Point-primitive generalised hexagons and octagons

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ABSTRACT. The only known examples of finite generalised hexagons and octagons arise from the finite almost simple groups of Lie type G_2, $^3\mathsf{D}_4$, and $^2\mathsf{F}_4$. These groups act transitively on flags, primitively on points, and primitively on lines. The best converse result prior to the writing of this paper was that of Schneider and Van Maldeghem (2008): if a group G acts flag-transitively, point-primitively, and line-primitively on a finite generalised hexagon or octagon, then G is an almost simple group of Lie type. We establish a stronger result by removing any symmetry hypothesis on lines: a group acting primitively on the points of a finite generalised hexagon or octagon is almost simple of Lie type.

1. Introduction

Generalised polygons were introduced by Tits [26] in an attempt to find geometric models for simple groups of Lie type. In particular, the group $\mathsf{PSL}(3, q)$ is admitted by the Desarguesian projective plane $\mathsf{PG}(2, q)$; the groups $\mathsf{PSp}(4, q)$, $\mathsf{PSU}(4, q)$, $\mathsf{PSU}(5, q)$ act on certain generalised quadrangles; and $\mathsf{G}_2(q)$, $^3\mathsf{D}_4(q)$, $^2\mathsf{F}_4(q)$ arise as automorphism groups of two generalised hexagons and a generalised octagon, respectively (up to point–line duality). These generalised polygons are called the *classical* generalised polygons [27, Chapter 2], and they are the examples that have the greatest degree of symmetry: their automorphism groups act primitively and distance-transitively on both points and lines, and transitively on flags. Although there exist many non-classical projective planes and generalised quadrangles, the existence of non-classical generalised hexagons and octagons remains an open question. In 1985, Cohen and Tits [7] proved the uniqueness of the generalised hexagons of orders $(2, 2)$ and $(2, 8)$ (up to duality in the former case). In 2009, De Medts and Van Maldeghem [9] proved the uniqueness of a generalised hexagon of order $(3, 3)$ subject to a certain additional assumption, namely the existence of a sub-hexagon of order $(1, 3)$. In 2015, De Bruyn [8] established the uniqueness of a generalised octagon of order $(2, 4)$ containing a sub-octagon of order $(2, 1)$, and Cohen et al. [6] proved the uniqueness of a generalised octagon of order $(2, 4)$ under a certain local symmetry assumption. Buekenhout and Van Maldeghem [5] showed that distance-transitivity on points implies primitivity on points for a generalised hexagon or octagon, and that there exist no point-distance-transitive generalised hexagons or octagons apart from the classical ones. However, it is not yet known whether a non-classical generalised hexagon or octagon could have an automorphism group acting primitively on points or lines. We make progress towards resolving this question by showing that if a generalised hexagon or octagon admits an automorphism group G that acts primitively on points, then G must be an almost simple group of Lie type, regardless of its action on lines.

A *generalised d-gon* is a point–line geometry whose bipartite incidence graph has diameter d and girth $2d$. An *automorphism* (or *collineation*) of a generalised d-gon is a permutation of the point set,
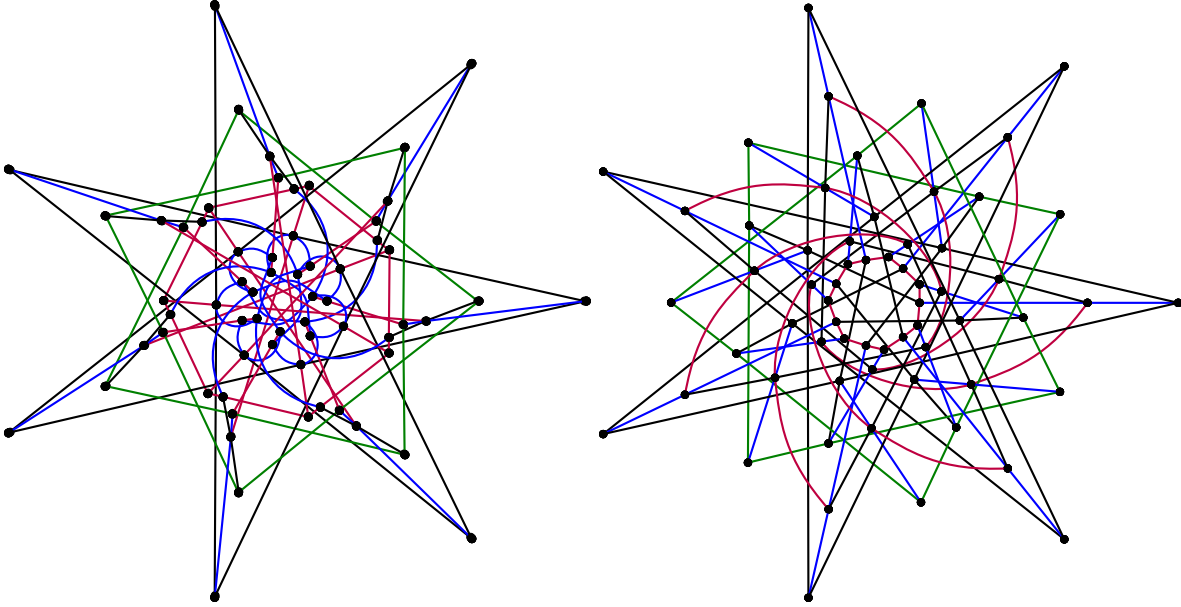
FIGURE 1. The two generalised hexagons of order $(2, 2)$. Each is the point–line dual of the other. There are $(2 + 1)(2^4 + 2^2 + 1) = 63$ points and lines, and each point (respectively line) is incident with exactly $2 + 1 = 3$ lines (respectively points). The Dickson group $\mathsf{G}_2(2)$ acts primitively and distance-transitively on both points and lines. These pictures were inspired by a paper of Schroth [23].

together with a permutation of the line set, such that incidence is preserved. To exclude trivial cases, we require that the geometry is *thick*, namely that each line contains at least three points and each point lies on at least three lines. In this case, there are constants $s \geqslant 2$ and $t \geqslant 2$ such that each line contains exactly $s+1$ points and each point lies on exactly $t+1$ lines, and (s, t) is called the *order* of the generalised d-gon [27, Corollary 1.5.3]. For illustration, the two generalised 6-gons of order $(2, 2)$ are shown in Figure 1. The celebrated theorem of Feit and Higman [12] shows that a thick generalised d-gon can only exist when $d \in \{2, 3, 4, 6, 8\}$, and as generalised 2-gons are simply geometries whose incidence graphs are complete bipartite, they can also be regarded as trivial. This leaves four distinct types of thick generalised polygon. A generalised 3-gon is precisely a projective plane, and it has long been conjectured [10, p. 208] that the mild condition of transitivity on the set of points characterises the classical projective plane $\mathrm{PG}(2, q)$. Currently the best result is to due to Gill [15], who proved that all minimal normal subgroups of a group G acting transitively on a non-classical projective plane are elementary abelian. Gill [14] also proved that the Sylow 2-subgroups of G are cyclic or generalised quaternion, and, in particular, that the only possible insoluble composition factor is A_5. Kantor [18] showed that a group acting primitively on a non-classical projective plane contains a cyclic normal subgroup of prime order acting regularly on points. This leads to severe number-theoretic restrictions on the possible size of the projective plane, and Thas and Zagier [25] have shown that these restrictions are not satisfied for any non-classical projective plane with fewer than 4×10^{22} points.

Both Kantor's results, and the characterisation by Buekenhout and Van Maldeghem, rely heavily on fundamental results regarding the structure of primitive permutation groups. While Buekenhout and Van Maldeghem show that primitivity is a consequence of distance-transitivity, it is not necessarily the case that a flag-transitive group of automorphisms of a generalised polygon with $d \geqslant 4$ is primitive on points and lines: the generalised quadrangles (4-gons) arising from transitive hyperovals in $\mathrm{PG}(2, 4)$ and $\mathrm{PG}(2, 16)$ admit flag-transitive, point-primitive, but line-imprimitive automorphism groups. Bamberg et al. [1] proved that if G is a group of automorphisms of a finite thick generalised quadrangle acting primitively on both points and lines, then G is almost simple. Moreover, if G is also flag-transitive then G is almost simple of Lie type. Schneider and Van Maldeghem [22] had previously proved the following result for generalised hexagons (6-gons) and octagons (8-gons).

THEOREM 1.1 (Schneider and Van Maldeghem [**22**, Theorem 2.1]). *Let $\mathcal{S}$ be a finite thick generalised hexagon or octagon. If $G \leqslant \mathrm{Aut}(\mathcal{S})$ acts flag-transitively, point-primitively, and line-primitively, then G is an almost simple group of Lie type.*

In this paper, we strengthen the result of Schneider and Van Maldeghem by showing that point-primitivity alone is sufficient for the same conclusion. That is, we prove the following theorem.

THEOREM 1.2. *Let $\mathcal{S}$ be a finite thick generalised hexagon or octagon. If $G \leqslant \mathrm{Aut}(\mathcal{S})$ acts point-primitively, then G is an almost simple group of Lie type.*

Theorem 1.2 is proved by analysing the possible O'Nan–Scott types for G considered as a primitive permutation group on the point set of $\mathcal{S}$. We first prove that G must be an almost simple group, that is, G must have a unique minimal normal subgroup T, and T must be a nonabelian simple group. It has been shown by Buekenhout and Van Maldeghem [**4**] that T cannot be a sporadic simple group, and we prove that T is also not an alternating group. We note that our arguments do not use the Classification of Finite Simple Groups (CFSG), though the CFSG is used to conclude that T must be a simple group of Lie type. To work around the absence of a symmetry condition on lines, we required some new ideas, especially in the almost simple case with T an alternating group. In particular, we use Jordan's Theorem [**16, 19**] about primitive groups with cycle type 3^1 or 3^2, and Neumann's Separation Lemma [**3**]; two powerful yet elementary results in permutation group theory.

Combining Theorem 1.2 with the previously mentioned results for projective planes and generalised quadrangles yields the following unified result for point- and line-primitive generalised polygons. The proof is summarised in Section 3.4, but we note here that the result of Bamberg et al. [**1**] for generalised quadrangles can be recast as below without the assumption of flag-transitivity. Recall also that the *socle* of a group G, denoted by $\mathrm{soc}(G)$, is the subgroup of G generated by the minimal normal subgroups.

COROLLARY 1.3. *Let $\mathcal{S}$ be a finite thick generalised d-gon, $d \geqslant 3$, of order (s, t). If $G \leqslant \mathrm{Aut}(\mathcal{S})$ acts point-primitively and line-primitively, then one of the following holds:*
  (i) *G is almost simple of Lie type,*
  (ii) *$d = 3$ and G is soluble,*
(iii) *$d = 4$, G is almost simple with $\mathrm{soc}(G) \cong A_n$, $n \geqslant 5$, and $\gcd(s, t) = 1$.*

2. Background

We first recall some basic facts about generalised polygons, referring the reader to the book by Van Maldeghem [**27**] for proofs. Let $\mathcal{S}$ be a generalised d-gon, with point set $\mathcal{P}$, and line set $\mathcal{L}$. The requirement that the incidence graph of $\mathcal{S}$ have diameter d and girth $2d$ is equivalent to the condition that
  (i) there are no ordinary k-gons in $\mathcal{S}$ for $2 \leqslant k < d$, and
  (ii) any two elements of $\mathcal{P} \cup \mathcal{L}$ are contained in some ordinary d-gon.
In particular, if $d > 4$ then there are no triangles (3-gons) and no quadrangles in the geometry. If $\mathcal{S}$ is thick with order (s, t) (as defined in Section 1), then we have the following properties.
  (PH) If $\mathcal{S}$ is a generalised hexagon, then st is a square, and $|\mathcal{P}| = (s + 1)(s^2 t^2 + st + 1)$.
  (PO) If $\mathcal{S}$ is a generalised octagon, then $2st$ is a square, and $|\mathcal{P}| = (s + 1)(s^3 t^3 + s^2 t^2 + st + 1)$.

Given points x and y of a generalised hexagon or octagon, we write $x \sim y$ if $x \neq y$ and x, y lie on a common line, and in this case we denote this (unique) line by $\langle x, y \rangle$. For several of the arguments in the proof of Theorem 1.2, we begin by constructing (or otherwise deducing the existence of) an automorphism g such that $x \sim xg$ for some point x. The idea is then to obtain a contradiction by having g fix the line $\langle x, xg \rangle$ while showing that, on the other hand, the stabilisers of x and $\langle x, xg \rangle$ are equal. The following lemma is extremely useful for these sorts of arguments.

LEMMA 2.1. *Let $\mathcal{S}$ be a finite thick generalised hexagon or octagon of order (s, t), and let $\mathcal{P}$ denote the set of points of $\mathcal{S}$.*
  (i) *If $\gcd(s, t) \neq 1$ and $g \in \mathrm{Aut}(\mathcal{S})$ is fixed-point free, then there exists $x \in \mathcal{P}$ such that $x \sim xg$.*
  (ii) *Let $x \in \mathcal{P}$ and $g_1, g_2 \in \mathrm{Aut}(\mathcal{S})$ such that $x \sim xg_1$, $x \sim xg_2$, and $xg_1g_2 = xg_2g_1$. If $xg_1g_2 \neq x$, then x, xg_1, xg_2 all lie on a common line.*
(iii) *If $H \leqslant \mathrm{Aut}(\mathcal{S})$ is transitive on $\mathcal{P}$, then the centraliser of H in $\mathrm{Aut}(\mathcal{S})$ is intransitive on $\mathcal{P}$.*

Type	Description
AS (almost simple)	$M \cong T \leqslant G \leqslant \mathrm{Aut}(T)$.
TW (twisted wreath)	$M \cong T^k$, $k \geqslant 2$, acts regularly on Ω.
SD (simple diagonal)	$M \cong T^k$, $k \geqslant 2$, M_ω ($\omega \in \Omega$) is a full diagonal subgroup of M, $|\Omega| = |T|^{k-1}$, and G acts primitively on the set of k simple direct factors of M.
CD (compound diagonal)	$\Omega = \Gamma^\ell$ and $G \leqslant H$ wr S_ℓ, $H \leqslant \mathrm{Sym}(\Gamma)$ primitive of type SD, $\mathrm{soc}(H) \cong T^{k/\ell}$, $k \geqslant 2$ and $k/\ell \geqslant 2$; G acts transitively on the simple direct factors of $M \cong T^k$.
PA (product action)	$\Omega = \Gamma^k$ and $G \leqslant H$ wr S_k, $H \leqslant \mathrm{Sym}(\Gamma)$ primitive of type AS, $\mathrm{soc}(H) \cong T$; G acts transitively on the simple direct factors of $M \cong T^k$, $k \geqslant 2$.

TABLE 1. Five of the possible O'Nan–Scott types of primitive groups $G \leqslant \mathrm{Sym}(\Omega)$. Here M is the unique minimal normal subgroup of G, and T denotes a nonabelian finite simple group.

(iv) *Let $x, y_1, y_2 \in \mathcal{P}$ such that $x \sim y_1$ and $x \sim y_2$, and let $g \in \mathrm{Aut}(\mathcal{S})$ such that $xg \neq x$. If g fixes y_1 and y_2, then x, y_1, y_2, xg all lie on a common line.*

PROOF. Part (i) is proved in [**24**, Corollary 5.2 and Lemma 6.2], and parts (ii) and (iii) are proved in [**22**, Lemmas 3.2(iii) and (iv)]. It remains to prove (iv). First suppose, towards a contradiction, that $y_1 \nsim y_2$. Since $x \sim y_1$ and $x \sim y_2$ and g is an automorphism, we have $xg \sim y_1 g = y_1$ and $xg \sim y_2 g = y_2$. If xg lies on the line $\langle x, y_1 \rangle$, then x, xg, y_2 form a triangle, which is impossible. Similarly, xg cannot lie on the line $\langle x, y_2 \rangle$. However, this implies that x, y_1, xg, y_2 form a quadrangle, which is also impossible. Hence $y_1 \sim y_2$, and so x, y_1, y_2 all lie on the common line $\lambda = \langle y_1, y_2 \rangle$ because $\mathcal{S}$ contains no triangles. Since g fixes y_1 and y_2, it fixes λ setwise; that is, $\lambda g = \langle y_1 g, y_2 g \rangle = \langle y_1, y_2 \rangle = \lambda$. Hence xg also lies on λ. $\square$

3. The proof

We now work under the following hypothesis.

HYPOTHESIS 3.1. *Let $\mathcal{S}$ be a finite generalised hexagon or octagon, with point set $\mathcal{P}$, and order (s, t), where both s and t are at least 2. Suppose that $G \leqslant \mathrm{Aut}(\mathcal{S})$ acts primitively on $\mathcal{P}$, and let M be a minimal normal subgroup of G.*

Recall that the structure of a primitive permutation group is described by the O'Nan–Scott Theorem. We follow the version of the O'Nan–Scott Theorem given by Praeger [**21**, Section 5], which splits the primitive permutation groups into eight types. It was shown by Schneider and Van Maldeghem [**22**, Lemma 4.2(i)] that if Hypothesis 3.1 holds then G cannot have O'Nan–Scott type HA (affine), HS (holomorph simple), or HC (holomorph compound). Indeed, if G has one of these three types then the centraliser of M in G is transitive on $\mathcal{P}$, which is impossible by Lemma 2.1(iii). The remaining five O'Nan–Scott types are described in Table 1, and we note that in these cases G has a unique minimal normal subgroup M, and M has the form

$$M \cong T^k, \quad \text{where } T \text{ is a nonabelian finite simple group and } k \geqslant 1.$$

Here we also recall that (in general) a permutation group G on a set Ω is said to be *semiregular* if, for every $x \in \Omega$, the stabiliser $G_x := \{g \in G \mid xg = x\}$ of x is the trivial subgroup. The group is said to be *regular* if it is semiregular and transitive, and in this case the cardinalities of G and Ω are equal.

We show in Section 3.1 that G cannot have O'Nan–Scott type PA or CD under Hypothesis 3.1, and in Section 3.2 we show that G cannot have type SD or TW. This leaves the possibility that G is an almost simple group (type AS). An existing result of Buekenhout and Van Maldeghem [**4**] shows that an almost simple group with socle a sporadic simple group cannot act primitively (or even transitively) on the points of a finite thick generalised hexagon or octagon. Thus, to complete the proof of Theorem 1.2, it remains to show that the socle of G cannot be an alternating group. This is done in Section 3.3.

3.1. Types PA and CD.
Suppose that G has O'Nan–Scott type PA or CD under Hypothesis 3.1. Then $\mathcal{P}$ can be identified with a Cartesian product Γ^ℓ in such a way that G embeds in the wreath product H wr S_ℓ, where H is a primitive subgroup of $\mathrm{Sym}(\Gamma)$ and G induces a transitive subgroup of S_ℓ. Write

$N = \mathrm{soc}(H)$ and note that $N^\ell = \mathrm{soc}(G) = M$. Let α be an arbitrary element of Γ, and consider the point $x \in \Gamma^\ell$ represented by the ℓ-tuple $(\alpha, \ldots, \alpha)$. Our argument is in three steps, marked (i)–(iii) below. In step (i) we show that there exists an element y collinear with x such that y is represented by the ℓ-tuple $(\beta, \alpha, \ldots, \alpha)$ for some $\beta \in \Gamma \backslash \{\alpha\}$. In step (ii) we show that $G_\lambda = G_x$, where λ is the line incident with both x and y. Finally, in step (iii) we construct an automorphism g that fixes λ but not x, thereby obtaining a contradiction and hence proving that G in fact cannot have type PA or CD.

(i) *There exists a point y such that $y \sim x$ and $y = (\beta, \alpha, \ldots, \alpha)$ for some $\beta \in \Gamma \backslash \{\alpha\}$.* This is established by Schneider and Van Maldgehem [**22**, proof of Lemma 4.2(ii)] without using flag-transitivity, but we include a proof to make it clear that flag-transitivity is not needed. Let $y = (\beta_1, \beta_2, \ldots, \beta_\ell)$ be a point collinear with x, and suppose that y has i entries different from α. If $i = 1$ then, without loss of generality, y differs from x in the first component and we are done. Now assume that $i \geqslant 2$. Then, without loss of generality, $\beta_1 \neq \alpha \neq \beta_2$ and, if $i < \ell$, $\beta_{i+1} = \cdots = \beta_\ell = \alpha$. Since G has type PA or CD, the primitive group H has type AS or SD (respectively), so $N = \mathrm{soc}(H)$ is not regular and it follows from [**2**, Corollary 2.2(a)] that the only point of Γ fixed by N_α is α. In particular, $N_\alpha \neq N_{\beta_1}$, and hence there exists $g \in N_\alpha$ such that $\beta_1' := \beta_1 g \neq \beta_1$. Set $\bar{g} = (g\ 1 \ldots 1)$ and $y' = y\bar{g} = (\beta_1', \beta_2, \ldots, \beta_\ell)$. Then $\bar{g} \in M \leqslant G$, and $\bar{g}$ fixes x so $x \sim y'$. Similarly, we can choose $h \in N_{\beta_2}$ such that $\alpha' := \alpha h \neq \alpha$, and we set $\bar{h} = (1\ h\ 1 \ldots 1) \in M$. Then $x \neq x\bar{h} = (\alpha, \alpha', \alpha, \ldots, \alpha)$. That is, $x\bar{h}$ differs from x in only one component. To complete the proof of (i), it suffices to check that $x \sim x\bar{h}$. Since $\bar{h}$ fixes both y and y' but not x, Lemma 2.1(iv) implies that $x, x\bar{h}, y, y'$ all lie on a common line. In particular, $x \sim x\bar{h}$.

(ii) *$G_\lambda = G_x$, where $\lambda := \langle x, y \rangle$ with y given by* (i). The following argument is also adapted from [**22**, proof of Lemma 4.2(ii)]. We show that $G_x \leqslant G_\lambda$, which implies that $G_\lambda = G_x$ because G_x is a maximal subgroup of the primitive group G and because $G_\lambda \neq G$ (since G acts transitively on the set of all points of $\mathcal{S}$, it cannot stabilise a single line). Let $g \in G_x$ and write $g = (g_1, \ldots, g_\ell)\sigma \in G$, where $g_1, \ldots, g_\ell \in H$ and $\sigma \in S_\ell$, so that $\alpha g_i = \alpha$ for all $i \in \{1, \ldots, \ell\}$. If g fixes y then it fixes λ setwise as required, so assume that $yg \neq y$. We show that yg lies on λ, which also implies that g fixes λ because then $\lambda g = \langle x, y \rangle g = \langle xg, yg \rangle = \langle x, yg \rangle = \lambda$. First suppose that $1\sigma \neq 1$. Then all components of yg are equal to α, except the component in position 1σ, which is equal to βg_1. Choose $h, h' \in N$ such that $\alpha h = \beta$ and $\alpha h' = \beta g_1$, and set $\bar{h} = (h\ 1\ \ldots\ 1)$ and $\bar{h}' = (1\ \ldots\ 1\ h'\ 1\ \ldots\ 1)$, where h' appears in position 1σ of $\bar{h}'$. Then $x \sim x\bar{h} = y$, $x \sim x\bar{h}' = yg$, and $x\bar{h}\bar{h}' = x\bar{h}'\bar{h} \neq x$, so Lemma 2.1(ii) implies that x, y, yg all lie on a common line. That is, yg lies on λ. Now suppose that $1\sigma = 1$. In this case, $yg = (\beta g_1, \alpha, \ldots, \alpha)$. Take any $g' = (g_1', \ldots, g_\ell')\sigma' \in G_x$ such that $1\sigma' \neq 1$. We have just shown that g' fixes λ, so in particular yg' lies on λ. Without loss of generality, we may assume that $1\sigma' = 2$, so that $yg' = (\alpha, \beta g_1', \alpha, \ldots, \alpha)$. Now let $\bar{h}'' = (h'\ 1\ \ldots\ 1)$, where $\alpha h' = \beta g_1$ as above, and let $\bar{h}''' = (1\ h''\ 1\ \ldots\ 1)$, where $\alpha h'' = \beta g_1'$. Then $x\bar{h}'' = yg$, $x\bar{h}''' = yg'$, and $x\bar{h}''\bar{h}''' = x\bar{h}'''\bar{h}'' \neq x$, so Lemma 2.1(ii) implies that yg, yg', x all lie on a common line, and hence yg lies on λ as required.

(iii) *A contradiction: there exists $g \in G_\lambda$ such that $g \notin G_x$.* Choose $g \in G_y$ such that $g = (g_1, \ldots, g_\ell)\sigma$ with $1\sigma = 2$. Such an element exists because the stabiliser of any point in G is transitive on the simple direct factors of M. Let i satisfy $i\sigma = 1$, and note that $i \neq 1$. Then $(\beta, \alpha, \ldots, \alpha) = y = yg = (\beta, \alpha, \ldots, \alpha)g = (\alpha g_i, \beta g_1, \ldots)$, where the components from the third position onwards are of the form αg_j with $j \notin \{1, i\}$. That is, $\alpha g_i = \beta$, $\beta g_1 = \alpha$, and $\alpha g_j = \alpha$ for all $j \notin \{1, i\}$. Next, observe that $xg \neq x$, $xg \neq y$, and $xg \sim y$. Indeed, $xg = (\alpha g_i, \alpha g_1, \alpha, \ldots, \alpha) = (\beta, \alpha g_1, \alpha, \ldots, \alpha)$ is not equal to x or y because $\beta g_1 = \alpha$ and hence $\alpha g_1 \neq \alpha$, and $x \sim y$ implies $xg \sim yg = y$. Choose $h', h'' \in N = \mathrm{soc}(H)$ such that $\beta h' = \alpha$ and $\alpha h'' = \alpha g_1$, and write $\bar{h}' = (h'\ 1 \ldots 1)$ and $\bar{h}'' = (1\ h''\ 1 \ldots 1)$. Then $y\bar{h}' = x$, $y\bar{h}'' = xg$, $\bar{h}'$ and $\bar{h}''$ commute, and $y\bar{h}'\bar{h}'' \neq y$, so Lemma 2.1(ii) implies that xg lies on λ. It follows that $\lambda g = \langle x, y \rangle g = \langle xg, yg \rangle = \langle xg, y \rangle = \lambda$, namely that $g \in G_\lambda$. However, $G_\lambda = G_x$ from (ii), so we have a contradiction because $g \notin G_x$.

To summarise, we have proved the following result.

LEMMA 3.2. *If Hypothesis 3.1 holds then the O'Nan–Scott type of G is not PA or CD.*

3.2. Types SD and TW. We begin with two lemmas, from which it is then deduced that G cannot have O'Nan–Scott type SD or TW under Hypothesis 3.1. For the second lemma, recall again that a permutation group is said to be *semiregular* if every point stabiliser is trivial.

LEMMA 3.3. *Let s and t be positive integers such that $\gcd(s, t) = 1$.*

  (i) *If st is a square then $(1 + s)(1 + st + s^2t^2)$ is not divisible by 4.*
 (ii) *If $2st$ is a square then $(1 + s)(1 + st)(1 + s^2t^2)$ is not divisible by 4.*

PROOF. (i) If 4 divides $(1 + s)(1 + st + s^2t^2)$ then $s \equiv 3 \pmod 4$ because $(1 + st + s^2t^2)$ is odd. However, st is a square and $\gcd(s, t) = 1$, so s must be a square and hence $s \not\equiv 3 \pmod 4$.

(ii) If 4 divides $(1 + s)(1 + st)(1 + s^2t^2)$ then $s \equiv 3 \pmod 4$ because $(1 + st)(1 + s^2t^2)$ is odd, since st is even if $2st$ is a square. In particular, s is odd. However, if s odd is then s must be a square because $2st$ is a square and $\gcd(s, t) = 1$, so $s \not\equiv 3 \pmod 4$. $\square$

LEMMA 3.4. *Suppose that Hypothesis 3.1 holds and that $M \cong T_1 \times \cdots \times T_k$ for some pairwise isomorphic nonabelian finite simple groups $T_1, \ldots, T_k$, with $k \geqslant 2$. Then, if $\gcd(s, t) \neq 1$, there exist distinct $i, j \in \{1 \ldots, k\}$ such that $T_i \times T_j$ is not semiregular on $\mathcal{P}$.*

PROOF. Suppose that $\gcd(s, t) \neq 1$ but that $T_i \times T_j$ is semiregular for all distinct i, j. Then, in particular, T_1 is semiregular. Choose an involution $h \in T_1$, namely an element of order 2. Such an element exists because the nonabelian finite simple group T_1 has even order, by the Feit–Thompson Theorem [13]. Since h does not fix any point, Lemma 2.1(i) says that there exists $x \in \mathcal{P}$ such that $x \sim xh$, and so h fixes the line $\lambda = \langle x, xh \rangle$ setwise. We now show that $G_\lambda = G_x$, which is a contradiction because h does not fix x. We first claim that λ is fixed (setwise) by every element of G_x that does not normalise T_1. Let $g \in G_x$ be such an element. Since $x = xg$ and $x \sim xh$, we have $x = xg \sim xhg = xgg^{-1}hg = xh^g$. Since g does not normalise T_1, there exists $i \neq 1$ such that $T_1^g = T_i$. Thus h^g commutes with h, and moreover, hh^g lies in the semiregular group $T_1 \times T_i$ and hence does not fix x. Lemma 2.1(ii) therefore implies that x, xh, xh^g all lie on a common line. Therefore, xh^g lies on λ, and hence $\lambda g = \langle x, xh \rangle g = \langle xg, xhg \rangle = \langle x, xh^g \rangle = \lambda$. That is, $g \in G_\lambda$ as claimed. Now take $a \in \mathbb{N}_{G_x}(T_1)$ and $b \in G_x \backslash \mathbb{N}_{G_x}(T_1)$. By the claim, both ab and b belong to G_λ, since $ab \notin \mathbb{N}_{G_x}(T_1)$, and hence $a = (ab)b^{-1} \in G_\lambda$. Thus $G_x \leqslant G_\lambda$, and since G_x is a maximal subgroup of G and $G_\lambda \neq G$ (as noted in Section 3.1), it follows that $G_x = G_\lambda$ as required. $\square$

LEMMA 3.5. *If Hypothesis 3.1 holds then the O'Nan–Scott type of G is not SD or TW.*

PROOF. Write $M = T^k$, where T is a nonabelian finite simple group (as in Table 1). If G has type TW then M acts regularly on $\mathcal{P}$, and if G has type SD then $M = R \times T$, where $R := T^{k-1}$ acts regularly on $\mathcal{P}$. In either case, the cardinality of $\mathcal{P}$ is divisible by 4, because the order of every nonabelian finite simple group is divisible by 4 (this is a well-known consequence of the Feit–Thompson Theorem, as explained in [22, pp. 1443–1444]). It therefore follows from Lemma 3.3 and properties (PH) and (PO) in Section 2 that $\gcd(s, t) > 1$, and Lemma 3.4 then contradicts the regularity of M in the TW case, and of R in the SD case provided that $k \geqslant 3$. It remains to consider the case where G has type SD with $k = 2$. Here, T_1 and T_2 are both transitive minimal normal subgroups of M, and they centralise each other. This contradicts Lemma 2.1(iii), and hence this case also cannot occur. $\square$

REMARK 3.6. We note that a more general version of Lemma 2.1(i) is given by Parkinson et al. [20, Theorem 10.2]. It says that, without the assumption $\gcd(s, t) > 1$, every fixed-point free automorphism will either map some point x to a point collinear with x, or it will map some point x to a point at distance 4 from x in the incidence graph of $\mathcal{S}$. This implies the conclusion of Lemma 3.4 without having to assume that $\gcd(s, t) > 1$: if the point xh in the proof is instead at distance 4 from x, then there is a point y with $x \sim y \sim xh$, but the fixed-point free involution h swaps x and xh without fixing y, so a similar argument to that in the proof of Lemma 2.1(iv) implies that $x \sim xh$, a contradiction. On the other hand, we actually know that $\gcd(s, t) > 1$ in the situation considered above (via Lemma 3.3), so in a sense it is more natural to argue as we have done. (It is an open question whether the order (s, t) of a finite thick generalised hexagon or octagon always satisfies $\gcd(s, t) > 1$.)

3.3. Type AS with socle an alternating group. By Lemmas 3.2 and 3.5, if Hypothesis 3.1 holds then G must be an almost simple group. We now treat the case where $M = T$ is an alternating group. That is, we prove the following result.

LEMMA 3.7. *If Hypothesis* 3.1 *holds with G an almost simple group, then the socle of G is not an alternating group.*

For the proof, suppose towards a contradiction that Hypothesis 3.1 holds with G almost simple and $\mathrm{soc}(G) \cong A_n$ for some $n \geqslant 5$. Buekenhout and Van Maldeghem [4] have shown that such a group G cannot act transitively on the points of a generalised hexagon or octagon if $n < 14$, so we may assume that $n \geqslant 14$. In particular, we have $G = A_n$ or S_n because $n \neq 6$. Our analysis splits into three cases, depending on whether the stabiliser G_x of a point $x \in \mathcal{P}$ is an intransitive, transitive but imprimitive, or primitive subgroup of S_n in the natural action on $\{1, \ldots, n\}$.

3.3.1. *Intransitive point stabiliser.* Let $x \in \mathcal{P}$ and suppose that G_x acts intransitively on $\{1, \ldots, n\}$. Then G_x stabilises a partition of $\{1, \ldots, n\}$ into two blocks, one of size k, say, and one of size ℓ, where $k + \ell = n$. If $k = \ell$ then $G_x < G \cap (S_k \text{ wr } S_2) < G$, so G_x is not a maximal subgroup, contradicting the primitivity of G on $\mathcal{P}$. We may therefore assume, without loss of generality, that $k < \ell$. We then have $(A_k \times A_\ell).2 \leqslant G_x \leqslant S_k \times S_\ell$, and the points of $\mathcal{S}$ can be labelled by k-element subsets of $\{1, \ldots, n\}$. We note also that G_x must have at least four orbits on $\mathcal{P}$, as it preserves distance in the incidence graph of $\mathcal{S}$ and there are points in $\mathcal{P}$ at distances 0, 2, 4, and 6 (and 8 in the case of a generalised octagon) from x. If $k < 3$ then the number of orbits of G_x is less than four, and hence we may assume that $k \geqslant 3$.

The following facts are proved by Bamberg et al. [1, Lemmas 5.5 and 5.6] in the case of a generalised quadrangle (with the same assumptions as above). We note that the proofs given there are also valid for generalised hexagons and octagons, but we include proofs to make this clear.

(F1) For every $i \in \{1, \ldots, k\}$, if $x, y \in \mathcal{P}$ are collinear and $|x \cap y| = i$, then any $x', y' \in \mathcal{P}$ with $|x' \cap y'| = i$ are also collinear.

(F2) For every $i \in \{1, \ldots, k\}$, if $x, y \in \mathcal{P}$ are collinear and $|x \cap y| = i$, then there exists $y' \in \mathcal{P}$ such that $|x \cap y'| = i$ and $y' \not\sim y$.

PROOF OF (F1) AND (F2). For (F1), it suffices to observe that $G = A_n$ or S_n preserves collinearity and is transitive on pairs of k-subsets of $\{1, \ldots, n\}$ with intersection size i. For (F2), suppose towards a contradiction that every point y' with $|x \cap y'| = i$ is collinear with y. By (F1), every such point y' is also collinear with x, and hence lies on the line $\lambda := \langle x, y \rangle$ (because $\mathcal{S}$ contains no triangles). Let J denote the generalised Johnson graph with vertices the k-subsets of $\{1, \ldots, n\}$ and two vertices adjacent if and only if their labels intersect in i elements. If $n \neq k/2$ then G acts primitively on the point set of J, and a partition into connected components is G-invariant, so J is a connected graph. We prove by induction on the distance $\delta(x, x_1)$ between x and $x_1 \in J$ that all vertices of J lie on the line λ. The inductive hypothesis is true for distance 0 because x lies on λ, and for distance 1 by our assumption that every point y' with $|x \cap y'| = i$ is collinear with y. Assume that it is true for distance d, and suppose that $\delta(x, x_1) = d + 1$. Then x_1 has a neighbour $x_2 \in J$ with $\delta(x, x_2) = d$, and thus, by the inductive hypothesis, x_2 lies on λ. Also, x_2 has a neighbour $x_3 \in J$ such that $\delta(x, x_3) = d - 1$, and hence x_3 lies on λ. If x_1 did not lie on λ then x_2 would have two neighbours, x_1 and x_3, such that $|x_2 \cap x_1| = |x_2 \cap x_3| = i$ but x_1 and x_3 are not on the same line, so by vertex-transitivity the same would have to be true for x, a contradiction. Hence all vertices of J lie on λ. However, this is a contradiction because the points of $\mathcal{S}$ do not all lie on a single line. $\square$

Now let x denote the point with label $\{1, \ldots, k\}$, and let $k_1 < k$ be maximal such that there exists a point $y \sim x$ with $|x \cap y| = k_1$. We claim that $k_1 = 0$. We first show that $k_1 < k - 1$ by adapting an argument from [1, Section 5]. Suppose, towards a contradiction, that $k_1 = k - 1$. Then, without loss of generality, y has label $\{1, \ldots, k - 1, k + 1\}$. By (F2), there exists $y' \in \mathcal{P}$ such that $|x \cap y'| = k - 1$ and $y' \not\sim y$, and by (F1) we have $y' \sim x$ and $|y \cap y'| \neq k - 1$. In particular, without loss of generality we can write $y' = \{2, \ldots, k, k + 2\}$. However, the automorphism $(1 \ k + 1)(k \ k + 2)$ fixes both y and y' but does not fix x, so Lemma 2.1(iv) implies that $y \sim y'$, a contradiction. Hence $k_1 < k - 1$. In particular, if $k = 2$ then $k_1 = 0$, so we can now assume that $k \geqslant 3$ and complete the proof of the claim that $k_1 = 0$ by adapting an argument from [22, Section 5]. Without loss of generality,

$y = \{1, \ldots, k_1, k+1, \ldots, 2k - k_1\}$. Let $z = \{1, \ldots, k_1, k+2, \ldots, 2k - k_1 + 1\}$. Then $y \not\sim z$ because $|y \cap z| = k - 1 > k_1$. On the other hand, $|x \cap z| = k_1$, and hence $x \sim z$ by (F1). If $k_1 > 0$ then the automorphism $(1 \ k + 2)(k - 1 \ k)$ fixes y and z but not x, so Lemma 2.1(iv) implies that $y \sim z$, a contradiction. Therefore, $k_1 = 0$ as claimed. However, now if $2k + 1 < n$ then the automorphism $g = (1 \ 2k + 2)(2 \ 3)$ fixes $y = \{k+1, \ldots, 2k\}$ and $z = \{k+2, \ldots, 2k+1\}$ but not x, a contradiction according to Lemma 2.1(iv). Therefore, $n = 2k + 1$. However, now by maximality of $k_1 = 0$ and transitivity of A_k, there are precisely $k + 1$ points collinear with x, and G_x acts 2-transitively on this set of $k + 1$ points. This implies that there is either only one line incident with x, or $k + 1$ such lines, each incident with only two points. Either situation contradicts the thickness of $\mathcal{S}$.

3.3.2. *Transitive but imprimitive point stabiliser.* In this case, G_x is the stabiliser of a partition of $\{1, \ldots, n\}$ into ℓ blocks of size k, where $n = k\ell$. First suppose that $\ell = 2$, and let H denote the stabiliser of the point n in the natural action of G on $\{1, \ldots, n\}$. Since G_x is transitive on $\{1, \ldots, n\}$, we have $G = HG_x$, and this in turn implies that H is transitive on $\mathcal{P}$. Moreover, $H_x = H \cap G_x = (S_k \times S_{k-1}) \cap G$, which is a maximal subgroup of H. Therefore, H is primitive on $\mathcal{P}$. However, by Section 3.3.1, $\mathcal{S}$ cannot admit a point-primitive action of H with stabiliser intransitive on $\{1, \ldots, n-1\}$. Therefore, $\ell \geqslant 3$.

Now, given points $x, y \in \mathcal{P}$, we use the notation $|x \cap y|$ to mean the number of partition classes common to x and y. We consider, in particular, points for which $|x \cap y| = \ell - 2$, and for this case we define the following additional notation.

DEFINITION 3.8. *Let $x, y \in \mathcal{P}$ such that $|x \cap y| = \ell - 2$. Then there are exactly two partition classes B_1, B_2 of x that do not belong to y, and exactly two partition classes B_1', B_2' of y that do not belong to x. By appropriate labelling, we may assume that $|B_1 \cap B_1'| = |B_2 \cap B_2'| \geqslant \lceil k/2 \rceil$, and we then write $|x - y|_{\ell-2} := |B_1 \cap B_1'| = |B_2 \cap B_2'|$.*

We first claim that there exist collinear points $x, y \in \mathcal{P}$ such that $|x \cap y| = \ell - 2$ and $|x - y|_{\ell-2} = k - 1$. The proof of this claim is via steps (i)–(iii) below, with steps (i) and (ii) adapted from Schneider and Van Maldeghem [22, p. 1447] but repeated here to make it clear that flag-transitivity is not required.

(i) *There exist collinear points $x, y \in \mathcal{P}$ with $|x \cap y| \geqslant 1$.* To prove this, first choose a point $x \in \mathcal{P}$ and suppose, without loss of generality, that x is labelled by

$$x = \{B_1, \ldots, B_\ell\}, \quad \text{where } B_i = \{(i-1)k + 1, (i-1)k + 2, \ldots, ik\} \text{ for } i = 1, \ldots, \ell - 1.$$

Choose $y \in \mathcal{P}$ such that $x \sim y$. If $|x \cap y| \geqslant 1$ then we are done, so suppose that $|x \cap y| = 0$. First consider the case $k = 2$. Then $\ell \geqslant 7$ because $n = k\ell \geqslant 14$. Observe that the automorphism $g = (1 \ 2)(3 \ 4)$ fixes x and fixes at least $\ell - \mathrm{supp}(g) = \ell - 4 \geqslant 3$ partition classes of y (here $\mathrm{supp}(g)$ is the *support* of g, namely the subset of $\{1, \ldots, n\}$ of elements moved by g). In particular, $|y \cap yg| \geqslant 3$, and we show that $y \sim yg$. Let $\{i_1, i_2\}, \{i_3, i_4\}$ be two (partition) classes common to y and yg. Since $\ell - \mathrm{supp}(g) \geqslant 3$, we may assume that $\{i_1, i_2, i_3, i_4\}$ is not a union of two classes of x. Then the automorphism $g' = (i_1 \ i_2)(i_3 \ i_4)$ fixes both y and yg, but does not fix x (because i_1, i_2 lie in the same class of y and hence in different classes of x, and these two classes of x are not fixed setwise by g'), so Lemma 2.1(iv) implies that $y \sim yg$ as required. Now suppose that $k \geqslant 3$. The automorphism $h = (1 \ 2 \ 3)$ fixes x and fixes at least $\ell - 3$ classes of y. Hence, if $\ell \geqslant 4$, then $|y \cap yh| \geqslant 1$. If $\ell = 3$ then $k \geqslant 5$ since $n \geqslant 14$, so, in particular, there is a class of y that shares at least two elements with some class of x. Without loss of generality, we may assume that $1, 2$ lie together in a class of y, and hence that h fixes at least one class of y. Thus we have $|y \cap yh| \geqslant 1$ for all $\ell \geqslant 3$, and we check that $y \sim yh$. Let $\{i_1, \ldots, i_k\}$ be a class common to y and yh, and suppose without loss of generality that i_1, i_2, i_3 do not all lie together in a class of x. Then the automorphism $(i_1 \ i_2 \ i_3)$ fixes y and yh but does not fix x, so Lemma 2.1(iv) implies that $y \sim yh$.

(ii) *There exist collinear points $x, y \in \mathcal{P}$ with $|x \cap y| = \ell - 2$.* This follows immediately from part (i) if $\ell = 3$, so suppose that $\ell \geqslant 4$. Let $x, y \in \mathcal{P}$ be collinear points with $|x \cap y| \geqslant 1$, and label x as in part (i). If $|x \cap y| = \ell - 2$ then we are done, so suppose the contrary. Suppose, without loss of generality, that y contains the class $B_1 = \{1, \ldots, k\}$. Choose a class B_i of x that does not belong to y, and choose $j_1, j_2 \in B_i$ such that j_1, j_2 do not lie in the same class of y. Then the automorphism $(1 \ 2)(j_1 \ j_2)$ fixes x and maps y to a point z such that $|y \cap z| = \ell - 2$, and we check that $y \sim z$. Since $\ell \geqslant 4$, y and z contain a common class B' that is not a class of x (because otherwise $|x \cap y| = \ell - 2$, contrary to our assumption). Hence, taking $j_3, j_4 \in B'$ lying in different

classes of x, the automorphism $(1\ 2)(j_3\ j_4)$ fixes y and z but does not fix x, so Lemma 2.1(iv) implies that $y \sim z$.

(iii) *With notation as in Definition 3.8, there exist collinear points $x, y \in \mathcal{P}$ with $|x \cap y| = \ell - 2$ and $|x - y|_{\ell-2} = k - 1$.* This follows immediately from part (ii) if $k \in \{2, 3\}$, so suppose that $k \geqslant 4$. Let $x, y \in \mathcal{P}$ be collinear points with $|x \cap y| = \ell - 2$, and again label x as in part (i). Assume without loss of generality that y also contains the classes $B_3, \ldots, B_\ell$. Let B_1', B_2' denote the remaining two classes of y, and label these classes such that $k_1 := |B_1 \cap B_1'| = |B_2 \cap B_2'| \geqslant \lceil k/2 \rceil$. That is, $k_1 = |x - y|_{\ell-2}$. Consider first the case $k \geqslant 5$. If $k_1 = k - 1$ then we are done, so suppose the contrary. Then $\lceil k/2 \rceil \leqslant k_1 \leqslant k - 2$, and since $k \geqslant 5$ we have $k_1 \geqslant 3$. Without loss of generality, $B_1' = \{1, \ldots, k_1, k+1, \ldots, 2k - k_1\}$ and $B_2' = \{k_1 + 1, \ldots, k, 2k - k_1 + 1, \ldots, 2k\}$. Now consider the automorphism $g = (1\ 2)(k_1\ k)$. Then g fixes x, and the point yg contains the classes $B_3, \ldots, B_\ell$ and its other two classes are $B_1'' = \{1, \ldots, k_1 - 1, k, \ldots, 2k - k_1\}$ and $B_2'' = \{k_1, \ldots, k - 1, 2k - k_1 + 1, \ldots, 2k\}$. In particular, $|y \cap yg| = \ell - 2$ and $|y - yg|_{\ell-2} = |B_1'' \cap B_1'| = |B_2'' \cap B_2'| = k - 1$. We now show that $y \sim yg$. Since $x \sim y$, we have $x = xg \sim yg$. Consider the automorphism $h = (1\ 2)(k_1 - 1\ k + 2)$. Since the points $1, 2, k_1 - 1, k + 2$ all lie in $B_1' \cap B_1''$, h fixes both y and yg (note that this holds even if $k_1 = 3$). However, h does not fix x, so Lemma 2.1(iv) implies that $y \sim yg$. Finally, if $k = 4$ then $2 = \lceil k/2 \rceil \leqslant k_1 \leqslant k - 2 = 2$, so $k_1 = 2$ and without loss of generality we have $B_1' = \{1, 2, 5, 6\}$ and $B_2' = \{3, 4, 7, 8\}$. Then $|yg' - y|_{\ell-2} = 3$, where $g' := (2\ 4)(5\ 6)$, and $h' := (1\ 5)(3\ 7)$ fixes y and yg' but not x, so Lemma 2.1(iv) implies that $y \sim yg'$.

Let us now fix collinear points x and y such that $|x \cap y| = \ell - 2$ and $|x - y|_{\ell-2} = k - 1$, and establish some further notation. Specifically, we assume that $x, y \in \mathcal{P}$ are collinear points labelled by

$$x = \{B_1, B_2, B_3, \ldots, B_\ell\},$$
$$y = \{\{1, \ldots, k - 1, k + 1\}, \{k, k + 2, \ldots, 2k\}, B_3, \ldots, B_\ell\},$$

where $B_i = \{(i-1)k + 1, (i-1)k + 2, \ldots, ik\}$ for $i = 1, \ldots, \ell - 1$ as before. We let λ denote the line incident with both x and y. Further, we define an automorphism h and a subset Δ of $\{1, \ldots, n\}$ as follows:

$$h := \begin{cases} (1\ 2)(k\ k+1) & \text{if } k \geqslant 3 \\ (1\ 3\ 2) & \text{if } k = 2 \end{cases}, \qquad \Delta := \begin{cases} \{1, 2, k, k+1\} & \text{if } k \geqslant 3 \\ \{1, 2, 3\} & \text{if } k = 2. \end{cases}$$

First observe that $y = xh$. In particular, $x \sim xh$. If $k \geqslant 3$ then h has order 2, and hence h fixes λ. If $k = 2$ then h has order 3 and we have $xh \sim xh^2$ and hence $xh^2 \sim xh^3 = x$, so again h fixes λ because $\mathcal{S}$ contains no triangles. In either case, we have $h \in G_\lambda \backslash G_x$. To complete the proof that G_x cannot be transitive but imprimitive in its natural action on $\{1, \ldots, n\}$, we now obtain a contradiction by showing that $G_x = G_\lambda$. The cases (i) $k = 2$ and (ii) $k \geqslant 3$ are treated separately.

(i) Suppose that $k = 2$, and recall that in this case $\ell \geqslant 7$, because $n \geqslant 14$. In particular, we do not need to consider the case $\ell = 3$, for which the following argument does not work. We first claim that if $g \in G_x$ is an element satisfying $\Delta g \cap \Delta = \varnothing$ and $yg \neq yh$, then $g \in G_\lambda$. To prove this, first note that $h = (1\ 3\ 2)$ commutes with h^g for such g, because $\Delta g \cap \Delta = \varnothing$. Note also that $h^g h$ does not fix x: since $g \in G_x$, we have $xh^g h = xhgh = ygh$, so $h^g h$ fixes x if and only if yg equals xh^2, which equals yh, but $yg \neq yh$. Therefore, and since $x = xg \sim xhg = xgg^{-1}hg = xh^g$, Lemma 2.1(ii) implies that x, xh, xh^g all lie on a common line. That is, $x = xg$ and $xh^g = xhg$ both lie on λ, so g fixes λ. The claim is proved. Now consider the elements $g_1, g_2, g_3 \in G_x$ given by

$$g_1 := (1\ 5)(2\ 6)(3\ 4)(7\ 8),$$
$$g_2 := (1\ 7)(2\ 8)(3\ 4)(5\ 6),$$
$$g_3 := (1\ 5)(2\ 6)(3\ 7)(4\ 8).$$

Then $\Delta g_i \cap \Delta = \varnothing$ and $yg_i \neq yh$ for $i = 1, 2$ and 3. (To check that $yg_i \neq yh$, observe that each yg_i contains the partition class $B_1 = \{1, 2\}$ but that yh does not contain B_1.) Hence each g_i lies in G_λ by the claim. Now consider the setwise stabiliser $(G_x)_{B_1}$ of B_1 in G_x. Then, in particular, g_1 does not lie in $(G_x)_{B_1}$, because g_1 maps B_1 to B_3. Since $(G_x)_{B_1}$ is a maximal subgroup of G_x, this

implies that $G_x = \langle (G_x)_{B_1}, g_1 \rangle$. We show that $(G_x)_{B_1}$ is contained in G_λ, which implies that G_x is contained in G_λ, and hence, by maximality of G_x in G, that $G_x = G_\lambda$. Let $a \in (G_x)_{B_1}$. Since all of the yg_i contain B_1, it follows that all of the $y(g_i a)$ contain B_1, and hence that none of these elements is equal to yh. We now show that for i equal to one of 1, 2, or 3, we have $\Delta(g_i a) \cap \Delta = \varnothing$. By the claim, this implies that $g_i a \in G_\lambda$, and it follows that $a = g_i^{-1}(g_i a) \in G_\lambda$ as required. Since a fixes $B_1 = \{1, 2\}$, we just need to choose i such that a does not map any of the elements of Δg_i to the element 3. If $3 \notin \{4a, 5a, 6a\}$ then $\Delta(g_1 a) = \{1, 2, 3\}(g_1 a) = \{4a, 5a, 6a\}$ and hence $\Delta(g_1 a) \cap \Delta = \varnothing$. If $3 \in \{5a, 6a\}$ then $\Delta(g_2 a) = \{4a, 7a, 8a\}$ and hence $\Delta(g_2 a) \cap \Delta = \varnothing$. Finally, if $3 = 4a$ then $\Delta(g_3 a) = \{5a, 6a, 7a\}$ and hence $\Delta(g_3 a) \cap \Delta = \varnothing$.

(ii) Now suppose that $k \geqslant 3$. We first claim that if $g \in G_x$ is an element satisfying $\Delta g \cap \Delta = \varnothing$, then $g \in G_\lambda$. To prove this, begin by observing that $h = (1\ 2)(k\ k+1)$ commutes with h^g for such g, and that $x = xg \sim xhg = xgg^{-1}hg = xh^g$. If $hh^g = h^g h$ does not fix x, then Lemma 2.1(ii) implies that x, xh, xh^g all lie on a common line. That is, both $x = xg$ and $xh^g = xhg$ lie on λ, and so g fixes λ. If $hh^g = h^g h$ does fix x then we have $x = xh^g h = (xg^{-1})hgh = (xh)gh = ygh$, and hence $y = xh = ygh^2 = yg$ (because $h^2 = 1$), so in this case g fixes both x and y, and hence also fixes λ. The claim is proved. Now consider the elements $g_1, g_2 \in G_x$ given by

$$g_1 := \begin{cases} (1\ 2k+1)(2\ 2k+2)\cdots(k\ 3k)(k+1\ k+2) & \text{if } k \text{ is odd} \\ (1\ 2k+1)(2\ 2k+2)\cdots(k\ 3k)(k+1\ k+2)(k+3\ k+4) & \text{if } k \text{ is even,} \end{cases}$$

$$g_2 := \begin{cases} (1\ 2k+1)(2\ 2k+2)\cdots(k\ 3k)(k+1\ k+3) & \text{if } k \text{ is odd} \\ (1\ 2k+1)(2\ 2k+2)\cdots(k\ 3k)(k+1\ k+3)(k+2\ k+4) & \text{if } k \text{ is even.} \end{cases}$$

Then $\Delta g_1 \cap \Delta = \Delta g_2 \cap \Delta = \varnothing$, and hence $g_1, g_2 \in G_\lambda$ by the claim. Consider the setwise stabiliser $(G_x)_{B_3}$ of B_3 in G_x. Since $g_1 \notin (G_x)_{B_3}$, and since $(G_x)_{B_3}$ is a maximal subgroup of G_x, we have $\langle (G_x)_{B_3}, g_1 \rangle = G_x$. We now show that $(G_x)_{B_3} \leqslant G_\lambda$, which implies that $G_x \leqslant G_\lambda$, and hence, by maximality of G_x in G, that $G_x = G_\lambda$. Let $a \in (G_x)_{B_3}$. We show that for one of $i = 1$ or $i = 2$ we have $\Delta(g_i a) \cap \Delta = \varnothing$. By the claim, this implies that $g_i a \in G_\lambda$, and hence that $a = g_i^{-1}(g_i a) \in G_\lambda$, as required. We have $\Delta g_1 = \{2k+1, 2k+2, 3k, k+2\}$ and $\Delta g_2 = \{2k+1, 2k+2, 3k, k+3\}$. Since a fixes B_3 setwise, we have $\Delta(g_1 a) \cap \Delta = \varnothing$ unless $(k+2)a = k+1$, and in this case we have instead $\Delta(g_2 a) \cap \Delta = \varnothing$.

3.3.3. *Primitive point stabiliser.* Now suppose that the stabiliser G_x of a point $x \in \mathcal{P}$ is a primitive subgroup of S_n in its action on $\{1, \ldots, n\}$. We need the following lemma about the index of a primitive permutation group. The result is likely to be well known, but we have been unable to find a reference for it, so we include a proof.

LEMMA 3.9. *Let $G = A_n$ or S_n, where $n \geqslant 9$. If H is a primitive maximal subgroup of G that does not contain A_n, then $|G : H|$ is divisible by 4.*

PROOF. Let Q and P be Sylow 2-subgroups of H and S_n, respectively, such that $Q \leqslant P$. Write $Q^+ = Q \cap A_n$ and $P^+ = P \cap A_n$. Then P^+ is a Sylow 2-subgroup of A_n, and $|P : P^+| = 2$. If $G = S_n$ then H is not contained in A_n, by maximality of H, so $|Q : Q^+| = 2$ and hence $|G : H|_2 = |P : Q| = |P^+ : Q^+|$, where $|G : H|_2$ is the largest power of 2 dividing $|G : H|$. If $G = A_n$ then $P = P^+$ and $Q = Q^+$, and again $|G : H|_2 = |P : Q| = |P^+ : Q^+|$. We now show that the 2-power $|P^+ : Q^+|$ is at least 4. If $|P^+ : Q^+| = 1$ then $Q^+ = P^+$, so in particular Q^+ is a Sylow 2-subgroup of A_n and hence contains a double transposition (a product of two disjoint transpositions). However, this means that the primitive group H contains a double transposition, and since $n \geqslant 9$, a theorem of Jordan [17] (see also [11, Example 3.3.1]) then implies that H contains A_n, a contradiction. Now suppose that $|P^+ : Q^+| = 2$. Let $K = \langle (1\ 2), (3\ 4), (5\ 6) \rangle$. By conjugating P and H simultaneously (and conjugating Q along with H), we may assume that $K \leqslant P$. Let $K_0 = K \cap A_n$. Then $|K| = 8$, $|K_0| = 4$, $K_0 \leqslant P^+$, and K_0 contains three double transpositions. If $K_0 \leqslant Q^+$ then Q^+ contains a double transposition, and we are done. Otherwise, $P^+ = Q^+ K_0$, and so $2 = |P^+ : Q^+| = |Q^+ K_0 : Q^+| = |K_0 : Q^+ \cap K_0|$. Thus $Q^+ \cap K_0$ is nontrivial, so Q^+ contains a double transposition, and hence so does H, a contradiction. □

We have $|\mathcal{P}| = |G : G_x|$ with G_x a primitive maximal subgroup of G, and G_x does not contain A_n because this would imply that $|\mathcal{P}| = 1$ or 2. Therefore, and since we are assuming that $n \geqslant 14 > 9$,

Lemma 3.9 implies that $|\mathcal{P}|$ is divisible by 4, and Lemma 3.3 together with properties (PH) and (PO) in Section 2 therefore implies that $\gcd(s,t) > 1$. We note also that, since G_x is a maximal subgroup of A_n or S_n and $n \geqslant 14$, G_x is not cyclic of prime order and, in particular, the stabiliser $(G_x)_j$ of an element $j \in \{1, \ldots, n\}$ is nontrivial. Moreover, we note that by a theorem of Jordan [16, 19], the primitive group G_x contains no elements with cycle type 3^1 or 3^2 if $n \geqslant 10$ (and hence, in particular, for $n \geqslant 14$).

Now consider the 3-cycle $h = (1\ 2\ 3) \in G$. Then, as noted above, h is fixed-point free in its action on $\mathcal{P}$, so Lemma 2.1(i) implies that there exists $x \in \mathcal{P}$ such that $x \sim xh$. This implies that $xh \sim xh^2$, and hence that $xh^2 \sim xh^3 = x$. Since $\mathcal{S}$ contains no triangles, it follows that h fixes the line $\lambda = \langle x, xh \rangle$. Write $\Delta = \{1, 2, 3\}$. We first claim that if $g \in G_x$ satisfies $\Delta g \cap \Delta = \varnothing$, then $g \in G_\lambda$. To prove this, first note that h and h^g commute. Since $x \sim xh$, we have $x = xg \sim xhg = xgg^{-1}hg = xh^g$; that is, $x \sim xh^g$. Moreover, hh^g is a permutation of type 3^2 and is therefore fixed-point free on $\mathcal{P}$, as noted above. In particular, hh^g does not fix x, so Lemma 2.1(ii) implies that x, xh, xh^g lie on a common line. That is, $x = xg$ and $xh^g = xhg$ both lie on $\lambda = \langle x, xh \rangle$, and hence g fixes λ as claimed. Let us assume at this point that $n > 15$. We show that $(G_x)_1 \leqslant G_\lambda$. Let $a \in (G_x)_1$ and write $\Gamma = \Delta \cup \{2a^{-1}, 3a^{-1}\}$. Suppose that $b \in G_x$ satisfies $\Delta b \cap \Gamma = \varnothing$. Then, in particular, $\Delta b \cap \Delta = \varnothing$, and moreover, $\Delta(ba) \cap \Delta = (\Delta b \cap \Delta a^{-1})a \subseteq (\Delta b \cap \Gamma)a = \varnothing$. Therefore, by the above argument, b and ba both lie in G_λ, and hence $a = b^{-1}(ba) \in G_\lambda$. By Neumann's Separation Lemma [3, Theorem 2], such an element $b \in G_x$ exists if $n > |\Delta||\Gamma| = 15$, and so $(G_x)_1 \leqslant G_\lambda$ as claimed. By an analogous argument, $(G_x)_2 \leqslant G_\lambda$, so $G_x = \langle (G_x)_1, (G_x)_2 \rangle \leqslant G_\lambda$ (because $(G_x)_1$ is nontrivial and is a maximal subgroup of G_x) and hence $G_x = G_\lambda$ (because G_x is a maximal subgroup of G). However, h fixes λ but not x, so this is a contradiction.

It remains to consider the cases $n = 14$ and $n = 15$. We require a primitive maximal subgroup H of $G = A_n$ or S_n such that H does not contain A_n, and such that $|G : H|$ is equal to the number of points of a (finite thick) generalised hexagon or octagon. For $n = 14$, we have two candidates: $\mathsf{PSL}(2, 13)$ and $\mathsf{PGL}(2, 13)$. The first is maximal in A_{14}, and the second is maximal in S_{14}. We have $|A_{14} : \mathsf{PSL}(2, 13)| = |S_{14} : \mathsf{PGL}(2, 13)| = 39\ 916\ 800$, and one checks computationally that this cannot be the number of points of a generalised hexagon or octagon (using properties (PH) and (PO) in Section 2). For $n = 15$, we have four candidates: A_7, A_6, S_6, and $\mathsf{PSL}(4, 2)$. All are contained in A_{15}, and only $\mathsf{PSL}(4, 2)$ is maximal in A_{15} (it contains the other three). We have $|A_{15} : \mathsf{PSL}(4, 2)| = 32\ 432\ 400$, which also cannot be the number of points of a generalised hexagon or octagon.

3.4. Proof of Theorem 1.2 and Corollary 1.3. Let us summarise the proof of Theorem 1.2 and deduce Corollary 1.3. Theorem 1.2 asserts that if Hypothesis 3.1 holds, then G must be an almost simple group of Lie type. Schneider and Van Maldgehem [22, Lemma 4.2(i)] already proved that G cannot have O'Nan–Scott type HA, HS, or HC under Hypothesis 3.1, and Lemmas 3.2 and 3.5 further imply that G cannot have type PA, CD, SD, or TW. Therefore, G must be an almost simple group. By a result of Buekenhout and Van Maldeghem [4], the socle of G cannot be a sporadic group, and by Lemma 3.7, it cannot be an alternating group. Therefore, the only remaining possibility is that $\mathrm{soc}(G)$ is a simple group of Lie type, namely that G is an almost simple group of Lie type.

Now, the conclusion of Theorem 1.2 holds in particular if G also acts primitively on lines, so case (i) of Corollary 1.3 is the only possibility for $d \in \{6, 8\}$. If $d = 3$ then, by the work of Gill [15], the only other possibility is case (ii). Finally, suppose that $d = 4$. Bamberg et al. [1, Theorems 1.1 and 1.2 (a)] proved that if G acts primitively on both the points and the lines of $\mathcal{S}$, then G must be an almost simple group whose socle is not a sporadic simple group; so either $\mathrm{soc}(G)$ is of Lie type, as in (i), or $\mathrm{soc}(G) \cong A_n$. They then considered [1, Sections 5.1 and 5.2] the case where $\mathrm{soc}(G) \cong A_n$ and showed, using only the assumption of point-primitivity, that the stabiliser G_x of a point x of $\mathcal{S}$ must be a primitive subgroup of S_n in the natural action on $\{1, \ldots, n\}$. By duality, the same is true for lines under the assumption of line-primitivity. In particular, the automorphism $h = (1\ 2\ 3)$ cannot fix any point or line of $\mathcal{S}$, because this would force the stabiliser to contain A_n and hence imply that $\mathcal{S}$ has either at most two points or at most two lines. However, if $\gcd(s,t) > 1$ then this contradicts [1, Lemma 3.4], which says that an automorphism of order 2 or 3 must fix either a point or a line. Therefore, the only remaining possibility is that $\gcd(s,t) = 1$, as in (iii).

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