Exact cosmological solutions of scale-invariant gravity theories

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Abstract. We have found new anisotropic vacuum solutions for the scale-invariant gravity theories which generalise Einstein's general relativity to a theory derived from the Lagrangian $R^{1+\delta}$. These solutions are expanding universes of Kasner form with an initial spacetime singularity at $t = 0$ and exist for $-1/2 < \delta < 1/4$ but possess different Kasner index relations to the classic Kasner solution of general relativity to which they reduce when $\delta = 0$. These solutions are unperturbed by the introduction of non-comoving perfect-fluid matter motions if $p < \rho$ on approach to the singularity and should not exhibit an infinite sequence of chaotic Mixmaster oscillations when $\delta > 0$.

There have been many studies of gravity which consider generalisations of the Einstein-Hilbert action of general relativity by the addition of terms of higher-order in the curvature to the Lagrangian, see refs in [1, 3]. These are of interest for the evolution of cosmological models close to the initial singularity and at late times, for the study of quantum gravitational phenomena, inflation, and the observational consequences on solar system scales. In particular, it is possible to study the structure of gravity theories in which the Lagrangian is quadratic in the curvature or an arbitrary analytic function of the curvature [4, 1, 5]. Unfortunately, such studies are complexified by the lack of exact solutions of the field equations for these theories other than the special cases provided by the solutions of general relativity. The field equations are typically fourth order and difficult to deal with except for spacetimes with isotropy and homogeneity or static spherical symmetry. We have recently considered in detail a different type of generalisation of Einstein's theory which derives from a power-law Lagrangian of the form $R^{1+\delta}$, where δ is constant. This theory reduces to general relativity when $\delta \to 0$, and both the flat Friedmann universes [3, 6, 7] and static spherically symmetric solutions [6] can be found, tested for stability, and confronted with cosmological and perihelion-precession observations to produce tight observational limits on δ of [6]

$$0 \le \delta < 7.2 \times 10^{-19}.$$

In this letter we will show that it is also possible to find exact anisotropic and spatially inhomogeneous cosmological solutions of this theory of Kasner type. These seem to be the first exact anisotropic solutions of higher-order Lagrangian theories of gravity and provide a testing ground for a range of interesting investigations. Kasner solutions [8, 9] form the building blocks for an understanding of the behaviour of the most general known solutions of the Einstein equations and allow us to understand the conditions under which chaotic behaviour arises and persists on approach to the initial cosmological singularity [10, 11, 13]. They also provide a simple environment in which to study quantum gravitational effects like particle production [14], and the conditions under which shear anisotropy dominates the expansion of the universe at early and late times. The simplicity of these new solutions is appealing and allows for a complete understanding of their behaviour as δ varies. They also provide an instructive context in which to evaluate the nature of the general relativistic evolution and to determine whether it is typical or atypical within this wide class of theories.

We consider here a gravitational theory derived from the Lagrangian density

$$\mathcal{L}_G = \frac{1}{\chi}\sqrt{-g}R^{1+\delta}, \qquad (1)$$

where δ and χ are constants. The limit $\delta \to 0$ gives us the familiar Einstein–Hilbert Lagrangian of general relativity and we are interested in the observational consequences of $|\delta| > 0$. There is a conformal equivalence between this theory and general relativity with a scalar field possessing an exponential self-interaction potential [15, 16].

We denote the matter action as S_m and ignore the boundary term. Extremizing

$$S = \int \mathcal{L}_G d^4x + S_m,$$

with respect to the metric g_{ab} then gives [17]

$$\delta(1-\delta^2)R^\delta \frac{R_{,a}R_{,b}}{R^2} - \delta(1+\delta)R^\delta \frac{R_{;ab}}{R} + (1+\delta)R^\delta R_{ab} \tag{2}$$

$$-\frac{1}{2}g_{ab}RR^\delta - g_{ab}\delta(1-\delta^2)R^\delta \frac{R_{,c}R_{,}^{\ c}}{R^2} + \delta(1+\delta)g_{ab}R^\delta \frac{\Box R}{R} = \frac{\chi}{2}T_{ab}, \tag{3}$$

where T_{ab} is the energy–momentum tensor of the matter, and is defined in the usual way. We take the quantity R^δ to be the positive real root of R throughout.

We seek solutions of these equations for the Bianchi type I metric

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \tag{4}$$

where the Kasner indices, p_i, are constants. This metric is a solution of the vacuum field equations of the $R^{1+\delta}$ theory if and only if the Kasner indices satisfy the algebraic relations

$$(1-\delta)(p_1^2+p_2^2+p_3^2) + 3\delta(1+2\delta) = (1+2\delta^2)(p_1+p_2+p_3), \tag{5}$$

$$(1-\delta)(p_1p_2+p_1p_3+p_2p_3) + 3\delta^2(1+2\delta) = \delta(2+\delta)(p_1+p_2+p_3) \tag{6}$$

These constraints can be solved to yield two classes of solutions. The first has

$$\sum_{i=1}^{3} p_i = \frac{3\delta(1+2\delta)}{(1-\delta)}$$

$$\sum_{i=1}^{3} p_i^2 = \frac{3\delta^2(1+2\delta)^2}{(1-\delta)^2}.$$

These are only solved by the isotropic solution with

$$p_1 = p_2 = p_3 = \delta\frac{(1+2\delta)}{(1-\delta)}.$$

This is the zero-curvature vacuum Friedmann universe found by Bleyer and Schmidt [2, 3]. The second class of solutions to (5)-(6) is new and generalises the anisotropic Kasner universe of general relativity to

$$\sum_{i=1}^{3} p_i = 1+2\delta \tag{7}$$

$$\sum_{i=1}^{3} p_i^2 = 1-4\delta^2. \tag{8}$$

Note that when $\delta = 0$ this reduces to the standard Kasner solution of general relativity [8]. The first constraint (7) means that we must have $\delta > -1/2$ for an expanding universe. The second constraint (8) requires $-1/2 < \delta < 1/2$ for consistency. When δ falls outside this range then anisotropic solutions with this simple power-law form do

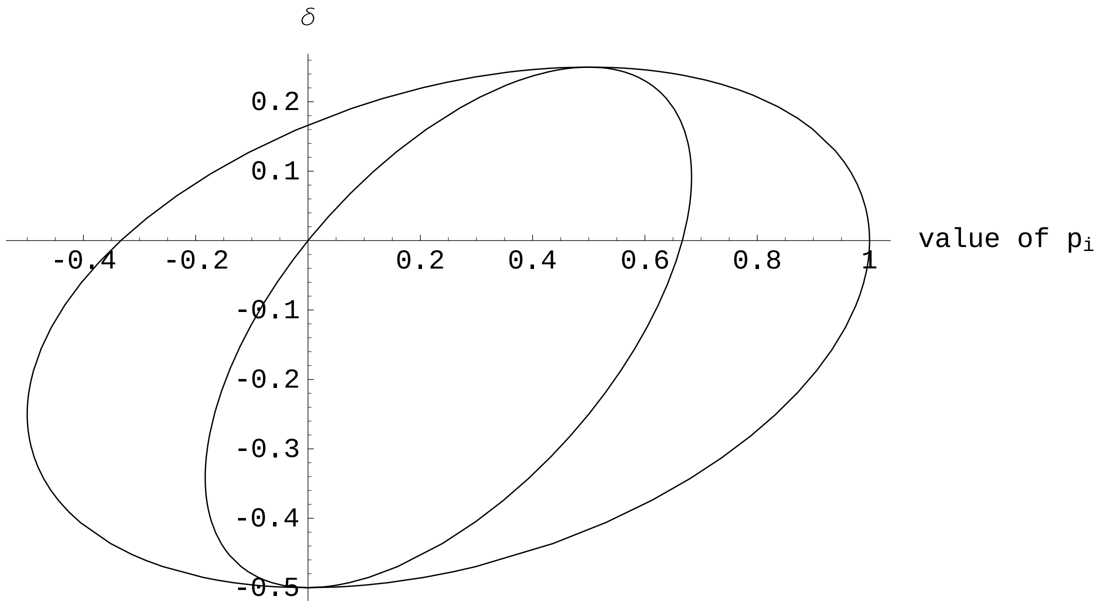

Figure 1. *The intervals in which the Kasner indices p_i lie can be read off this graph. For any value of δ in the range $-\frac{1}{2} < \delta < \frac{1}{4}$ a horizontal line is drawn; the boundaries of the intervals in which the p_i lie are then given by the four points at which the horizontal line crosses the two closed curves defined by (9)-(11). For $\delta = 0$ these boundaries can be seen to be $-\frac{1}{3}$, 0, $\frac{2}{3}$ and 1, as expected for the Kasner solution in general relativity.*

not exist. These constraints restrict the range of variation of the individual p_i as follows (assuming with out loss of generality that they are ordered $p_1 < p_2 < p_3$):

$$\frac{1+2\delta-2\sqrt{(1+2\delta)(1-4\delta)}}{3} \leq p_1 \leq \frac{1+2\delta-\sqrt{(1+2\delta)(1-4\delta)}}{3} \tag{9}$$

$$\frac{1+2\delta-\sqrt{(1+2\delta)(1-4\delta)}}{3} \leq p_2 \leq \frac{1+2\delta+\sqrt{(1+2\delta)(1-4\delta)}}{3} \tag{10}$$

$$\frac{1+2\delta+\sqrt{(1+2\delta)(1-4\delta)}}{3} \leq p_3 \leq \frac{1+2\delta+2\sqrt{(1+2\delta)(1-4\delta)}}{3}. \tag{11}$$

These ranges are plotted in Figure 1 for different values of δ. They impose the further constraint $-1/2 < \delta < 1/4$ on the range of allowed values of δ for which real-valued solutions exist. A horizontal line of constant δ intersects the two closed curves at four points which give three line segments covering the allowed ranges of the Kasner indices for all $-1/2 \leq \delta \leq 1/4$. For $\delta = 0$, the intersects give the ranges for the Kasner solution in general relativity: $-1/3 \leq p_1 \leq 0 \leq p_2 \leq 2/3 \leq 1$.

We note the following features of these solutions, they have a Weyl curvature singularity at $t = 0$ so long as $\delta > -1/2$. Solutions with $\delta < -1/2$ are contracting universes and have a 'big rip' singularity as $t \to \infty$. When $\delta > 0$ it is possible for all the Kasner indices to take positive values for every positive value of δ. When $\delta < 0$ we see that $p_1 < 0$ always and p_2 can also take negative values but the universe expands in volume. The measure of the positive values for p_1 increases as δ increases to its maximum allowed value. This situation is of particular interest in connection with the behaviour of more general Bianchi type IX Mixmaster cosmologies in these gravity theories. In general relativity we know that these cosmologies exhibit chaotic oscillatory behaviour on approach to the initial Weyl curvature singularity at $t = 0$ in vacuum [10, 11, 12, 13]. In order for an infinite number of chaotic oscillations to occur we need one of the Kasner indices to take only negative values. This is the case in vacuum in general relativity. When a scalar field is added the Kasner relations change and it is possible for all indices to be positive and the oscillatory sequence of permutations of the scale factors will cease after a finite number of oscillations [18]. The subsequent evolution will be in general of Kasner type. We expect a similar sequence of events in the Mixmaster cosmologies in $R^{1+\delta}$ gravity theories when $\delta > 0$. Oscillations of the scale factors as $t \to 0$ will eventually permute the Kasner indices into one of the combinations in which they are all positive and oscillations will cease leaving the evolution to proceed in Kasner form. By contrast, we expect chaotic oscillations to persist when $-1/2 < \delta < 0$ although their detailed structure [19] will differ from those that occur in general relativity and will be investigated elsewhere.

The new form of the Kasner index constraints found here differs from the form found in general relativity when a massless scalar field is added [20, 23] because a change is made to the sum of the indices (7) as well as to the sum of their squares, (8).

We can also investigate the extent to which this vacuum metric (4) is a stable asymptotic solution of the field equations in the presence of perfect-fluid matter as $t \to 0$ as it is in general relativity. We need to determine whether the form of the exact solution leads to fluid stresses which grow faster than the vacuum anisotropy terms (which are all $O(t^{-2(1+\delta)})$ as $t \to 0$). Similar investigations can be made *mutatis mutandis* for the case of fluids with anisotropic pressures [21].

Consider fluid motions in the background Kasner metric and assume that the material content of the universe is a perfect fluid with equation of state $p = (\gamma - 1)\rho$ with $1 \leq \gamma < 2$. The continuity equation is [22, 20, 24]

$$\frac{\partial}{\partial x^i}(t^{p_1+p_2+p_3}\rho^{1/\gamma}u^i) = 0, \tag{12}$$

where u^i is the normalised 4-velocity ($u_a u^a = 1$), and the momentum conservation equation is

$$(\rho + p)u^k(u_{i;k} - \frac{1}{2}u^l g_{kl,i}) = -\frac{1}{3}\rho_{,i} - u_i u^k p_{,k}. \tag{13}$$

Neglecting the space derivatives with respect to the time derivatives, so that we confine attention to scales larger than the particle horizon in the velocity-dominated approximation, we have

$$t^{p_1+p_2+p_3}u_0\rho^{1/\gamma} = const\ ; \qquad u_\alpha \rho^{(\gamma-1)/\gamma} = const.$$

For relativistic motions, keeping the dominant velocity component ($u^3 = u_3 t^{-2p_3}$ and $u_0^2 \sim u_3 u^3 \sim (u_3)^2 t^{-2p_3}$) as $t \to 0$ gives to leading order $\rho \sim t^{-\gamma(p_1+p_2)/(2-\gamma)}$ and $u_\alpha \sim t^{(p_1+p_2)(\gamma-1)/(2-\gamma)}$. Using these asymptotic forms we can now check that all the components of T_a^b which they induce on the right-hand side of the field equations (3) diverge more slowly than the vacuum terms, $t^{-2(1+\delta)}$, as $t \to 0$. This is the condition for the Kasner solution to be unperturbed by the metric effects of the fluid motions. We have to leading order that

$$T_0^0 \sim \rho u_0^2 \sim t^{-1-2\delta-p_3} \quad ; \quad T_1^1 \sim \rho \sim t^{-\frac{\gamma}{(2-\gamma)}(1+2\delta-p_3)} \tag{14}$$

$$T_2^2 \sim \rho u_2 u^2 \sim t^{-2p_2 \ -(1+2\delta-p_3)} \ ; \quad T_3^3 \sim \rho u_3 u^3 \sim t^{-1-2\delta-p_3}. \tag{15}$$

In the simple case of $\gamma = 4/3$ where the 4-velocity of the fluid is comoving ($u_i = \delta_i^0$) we require only that $\rho \sim t^{-2-4\delta+2p_3}$ diverges slower than $t^{-2(1+\delta)}$ and, this requires only that $p_3 > \delta$, which is always satisfied. For general γ the worst divergence created by relativistic motions is in $\rho u_3 u^3 \sim t^{-1-2\delta-p_3}$ and this is always slower than $t^{-2(1+\delta)}$ since $p_3 < 1$. But for stiffer equations of state we have to careful that the assumption that the velocities are increasingly relativistic ($u^0 >> 1$) as $t \to 0$ continues to hold. This requires $\gamma-1+2\delta < p_3$. If this inequality fails then $u^\alpha u_\alpha \to 0$ and $u^0 \sim 1$ as $t \to 0$ and we need to re-solve (12)-(13) in the approximation where the velocities $u_\alpha u^\alpha$ go to zero and $u^0 \simeq 1$ because the ultra-stiff fluid makes the motions grind to a halt [25]. In this case $\rho \sim t^{-\gamma(1+2\delta)}$ and $u_\alpha \sim t^{(\gamma-1)(1+2\delta)}$. Since we have $u_\alpha u^\alpha \sim (u_3)^2 t^{-2p_3} \sim t^{2(\gamma-1)(1+2\delta)-2p_3}$ this behaviour occurs when

$$(\gamma-1)(1+2\delta) > p_3. \tag{16}$$

Since $u_\alpha u^\alpha \to 0$ as $t \to 0$ we will always have $\rho >> \rho u_\alpha u^\beta$ as $t \to 0$ and we need only check that ρ diverges more slowly than the vacuum terms $O(t^{-2(1+\delta)})$ in order to check that the Kasner solution for the metric is unperturbed by the fluid. This requires $(\gamma-2)(2\delta+1) < 0$ and since $\gamma < 2$ we require $2\delta + 1 > 0$ for the vacuum term to dominate. This is just the condition for the universe volume to be expanding as t increases, (7), and always holds.

Hence, the Kasner solutions we have found will provide a good description of the general spatially homogeneous perfect-fluid solutions of the $R^{1+\delta}$ gravity theories in the vicinity of an initial cosmological solution. They may also provide a useful approximation to the time dependence of a general inhomogeneous cosmological solution to these theories.

In summary, we have found new anisotropic vacuum solutions for the scale-invariant gravity theories which generalise Einstein's general relativity to a Lagrangian $R^{1+\delta}$. These solutions are expanding universes of Kasner form with an initial spacetime singularity at $t = 0$ and exist if $-1/2 < \delta < 1/4$ but have different Kasner-index relations to the classic Kasner solution of general relativity when $\delta \neq 0$. These solutions are unperturbed by the introduction of non-comoving fluid matter motions if $p < \rho$ on approach to the singularity for this range of δ, and do not exhibit an infinite sequence of chaotic Mixmaster oscillations when $\delta > 0$. They should provide a simple new testing ground for quantum cosmological processes and late-time behaviour in theories 'close' to Einstein's general relativity.

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References

[1] Barrow J D and Ottewill A C 1983 *J. Phys. A* **16** 2757
[2] Bleyer U and Schmidt H J 1990 *Int. J. Mod. Phys. A* **5** 4671
[3] Schmidt H J 2004 *gr-qc/0407095*
[4] Kerner R 1982 *Gen. Rel. Grav.* **14** 453
[5] Ruzmaikina T V and Ruzmaikin A A 1969 *Sov. Phys. JETP* **57** 680
[6] Clifton T and Barrow J D 2005 *gr-qc/0509059*
[7] Carloni S, Dunsby P K S, Capoziello S and Troisi A 2004 *gr-qc/0410046*
[8] Kasner E 1925 *Trans. Am. Math. Soc.* **27** 101
[9] Lifshitz E M and Khalatnikov I M 1963 *Adv. Phys.* **12** 185
[10] Misner C W 1969 *Phys. Rev. Lett.* **22** 1071
[11] Belinskii V A, Khalatnikov I M and Lifshitz E M 1971 *Sov. Phys. Usp.* **13** 745
[12] Barrow J D 1981 *Phys. Rev. Lett.* **46** 963
[13] Barrow J D 1982 *Physics Reports* **85** 1
[14] Zeldovich Y B 1970 *Sov. Phys. JETP Lett.* **12** 307
[15] Barrow J D and S. Cotsakis 1994 *Phys. Lett. B* **214** 515
[16] Maeda K I 1989 *Phys. Rev. D* **39** 3159
[17] Buchdahl H A 1970 *Mon. Not. R. Astron. Soc.* **150** 1
[18] Belinskii V A and Khalatnikov I M 1973 *Soviet Physics JETP* **36** 591
[19] Chernoff D and Barrow J D 1983 *Phys. Rev. Lett.* **50** 134
[20] Barrow J D 1978 *Nature* **272** 211
[21] Barrow J D 1997 *Phys. Rev. D* **55** 7451
[22] Landau L and Lifshitz E M 1975 *The Classical Theory of Fields* 4th rev. edn. Pergamon
[23] Barrow J D and J Stein-Schabes, 1985 *Phys. Rev. D* **32** 1595
[24] Barrow J D 1977 *Mon. Not. Roy. astr. Soc.* **178** 625
[25] Barrow J D 1977 *Mon. Not. Roy. astr. Soc.* **179** 47P