Efficient Nonlinear Total Least Squares Estimation For Differentially Flat Systems

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Abstract— **This paper proposes a computationally efficient framework for nonlinear total least square (TLS) estimation problems by exploiting differential flatness with unknown initial conditions. Classical ordinary least squares (OLS) assumes only the dependent variables have noise and independent variables are perfectly known. However, it is more realistic to formulate TLS estimation problems because both dependent variable and independent variables are assumed to be noisy. It is challenging to solve general nonlinear TLS estimation problems due to the fact that there is no analytical solutions and its numerical solutions are computationally very expensive. This paper addresses this challenge by exploiting differential flatness and using pseudospectral methods. As a result, nonlinear TLS problems are transformed into unconstrained nonlinear programming problems with a small number of optimization variables. This paper demonstrates the framework by solving a state and input estimation problem for a mass-spring-damper system. The results show that estimation errors are bounded within one standard deviation, which is very accurate.**

I. INTRODUCTION

This paper proposes a framework to solve general nonlinear total least squares (TLS) estimation problems in a computationally efficient manner by exploiting the differential flatness property. TLS estimation problems in this paper refer to state estimation problems that minimize squared estimation errors and is subject to system dynamics given noisy input and output measurement. Least squares estimation is widely used in the literature, such as for curve fitting, state and parameter estimation [1]–[4]. There are two categories of least squares problems: ordinary least squares (OLS) and TLS. The ordinary least squares optimization assumes only the dependent variables (i.e., output variables in this paper) have measurement noise and independent variables (i.e., input variables in this paper) are assumed to have no noise and hence have true values. Because all data is corrupted by noise especially in engineering field [5], it is more realistic to use TLS which assumes all data is noisy [6].

However, it is challenging to solve TLS estimation problems due to two reasons. First, classical TLS problems focus on linear curve fitting and can be solved using well-developed singular value decomposition-based approaches [4], [6]. For dynamic systems, even for linear systems, the TLS problems is nonlinear in the sense that the output variables (i.e., dependent variables) have nonlinear relationship with input variables (i.e., independent variables) [7]. Therefore, there is no analytical solution to solve estimation problems for dynamic systems using TLS. Thus, numerical methods have to be used to solve TLS estimation problems. This brings the second challenge: it is computationally expensive to solve TLS estimation problems for general dynamic systems. Since both input and output variables are corrupted with noise in TLS problems, optimization variables include all the input and state variables at each sampling points. Therefore, this can easily result in a optimization problem with hundreds of optimization variables and therefore can be infeasible to solve.

This paper proposes a computationally efficient framework by exploiting differential flatness and pseudospectral methods. We formulate estimation problems as an optimization problem with the cost function in TLS form and solve the resulting optimization problems numerically in a computationally efficient manner. First, this paper exploits the differential flatness property, which makes it possible to express system dynamics using only the trajectory of one fictitious variable (with the same dimension as the input). The concept of differential flatness is introduced by Fliess *et al.* [8]. A system is differentially flat if the state and input variables can be expressed by the flat output and a finite number of its derivatives and if the flat output can be expressed in terms of state, input, and a finite number of its derivatives. The differential flatness can be seen as the nonlinear extension of controllability for nonlinear systems and for linear system it is equivalent to the controllability [8], [9]. Transforming a system from the state space to the flat output space enables one to represent system dynamics (i.e., trajectories of states and inputs) using only the trajectory of flat output. Since the transformation process implicitly implies system dynamics, one major benefit of flat systems is that transformed optimization problems do not have dynamic equality constraint, which makes problems numerically easier to solve.

Second, this paper uses pseudospectral methods to optimize flat output trajectory to achieve similar accuracy with a much smaller number of discretization points compared to traditional methods. Pseudospectral methods transform original problems into nonlinear programming (NLP) problems which are solved using well-developed NLP algorithms [9], [10]. The original problems are discretized at unevenly distributed collocation points where constraints are enforced. It is shown that the discretization using collocation points is more efficient compared to traditional discretization methods, such as finite difference methods, in the sense that it demonstrates significantly faster convergence rate [10].

To the best of authors' knowledge, this is the first paper

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to solve general nonlinear TLS problems in the flat output space using pseudospectral methods. The end product of applying the proposed framework is unconstrained optimization problems with a smaller number of optimization variables. In addition, since the initial condition is included as optimization variables, it does not need accurate initial condition to get accurate estimation results. This paper demonstrates it by estimating the state and input variables for a mass-spring-damper system given noisy input and output measurement. The results show that the estimated state and input variables converge to true values accurately.

The remainder of this paper is organized as follows. Section 2 introduces general total least squares problems and formulates the estimation problem. Section 3 presents the differentially flat systems and their properties. The Legendre pseudospectral method is introduced in section 4. In section 5, this paper demonstrates the proposed idea by estimating the state and input variables for a mass-spring-damper system. Finally, the paper is concluded in section 6.

II. GENERAL TOTAL LEAST SQUARES ESTIMATION

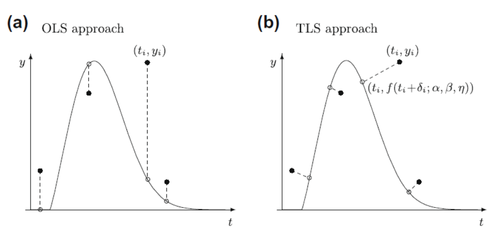


Fig. 1: Comparison between OLS (left) and TLS (right)

This section introduces TLS estimation problems and formulate the TLS estimation problem. In this paper all noise is assumed to be independently, identically distributed (iid).

First, consider a general continuous dynamic system as follows

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= g(x,u) \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$ and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_x}$ and $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_y}$.

The difference of OLS and TLS is apparent in Fig. 1 which is to estimate a nonlinear curve using least squares. OLS assumes that only the output variables are corrupted with noise and the input variables are exactly as demanded. The discretized system with the same dynamics as system (1) is

$$\begin{aligned} x_{k+1} &= f_k(x_k,u_k) \\ y_k &= g(x_k,u_k) \end{aligned} \tag{2}$$

where $k = 1,2,\ldots,n$ with n is the number of data points and v_k is the observation noise. The state x_k is the state evaluated at sampling time t_k, i.e., $x_k = x(t_k)$. This paper simplifies the notation using subscript k to indicate this variable is evaluated at time t_k. The measured output $y_{m,k}$ at sampling time t_k is the true but unknown output y_k plus measurement noise v_k, i.e., $y_{m,k} = y_k + v_k$. Since the input is accurate and noise is iid, each data point can move around itself in a the y direction. As a result, OLS minimizes the sum of squared vertical distance from each data point to the estimated curve as shown in Fig. 1. The OLS problem then can be formulated as

$$\begin{aligned} \min_{x_k} J &= \sum_{k=1}^{n} (y_k - y_{m,k})^2 \\ &\text{subject to: system dynamics (2)} \end{aligned} \tag{3}$$

In contrast, the TLS, also called orthogonal distance regression, makes more realistic assumption on noise that both the input and output variables cannot be measured perfectly and there are noises associated with these variables [6]. The measured input $u_{m,k}$ is the true but unknown input u_k corrupted with noise w_k, i.e., $u_{m,k} = u_k + w_k$. Because all data are noisy and the noise is iid, the maximum possible location of each data each data point can move in both u and y direction. TLS minimizes the sum of orthogonal squared distance from data points to the estimated curve. TLS can be applied to different fields, such as state/parameter estimation in optimal control and curve fitting in statistics.

This paper focus on solving the following TLS estimation problem for dynamic systems given noisy input and output with unknown initial condition

$$\begin{aligned} \min_{x_k,u_k} J &= \frac{\sum_{k=1}^{n} (y_k - y_{m,k})^2}{\delta_y^2} + \frac{\sum_{k=1}^{n} (u_k - u_{m,k})^2}{\delta_u^2} \\ &\text{subject to: equatiuon (2)} \end{aligned} \tag{4}$$

The cost function J has two weighted squared errors: (i) the difference between the estimated output y_k and the measured output $y_{m,k}$ (ii) the difference between true but unknown input u_k and the measured input $u_{m,k}$. The sum of squares of output and input are weighted by the their variances δ_y^2 and δ_u^2 respectively. The weights represent the confidence the estimator puts on input and output data. For instance, with a very noisy output (i.e., larger δ_y) the estimator tends to trust more on the input data to extract more information. The optimization tries to minimize the weighted and squared estimation error subject to system dynamics.

There are two challenges when one solves the estimation problem (4). First, the optimization problem always has a high dimension, which makes it infeasible to solve. The optimization variables are always all of the input u_k and state x_k at each sampling time, which results an optimization problem with $n \times (n_u + n_x)$ variables and it can easily be hundreds of variables. Second, the dynamic equality constraint (2) makes the problem more difficult to be solved, especially for nonlinear systems. The proposed framework can transform the problem (4) into an unconstrained optimization problem with only $N \times n_u$ variables, where N is the number of collocation points and is typically much smaller than the number of data points.

III. DIFFERENTIALLY FLAT SYSTEMS

The system (1) is differentially flat if there exists a fictitious variable, flat output z, such that [8], [11]

1) the state x and input u can be expressed in terms of the flat output z and a finite number of its derivatives as

$$x = \phi_x(z, \dot{z}, \ldots, z^{(\alpha)}) \tag{5a}$$

$$u = \phi_u(z, \dot{z}, \ldots, z^{(\beta)}) \tag{5b}$$

2) the flat output z can be expressed in terms of state x, input u, and a finite number of input's derivatives

$$z = \phi_z(x, u, \dot{u}, \ldots, u^{(\gamma)}), \tag{6}$$

where α, β, γ are integers which vary for different systems and $z^{(r)}$ is the r^{th} derivative with respect of time.

Suppose system (1) is flat and we simplify the notation in (5) using z to express the flat output z and its derivatives, since if the derivatives are known given the trajectory of z. According to the description of flat systems, the optimization problem (4) can be transformed into the following

$$\begin{aligned}\min_z J &= \frac{\sum_{k=1}^{n}(g_k(\phi_x(z), \phi_u(z)) - y_{m,k})^2}{\delta_y^2} \\ &+ \frac{\sum_{k=1}^{n}(\phi_{u,k}(z) - u_{m,k})^2}{\delta_u^2} \\ &= \frac{\sum_{k=1}^{n}(G_{z,k}(z) - y_{m,k})^2}{\delta_y^2} + \frac{\sum_{k=1}^{n}(\phi_{u,k}(z) - u_{m,k})^2}{\delta_u^2}\end{aligned} \tag{7}$$

where the function $g_k(\cdot)$ and $\phi_{u,k}(\cdot)$ are the functions $g(\cdot)$ and $\phi_u(\cdot)$ evaluated at sampling time t_k and are the functions of flat output z. Note that $g_k(z) \neq g(z_k)$ and $\phi_{u,k}(z) \neq \phi_u(z_k)$ because that the information of the whole trajectory of flat output is required to calculate the derivatives of z.

Essentially, the problem (4) is transformed into the flat output space where one only needs to optimize z, the trajectory of flat output, with which one can express the derivatives of flat output. Since the exploitation of differential flatness implies system dynamics, the system dynamics are automatically satisfied and there is no system dynamic equality constraint. The end product is the problem (7), an unconstrained optimization problem.

IV. LEGENDRE PSEUDOSPECTRAL METHODS WITH THE FLATNESS

This paper uses pseudospectral methods to solve the trajectory of flat output z, which furnishes a more efficient discretization method compared to traditional evenly distributed discretization points. This section briefly introduces the implementation of the Legendre pseudospectral method (LPM) for differentially flat systems.

As a class of efficient direct methods, Pseudospectral methods are widely used in different fields [9], [12]. One advantage of pseudospectral methods is that pseudospectral methods have a exponential convergence rate for smooth problems [13]. These methods discretize the optimization problem using collocation points which are unevenly distributed in temporal domain. The choice of collocation points features different versions of pseudospectral methods, such as the LPM and the Gauss pseudospectral method (GPM). Finally, original problems are transformed into NLP problems.

This paper adopts the LPM. The LPM uses Legendre-Gauss-Lobatto (LGL) points as collocation points which are the roots of the first derivative of Nth-degree of Legendre polynomial, P_N [10], [12], where N is the number of discretization points (which is the same set as collocation points). The LGL points are located in the range $[-1, 1]$. Thus, to use this set of collocation points as discretization points, time should be mapped from $t \in [t_0, t_f]$ to $\tau \in [-1, 1]$

$$t = \frac{(t_f - t_0)\tau + (t_f + t_0)}{2} \tag{8}$$

where t_0 and t_f are the initial time and final time of optimization.

The flat output $z(\tau)$ is approximated by a basis of N Lagrange polynomials based on the N collocation points

$$z(\tau) \approx \mathbf{z}(\tau) = \sum_{j=1}^{N} L_j(\tau) z(\tau_j) \tag{9}$$

where $\mathbf{z}$ is the interpolated flat output trajectory and the Lagrange polynomial bases are

$$L_j(\tau) = \prod_{\substack{i=0 \\ i \neq j}}^{N} \frac{\tau - \tau_i}{\tau_j - \tau_i} \tag{10}$$

These Lagrange polynomials have the property

$$L_j(\tau_i) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{11}$$

This gives accurate interpolated flat output at collocation points, that is

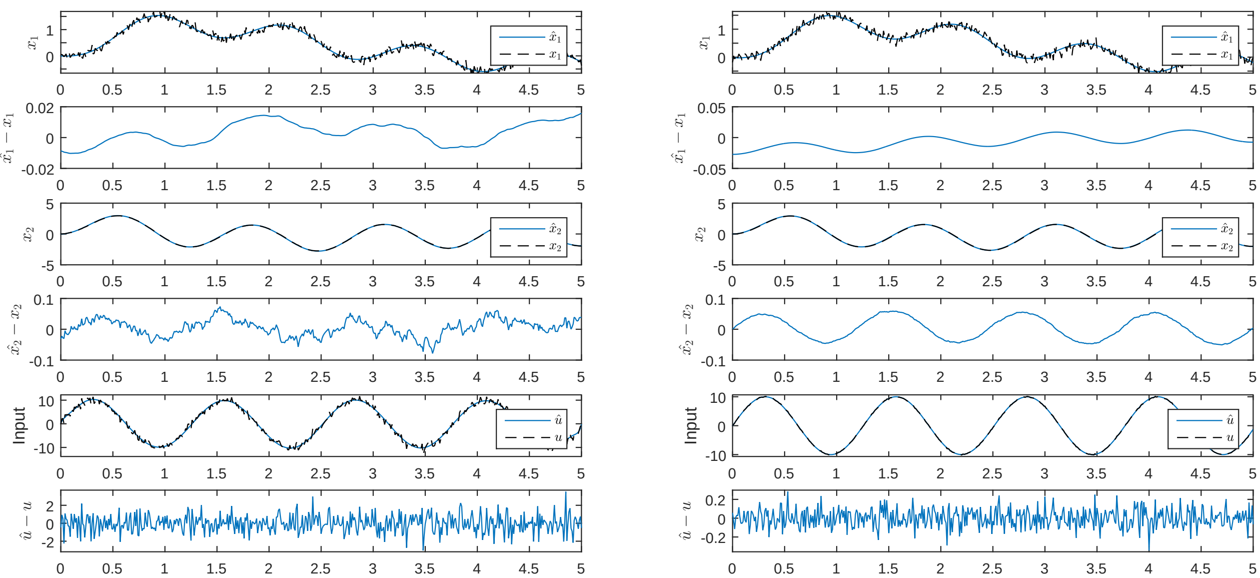
$$\mathbf{z}(\tau_j) = z(\tau_j) \tag{12}$$

One benefit of the LPM is that the derivatives can be calculated analytically using the Lagrange polynomials. The first order derivation of flat output z can be expressed as

$$\dot{\mathbf{z}}(\tau_i) = \sum_{j=1}^{N} \dot{L}_j(\tau_i) \mathbf{z}(\tau_j) = D_1(j, i) Z \tag{13}$$

with $Z := [\mathbf{z}(\tau_1), \ldots, \mathbf{z}(\tau_N)]^T$ and $D_1 \in \mathbb{R}^{N \times N}$ is the pseudospectral differentiation matrix which has the relationship $D_1 = \dot{L}_j(\tau_i)$. The differentiation matrix can be calculated either using (10) or using the following formula

$$D_1(j, i) = \begin{cases} \frac{P_{N-1}(\tau_k)}{P_{N-1}(\tau_i)} \frac{1}{\tau_k - \tau_i} & \text{if} & k = i \\ -\frac{(N-1)N}{4} & \text{if} & k = i = 1 \\ \frac{(N-1)N}{4} & \text{if} & k = i = N \\ 0 & \text{otherwise} & \end{cases} \tag{14}$$



(a) Estimation results with $\delta_u = 1$ and $\delta_y = 0.1$

(b) Estimation results with $\delta_u = 0.1$ and $\delta_y = 0.1$

Fig. 2: Estimation results using sine wave sin(5t) with noise

The r^{th} order derivatives of flat output z can be expressed as

$$\mathbf{z}^r(\tau_i) = \sum_{j=1}^{N} L_j^{(r)}(\tau_i)\mathbf{z}(\tau_j) = D_r(j,i)Z \tag{15}$$

It is shown that the high order differentiation matrix D_r can be expressed as the r^{th} power of D_1, that is, $D_r = D_1^r$.

The l^{th} order derivative of the flat output $\mathbf{z}^{(l)}$ can be derived by iteratively calculating the first derivative of Eqn. (9) l times with the consideration of the initial state $x(\tau_0)$. The first derivative of the flat output evaluating at the collocation points gives

where $z(\tau_0)$ is the initial flat output which can be derived based on the given initial state $x(t_0)$. And $\dot{L}_j(\tau)$ represents the first derivative of Lagrange polynomials $L_j(\tau)$ for $j = 0$, $\ldots$, N.

Substituting Eqn. (13) into Eqn. (5) gives the mappings from z to x and u

$$x(\tau_i) = \phi_{x,i}(\mathbf{z}(\tau_1), \ldots, \mathbf{z}(\tau_N)) \tag{16a}$$

$$u(\tau_i) = \phi_{u,i}(\mathbf{z}(\tau_1), \ldots, \mathbf{z}(\tau_N)) \tag{16b}$$

where the mappings are $\phi_{x,i} : \mathbb{R}^N \mapsto \mathbb{R}_x^n$ and $\phi_{u,i} : \mathbb{R}^N \mapsto \mathbb{R}_u^n$.

Then the cost function is then calculated. For traditional pseudospectral methods which is used to solve optimal control input, the cost function is calculated using quadrature rule at collocation points. This requires the data of state and input variables at collocation points and typically optimal control problems have this data because the state and input variables evaluated at collocation points are the optimization variables. The measurement data in estimation problems, however, is only available at sampling time, unlike collocation points, is always evenly distributed. There are two ways to calculate the cost function in problem (4): (i) adopts the quadrature rule at collocation points by interpolating the data [14]; (ii) calculates the summation of squared errors at each sampling time by interpolating the flat output z. This paper chooses the second way, since this guarantees that the data used for cost function calculation is real data rather than interpolated and approximated data. The flat output z is interpolated using Lagrange polynomial following (9).

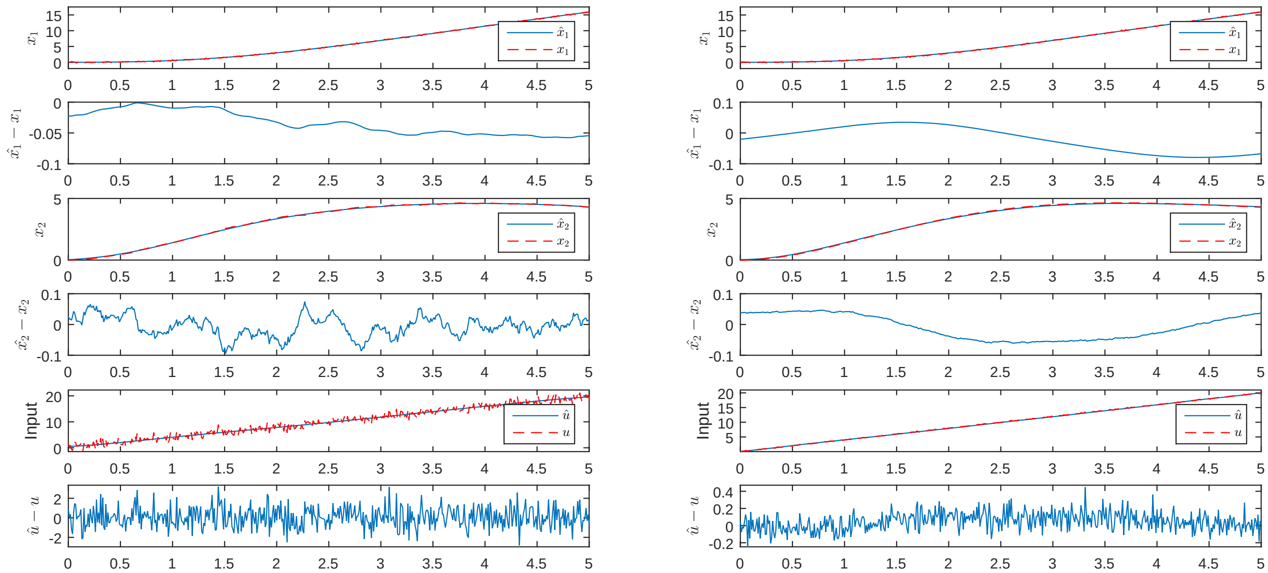
Finally, the constrained optimization problem can be transformed as an unconstrained NLP problem

$$\min_{Z} J = \frac{\sum_{k=1}^{n}(G_{z,k}(z) - y_{m,k})^2}{\delta_y^2} + \frac{\sum_{k=1}^{n}(\phi_{u,k}(z) - u_{m,k})^2}{\delta_u^2} \tag{17}$$

where the optimization variable is $Z \in \mathbb{R}^{Nn_u}$. Compared to the original problem (4), the resulting problem (17) is an unconstrained NLP problem with $N \times n_u$ optimization variables and can be solved with well-developed algorithms. System dynamics are automatically satisfied by the exploitation of differential flatness using (5). Note that if the problem (4) is solved using the traditional way (i.e., using the state and input variables at each sampling time as the optimization variables), the total number of optimization variables would be $n \times (n_u + n_x)$.

V. AN ILLUSTRATIVE EXAMPLE

This paper demonstrates the proposed flatness-based pseudospectral methods for TLS estimation using a second order mass-spring-damper system.



(a) Estimation results with $\delta_u = 1$ and $\delta_y = 0.1$

(b) Estimation results with $\delta_u = 0.1$ and $\delta_y = 0.1$

Fig. 3: Estimation results using affine input with noise

A. Estimation Problem Formulation

The estimation problem can be formulated as follows. First, the dynamics of the mass-spring-damper system in the state-space representation are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u \\ y &= x_1 \end{aligned} \tag{18}$$

where x_1 and x_2 are the displacement and velocity of the mass with the mass $m = 1kg$, spring constant $k = 1N/m$, and the damping coefficient $c = 1N \cdot s/m$. The measured output $y_{m,k}$ is the true but unknown displacement y_k corrupted by measurement noise v_k. Similarly, the measured output $u_{m,k}$ is u_k the true force acted on the system with noise w_k.

Then the TLS estimation problems can be formulated as

$$\min_{x_k, u_k} J = \frac{\sum_{i=1}^{n}(y_k - y_{m,k})^2}{\delta_y^2} + \frac{\sum_{i=1}^{n}(u_k - u_{m,k})^2}{\delta_u^2} \quad \text{subject to: dynamic model (18)} \tag{19}$$

The goal of this estimation problem (19) is to estimate the displacement x_1, velocity x_2, and true input u_k, given measured noisy output $y_{m,k}$. The noise is assumed to be white Gaussian noise. To solve this estimation problem, the flatness-based pseudospectral methods are adopted. First, since the system (18) is in controllable canonical form, the structure of controllable canonical form automatically implies the system is differentially flat. Thus, it is possible to transform the problem (19) into the flat output space. The flat output is chosen to be the displacement, i.e., $z = x_1$. Second, the trajectory of the flat output is optimized using the pseudospectral method.

B. Results

This paper adopts two kinds of input to test the TLS estimator: the first input is a sine wave and second is an affine input. First, Fig. 2 depicts the estimation results using sine wave are with different standard deviation on input and output. The variables $\hat{x}_1, \hat{x}_2, \hat{u}$ are estimated displacement, velocity, and input, respectively. The duration of the problem is $t \in [0s, 5s]$ with the sampling time $\Delta_t = 0.01s$, which implies the number of sampling points $n = 501$. The number of collocation points is set to be $N = 30$. The computational advantage of adopting the proposed framework for TLS estimation is apparent: the resulting problem is an unconstrained NLP problem with $N \times n_u = 30$ optimization variables. However, if the problem is discretized using the finite difference method, it would result in a problem with $n \times (n_u + n_x) = 1503$ optimization variables and $n \times n_x = 1002$ dynamical equality constraints.

Additionally, the other advantage of the proposed framework is that the estimator does not require accurate initial condition. Actually, the initial condition is included as optimization variable and is optimized. This paper adopts the true initial condition as $x(t_1) = [0, 0]^T$ and the initial guess for TLS estimator is set to be $\hat{x}(t_1) = [-10, -10]^T$. Although the initial guess for the estimator is incorrect, the estimator can estimate the initial states and input relatively accurately.

The estimation results shown in Fig. 3 and Fig. 2 estimate state and input variables relatively accurately. Since the displacement has slower dynamics than the velocity and the output measures the displacement, the estimated displacement $\hat{x}_1$ always has higher accuracy than the estimated velocity $\hat{x}_1$ Additionally, the Fig. 2a and Fig. 3a have large uncertainty in the input and therefore the estimated input has larger absolute error, which is still bounded in the one standard deviations.

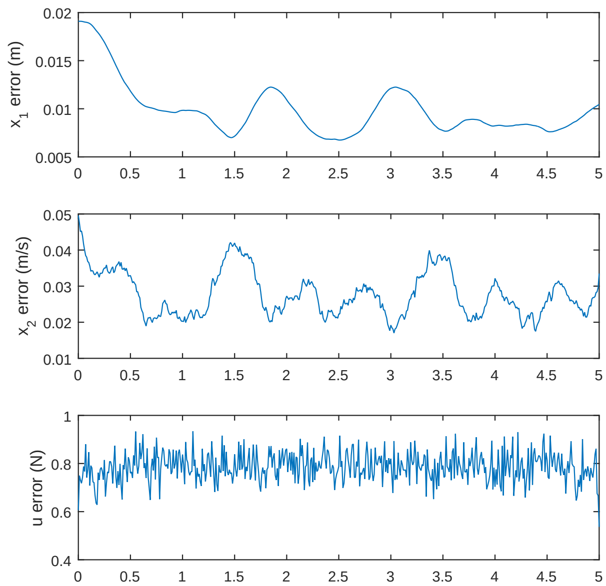


Fig. 4: Monte Carlo simulation of averaged absolute errors

This paper also conducts a Monte-Carlo simulation to show that the TLS estimator together with the proposed flatness-based LPM works as expected by showing the errors are within three standard deviations (actually, it is bounded within one standard deviation). The Monte Carlo simulation has 100 estimation runs which use the same statistical parameter set with the one used in Fig. 2. Each simulation run records the absolute values of estimation errors and at the end of Monte Carlo simulation the errors are averaged by dividing the total number of simulation. The final results shown in Fig. 4 represents the general statistical property of the TLS estimator. It can be seen that the TCL estimator using the proposed framework is very accurate.

VI. CONCLUSION

This paper proposes a computationally efficient framework for total least squares (TLS) estimation for dynamics systems. This framework exploits the differential flatness property of flat systems, which transforms the original system in flat output space. The trajectory of the resulting flat output is then optimized using pseudospectral methods. For TLS estimation problem, the advantages using the proposed framework are the following. (i) it significantly reduces the number of optimization variables to make TLS hesitation problems feasible to solved with high speed; (ii) the end product is an unconstrained nonlinear programming (NLP) problem which is relatively easy to solve; (iii) it does not require accurate initial condition, since the initial condition is included in the optimization process. Moreover, this paper uses the proposed framework to solve an TLS estimation problem with a second order mass-spring-damper system. The results demonstrate the accuracy of the estimation process by showing estimation errors are within the tight one standard deviations.

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REFERENCES

[1] M. J. Rothenberger, J. Anstrom, S. Brennan, and H. K. Fathy, "Maximizing parameter identifiability of an equivalent-circuit battery model using optimal periodic input shaping," in *ASME 2014 Dynamic Systems and Control Conference*. American Society of Mechanical Engineers, 2014, pp. V001T19A004–V001T19A004.

[2] M. Aoki, *Optimization of stochastic systems*. Academic Press New York, 1967.

[3] G. L. Plett, "Recursive approximate weighted total least squares estimation of battery cell total capacity," *Journal of Power Sources*, vol. 196, no. 4, pp. 2319–2331, 2011.

[4] I. Markovsky and S. Van Huffel, "Overview of total least-squares methods," *Signal processing*, vol. 87, no. 10, pp. 2283–2302, 2007.

[5] S. Van Huffel, "Total least squares and errors-in-variables modeling: bridging the gap between statistics, computational mathematics and engineering," in *COMPSTAT 2004Proceedings in Computational Statistics*. Springer, 2004, pp. 539–555.

[6] S. Van Huffel and J. Vandewalle, *The total least squares problem: computational aspects and analysis*. Siam, 1991, vol. 9.

[7] C.-T. Chen, *Linear system theory and design*. Oxford University Press, Inc., 1995.

[8] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Flatness and defect of non-linear systems: introductory theory and examples," *International journal of control*, vol. 61, no. 6, pp. 1327–1361, 1995.

[9] J. Liu, G. Li, and H. Fathy, "Efficient lithium-ion battery model predictive control using differential flatness-based pseudospectral methods," in *Dynamic Systems and Control Conference*. ASME, 2015, to be appear.

[10] D. Benson, "A gauss pseudospectral transcription for optimal control," Ph.D. dissertation, Massachusetts Institute of Technology, 2005.

[11] I. M. Ross and F. Fahroo, "Pseudospectral methods for optimal motion planning of differentially flat systems," *Automatic Control, IEEE Transactions on*, vol. 49, no. 8, pp. 1410–1413, 2004.

[12] G. T. Huntington, "Advancement and analysis of a gauss pseudospectral transcription for optimal control problems," Ph.D. dissertation, University of Florida, 2007.

[13] I. M. Ross and F. Fahroo, "Legendre pseudospectral approximations of optimal control problems," in *New Trends in Nonlinear Dynamics and Control and their Applications*. Springer, 2003, pp. 327–342.

[14] Q. Gong, I. M. Ross, and W. Kang, "A pseudospectral observer for nonlinear systems," *DISCRETE AND CONTINUOUS DYNAMICAL SYSTEMS SERIES B*, vol. 8, no. 3, p. 589, 2007.