

The Ups and Downs of Cyclic Universes

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Abstract

We investigate homogeneous and isotropic oscillating cosmologies with multiple fluid components. Transfer of energy between these fluids is included in order to model the effects of non-equilibrium behavior on closed universes. We find exact solutions which display a range of new behaviors for the expansion scale factor. Detailed examples are studied for the exchange of energy from dust or scalar field into radiation. We show that, contrary to expectation, it is unlikely that such models can offer a physically viable solution to the flatness problem.

1 Introduction

In this paper we consider a wide range of homogeneous and isotropic cosmological models containing two fluids which are able to exchange energies and manifest non-equilibrium behavior. Recently [1], we studied a class of such examples in the dynamical context of a flat Friedmann universe. A simple ansatz was used to model energy transfer between two fluids, which would be separately conserved perfect fluids in the absence of the energy transfer. By means of this simple formulation we were able to find a single master equation whose range of solutions allowed us to study the cases of massive particles decaying into radiation, particle-antiparticle annihilation into radiation, the decay of classical vacuum energy, and the formation and Hawking evaporation of black holes. In this paper, we add some further ingredients: the effects of spatial curvature are included in the cosmological dynamics and a wider range of energy exchanges

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are considered. The most interesting situation allowed by the introduction of spatial curvature is the possibility of an oscillating closed universe. Following Tolman's [2] identification of the important role that the second law of thermodynamics could play in the long term evolution of a cyclic universe in which entropy production takes place, there have been further detailed studies of how thermodynamics leads to differences in the cycle-to-cycle evolution. Landsberg and Park [4] studied the approach to flatness created by the growing cycles; Barrow and Dąbrowski [5] carried out a detailed study of the evolution of anisotropies and black holes, and also showed that if a positive cosmological constant is present then Tolman's growing oscillations will eventually be ended – replaced by unending expansion towards the de Sitter space-time. However, these studies did not attempt to model the action of energy exchanges between matter and radiation; they simply injected entropy (in the form of radiation) into the universe at each moment of bounce. A more realistic modelling of energy exchange was introduced by Barrow, Kimberly, and Magueijo [6], who included energy transfer between scalar fields and radiation in some cosmological models arising in general relativity, Brans-Dicke theory, and theories in which variations in the fine structure 'constant' are driven by appropriately coupled scalar fields [7]. This revealed the interesting feature that oscillating universes, in which time variations in the gravitational 'constant' and the fine structure 'constant' can occur, saw those variations continuing in an almost monotonic fashion from cycle to cycle. Unlike the expansion scale factor, these varying 'constants' did not oscillate. A further variant was considered by the present authors, in ref [8], where they considered energy exchange between radiation and the Brans-Dicke scalar field so that energy was drained from the gravitational 'constant' as the universe expanded. Here, we extend these investigations and enlarge the gallery of possible cosmological evolutions of the scale factor that can occur in oscillating closed universes in general relativity.

2 Cosmological Models

Consider a spatially homogeneous and isotropic Friedmann universe, with expansion scale factor $a(t)$, containing two fluids with equations of state

$$\begin{aligned} p &= (\gamma - 1)\rho, \\ p_1 &= (\Gamma - 1)\rho_1, \end{aligned}$$

where the γ and Γ are constants, p and p_1 are the fluid pressures and ρ and ρ_1 are their densities. The Friedmann equation is then given by

$$H^2 = \rho + \rho_1 - \frac{k}{a^2}, \tag{1}$$

where $8\pi G/3 \equiv 1$, $H = \dot{a}/a$ is the Hubble expansion rate of the universe, and t is comoving proper time. We write the evolution equations for these two fluids as

$$\dot{\rho} + 3H\gamma\rho = s, \tag{2}$$

$$\dot{\rho}_1 + 3H\Gamma\rho_1 = -s \tag{3}$$

where s is a function parameterising any exchange of energy between them.

In general, s could be any function of H , ρ , ρ_1 , t and a . In a previous work [1], we considered the case where s was prescribed by

$$s = -\alpha H\rho + \beta H\rho_1, \quad (4)$$

with α and β both constant, and $k = 0$. This case can be used to model the two-way exchange of energy between fluids in a number of physically interesting situations involving particle decays, vacuum decay, particle-antiparticle annihilation, or black hole formation and evaporation, and is particularly appealing because it can be solved completely in terms of simple functions. Here, we will consider more general formulations of this problem, particularly those that arise when there is non-zero spatial curvature ($k \neq 0$) and with different dependence of the energy exchange parameter, s , on the other physical variables.

The inclusion of spatial curvature allows particular cases of special interest, such as oscillating universes with entropy production, to be investigated. In these models a universe of positive spatial curvature repeatedly expands to a maximum and recollapses towards a big crunch. In the absence of non-adiabatic processes, these cyclic universes will all have the same maximum of expansion and total lifetime. However, it was first argued by Tolman [2] that an increase of entropy at the moment of each crunch-to-bang transition should create a growth in the scale of successive maxima of expansion. In this way, the universe may approach spatial flatness at asymptotically late times, allowing a potential solution of the flatness problem. These growing oscillations will continue until such time as any positive cosmological constant, or quintessence field that violates the strong energy condition, dominates the expansion, after which the oscillations will cease, as first shown in ref [5]. If the entropy production occurs in relatively small steps then the final state should also be one in which the energy density in the quintessence field is just slightly greater than that in the cold dark matter fields.

3 Cosmological Thermodynamics

The exchange of energy and momentum between different components of a universe containing multiple fluids requires us to reconsider the thermodynamics of the universe, especially the effects of the second law of thermodynamics. We write the fundamental law of thermodynamics as

$$Td\mathbf{S} = d(\rho V) + pdV = d[(\rho + p)V] - Vdp \quad (5)$$

where \mathbf{S} (not to be confused with s) is the entropy of the fluid and $V \equiv a^3$ is the comoving volume of the universe. The integrability condition

$$\frac{\partial^2 \mathbf{S}}{\partial T \partial V} = \frac{\partial^2 \mathbf{S}}{\partial V \partial T}$$

then gives the relation

$$dp = \frac{(\rho + p)}{T} dT$$

which can be substituted into equations (2) and (5) to give

$$\frac{d}{dt} \left[\frac{(\rho + p)V}{T} \right] = \frac{Vs}{T} \quad \text{and} \quad d\mathbf{S} = d \left[\frac{(\rho + p)V}{T} \right].$$

We can then read off the expression for the density ρ fluid

$$\dot{\mathbf{S}} = \frac{sV}{T} \quad (6)$$

and similarly, for ρ_1 ,

$$\dot{\mathbf{S}}_1 = -\frac{sV}{T_1}. \quad (7)$$

It can be seen directly from these expressions that allowing a transfer of energy between different components of the universe ($s \neq 0$) results in the entropy of those components no longer staying constant. In the limit that this interaction disappears, $s \rightarrow 0$, the rate of change of entropy also disappears and entropy is again conserved. These expressions for the rate of change in entropy of the two fluids enable us to assess the implications of the second law of thermodynamics. According to the second law:

$$0 \leq \dot{\mathbf{S}}_T \equiv \dot{\mathbf{S}} + \dot{\mathbf{S}}_1 = sV \left(\frac{1}{T} - \frac{1}{T_1} \right).$$

This equation tells us that for energy to be transferred from ρ to ρ_1 (i.e. $s < 0$) we must have $T \geq T_1$ and that, conversely, for energy to be transferred from ρ_1 to ρ ($s > 0$) we must have $T_1 \geq T$. In other words: energy is always transferred from the hotter to the cooler fluid.

4 Exact Solutions

There are a number of exact solutions that can be found to the system of equations (1), (2) and (3). In this section we will present methods of integrating these equations, the analysis of which will follow in the subsequent sections of the paper.

4.1 Solutions with $\rho = \rho(\mathbf{a})$

Taking the form of s given by (4) and making the definitions

$$\begin{aligned} A &\equiv \alpha + \beta + 3(\gamma + \Gamma), \\ B &\equiv \frac{3}{2}(\alpha\Gamma + \beta\gamma + 3\gamma\Gamma), \end{aligned}$$

the equations (2) and (3) can be put in the simple form

$$\frac{\rho''}{\rho} + A\frac{\rho'}{\rho} + 2B = 0,$$

where primes denote differentiation with respect to the new time coordinate,

$$\eta \equiv \ln a.$$

The solution to this equation is then given by

$$\begin{aligned}\rho &= c_1 e^{\frac{1}{2}(-A-\sqrt{A^2-8B})\eta} + c_2 e^{\frac{1}{2}(-A+\sqrt{A^2-8B})\eta} \\ &= c_1 a^{\frac{1}{2}(-A-\sqrt{A^2-8B})} + c_2 a^{\frac{1}{2}(-A+\sqrt{A^2-8B})}\end{aligned}$$

where c_1 and c_2 are constants. Substitution into (3) then gives $\rho_1 \propto \rho$ and so (1) can be rewritten as

$$H^2 = c_3 a^{\frac{1}{2}(-A-\sqrt{A^2-8B})} + c_4 a^{\frac{1}{2}(-A+\sqrt{A^2-8B})} - \frac{k}{a^2}.$$

It can be seen that this equation reduces to the solution found in ref [1] in the limit $k \rightarrow 0$, and that the inclusion of spatial curvature does not add any new features to the form of a that are not already present in the standard evolution without interaction. Specifically, it can be seen that in this case an oscillating universe with $k > 0$ will have the same maximum of expansion in each of its cycles.

4.2 Solutions with $\rho = \rho(\mathbf{a}, \mathbf{t})$

The Friedmann equation (1) shows that, for any functions $\rho = \rho(a)$ and $\rho_1 = \rho_1(a)$, the evolution of the universe shares the same property as the example above. Oscillating $k > 0$ universes have constant amplitude. In order to find new behaviors, we will consider a more general situation where we parameterise $\rho = \rho(a, t)$ and $\rho_1 = \rho_1(a, t)$. This will allow new exact solutions to be found with two interacting fluids, and non-zero spatial curvature. Integrating equations (2) and (3), we obtain

$$\rho = \frac{\int s a^{3\gamma} dt}{a^{3\gamma}} \equiv \frac{m}{a^{3\gamma}} \quad \text{and} \quad \rho_1 = \frac{-\int s a^{3\Gamma} dt}{a^{3\Gamma}} \equiv \frac{m_1}{a^{3\Gamma}}, \quad (8)$$

which defines the quantities m and m_1 . In the absence of energy exchange ($s = 0$), these two quantities are constant and we have the same form for ρ and ρ_1 as in the usual case. As soon as energy exchange is allowed, m and m_1 become non-constant, corresponding to energy being exchanged between the two fluids.

4.2.1 A universe containing radiation and dust or massless scalar field

For the cases of universes containing radiation and dust, or radiation and a scalar field, the field equations (1), (2) and (3) can be transformed into the equation of motion for a forced harmonic oscillator. Such equations can be solved exactly using standard techniques. These cases are of particular interest for modelling an oscillating universe as they correspond to physical significant situations. It has often been hypothesised that an oscillating universe containing these fluids should allow for energy to be transferred, and entropy to be increased, at the moment of crunch-to-bang. The case of exchange between radiation and a scalar field is of particular interest as the kinetic parts of scalar fields may dominate the earliest stages of the universe, and non-equilibrium behavior can arise through slow or fast decays, as expected at the end of inflation or other phase transitions involving these fields. Negative-energy scalar fields are often used in oscillating universe models to produce a non-zero minimum of expansion, and a detailed study

of their effects, especially on any ‘constant’ of Nature which may vary in time was given in ref [6].

Differentiation of the Friedmann equation (1), and some manipulation and substitution using (2) and (3) allows us to write the evolution equation for $H(t)$ as

$$\dot{H} + \frac{3}{2}\Gamma H^2 = \frac{3}{2}(\Gamma - \gamma)\rho + (1 - \frac{3}{2}\Gamma)\frac{k}{a^2}.$$

Following the prescription given in ref [9] to turn non-dissipative Friedmann universes into simple harmonic oscillators in conformal time, we make a transformation to the conformal time coordinate $ad\tau = dt$ and define the new variable $b = a^{3\gamma-2}$, to obtain the equation of motion of a forced harmonic oscillator:

$$b'' + (2 - 3\gamma)^2 kb = \frac{1}{2}(2 - 3\gamma)^2 m, \quad (9)$$

where primes here denote differentiation with respect to τ . In deriving this equation we have used the definition of m given by (8), and assumed the relation $3\Gamma + 2 = 6\gamma$: This includes the important cases of radiation and dust ($\Gamma = 4/3$ and $\gamma = 1$) and radiation and scalar field ($\gamma = 4/3$ and $\Gamma = 2$), which this section focuses upon. The well known solution to equation (9), for general m and $k \neq 0$, is given by

$$\begin{aligned} b(\tau) = & c_1 \sin\left(\sqrt{(2-3\gamma)^2 k \tau}\right) + c_2 \cos\left(\sqrt{(2-3\gamma)^2 k \tau}\right) \\ & + \frac{\sqrt{(2-3\gamma)^2}}{2k} \sin\left(\sqrt{(2-3\gamma)^2 k \tau}\right) \int m \cos\left(\sqrt{(2-3\gamma)^2 k \tau_1}\right) d\tau_1 \\ & - \frac{\sqrt{(2-3\gamma)^2}}{2k} \cos\left(\sqrt{(2-3\gamma)^2 k \tau}\right) \int m \sin\left(\sqrt{(2-3\gamma)^2 k \tau_1}\right) d\tau_1, \end{aligned}$$

where c_1 and c_2 are constants of integration. By specifying a particular energy-exchange function, s , the value of m can be found from equation (8), and the integrals above can then be performed. Methods of obtaining various forms of $m(t)$ are outlined in the appendix.

If we take, for example, the forms of s to be

$$s_A = s_0 \rho a^{-1} \quad \text{and} \quad s_B = s_0 a^{-(1+3\gamma)},$$

where s_0 is a constant, then the corresponding functions m are found in the appendix to take the particularly simple forms

$$m_A = m_0 e^{s_0 \tau} \quad \text{and} \quad m_B = m_0 + s_0 \tau,$$

where m_0 is a constant of integration. The solutions, for $k \neq 0$, can then be written as

$$b_A = c_1 \sin\left(\sqrt{(2-3\gamma)^2 k (\tau - \tau_0)}\right) + \frac{(2-3\gamma)^2 m_0 e^{s_0 \tau}}{2((2-3\gamma)^2 k + s_0^2)} \quad (10)$$

and

$$b_B = \bar{c}_1 \sin\left(\sqrt{(2-3\gamma)^2 k (\tau - \bar{\tau}_0)}\right) + \frac{(m_0 + s_0 \tau)}{2k} \quad (11)$$

where c_1 , \bar{c}_1 , τ_0 and $\bar{\tau}_0$ are constants, and subscripts A and B denote solutions corresponding to s_A and s_B , respectively. The behavior of these solutions will be investigated in the next section. The case of no energy being exchanged can be read off from equation (11) as the sub-class of solutions with $s_0 = 0$.

4.2.2 A universe containing radiation and non-zero vacuum energy

A second mechanism for obtaining an oscillating universe is a negative vacuum energy. Due to the lack of dissipation of vacuum energy as the universe expands, this fluid always comes to dominate at late-times (if the universe survives that long, and in the absence of phantoms). Negative vacuum energy then plays the role that was previously played by positive spatial curvature, and causes the universe to collapse at late-times as anti-de Sitter behavior is approached.

We will now proceed to find solutions for a universe containing radiation and non-zero vacuum energy ($\rho = -p$). Again, the Friedmann equations can be cast into the form of the equation of motion for a harmonic oscillator, but now the forcing term will be constant and the analogue of the mass will be non-constant.

Taking $\Gamma = 4/3$, $\gamma = 0$ and defining the new variable $\alpha \equiv a^2$, allows the equations (1), (2) and (3) to be reduced to the form

$$\ddot{\alpha} - \frac{4\Lambda}{3}\alpha = -2k, \quad (12)$$

where use has been made of (8), and $\Lambda \equiv 3m = 3\rho$ has been defined in analogy to the usual notation of the cosmological constant. (The reader should bear in mind that Λ is *not* a constant here, as energy is being exchanged between it and the radiation fluid). This equation can be solved once s has been specified, and a solution obtained for Λ .

If we now take, for example, the energy-exchange parameters

$$s_C = s_0\Lambda \quad \text{and} \quad s_D = s_0$$

then, from the appendix, we obtain the functions

$$\Lambda_C = \Lambda_0 e^{3s_0 t} \quad \text{and} \quad \Lambda_D = \Lambda_0 + 3s_0 t,$$

where s_0 and Λ_0 are constants. The equation (12) can now be solved for these s and m , to give

$$\alpha_C = c_3 I_0(x) + c_4 K_0(x) - \frac{2k}{3s_0} I_0(x) \int K_0(x_1) dt_1 + \frac{2k}{3s_0} K_0(x) \int I_0(x_1) dt_1 \quad (13)$$

and

$$\alpha_D = \bar{c}_3 A_i(y) + \bar{c}_4 B_i(y) + ks_0 \sqrt[3]{\frac{\pi^3}{12s_0^4}} A_i(y) \int B_i(y_1) dt_1 - ks_0 \sqrt[3]{\frac{\pi^3}{12s_0^4}} B_i(y) \int A_i(y_1) dt_1 \quad (14)$$

where c_3 , \bar{c}_3 , c_4 and \bar{c}_4 are constants, I_0 and K_0 are Bessel functions, A_i and B_i are Airy functions and x and y are defined by

$$x = \sqrt{\frac{16|\Lambda_0|}{27s_0^2}} e^{3s_0 t/2} \quad \text{and} \quad y = \sqrt[3]{\frac{4}{27s_0^2}} (\Lambda_0 + 3s_0 t).$$

The form of α given in (13) corresponds to $\Lambda_0 > 0$; making the substitutions $K_0 \rightarrow \pi J_0$ and $I_0 \rightarrow Y_0$ gives the solution for $\Lambda_0 < 0$. For $k = 0$ the unsightly terms involving integrals of Bessel and Airy functions in (13) and (14) vanish.

In the absence of any energy exchange, Λ is constant, and the solution to equation (12) is

$$\alpha = c_5 \sin \left\{ \sqrt{\frac{-4\Lambda}{3}}(t - t_1) \right\} + \frac{3k}{2\Lambda}$$

where c_5 and t_1 are constants. In the next section we will proceed to investigate the physical behavior of these solutions.

5 Behavior of the Cosmological Solutions

Having found exact solutions for oscillating universes with energy exchange, in the previous section, we will now investigate their behavior. The oscillatory nature of these solutions is created either by positive spatial curvature, or a negative vacuum energy, either of which will cause a maximum of expansion, after which the universe collapses. By introducing energy exchange between the fluids in these models, we allow for the possibility of different cycles having different expansion maxima.

5.1 A closed universe containing radiation and scalar field

From equations (10) and (11) above, we see that the scale-factor for a universe containing radiation ($\gamma = 4/3$) and scalar field ($\Gamma = 2$) with positive spatial curvature ($k = 1$) can be written as

$$a_A^2 = c_1 \sin \{2(\tau - \tau_0)\} + e^{s_0(\tau - \tau_1)} \quad (15)$$

when $s = s_0 \rho a^{-1}$, and as

$$a_B^2 = \bar{c}_1 \sin \{2(\tau - \bar{\tau}_0)\} + \frac{1}{2} s_0 (\tau - \bar{\tau}_1) \quad (16)$$

when $s = s_0 a^{-5}$. Here we have absorbed some of the constants of (10) and (11) into the new constants τ_1 and $\bar{\tau}_1$.

The two functions (15) and (16) above have the undesirable feature of allowing a^2 to be both positive and negative, for any set of the constants c_1 , τ_0 and τ_1 . However, only two of these three will be fixed by specifying the energies of the two fluids at the moment of collapse; the third is a free constant which can be used to match together different cycles, at the moment of crunch-to-bang. In this way the scale-factor can be made real and positive semi-definite, throughout its evolution. Figures 1 and 2 show the evolution of these universes after such a matching.

It can be seen from figures 1 and 2 that, as time progresses, the expansion maximum of each cycle increases. This is because we chose $s_0 > 0$ to construct these plots, corresponding to energy being transferred from the scalar field to radiation. This causes successive maxima of expansion to increase in magnitude, due to the extra radiation. This is in accord with Tolman's original model [2]. However, as well as the maxima increasing from cycle to cycle, we also see that after a certain number of cycles there exist real non-zero minima of expansion, instead of

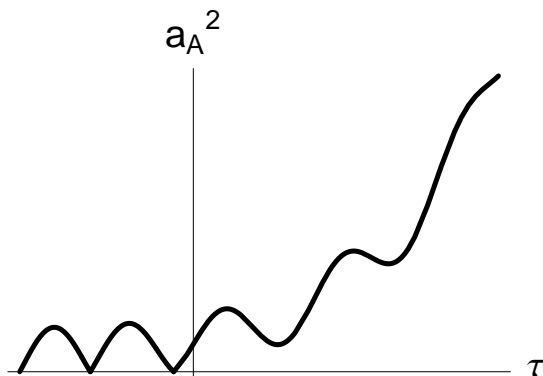


Figure 1: *The evolution of the scale factor in a closed universe with radiation and scalar field exchanging energy, as prescribed in equation (15) when $0 < s \propto \rho a^{-1}$.*

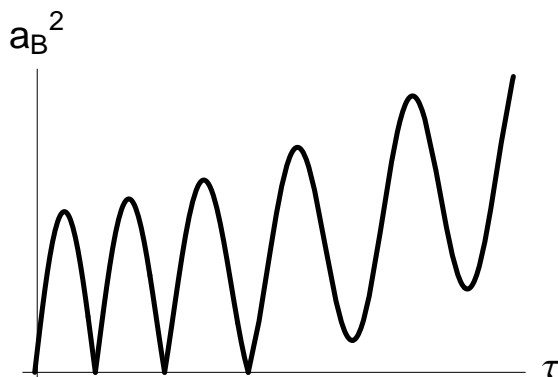


Figure 2: *The evolution of the scale factor in a closed universe with radiation and scalar field exchanging energy, as prescribed in equation (16) when $0 < s \propto a^{-5}$.*

collapse to a singularity. After this point the evolution of the universe is non-singular. It can clearly be seen that these minima of expansion also increase in magnitude as τ increases. This is an effect that has not been recognised in earlier discussions of oscillating universes subject to the second law of thermodynamics. We interpret this behavior as being due to the scalar field transferring so much energy to the radiation field that its own energy density becomes negative, allowing a non-zero minimum of expansion. Negative energy scalar fields are often used in this way to create oscillating universe models without singularities; however, we can include a cut-off to the energy transfer that would prevent this from occurring.

In addition to this common behavior, significant differences can also be seen to occur between figures 1 and 2 at both early and late-times. Such differences in behavior clearly illustrate that the evolution of these universes is sensitive to the particular form of energy exchange that has been chosen.

At late-times, figure 1 can be seen to exhibit a runaway behavior. As the energy density of the radiation in this universe increases, so does the rate of energy transfer (as it is proportional to ρ). This results in the endless transfer of energy, at ever increasing rates, from the unbounded negative energy scalar to the radiation. In fact, after a sufficient number of cycles the runaway

transfer of energy eventually becomes so great that it overwhelms the oscillatory nature of the solutions, after which these universes expand eternally. The late-time behavior of figure 2 also allows an endless transfer of energy from the negative energy scalar to radiation. However, now the energy transfer is no longer increased by the ever-increasing energy density in radiation, ρ , and so does not display the runaway behavior seen in figure 1. The increase of each expansion maximum and minimum happens at a steadier rate, and at no point does the energy exchange become rapid enough to overwhelm the oscillatory nature of the solutions.

At early-times, these solutions exhibit cycles which are separated by catastrophic collapse to a singularity. As time is run backwards, the maximum of each cycle decreases, as the energy of the radiation decreases. Figure 1 shows approach to a regime where each cycle is of the same amplitude, in the limit, as $\rho \rightarrow 0$ and hence $s_A \rightarrow 0$. Expansion is then due to the scalar field, until it is inevitably halted by the positive spatial curvature. By contrast, in figure 2 we see that the expansion maximum and the total duration of each cycle continues to decrease, as time is run backwards. This behavior occurs because the energy density of radiation is no longer bounded from below, and continues to decrease until it becomes negative. After this point, the radiation contributes to the onset of collapse, increasingly so as its energy density becomes more negative. Of course, negative energy radiation may not be particularly realistic, and we can include a cutoff in the energy transfer function to prevent this from occurring. Such a cutoff will eventually result in cycles of constant amplitude, when energy is no longer being transferred. It might also be avoided by picking forms for s which are more physically motivated, and may have a more complicated evolution in time, rather than the simple choices we have made for illustrative purposes.

The case of energy transfer from radiation to the scalar field can be pictured using figures 1 and 2 by reversing the direction of time.

5.2 A closed universe containing radiation and dust

From equations (10) and (11) we now see that a closed universe containing radiation ($\Gamma = 4/3$) and pressureless dust ($\gamma = 1$) has scale-factor

$$a_A = \hat{c}_1 \sin(\tau - \hat{\tau}_0) + e^{s_0(\tau - \hat{\tau}_1)} \quad (17)$$

when $s = s_0 \rho a^{-1}$, and

$$a_B = \tilde{c}_1 \sin(\tau - \tilde{\tau}_0) + \frac{1}{2} s_0 (\tau - \tilde{\tau}_1) \quad (18)$$

when $s = s_0 a^{-4}$. Again, we have absorbed some of the constants of (10) and (11) into the new constants $\hat{\tau}_1$ and $\tilde{\tau}_1$.

The behavior of the scale-factor is qualitatively the same as that shown in figures 1 and 2 (with the axis label a^2 being replaced by a). The main difference here is that $s_0 > 0$ (the direction of time going from left to right in the figures above) now corresponds to energy being transferred from radiation to dust. This results in the subsequent maxima and minima of expansion increasing as the energy density of the dust increases. For the more realistic situation of energy going from dust to radiation, one should follow the graphs backwards, from right to left. Real non-zero minima of expansion are taken here to arise from the energy density in the

radiation field taking a negative value. Again, a cut-off in energy transfer may be needed to prevent this from occurring.

5.3 A flat universe containing radiation and vacuum energy

From (13) and (14) it can be seen that a spatially flat universe containing radiation ($\Gamma = 4/3$) and negative vacuum energy ($\gamma = 0$) evolves as

$$a_C^2 = c_2 Y_0(x) + c_3 \pi J_0(x) \quad (19)$$

when $s = s_0 \Lambda$, and as

$$a_D^2 = \bar{c}_2 A_i(y) + \bar{c}_3 B_i(y) \quad (20)$$

when $s = s_0$, where x and y are the same as before. The form of these solutions are shown in figures 3 and 4.

Again, these solutions will take both positive and negative values for any set of c_2 , c_3 and Λ_0 , and likewise, solutions can be matched at the moment of crunch-to-bang to ensure that a remains real and positive. This is done in figures 3 and 4.

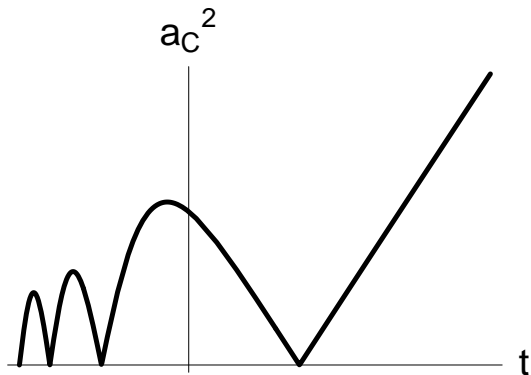


Figure 3: *The evolution of the scale factor in a flat universe with radiation and vacuum exchanging energy, as prescribed in equation (19) when $0 < s \propto \Lambda$.*

Both of figures 3 and 4 show universes which are initially oscillating, with scale factor oscillations of increasing amplitude that eventually end in an asymptotic period of continual expansion. This is due to the choice $s > 0$, which corresponds to energy being transferred from the radiation to the negative vacuum energy. Once again, the contrary case of energy transfer from vacuum energy to radiation can be considered by reversing the direction of time in these plots.

In figure 3 we have the energy density of the vacuum evolving as $\Lambda = -|\Lambda_0|e^{-3|s_0|t}$, so that as $t \rightarrow \infty$ the energy density of the vacuum goes to zero. This has the effect of leaving a radiation-dominated universe ($a^2 \propto t$) in the late-time limit, as can clearly be seen from the plot. The exchange of energy then becomes unimportant as subsequent evolution continues in the standard way, with the vacuum energy density having been forced to zero. At earlier times, however, the influence of the negative vacuum energy becomes increasingly important,

resulting in oscillations of ever-decreasing amplitude. In contrast to some of the previous cases, the energy density of the radiation is always positive in this scenario, because increasing the magnitude of the negative vacuum energy results in an increase in the positive energy density of the radiation.

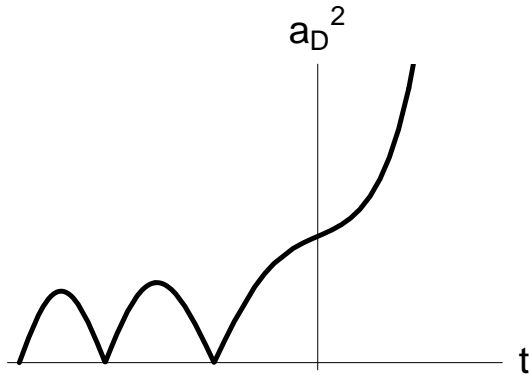


Figure 4: *The evolution of the scale factor in a flat universe with radiation and vacuum exchanging energy, as prescribed in equation (20) when $0 < s = \text{constant}$.*

In figure 4, the transfer of energy from the vacuum to radiation occurs at a constant rate, resulting in an evolution of the vacuum energy density as $\Lambda = -|\Lambda| + 3s_0t$. For $s_0 > 0$, energy is transferred from the radiation to the vacuum, as in the previous example. Now, however, the energy density of the vacuum is allowed to become positive at sufficiently large t . This results in a period of rapidly accelerating expansion as $t \rightarrow \infty$: not only is the evolution of this universe dominated by a positive cosmological constant, but the value of this “constant” is itself increasing with time. The early-time behavior is qualitatively similar in this example to the previous one: as t becomes increasingly negative, oscillations are increasingly damped. Once again, we are guaranteed a positive energy density of radiation, via the same mechanism as before. The principal difference here is that the decrease in amplitude of oscillations, as t decreases, is slower than before because the transfer of energy is now constant, and not proportional to Λ .

Inclusion of non-zero spatial curvature in these models results in a more elaborate form of $a(t)$, as given in equations (13) and (14). However, although more difficult to express in a concise analytic form, the behavior of these solutions can be simply understood. In limits where radiation previously dominated the evolution (such as the $t \rightarrow \infty$ limit of figure 3) spatial curvature now eventually dominates. Positive curvature causes a maximum of expansion, and a subsequent collapse, while negative curvature causes evolution towards a Milne universe, $a \propto t$. In limits where the vacuum energy dominates expansion, the spatial curvature has a negligible effect, unless it is large enough to cause collapse before vacuum domination can occur.

6 Discussion

We have investigated the behavior of spatially homogeneous and isotropic oscillating universes when non-equilibrium behavior is present. We have found exact solutions for universes con-

taining multiple fluids and non-zero spatial curvature. These fluids include radiation and dust, scalar field, or vacuum energy.

In the usual treatment, cosmologies with more than one fluid component are assumed to evolve with these fluids staying non-interacting. The energy of each fluid is then separately conserved, and the evolution of the scale factor can be straightforwardly obtained by solving the Friedmann equation. Here, we have considered a more general situation in which the different fluid components of the universe are allowed to interact. Total energy is still conserved in these models, but we have allowed energy to be transferred between the different components. This situation has often been discussed in the literature, with cosmological models in which entropy increases only at each moment of crunch-to-bang. Here, we have formulated the problem consistently and found exact solutions in which entropy production occurs realistically and we have used them to model a number of different possibilities.

Firstly, we considered universes containing radiation and dust and universes containing radiation and scalar field. In these models the oscillatory nature of the solutions is due to positive spatial curvature. By allowing interactions between these components, we find oscillating solutions where the amplitude of successive cycles is allowed to vary. The case of radiation and dust is of obvious interest as these components are required for the nucleosynthesis of the light elements and formation of large-scale structure in the standard cold dark matter model. The case of radiation and scalar field is also of interest as scalar fields, if any exist in nature, will be influential in the early evolution of the universe. We find that the evolution of these universes is highly dependent upon the exact form of interaction between the fluids. For some interactions, such as those of the form considered in [1], oscillating universe models are found to progress with each of their cycles having identical amplitude. For other interactions, it is found that energy transfer leads to cycles of varying amplitude. Again, the precise form of this variation is heavily dependent on the precise form of interaction. We find general methods for solving the Friedmann equations, and investigate explicitly two particularly simple models. It has often been conjectured that oscillating anti-damped universes of growing amplitude may offer a solution to the flatness problem, without the need for inflation. By allowing the amplitude of each cycle to increase monotonically, it is suggested [4, 5] that eventually the universe will end up being pushed closer and closer to spatial flatness, with each successive cycle being longer and longer lived. Our toy models show us that this may not be a particularly realistic expectation, even if there is no cosmological constant. We find that the increase in amplitude of the oscillations is halted in cases where the energy density of fluids is bounded from below: there is simply not enough energy to allow the cycles to become indefinitely large and long lived. In cases where the energy densities are not bounded from below, such as are often considered with scalar field models, we also find problems. When the energy density of these components become negative the universe experiences a non-zero minimum of expansion, instead of a crunch to singularity. This behavior is well known, and is often used to model non-singular oscillating cosmologies. However, we find that the continual transfer of energy required to create cycles of indefinite size and duration results in subsequent increases in the non-zero minimum of expansion. Thus, by sourcing energy from this negative energy scalar field it becomes more negative, and increases in the maximum of expansion are accompanied by increases in the minimum of expansion. This behavior does not appear to be consistent with a physically viable cosmology – that is, one which starts small and lives long enough for matter domination to occur.

In addition to these models, we also consider universes containing radiation and non-zero vacuum energy. The oscillatory nature of these solutions is either produced by positive spatial curvature, or by a negative vacuum energy. For spatial curvature to have any non-negligible effect on these cosmologies it must be strong enough to dominate the evolution of the universe before the effects of vacuum energy become significant, otherwise vacuum effects dominate. Again, we provide a general prescription for solving this problem for arbitrary functions parameterising the exchange of energy. As before, we give two particularly simple toy models explicitly. In these simple models we neglect spatial curvature and allow a negative vacuum energy to collapse the universe. We find that if the vacuum energy is allowed to become positive then it quickly comes to dominate, and accelerating expansion rapidly ensues. On the other hand, if the vacuum energy becomes increasingly negative then the amplitude and duration of cycles is found to decrease as the vacuum energy comes to dominate, and collapses the universe increasingly rapidly. Finally, in examples where the exchange of energy causes the vacuum energy to approach zero, we find that the subsequent evolution of the universe progresses as a standard radiation-dominated Friedmann universe. Including the effects of spatial curvature complicates the precise analytic form of the solutions, but otherwise acts in the expected way.

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Appendix: From $s(\mathbf{a}, \mathbf{t})$ to $m(\mathbf{t})$

It can be seen from (2) and (3) that we can obtain any invertible function $m = m(\tau)$ by choosing the energy exchange parameter to take a form

$$s = [a^{3\gamma} f'(\rho a^{3\gamma}) g(a)]^{-1},$$

where f is the inverse function of $m = m(t)$ and $g(a)dT \equiv dt$ defines the time coordinate τ . Substituting this into (2) gives

$$f'(\rho a^{3\gamma}) \frac{d(\rho a^{3\gamma})}{d\tau} = 1,$$

which integrates to

$$f(\rho a^{3\gamma}) = \tau - \tau_0$$

or

$$\rho a^{3\gamma} = m(\tau - \tau_0) = f^{-1}(\tau - \tau_0),$$

where τ_0 is a constant of integration, which can be trivially absorbed into a redefinition of the time coordinate $\tau \rightarrow \tau + \tau_0$. This gives us a prescription for the form of $s(\rho, a)$ required to derive any invertible function $m = m(\tau)$.

We will often be interested in the simple forms

$$m_1 = m_0 e^{s_0 \tau} \quad \text{and} \quad m_2 = m_0 + s_0 \tau \quad (21)$$

where m_0 and s_0 are constants. It can now be seen from the above that these forms for m correspond to

$$s_1 = \frac{s_0 \rho}{g} \quad \text{and} \quad s_2 = \frac{s_0}{a^{3\gamma} g}, \quad (22)$$

respectively.

These results can be used to find exact solutions for $a(t)$ in universes with two interacting fluids, and non-zero spatial curvature.

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