Further Exact Cosmological Solutions to Higher-Order Gravity Theories.

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Abstract.
We investigate the effect of deviations from general relativity on approach to the initial singularity by finding exact cosmological solutions to a wide class of fourth-order gravity theories. We present new anisotropic vacuum solutions of modified Kasner type and demonstrate the extent to which they are valid in the presence of non-comoving perfect-fluid matter fields. The infinite series of Mixmaster oscillations seen in general relativity will not occur in these solutions, except in unphysical cases.

1. Introduction

There have been numerous studies in the literature on generalisations of the usual Einstein-Hilbert action of general relativity to more complicated functions of the curvature (see refs. in [1, 2]). The motivation for such studies comes from a variety of different sources ranging from attempts to include quantum effects in the gravitational action [3] to the investigation of phenomena that are presently inadequately explained in the standard model, such as dark energy [4, 5], and violations of the cosmic no hair theorem [6, 7]. Of particular interest is the behaviour of these modified theories of gravity in the high-curvature limit, where quantum corrections to general relativity are expected to become important and their influence on the occurrence of singularities and possible bounces of the universe at high curvatures can be studied [8]. It is the investigation of these modified theories of gravity on approach to the initial cosmological singularity that concerns us here. Previous studies have focussed on the behaviour of isotropic cosmologies but, as we know from the situation in general relativity, their behaviour can be misleading. Anisotropies diverge faster than isotropic densities at high curvatures and will generally dominate the behaviour of the cosmology at early (and in some cases even late) times. The most important anisotropic solutions of general relativistic cosmologies are the vacuum Kasner solutions [9] and their fluid-filled counterparts that form Type I of the Bianchi classification of three-dimensional homogeneous spaces. These universes are geometrically special, in vacuum (or perfect-fluid) cases they are defined by just one (or two) free constant(s) compared to the four (eight) that specify the most general spatially homogeneous vacuum (or perfect fluid) solutions. However, they have proved to provide an excellent dynamical description of the evolution of the most general models over finite time intervals. The chaotic vacuum Mixmaster universe of Bianchi Type IX undergoes a infinite sequence of chaotic space-time oscillations on approach to its initial or final singularities which is well approximated by a sequence of different Kasner epochs which form a Poincaré return mapping for the chaotic dynamical system [10, 11, 12, 13, 14, 15].

The way in which the Kasner metric has played a central role in the elucidation of the existence and structure of anisotropic cosmological models and their singularities in general relativity makes it the obvious starting point for an extension of that understanding to cosmological solutions of higher-order gravity theories. In an earlier Letter [16] we have reported the discovery of a new class of exact Kasner-like solutions for gravity theories derived from Lagrangians that are an arbitrary power of the scalar curvature. In this paper we will generalise that study to a much wider class of higher-order Lagrangian theories. We will determine the conditions for the existence of Kasner solutions and find their exact forms. In some cases these solutions are required to be isotropic and correspond to exact Friedmann-Robertson-Walker (FRW) vacuum solutions with zero spatial curvature. By studying gravity theories whose Lagrangians are derived from arbitrary powers of the curvature invariants we are able to find simple exact solutions. Past studies have usually focussed on the addition of higher-order curvature terms to the Einstein-Hilbert Lagrangian. This results in enormous algebraic complexity and exact solutions cannot be found. The results presented here provide a tractable route into understanding the behaviours of anisotropic cosmological models in situations where the higher-order curvature corrections are expected to dominate. The solutions we present are vacuum solutions but we provide a simple analysis which determines when the introduction of perfect fluids with non-coming velocities has a negligible effect on the cosmological evolution at early times.

2. Field Equations

We consider field equations derived from an arbitrary analytic function of the three possible linear and quadratic contractions of the Riemann curvature tensor; R, $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ [17]. The weight-zero Lagrangian density for such theories is

$$\mathcal{L}_G = \chi^{-1}\sqrt{g}f(X,Y,Z) \tag{1}$$

where χ is a constant and $f(X,Y,Z)$ is an arbitrary function of the variables X, Y and Z defined as $X = R$, $Y = R_{ab}R^{ab}$ and $Z = R_{abcd}R^{abcd}$. We ignore the boundary terms, for a discussion of which see ref. [18]. The field equations derived from the variation of the corresponding action are [18, 19]

$$\begin{aligned} P^{ab} = & -\frac{1}{2}fg^{ab} + f_X R^{ab} + 2f_Y R^{ac}R^{b}_{\ c} + 2f_Z R^{acde}R^{b}_{\ cde} \\ & - \Box(f_Y R^{ab}) - g^{ab}(f_Y R^{cd})_{;cd} + 2(f_Y R^{c(a})_{;\ \ c}^{\ \ b)} \\ & - f_{X;cd}(g^{ab}g^{cd} - g^{ac}g^{bd}) - 4(f_Z R^{d(ab)c})_{;cd} \\ = & \frac{\chi}{2}T_{ab} \end{aligned} \tag{2}$$

where a subscript X, Y or Z denotes differentiation with respect to that quantity.

We will be looking for spatially homogeneous, vacuum solutions of Bianchi type I, described by the line–element

$$ds^2 = -dt^2 + \sum_{i=1}^{3} t^{2p_i}dx_i^2 \tag{3}$$

where p_1, p_2 and p_3 are constants to be determined. For the special case $p_1 = p_2 = p_3$ these solutions correspond to spatially flat FRW metrics, which will be found to exist for various higher-order theories both in vacuum (where none exist in general relativity) and in the presence of a perfect fluid.

We can determine the number of independently arbitrary functions of three space variables that will characterise the *general* vacuum solution of the field equations on a Cauchy surface of constant time in these higher-order theories. The field equations are in general 4^{th} order in time; so if we choose a synchronous reference system then we need 6 functions each for the symmetric 3×3 tensors $g_{\alpha\beta}$, $\dot{g}_{\alpha\beta}$, $\ddot{g}_{\alpha\beta}$ and $\dddot{g}_{\alpha\beta}$. This gives 24 functions, but they may be reduced to 20 by using the 4 constraint equations, and finally again to 16 by using the 4 coordinate covariances. If a perfect fluid were included as a matter source the final number would rise to 20 due to the inclusion of the density and 3 non-comoving velocity components in the initial data count. We note that in general relativity the general vacuum solution is prescribed by 3 arbitrary functions of 3 space variables. In general relativity, the Kasner vacuum solution is prescribed by one free constant, as the three p_i satisfy two algebraic constraints.

3. Exact Solutions

For a realistic theory we should expect the dominant term of the analytic function $f(X, Y, Z)$ to be of the Einstein-Hilbert form in the Newtonian limit. However, there is no reason to expect such a term to dominate in the high curvature limit - in fact, this is the limit in which quantum corrections should become dominant. We therefore allow the dominant term in a power series expansion of $f(X, Y, Z)$ to be of the form R^n, $(R_{ab}R^{ab})^n$ or $(R_{abcd}R^{abcd})^n$ on approach to the singularity, where n is a constant. These three different cases will be investigated separately below.

3.1. $f = R^n$

In a previous work [16] we found Bianchi type I solutions to the theory defined by the choice $f = f(X) = R^n$. These solutions have a particularly simple form and they could be used to show that anisotropic universes in such theories do not exhibit an infinite sequence of Mixmaster oscillations on approach to the initial singularity, if $n > 1$. We also include these solutions here for completeness.

Substituting $f = f(X) = R^n$ into the field equations (2) together with the line-element (3) gives the two independent algebraic constraints,

$$(2n^2 - 4n + 3)P + (n - 2)Q - 3(n - 1)(2n - 1) = 0$$

and

$$2(n^2 - 1)P + (n - 2)(P^2 - Q) - 3(n - 1)^2(2n - 1) = 0,$$

where we have defined

$$P \equiv \sum_i p_i \quad \textit{and} \quad Q \equiv \sum_i p_i^2. \tag{4}$$

The constraint equations have two classes of solution. The first is given by

$$P = \frac{3(n-1)(2n-1)}{(2-n)}$$
$$Q = \frac{P^2}{3}$$

which is only solved by the isotropic solution $p_1 = p_2 = p_3 = P/3$. This is the zero-curvature isotropic vacuum universe found by Bleyer and Schmidt [2, 20]. It can be seen that these isotropic cosmologies are valid for all $n \neq 2$ and correspond to expanding universes for $n < 1/2$ or $1 < n < 2$ and to collapsing universes for $1/2 < n < 1$ or $2 < n$.

The second class of solutions to (4) is given by

$$P = 2n - 1 \tag{5}$$
$$Q = (2n - 3)(1 - 2n).$$

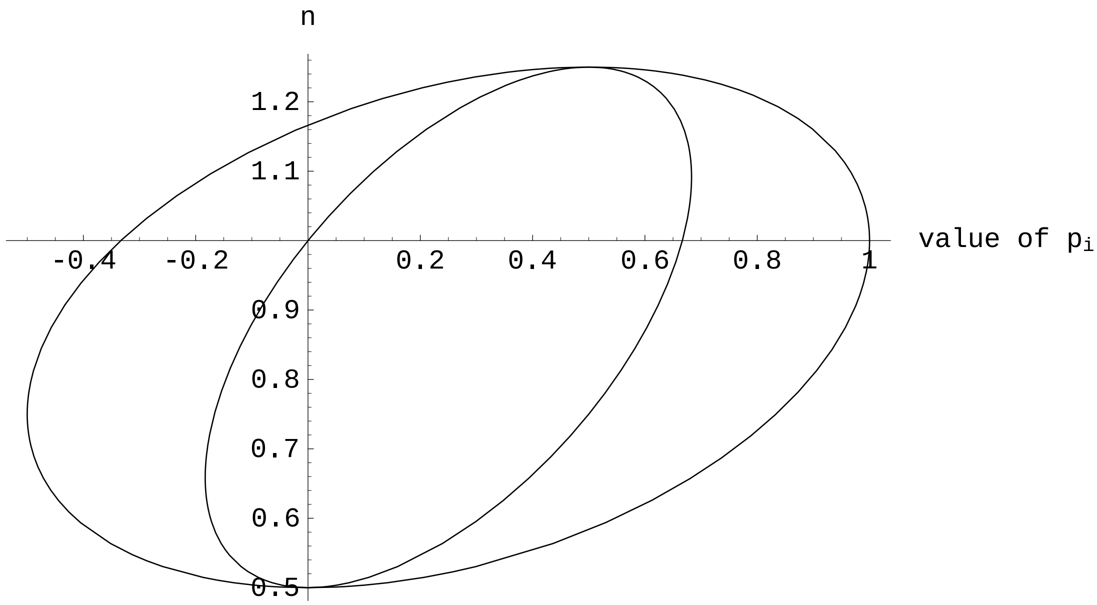

Figure 1. *For the solution (5), the intervals in which the Kasner indices p_i are allowed to lie can be read off from this graph. For any value of n in the range $\frac{1}{2} \leqslant n \leqslant \frac{5}{4}$ a horizontal line is drawn; the boundaries of the intervals in which the p_i lie are then given by the four points at which the horizontal line crosses the two closed curves. For $n = 1$ these boundaries can be seen to be $-\frac{1}{3}$, 0, $\frac{2}{3}$ and 1, as expected for the Kasner solution of general relativity.*

This class of solutions represents a generalisation of the Kasner metric of general relativity and was the basis of the study [16]. It was shown in [16] that this class of solutions only exists for n in the range $1/2 \leqslant n \leqslant 5/4$ and that the values of the constants p_i must then lie within the ranges

$$\begin{aligned} 2n-1-2\sqrt{(2n-1)(5-4n)} &\leqslant 3p_1 \leqslant 2n-1-\sqrt{(2n-1)(5-4n)}, \\ 2n-1-\sqrt{(2n-1)(5-4n)} &\leqslant 3p_2 \leqslant 2n-1+\sqrt{(2n-1)(5-4n)}, \\ 2n-1+\sqrt{(2n-1)(5-4n)} &\leqslant 3p_3 \leqslant 2n-1+2\sqrt{(2n-1)(5-4n)}, \end{aligned}$$

where it has been assumed without loss of generality that $p_1 \leqslant p_2 \leqslant p_3$. These ranges are shown on Figure 3.1 and can be read off by drawing a horizontal line of constant n on the plot. The four points at which the horizontal line crosses the two closed curves gives the four boundary values for the allowed ranges of the constants p_i. It can be seen that the points at which the curves cross the abscissa, which corresponds to $n = 1$, give the boundary values $-1/3 \leqslant p_1 \leqslant 0 \leqslant p_2 \leqslant 2/3 \leqslant p_3 \leqslant 1$, in agreement with the Kasner solution of general relativity. For $n > 1/2$ these solutions correspond to expanding universes with a curvature singularity at $t = 0$. For $n = 1/2$ the only solution is Minkowski spacetime.

For a universe filled with a perfect fluid having an equation of state $p = (\gamma - 1)\rho$, γ constant, relating the fluid pressure p to its density ρ, the field equations (2) have the isotropic solution

$$p_1 = p_2 = p_3 = \frac{2n}{3\gamma}, \tag{6}$$

for $\gamma \neq 0$. This reduces to the spatially-flat FRW solution of general relativity in the limit $n \to 1$. For $n > 0$, these isotropic cosmologies are expanding, and for $n < 0$ they are contracting, with $n = 0$ giving Minkowski spacetime. The stability and observational consequences of cosmologies of this type were investigated in [19, 21], where primordial nucleosynthesis and the microwave background were used to impose observational constraints on the admissible values of n.

3.2. $f = (R_{ab}R^{ab})^n$

Substituting $f = f(Y) = (R_{ab}R^{ab})^n$ into the field equation (2) along with the metric ansatz (3) gives the two independent equations,

$$\begin{aligned} Y^{n-1}\Big(&(P^2 + Q - 4PQ + P^2Q + Q^2) - \\ &2(3P^2 + P^3 + 3Q - 9PQ + 2Q^2)n + 2(4P^2 + 4Q - 8PQ)\Big) = 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} Y^{n-1}\Big(&(24P - 2P^2 - 2P^3 - 30Q + 10PQ)n + (8P^2 - 32P + 24Q)n^2 \\ &+ (P^2 - 4P + 2P^3 + 9Q - 10PQ - P^2Q + 3Q^2)\Big) = 0, \end{aligned} \tag{8}$$

where P and Q are defined as before. Equations (7) and (8) have five classes of solutions.

The first of these classes is given by

$$P = Q = 0.$$

This class of solutions is only satisfied by $p_1 = p_2 = p_3 = 0$, which is Minkowski spacetime. It can be seen from (7) and (8) that as this solution corresponds to $Y = 0$ it only exists for $n \geqslant 0$ (for $n < 0$ the premultiplicative factor in (7) and (8) causes the left-hand side of those equations to diverge).

The second class of solutions is given by

$$P = Q = 1.$$

This is just the Kasner solution of general relativity, for which the values of the constants p_i are constrained to lie within the ranges $-1/3 \leqslant p_1 \leqslant 0 \leqslant p_2 \leqslant 2/3 \leqslant p_3 \leqslant 1$. Again, this solution corresponds to $Y = 0$ and so is only valid for $n \geqslant 0$.

The third class of solutions is given by

$$P = \frac{3(1 - 3n + 4n^2) \pm \sqrt{3(-1 + 10n - 5n^2 - 40n^3 + 48n^4)}}{2(1 - n)}$$

$$Q = \frac{P^2}{3}.$$

The only solution belonging to this class corresponds to an isotropic and spatially flat vacuum FRW cosmology. The values of the constants p_i are all equal to $P/3$ in this case and the solution is valid for all $n \neq 1$.

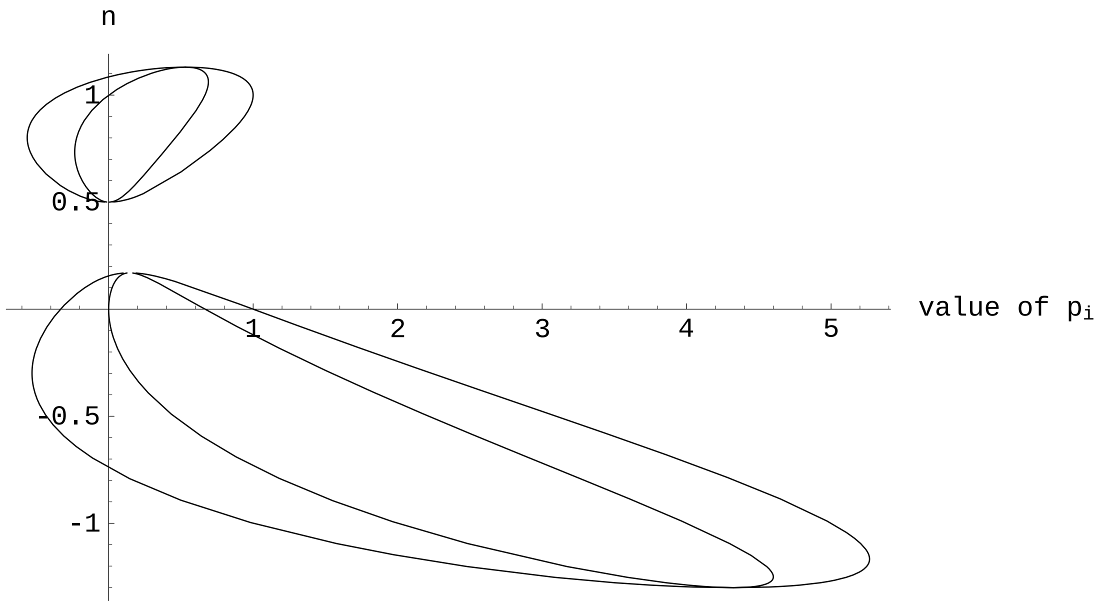

Figure 2. *For the solution (9) the intervals in which the Kasner indices p_i are constrained to lie can be read off this graph, as with figure 3.1. A horizontal line of constant n is drawn; the boundaries of the intervals in which the p_i lie are then given by the four points at which the horizontal line crosses the curves. For $n = 1$ these boundaries can be seen to be $-\frac{1}{3}$, 0, $\frac{2}{3}$ and 1, as expected for the Kasner solution of general relativity.*

The fourth class of solutions is given by

$$
\begin{aligned}
P &= (1-2n)^2 \\
Q &= 1 - 8n + 16n^2 - 8n^3.
\end{aligned}
\tag{9}
$$

The solutions belonging to this class are anisotropic cosmologies which reduce to the standard Kasner form, $P = Q = 1$, in the limit $n \to 1$. For this class of solutions, the values of the constants p_i are in general constrained to lie within the ranges

$$
\begin{aligned}
(1-2n)^2 - 2A &\leqslant 3p_1 \leqslant (1-2n)^2 - A \\
(1-2n)^2 - A \ &\leqslant 3p_2 \leqslant (1-2n)^2 + A \\
(1-2n)^2 + A \ &\leqslant 3p_3 \leqslant (1-2n)^2 + 2A
\end{aligned}
$$

where $A = \sqrt{(1-2n)(1-6n+4n^3)}$ and the p_i have again been ordered so that $p_1 \leqslant p_2 \leqslant p_3$. These boundaries are shown in figure 3.2 which can be read in the same way as figure 3.1, by taking a horizontal line of constant n and noting the four points at which that line intersects the curves. These intercepts determine the allowed intervals for the values of the p_i. For real-valued p_i, the value of n must lie either in the range $n_1 \leqslant n \leqslant n_2$ or in the range $1/2 \leqslant n \leqslant n_3$, where n_1, n_2 and n_3 are the roots of the cubic polynomial $1 - 6n + 4n^3 = 0$ and are chosen such that $n_1 < n_2 < n_3$. These generalisations of the Kasner metric always correspond to expanding cosmologies, independent of the value of n.

The fifth class of solutions to (7) and (8) is given by

$$P = 4n - 1$$
$$Q = -3 + 12n - 8n^2 \pm 2(1-2n)\sqrt{2(1-4n+2n^2)}.$$

This class describes complex-valued p_i for all values of n (except for $n = 1/4$, for which this class of solutions reduces to the first class) and is therefore of limited interest.

The isotropic solution for a universe filled with a fluid with equation of state $p = (\gamma - 1)\rho$ is given by the choice

$$p_1 = p_2 = p_3 = \frac{4n}{3\gamma}.$$

This solution reduces to the spatially-flat FRW cosmology of general relativity in the limit $n \rightarrow 1/2$ and corresponds to an expanding universe for $n > 0$ and to a contracting universe for $n < 0$.

3.3. $f = (R_{abcd}R^{abcd})^n$

Substituting $f = f(Z) = (R_{abcd}R^{abcd})^n$ and (3) into the field equations (2) gives the two independent equations

$$Z^{n-1}\Bigg(\Big(\frac{P^2Q}{2} - \frac{P^4}{12} - Q - \frac{3Q^2}{4} + 2S - \frac{2PS}{3}\Big) + \Big(\frac{P^4}{3} + 6Q + 2PQ - 2P^2Q + 3Q^2 - 10S + \frac{2PS}{3}\Big)n + 8(S-Q)n^2\Bigg) = 0 \tag{10}$$

and

$$Z^{n-1}\Bigg(\Big(\frac{P^4}{4} - P - \frac{3P^2}{2} - \frac{P^3}{2} + \frac{13Q}{2} + \frac{7PQ}{2} - \frac{3P^2Q}{2} + \frac{9Q^2}{4} - 8S\Big) + (6P + 4P^2 - 16Q - 2PQ + 8S)n + 8(Q-P)n^2\Bigg) = 0 \tag{11}$$

where P and Q are defined as before and we have also now defined

$$S \equiv \sum_i p_i^3.$$

Equations (10) and (11) have four different classes of solution.

The first class of solutions is given by

$$P = Q = S = 0.$$

The only solution that belongs to this class is Minkowski spacetime. As $R_{abcd}R^{abcd} = 0$ for Minkowski spacetime it can only be a solution for $n \geqslant 0$, due to the premultiplier in equations (10) and (11).

The second class of solutions is given by

$$P = Q = S = 1.$$

The only solution that belongs to this class corresponds to $p_1 = p_2 = 0$ and $p_3 = 1$. Making the coordinate transformations $\bar{z} = t \sinh z$ and $\bar{t} = t \cosh z$ [24] allows the line-element (3) to then be written in the form

$$ds^2 = -d\bar{t}^2 + dx^2 + dy^2 + d\bar{z}^2$$

which is clearly Minkowski spacetime again. This is only a solution for $n \geq 0$, as with the first class of solutions.

The third class of solutions to (10) and (11) is given by

$$P = \frac{3(1 - 2n + 4n^2 \pm \sqrt{-1 + 10n - 16n^2 + 16n^4}}{2(1-n)},$$
$$Q = \frac{P^2}{3},$$
$$S = \frac{P^3}{9},$$

which only has the isotropic and spatially flat vacuum FRW cosmology as a solution, where $p_1 = p_2 = p_3 = P/3$. This is a solution for all $n \neq 1$.

The fourth and last class of solutions is given by

$$P = 4n - 1$$
$$Q = \frac{1}{3}\{16n^2 - 8n - 1 \pm 4\sqrt{2[n(1-2n) + S(1-n)]}\}.$$

This class of solutions corresponds to anisotropic metrics with $Z = 0$ and is unusual in that it cannot be expressed in the form $P =$ constant and $Q =$ constant. This feature means that the standard picture of a plane intersecting an ellipsoid is no longer a valid one for this class. Solutions in this class do not appear to have any range of n for which the constants p_i take real values, and so are of limited physical interest.

The isotropic solution for a universe filled with a fluid with equation of state $p = (\gamma - 1)\rho$ is given by

$$p_1 = p_2 = p_3 = \frac{4n}{3\gamma}.$$

As before, this reduces to the spatially flat FRW solution of general relativity in the limit $n \to 1/2$ and corresponds to an expanding universe for $n > 0$ and to a contracting universe for $n < 0$.

4. Investigation of the effects of matter

We have identified above two generalisations of the Kasner metric of general relativity which are solutions to the scale-invariant theories of gravity we are investigating. The first of these is a solution to the theory defined by $\mathcal{L} = R^n$ and was the subject of investigation in [16]; the second is a new solution to the theory $\mathcal{L} = (R_{ab}R^{ab})^n$. These are the first exact anisotropic solutions to be found for higher-order gravity theories. In this section we will investigate some of the properties of these cosmologies.

The behaviour of these solutions is of particular interest when considering the Bianchi type VIII or type IX 'Mixmaster' cosmologies. The field equations for these cosmologies can be cast into the form of the equations of motion of a particle moving inside an exponentially steep triangular potential well with open channels in the corners [11]. The three steep-sided walls are created by the 3-curvature anisotropies. In the region where the potential is negligible (far from the walls) the behaviour of the solutions approaches that of the Kasner metric. As the exponentially steep potential wall is approached the universe 'particle' is reflected and re-enters a Kasner–like regime with the Kasner indices p_i systematically changed to some new values by the rule of reflection from the potential wall. In general relativistic cosmologies of Bianchi types VIII and IX this process is repeated an infinite number of times as the singularity is approached [10, 11, 12, 13, 14, 15] so long as matter obeys an equation of state with $p < \rho$. After reflection from the potential wall the Kasner index that was previously negative is permuted to a positive value and the lowest-valued positive Kasner index is permuted to a negative value. This is repeated *ad infinitum* in the general relativistic cosmology as one of the Kasner indices must be kept negative while the other two are positive. In the generalisations of the Kasner metric presented above this is no longer the case; for some solutions it is possible for all of the Kasner indices to be positive. If the Kasner indices are being permuted in a chaotic fashion for long enough then, eventually, they will end up in such a configuration. Once this occurs the oscillations will end, as all spatial dimensions will be contracting, and the singularity will reached monotonically without further oscillation of the scale factors [22, 23].

From figure 3.1 it can be seen that for solution (5) all Kasner indices can be made positive for values of $1 < n < 5/4$, in the $\mathcal{L} = R^n$ theories. Similarly, from figure 3.2 it can be seen that for the solution (9), with $n_1 < n < n_2$ or $1 < n < n_3$, that all the Kasner indices can be made positive, in the theory $\mathcal{L} = (R_{ab}R^{ab})^n$ (where the n_i are defined as before). For both of these solutions, with these ranges of n, there is therefore no infinite series of chaotic Mixmaster oscillations as the singularity is approached. However, the solutions to both of these theories are still expected to exhibit an infinite number of chaotic oscillations if $1/2 < n < 1$, as at least one of the Kasner indices must be kept negative whilst another is kept positive. A separate detailed analysis is require to determine where there are differences between this behaviour and the chaotic oscillations found in general relativity.

It now remains to show that the vacuum solutions found above are the asymptotic attractor solutions in the presence of non-comoving matter motions as $t \to 0$. This analysis will follow closely that of [16] and [24]. We aim to show that the fluid stresses diverge more slowly than the vacuum terms as $t \to 0$ and so produce negligible metric perturbations to an anisotropic Kasner universe. If this is the case, then the vacuum solutions above can indeed be considered as the asymptotic attractor solutions even in the presence of matter. Matter will just be carried along by the expansion and behave like a test fluid.

We now consider a perfect fluid with equation of state $p = (\gamma - 1)\rho$ where $1 \leqslant \gamma < 2$ and non-zero 4-velocity components, u^i normalised so that $u^i u_i = 1$. The conservation equations $T^{ab}{}_{;b} = 0$ on the metric background (3) can then be written in the form [24, 23, 25]

$$\frac{\partial}{\partial x^i}(t^p \rho^{\frac{1}{\gamma}} u^i) = 0$$
$$\gamma \rho u^k (u_{i,k} - \frac{1}{2} u^l g_{kl,i}) = -\frac{1}{3}\rho_{,i} - (\gamma - 1) u_i u^k \rho_{,k}.$$

Neglecting spatial derivatives with respect to time derivatives these equations integrate to

$$t^p u_0 \rho^{\frac{1}{\gamma}} \quad = const.$$
$$u_\alpha \rho^{\frac{(\gamma-1)}{\gamma}} = const.$$

The neglecting of spatial derivatives means that these equations are valid on scales larger than the particle horizon as in the velocity-dominated approximation (although we have not so far restricted the fluid motions to be non-relativistic) [24, 26, 27].

From the second of the integrals above, it can be seen that all the covariant components of the spatial 3-velocity, u_α, are approximately equal. This is not true of the contravariant components as the Kasner indices, p_i, in the metric elements used to raise indices are not equal in these solutions. The contravariant component that diverges the fastest, and dominates the others in the $t \to 0$ limit, is therefore $u^3 = u_3 t^{2p_3}$, as p_3 is the largest of the p_i. If the 4-velocity is normalised, so that $u_a u^a = 1$, and the contravariant 3-velocity component u^3 diverges the fastest as $t \longrightarrow 0$, then we must have $u_0 u^0 \sim u_3 u^3 = (u_3)^2 t^{-2p_3}$ in that limit. The integrated conservation equations above can now be solved approximately in this limit to give

$$\rho \sim t^{-\gamma(p_1+p_2)/(2-\gamma)}$$
$$u_\alpha \sim t^{(p_1+p_2)(\gamma-1)/(2-\gamma)}$$

as $t \longrightarrow 0$.

It is now possible to calculate the leading-order contributions to the energy-momentum tensor, $T^a_b = (\rho + p)u^a u_b - p\delta^a_b$, as being

$$T^0_0 \sim \rho u^0 u_0 \sim t^{-P-p_3}$$
$$T^1_1 \sim \rho \qquad \sim t^{-\gamma(P-p_3)/(2-\gamma)}$$
$$T^2_2 \sim \rho u^2 u_2 \sim t^{-2p_2-P+p_3}$$
$$T^3_3 \sim \rho u^3 u_3 \sim t^{-P-p_3}.$$

The component which diverges the fastest here as $t \longrightarrow 0$ is $T^3_3 \sim t^{-P-p_3}$, for general γ. For the case $\gamma = \frac{4}{3}$ where the 4-velocity of the fluid is comoving, $u_i = \delta^0_i$, all the components of T^a_b diverge as $\rho \sim t^{-2p+2p_3}$.

For the generalisation of the Kasner metric in the $\mathcal{L} = R^n$ theory, (5), the vacuum terms diverge as t^{-2n}. We therefore require that $2n > P + p_3$, or $p_3 < 1$, in order for the vacuum terms to dominate over the matter terms for a non-comoving perfect fluid, with general γ, in the limit $t \to 0$. For the $\gamma = \frac{4}{3}$, comoving perfect fluid ($u_i = \delta_i^0$), the condition for vacuum domination is $2n > 2P - 2p_3$, or $p_3 > n - 1$. Both of these conditions are ensured by the boundary values on p_3 given above. In the velocity-dominated limit, the vacuum solutions given here are therefore the appropriate asymptotic solutions on approach to the singularity, as was found in [16].

For the generalised Kasner metric in the $\mathcal{L} = (R_{ab}R^{ab})^n$ theory, (9), the vacuum terms diverge as t^{-4n}. For vacuum domination as $t \to 0$ in this solution we therefore require $4n > P + p_3$, or $p_3 < -1 + 8n - 4n^2$, for the general non-comoving γ fluid; and $4n > 2P - 2p_3$, or $p_3 > 1 - 6n + 4n^2$, for the $\gamma = 4/3$ comoving fluid. From figure 3.2 it can be seen that solutions of this class can be in one of two regions, $n_1 \leqslant n \leqslant n_2$ or $1/2 \leqslant n \leqslant n_3$. The validity of the vacuum solution as $t \to 0$ is different in these two different regions. For the region $1/2 \leqslant n \leqslant n_3$ the boundary conditions on the index p_3 mean that the conditions above are automatically satisfied. For the region $n_1 \leqslant n \leqslant n_2$ these inequalities are not always satisfied. For general γ, the condition for vacuum domination is only met if n lies in the narrow range $1/6 < n < n_2$, for any other value of n, in this region, the index p_3 can be such that the fluid diverges faster than the vacuum terms. Similarly, for the $\gamma = 4/3$ comoving fluid the condition for vacuum domination is only satisfied if n lies in the narrow range $n_4 < n < n_2$, where n_4 is the real root of the cubic

$$4n^3 - 12n^2 + 10n - 1 = 0.$$

To understand the evolution of an anisotropic solution of the form (3) in the theory $\mathcal{L} = (R_{ab}R^{ab})^n$, where $n_1 \leqslant n \leqslant n_2$, it is therefore necessary to take into account the relativistic motions of any fluid that is present (except in the narrow ranges of n identified above). This range of n can, however, be regarded as not belonging to physically interesting theories on other grounds. In order to agree with weak-field experiments it is necessary for a gravitational theory to contain a term which approximates the Einstein-Hilbert action in the weak-field limit (see ref. [19]). Any theory containing such a limit must therefore have a term in its Lagrangian that diverges as t^{-2}. In considering the behaviour of alternative theories close to the singularity it is necessary for the extra terms in the Lagrangian to diverge faster than this if they are to be influential as $t \to 0$. For the theory $\mathcal{L} = (R_{ab}R^{ab})^n$ this requires $n > 1/2$. For $n < 1/2$ such a term will diverge slower than the Einstein-Hilbert term which will then dominate and display the standard Kasner behaviour of general relativity. For the anisotropic solution (9) we are therefore only interested in the region lying in the range $1/2 \leqslant n \leqslant n_3$, which has vacuum terms that dominate the fluid stresses in the velocity-dominated approximation.

It remains to investigate the effects of fluids with stiff equations of state, $\gamma > 4/3$. For sufficiently stiff fluids the velocity-dominated approximation is not valid and $u^\alpha u_\alpha \to 0$ as $t \to 0$ due to the very high inertia of the fluid producing a slow down under contraction [28]. In such a limit $u_0 \to 1$ and the conservation equations can be solved to give

$$\rho \sim t^{-\gamma P}$$
$$u_\alpha \sim t^{(\gamma-1)P}.$$

In this approximation we can write $u_\alpha u^\alpha \sim (u_3)^2 t^{-2p_3} \sim t^{2(\gamma-1)P-2p_3}$. It can now be seen that this behaviour occurs when

$$p_3 \leqslant (\gamma - 1)P. \tag{12}$$

In this limit it is only required that ρ diverges more slowly than the vacuum terms, as $\rho u_\alpha u^\alpha << \rho$ when $t \to 0$.

For the solution (5) to the theory $\mathcal{L} = R^n$ it can be seen from the condition (12) that it is necessary for $\gamma \geqslant 4/3$ in order for the velocity-dominated approximation to break down (this is derived using the upper limit on n and the lower limit on p_3). The condition that the fluid stresses diverge more slowly than the vacuum terms is now

$$\gamma < \frac{2n}{(2n-1)} \tag{13}$$

where it has been assumed $n > 1/2$. A sufficient condition to satisfy this for all allowed values of n is $\gamma < 5/3$. For fluids with a stiffer equation of state, $5/3 \leqslant \gamma < 2$, the vacuum terms dominate over the fluid motions provided that $n < 1$. For $n < 1$, however, the Einstein-Hilbert term will be the leading one in the gravitational action, as discussed above.

Similarly, for the solution (9) to the theory $\mathcal{L} = (R_{ab}R^{ab})^n$ it can be seen from the condition (12) that it is again necessary for $\gamma \geqslant 4/3$ in order for the velocity-dominated approximation to break down. In the region $1/2 \leqslant n \leqslant n_3$ the condition for the vacuum solution to be unperturbed by the stiff fluid is always satisfied for any fluid with equation of state $\gamma < 2$. However, in the region $n_1 < n < n_2$ the condition for the vacuum to diverge faster than the energy density of the stiff fluid is never satisfied for a fluid which satisfies the necessary condition for the velocity-dominated approximation to break down (except in the narrow range $(7 - \sqrt{33})/8 < n < n_2$ where there are some values of γ for which the vacuum terms can diverge fastest).

5. Conclusions

We have investigated some anisotropic cosmological solutions to higher-order Lagrangian theories of gravity. Whist the standard general relativistic theory, defined by the Einstein-Hilbert action, appears to be consistent with all weak-field tests, there is less reason to think that it should be valid in high curvature regimes, such as in the vicinity of a possible initial cosmological curvature singularity. In fact, it is in the high-curvature limit that quantum effects should become important and we should expect to see deviations from the standard theory. Without knowing the exact form of such deviations, we have approached this problem by considering a general class of theories that can be derived from an arbitrary analytic function the three curvature invariants

R, $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$. Expanding this function as a power series in these variables we then expect the dominant term in the Lagrangian to be of the form R^n, $(R_{ab}R^{ab})^n$ or $(R_{abcd}R^{abcd})^n$ as the singularity is approached. We have found all of the solutions to these theories that can be expressed in terms of the Bianchi type I line–element (3) in vacuum. These solutions provide simple testing grounds for the exploration of quantum cosmological effects in higher-order gravity theory. We have also found the homogeneous, isotropic and spatially flat solutions to these theories in the presence of a perfect fluid.

Exact vacuum anisotropic solutions of the form (3) were found for the theories $\mathcal{L} = R^n$ and $\mathcal{L} = (R_{ab}R^{ab})^n$, whilst it was shown that no such solutions exist for theories defined by $\mathcal{L} = (R_{abcd}R^{abcd})^n$. The properties of these solutions, with respect to their relation to the more general Bianchi type VIII and IX cosmological behaviour, has been investigated. We have argued that for all of the physically relevant new solutions of these theories, the universe will not experience an infinite number of Mixmaster oscillations as the singularity is approached. This is an extension of what was found in the earlier work [16]. The extent to which these vacuum solutions can be considered as realistic in the presence of a perfect fluid has also been investigated. It has been shown in the velocity-dominated approximation that all the anisotropic vacuum solutions found for plausible theories are valid in the limit $t \to 0$. The case of stiff fluids that do not satisfy the velocity-dominated approximation as $t \to 0$ has also been investigated. For the solutions to the theories $\mathcal{L} = R^n$ it was found that for a fluid with equation of state $\gamma < 5/3$ the vacuum solutions are good approximations in the vicinity of the singularity. For the theories $\mathcal{L} = (R_{ab}R^{ab})^n$ it was found that the vacuum solutions are good approximations for all $\gamma < 2$.

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