

# Spherically Symmetric Solutions to Fourth-Order Theories of Gravity

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PACS numbers: 04.20.-q, 04.20.Jb, 04.25.Nx

**Abstract.**

Solutions to the field equations generated from Lagrangians of the form  $f(R)$  are considered. The spherically symmetric solutions to these equations are discussed, paying particular attention to features that differ from the standard Schwarzschild solution. The asymptotic form of solutions will be described, as will the lack of validity of Birkhoff's theorem. Exact solutions are presented which illustrate these points and the stability and equations of motion of particles in these space-times are investigated.

## 1. Introduction

It has been known for some time that Birkhoff's theorem and asymptotic flatness are not ensured in the vacuum solutions of generalised fourth-order theories of gravity, as they are in general relativity (GR) (see e.g. [1, 2]). The standard approach to overcoming this problem is to consider theories which contain in their generating Lagrangian an Einstein-Hilbert term which dominates the field equations in the low-curvature limit. In this way the usual behaviour of approaching Minkowski space at asymptotically large distances from sources is assumed to occur and perturbative expansions about a Minkowski background are then possible. The weak-field limit of the theory in question can then be investigated in a straightforward way and compared with experiment.

Whilst the existence of a Minkowski background is a great simplification, and very useful in investigations of the weak-field, it is unclear whether or not a theory should be disregarded solely on the grounds of this limit not existing. There are a number of gravitational theories that can be conceived of which do not always admit Minkowski space as a solution. These include theories derived from Lagrangians of the form  $R^{1+\delta}$  (where  $\delta \neq 0$ ) [3, 4, 5, 6, 7, 8], as well as those derived from  $R + \alpha/R^n$  (where  $n > 0$ ) [10, 11]. The latter has caused considerable debate as to whether or not it is compatible with solar system experiments. Studies on this subject usually follow the prescribed analysis for computing the weak field geometry in GR; picking a highly symmetric background<sup>‡</sup> and calculating the form of spherically symmetric perturbations by linearising the field equations and solving them, to first order in perturbations. However, it is often unclear how an appropriate background to expand about should be chosen, and how the lack of Birkhoff's theorem or asymptotic flatness should be taken into account when performing these analyses. We will discuss these points here and attempt to make some progress into understanding the spherically symmetric situation by using explicit exact vacuum solutions of the field equations.

In section 2 we present the field equations for the theories we will be considering and two exact spherically symmetric solutions to these field equations. In section 3 we perform a linear perturbation analysis about the backgrounds of the two exact solutions and in section 4 we find the equations of motion for test-particles in these space-times. In section 5 we summarise our results.

## 2. Field equations and exact solutions

We will consider here gravitational theories derived from a Lagrangian of the form

$$\mathcal{L} = f(R) \tag{1}$$

where  $f(R)$  is an arbitrary power-series expandable function of the Ricci curvature scalar  $R$ . Extremizing the action that is obtained by integrating (1) over all space then gives

<sup>‡</sup> often Minkowski space, even though it is not always a solution of the field equations

the field equations

$$f'R_{ab} - \frac{1}{2}fg_{ab} + f';^{cd}(g_{ab}g_{cd} - g_{ac}g_{bd}) = 8\pi GT_{ab} \quad (2)$$

where matter sources have been included, primes denote differentiation with respect to  $R$  and we have disregarded boundary terms. It can be seen that in the low-curvature limit,  $R \ll 1$  (assuming this limit exists), the term in the power-series expansion of  $f(R)$  with the lowest power will be the one which dominates the field equations. For this reason we specialise our considerations, from this point on, to the choice  $f(R) = R^{1+\delta}$  which reduces to general relativity in the limit  $\delta \rightarrow 0$ . This choice is the theory considered in [3, 4, 5] and will be the effective theory at low curvatures for  $f(R) = R + \alpha/R^n$  when  $\delta = -(1+n)$ . The field equations are then given by

$$\begin{aligned} \delta(1-\delta^2)R^\delta \frac{R_{,a}R_{,b}}{R^2} - \delta(1+\delta)R^\delta \frac{R_{;ab}}{R} + (1+\delta)R^\delta R_{ab} - \frac{1}{2}g_{ab}RR^\delta \\ - g_{ab}\delta(1-\delta^2)R^\delta \frac{R_{,c}R_{,c}}{R^2} + \delta(1+\delta)g_{ab}R^\delta \frac{\square R}{R} = 8\pi T_{ab}. \end{aligned} \quad (3)$$

We will now present two exact spherically symmetric vacuum solutions to this set of equations. The first of these was found previously in [3] and is given by the line-element

$$\textbf{Solution 1} \quad ds^2 = -A_1(r)dt^2 + \frac{dr^2}{B_1(r)} + r^2d\Omega^2 \quad (4)$$

where

$$\begin{aligned} A_1(r) &= r^{2\delta \frac{(1+2\delta)}{(1-\delta)}} + \frac{C_1}{r^{\frac{(1-4\delta)}{(1-\delta)}}} \\ B_1(r) &= \frac{(1-\delta)^2}{(1-2\delta+4\delta^2)(1-2\delta(1+\delta))} \left( 1 + \frac{C_1}{r^{\frac{(1-2\delta+4\delta^2)}{(1-\delta)}}} \right) \end{aligned}$$

and  $C_1$  is a constant. This solution is conformally related to the  $Q = 0$  limit of a solution found by Chan, Horne and Mann [12] and reduces to the Schwarzschild solution in the limit  $\delta \rightarrow 0$ . The second solution is given by

$$\textbf{Solution 2} \quad ds^2 = -A_2(r)dt^2 + a^2(t)B_2(r)(dr^2 + r^2d\Omega^2) \quad (5)$$

where

$$\begin{aligned} A_2(r) &= \left( \frac{1 - \frac{C_2}{r}}{1 + \frac{C_2}{r}} \right)^{\frac{2}{q}} & a(t) &= t^{\delta \frac{(1+2\delta)}{(1-\delta)}} \\ B_2(r) &= \left( 1 + \frac{C_2}{r} \right)^4 A(r)^{q+2\delta-1} & q^2 &= 1 - 2\delta + 4\delta^2 \end{aligned}$$

and  $C_2$  is a constant. This solution is conformally related to one found by Fonarev [13] and again reduces to the Schwarzschild solution in the limit  $\delta \rightarrow 0$ .

These two solutions can be seen to exhibit features not present in the Schwarzschild solution of GR. Both of these solutions are strongly curved, but each displays this curvature in a different way. Solution 1 is static and does not reduce to an  $r$  independent form in the limit  $r \rightarrow \infty$  (despite the Ricci scalar approaching 0 in this limit). Solution

2 displays more conventional behaviour in the limit  $r \rightarrow \infty$ , but shows explicitly the lack of validity of Birkhoff's theorem. This solution becomes  $r$  independent in the limit  $r \rightarrow \infty$ , but still displays strong curvature in this limit as the metric reduces to the spatially flat vacuum Friedmann-Robertson-Walker cosmology found in [3].

We will now continue to find the general form of spherically symmetric perturbations to the backgrounds ( $r \rightarrow \infty$  limit) of the two exact solutions above. It will be seen that there exist extra modes which are not excited in the exact solutions, but that the modes corresponding to the linearised exact solutions above are the ones most interesting for performing gravitational experiments in these space-times.

### 3. Linear perturbation analysis

Perturbative analyses in the literature are often performed about Minkowski space or de-Sitter space. This is perfectly acceptable practise in GR and fourth-order theories in which an Einstein-Hilbert term dominates in the low curvature regime. In other fourth-order theories, of the type considered here, in which the Einstein-Hilbert term does not dominate the low curvature regime then there is good reason to consider perturbing about other backgrounds. We have shown explicitly, with exact solutions, the existence of other spherically symmetric space-times. We will now proceed to perform a linear perturbation analysis about the backgrounds ( $r \rightarrow \infty$  limit) of these two space-times. The general solution to first order in perturbations will be found and it will be shown that the form of these linearised solutions will be strongly dependant on the background.

#### 3.1. Solution 1

An analysis of the linear perturbations about the background prescribed by equation (4) has already been performed in [3]. We will simply quote the result of this study here, the reader is referred to [3] for details of the derivation.

Writing the perturbed line-element as

$$ds^2 = -r^{2\delta\frac{(1+2\delta)}{(1-\delta)}}(1+V(r))dt^2 + \frac{(1-2\delta+4\delta^2)(1-2\delta-2\delta^2)}{(1-\delta)^2}(1+W(r))dr^2 + r^2d\Omega^2,$$

substituting this into the field equations and linearising them to first order in  $V$  and  $W$  allows the general solution to first order in perturbations to be found as

$$\begin{aligned} V(r) &= c_1V_1(r) + c_2V_2(r) + c_3V_3(r) + \text{constant} \\ W(r) &= -c_1V_1(r) + c_2W_2(r) + c_3W_3(r) \end{aligned}$$

where

$$\begin{aligned} V_1 &= -r^{-\frac{(1-2\delta+4\delta^2)}{(1-\delta)}} \\ V_2 &= \frac{(1+2\delta)r^{-\frac{(1-2\delta+4\delta^2)}{2(1-\delta)}}}{2(2-3\delta+12\delta^2+16\delta^3)} \left( (1+2\delta)^2 \sin(A \log r) + 2A(1-\delta) \cos(A \log r) \right) \end{aligned}$$

$$\begin{aligned}
W_2 &= r^{-\frac{(1-2\delta+4\delta^2)}{2(1-\delta)}} \sin(A \log r) \\
V_3 &= \frac{(1+2\delta) \left( (1+2\delta)^2 \cos(A \log r) - 2A(1-\delta) \sin(A \log r) \right) r^{-\frac{(1-2\delta+4\delta^2)}{2(1-\delta)}}}{2(2-3\delta+12\delta^2+16\delta^3)} \\
W_3 &= r^{-\frac{(1-2\delta+4\delta^2)}{2(1-\delta)}} \cos(A \log r)
\end{aligned}$$

and

$$A = -\frac{\sqrt{7-28\delta+36\delta^2-16\delta^3-80\delta^4}}{2(1-\delta)}.$$

The extra constant in  $V$  can be absorbed into the definition of the time coordinate. This procedure of solving the linearised field equations does not ensure that the solution obtained will be the linearisation of the general solution to the full non-linear field equations, but it is encouraging to note that the  $c_2$  mode corresponds to the linearisation of the exact solution (4).

### 3.2. Solution 2

The corresponding perturbative analysis about the background given by (5) will now be performed. Writing the line-element as

$$ds^2 = -(1+P(r))dt^2 + b^2(t)(1+Q(r))(dr^2 + r^2d\Omega^2) \quad (6)$$

allows the vacuum field equations to be linearised in  $P$  and  $Q$ . These linearised field equations are given in the appendix, and are solved to give the solutions

$$\begin{aligned}
P &= -\frac{c_4}{r} + \frac{2c_5(1-6\delta+4\delta^2+4\delta^3)}{(5-14\delta-12\delta^2)}r^2 + \text{constant} \\
Q &= \frac{(1-2\delta)c_4}{r} + \delta c_5 r^2 + \text{constant}
\end{aligned}$$

where  $c_4$  and  $c_5$  are constants and the two other constant terms in  $P$  and  $Q$  are independent of each other and can be absorbed into  $t$  and  $s$  by redefinitions. Again, it can be seen that one of the modes,  $c_4$ , corresponds to the linearised version of the exact solution, (5).

It can immediately be seen that the form of these linearised solutions are quite different to those obtained by expanding around the background of solution 1, (4). Whilst the expansion about (4) produces damped oscillatory modes, as well as the mode corresponding to the linearised exact solution, the expansion about (5) produces more familiar looking terms proportional to  $r^2$ . Aside from the different form of these extra modes there is also a noticeable difference in the terms corresponding to the linearised exact solutions, which both go as  $r^{-1}$  in the limit  $\delta \rightarrow 0$ , but behave differently from each other when  $\delta \neq 0$ . This shows explicitly the differences that can arise when linearising about different backgrounds. Not only are there extra modes which can take different functional forms, but even the modes which reduce to the Schwarzschild limit as  $\delta \rightarrow 0$  are different, depending on the background that has been chosen.

We will now proceed to calculate the equations of motion of particles following geodesics of these space-times, to Post-Newtonian order. It will be shown that not only are the terms due to the linear perturbations different (as should be expected as the perturbations themselves have been shown to be background dependant) but that the background itself also contributes extra terms to the post-Newtonian equations of motion.

#### 4. Equations of motion

We will now proceed to calculate the geodesics of the two space-times, (4) and (5). In doing this we will neglect the contributions from the oscillating modes to (4) and the contribution of the  $r^2$  mode to (5) so that we only take into account the modes which go as  $r^{-1}$ , in the limit  $\delta \rightarrow 0$ . These are the modes corresponding to the linearisation of the exact solutions (4) and (5). In performing this computation we will transform the solution (4) into isotropic coordinates (details of this transformation are given in [3]).

The geodesic equation can be written, as usual, in the form

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0,$$

where  $\lambda$  can be taken as proper time for a time-like geodesic, or as an affine parameter for a null geodesic. In terms of coordinate time this can be re-written as

$$\frac{d^2 x^\mu}{dt^2} + \left( \Gamma^\mu_{ij} - \Gamma^0_{ij} \frac{dx^\mu}{dt} \right) \frac{dx^i}{dt} \frac{dx^j}{dt} = 0. \quad (7)$$

Substituting the linearised solutions into this equation will then give the equations of motion for test particles in these space-times.

##### 4.1. Solution 1

In isotropic coordinates the linearised version of (4) can be written as

$$ds^2 = -A_3(r)dt^2 + B_3(r)(dr^2 + r^2 d\Omega^2)$$

where

$$A_3(r) = r^{\frac{2\delta(1+2\delta)}{n}} \left( 1 - \frac{(1-2\delta)c_6}{r^m} \right)$$

$$B_3(r) = r^{-2 + \frac{2(1-\delta)}{n}} \left( 1 + \frac{c_6}{r^m} \right)$$

and

$$n^2 = (1 - 2\delta - 2\delta^2)(1 - 2\delta + 4\delta^2)$$

$$m^2 = \frac{(1 - 2\delta + 4\delta^2)}{(1 - 2\delta - 2\delta^2)}$$

and  $c_6$  is constant. Substituting this metric into the geodesic equation (7) gives, to post-Newtonian order,

$$\frac{d^2 \mathbf{x}}{dt^2} = -\frac{\delta(1+2\delta)}{nr^{\frac{2}{m}-1}} \mathbf{e}_r - \frac{(1-8\delta+4\delta^2)c_6}{2nr^{\frac{3(1-2\delta)}{n}-1}} \mathbf{e}_r + \frac{(1-8\delta+4\delta^2)c_6^2}{2nr^{\frac{4(1-\delta)^2}{n}-1}} \mathbf{e}_r \quad (8)$$

$$\begin{aligned}
& + \left( \frac{(1 - \delta - n)}{nr} - \frac{mc_6}{2r^{1+m}} \right) \left( \frac{d\mathbf{x}}{dt} \right)^2 \mathbf{e}_r \\
& + \left( \frac{2(m-1)}{mr} + \frac{2(1-\delta)mc_6}{r^{1+m}} \right) \mathbf{e}_r \cdot \frac{d\mathbf{x}}{dt} \frac{d\mathbf{x}}{dt},
\end{aligned}$$

which to first order in  $\delta$  is

$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{Gm}{r^2} \left( 1 + \left| \frac{d\mathbf{x}}{dt} \right|^2 \right) \mathbf{e}_r + 4\frac{G^2m^2}{r^3} \mathbf{e}_r + 4\frac{Gm}{r^2} \mathbf{e}_r \cdot \frac{d\mathbf{x}}{dt} \frac{d\mathbf{x}}{dt} - \frac{\delta}{r} \mathbf{e}_r \quad (9)$$

where the Newtonian limit has been used to set  $c_6$ . It can be seen that these equations of motion are modified from their usual form in GR, both in the pre-multiplicative factors of the terms with GR counterparts as well as modifications to the powers of  $r$  and the existence of entirely new terms which vanish in the limit  $\delta \rightarrow 0$ . The first order corrections due to small, but non-zero,  $\delta$  are primarily due to the new term arising in (9). This term corresponds to a new force which drops off as  $r^{-1}$  and is the term which was used in [3] to impose upon  $\delta$  the tight constraint

$$\delta = 2.7 \pm 4.5 \times 10^{-19} \quad (10)$$

from observations of the perihelion precession of Mercury [14].

#### 4.2. Solution 2

Solution 2 has already been given in isotropic coordinates in equation (5), which on substitution into (7) gives to post-Newtonian order the equation of motion

$$\begin{aligned}
\frac{d^2\mathbf{x}}{d\tau^2} = & -\frac{Gm}{r^2} \mathbf{e}_r + 4(1-\delta) \frac{G^2m^2}{r^3} \mathbf{e}_r - (1-2\delta) \frac{Gm}{r^2} \left( \frac{d\mathbf{x}}{d\tau} \right)^2 \\
& + 4(1-\delta) \frac{Gm}{r^2} \mathbf{e}_r \cdot \frac{d\mathbf{x}}{d\tau} \frac{d\mathbf{x}}{d\tau} - \mathcal{H} \frac{d\mathbf{x}}{d\tau} \left( 1 - \left( \frac{d\mathbf{x}}{d\tau} \right)^2 \right)
\end{aligned} \quad (11)$$

where

$$\mathcal{H} \equiv \frac{1}{a} \frac{da}{d\tau},$$

the conformal time coordinate  $\tau$  is defined by  $dt \equiv a d\tau$  and  $c_4$  has been set by the appropriate Newtonian limit. The equations of motion for this solution are considerably simpler than those of solution 1, but still differ from those of GR in significant ways. All of the terms except the last in equation (11) have GR counterparts and the powers of  $r$  in these terms are all the same as in the general relativistic case. The premultiplicative factors of these terms are, however, modified and can be described adequately within the frame-work of the parameterised post-Newtonian approach [15] by assigning

$$\beta = 1 \quad \text{and} \quad \gamma = 1 - 2\delta.$$

By making this identification the constraints on  $\gamma$  from observations of the Shapiro time delay of radio signals from the Cassini space probe [16] can be used to impose upon  $\delta$  the constraint

$$\delta = -1.1 \pm 1.2 \times 10^{-5}. \quad (12)$$

As well as the usual effects associated with  $\gamma - 1$  being non-zero there are extra effects in this space-time due to the last term in (11). This term is proportional to the velocity of the test-particle (when  $v \ll c$ ) and is zero for photons. For this reason we identify it as a friction term, with the friction coefficient being given by  $\mathcal{H}$ . This ‘friction’ is a purely gravitational effect and is not due to any non-gravitational interaction of test particles with any other matter.

## 5. Conclusions

We have considered here the problem of finding spherically symmetric vacuum solutions of fourth-order theories of gravity. In these theories the Einstein-Hilbert term may not be the dominating one at low curvatures in the generating Lagrangian and the validity of taking a Minkowski or de-Sitter background to expand about cannot be assumed so readily. As well as this, Birkhoff’s theorem is not valid in fourth-order theories, which allows for a much larger manifold of solutions, when spherical symmetry is imposed. These features allow for the interesting possibility of having a choice of backgrounds to expand about, both static and non-static, which can vary considerably in form from their general relativistic counterparts.

We have presented two exact solutions which are spherically symmetric and reduce to the Schwarzschild solution in the limit of the fourth-order theory reducing to general relativity. These solutions have very different backgrounds ( $r \rightarrow \infty$  limits) and gravitational physics in each of them is correspondingly different. The first of these solutions is static and has a non-trivial dependence on the radial coordinate  $r$ , as  $r \rightarrow \infty$ . The second is non-static and has an  $r$  independent form in the limit  $r \rightarrow \infty$ . In some sense these solutions can be considered as being extremes of the general solution; in the first case the strong curvature at large distances from the centre of symmetry is only dependent on  $r$  and in the second the curvature at large distances is only a function of  $t$  (in the limit  $r \rightarrow \infty$ ).

Using these exact solutions we have shown how different choices of background lead to very different solutions of the linearised field equations, when spherically symmetric perturbations are introduced. By taking the perturbation modes which reduce to something approximating Newtonian gravity, in the appropriate limit, we have derived the equations of motion for test particles following geodesics of these space times. The equations of motion are found to be strongly dependent on the choice of background. In both cases the form of the terms that have counterparts in GR are dependent on the background. As well as this dependence there are additional terms due to the background itself which have no counterparts in the general relativistic solutions. For the static case these extra terms look like a fifth force which drops off as  $\sim r^{-1}$  and for the time dependent case they look like a friction term  $\propto v$ , when  $v \ll c$ .

These considerations show explicitly that there are a number of different backgrounds about which one may choose to perform a perturbative expansion, in fourth-order theories, and that these backgrounds can display behaviour which is

not permitted in general relativity. Moreover, it has been shown that the choice of background is highly non-trivial when performing a perturbative expansion. The form of the perturbations are background dependent and, correspondingly, so are the geodesics of the perturbed space-times.

The existence of multiple backgrounds, all of which appear to be valid solutions of the field equations, prompts the question: which is the appropriate solution for physical situations? We will not attempt to address this question fully here, but will offer a few suggestions and leave a more comprehensive analysis for a future study. Firstly, a physically relevant background solution should be stable to perturbations. Imposing the extra symmetry of time-independence it was shown in [3] that the  $r \rightarrow \infty$  limit of solution (4) is the attractor of the general spherically symmetric and time independent solution, in this limit. It was also shown in reference [3] that the  $r \rightarrow \infty$  limit of solution (5) is the attractor of the general homogeneous and isotropic vacuum solution in the limit  $t \rightarrow \infty$  (at least for some range of  $\delta$ ). In this sense both of these solutions can be considered as stable, but this does not allow us to differentiate between the two of them which is the physically more relevant. In order to establish the appropriate solution for any realistic physical situation it will be necessary to apply boundary conditions to either or both of an interior solution for an energy-momentum distribution and/or the relevant cosmological solution. Whilst a number of cosmological solutions are known [3, 4, 5, 9] this problem is made more difficult by a lack of exact solutions which may be valid in the interior of, for example, a star. A more comprehensive analysis is also required in order to establish the effect of allowing the perturbations themselves to be time dependent.

## 6. Appendix

Substituting the perturbed line-element (6) into the vacuum field equations (3) allows the following set of coupled, linearised differential equations to be found.

The  $t - t$  equation is

$$\begin{aligned} &6\delta(1 - 2\delta - 2\delta^2)(5 - 14\delta - 12\delta^2)\nabla P \\ &-12(1 - 2\delta - 2\delta^2)(1 - 6\delta + 4\delta^2 + 4\delta^3)\nabla Q \\ &+(1 + 2\delta)(1 - \delta)^2 t^{\frac{2(1-2\delta-2\delta^2)}{(1-\delta)}}\nabla\psi = 0 \end{aligned}$$

the  $r - r$  equation is

$$\begin{aligned} &(2 - 17\delta + 18\delta^2 + 52\delta^3 + 8\delta^4)P'' - 2(1 - \delta + 3\delta^2 - 34\delta^3 - 32\delta^4)\frac{P'}{r} \\ &+2(2 - 8\delta - 9\delta^2 + 16\delta^3 + 8\delta^4)Q'' + 2(1 - 4\delta - 15\delta^2 + 20\delta^3 + 16\delta^4)\frac{Q'}{r} \\ &+3(1 - \delta)^2 t^{\frac{2(1-2\delta-2\delta^2)}{(1-\delta)}}\left(\frac{\psi''}{2(1 + 2\delta)} + \frac{\psi'}{3r}\right) = 0 \end{aligned}$$

and the  $t - r$  equation is

$$\psi' = 0$$

where

$$\psi = \nabla P + 2\nabla Q,$$

primes here denote differentiation with respect to  $r$  and  $\nabla$  is the Laplacian on the three dimensional subspace. The  $\theta - \theta$  and  $\phi - \phi$  equations are then linear combinations of the above equations.

## Acknowledgments

The author would like to acknowledge John Barrow for helpful discussions and the PPARC for financial support.

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