertial frame, and obtained by iterating the expressions. $T_1$ is calculated using the same iterative procedure as in section II.2.2.1.

All iterations are continued until the difference between two consecutive terms is less than $10^{-16}$ s$_{TCG}$. It is in this sense that the numerical method can be considered more exact than the analytic expression (II.2.21).

The satellite was assumed to have a residual velocity of 1 m/s$_{TCG}$.

The difference between (II.2.21) and the numerical method is typically of the same order as the missing terms mentioned in section II.2.2.4 (some $10^{-3}$ ps$_{TCG}$). Hence the
A numerical application represents a validation of (II.2.21) for particular cases at the level of $10^2$ ps$_{TCG}$.

The difference between (II.2.27) and the numerical results, corresponding to the second term in (II.2.21), depends on the initial desynchronisation of the stations $\Delta t$, the residual velocity $\nu$, and the geometry of the particular case. Some of the results are summarised in table II.4.

<table>
<thead>
<tr>
<th>Transfer</th>
<th>num - (II.2.21)</th>
<th>num. - (II.2.27) (ps$_{TCG}$)</th>
<th>$\Delta t$ (ideal) (ms$_{TCG}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>($10^{-3}$ ps$_{TCG}$)</td>
<td>---------------------------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>Toulouse (France) -</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Paris (France)</td>
<td>0.02</td>
<td>-5</td>
<td>-172</td>
</tr>
<tr>
<td>via INMARSAT2</td>
<td></td>
<td>$\Delta t = 0$</td>
<td>$\Delta t = 50$ ms$_{TCG}$</td>
</tr>
<tr>
<td>FTZ (Germany)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PTB (Germany)</td>
<td>-0.03</td>
<td>-2</td>
<td>-169</td>
</tr>
<tr>
<td>Kouru (F. Guyana)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HBK (S-Africa)</td>
<td>-1.6</td>
<td>-11</td>
<td>-177</td>
</tr>
<tr>
<td>Graz (Austria)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>USNO (USA)</td>
<td>1.3</td>
<td>35</td>
<td>-131</td>
</tr>
</tbody>
</table>

Table II.4: Differences between the results of the numerical calculation of the relativistic correction for two way time transfers and applications of the analytical formulae (II.2.27) and (II.2.21). The differences between the numerical results and (II.2.27) are given for different values of $\Delta t$. The last column gives the values of $\Delta t$ for which the second term in (II.2.21) vanishes, which is the case when the two signals arrive at the satellite at the same time. In this case the differences between the numerical results and (II.2.27) are of the order of some $10^{-3}$ picoseconds.
II.3 Realisation of Coordinate Time Scales

Coordinate time scales were defined within the theoretical framework of general relativity by the IAU in its 1991 general assembly in Buenos Aires (IAU 1991, see also section I.3). It was clearly stated that all coordinate time scales be derived from atomic clocks operating on the Earth.

In order to ensure reliability and to enhance stability and accuracy it is preferable to use an ensemble of clocks rather than individual ones. Hence the atomic time from which coordinate time scales are derived should be based on such an ensemble. The internationally used such ensemble time is the so called Echelle Atomique Libre (EAL) produced by the Bureau International des Poids et Mesures (BIPM in collaboration with a number of timing laboratories spread around the world. In 1995 46 timing laboratories participated, providing data from about 190 individual atomic clocks most of which were caesium clocks with a few (about 20) hydrogen masers. International Atomic Time (TAI) is a realised coordinate time scale derived from EAL.

In the following sections the construction of EAL is briefly explained, together with its application to obtain realisations of the coordinate time scales TCG, TT and TCB. These three coordinate time scales are related by transformations which are described in sections (II.3.3) and (II.3.4) together with their limitations mainly due to uncertainties in our knowledge of geophysical and astronomical constants.

II.3.1 The Free Atomic Time Scale (EAL)

The clocks participating in EAL are compared to each other using either a local interval counter (when they are in the same laboratory) or GPS for distant clocks, providing a set of $N-1$ non-redundant time comparisons (desynchronisations) $x_i(TCG)$, where $N$ is the total number of participating clocks. All $x_i(TCG)$ are provided for the same date $TCG$ once every 10 days (in 1995) and once every 5 days (since Jan. 1996). The periodicity of the data is chosen in view
of the measurement noise of the GPS comparisons which at present can be smoothed out to a
level below the intrinsic noise of the clocks even for intercontinental links by averaging over 2-
3 days of data (see e.g. Thomas, Wolf & Tavella 1994).

The values of the $x_i(TCG)$ are used to calculate the ensemble time $EAL(TCG)$ with a
periodicity of $T$ (currently $T$ is 60 days with plans to shorten it to 30 days). After every
calculation EAL can be accessed from any one of the participating clocks by the $N$ time
differences

$$x_i(TCG) = EAL(TCG) - h_i(TCG), i = 1, ..., N$$  \hspace{1cm} (II.3.1)

the $x_i(TCG)$ being the results of the calculation for all comparison dates $TCG$ during the
previous period $T$. So EAL is obtained in deferred time with the update period $T$ chosen as a
function of user requirements and the characteristics of the participating clocks. EAL is
designed for maximum long term stability with real-time or near real-time access not being a
major requirement. Hence the update interval is chosen close to the averaging time for which
the participating clocks are most stable. For most clocks participating in EAL this averaging
time is of the order of 40 days.

The solutions $x_i(TCG)$ are obtained using a time scale algorithm called ALGOS(BIPM)
(Guinot & Thomas 1988, Tavella & Thomas 1991a) which relies upon two basic assumptions:

* Measurement results $x_i(TCG)$ are affected by measurement noise which is negligible
  with respect to the clock noise.

* Clocks are independent and the corresponding data series are uncorrelated. This
  assumption was tested through a survey on the behaviour of the clocks contributing to EAL
  (Tavella & Thomas 1990, 1991b) which detected some correlations corresponding mainly to
  responses to changes in the environmental conditions experienced by clocks. Since several
  years efforts have been pursued to improve clock independence either through better control of
  the environment or through the realisation of less sensitive atomic clocks (De Marchi 1988).

Assume EAL is calculated for coordinate time $t$ (replacing "$TCG$" by "$t$"). The
ensemble time is defined as a weighted average of the readings of the contributing clocks:
Relative weights $\omega_i(t)$, $i = 1, ..., N$, are introduced in order to discriminate between clocks according to their intrinsic qualities. They satisfy the relation

$$\sum_{i=1}^{N} \omega_i(t) = 1.$$  \hspace{1cm} (II.3.3)

Since time scale algorithms are designed to optimise frequency stability, each clock should be weighted according to its own frequency stability. For EAL the weight attributed to a given clock is chosen to be inversely proportional to the classical variance, $\sigma_i^2$, of its 60 day frequency with respect to EAL, determined over the last six update intervals $T$:

$$\omega_i = \frac{1/\sigma_i^2}{\sum_{k=1}^{N} 1/\sigma_k^2}, \quad i = 1, ..., N.$$  \hspace{1cm} (II.3.4)

If the contributing clocks are independent and if weights are not artificially limited, the frequency variance of the resulting time scale may be written as

$$\frac{1}{\sigma_{EAL}^2} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2},$$  \hspace{1cm} (II.3.5)

which means that the time scale is, in principle, more stable than any contributing element. The frequencies of the individual clocks are determined with respect to EAL because its stability is supposed to be better than that of the contributing clocks. It follows that the computed variance is inherently biased (Yoshimura 1980) and ceases to represent the true quality of the clock. This is the so-called "clock-ensemble correlation" effect. An approach to the correction of this effect has been published (Tavella et al. 1991), but for EAL the contribution of individual clocks is small (about 1%) and this correction can be neglected.

To ensure reliability an upper limit of weight is applied in the calculation of EAL, necessary to make the time scale rely on the best clocks and yet avoid giving a predominant
role to any one of them. Additionally clocks are attributed zero weight in the case of abnormal behaviour (for a detailed discussion of the weighting procedure see Thomas & Azoubib 1996). It is important to stress that the existence of an upper limit of weight safeguards reliability but invalidates (II.3.5). It can thus lead to a time scale that is no better than the best single contributing clock.

As in general clocks enter and leave the ensemble or change their characteristics between consecutive updates, the clock weights can vary significantly, leading to time and frequency discontinuities in (II.3.2) which is therefore completed

\[ EAL(t) = \sum_{i=1}^{N} \omega_i(t) [h_i(t) + h'_i(t)], \quad \text{(II.3.6)} \]

where \( h'_i(t) \) is a time correction designed to ensure time and frequency continuity of EAL at the border date, \( t_0 \), of two update intervals \( T(t-t_0 = 5n \text{ days}, n=0,\ldots,12) \) written as (Guinot and Thomas 1988):

\[ h'_i(t) = x_i(t_0) + B_{ip}(t - t_0), \quad \text{(II.3.7)} \]

where \( x_i(t_0) = EAL(t_0) - h_i(t_0) \) is known, since it results from the computation of EAL at date \( t_0 \), and where \( B_{ip}(t) \) is the predicted frequency of clock \( i \), relative to EAL, over the interval \([t_0, t]\). For EAL the predicted frequency \( B_{ip} \) over an interval \( T \) is chosen to be the frequency determined over the 60 day interval preceding the last update \( t_0 \).

Equations (I.2.4), (II.3.1), (II.3.6) and (II.3.7) lead to a system of \( N \) equations

\[ \sum_{i=1}^{N} \omega_i(t)x_i(t) = \sum_{i=1}^{N} \omega_i(t)x_i(t_0) + \sum_{i=1}^{N} \omega_i(t)B_{ip}(t)(t-t_0), \]

\[ x_j(t) - x_i(t) = x_{i,j}(t), \quad \text{(II.3.8)} \]

The solution is unique and the results are the time differences \( x_i(t), i = 1, \ldots, N \), which give access to EAL for date \( t \). The difference between clock \( j \) and EAL is explicitly given by:
\[ x_j(t) = \sum_{i=1}^{N} \omega_i(t) [h'_i(t) + x'_i(t)]. \]  

(II.3.9)

So (II.3.9) provides access to EAL from the readings of any clock of the ensemble for all dates \( t \) for which values of \( x'_j(t) \) have been measured. Final values of the \( x_j(t) \) are calculated in deferred time, every two months for the 60 day period preceding the calculation.

II.3.2 Realisation of TCG

Geocentric Coordinate Time (TCG) can be realised from EAL by frequency steering. First the relative rate of EAL with respect to TCG, \( \frac{d\text{EAL}}{dT\text{CG}} \) has to be determined. Then intentional frequency steps are applied to EAL, the resulting time scale being designated as TCG(EAL), in order to drive \( \frac{dT\text{CG(EAL)}}{dT\text{CG}} \) towards 1 within the uncertainties of the determination of \( \frac{d\text{EAL}}{dT\text{CG}} \). The magnitude of the frequency steps is kept smaller than the instability of EAL in order to avoid degradation of the stability of the resulting time scale.

The relative rate of EAL with respect to a few chosen primary frequency standards is determined routinely on a 60 day basis if the standards operate continuously and participate in the generation of EAL (which is the case, for example, for PTB CS1, PTB CS2, PTB CS3), and obtained punctually for standards that operate discontinuously by comparing them, when in operation, to some other, highly stable but less accurate, atomic clock that participates in EAL (the case for NIST-7 and LPTF-FO1). The uncertainty of the measurement of the rate of EAL with respect to the standard is the quadratic sum of the uncertainty of the standard, its instability and the instability of EAL over the period of measurement, and the instability of the clock used to link the two (in the case of NIST-7 and LPTF-FO1). In all cases except LPTF-FO1 the uncertainty of the standard is by far dominant.

Each measurement of the rate of EAL with respect to a standard provides an estimation of \( \frac{d\text{EAL}}{d\tau_A}(TCG) \) where \( A \) refers to the standard in question. Additionally \( \frac{d\tau_A}{dT\text{CG}}(TCG) \) can be
determined for each standard as detailed in section II.1.1. Combining the two one obtains a value of $\frac{dEAL}{dTCG}(TCG)$ for each measurement, often referred to as a calibration of EAL.

Finally the rate of EAL with respect to TCG can be estimated using a number of calibrations from several primary standards and transferring them to the same date $TCG_E$. Consequently the instability of EAL over the transfer time interval (the time elapsed between the calibration and $TCG_E$) has to be added to the uncertainty of the individual calibrations. The rate $\frac{dEAL}{dTCG}(TCG_E)$ can then be obtained by combining the individual calibrations (Azoubib et al. 1977, Thomas 1996) taking into account their individual uncertainties after transfer and correlations between subsequent calibrations of standards that operate continuously.

At present TCG is not realised by the BIPM, instead EAL is used for realisations of Terrestrial Time (TT). However, TCG can be obtained from TT (and vice versa) by a transformation, given in the next section.

II.3.3 TCG - TT Transformation

The IAU defined TT as a geocentric coordinate time scale differing from TCG by a constant rate, the scale unit of TT being chosen so that it agrees with the SI second on the geoid (IAU 1991, cf. section I.3.3.1). So TT is related to TCG by the transformation

$$\frac{dT_T}{dT_C} = [1 - L_s],$$  \hspace{1cm} (II.3.10)

where $L_s = W_0/c^2 = 6.9692903 \times 10^{-10} \pm 1 \times 10^{-17}$, $W_0$ being the potential on the rotating geoid.

Integration of (II.3.10) yields an explicit expression, with the integration constant equal to zero, in accordance with the definition of the TCG and TT origins (IAU 1991, cf. section I.3).
At present the above transformation induces an uncertainty of $1 \times 10^{-17} \, s_{TT}/s_{TCG}$ due to the uncertainty in the determination of $W_0$. It follows that at present the accuracy of synchronisation with respect to TT is limited to $1 \times 10^{-17} \, s_{TT}/s_{TT}$ by uncertainties in the determination of the potential on the geoid $W_0$, even for clocks onboard terrestrial satellites.

This limit is inherent in the definition of TT and can therefore only be improved by reducing the uncertainty of $W_0$. If highly stable clocks on terrestrial satellites are to be used for the realisation of TT at accuracies exceeding this limit it may prove necessary to change this definition. One possibility would be to turn $L_s$ into a defining constant with a fixed value. This would also provide a relativistic definition of the geoid (Bjerhammar 1985, Soffel et al. 1988).

**II.3.3.1 Realisation of TT**

Realisations of TT are based on EAL and obtained using the same principles as for TCG (see section II.3.2) with the difference that calibrations by the individual standards are now used to determine the rate of EAL with respect to TT rather than TCG

$$\left( \frac{dEAL}{dT} = \frac{dEAL}{dT} \frac{dT}{dTT} \right)$$

making use of the TCG - TT transformation (see previous section).

At present two realisations exist, International Atomic Time (TAI) and TT(BIPMxx) where xx denotes the year of the last update. Apart from the $32.184 \, s_{TT}$ offset (see section I.3.3) the two differ essentially by the way the steering corrections are applied to EAL.

Steering corrections for TAI are applied at the end of 60 day intervals when deemed necessary, without any further post-processing. As a result some of the corrections are recognised as unnecessary or counterproductive in retrospect, but are not changed as changes of TAI "after the fact" are unacceptable for the users.

On the other hand, each update of TT(BIPMxx) redetermines the time scale for all previous dates (Guinot 1988). As a consequence the same event may receive different time coordinates in different updates of TT(BIPMxx).
By its definition UTC differs from TAI by an integer number of seconds (see section I.3.3.3), therefore UTC can be considered a realisation of TT apart from an offset equal to \((n+32,184)\) \(s_{TT}\) where \(n\) is an integer.

II.3.4 TCG - TCB Transformation

TCB is related to TCG by the relativistic transformation (Kopejkin 1988, Brumberg 1991a, IAU 1991)

\[
\frac{dT_{TCG}}{dT_{TCB}} = [1 - \frac{1}{c^2}U(TCB, x_e) + \frac{1}{2}v_e^2(TCB) + v_e(TCB) \frac{d\phi}{dT_{TCB}}(TCB) + a_e^k(TCB)\phi_k(TCB)] + O(c^{-4})
\] (II.3.11)

where \(U(TCB, x_e)\) is the Newtonian gravitational potential at the Earth's centre of mass of all celestial bodies except the Earth, \(v_e^k\) and \(a_e^k\) are the barycentric velocity and acceleration of the Earth's centre of mass, and \(\phi_k = x_o^k - x_e^k\) with \(x_o^k\) being the barycentric position of the observer (of the clock). Note that all terms in (II.3.11) are functions of TCB and need to be obtained from solar system ephemerides for the time at which the transformation takes place.

Terms of order \(c^{-4}\) are \(\leq 10^{-16} \) \(s_{TT}/s_{TCB}\) in the vicinity of the Earth. They can be calculated (Kopejkin 1988) but require the specification of coordinate conditions for the reference systems used.

Orders of magnitude and present day uncertainties of the individual terms in (II.3.11) are listed in table II.5.

The transformation (II.3.11) (and consequently the realisation of TCB) is limited at the level of a few \(10^{-18} \) \(s_{TT}/s_{TCB}\) by uncertainties in the determination of the solar potential in the vicinity of the Earth and of the barycentric velocity of the Earth. These are mainly due to uncertainties in our knowledge of the heliocentric constant for gravitation \(GM_{Sun}\) and of the
Earth-Sun distance $r_{\text{ES}}$ which are of the order $4\times10^{10}$ m$^3$/s$^2$ and 50 m respectively (Seidelmann & Fukushima 1992).

<table>
<thead>
<tr>
<th>Effect</th>
<th>Order of magnitude</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solar and planetary pot.</td>
<td>$1\times10^{-8}$</td>
<td>$4\times10^{-18}$</td>
</tr>
<tr>
<td>2nd order Doppler ($vE^2/2c^2$)</td>
<td>$5\times10^{-9}$</td>
<td>$2\times10^{-11}$</td>
</tr>
<tr>
<td>4th term</td>
<td>$\leq 2\times10^{-9}$</td>
<td>$&lt;10^{-18}$</td>
</tr>
<tr>
<td>5th term</td>
<td>$\leq 2\times10^{-11}$</td>
<td>$&lt;10^{-18}$</td>
</tr>
<tr>
<td>Higher order ($c^4$)</td>
<td>$\leq 10^{-16}$</td>
<td>$&lt;10^{-18}$</td>
</tr>
</tbody>
</table>

Table II.5: Orders of magnitude and present day uncertainties of the terms participating in the TCG-TCB transformation (II.3.11) for events in the vicinity of the Earth.
II.4 Conclusion

In section II.1.1 a theory for the syntonisation of clocks with respect to Geocentric Coordinate Time (TCG) is presented, including all terms greater than $10^{-18}$ s/s_{TCG} for clocks onboard satellites at altitudes exceeding 10000 km. For clocks on the Earth's surface, syntonisation with respect to TCG is limited to $(2-3)10^{-17}$ s/s_{TCG} by uncertainties in the determination of the geopotential at the location of the clock. Syntonisation with respect to Terrestrial Time (TT), an ideal form of TAI, is limited to $10^{-17}$ s/s_{TT} (even for clocks onboard satellites) by uncertainties in the determination of the potential on the geoid, $W_0$, inherent to its definition (section II.3.3). For the comparison of the stability of two distant clocks one is only interested in time varying effects. These can be determined at $10^{-18}$ accuracy even for clocks on the Earth's surface (section II.1.2).

In section II.2.2.1 the relativistic correction for a one way time transfer between two stations that have their position given in a geocentric reference frame rotating with the Earth is explicitly derived (equation (II.2.15)) including all terms in $c^3$ and larger. For time transfer with a geostationary satellite the terms in $c^3$ can amount to around $10$ ps_{TCG} for the Sagnac correction (a term not found in previous publications) and $80$ ps_{TCG} for the gravitational delay. At present, one way time transfers are not accurate enough to necessitate the consideration of these terms. However, with accuracy expected to increase in the near future, and in view of possible satellite to satellite transfers (which would eliminate uncertainties due to atmospheric delays) these terms might well become significant. In sections II.2.2.2, II.2.2.3 the relativistic corrections that need to be applied to two way techniques like TWSTT and LASSO are derived, including all terms greater than one picosecond. It is shown that the main errors in computing these corrections are due to the uncertainties in the position and the residual velocity of the geostationary satellite (section II.2.2.4). The uncertainty in the position leads to an error in the computation of $2\omega A\Delta s/c^2$ of the order of $10$ ps_{TCG} for both techniques. The uncertainty in the residual velocity of the satellite affects the two techniques differently. For LASSO the second term in (II.2.25) is typically of the order of $10$ ps_{TCG}, hence reducing the overall uncertainty for LASSO requires better knowledge of the satellite position as well as consideration of this term. For TWSTT, on the other hand, the second term in (II.2.21) can reach $80$ ps_{TCG} (for simultaneous emission of the signals, i.e. $\Delta t = 0$). Hence reducing this term
by an appropriate choice of $\Delta t$ will improve the overall accuracy of the two way time transfer even in the case where the residual velocity is unknown.

These results amount to a complete relativistic theory for the synchronisation and syntonisation of clocks and the realisation of geocentric coordinate time scales with uncertainties of $10^{-18}$ and one picosecond respectively, which should be sufficient to accommodate future developments in time transfer and clock technology.
"But what men consider reasonable or unreasonable alters. At certain periods men find reasonable what at other periods they found unreasonable. And vice versa."

Ludwig Wittgenstein (1969) §336/337

At present, of all the SI base units, the second can be realised with the smallest relative uncertainty ($\approx 3 \times 10^{-15}$) and therefore time measurements provide one of the most accurate ways of probing the fundamental laws and theories of nature. Furthermore time metrology has traditionally been the domain of astronomers (before the advent of atomic clocks) and still has a global character (long distance clock comparisons, pulsar timing etc...). Consequently the methods of time metrology are well suited for experimental tests of relativistic theories as the predicted effects on space-time are more appreciable over large spatial domains and can often be measured using clocks and the transmission of electro-magnetic signals (the technology used in time metrology). Of the many tests of general relativity based on state of the art time metrology three of the more important examples are briefly introduced below: the Gravity Probe A (GP-A) experiment (Vessot & Levine 1979, Vessot et al. 1980), radar ranging to the Viking spacecraft (Reasenberg et al. 1979), and the timing of binary pulsars (Taylor 1992, Damour 1992). For a more comprehensive review of these and other tests see Will (1993).

When carrying out an experimental test of a physical theory which turns out to confirm the theoretical prediction, it is often desirable to quantify this result (e.g. in form of an uncertainty) in order to be able to compare and evaluate different experiments that test the same theory. Simply stating the uncertainty of the raw measurement is in general unsatisfactory as this does not allow the intercomparison of experiments that use different methods and measurements and yet test the same theory. So most experiments are evaluated with respect to
an alternative or test theory that differs from the theory which is tested in a number of observable effects. All experiments that are sensitive to these differences can then be evaluated via the relative uncertainties of their measurement of these effects. In general the comparison of different experiments by this method depends on the test theory used, and has therefore no "absolute" character. However, the number of "sensible" test theories is usually fairly limited and generally a consensus exists concerning the one which is used.

For tests of general relativity the most common alternative or test theory is simply Newtonian theory. A certain number of observable effects (the so called post-Newtonian effects, e.g. the gravitational redshift or the Shapiro effect) distinguish these two theories. The accuracy of an experiment sensitive to these differences is then quantified by the relative uncertainty of the measurement of the post-Newtonian effects. This is the case, for example, for the three experiments mentioned below.

In the GP-A experiment (Vessot & Levine 1979, Vessot et al. 1980) a hydrogen maser clock was flown on a parabolic flight onboard a scout rocket and compared to a ground hydrogen maser using a microwave link. The comparison yielded a test of the relativistic prediction for the shift in the relative rate of the two clocks due to the combined effect of the gravitational redshift and the second order doppler shift. The theoretical prediction was confirmed with a relative uncertainty of $7 \times 10^{-5}$.

Radar ranging to the Viking spacecraft that orbited and landed on Mars consisted of measurements of the round trip travel time of a radar signal from the Earth to the satellite and back. The variation of the round trip travel time as the satellite passed behind the sun, combined with a knowledge of the satellite position (obtained from ranging data when the satellite was far from the sun) provided a test of the relativistic time delay of a signal that passes through a gravitational field (Shapiro delay). The measurements allowed a verification of the theoretical prediction with a relative uncertainty of $1 \times 10^{-3}$ (Reasenberg et al. 1979).

The timing of pulsars in a binary system with a companion (e.g. another neutron star) provides, at present, the only possibility for tests of relativistic theories in the regime of strong gravitational fields (the gravitational field on the surface of a neutron star is around five orders of magnitude stronger than the one on the surface of the sun). The general relativistic timing
model that is fitted to the raw observations includes a number of post-Keplerian (PK) parameters. The values of these parameters (or of their combination) determined by fitting the model to the observed arrival times of the pulses can be compared to the theoretical predictions, thereby yielding a test of general relativity in the strong field regime. Up to now two binary pulsars (PSR 1913+16 and PSR 1534+12) have been used for such tests (Taylor 1992, Damour 1992). The experiment using PSR 1913+16 verified the theoretical prediction for a combination of three PK parameters with a relative uncertainty of $5 \times 10^{-3}$. Apart from confirming the validity of general relativity in the strong field regime this result presents the first evidence for the existence of gravitational waves, as one of the three PK parameters is the binary period derivative linked to the dissipation of energy due to the emission of gravitational waves. Although less accurate the results using PSR 1534+12 are also of interest as they are complementary to the PSR 1913+16 test, determining two independent combinations of four PK parameters, thereby providing two new tests of general relativity in the strong field regime both of which also confirm the validity of the theory.

In part III of this thesis two original experiments which test the theory of special relativity using state-of-the-art time metrology are presented. The first experiment (Wolf & Petit 1996) uses data from clock comparisons between ground clocks and clocks onboard the Global Positioning System (GPS) satellites to test the second postulate of special relativity (the universality of the speed of light). The experiment is sensitive to a possible anisotropy of the one-way speed of light in any spatial direction, and on a non-laboratory scale (baselines $\geq 20000$ km) and provides the most stringent limits for the anisotropy published up to date. The second (Wolf 1995) is a proposal for a test of special relativity using a spacecraft that carries an onboard atomic clock and uses a two way time transfer system. The potential accuracy of such a test is evaluated for the ESA/RSA ExTRAS (Experiment on Timing Ranging and Atmospheric Sounding) experiment which was planned for launch in 1997 but is now "on hold".

Both of these experiments demonstrate how research in fundamental science can profit from the technology and methods of modern time metrology, even though these were conceived, and are operated for other purposes. In days of continuous cuts in science budgets such "opportunistic" experiments that take advantage of systems conceived primarily for use in other fields seem to be a good alternative to experiments devoted solely to fundamental
science. They require scientists that are familiar with issues of fundamental research as well as the most recent developments in the fields of applied science and technology, and/or extensive communication and collaboration between the scientists working in the different fields.
III.1 A Test of Special Relativity Using GPS

Experiments that test the second postulate of special relativity (the universality of the speed of light), like the ones presented in part III of this thesis, are usually interpreted using an "aether" theory as the test theory. In such an aether theory the speed of light is anisotropic along a particular spatial direction (in an inertial frame) i.e. its component along this preferred axis is \( c + \delta c \) in one direction and \( c - \delta c \) in the opposite direction. The experiments then determine whether the special relativistic prediction \( \delta c = 0 \) is confirmed within the uncertainty of the experiment and set an upper limit on the anisotropy of the speed of light i.e. on the parameter \( \delta c / c \).

A more sophisticated theoretical approach to all tests of special relativity (including those testing the second postulate) was developed by Mansouri & Sexl (1977 a,b,c). In their test theory a generalised form of the Lorentz transformation is used to link a moving frame to a privileged frame in which the speed of light is isotropic (expressed by the fact that clock synchronisation by slow clock transport and Einstein synchronisation are equivalent). If special relativity is correct, this generalised transformation reduces to the Lorentz transformation. Experiments that test special relativity can therefore be evaluated via the limits they set on the difference between the two transformations.

The experiment presented here is interpreted at first using an aether theory as the test theory (as in Wolf & Petit 1996), with an interpretation in the framework of Mansouri & Sexl given in section III.1.5.

III.1.1 Introduction

Einstein's second postulate, affirming the universality of the speed of light for inertial frames, is fundamental to the theories of special and general relativity. It can be tested directly by comparing the one way propagation times of light signals along known paths, but in different spatial directions (often referred to as a test of the isotropy of the one-way speed of...
light). The only such test, was carried out by Krisher et al. (1990), who compared the phases of two hydrogen masers separated by a distance of 21 km and linked via an ultrastable fibre optics link of the NASA deep space network. The sensitivity of this experiment, expressed as a limit on the anisotropy of the speed of light, was $\delta c/c < 3.5 \times 10^{-7}$. Riis et al. (1988) tested the isotropy of the first order Doppler shift of light emitted by an atomic beam (and indirectly thereby the second postulate) using fast-beam laser spectroscopy obtaining the currently best limit on the anisotropy, $\delta c/c < 3 \times 10^{-9}$. Both of these experiments relied on the rotation of the Earth for a change in direction of the light signals and were therefore only sensitive to an anisotropy with a component in the equatorial plane. The GP-A rocket experiment (Vessot & Levine 1979, Vessot et al. 1980) can be interpreted as testing the isotropy of the first order Doppler shift of the link between the ground and onboard masers, giving a limit of $\delta c/c < 3.2 \times 10^{-9}$ in one particular spatial direction. The only experiment sensitive to anisotropy in any spatial direction was carried out by Turner and Hill (1964) who tested the isotropy of the first order Doppler shift in a Mössbauer rotor, obtaining a limit of $\delta c/c < 3 \times 10^{-8}$. We present here the results of a test of Einstein's second postulate sensitive to an anisotropy of the one-way speed of light in any spatial direction. Using the clocks onboard the Global Positioning System (GPS) satellites (providing baselines $\geq 20000$ km) we obtain a limit of $\delta c/c < 4.9 \times 10^{-9}$ when considering all spatial directions and $\delta c/c < 1.6 \times 10^{-9}$ for the component of the anisotropy that lies in the equatorial plane. These results, together with those obtained by previous experiments are summarised in Tab. III.1.
Limits on the anisotropy of the one-way speed of light

Direct measurements:

Krisher T.P. et al. (1990) \( \frac{\delta c}{c} < 3.5 \times 10^{-7} \) component in equatorial plane

GPS Test (this experiment) \( \frac{\delta c}{c} < 4.9 \times 10^{-9} \) all spatial directions

GPS Test (this experiment) \( \frac{\delta c}{c} < 1.6 \times 10^{-9} \) component in equatorial plane

Indirect measurements:

Riis E. et al. (1987) \( \frac{\delta c}{c} < 3 \times 10^{-9} \) component in equatorial plane

Vessot R.F.C. et al. (1979) \( \frac{\delta c}{c} < 3 \times 10^{-9} \) component in one particular direction

Turner K.C. & Hill H.A. (1964) \( \frac{\delta c}{c} < 3 \times 10^{-8} \) all spatial directions

Tab. III.1: Tests of the second postulate of special relativity showing the limits they set on anisotropy of the one-way speed of light and their respective spatial sensitivities.

III.1.2 Principle of the Experiment

Satellites of the GPS constellation are distributed in six orbital planes, at an inclination of 55° in near circular orbits with a period corresponding to 0.5 sidereal days (718 min) (NATO 1990). Each satellite is equipped with an onboard atomic clock and a dual-frequency signal transmission system.

The emission time of a signal as measured by the onboard clock \( \tau_{o} \) and its reception time as measured by the ground-clock \( \tau_{g} \) are recorded. The difference \( T = \tau_{g} - \tau_{o} \) represents the transmission time of the signal plus some initial phase difference of the clocks. Note that no synchronisation convention or procedure is assumed. Defining \( D \) as the distance along a
straight line from the satellite (at the moment of emission) to the ground station (at the moment of reception) in a geocentric, inertial (non-rotating) coordinate system one can write

\[ T - \frac{D}{c} = \Delta_0 \]

(III.1.1)

where \( \Delta_0 \) is a constant characterising the initial phase difference of the two clocks. Einstein’s second postulate requires that, for a series of measurements in different directions (e.g. during a complete passage of the satellite), \( T - D/c \) should remain constant, after correction for the relative rate of the two clocks due to the gravitational redshift, second order Doppler shift and the difference in normalised frequencies (c.f. section I.2.1 equation (I.2.2)) of the clocks.

A possible anisotropy of the one-way speed of light would affect the value of \( T \) as a function of direction, but it might also affect the determination of the satellite ephemerides, and therefore the value of \( D \), leaving the difference \( T-D/c \) unchanged. Therefore a meaningful test of special relativity using the above principle necessitates a method of satellite orbit determination which is insensitive to a possible anisotropy of the one-way speed of light.

This is the case for the GPS ephemerides obtained by the IGS-CODE processing centre (IGS 1994). The method used adjusts a post-Keplerian, non-relativistic orbit model to doubly differenced GPS timing data (see Fig. III.1). The effect of a possible anisotropy of magnitude \( \delta c/c \) on an individual link would be \( (\delta c/c)(D/c)\cos \alpha \), where \( \alpha \) is the angle between the direction of the anisotropy and the link, but these effects cancel when the double differences are formed (see Fig. III.1). The IGS-CODE method is used to simultaneously adjust a number of parameters, including the satellite ephemerides and the ground station coordinates, thereby providing values of \( D \) which are unaffected by a possible anisotropy of the one-way speed of light.
Fig. III.1: Double difference (Y) for pairs of stations (A and B) and satellites (1 and 2),

\[ Y = (T_{1A} - T_{1B}) - (T_{2A} - T_{2B}) \]

The effect of an anisotropy on Y given by

\[ \frac{(D_{1A} \cos \alpha_{1A} - D_{1B} \cos \alpha_{1B}) - (D_{2A} \cos \alpha_{2A} - D_{2B} \cos \alpha_{2B})}{c^2} \]

where \( \alpha \) is the angle between the directions of the anisotropy and of the transmitted signal, vanishes.

Additionally one has to ensure that corrections applied to the raw timing data used for orbit determination and the measurement of \( T \) do not presuppose the second postulate. In fact two corrections are routinely applied to GPS timing data which are of relativistic origin and therefore do imply the isotropy of \( c \) (NATO 1990): the correction for the gravitational redshift and the second order Doppler shift of the rate of the satellite clock with respect to coordinate time, and the correction for the so called Sagnac effect which is due to the rotation of the Earth during signal transmission. Both of these are small corrections of order \( c^{-2} \) hence the effect of an error in these corrections due to an anisotropy would be negligible with respect to the first order effect on \( T \).
So the observation of GPS satellites in varying spatial directions provides a meaningful test of the second postulate of special relativity via relation (III.1.1) with $D$ obtained using IGS-CODE ephemerides and station coordinates.

III.1.3 Experimental Procedure

The IGS is a global network of ground stations that continuously observe the GPS satellites for civil, geodetic purposes (IGS 1994). From the raw data the IGS processing centres calculate (among other parameters) precise satellite ephemerides and ground station coordinates. These, together with the raw observations, are freely available through the internet via anonymous ftp (IGS 1994). We use data from eight ground stations for our experiment: Brussels (Belgium), Algonquin (Canada), Yellowknife (Canada), Fairbanks (Alaska, USA), Kokee Park (Hawaii, USA), Fortaleza (Brazil), Santiago (Chile) and Hobart (Australia). The motivation for this choice of ground stations is to ensure global coverage whilst providing maximum ground clock stability (for averaging times $\approx 6$ h (one passage)) by using only stations which are equipped with hydrogen-maser clocks. The GPS receivers used are all AOA Rogue or Turbo-Rogue geodetic receivers providing raw phase measurements of the two GPS carrier frequencies at a sample interval of 30 seconds. The data sets cover six days (1994 September 18 to 23) and contain observations of all 25 GPS satellites available at the time. During this period (coinciding with the military intervention in Haiti) all GPS signals were free of the intentional degradation (Selective Availability, SA) which is imposed by the US military. In general, this affects all but two satellites, making them unusable for the experiment described here. The satellites used are all equipped with caesium clocks, except 6 older, block I, satellites which carry rubidium clocks.

From the raw data the differences $T-D/c$ are formed, taking into account corrections for the variable part of the gravitational redshift and second order Doppler shift, the Sagnac effect, the ionospheric delay (using an ionosphere-free combination of the two frequencies), and the tropospheric delay (using the STANAG (NATO 1990) tropospheric model). The amount of data is reduced by using one measurement every 5 min, in order to save computer space and to
make the data more manageable. Fig. III.2 shows a typical example of a resulting data set for a station-satellite pair.

For a test of the isotropy of $c$ one is interested in the variation of the difference $T - D/c$ during individual passages of the satellite over the ground station i.e. variations over time scales of less than 6 hours. Therefore we first filter the data, excluding all long term variations (greater $\approx$ 6 days) effectively subtracting the relative rate of the two clocks. Then an arbitrary offset per passage is adjusted as only the variation of $T - D/c$ during the passage is of interest.

As is well known, measurements of the GPS carrier phase are subject to an unknown phase ambiguity error of an integer number of cycles. This does not present a problem for our purposes as long as the induced error remains constant during each passage, which is the case if the receiver stays locked onto the satellite over the complete passage. Therefore all passages that are incomplete (data gaps indicating a possible loss of the satellite) are excluded.
The 0.5 sidereal day period of the GPS satellites implies that a station sees each passage of a particular GPS satellite at the same time of day (in sidereal days) and in the same directions (in a geocentric inertial frame). So, to get a better view of the data, all passages can be projected onto the same day, by shifting each of them by an integer number of sidereal days. Fig. III.3 shows a typical data set after filtering and adjustment of an offset per passage, and with all passages shifted onto the same day.

![Fig. III.3: Residuals of (T-D/c) after filtering and adjustment of an offset per passage. The graph shows 6 passages of PRN22 over Brussels(B) shifted onto the same day.](image)
III.1.4 Results

Fig. III.4 shows the spatial directions of the individual links in an inertial geocentric frame. There are no links at colatitudes below ≈20° and above ≈163° which is due to the 55° inclination of the satellite orbits and implies that the experiment was least sensitive in the N-S direction.

The effect of an anisotropy of c on the transmission time T for a particular link is given by $(D/c) (\delta c/c) \cos \alpha$ where $\alpha$ is the angle between the direction of signal transmission and the direction of the anisotropy. This model was fitted to the data using the least squares method, adjusting an offset and the magnitude of the anisotropy $(\delta c/c)$. The adjustment was performed for a range of anisotropy directions, spanning colatitudes and longitudes from 0 to $\pi$ in a 0.1 rad x 0.1 rad grid. It is sufficient to cover half of all possible spatial directions, as opposing directions correspond to the same anisotropy with the opposite sign. Directions are given in
the non-rotating geocentric frame that is coincident with the ITRF Earth fixed frame at MJD 49755 (March 7, 1995) 0h 00 (UTC).

Fig. III.5 shows the adjusted anisotropy magnitudes as a function of the direction of the anisotropy. The extremum value of $\frac{\Delta c}{c}$ is $4.9 \times 10^{-9}$ at a colatitude of 2.9 rad and a longitude of 0.5 rad. In the equatorial plane (colatitude of 1.6 rad) the extremum value of $\frac{\Delta c}{c}$ is $1.6 \times 10^{-9}$ at a longitude of 0.6 rad.

Fig. III.5: Magnitude $\frac{\Delta c}{c}$ of the adjusted anisotropy as a function of its direction in a geocentric non-rotating frame.

The "goodness of fit" as characterised by the diminution of the standard deviation of the residuals before and after the adjustment is shown in Fig. III.6. The best fit is obtained in the direction 2.9 rad (colatitude), 0.6 rad (longitude), and diminishes the residuals by 0.27 %. The standard deviation of the pre-fit residuals is 2.2 ns. The value of $\frac{\Delta c}{c}$ in the direction of best fit is $4.9 \times 10^{-9}$. 
Fig. III.6: Diminution of the standard deviation of the residuals before and after the adjustment as a function of direction (in a geocentric non-rotating frame) of the adjusted anisotropy.

III.1.5 Interpretation within the Theoretical Framework of Mansouri & Sexl

The test theory developed by Mansouri & Sexl (1977a,b,c) is based on a general transformation between a universal reference frame $\Sigma:(T,\vec{X})$ and a moving inertial frame $S:(t,\vec{r})$. The universal frame $\Sigma$ is distinguished by the fact that in it Einstein and slow clock-transport synchronisation are equivalent, implying isotropy of the speed of light, which is not necessarily the case in $S$ (in $S$ the equivalence only holds for the special case of special relativity). With $v$ being the velocity of $S$ as measured in $\Sigma$ the transformation reads
\[ T = a^{-1}(t - \vec{v} \cdot \vec{x}) \]  
\[ \vec{X} = d^{-1}\vec{x} - (d^{-1} - b^{-1}) \frac{\vec{v} \cdot \vec{x}}{v^2} + \nu T \]  

where \( a, b, \) and \( d \) are dimensionless parameters, functions of \( v^2/c^2 \), and \( \vec{e} \) is a vector determined by the procedure adopted for the global synchronisation of clocks in \( S \). In special relativity, with either Einstein or clock-transport synchronisation in \( S \), \( a^1 = b = \gamma = (1-v^2/c^2)^{-1/2} \), \( d = 1 \), \( \vec{e} = -\vec{v} / c^2 \), and (III.1.2) reduces to the Lorentz transformation of special relativity.

In the limit where \( v^2/c^2 \ll 1 \) the three parameters \( a, b, \) and \( d \) are often expanded

\[ a = 1 + \alpha \frac{v^2}{c^2} + \ldots \]  
\[ b = 1 + \beta \frac{v^2}{c^2} + \ldots \]  
\[ d = 1 + \delta \frac{v^2}{c^2} + \ldots \]  

where in special relativity \( \alpha = -\frac{1}{2}, \beta = \frac{1}{2} \) and \( \delta = 0 \). Tests of special relativity are then evaluated by the limits they set on the deviation of the parameters \( \alpha, \beta, \) and \( \delta \) from their special relativistic values.

III.1.5.1 The Variation of the Transmission Time, \( T \), over one Passage

Choose the moving inertial frame \( S \) such that the ground clock of the experiment, \( O \), is at its origin, the signal transmission is in the x-y plane and \( \vec{v} \) is in the positive x direction. The transmission time (in \( S \)) of a signal emitted from some point, \( P \), and received at \( O \) is obtained by setting \( ds^2 = -c^2dT^2 + d\chi^2 = 0 \) for a light signal in \( \Sigma \) and transforming according to (III.1.2). One obtains

\[ t_r = t_* + \frac{D}{c} \left( 1 - \frac{v}{c} \cos \theta \right) - D(\epsilon_x \cos \theta + \epsilon_y \sin \theta) + o\left( \frac{Dv^2}{c^2} \right) \]  

(III.1.4)
where $t_e$ and $t_t$ are the emission and reception time in $S$, $D = \bar{x}_p(t_e)$, $\theta$ is the angle between the vectors $D$ and $v$ (as measured in $S$), and the expansion (III.1.3) has been used.

Consider now two points, $A (D_A, \theta_A=0)$ and $B (D_B, \theta_B \neq 0)$, in the x-y plane of $S$ as shown in Fig. III.7. A clock $M$ (modelling the space clock) situated at $A$ emits a signal that is received at $O$, and then moves to $B$ where it emits another signal that is also received at $O$. The variation of the transmission time (measured by the clocks $M$ and $O$) is then given by the difference in transmission times in $S$ (given by (III.1.4) applied to points $A$ and $B$) and the time difference between $M$ and the coordinate time of $S$ accumulated during the transport of $M$ from $A$ to $B$.

![Fig. III.7: Positions A and B of transported clock M in frame S.](image)

The clock $M$ is transported in $S$ with a velocity $w$ ($w \ll v$). Its velocity in $\Sigma$, $W$, is then given by the law of addition of velocities

\[
\begin{align*}
W_x &= \frac{ab^{-1}w_x + v(1-\vec{e} \cdot \vec{w})}{1-\vec{e} \cdot \vec{w}} \\
W_y &= \frac{ad^{-1}w_y}{1-\vec{e} \cdot \vec{w}} \\
W_z &= \frac{ac^{-1}w_z}{1-\vec{e} \cdot \vec{w}}
\end{align*}
\]  

(III.1.5)
obtained directly from (III.1.2) with \( \bar{v} \) in the positive x direction.

The time difference between M and the coordinate time of S accumulated during the transport of M from A to B is equivalent to the time difference accumulated by a clock M' travelling from O to a point C as shown in Fig. III.7. A third frame S':(\( t', \bar{x}' \)) is co-moving with M' its origin coinciding with the clock M' and the origins of the frames \( \Sigma \) and S at \( T = t = t' = 0 \). The frame S' is related to the universal frame \( \Sigma \) by the transformation (III.1.2) with \( \bar{W}, a', b', d' \) and \( \bar{e}' \), substituted for \( \bar{v}, a, b, d, \) and \( \bar{e} \). Consider now the event "arrival of M' at point C" (denoted by the subscript \( a \)). Its time coordinate \( t'_a \) in S' corresponds to the reading of M' at this event (by definition). Therefore the time difference with respect to S accumulated by M' during transport can be obtained by expressing \( t'_a \) in terms of \( t_a \) and \( \bar{x}_a \) (the coordinates of a in S). Using the transformation (III.1.2) between the different frames one immediately obtains

\[
t'_a = \frac{a'}{a}(t_a - \bar{e} \cdot \bar{x}_a). \tag{III.1.6}
\]

But the parameter \( a' \) can be expanded in terms of \( \bar{W}^2/c^2 \) using (III.1.3) with \( \bar{W}^2 \) obtained from (III.1.5) so that

\[
a' = a \left( 1 + \frac{2 \alpha \bar{v} \bar{x}_a}{c^2 (1 - \bar{e} \cdot \bar{w}) t_a} \right) + O\left( \frac{\bar{w}^2}{c^2} \right). \tag{III.1.7}
\]

Substituting this result into (III.1.6), replacing \( \bar{x}_a \) by the coordinates of C in S (see Fig. III.7) and applying the expansion (III.1.3) one finally obtains

\[
t'_a = t_a + \frac{2 \alpha \bar{v}}{c^2} (D_a \cos \theta_a - D_a)
- \left[ e_a(D_a \cos \theta_a - D_a) + e_D \sin \theta_a \right] + O\left( \frac{Dw}{c^2} \right). \tag{III.1.8}
\]

which gives the time difference between M and the coordinate time of S accumulated during the transport of M from A to B.
So using (III.1.4) and (III.1.8) to model the experiment one obtains the variation in transmission times

$$\Delta T = T_B - T_A = (t_r - t_0) - (t'_r - t_s) - (t_r - t_s)_A$$

$$= -\frac{v}{c}(1 + 2\alpha) D_B \cos \theta_B + \frac{D_B - D_A}{c}$$

$$+ \frac{v}{c}(1 + 2\alpha) D_A + O\left(\frac{Dw}{c^2}\right)$$

(III.1.9)

where $T_i$ designates the transmission time of a light signal from point $i$ to $O$ as measured by the transported clock $M$ and the clock at $O$. The first two terms on the right hand side of (III.1.9) vary during the passage of the satellite, while the third term is a constant for each passage. Note also that (III.1.9) is independent of $\vec{E}$ which reflects the fact that no synchronisation convention or procedure needs to be employed.

Applied to our experiment, the third term is absorbed by the constant adjusted to each passage (see section III.1.3) and so the effect on the difference $T-D/c$ for one particular link between the space and the ground clock is given by the first term of (III.1.9). By comparison, the effect adjusted using the aether theory (section II.1.4) is given by $(D/c) (\delta \gamma/c) \cos \alpha$, therefore, the value obtained for $\delta \gamma/c$ from the adjustment is related to the parameter $\alpha$ by:

$$\frac{\delta \gamma c}{c} = (1 + 2\alpha) \frac{v}{c}$$

(III.1.10)

and the "direction of anisotropy" corresponds to the direction of $v$.

At first order in $v/c$ the dependence on $\alpha$ of (III.1.9) characterises solely the effect of the transport of clock $M$ (all terms in $\alpha$ arise from (III.1.8)) and not the difference in transmission times as measured in frame $S$ (as (III.1.4) is independent of $\alpha$). More particularly, the effect of $\alpha$ in (III.1.9) is directly proportional to $D_A - D_B \cos \theta_B$ i.e. only the component of the trajectory of $M$ that is parallel to $\vec{v}$ gives rise to terms in $\alpha$ at first order in $v/c$.  

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III.1.5.2 The Effect on Double Differences

The experiment can only provide meaningful results if the determination of the distances, $D$, is independent of the parameter $\alpha$ (to first order in $v/c$). As mentioned before (section III.1.2) satellite ephemerides and station coordinates are obtained from double differences. In the following it is shown that the expression obtained for a double difference using the theory of Mansouri & Sexl is identical to the special relativistic expression (to first order in $v/c$) independently of the value of $\alpha$, as required.

All terms in $\alpha$ arise from the components of the trajectories of the transported clocks that are parallel to $\vec{v}$. Therefore it is sufficient, without loss of generality, to only consider the projection onto a plane containing $\vec{v}$ of the satellite and signal trajectories (as shown in Fig. III.8 for two ground stations, A and B, and two satellites, C and D). The double difference, $Y$, is given by $Y = (T_1 - T_2) - (T_3 - T_4)$ where the $T_i$ are the transmission times measured by the different pairs of clocks (see Fig. III.8).

![Fig. III.8: Double difference: projection onto a plane (containing $\vec{v}$) of satellite and station positions and of the signal trajectories.](image)
Denoting by $D_{vl}$ and $T_{v}$ the distances and transmission times when the satellites are at points $C'$, $D'$ and $C''$, $D''$ and applying (III.1.9) to the individual links the double difference, $Y$, is expressed as:

$$Y = \frac{1}{c}(D_1 - D_2 - D_3 + D_4) + \left[\left(T_{v1} - \frac{D_{v1}}{c}\right) - \left(T_{v2} - \frac{D_{v2}}{c}\right) - \left(T_{v3} - \frac{D_{v3}}{c}\right) + \left(T_{v4} - \frac{D_{v4}}{c}\right)\right] + \frac{v}{c^2}(1+2\alpha)(D_{v1} - D_{v2} - D_{v3} + D_{v4}) + O\left(\frac{Dw}{c^2}\right)$$

(III.1.11)

The transmission times $T_{v}$ can be expressed using (III.1.4) and (III.1.8). Applying (III.1.4) to a signal transmitted from $C'$ to $A$ gives

$$T_{v1} = \frac{D_{v1}}{c}\left(1 - \frac{v}{c}\right) - D_v\varepsilon_x + O\left(\frac{Dv^2}{c^2}\right)$$

(III.1.12a)

and taking into account the time difference accumulated by clock $C$ during the transport from $C'$ to $C''$ (using (III.1.8)) gives for the signal from $C''$ to $B$

$$T_{v2} = \frac{D_{v2}}{c}\left(1 - \frac{v}{c}\right) - D_v\varepsilon_x - 2\alpha\frac{v}{c^2}(D_2\cos\theta_2 - D_v) + \varepsilon_x(D_2\sin\theta_2) + \frac{Dw}{c^2}$$

(III.1.12b)

where $D_{2''}$ is the distance $AC''$ and $\theta_{2''}$ is the angle $C'AC''$. Similarly

$$T_{v3} = \frac{D_{v3}}{c}\left(1 - \frac{v}{c}\right) - D_v\varepsilon_x + O\left(\frac{Dv^2}{c^2}\right)$$

(III.1.12c)
\[
T_{\text{in}} = \frac{D_{\text{in}}}{c} \left(1 - \frac{v}{c}\right) - D_{\text{in}} \varepsilon_x - 2\alpha \frac{v}{c^2} \left[D_1 \cos \theta_1 - D_{\text{in}}\right] \\
+ \varepsilon_x \left(D_1 \cos \theta_1 - D_{\text{in}}\right) + \varepsilon_y \left(D_1 \sin \theta_1\right) + \mathcal{O}\left(\frac{D_w}{c^2}\right)
\]

(III.1.12d)

where \(D_{\text{in}}\) is the distance \(AD_{\text{in}}\) and \(\theta_1\) is the angle \(D'AD_{\text{in}}\).

Substituting from (III.1.12a-d) into the second term of (III.1.11) one finally obtains

\[
Y = \frac{1}{c} \left(D_1 - D_2 - D_3 + D_4\right) + \mathcal{O}\left(\frac{D_w}{c^2}\right)
\]

(III.1.13)

which is identical to the special relativistic expression and independent (to first order in \(v/c\)) of the value of \(\alpha\), as required.

Therefore, the experiment, when interpreted in the theoretical framework of Mansouri & Sexl, provides a meaningful test of special relativity with the limits on the parameter \(\alpha\) given by (III.1.10) using the adjusted value of \(\delta c/c\) (section II.1.4) and assigning a particular value to \(v\) (see section II.1.7).

III.1.6 Systematic Effects

The three main systematic effects that could affect the data are the satellite clock instability, ephemerides errors and uncertainties in the estimated tropospheric delays.

Pre-launch measurements of the GPS caesium clock relative frequency stability showed an instability of \(\sigma_f(\tau) \approx 1.5 \times 10^{-13}\) (standard Allan deviation) for integration times of \(\approx 6\ h\) (Wisnia 1992). On-orbit measurements showed satellite clock instabilities of order \(10^{-13}\) for integration times of 1 day (McCaskill et al. 1991), which corresponds to \(\sigma_f(6\ h) \approx 2 \times 10^{-13}\) when extrapolated assuming a \(\tau^{1/2}\) dependence. This translates into an accumulated time error over one passage of the satellite of \(\delta \tau \approx \tau \sigma_f(\tau) \approx 4.3\ ns\).
Beutler et al. (1996) estimate the uncertainty of the IGS-CODE satellite ephemerides to be 15-20 cm corresponding to a timing error of $\delta r \approx 0.7$ ns.

Estimations of tropospheric delays using the standard STANAG model and a more accurate model which uses meteorological data show differences at the sub-nanosecond level for temperate regions, which can increase to several ns for tropical conditions and low elevations (Lewandowski et al. 1992). Therefore, the STANAG model may not estimate the zenithal tropospheric delay correctly. In addition both models use mapping functions which depend only on elevation, so the calculated value for the tropospheric delay cannot account for a possible spatial variation (at constant elevation) of the troposphere. This would have an effect on our measurements as we consider the variation of the signal transmission time over individual passages of the satellite. It is therefore difficult to estimate the effect of uncertainties in tropospheric delay but, assuming the delay varies only slightly with spatial direction (at constant elevation) the timing error due to the troposphere should not exceed a few nanoseconds. Using tropospheric delays that are estimated in a regional or global network (as done for example by the IGS) would decrease these errors. We did not use such a method because we do not think that uncertainties in the modelled tropospheric delays significantly limit the sensitivity of our experiment as the dominant limitation is more likely to arise from satellite clock instabilities.

### III.1.7 Discussion and Conclusion

Timing errors due to the systematic effects discussed in the previous section could give rise to values of $\Delta/c$ of order $10^{-8}$ which is an order of magnitude larger than the maximum value observed. The experiment therefore does not suggest a violation of the second postulate of special relativity.

It is unlikely that, in a global treatment, the systematic errors are correlated with the signature of an anisotropy, as they are expected, in general, to correlate differently with the effect of an anisotropy for each satellite-station pair and, to some extent, for each passage. For
this reason, increasing the number of stations and/or satellites should decrease the importance of the systematic effects in a global treatment. This is confirmed when using subsets of the data which, in all cases, give limits on $\delta c/c$ which are less severe than that obtained from the complete data set.

So, assuming no correlation (and resulting cancellation) between the systematic effects and the effect of a possible anisotropy of the one-way speed of light, we can set a limit of $\delta c/c < 4.9 \times 10^{-9}$ on the spatial variation of $c$ when considering all spatial directions and $\delta c/c < 1.6 \times 10^{-9}$ for the component of the anisotropy that lies in the equatorial plane.

The results can be "translated" into a limit on the parameter $\alpha$ of the test theory by Mansouri & Sexl using (III.1.10) and taking $v$ as the velocity of the Earth with respect to the "mean rest frame of the universe" ($v \approx 300$ km/s) in the direction of the dipole anisotropy of the cosmic microwave background (declination = -6.1°, right ascension = 11.2 h) (Fixsen et al. 1983, Lubin et al. 1983). This results in the limit $|\alpha + 1/2| < 9.7 \times 10^{-7}$ ($\delta c/c < 1.94 \times 10^{-9}$ in this direction) which, to my knowledge, is the smallest limit for the parameter $\alpha$ published up to date.

Finally it should be emphasised that an experiment like the one described in this section uses existing operational technology and hence requires a minimum of financial investment. In fact it can be carried out by virtually anyone as all the IGS data is freely available on the internet via anonymous ftp. This is of particular interest in view of a recent US decision (US 1996) to switch off SA completely on all GPS satellites within the next ten years.
III.2 Proposed Satellite Test of Special Relativity

In the GPS test of special relativity (section III.1) the electro-magnetic signals are transmitted using a one way system (transmissions from the satellite to the ground station). Satellites equipped with two way systems (transmissions from the ground station to the satellite and vice versa) can be used for a test of special relativity by comparing the propagation times of two light signals travelling along the same path but in opposing directions. Such a test requires only a very rough knowledge of the distance $D$ (satellite to ground station) to correct for path asymmetries, hence uncertainties due to ephemerides and station coordinate errors become negligible. Similarly any atmospheric delay uncertainties can be neglected (assuming identical up and down link frequencies) as these cancel when the difference between the two links is formed. Consequently any satellite carrying an atomic clock and equipped with a two way time transfer system is a likely candidate for an improved test of the second postulate of special relativity. In this section the possible use of the ESA/RSA ExTRAS (Experiment on Timing Ranging and Atmospheric Sounding) project for such an experiment is examined. The ExTRAS mission was planned for launch on board the Russian Meteor-3M satellite in 1997 but is now "on hold".

The ExTRAS payload consists of two active, auto-tuned hydrogen masers communicating with ground stations via a PRARE (Precise Range and Range-Rate Equipment) microwave link and a T2L2 (Time Transfer by Laser Light) laser link. Once operational, the system should reflect laser pulses, emit and receive microwave signals, and date all such events on the onboard time scale provided by the hydrogen masers. The Meteor-3M satellite will follow polar orbit, at an altitude of 1000 km with a period of order 100 min and a duration of one passage of $\approx 17$ min.

In section III.2.1 the principle of the experiment is explained while section III.2.2 provides an evaluation of its sensitivity aimed at including all error sources that may exceed one picosecond and based on the uncertainty budget for the T2L2 method by Thomas, Wolf, et al. (1994). The results are interpreted in terms of an "aether theory", setting a limit on the variation of the speed of light $\Delta c/c$ along a privileged direction, and in the framework of the test theory of Mansouri & Sexl (1977a,b,c), setting a limit on the parameter $\alpha$ of that theory.
### III.2.1 Experimental Principle

In principle, the proposed experiment is similar to that performed by Krisher et al. (1990). A laser signal emitted from the station E is reflected at the satellite S and returned to E (see Fig. III.9).

The readings of the ground hydrogen maser at emission ($\tau_0$) and reception ($\tau_2$) and that of the space maser at the moment of reflection ($\tau_1$) are recorded. The differences $\tau_1 - \tau_0$ and $\tau_2 - \tau_1$ represent the up and down transmission times $T_1$ and $T_2$ respectively plus some initial phase difference of the clocks. Note that no synchronisation convention or procedure is assumed. Einstein's second postulate would require that for a series of measurements, after accounting for path asymmetries, the difference $T_1 - T_2$ should be equal to a constant $\Delta_0$ (due to the initial clock offset) independent of the spatial orientation of the individual links. More particularly one obtains for a single link (see Petit & Wolf 1994 for more detail),

$$T_1 - T_2 = \Delta_0 + \frac{2D(t_s) \cdot \gamma(t_s)}{c^2} + \left( A_{\text{up}} - A_{\text{down}} \right) + O(c^{-3})$$

(III.2.1)
where $D(t_s)$ is the vector from E to S at the coordinate time of emission of the signal $t_s$ in a geocentric, inertial reference frame, $v(t_s)$ is the velocity of the ground station at signal emission in the same frame and $\Delta_i$ are internal delays (cables etc.).

The initial clock offset $\Delta_0$ is a constant, provided that the two clocks are synonised. This can be achieved using time transfer data over a sufficiently long integration period (> 1 day) and taking into account all known effects (gravitational redshift, second order Doppler, maser drift). The accuracy of synonisation will be limited by the stability of the masers for integration times of the order of one day which is $\approx 2 \times 10^{-15}$ (Thomas, Wolf et al. 1994). One would expect the effect on the synonisation, of an eventual anisotropy of the propagation time of the light signals, to average out in a global treatment using time transfers in all spatial directions.

Terms of order $c^{-2}$ amount to a few tens of nanoseconds and can be calculated to picosecond accuracy if $D(t_s)$ and $v(t_s)$ are known to within $\approx 50$ m and $\approx 0.01$ m/s respectively, which represents no difficulty for modern satellite orbitography. Of course, a possible anisotropy would also have an effect on the satellite orbit determination, but as the range $D$ cancels to first order in (III.2.1) this effect would be negligible. Furthermore, the satellite orbit is obtained from round-trip ranging measurements, which should, again to first order, be insensitive to anisotropy of the propagation time of the light signals.

Terms of order $c^{-3}$ can amount to several picoseconds but can be calculated to picosecond accuracy without difficulty (Petit & Wolf 1994). The effect of asymmetry in the atmospheric delays for the up and down links is below one picosecond.

Hence, after accounting for path asymmetry, any variation of the difference $T_1 - T_2$ with the spatial orientation of the laser link should be due to a violation of the second postulate.
III.2.2 Estimation of the Experiment Sensitivity

The sensitivity of the proposed test can be estimated by considering two individual laser links as shown in Fig. III.10.

![Diagram of two-way laser links](image)

**Fig. III.10:** A pair of two-way laser links between an Earth station and the satellite, as viewed in a geocentric, inertial frame.

The time intervals $\tau_2-\tau_0$, $\tau_3-\tau_3$ and $\tau_3-\tau_0$ are measured using the ground hydrogen maser with the interval $\tau_4-\tau_1$ obtained from the space hydrogen maser. Designating the individual transmission times by $T_1$, $T_2$, $T_3$ and $T_4$ as shown in Fig. III.10 and assuming that one of the links is colinear with the direction of the presumed anisotropy, the difference between the two links is given by,

$$(T_1 - T_2) - (T_3 - T_4) + \Delta = 2\Delta(1 - \cos \theta). \tag{III.2.2}$$

Here $\Delta_i$ represents the correction due to the path asymmetries of the individual links, $\Delta_s$ is the maximum delay for a single transmission due to the anisotropy, and $\theta$ is the angle between the two links in the inertial geocentric frame.

If Einstein's second postulate is true the right hand side of equation (III.2.2) should be equal to zero within the measurement error.
The experiment should be capable of detecting an anisotropy under the condition

$$\varepsilon < 2A(1-\cos\theta),$$  \hspace{1cm} (III.2.3)

where $\varepsilon$ represents the total measurement uncertainty.

The sensitivity of the experiment is therefore given by,

$$\frac{\Delta c}{c} = \frac{A}{T} = \frac{\varepsilon}{2T(1-\cos\theta)}$$  \hspace{1cm} (III.2.4)

where $T$ is a typical transmission time.

Consider the measurements taken at the beginning and the end of a single passage of the satellite directly above the station. Laser measurements are generally limited to elevation angles greater than 20° in which case $\theta \approx 140^\circ$, $T \approx 7.1$ ms, and the error accumulated due to the instability of the hydrogen masers is very small because of the short integration time of $\approx 550$ s. Table III.2 lists the individual sources of uncertainty that are estimated to exceed 1 ps. Four sources of uncertainty are listed in the table:

<table>
<thead>
<tr>
<th>Source of uncertainty</th>
<th>$\sigma$/ps</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen masers(i)</td>
<td>$3\sqrt{2}$</td>
</tr>
<tr>
<td>Onboard payload(ii)</td>
<td>20</td>
</tr>
<tr>
<td>Earth station(iii)</td>
<td>$10\sqrt{2}$</td>
</tr>
<tr>
<td>Counters(iv)</td>
<td>$20\sqrt{2}$</td>
</tr>
<tr>
<td><strong>Total (quadratic sum)</strong></td>
<td>$\varepsilon = 38$</td>
</tr>
</tbody>
</table>

Table III.2: Anticipated uncertainty budget for measurement of an anisotropy whose direction lies in the orbital plane. All uncertainties are in picoseconds and correspond to an estimated one standard uncertainty, $\sigma$. Uncertainties that affect two independent measurements are multiplied by a factor $\sqrt{2}$. 
(i) The stability of the hydrogen masers for integration times of 500 s is of the order 2-3 parts in $10^{15}$ (Thomas, Wolf, et al. 1994) which gives an accumulated uncertainty of $\approx 1.5$ ps per maser over one passage.

(ii) As systematic errors in the onboard payload cancel when the two links are differenced, only its instability over the passage contributes. Ten picoseconds (Thomas, Wolf, et al. 1994) seems a conservative estimate for such a short integration time.

(iii) Only the instability of the Earth station during the experiment contributes. Degnan (1993) states that the precision of satellite laser ranging stations is of order 1 to 3 mm, which corresponds to an uncertainty of less than ten picoseconds.

(iv) Information on the counter uncertainties (the uncertainty in the datation of a laser pulse on the time scale provided by the hydrogen maser) is provided by the T2L2 proposing team.

In the calculation of $(T_1 - T_2) - (T_3 - T_4)$ the differences $\tau_1 - \tau_1$ and $\tau_3 - \tau_0$ measured by the space and ground clock respectively appear with a factor of 2. Hence all uncertainty sources participating in the measurement of these intervals ((i),(ii),(iv)) have been multiplied by this factor.

For the measurement of anisotropy in a direction which is not in the plane of orbit, the two links are separated by the time necessary for the Earth station to change its position with the rotation of the Earth so as to see the satellite from opposing directions. For observations an orbital period apart the hydrogen maser stability is of the order 1,5 parts in $10^{15}$ (Thomas, Wolf, et al. 1994), but the limiting factor will be the uncertainty in the syntonisation of the clocks ($\approx 2 \times 10^{15}$) giving an uncertainty in (i) of $\approx 25$ ps. Contributions from other error sources are those given in Table III.2. Hence the value for the total measurement uncertainty is $\varepsilon \approx 51$ ps. Note also that in this case $\theta \approx 102^\circ$ and $T \approx 6.2$ ms,
Substituting these values for \( \varepsilon \), into (III.2.4) gives an experimental sensitivity of \( \Delta c/c = 1.5 \times 10^{-9} \) when the direction of the anisotropy lies in the orbital plane of the satellite and \( \Delta c/c = 3.4 \times 10^{-9} \) otherwise.

The experiment can be interpreted in the theoretical framework of Mansouri & Sexl using the same model as for the GPS experiment (section II.1.5), in particular equations (III.1.4) and (III.1.8). The relation between \( \Delta c/c \) and \( \alpha \) is again given by (III.1.10) with \( v = 300 \text{ km/s} \) (section III.1.7), so the experiment yields limits of \(|\alpha + 1/2| < 7.5 \times 10^{-7} \) and \(|\alpha + 1/2| < 1.7 \times 10^{-6} \) for the two cases.

### III.2.3 Conclusion

The proposed test of special relativity is expected to yield an upper limit for anisotropy of the one-way speed of light of \( \Delta c/c = 1.5 \times 10^{-9} \) when the direction of the anisotropy lies in the orbital plane of the satellite and \( \Delta c/c = 3.4 \times 10^{-9} \) otherwise.

In spite of the advantages of two way time transfer systems and of significantly improved space clock stability these limits are of the same order as those obtained from the GPS experiment (Wolf & Petit 1996, section III.1 of this thesis). This is due to the comparatively low altitude of the Meteor-3M satellite (a factor of 20 lower than GPS). However, the two experiments are complementary as the GPS test is least sensitive in the N-S direction, which is not the case for ExTRAS due to the polar orbit of the satellite.

Similarly to the GPS experiment the test of special relativity proposed here uses systems that are intended primarily for use in other fields of science (metrology, navigation, geodesy, atmospheric studies) with no need for additional equipment specific to this experiment. Hence it can be considered an essentially low-cost experiment which is generally an important factor for research in fundamental science.

The same experiment could be performed using the PRARE microwave transfer system in the two-way ranging mode (Thomas, Wolf, et al 1994) rather than the T2L2 link. This might
be of advantage as the PRARE method is not weather dependent. However, uncertainties in 
the ionospheric propagation delays due to different up and down link frequencies introduce an 
additional uncertainty of $\approx 20$ ps per link, which slightly decreases the overall sensitivity of the 
experiment to $\Delta c/c = 1.7 \times 10^{-9}$ for the case where the direction of the anisotropy lies in the 
orbital plane of the satellite and to $\Delta c/c = 3.7 \times 10^{-9}$ otherwise.

It is likely that the sensitivity of the experiment can be improved if data taken 
continuously during the passage of the satellite is used to search for the sinusoidal variation 
with $\theta$ of the signal due to anisotropy. Furthermore, for any particular orientation of the 
presumed anisotropy, it should be possible to improve the experimental sensitivity by statistical 
treatment of data from different ground stations and from repeated measurements.
IV CONCLUSION

This thesis presents the results of a study of the application of general relativity to the metrology of time and of the use of the methods and technology of time metrology for tests of relativity. Such a parallel study of applied and fundamental science provides a "global" view which, in my opinion, leads to new possibilities and ideas like, for example, the new test of special relativity described in section III.1.

In Part II a detailed theory for the treatment of the metrology of time in a relativistic context is developed. It provides mathematical expressions for application to the syntonisation and synchronisation of clocks and the realisation of the time coordinates of space-time reference systems at $10^{-18}$ syntonisation and picosecond synchronisation accuracy. These limits should be sufficient to accommodate present and near future developments in time transfer and clock technology. In the longer term clock stabilities of order $10^{-19}$ and better together with sub-picosecond time transfers will necessitate more accurate theoretical developments which will imply a number of new conventions and definitions: at these accuracy levels terms of order $\varepsilon^3$ and $\varepsilon^4$ (terms in $c^3$ and $c^4$) in the metric become significant. These require the choice (by convention) of coordinate conditions and of the state of rotation (dynamically or kinematically non-rotating) of the reference system. For the realisation of TT such accuracies imply either a knowledge of the geoid at the millimetric level or a change of the definition of TT (c.f. section II.3.3). In my opinion it is unlikely that the geopotential on the surface of the Earth will be determined with sufficient accuracy for such applications, therefore future reference clocks are likely to operate in space with either TCG or a newly defined TT as the reference coordinate time scale.

Part III presents two original experiments which test the theory of special relativity using state-of-the-art time metrology. The results set the most stringent limits for the anisotropy of the one-way speed of light published up to date (c.f. table III.1). The GPS test (section III.1) is likely to be slightly improved within the next decade when SA will be switched off definitely on all GPS satellites. In the longer term new space missions with highly stable and accurate onboard clocks and two way optical and/or microwave links should provide new opportunities not only for tests of special relativity but also for measurements of
the gravitational red shift and the Shapiro delay predicted by general relativity. Much will depend on the orbits chosen for such missions as, in general, low terrestrial orbits are not favourable for tests of relativity.

Another important aspect of this thesis is, in my opinion, the introduction of a vocabulary and of a system of definitions and notations for relativistic time metrology, in particular of a notation that explicitly distinguishes the unit of proper time from the scale units of coordinate time scales (section 1.2.1 and appendix). Although, to specialists this might seem unnecessary, a clear vocabulary and system of definitions and notations would, if generally used, enhance communication and understanding between scientists working in different fields. This in turn supports collaboration and "global" studies which, as shown in this thesis, can be sources of new ideas and developments.
APPENDIX

Quantity Calculus in Relativistic Time Metrology

The distinction between proper and coordinate quantities in general relativity (chapter 1.1.1) and its application to the concepts of the metrology of time (chapter 1.2.1) leads to the introduction of several units for time: the SI second "s" for proper time and the time coordinate scale units "sRF". This situation is different from the Newtonian case where there was only one temporal quantity (absolute time) and therefore a single unit was in principle sufficient.

The purpose of this appendix is to try and clarify this new situation using the concepts and principles of quantity calculus pioneered by Maxwell and developed by Wallot in the 1920s (see De Boer 1994/95 for a comprehensive introduction). Although not indispensable for the practical application of the results presented in this thesis, the investigation into quantity calculus in this chapter is, in my view, helpful for a better understanding of the "co-existence" of the different quantities and units and the interpretation of expressions like, for example, (1.2.7) and (1.2.8).

A.1 Elements of Quantity Calculus

A.1.1 Quantities, Units, Numerical Values

The concept of a quantity was first explicitly introduced by James Clerk Maxwell in his 1873 Treatise on Electricity and Magnetism (Maxwell 1873). In the opening sentences of the Preliminary to the Treatise he writes:

"Every expression of a quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind as the quantity to be expressed, which is
taken as a standard of reference. The other component is the number of times the standard is to be taken to make up the required quantity. The standard quantity is technically called the *Unit* and the number is called the *Numerical Value of the quantity.*

In the notation generally used when discussing quantities and units this may be expressed as

\[
Q = \{Q\}[Q]
\] (A.1)

where \(Q\) represents the quantity, \(\{Q\}\) the numerical value and \([Q]\) the unit.

In present day terminology the "unit" is usually associated with the ideal reference quantity (the definition) whereas the term "standard of reference" generally designates a realisation of the unit.

Defining two different units \([Q]'\) and \([Q]'\) of the same kind their relationship is written as

\[
[Q]' = k[Q]''
\] (A.2)

where \(k\) is a numerical constant (a pure number) called the conversion factor. Then the same quantity \(Q\) can be expressed by either of the two units giving

\[
\{Q\}'[Q]' = \{Q\}''[Q]''
\] (A.3)

which leads to the numerical value equation

\[
\{Q\}' = \frac{1}{k}\{Q\}''.
\] (A.4)

Note the inverse proportionality relation between (A.2) and (A.4) and the invariance of the quantity \(Q\) (expressed in the form (A.1)) with respect to the choice of units.
A.1.2 Quantity Equations

When expressing physical phenomena by mathematical relationships it is usually assumed that the symbols, expressions and equations have a general character, quite independent of the choice of units. Therefore the symbols used in mathematical equations which represent physics are not pure numbers, or numerical values (in this case the equations would not be independent of the choice of units) but quantities that can be expressed in the form of (A.1). The calculus with quantities (quantity calculus) rather than pure numbers was first explicitly introduced by Wallot (1926):

"The calculus with quantities, and not that with numerical values, is for most - or at least for very many - people the natural situation" (Wallot 1957).

So, according to Wallot, the mathematical equations used to represent physics give relations between quantities which can be displayed explicitly by substituting (A.1) for the symbols used. The relation between the units used (the unit equation) then provides the relation between the numerical values (the numerical value equation). This is demonstrated for the examples of addition and multiplication of quantities:

For the addition of two quantities A and B one writes the quantity equation

\[ A + B = C \]  \hspace{1cm} (A.5)

or, substituting (A.1),

\[ \{A\}[A] + \{B\}[B] = \{C\}[C]. \]  \hspace{1cm} (A.6)

Choosing units that obey the relation (the unit equation)

\[ [A] = [B] = [C] \]  \hspace{1cm} (A.7)

the numerical value equation

\[ \{A\} + \{B\} = \{C\} \]  \hspace{1cm} (A.8)
is obtained. Note that under condition (A.7) the quantity equation (A.5) has the same form as the numerical value equation (A.8). This is the case when a coherent system of units is used i.e. when the unit equations do not contain any conversion factors other than 1.

Similar equations can be written for the multiplication of quantities A and B. The quantity equation is

\[ A \cdot B = C \]  \quad \text{(A.9)}

or

\[ [A][A] \cdot [B][B] = [C][C]. \]  \quad \text{(A.10)}

Choosing units such that

\[ [A] \cdot [B] = [C] \]  \quad \text{(A.11)}

gives the numerical value equation

\[ \{A\} \cdot \{B\} = \{C\}. \]  \quad \text{(A.12)}

Again the numerical value equation (A.12) has the same form as the quantity equation (A.9), as the chosen units are coherent (the unit equation (A.11) does not contain any conversion factors other than 1).

So in the perspective of quantity calculus mathematical physics consists of laws and theories of physics expressed in the form of mathematical expressions which are independent of the choice of units (quantity equations). A particular choice of units then provides the corresponding unit equations which in turn allow the derivation of numerical value equations. In experiments the units are realised and used for measurements the results of which can then be compared to the numerical value equations obtained from theory and the unit equations.
A.2 Application to Relativistic Time Metrology

A.2.1 Quantities and Units

In chapter 1.1.1 proper and coordinate quantities were introduced, in particular proper and coordinate time which are of interest for the metrology of time. In the following a more detailed description of these quantities and the corresponding units is given. For a better understanding the concept of coordinate time is first examined in the context of special relativity and then generalised to the more complex case of general relativity in the weak field approximation.

Proper Time

Proper Time \( t \) is defined along a particular time-like (\( ds^2<0 \)) world-line. A proper time interval \( \Delta t_A \) between two events that lie on some world-line \( A \) can be obtained from the readings of a clock that moves along this world-line. The proper time interval between two events and along a world line \( A \) can be expressed as

\[
\Delta t_A = \{ \Delta t_A \} [\tau].
\]  

(A.13)

Note that this quantity is a function of the two events and the world-line \( A \) as, according to general relativity, for the same two events and two different paths \( A \) and \( B \) \( \Delta t_A \neq \Delta t_B \) in general.

The unit of proper time recommended by the IAU (1991) is the second of the international system of units (SI) (see section 1.4.1 or (BIPM 1991)), denoted by the symbol "s".

Coordinate Time in Special Relativity

Coordinate time in special relativity can be defined by an array of ideal clocks distributed throughout space and linked to each other via the exchange of light signals. It is made sure that all clocks are at rest with respect to each other, meaning that the return travel
time of a signal between any two clocks remains constant. The clocks are then synchronised using the Einstein synchronisation convention (Einstein 1905). After synchronisation these clocks (henceforth called “coordinate clocks”) provide coordinate time \( u \), with its scale unit denoted \([u]\). Note that there exist, in principle, an infinite number of such clock arrays - each associated with a different coordinate system - moving at a constant velocity with respect to each other.

If the chosen unit of proper time is the SI second “s” the corresponding scale unit of coordinate time will be denoted “s\(_{\text{RF}}\)”, where RF denotes the particular coordinate system.

The proper time interval between two events measured by a clock \( A \), and the coordinate time interval between the same two events in some space-time coordinate system are related by

\[
\{\Delta t_A\}[\tau] = \{f\}[f] \cdot \{\Delta u\}[u]
\]

(A.14)

where \( f \) is a function of the coordinate velocity \( v' = dx_A'/du \) of \( A \).

If \( A \) is colocated (moving along the same world-line) with a coordinate clock of the coordinate system in question (\( v = 0 \)) and the units are chosen such that \([f] = [\tau]/[u]\) then, by definition, the numerical value \( \{f\} = 1 \).

But consider another clock \( B \) which is not at rest with respect to the coordinate clocks. It measures a proper time interval \( \Delta t_B \) between two events that lie on its world-line. These two events are colocated with two (different) coordinate clocks. The difference between the readings of these two clocks when the respective events take place is defined as the coordinate time interval \( \Delta u \) between the two events and related to \( \Delta t_B \) by (A.14). But in this case again choosing units such that \([f] = [\tau]/[u]\) leads to

\[
\{f\} = \sqrt{1 - \frac{v'^2}{c^2}}
\]

(A.15)
where \( v \) is the coordinate velocity of \( B \).

In summary, coordinate time in special relativity can be defined by an array of coordinate clocks that are at rest (in the sense defined above) with respect to each other and synchronised according to the Einstein synchronisation convention. The coordinate time interval between two events might differ from the proper time interval between the same two events (obtained from a clock on whose world-line they lie) and therefore the two quantities should be distinguished explicitly. The scale unit of coordinate time [\( u \)] (e.g. "sRF") indicates that the quantity refers to the difference of readings of one and the same or two different coordinate clocks of a space-time coordinate system (the system RF) established according to the conventions given above.

**Coordinate Time in General Relativity**

When trying to generalise the special relativistic concept of coordinate time to the case of general relativity one encounters several problems:

(i) In the general case the Einstein synchronisation convention becomes non-transitive, meaning that when clocks \( A \) and \( B \) are synchronised, and clocks \( B \) and \( C \) are synchronised, clocks \( A \) and \( C \) are not, in general, synchronised.

(ii) The relative rate of two distant clocks (as determined, for example by the exchange of light signals) is a function of the gravitational fields at the respective positions of the clocks, which implies that syntonisation as well as synchronisation is required.

(iii) In general the metric coefficients, and therefore the travel time of a light signal, can be functions of space and time, so a simple definition of two clocks "at rest" with respect to each other (as in the case of special relativity) is no longer possible.

Within the solar system the weak-field approximation of general relativity is usually sufficient (see section 1.1.4). In this approximation and considering present and near future time transfer and syntonisation accuracies the problems mentioned above can be solved without too much difficulty.

For the applications considered here the temporal variations of the metric coefficients (due, for example, to variations of the Earth's gravitational field) are small (amplitudes \(< 10^{-16} \)
see section II.1.2 or Wolf & Petit 1995) and their effect on the return travel time of a light signal can be neglected or calculated to sufficient accuracy, so the special relativistic criteria for clocks at rest with respect to each other can be used.

The relative rate of two distant clocks can be calculated using the metric coefficients obtained from theory. For two ideal clocks this rate is equal to $1 + \varepsilon$ where $\varepsilon$ is a small correction $<10^{-9}$ in the vicinity of the Earth. To correct for this rate shift the position of the clocks, and the gravitational potential needs to be known with sufficient accuracy. This is the case at present, where all such corrections can be calculated at sufficient accuracy compared to current and expected clock stabilities (see section II.1 or Wolf & Petit 1995).

To ensure transitivity using the Einstein synchronisation convention it is necessary that the coordinate travel-time of a light signal exchanged by the two clocks should be the same in both directions. This is not the case when the clocks are accelerated (e.g. for two clocks on a rotating disc or on the surface of the Earth) or when the gravitational delay of the light signal is different for the two paths (when the metric includes terms in $g_{0i}$). In the weak-field approximation and at the accuracies required in present and near future time metrology, all $g_{0i}$ terms of the metric in a non-rotating (with respect to distant extragalactic objects) coordinate system can be neglected. Hence for clocks that are at rest in such a coordinate system the Einstein synchronisation convention can be used. For clocks that are not at rest in the non-rotating coordinate system (i.e. all clocks used in practice, on the surface of the Earth or on board terrestrial satellites) a small correction has to be applied to the synchronisation (the so called Sagnac correction) which requires the knowledge of the positions and velocities of the clocks. This correction can be calculated at sufficient accuracy for all present and near future applications (c.f. section II.2.2 or Petit & Wolf 1994).

So in the weak-field approximation of general relativity and at the accuracies required in present and near future time metrology, coordinate time can be pictured as an array of clocks that are at rest with respect to each other (by the definition given for special relativity) and synchronised using the Einstein convention. Two additional conditions have to be applied: (i) The array should show no rotation with respect to the fixed stars.
(ii) As two clocks of this array will in general display a relative rate, a particular clock, or a
subset of the clocks, has to be chosen (by convention) to which all other clocks will be
syntonised using the small corrections obtained from the metric.

In practice such coordinate times are realised using clocks that are not at rest with
respect to this ideal array of clocks. Consequently additional corrections to the synchronisation
and the syntonisation of such clocks need to be applied. For example, Terrestrial Time (TT) is
defined as a geocentric coordinate time with its scale unit equal to the SI second on the
rotating geoid. The clocks participating in the realisation of TT are fixed on the surface of the
Earth i.e. not at rest in the non-rotating system. These clocks are synchronised taking into
account the correction due to their motion in the non-rotating system (the so called Sagnac
correction) and syntonised with clocks on the rotating geoid using the appropriate metric.
These tasks are performed routinely at the BIPM in the construction of TAI and TT(BIPM)
which are both realisations of the geocentric coordinate time TT.

Consider again a clock A which measures a proper time interval $\Delta \tau_A$ between two
events that lie on its world-line. These two events are colocated with one and the same or two
different coordinate clocks. The difference between the readings of the coordinate clock(s)
when the respective events take place is defined as the coordinate time interval $\Delta \mu$ between the
two events. Similarly to (A.14)

$$\{\Delta \tau_A\} = \{g\} \{g\} \cdot \{\Delta \mu\} \{\mu\}$$  \hspace{1cm} (A.16)

where $g$ is now not only a function of the coordinate velocity $v$ of A but also of the
gravitational potential at the location of A, and has to be integrated along the path of A.

So, in contrast to special relativity, for the case where $v = 0$ and choosing units such
that $[g]=[\tau]/[\mu]$, in general, $\{g\} \neq 1$. The equality only holds when A is colocated with the
(conventionally chosen) clock to which all others are syntonised. This is the case, for example,
for A on the rotating geoid and $\nu=T$. So analogously to the case of special relativity the scale unit of coordinate time $\{\mu\}$ (e.g.
"srF") indicates that the quantity refers to the difference of readings of one and the same or two
different coordinate clocks of a space-time coordinate system (the system RF) established according to the conventions given four paragraphs earlier.

### A.2.2 Quantity Equations

Using the principles of quantity calculus (section A.1) and the quantities of proper and coordinate time with their appropriate units (as defined in section A.2.1) equations like (1.2.7) and (1.2.8) can be readily interpreted as quantity equations.

For two infinitesimally close events separated by a time-like interval \((ds^2<0)\) the metric equation (I.1.2) in an arbitrary coordinate system can be written in the form

\[
d\tau = g(u, x, \dot{x}) du \tag{A.17}
\]

where a dot signifies differentiation with respect to \(u\) and the function \(g\) is determined by the metric tensor and the coordinates of the two events.

Interpreting this as a quantity equation, and substituting (A.1) gives

\[
\{d\tau\}[\tau] = \{g(u, x, \dot{x})\}[g][du][u] \tag{A.18}
\]

where the unit of proper time \([\tau]\) and the scale unit of coordinate time \([u]\) were defined in the previous section. Choosing the unit of \(g\) provides the unit equation, for example

\[
[\tau] = [g][u] , \tag{A.19}
\]

which leads to the numerical value equation

\[
\{d\tau\} = \{g(u, x, \dot{x})\}[du] . \tag{A.20}
\]

For an ideal clock, \(A\), (A.18) is the equivalent of (I.2.8) interpreted as a quantity equation with the involved quantities expressed using (A.1).
Defining two different space-time coordinate systems A and B the relation between their coordinate times is expressed by (1.2.7) which, interpreted as a quantity equation, is written as

\[
\frac{\{du_A\}}{\{du_B\}} = \{1+g(x^\lambda)\}[s] \tag{A.21}
\]

and where, analogously to the previous example, the numerical value equation can be obtained from the unit equation.

So the expressions used in time metrology within the framework of general relativity can be interpreted as quantity equations in accordance with the principles of quantity calculus. The different quantities involved (proper and coordinate time) should be distinguished explicitly in order to avoid confusion which at present day levels of observational accuracy can lead to non-negligible errors. It is the authors opinion that the consistent use of different notations for the unit of proper time "s" and the coordinate time scale-units "sRF" (as done throughout the thesis) can help to avoid such errors whilst clarifying, at least for the non-specialist, the concepts involved.
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