Decaying Gravity

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We consider the possibility of energy being exchanged between the scalar and matter fields in scalar-tensor theories of gravity. Such an exchange provides a new mechanism which can drive variations in the gravitational 'constant' G. We find exact solutions for the evolution of spatially flat Friedman-Robertson-Walker cosmologies in this scenario and discuss their behaviour at both early and late times. We also consider the physical consequences and observational constraints on these models.

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I. INTRODUCTION

Scalar-tensor theories of gravity provide a convenient framework within which to model space-time variations of the Newtonian gravitational constant, G. They feature a scalar field, ϕ, which is non-minimally coupled to the space-time curvature in the gravitational action. It is this scalar, or more usually its reciprocal, that drives variations in G. Non-minimally coupled scalar fields arise in a variety of different theories, including Kaluza-Klein theory [1], string theories [2] and brane-worlds [3]. The same mechanism that creates a scalar field non-minimally coupled to the curvature in these theories can also lead to a coupling between the scalar and matter fields. This coupling manifests itself through the matter Lagrangian becoming a function of ϕ. The possibility of such a coupling is usually neglected in the literature, where the matter Lagrangian is *a priori* assumed to be independent of ϕ. It is the possibility of a coupling between the scalar and matter fields in scalar-tensor theories that will be the subject of this work.

The introduction of a coupling between ϕ and matter greatly enlarges the phenomenology of the theory. Potentially, this allows greater variability of G in the early universe whilst still satisfying the solar system bounds on time-varying G [4]. We will consider spatially-flat Friedmann-Robertson-Walker (FRW) cosmologies and investigate the extent to which G can vary when energy is exchanged between the ϕ field and ordinary matter. As well as giving a window into the four-dimensional cosmologies associated with higher-dimensional theories, we hope that this direction of study might also be useful in understanding why the present value of G is so small compared to the proton mass scale ($Gm_{pr}^2 \sim 10^{-39}$). The direct exchange of energy between ϕ and the matter fields offers a non-adiabatic mechanism for G to 'decay' towards its present value from a potentially different initial value. There have been a variety of studies which investigate the drain of energy from ordered motion by entropy generation, due to bulk viscosity [5] or direct decay [6, 7] or energy exchange [8], but few studies of the drain of energy by non-adiabatic processes from a scalar field that defines the strength of gravity [9, 10, 11]. This creates a range of new behaviours in scalar-tensor cosmologies.

In considering a coupling between ϕ and matter we are forced to reconsider the equivalence principle. The energy-momentum tensor of perfect-fluid matter fields will no longer be covariantly conserved and the trajectories of test-particles will no longer follow exact geodesics of the metric. These violations of the experimentally well verified weak equivalence principle exclude most possible couplings between ϕ and matter [12]. Such violations are not necessarily fatal though. We show that whilst energy-momentum is not separately conserved by the matter fields there is still an exact concept of energy-momentum conservation when the energy density of the scalar field is included. Furthermore, the non-geodesic motion of test particles is only problematic if the coupling increases above experimentally acceptable levels as the Universe ages. The theory we consider is still a geometric one and it remains true that at any point on the space-time manifold it is possible to choose normal coordinates so that it looks locally flat, ensuring that it is always possible to transform to a freely-falling frame in which the effects of gravity are negligible (up to tidal forces).

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II. FIELD EQUATIONS

The simplest scalar-tensor theory is the Brans-Dicke theory [13], defined by the Lagrangian density

$$\mathcal{L} = \phi R - \frac{\omega}{\phi}\partial^a\phi\partial_a\phi + 16\pi\mathcal{L}_m[g_{ab};\Psi] \tag{1}$$

where ω is the Brans-Dicke coupling constant, R is the scalar curvature of space-time, and $\mathcal{L}_m$ is the Lagrangian density of the matter fields, denoted by Ψ. As $\omega \to \infty$ this theory reduces to general relativity and $G \sim \phi^{-1}$ becomes constant (for exceptions see [14, 15]). It is an important feature of these theories that $\mathcal{L}_m$ is independent of ϕ. This ensures that the matter fields do not interact with the scalar field directly and therefore that the energy-momentum tensor, T^{ab}, derived from $\mathcal{L}_m$ is conserved (${T^{ab}}_{;b} = 0$).

This conservation of T^{ab}, whilst appealing, is not absolutely necessary in deriving a theory in which G can vary. There are numerous examples where one might expect ${T^{ab}}_{;b} \neq 0$. For example, when considering two fluids the energy-momentum tensor of each fluid is not separately conserved unless the fluids are completely non-interacting. It is only required that the energy-momentum being lost by one of the fluids is equal to the energy-momentum being gained by the other.

In what follows we will consider the scalar field and the matter fields as two fluids (or more than two fluids if there is more than one matter fluid present) and introduce a transfer of energy and momentum between them. Such an interaction can be introduced by allowing $\mathcal{L}_m$ to be a function of ϕ and will change the nature of the resulting FRW cosmologies.

The field equations are derived from (1) by extremizing the corresponding action with respect to the metric. Defining T^{ab} for the matter in the usual way, the field equations take their standard Brans-Dicke form [13] independent of the presence of interactions between ϕ and matter, and the Einstein tensor is given by

$$G^{ab} = \frac{\omega}{\phi^2}({\phi_;}^a{\phi_;}^b - \frac{1}{2}g^{ab}\phi_{;c}{\phi_;}^c) + \frac{1}{\phi}({\phi_;}^{ab} - g^{ab}\Box\phi) + \frac{8\pi}{\phi}T^{ab}_m. \tag{2}$$

The scalar-field propagation equation and matter energy-momentum conservation equations are

$$\Box\phi = \frac{8\pi T}{(2\omega+3)} - \frac{16\pi\phi}{(2\omega+3)}\frac{\sigma^a}{{\phi_;}^a} \tag{3}$$

$${T^{ab}}_{;b} = \sigma^a \tag{4}$$

where T is the trace of the energy-momentum tensor and σ^a is an arbitrary vector function of the space-time coordinates x^b that determines the rate of transfer of energy and momentum between the scalar field ϕ and the ordinary matter fields. The precise form of σ^a depends on the detailed form of the interaction between the scalar and matter fields in $\mathcal{L}_m$. For example, a conformal transformation of the form $g_{ab} \to A^2(\phi)g_{ab}$ from a frame in which ${T^{ab}}_{;b} = 0$ gives

$$\sigma^a = \frac{T}{A}\frac{dA}{d\phi}{\phi_;}^a.$$

This particular choice of energy transfer can be interpreted as a space-time variation of the rest masses of matter described by $\mathcal{L}_m$. For the moment, we consider the case of more general interactions by leaving σ^a as an arbitrary function. Later, we will consider specific forms of σ^a that allow direct integration of the field equations.

We specialise the metric to the spatially flat, isotropic and homogeneous FRW line-element with expansion scale factor $a(t)$:

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)). \tag{5}$$

Substituting this metric into the field equations (2), (3) and (4) gives the generalised Friedmann equations:

$$\left(\frac{\dot{V}}{V}\right)^2 = -3\frac{\dot{V}}{V}\frac{\dot{\phi}}{\phi} + \frac{3\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + 3(3+2\omega)\alpha\frac{\rho}{\phi} \tag{6}$$

$$\frac{(\dot{V}\phi)^{\cdot}}{\rho V} = 3\alpha((2-\gamma)\omega+1) + 3\frac{\alpha}{\rho}\frac{\phi}{\dot{\phi}}\sigma^0 \tag{7}$$

$$\frac{(V\dot{\phi})^{\cdot}}{\rho V} = \alpha(4-3\gamma) - 2\frac{\alpha}{\rho}\frac{\phi}{\dot{\phi}}\sigma^0 \tag{8}$$

$$\dot{\rho} + \gamma\frac{\dot{V}}{V}\rho = \sigma^0 \tag{9}$$

where we have defined a comoving volume $V = a^3$ and a constant $\alpha = \frac{8\pi}{(3+2\omega)}$; the energy-momentum tensor is assumed to be a perfect barotropic fluid with density ρ and pressure p which are linked by a linear equation of state $p = (\gamma - 1)\rho$, and over-dots denote differentiation with respect to the comoving proper time, t. It is this set of differential equations that we need to solve in order to determine the evolution of $a(t)$ and $G \propto \phi(t)^{-1}$ in cosmological models of this type.

III. TRANSFER OF ENERGY AND ENTROPY

The conservation of energy and momentum as well as the second law of thermodynamics are of basic importance to physics. In considering an interaction between a gravitational scalar field ϕ and the matter fields Ψ it is, therefore, necessary to investigate the extent to which we can consider energy and momentum to be conserved and the second law to be obeyed.

When we consider the thermodynamics of an exchange of energy between the scalar field and matter it is useful to define an effective energy density, ρ_ϕ, for the scalar field ϕ. Defining

$$\rho_\phi \equiv \frac{\dot{\phi}^2}{16\pi\omega\phi}, \tag{10}$$

the scalar-field propagation equation (7) can then be rewritten as

$$\dot{\rho}_\phi + 2\frac{\dot{V}}{V}\rho_\phi = -\frac{R}{16\pi}\dot{\phi} - \sigma^0. \tag{11}$$

Comparison of this equation with (9) shows that ϕ acts as a fluid with equation of state $\gamma = 2$ ($p_\phi = \rho_\phi$). The two terms on the right hand side of this equation act as sources for the energy density ρ_ϕ. The first is the standard Brans-Dicke source term for the scalar field and the second, $\sigma^0(t)$, is new and describes the energy exchange between ϕ and the matter fields. It can be seen that the second term is exactly the opposite of the source term in equation (6), and it is in this sense that the total energy is conserved in this theory.

It is also useful to consider the entropy. Contracting the divergence of the energy-momentum tensor with the comoving four-velocity U^a we obtain

$$\begin{aligned} U^a\sigma_a &= U^a T_a{}^b{}_{;b} \\ &= U^a p_{;a} + U^a((\rho+p)U_a U^b)_{;b} \\ &= U^a p_{;a} - ((\rho+p)U^b)_{;b}, \end{aligned}$$

where, in the last line, we have used the normalisation $U^a U_a = -1$ and $(U^a U_a)_{;b} = U^a{}_{;b}U_a + U^a U_{a;b} = 0$. Defining the particle current by $N^a \equiv nU^a$, where n is the number density in a comoving Lorentz frame, this expression can be rewritten as

$$\begin{aligned} U^a\sigma_a &= U^a\left[p_{;a} - n\left(\frac{(\rho+p)}{n}\right)_{;a}\right] - \frac{(\rho+p)}{n}N^a{}_{;a} \\ &= -nU^a\left[p\left(\frac{1}{n}\right)_{;a} + \left(\frac{\rho}{n}\right)_{;a}\right], \end{aligned}$$

where we have used the conservation of particle number, $N^a{}_{;a} = 0$. Recalling the first law of thermodynamics,

$$\Theta dS = pdV + dE = pd\left(\frac{1}{n}\right) + d\left(\frac{\rho}{n}\right),$$

where Θ is the temperature and S is the entropy, we now get

$$U^a\sigma_a = -n\Theta U^a S_{;a}$$

or, making use of our assumption of spatial homogeneity,

$$\dot{S} = \frac{\sigma^0}{n\Theta}.$$

This tells us that as energy is transferred from ϕ to the matter fields the entropy of the matter fields increases, as expected. Conversely, the matter fields can decrease their entropy by transferring energy into ϕ.

Unfortunately, there is currently no known way of defining the entropy of a non-static gravitational field so it is not possible to perform an explicit calculation of the entropy changes in ϕ and g_{ab}. We can only assume that if the Universe can be treated as a closed system, and the exchange of energy is an equilibrium process, then the entropy that is lost or gained by the matter through this exchange will be gained or lost by the gravitational fields. This direct interaction of the matter with ϕ then allows an additional mechanism for increasing or decreasing the entropy of the matter content of the Universe.

IV. GENERAL SOLUTIONS

It is convenient to define a new time coordinate τ by

$$d\tau \equiv \rho V dt \tag{12}$$

and to re-parametrise the arbitrary function σ^0 by

$$\sigma^0 = \rho^2 V \frac{\phi'}{\phi} \lambda' \tag{13}$$

where a prime denotes differentiation with respect to τ and $\lambda(\tau)$ is a new arbitrary function. This re-parameterisation of the interaction is chosen to enable a direct integration of the field equations and does not imply any loss of generality, as λ is an arbitrary function. The field equations (7) and (8) can now be integrated to

$$\rho\phi V V' = 3\alpha((2-\gamma)\omega + 1)(\tau - \tau_1) + 3\alpha\lambda \tag{14}$$

$$\rho V^2 \phi' = \alpha(4-3\gamma)\tau + \alpha\tau_2 - 2\alpha\lambda \tag{15}$$

where τ_1 and τ_2 are constants of integration. We have a freedom in where we define the origin of τ and can, therefore, absorb the constant τ_1 into τ and the definition of τ_2 by the transformations $\tau \to \tau + \tau_1$ and $\tau_2 \to \tau_2 - (4-3\gamma)\tau_1$. It can now be seen from (15) that ϕ' is sourced by three terms. The first corresponds to the source term in (4) and can be seen to disappear for $\gamma = 4/3$, as expected for black-body radiation. The second term is constant and is the contribution of the free scalar to the evolution of ϕ; it is this term which distinguishes the general spatially-flat Brans-Dicke FRW solutions [18] from the power-law late-time attractor solutions [19]. The third term is new and gives the effect of the energy transfer on the evolution of ϕ. This term is dependent on the arbitrary function λ, which specifies the interaction between ϕ and the matter fields.

The problem is now reduced to solving the coupled set of first-order ordinary differential equations (14) and (15) with the constraint equation (9). The remaining equation (6) is rewritten in terms of τ and λ as

$$\frac{\rho'}{\rho} + \gamma\frac{V'}{V} = \lambda'\frac{\phi'}{\phi}, \tag{16}$$

and can be solved for ρ once V and ϕ have been found for some λ.

We can decouple the set of equations (14) and (15) by differentiating (15) and substituting for (14) to get the second-order ordinary differential equation

$$\begin{aligned}&((4-3\gamma)\tau + \tau_2 - 2\lambda)\frac{\phi''}{\phi} - ((4-3\gamma) - 2\lambda')\frac{\phi'}{\phi} \\ &= -\left[((4-3\gamma)\tau + \tau_2 - 2\lambda)\lambda' + 3(2-\gamma)(((2-\gamma)\omega + 1)\tau + \lambda)\right]\left(\frac{\phi'}{\phi}\right)^2\end{aligned}$$

which can be integrated to

$$\frac{\phi'}{\phi} = \frac{(4-3\gamma)\tau + \tau_2 - 2\lambda}{(A\tau^2 + B\tau + C)}, \tag{17}$$

where

$$\begin{aligned}A &= 3\gamma^2\omega/2 - 3\gamma(1+2\omega) + (5+6\omega) \\ B &= \tau_2 + (4-3\gamma)\lambda \\ C &= -\lambda^2 + \lambda\tau_2 + D\end{aligned}$$

and D is a constant of integration. The three source terms for ϕ' appear in the numerator on the right-hand side of equation (17). The equations (14) and (15) can now be combined to give a'/a in terms of ϕ'/ϕ as

$$\frac{a'}{a} = \frac{((2-\gamma)\omega+1)\tau+\lambda}{(A\tau^2+B\tau+C)} \tag{18}$$

where A, B, C and D are defined as before. The constant D can be set using the constraint equation (9). Using (14) and (18) we obtain the expression

$$\rho V^2\phi = \alpha(A\tau^2+B\tau+C).$$

This can then be substituted into (9) which, in terms of τ and a, gives the generalised Friedmann equation:

$$3\left(\frac{a'}{a}\right)^2 + 3\frac{a'}{a}\frac{\phi'}{\phi} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 = \frac{(3+2\omega)}{(A\tau^2+B\tau+C)}.$$

Substituting (17) and (18) into this we find that

$$D = -\frac{\tau_2^2\omega}{2(3+2\omega)}.$$

V. PARTICULAR SOLUTIONS

If λ is specified in terms of τ we now have a set of two decoupled first-order ordinary differential equations for the two variables a and ϕ. It is the solution of these equations, for specific choices of $\lambda(\tau)$, that we give in this section.

A. $\lambda(\tau) = c_1 + c_2\tau$

A simple form for λ that allows direct integration of equations (17) and (18) is the linear function $\lambda = c_1 + c_2\tau$. From equations (14) and (15) it can be seen that the constant c_1 can be absorbed into τ_1 and τ_2 by simple redefinitions. The equations (17) and (18) then become

$$\frac{\phi'}{\phi} = \frac{(4-3\gamma-2c_2)\tau+\tau_2}{(\hat{A}\tau^2+\hat{B}\tau+\hat{C})}$$
$$\frac{a'}{a} = \frac{((2-\gamma)\omega+1+c_2)\tau}{(\hat{A}\tau^2+\hat{B}\tau+\hat{C})}$$

where $\hat{A} = A - c_2^2$, $\hat{B} = B + c_2\tau_2$ and $\hat{C} = D$. The solutions of these equations depend upon the sign of the discriminant

$$\Delta = \hat{B}^2 - 4\hat{A}\hat{C}. \tag{19}$$

For the case $\Delta = 0$, there exist simple exact power-law solutions

$$a(\tau) \propto \tau^{\frac{2(2-\gamma)\omega+2+2c_2}{3\gamma^2\omega-6\gamma(1+2\omega)+2(5+6\omega)-2c_2^2}} \tag{20}$$

$$\phi(\tau) \propto \tau^{\frac{2(4-3\gamma)-4c_2}{3\gamma^2\omega-6\gamma(1+2\omega)+2(5+6\omega)-2c_2^2}}. \tag{21}$$

Substituting these power-law solutions into (16) we can obtain the corresponding power-law form for ρ

$$\rho \sim \tau^{\frac{4c_2(2-3\gamma-c_2)-6\gamma(1+(2-\gamma)\omega)}{3\gamma^2\omega-6\gamma(1+2\omega)+2(5+6\omega)-2c_2^2}}.$$

The relationship between τ and the cosmological time t can now be obtained by integrating the definition $d\tau = \rho a^3 dt$ given in eq. (12). This gives (20) and (21) in terms of t time as

$$a(t) \sim t^{\frac{2+2(2-\gamma)\omega+2c_2}{4+3\gamma\omega(2-\gamma)-2c_2(7-6\gamma-c_2)}} \tag{22}$$

$$\phi(t) \sim t^{\frac{2(4-3\gamma)-4c_2}{4+3\gamma\omega(2-\gamma)-2c_2(7-6\gamma-c_2)}}. \tag{23}$$

The condition required for the occurrence of power-law inflation is obtained by requiring the power of time in equation (23) to exceed unity (for the case with out energy transfer see refs. [16, 17]). For $\omega > -3/2$ we always have $\Delta \geqslant 0$, and the case $\Delta > 0$ possesses the exact solutions

$$a(\tau) = a_0(\hat{A}\tau^2 + \hat{B}\tau + \hat{C})^{\frac{(2-\gamma)\omega+1+c_2}{2\hat{A}}} \left(\frac{2\hat{A}\tau + \hat{B} + \sqrt{\Delta}}{2\hat{A}\tau + \hat{B} - \sqrt{\Delta}}\right)^{\frac{\hat{B}((2-\gamma)\omega+1+c_2)}{2\hat{A}\sqrt{\Delta}}} \tag{24}$$

$$\phi(\tau) = \phi_0(\hat{A}\tau^2 + \hat{B}\tau + \hat{C})^{\frac{4-3\gamma-2c_2}{2\hat{A}}} \left(\frac{2\hat{A}\tau + \hat{B} + \sqrt{\Delta}}{2\hat{A}\tau + \hat{B} - \sqrt{\Delta}}\right)^{\frac{(4-3\gamma-2c_2)\hat{B}-\tau_2\hat{A}}{2\hat{A}\sqrt{\Delta}}}. \tag{25}$$

where a_0 and ϕ_0 are constants of integration. For $\omega < -3/2$ we have $\Delta \leqslant 0$, and the case $\Delta < 0$ has the exact solutions

$$a(\tau) = a_0(\hat{A}\tau^2 + \hat{B}\tau + \hat{C})^{\frac{(2-\gamma)\omega+1+c_2}{2\hat{A}}} \exp\left[-\frac{((2-\gamma)\omega+1+c_2)\hat{B}}{\hat{A}\sqrt{-\Delta}} \tan^{-1}\left(\frac{\hat{B}+2\hat{A}\tau}{\sqrt{-\Delta}}\right)\right] \tag{26}$$

$$\phi(\tau) = \phi_0(\hat{A}\tau^2 + \hat{B}\tau + \hat{C})^{\frac{4-3\gamma-2c_2}{2\hat{A}}} \exp\left[\frac{2\tau_2\hat{A} - 2(4-3\gamma-2c_2)\hat{B}}{\hat{A}\sqrt{-\Delta}} \tan^{-1}\left(\frac{\hat{B}+2\hat{A}\tau}{\sqrt{-\Delta}}\right)\right]. \tag{27}$$

These solutions have the same functional form as those found by Gurevich, Finkelstein and Ruban [18] for Brans-Dicke theory, in the absence of energy exchange ($\lambda =$ constant), and reduce to them in the limit $c_2 \to 0$. The behaviour of these solutions at early and late times will be discussed in the next section.

B. $\lambda(\tau) = c_3\tau^n$, $n \neq 1$

We now consider forms of $\lambda(\tau)$ that are more general than a simple linear function of τ. Making the choice $\lambda = c_3\tau^n$, where $n \neq 1$ and c_3 is constant, and setting the free scalar component to zero ($\tau_2 = 0$), we find that (17) and (18) can be integrated exactly. The form of the solutions again depends upon the roots of the denominator. For real roots we require $\omega > -3/2$, for which we find the solutions

$$\begin{aligned} a(\tau) = a_0\tau^{\frac{2+2(2-\gamma)\omega}{\kappa}} &\left[\pm 2c_3\tau^{n-1} \mp (4-3\gamma) \pm (2-\gamma)\sqrt{3(3+2\omega)}\right]^{-\frac{3+3(2-\gamma)\omega-\sqrt{3(3+2\omega)}}{3\kappa(n-1)}} \\ &\times \left[\pm 2c_3\tau^{n-1} \mp (4-3\gamma) \mp (2-\gamma)\sqrt{3(3+2\omega)}\right]^{-\frac{3+3(2-\gamma)\omega+\sqrt{3(3+2\omega)}}{3\kappa(n-1)}} \end{aligned} \tag{28}$$

$$\begin{aligned} \phi(\tau) = \phi_0\tau^{\frac{2(4-3\gamma)}{\kappa}} &\left[\pm 2c_3\tau^{n-1} \mp (4-3\gamma) \pm (2-\gamma)\sqrt{3(3+2\omega)}\right]^{-\frac{(4-3\gamma)+(2-\gamma)\sqrt{3(3+2\omega)}}{\kappa(n-1)}} \\ &\times \left[\pm 2c_3\tau^{n-1} \mp (4-3\gamma) \mp (2-\gamma)\sqrt{3(3+2\omega)}\right]^{-\frac{(4-3\gamma)-(2-\gamma)\sqrt{3(3+2\omega)}}{\kappa(n-1)}} \end{aligned} \tag{29}$$

where $\kappa \equiv 2(5-3\gamma) + 3(2-\gamma)^2\omega$. For a denominator with imaginary roots we require $\omega < -3/2$, for which we find

$$\begin{aligned} a(\tau) = a_0 &\left[\pm 2(4-3\gamma)c_3\tau^{1-n} \pm \kappa\tau^{2(1-n)} \mp 2c_3^2\right]^{\frac{1+(2-\gamma)\omega}{\kappa(1-n)}} \\ &\times \exp\left\{\frac{2\sqrt{-3(3+2\omega)}}{3\kappa(1-n)} \tan^{-1}\left(\frac{(4-3\gamma) - 2c_3t^{n-1}}{(2-\gamma)\sqrt{-3(3+2\omega)}}\right)\right\} \end{aligned} \tag{30}$$

$$\begin{aligned} \phi(\tau) = \phi_0 &\left[\pm 2(4-3\gamma)c_3\tau^{1-n} \pm \kappa\tau^{2(1-n)} \mp 2c_3^2\right]^{\frac{(4-3\gamma)}{\kappa(1-n)}} \\ &\times \exp\left\{-\frac{2(2-\gamma)\sqrt{-3(3+2\omega)}}{\kappa(1-n)} \tan^{-1}\left(\frac{(4-3\gamma) - 2c_3t^{n-1}}{(2-\gamma)\sqrt{-3(3+2\omega)}}\right)\right\} \end{aligned} \tag{31}$$

with κ defined as above. The $\pm$ and $\mp$ signs here indicate that there are multiple solutions that satisfy the field equations. These signs should be chosen consistently within each set of square brackets (solutions (28) and (29) can have upper or lower branches chosen independently in each set of square brackets, as long as a consistent branch is taken within each separate set of square brackets). The physical branch should be chosen as the one for which the quantity in brackets remains positive as $\tau \to \infty$, so ensuring the existence of a positive real root in this limit,

These solutions display interesting new behaviours at both early and late times, which will be discussed in the next section.

VI. BEHAVIOUR OF SOLUTIONS

These exact solutions for $a(\tau)$ and $\phi(\tau)$ can now be analysed at early and late times.

A. $\lambda(\tau) = c_1 + c_2\tau$

At late times, as $\tau \to \infty$, the solutions (24)-(27) all approach the exact power-law solutions (20) and (21). It can be seen that these solutions reduce to the usual spatially flat FRW Brans-Dicke power-law solutions [19] in the limit that the rate of energy transfer goes to zero, $c_2 \to 0$. It can also be seen that these solutions reduce to the spatially-flat FRW general relativistic solutions in the limit $\omega \to \infty$, irrespective of any (finite) amount of energy transfer.

The early-time behaviour of these solutions approaches that of the general Brans-Dicke solutions [18], without energy transfer. Generally, we expect an early period of free-scalar-field domination except in the case $\tau_2 = 0$, in which case the power-law solutions (20) and (21) are valid right up to the initial singularity. For $\omega > -3/2$, the scalar-field dominated phase causes an early period of power-law inflation. In this case there is always an initial singularity and the value of the scalar field diverges to infinity or zero as it is approached, depending on the sign of τ_2. For $\omega < -3/2$ there is a 'bounce' and the scale factor has a minimum non-zero value. In these universes there is a phase of contraction followed by a phase of expansion, with no singularity separating them. Solutions of this type were the focus of [6] where the evolution of ϕ through the bounce was used to model the variation of various physical constants in such situations. The energy exchange term does not play a significant role at early times in these models. The asymptotic solutions as the singularity (or bounce) is approached are the same as if the energy exchange term had been neglected, and are given by [20], up to the absorption of c_1 into τ_2 previously described.

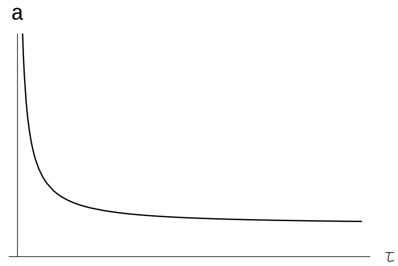
B. $\lambda(\tau) = c_3\tau^n$, $n \neq 1$

The behaviour of the solutions (28)-(31) depends upon the signs of $n-1$ and c_3, as well as on the sign of $\omega+3/2$. For illustrative purposes we will consider the radiation case $\gamma = 4/3$ which is appropriate for realistic universes dominated by asymptotically-free interactions at early times.

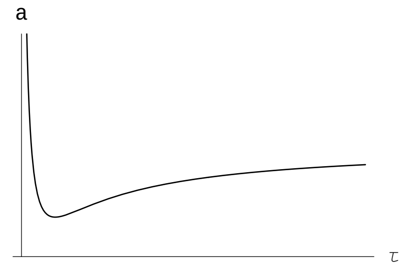
For $n > 1$ it can be seen that the late-time attractors of solutions (28)-(31) are $a \to$ constant and $\phi \to$ constant as $\tau \to \infty$, for both $\omega > -3/2$ and $\omega < -3/2$. At late times these universes are asymptotically static; the evolution of the scale-factor ceases as $\tau \to \infty$ and both ϕ and ρ become constant. Further analysis is required to establish whether these static universes are stable or not (we expect them to be stable as no tuning of parameters or initial conditions has been performed to obtain these solutions).

The early-time behaviour of solutions with $n > 1$ depends upon the sign of c_3 as well as whether ω is greater or less than $-3/2$. We will consider first the case of $\omega > -3/2$. For $c_3 > 0$, we see that $a \to \infty$ as $\tau^{n-1} \to \tau_0^+$; for $c_3 < 0$ we see that $a \to \infty$ as $\tau^{n-1} \to \tau_0^-$ (where $\tau_0^+ = ((2-\gamma)\sqrt{3(3+2\omega)} + (4-3\gamma))/2c_3$ and $\tau_0^- = -((2-\gamma)\sqrt{3(3+2\omega)} - (4-3\gamma))/2c_3$). For $n > 1$ and $\omega > -3/2$ we therefore have the generic behaviour that $a \to \infty$ at early times and $a \to$ constant at late times. The behaviour of a at intermediate times varies in form depending on the sign of c_3, as can be seen in Figures 1(a) and 1(b). The asymptotic form of ϕ for $n > 1$ and $\omega > -3/2$ depends critically on the sign of c_3. For $c_3 > 0$ it can be seen that $\phi \to 0$ as $\tau^{n-1} \to \tau_0^+$, whereas for $c_3 < 0$ it can be seen that $\phi \to \infty$ as $\tau^{n-1} \to \tau_0^-$. The behaviour of ϕ in these two cases is illustrated in Figures 1(c) and 1(d).

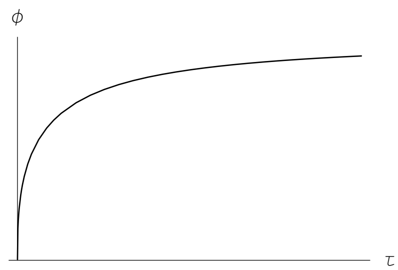
We now consider the early-time behaviour of solutions with $n > 1$ and $\omega < -3/2$. It can be seen that $a \to 0$ as $\tau \to 0$ irrespective of the sign of c_3, so that we find the generic behaviour $a \to 0$ at early times and $a \to$ constant at late times (this is in contrast to the standard theory where an initial singularity is avoided when $\omega < -3/2$). Again, the behaviour of a at intermediate times is dependent on the sign of c_3, as can be seen from Figures 2(a) and 2(b). As $\tau \to 0$ we see that ϕ has a finite non-zero value and is either increasing or decreasing depending on the sign of c_3. This behaviour is shown in Figures 2(c) and 2(d).



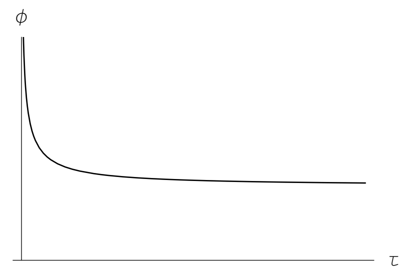
(a) $c_3 = 10$



(b) $c_3 = -10$

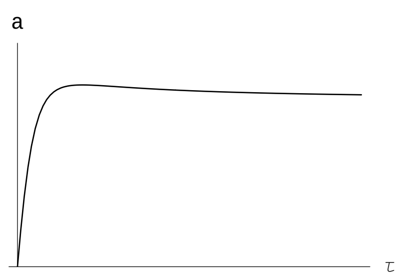


(c) $c_3 = 10$

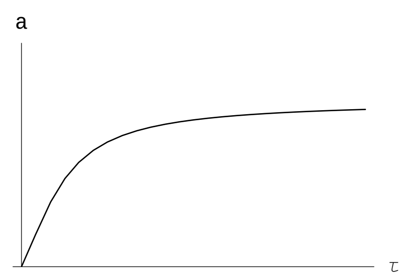


(d) $c_3 = -10$

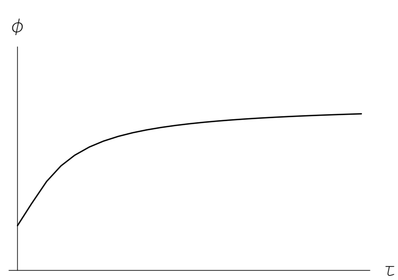
FIG. 1: The time evolution of a and ϕ for $n = 2$, $\omega = 10$ and $\gamma = 4/3$.



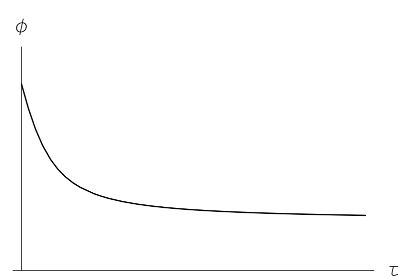
(a) $c_3 = 10$



(b) $c_3 = -10$

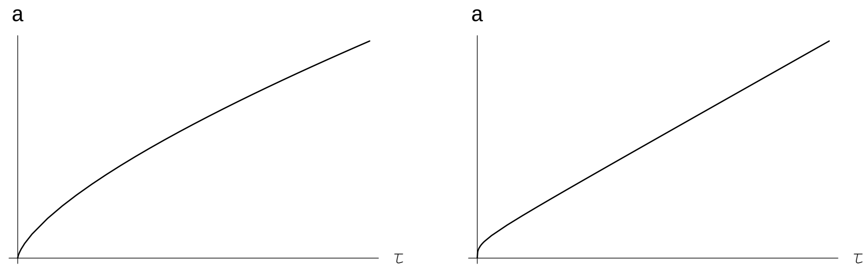


(c) $c_3 = 10$

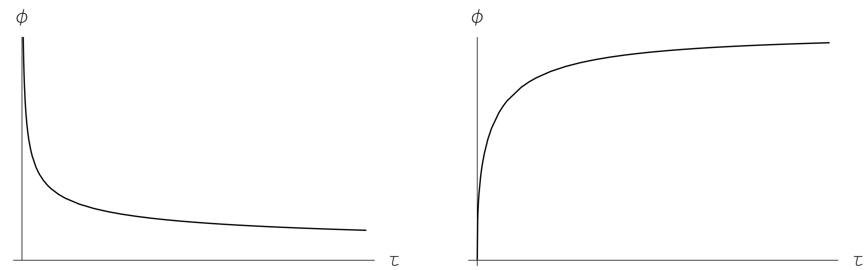


(d) $c_3 = -10$

FIG. 2: The time evolution of a and ϕ for $n = 2$, $\omega = -10$ and $\gamma = 4/3$.

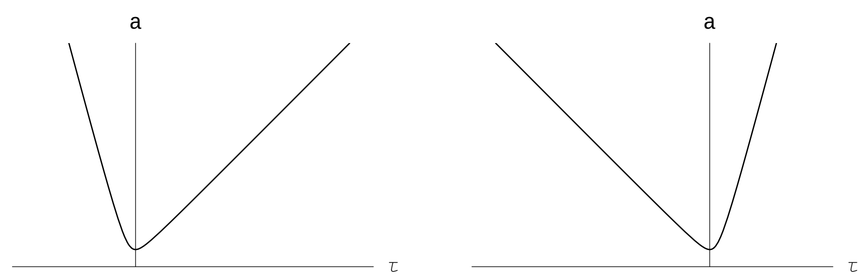


(a) $c_3 = 10$ (b) $c_3 = -10$

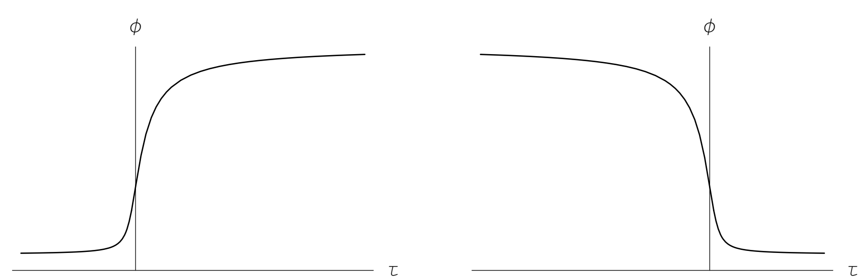


(c) $c_3 = 10$ (d) $c_3 = -10$

FIG. 3: The time evolution of a and ϕ for $n = 0$, $\omega = 10$ and $\gamma = 4/3$.



(a) $c_3 = 10$ (b) $c_3 = -10$



(c) $c_3 = 10$ (d) $c_3 = -10$

FIG. 4: The time evolution of a and ϕ for $n = 0$, $\omega = -10$ and $\gamma = 4/3$.

It remains to investigate the nature of the solutions with $n < 1$. At late times we see that

$$a \rightarrow \tau^{\frac{2+2(2-\gamma)\omega}{2(5-3\gamma)+3(2-\gamma)^2\omega}},$$
$$\phi \rightarrow \tau^{\frac{2(4-3\gamma)}{2(5-3\gamma)+3(2-\gamma)^2\omega}},$$

as $\tau \rightarrow \infty$, irrespective of the sign of c_3 or the value of ω. These late-time attractors are the flat FRW power-law Brans-Dicke solutions [19] which reduce to the standard general-relativistic solutions in the limit $\omega \rightarrow \infty$.

The early-time behaviour when $n < 1$ depends on the sign of c_3 and the sign of $\omega + 3/2$. We consider first the case $\omega > -3/2$. In this case it can be seen that $a \rightarrow 0$ as $\tau^{n-1} \rightarrow \tau_0^+$ or $\tau^{n-1} \rightarrow \tau_0^-$, for $c_3 > 0$ or $c_3 < 0$ respectively. The behaviour of a for both $c_3 > 0$ and $c_3 < 0$ is shown in Figure 3(a) and 3(b). The behaviour of ϕ at early times depends on the sign of c_3 and goes to ∞ for $c_3 > 0$ or to 0 for $c_3 < 0$. The behaviour of ϕ in these cases is shown in Figures 3(c) and 3(d). For $\omega < -3/2$ the scale factor a contracts to a finite, but non-zero, minimum value and then expands. The exact form of the minimum depends on the values of n, ω and c_3, but it is interesting to note that odd values of n produce symmetric bounces and even values of n produce asymmetric bounces, as illustrated in Figures 4(a) and 4(b). The evolution of ϕ through these bounces is smooth with a time direction prescribed by the value of c_3, as shown in Figures 4(c) and 4(d) (increasing for $c_3 > 0$ and decreasing for $c_3 < 0$). The effect of changing the sign of c_3 is seen to be a mirroring of the evolution of a and ϕ in the $y-$axis.

VII. PHYSICAL CONSEQUENCES

The solutions found in the previous sections are of physical interest for a number of reasons. The transfer of energy and momentum between a non-minimally coupled scalar field ϕ and matter fields is a prediction of a number of fundamental theories of current interest, including string theories, Kaluza-Klein theories and brane-worlds. The cosmologies produced by such an interaction, therefore, should be of direct interest in the consideration of these theories. Furthermore, the solutions we have found display modified behaviour at both early and late times. The investigation of modified theories of gravity at early times is of particular interest as it is in the high-energy limit that deviations from general relativity are usually expected. Modified behaviour at late times is also of interest as it is at these times that we can make direct observations which can be used to constrain deviations from the standard general-relativistic model. We will now summarise the behaviour of the solutions found in the previous section, highlighting the physically significant results and constraints that can be placed on the theory by observations.

For the case of λ linear in τ it was shown that the late-time attractors of the general solutions are no longer the power-law solutions of the Brans-Dicke theory, but are given by equations (22) and (23). These attractors are of special interest as they have a simple power-law form that reduces to the general relativistic result in the limit $\omega \rightarrow \infty$ and to the Brans-Dicke result in the limit $c_2 \rightarrow 0$. Observations of cosmic microwave background anisotropies and the products of primordial nucleosynthesis will therefore be able to constraint any potential late-time deviations of this kind, and hence the underlying model. The process of primordial nucleosynthesis in scalar-tensor theories has been used by a number of authors to place constraints on the coupling parameter $\omega(\phi)$ [21, 22, 23, 24, 25, 26, 27]. In these studies the different value of G during nucleosynthesis causes the weak interactions to freeze out at a different time and hence the proton to neutron ratio at this time is different to the standard case. This modification causes different abundances of the light elements to be produced, which can be compared with observations to constrain the underlying theory. Studies of this kind usually assume G to be constant during nucleosynthesis, which will not be the case when energy is allowed to be exchanged between ϕ and the matter fields. The effects of a non-constant G were studied in [27]. A similar study would be required to place constraints upon the parameters c_2 and ω in this theory. The cosmic microwave background power spectrum has also often been used to constrain scalar-tensor theories of gravity [28, 29, 30, 31, 32, 33]. In these studies the redshift of matter-radiation equality is different from its usual general relativistic value due to the modified late-time evolution of the Universe. This change in the redshift of equality is imprinted on the spectrum of perturbations as it is only after equality that sub-horizon scale perturbations are allowed to grow. The main effect is seen as a shift in the first peak of the power-spectrum, which can be compared with observations to constrain the theory. Again, the late time evolution of the Universe is modified from the usual Brans-Dicke case by the energy exchange that we consider, so that the previous constraints are not directly applicable.

For the case of a non-linear power-law exchange of energy, described by $\lambda \propto \tau^n$, the late-time evolution of a and ϕ can be significantly modified. For $n > 1$, the solutions do not continue to expand eternally, but are attracted towards a static state where the time-evolutions of a, ϕ and ρ cease. For $n < 1$ the generic late-time attractor is the power-law solution of a flat FRW Brans-Dicke universe. It appears that theories of energy exchange with $n > 1$ are ruled out immediately by observations of an expanding universe whilst the case of $n < 1$ is subject to the same late-time constraints as the standard Brans-Dicke theory [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33].

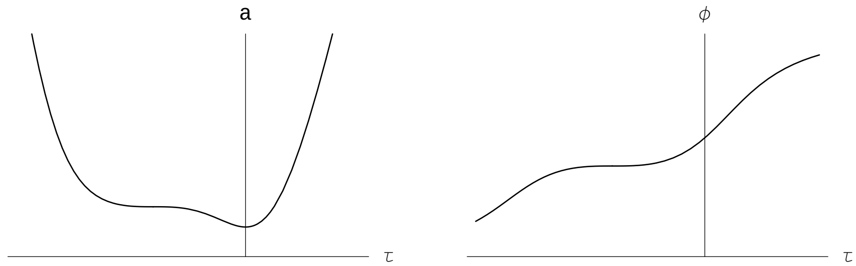


FIG. 5: The time evolution of a and ϕ for $n = -2$, $\omega = -10$, $c_3 = 10$ and $\gamma = 4/3$.

It remains to investigate the physical consequences of the early-time behaviour of our solutions. For λ linear in τ these solutions approach those of the standard Brans-Dicke theory as either the initial singularity or the minimum of the bounce are approached, according to the sign of $\omega - 3/2$. The physical significance of this behaviour has been discussed many times before, usually focusing on the avoidance of the initial singularity and the inflation that can result from the presence of the free component of the scalar field.

For $\lambda \propto \tau^n$ the early-time behaviour can be significantly changed from that of the standard theory. For $n > 1$ the scale factor a either approaches infinity or zero, depending on our choice of ω and c_3, as previously described. For the more realistic case of $n < 1$ the evolution of a at early times either undergoes a period of rapid expansion or a non-singular bounce, depending on whether ω is greater or less than $-3/2$. This behaviour is similar to that of the general solutions of the standard theory, but in this case the free scalar-field-dominated epoch has not been invoked and there is more freedom as to the exact form of the evolution. For example, with a suitable choice of parameters it is possible to create a universe that contracts and then is briefly static before 'bouncing' and continuing on to its late-time power-law evolution. This is shown in Figure 5 for the case $\omega = -10$, $n = -2$ and $c_3 = 10$. (It is interesting to note that Peter and Pinto-Neto remark that a static period followed by a bounce could potentially produce a scale-invariant spectrum of perturbations [34]).

For the physically reasonable models with $n \leqslant 1$ the evolution of ϕ, and hence of G, can be significantly altered at early times from what is generally assumed to be the case in scalar-tensor theories of gravity. For the case $\omega > -3/2$, the value of G can be made to diverge to infinity or to zero as the initial singularity is approached, independent of whether or not there was an early scalar-dominated phase to the universe's history. For the case $\omega < -3/2$, the value of G evolves smoothly through the bounce in the scale-factor, and is again independent of whether or not there was a scalar dominated phase. More complicated evolutions of ϕ can also be constructed, as can be seen in Figure 5.

VIII. DISCUSSION

We have considered spatially-flat FRW universes in scalar-tensor theories of gravity where energy is allowed to be exchanged between the Brans-Dicke scalar field that determines the strength of gravity and any perfect-fluid matter fields in the space-time. We have presented a prescription for integrating the field equations exactly for some unknown function λ which describes the rate at which energy is exchanged. For the case of λ being a linear function of τ we have found the general solutions to the problem and for the case of λ being a non-linear power law function of τ we have been able to find a wide class of exact solutions. These solutions display behaviours that can deviate substantially from the corresponding solutions in the standard case, where the exchange of energy is absent. Depending upon the values of the parameters defining the theory and the exchange of energy, deviations in the evolution of a and ϕ can occur at both early and late times, providing a richer phenomenology than is available in the standard theory.

We have found that the parameter n must be bounded by the inequality $n \leqslant 1$ if the Universe is to be expanding at late times. For $n = 1$ we have found late-time power-law attractor solutions which can be used to constrain the parameters c_2 and ω. For $n < 1$ we have seen that the late-time evolution will be that same as in the standard Brans-Dicke case, and so is subject to the same observational constraints as these theories. The parameters c_3 and τ_2 have been shown to be influential only in the vicinity of the initial singularity, or at the minimum of expansion in non-singular solutions. These parameters are therefore less accessible to constraint by late-time observations (see, however, [27] where the influence on primordial nucleosynthesis is used to constrain τ_2). The parameter ω is, as always, subject to the very tight solar system constraint $\omega > 40000$ to 2σ [4].

These results could be of interest in attempting to explain why G is so small in the present day Universe compared to the proton mass scale ($Gm_{pr}^2 \sim 10^{-39}$). In these models the value of G can decay away by a coupling between the scalar field ϕ and the matter fields which allows energy to be transferred. The small value of G is then due to the age of the Universe. It remains to see whether or not the late-time modifications found above are consistent with observations of the primordial abundance of light elements, microwave background formation and other late-time physical processes. Whilst being beyond the scope of this article, these studies should be able to be performed in an analogous way to the ones that already exist for the standard Brans-Dicke theory.

Using the late time solutions that have been found it is possible to comment on the case of FRW cosmologies with non-zero spatial curvature. At early times it is expected that the effect of any spatial curvature on the evolution of $a(t)$ should be negligible. From the solution (20) we can see that spatial curvature will dominate the late time evolution if the condition

$$\frac{2+2(2-\gamma)\omega+2c_2}{4+3\gamma\omega(2-\gamma)-2c_2(7-6\gamma-c_2)}<1$$

is satisfied. If this condition is not satisfied then the power-law solution (20) will be an attractor as $t\to\infty$ even in the case of non-zero spatial curvature, offering a potential solution to the flatness problem. This behaviour corresponds to power-law inflation and it can be seen that the condition for equation (20) to dominate over the spatial curvature at late times is, indeed, also the condition that power-law inflation should occur.

In conclusion, we have found that a direct coupling between ϕ and the matter fields in scalar-tensor cosmologies provides a richer frame-work within which one can consider variations of G. We have shown that it is possible to construct models where the late-time violations of the equivalence principle can be made arbitrarily small (for $\lambda \propto \tau$) or are attracted to zero (for $\lambda \propto \tau^n$ where $n<1$). This enlarged phenomenology is of interest for the consideration of the four-dimensional cosmologies associated with higher-dimensional theories as well as for more general considerations of the variation of G and its late-time value. This study has been limited to scalar-tensor theories with constant coupling parameters, to flat FRW cosmologies, and to special cases of σ^0 that allow direct integration of the field equations. Obvious extensions exist in which these assumptions are partially or completely relaxed.

ACKNOWLEDGEMENTS

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