The Parameterised Post-Newtonian Limit of Bimetric Theories of Gravity

Timothy Clifton[1], **Máximo Bañados**[2]‡,
and Constantinos Skordis[3]

[1]Department of Astrophysics, University of Oxford, OX1 3RH, UK.
[2]Departamento Física, P. Universidad Católica de Chile, Santiago, Chile.
[3]School of Physics and Astronomy, University of Nottingham, NG7 2RD, UK.

E-mail: t.clifton@cantab.net,
maxbanados@gmail.com,
skordis@nottingham.ac.uk

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Abstract.
We consider the post-Newtonian limit of a general class of bimetric theories of gravity, in which both metrics are dynamical. The established parameterised post-Newtonian approach is followed as closely as possible, although new potentials are found that do not exist within the standard frame-work. It is found that these theories can evade solar system tests of post-Newtonian gravity remarkably well. We show that perturbations about Minkowski space in these theories contain both massless and massive degrees of freedom, and that in general there are two different types of massive mode, each with a different mass parameter. If both of these masses are sufficiently large then the predictions of the most general class of theories we consider are indistinguishable from those of general relativity, up to post-Newtonian order in a weak field, low velocity expansion. In the limit that the massive modes become massless, we find that these general theories do not exhibit a van Dam-Veltman-Zakharov-like discontinuity in their γ parameter, although there are discontinuities in other post-Newtonian parameters as the massless limit is approached. This smooth behaviour in γ is due to the discontinuities from each of the two different massive modes cancelling each other out. Such cancellations cannot occur in special cases with only one massive mode, such as the Isham-Salam-Strathdee theory.

‡ JS Guggenheim Memorial Foundation Fellow

1. Introduction

There exists a long history in gravitational physics of considering theories that are generalisations of, or alternatives to, general relativity. These theories can take on a variety of different forms, and often involve the introduction of extra scalar [1], vector [2] or tensor fields [3] in the gravitational action. Here we consider the latter of these: Bimetric theories in which there are two dynamical rank-2 symmetric tensor fields, rather than the usual one. Our goal is to calculate the Parameterised Post-Newtonian (PPN) limit of a general class of these theories, so that their consequences for observational and experimental gravitational physics can be determined in a straightforward way.

The PPN formalism [4, 5] is now the standard frame-work within which investigations of the phenomenology of relativistic gravitational physics are performed [6]. It has at its heart a simple and compelling rationale: that by observationally constraining terms in a general 'test metric', it is then possible to impose corresponding constraints on a variety of different metric based gravitational theories without going through the rigmarole of directly calculating their physical consequences every time. This allows the space-time geometry of the physical environments in question to be constrained in a theory independent way, as well as supplying a quick and direct route to determining the observational validity (or otherwise) of particular theories.

The problem of determining the PPN limit of bimetric theories of gravity has been approached in the past. The PPN limit of the bimetric theory of Rosen [7] was found by Lee, Caves, Ni and Will [8]. The PPN limits of Rastall's bimetric theory [9, 10], and Lightman and Lee's bimetric theory [11] are also known. These theories, however, contain only one dynamical rank-2 tensor each, the other being *a priori* specified as being Riemann flat. Such non-dynamical fields can cause serious problems for these theories, particularly with respect to the emission of gravitational waves from binary systems. For example, Rosen's theory predicts an unobserved *increase* in the rotational period of the Hulse-Taylor binary pulsar system PSR 1913+16 [12], and it can be argued that similar behaviour should be expected in all theories with 'priori geometry' [6]. The bigravity theories considered in the present article do not have any such non-dynamical fields, and are therefore not expected to fall foul of binary pulsar observations in the way discussed above. To the best of our knowledge, the case of determining the PPN limit of a general class of bimetric theories with *two* dynamical rank-2 tensor fields has yet to be investigated in detail. It is this subject that we intend to investigate here.

As motivation for this study we use the recent astronomical observations that have revealed that most of the matter in the Universe is in the form of unknown "dark components" (i.e. particles or fields that are not present in the standard model, and that do not interact electromagnetically). Despite ongoing efforts these fields have yet to be observed directly, and it seems natural to consider extensions of general relativity that may explain them. In this context, bimetric gravity has recently been put forward as an alternative to both dark matter and dark energy. For details of the way in which the extra degrees of freedom in these theories can be made to reproduce the observations usually attributed to the dark components of the Universe we refer the reader to [13–19]. Here we will not concern ourselves with the cosmological implications of these theories directly, but will instead study their weak field limit. If theories of this type are to be considered observationally viable it is absolutely necessary that we properly understand their weak field limit, and how they are constrained by observations of post-Newtonian gravitational phenomena.

The plan of this paper is as follows. In Section 2 we introduce and discuss the bimetric theories of gravity that we will be investigating, giving their gravitational action and field equations. In Section 3 we briefly introduce the PPN formalism, the perturbative expansion it relies upon, and the way in which it is used to constrain gravitational theories. Section 4 contains a calculation of metric perturbations up to Newtonian accuracy. We then proceed in Sections 5 and 6 to derive the full post-Newtonian limit of the metric perturbations about both of the metrics involved in these theories. In Section 7 we transform into the standard post-Newtonian gauge, and calculate the form of the perturbations in the metric that couples to matter. Section 8 provides a discussion of the extent to which these theories can be subject to observational constraint, and in Section 9 we provide a brief discussion of our results.

2. Bimetric Theories of Gravity

One of the most straightforward ways to extend general relativity is to take the already existing concept of a dynamical rank-2 symmetric tensor field, and replace it with a multiplet of N_g such fields. Multigravity theories can then be constructed by considering an extension of the Yang-Mills approach to gauge theories. That is, if we write our N_g fields as $g^a_{\mu\nu}$, where $a = 1, 2, ..., N_g$, then our gravitational action can be written

$$S[g^a] \sim \int d^4x \left[\sum_{a=1}^{N_g} \sqrt{det(g^a)} R(g^a) - I_{int}(g^a) \right], \tag{1}$$

where I_{int} is an interaction term that can depend on any or all of the metrics. Although it is impossible to construct a non-trivial I_{int} term that preserves the original N_g-dimensional symmetries of the free theory [20], the action above is manifestly invariant under diagonal diffeomorphisms acting on all metrics. The potential therefore breaks the symmetry group down to the diagonal subgroup of diffeomorphisms acting on all metrics with the same parameter. This approach to gravity has a long history, especially the $N_g = 2$ incarnation known as 'bigravity'. It began in the early 1970s with the pioneering work of Isham, Salam and Strathdee [3], while today actions like (1) are mostly used as covariant actions for massive gravitons. A classification of different interaction terms, as well as a variety of different motivations for bigravity, are explored in [21].

In order to make progress we must specialise the general action (1) somewhat. The theory we will consider for the rest of this paper is therefore the $N_g = 2$ bigravity theory

$$S = \frac{1}{16\pi G}\int d^4x \Big[\sqrt{-g}(R-2\Lambda_0) + \sqrt{-q}(K-2\lambda) \\ - \frac{1}{l^2}\sqrt{-q}\left(\kappa_0 J + \kappa_1 J^2 + \kappa_2 (q^{-1})^{\mu\nu}(q^{-1})_{\mu\nu}\right)\Big]. \quad (2)$$

Here we have written the two metric fields as $g_{\mu\nu}$ and $q_{\mu\nu}$. The Ricci scalars constructed from these metrics are written as R and K, respectively, and the two constants Λ_0 and λ are the bare cosmological constants on $g_{\mu\nu}$ and $q_{\mu\nu}$. The three constants κ_0, κ_1 and κ_2 parameterise the interactions between $g_{\mu\nu}$ and $q_{\mu\nu}$. The inverse $(q^{-1})^{\mu\nu}$ is defined such that $(q^{-1})^{\mu\alpha}q_{\mu\beta} = \delta^{\alpha}{}_{\beta}$, and all raising or lowering of the indices of these tensors is done with $g_{\mu\nu}$, so that e.g. $(q^{-1})_{\mu\nu} \equiv (q^{-1})^{\alpha\beta}g_{\alpha\mu}g_{\beta\nu}$. We have also defined $J \equiv (q^{-1})^{\mu\nu}g_{\mu\nu} = (q^{-1})^{\mu}{}_{\mu}$.

The theory above is a generalization of the often considered Isham-Salam-Strathdee (ISS) action

$$S = \frac{1}{16\pi G}\int d^4x \left[\sqrt{-g}(R-2\Lambda_0) + \sigma\sqrt{-f}(\hat{K}-2\hat{\lambda}) - \frac{\sigma}{l^2}\sqrt{-f}I_{int}\right], \quad (3)$$

with the interaction term $I_{int} = (g_{\mu\nu}-f_{\mu\nu})(g_{\alpha\beta}-f_{\alpha\beta})((f^{-1})^{\mu\alpha}(f^{-1})^{\nu\beta}-(f^{-1})^{\mu\nu}(f^{-1})^{\alpha\beta})$. Here the scalar $\hat{K}$ is the Ricci scalar constructed from the metric $f_{\mu\nu}$, and $(f^{-1})^{\mu\alpha}f_{\mu\beta} = \delta^{\alpha}{}_{\beta}$. The generalised theory (2) can be seen to reduce to the ISS action (3) under the metric rescaling $q_{\mu\nu} = \sigma f_{\mu\nu}$, together with the identifications $\kappa_0 = 6$, $\kappa_1 = -\sigma$ and $\kappa_2 = \sigma$, and the redefinition $\lambda = \hat{\lambda}/\sigma - 6/(l^2\sigma)$.

Of course, we also want to couple matter to these theories. We do this by including a Lagrangian density of the form $\mathcal{L}_m = \mathcal{L}_m(\psi, \hat{g}_{\mu\nu})$, where the matter fields ψ are coupled to a linear combination of the metrics

$$\hat{g}_{\mu\nu} = m g_{\mu\nu} + n q_{\mu\nu}, \quad (4)$$

where m and n are constants. After varying with respect to each of the metric fields this then results in the field equations

$$l^2 R_{\mu\nu} - 8\pi G l^2 m\sqrt{\frac{\hat{g}}{g}}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\alpha}{}_{\alpha}\right) - \alpha_0 g_{\mu\nu} \quad (5)$$
$$= -\sqrt{\frac{q}{g}}\Big[(\kappa_0 + 2\kappa_1 J)\left((q^{-1})_{\mu\nu} - \frac{1}{2}g_{\mu\nu}J\right) \\ + 2\kappa_2\left((q^{-1})_{\mu\alpha}(q^{-1})^{\alpha}{}_{\nu} - \frac{1}{2}(q^{-1})_{\alpha\beta}(q^{-1})^{\alpha\beta}g_{\mu\nu}\right)\Big],$$

and

$$l^2 K_{\mu\nu} - 8\pi G l^2 n\sqrt{\frac{\hat{g}}{q}}\left(q_{\mu\alpha}q_{\nu\beta}T^{\alpha\beta} - \frac{1}{2}q_{\mu\nu}T^{\alpha\beta}q_{\alpha\beta}\right) - \alpha q_{\mu\nu} \quad (6)$$
$$= -\frac{1}{2}\left[\kappa_1 J^2 + \kappa_2 (q^{-1})_{\alpha\beta}(q^{-1})^{\alpha\beta}\right]q_{\mu\nu} + \left[\kappa_0 + 2\kappa_1 J\right]g_{\mu\nu} + 2\kappa_2 (q^{-1})_{\mu\nu},$$

where $\alpha_0 \equiv \Lambda_0 l^2$ and $\alpha \equiv \lambda l^2$, and where $T^{\mu\nu}$ is defined by

$$T^{\mu\nu} = -\frac{2}{\sqrt{-\hat{g}}}\frac{\delta \mathcal{L}_m}{\delta \hat{g}_{\mu\nu}}. \tag{7}$$

Expressing the field equations in this form, without R and K, the perturbation equations take on their simplest form.

Now, we should be aware that these theories contain within their spectrum a massive spin-2 field, and that such terms often have associated with them certain discontinuities and instabilities [22–28]. These include the van Dam-Veltman-Zakharov (vDVZ) discontinuity [29, 30], and the Boulware-Deser (BD) instability [31]. Both of these are potentially very serious problems. The vDVZ discontinuity involves the zero-mass limit of massive theories of gravity. Instead of recovering the solutions of general relativity in this limit, it can instead sometimes be found that space-time geometry approaches a limit which is inconsistent with observed gravitational phenomena. For example, the prediction for weak lensing by the Sun in the Pauli-Fierz theory [32] yields results which disagree with general relativity by 25%. The BD instability involves the possible existence of negative energy states at the non-linear level, and a lack of boundedness from below. We refer the reader to [33–35] for recent discussions on massive (single metric) gravitons.

While deserving of careful study, these problems are not necessarily fatal for the bimetric theories we are considering. The vDVZ discontinuity and the BD instability were both identified in the context of bimetric theories in which only one of the metrics is dynamical (the other being *a priori* specified as Riemann flat). It remains to be seen to what extent these problems endure when both tensor fields are dynamical. Furthermore, in the case of the vDVZ discontinuity it is thought that in the zero-mass limit the graviton can become strongly-coupled, such that the usual perturbative analysis can break down on scales smaller than the Vainshtein radius, so that general relativity is recovered [36, 37]. This 'Vainshtein mechanism' can be demonstrated numerically [38]. Also, in the theories we are considering the vDVZ discontinuity is not necessarily a problem as both massless and massive modes can be shown to exist when Minkowski space is perturbed. The mass parameters can therefore be assumed to be large, and there will still always exist long-ranged massless modes to carry the Newtonian gravitational force (as long as matter doesn't couple to the massive modes only). The relevant question in this case then becomes whether or not general relativity can be recovered in the limit $M \to \infty$, rather than $M \to 0$. This is primarily the question we will concern ourselves with here, although we will also pause in the final sections to consider the zero-mass limit of the theory.

3. The PPN Approach

Here we will briefly recap the essential elements of the PPN formalism, as required for the internal coherence of this article. The PPN formalism is a perturbative treatment, and requires a small parameter to expand in. An "order of smallness" is therefore defined by

$$U \sim v^2 \sim \frac{p}{\rho} \sim \Pi \sim O(2) \tag{8}$$

where U is the Newtonian potential, v is the velocity of a fluid element, p is the fluid pressure, ρ is its rest-mass density and Π is the ratio of energy to rest-mass densities. We also take time derivatives to add an extra order of smallness each:

$$\frac{\partial}{\partial t} \sim O(1). \tag{9}$$

The PPN formalism then proceeds using an expansion in this order of smallness. For our theories, we will now perturb our two fundamental tensors as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{10}$$

$$q_{\mu\nu} = \bar{\eta}_{\mu\nu} + \hbar_{\mu\nu}, \tag{11}$$

where $\eta_{\mu\nu}$ is the metric of Minkowski space, and where we take $\bar{\eta}_{\mu\nu} = X_0^2 \eta_{\mu\nu}$, where X_0 is a constant. This last equation expresses the fact that, in general, one set of coordinates cannot be used to correspond to the same proper separation between events in the geometries associated with each of the two metrics. To be fully general one could consider different constants in front of the different components of $\eta_{\mu\nu}$. We will not do this here.

Let us note that Minkowski space is not always a solution to the field equations (5) and (6), due to constant terms in the action. For $q_{\mu\nu} = X_0^2 g_{\mu\nu}$ these terms correspond to the 'cosmological constants' terms

$$\rho_\Lambda = \frac{1}{8\pi G \ell^2} \left(\alpha_0 + X_0^2 \kappa_0 + 8\kappa_1 + 2\kappa_2 \right) \tag{12}$$

$$\rho_\lambda = \frac{1}{8\pi G \ell^2} \left(\alpha + \frac{\kappa_0}{X_0^2} \right), \tag{13}$$

for g and q, respectively. These can, in general, be non-zero even in the absence of α and α_0, resulting in a de Sitter background§. For vanishingly small ρ_Λ and ρ_λ, however, we can approximate this as Minkowski space. This corresponds to the conditions

$$\alpha_0 \to -X_0^2 \kappa_0 - 8\kappa_1 - 2\kappa_2 \tag{14}$$

$$\alpha \to -\frac{\kappa_0}{X_0^2}, \tag{15}$$

which we will use to eliminate α_0 and α in the perturbed equations that follow.

Now consider the energy-momentum tensor, $T^{\mu\nu}$. For a perfect fluid this is given by $T^{\mu\nu} = [\rho(1+\Pi) + p]\, u^\mu u^\nu + p\hat{g}^{\mu\nu}$, where ρ is the density of rest mass, p is pressure, Π is internal energy per unit rest mass, and u^μ is the 4-velocity of matter. Neglecting $O(1)$ contributions from h_{0i} and $\hbar_{0i}$, the time-like normalisation $u^\mu u^\nu \hat{g}_{\mu\nu} = -1$ then gives

$$u^\mu = \frac{1}{\sqrt{m + nX_0^2}} \left(1 + \frac{1}{2}v^2 + \frac{(mh_{00} + n\hbar_{00})}{2(m + nX_0^2)} + O(4); v^i + O(3) \right), \tag{16}$$

§ An exception is the ISS theory, for which $\rho_\Lambda = \alpha_0 / 8\pi G \ell^2$.

to the relevant order in perturbations the components of the energy-momentum tensor are then given by

$$T^{00} = \frac{\rho}{(m+nX_0^2)}\left[1+\Pi+v^2+\frac{(mh_{00}+n\hbar_{00})}{(m+nX_0^2)}\right]+O(6) \tag{17}$$

$$T^{0i} = \frac{\rho v^i}{(m+nX_0^2)}+O(5) \tag{18}$$

$$T^{ij} = \frac{(\rho v^i v^j + p\delta^{ij})}{(m+nX_0^2)}+O(6). \tag{19}$$

The two expanded metrics (10) and (11) can now be substituted into the field equations (5) and (6), along with (17), (18) and (19). The field equations can then be solved for order by order in smallness of perturbations, and gauge transformations of the form

$$x^\mu \to x^\mu + \xi^\mu \tag{20}$$

can be used to transform it into the "standard post-Newtonian gauge", where the spatial part of the metric is diagonal, and where terms containing time derivatives are removed wherever possible. The metrics that result then allow us to determine the post-Newtonian parameters of these theories by comparing to the test metric

$$\begin{aligned}\hat{g}_{00} = &-1+2GU-2\beta G^2U^2-2\xi G^2\Phi_W+(2\gamma+2+\alpha_3+\zeta_1-2\xi)G\Phi_1\\ &+2(1+3\gamma-2\beta+\zeta_2+\xi)G^2\Phi_2+2(1+\zeta_3)G\Phi_3\\ &+2(3\gamma+3\zeta_4-2\xi)G\Phi_4-(\zeta_1-2\xi)G\mathcal{A}\end{aligned} \tag{21}$$

$$\hat{g}_{0i} = -\frac{1}{2}(3+4\gamma+\alpha_1-\alpha_2+\zeta_1-2\xi)GV_i-\frac{1}{2}(1+\alpha_2-\zeta_1+2\xi)GW_i \tag{22}$$

$$\hat{g}_{ij} = (1+2\gamma GU)\delta_{ij}, \tag{23}$$

where β, γ, ξ, ζ_1, ζ_2, ζ_3, ζ_4, α_1, α_2 and α_3 are the post-Newtonian parameters, U is the Newtonian gravitational potential that solves Poisson's equation, and Φ_W, Φ_1, Φ_2, Φ_3, Φ_4, $\mathcal{A}$, V_i and W_i are the post-Newtonian gravitational potentials given in [6]. The particular combination of parameters before each of the potentials in (21), (22) and (23) are given so that the parameters have particular physical significance, once gravitational phenomena have been computed. We have chosen to construct the test metric here in terms of the combined metric $\hat{g}_{\mu\nu}$, as this is the metric that couples to the matter fields, and hence is the one being constrained by observations of those fields.

In general relativity the PPN parameters in (21)-(23) are given by $\beta=\gamma=1$, with all other parameters equaling zero. The interpretation that is often given to these parameters is that γ is 'the spatial curvature per unit rest mass', β is 'the degree of nonlinearity in the law of gravity', the α_i are due to 'preferred-frame effects', and ξ is due to 'preferred-location' effects. The parameters ζ_i, as well as α_3, are sometimes associated with the violation of conservation of momentum. These parameters are all constrained by observations, and for the present study the two constraints below are of particular interest [39]:

$$\gamma - 1 = (2.1 \pm 2.3)\times 10^{-5} \tag{24}$$

$$|\alpha_2| \lesssim 1.2\times 10^{-7}. \tag{25}$$

The first of these comes from observations of the Shapiro time delay of radio signals from the Cassini space-craft as it passed behind the Sun [39], and the second is derived from the observation that the Sun's spin axis is closely aligned with the angular momentum vector of the solar system [41]. For further constraints the reader is referred to [6].

Finally, the equations of motion show that for time-like particles propagating along geodesics the Newtonian limit is given by $\hat{g}_{00}$ (and hence g_{00} and q_{00}) to $O(2)$ and $\hat{g}_{0i}$ to $O(1)$, with no other knowledge of the metric components beyond the background level being necessary‖. The post-Newtonian limit for time-like trajectories requires a knowledge of

$$\hat{g}_{00} \quad \text{to} \quad O(4) \tag{26}$$

$$\hat{g}_{0i} \quad \text{to} \quad O(3) \tag{27}$$

$$\hat{g}_{ij} \quad \text{to} \quad O(2), \tag{28}$$

where Latin letters are used to denote spatial indices. To obtain the Newtonian limit of trajectories followed by null particles we only need to know the metric to background order. The post-Newtonian limit of null trajectories requires $\hat{g}_{00}$ and $\hat{g}_{ij}$ both to $O(2)$, as well as $\hat{g}_{0i}$ to $O(1)$.

In order to proceed in calculating the form of the weak field metric $\hat{g}_{\mu\nu}$, to the orders of smallness specified above, it is useful to have expressions for various perturbed quantities involving the metric tensors, such as their determinants and inverses. These are given in Appendix A.

4. Newtonian Perturbations to $O(1)$ and $O(2)$

4.1. The g_{0i} and q_{0i} terms, to $O(1)$

Unlike the case of gravitational theories with a single metric, for the theories being considered here we cannot automatically set both $g_{0i} = 0$ and $q_{0i} = 0$ to $O(1)$ through gauge transformations. We must therefore calculate these terms explicitly.

Substituting the expressions derived above into the $0 - i$ components of the field equations (5) and (6), and eliminating α_0 and α using (14) and (15), gives

$$l^2 \left[-\frac{1}{2}\nabla^2 h_{0i} + \frac{1}{2} h_{0j,ji} \right] = \left[\kappa_0 + \frac{8\kappa_1}{X_0^2} + \frac{4\kappa_2}{X_0^2} \right] \left[\hbar_{0i} - X_0^2 h_{0i} \right] \tag{29}$$

$$l^2 \left[-\frac{1}{2}\nabla^2 \hbar_{0i} + \frac{1}{2} \hbar_{0j,ji} \right] = \left[\kappa_0 + \frac{8\kappa_1}{X_0^2} + \frac{4\kappa_2}{X_0^2} \right] \left[X_0^2 h_{0i} - \hbar_{0i} \right]. \tag{30}$$

The left-hand side of these equations vanishes when acted upon with $\nabla\cdot$, due to the Bianchi identities. We must therefore have $\hbar_{0i,i} = X_0^2 h_{0i,i}$. The equations (29) and (30) can then be decoupled by defining two new variables:

‖ The $O(1)$ terms in $\hat{g}_{0i}$ are usually omitted from the beginning in theories with a single metric, as they can be removed by a suitable gauge choice. In what follows we will find that the situation is somewhat more complicated in multigravity.

$$h^{(m)}_{0i} \equiv h_{0i} - \frac{\hbar_{0i}}{X_0^2} \tag{31}$$
$$h^{(0)}_{0i} \equiv h_{0i} + \hbar_{0i}, \tag{32}$$

to get

$$\nabla^2 h^{(m)}_{0i} = M^2 h^{(m)}_{0i} \tag{33}$$
$$\nabla^2 h^{(0)}_{0i} = h^{(0)}_{0j,ji}, \tag{34}$$

where we have defined

$$M^2 \equiv \frac{2(1+X_0^2)}{X_0^2 l^2}[X_0^2\kappa_0 + 8\kappa_1 + 4\kappa_2]. \tag{35}$$

We can now set $h^{(0)}_{0i} = 0$ by a gauge transformation; the $h^{(m)}_{0j}$ mode, however, is gauge invariant, and so cannot be made to vanish in this way. We now note that $h^{(m)}_{0i,i} = 0$, so $h^{(m)}_{0i}$ is the curl of some vector potential A_i, with the solution

$$h^{(m)}_{0i} = (\nabla \times A)_i = (c_1)_i \frac{e^{-M|x-x'|}}{|x-x'|} + (c_2)_i \frac{e^{M|x-x'|}}{|x-x'|}. \tag{36}$$

If $M \in \mathcal{R}$ then $h^{(m)}_{0i}$ is a hyperbolic function, and if $M \in \mathcal{I}$ then it is oscillatory. This solution does not depend on the matter content of the space-time, and exists in a vacuum where only $g_{\mu\nu}$ and $q_{\mu\nu}$ are present. It also has a preferred point in space, x', and $h^{(m)}_{0i}$, being the curl of a vector, looks like a rotation field. Finally, we impose $c_2 = 0$ as a boundary condition, in order to maintain asymptotic flatness at infinity. This is one of the PPN assumptions, up to cosmological terms.

4.2. The g_{00} and q_{00} terms, to $O(2)$

We will now proceed to calculate g_{00} and q_{00} to $O(2)$. To do this we will need both the 0-0 field equations to $O(2)$, and the trace of the i-j equations to $O(2)$. As before, in order to decouple the equations let us now define massive and massless combinations

$$h^{(m)}_{\mu\nu} \equiv h_{\mu\nu} - \frac{\hbar_{\mu\nu}}{X_0^2} \tag{37}$$
$$h^{(0)}_{\mu\nu} \equiv h_{\mu\nu} + \hbar_{\mu\nu}. \tag{38}$$

With the gauge choice $h^{(0)}_{0i} = 0$, and by using $h^{(m)}_{0i,i} = 0$, we can then write the relevant massless combination of (5) and (6) as

$$\begin{aligned}&\nabla^2 h^{(0)}_{00} + 8\pi G(m + nX_0^2)^2\rho \\ &= \frac{X_0^2}{(1+X_0^2)}\left(h^{(m)2}_{0i,j} - h^{(m)}_{0i,j}h^{(m)}_{0j,i}\right) + \frac{M^2 X_0^2}{2(1+X_0^2)}h^{(m)2}_{0i},\end{aligned} \tag{39}$$

while the massive combination is given by

$$
\begin{aligned}
&\nabla^2 h_{00}^{(m)} - M^2 h_{00}^{(m)} + 8\pi G(m+nX_0^2)(m-n)\rho \\
&= \frac{(1-X_0^2)}{(1+X_0^2)}\left(h_{0i,j}^{(m)}h_{0j,i}^{(m)} - h_{0i,j}^{(m)2}\right) + \frac{2(2\kappa_1+\kappa_2)(1+X_0^2)}{l^2 X_0^2}A \\
&\quad + \frac{(X_0^4\kappa_0 - 8(1-X_0^2)\kappa_1 - 2(1-3X_0^2)\kappa_2)}{l^2X_0^2}h_{0i}^{(m)2}. \qquad (40)
\end{aligned}
$$

Here we have defined

$$A \equiv h_{ii}^{(m)} - h_{00}^{(m)}, \qquad (41)$$

which is given by the solution of

$$
\begin{aligned}
&\left(X_0^2\kappa_0 + 12\kappa_1 + 6\kappa_2\right)\nabla^2 A - \kappa_0 M^2 X_0^2 A \\
&= -\frac{8\pi G X_0^2 M^2 l^2 (m+nX_0^2)(m-n)}{(1+X_0^2)}\rho + \frac{4M^2(8\kappa_1 + (3-X_0^2)\kappa_2)}{(1+X_0^2)}h_{0i}^{(m)2} \\
&\quad + \frac{(X_0^2(1-X_0^2)\kappa_0 + 8(3-X_0^2)\kappa_1 + 8(1-X_0^2)\kappa_2)}{(1+X_0^2)}h_{0i,j}^{(m)2} \\
&\quad + \frac{(X_0^2(1-X_0^2)\kappa_0 + 8(1-X_0^2)\kappa_1 + 8\kappa_2)}{(1+X_0^2)}h_{0i,j}^{(m)}h_{0j,i}^{(m)}. \qquad (42)
\end{aligned}
$$

To find the equations above we have made use of the expression $\nabla^2\left(h_{0i}^{(m)2}\right) = 2h_{0i,j}^{(m)2} + 2M^2 h_{0i}^{(m)2}$, as well as the Bianchi identities to $O(2)$:

$$
\begin{aligned}
\frac{M^2 l^2 X_0^2}{2(1+X_0^2)}h_{ij,ij}^{(m)} &= \frac{[X_0^2(1-X_0^2)\kappa_0 + 8(1-X_0^2)\kappa_1 + 2(3-X_0^2)\kappa_2]}{(1+X_0^2)}h_{0j,i}^{(m)}h_{0i,j}^{(m)} \\
&\quad - \frac{[X_0^2(1-X_0^2)\kappa_0 - 8X_0^2\kappa_1 + 2(1-X_0^2)\kappa_2]}{2(1+X_0^2)}\nabla^2\left(h_{0i}^{(m)2}\right) \\
&\quad + \frac{1}{2}(X_0^2\kappa_0 + 4\kappa_1 + 2\kappa_2)\nabla^2\left(h_{ii}^{(m)} - h_{00}^{(m)}\right). \qquad (43)
\end{aligned}
$$

It is interesting to note that in the ISS case the factor multiplying the differential operator in (42) vanishes. The equation above then becomes an algebraic relation between A, ρ and $h_{0i}^{(m)}$ (and its derivatives). It can also be seen from (42) that in the case $\kappa_0 = 0$ the Green's functions for A are those of Laplace's equation, rather than those of Helmholtz's equation. In this case A becomes long-ranged. In what follows we will therefore consider the ISS and $\kappa_0 = 0$ cases separately from the more general case, as they clearly have different behaviour.

We can now integrate Eq. (39) using the Green's function of Laplace's equation. The massless mode $h_{00}^{(0)}$ is then given by

$$
\begin{aligned}
h_{00}^{(0)} &= 2G(m+nX_0^2)^2 U - \frac{M^2X_0^2}{8\pi(1+X_0^2)}\mathcal{V}\left(h_{0i}^{(m)2}\right) \\
&\quad - \frac{X_0^2}{4\pi(1+X_0^2)}\left[\mathcal{V}\left(h_{0i,j}^{(m)2}\right) - \mathcal{V}\left(h_{0i,j}^{(m)}h_{0j,i}^{(m)}\right)\right] \qquad (44)
\end{aligned}
$$

where the Newtonian potential U is

$$U = \int \frac{\rho(x')}{|x-x'|}d^3x' \qquad (45)$$

and we have defined the additional functional

$$\mathcal{V}(\phi) \equiv \int \frac{\phi(x')}{|x-x'|} d^3x'. \tag{46}$$

If M is large, we are then left with the long-ranged part of this metric component being given by

$$h_{00}^{(0)} \doteq 2G(m+nX_0^2)^2 U. \tag{47}$$

Here, and throughout, the symbol $\doteq$ will be taken to mean 'equal up to exponentially suppressed terms'. Such suppression is guaranteed as long as M^{-1} is much smaller than the length scale over which observations are being made, and can be arranged by choice of the parameters κ_0, κ_1, κ_2, l and X_0.

Let us now consider the massive modes. These are different, depending on the theory in question.

ISS theory

For the ISS theory we can write the solution to (40) and (42) as

$$\begin{aligned} h_{00}^{(m)} = & \frac{8G(m+n\sigma)(m-n)}{3} \mathcal{W}_M(\rho) + \frac{(11-9\sigma)}{24\pi(1+\sigma)} \mathcal{W}_M\left(h_{0i,j}^{(m)2}\right) \\ & - \frac{(1-3\sigma)}{24\pi(1+\sigma)} \mathcal{W}_M\left(h_{0i,j}^{(m)} h_{0j,i}^{(m)}\right) + \frac{(1-4\sigma)}{6\pi l^2} \mathcal{W}_M\left(h_{0i}^{(m)2}\right), \end{aligned} \tag{48}$$

where we have defined the new functional

$$\mathcal{W}_c(\phi) \equiv \int \frac{\phi(x')}{|x-x'|} e^{-c|x-x'|} d^3x', \tag{49}$$

and suppressed the exponentially increasing mode. For sufficiently large M, the long-ranged part of this massive mode is zero:

$$h_{00}^{(m)} \doteq 0. \tag{50}$$

$\kappa_0 = 0$ theory

When $\kappa_0 = 0$ the solution to equations (40) and (42) is

$$\begin{aligned} h_{00}^{(m)} = & \frac{8G}{3}(m+nX_0^2)(m-n)\mathcal{W}_M(\rho) - \frac{A}{4} \\ & + \frac{(1-X_0^2)}{4\pi(1+X_0^2)} \left[\mathcal{W}_M\left(h_{0i,j}^{(m)2}\right) - \mathcal{W}_M\left(h_{0i,j}^{(m)} h_{0j,i}^{(m)}\right)\right] \\ & + \frac{(4(1-X_0^2)\kappa_1 + (1-3X_0^2)\kappa_2)}{2\pi l^2 X_0^2} \mathcal{W}_M\left(h_{0i}^{(m)2}\right), \end{aligned} \tag{51}$$

where

$$A = \frac{8G(m+nX_0^2)(m-n)}{3}U - \frac{((3-X_0^2)\kappa_1 + (1-X_0^2)\kappa_2)}{3\pi(1+X_0^2)(2\kappa_1+\kappa_2)}\mathcal{V}\left(h_{0i,j}^{(m)2}\right)$$
$$- \frac{((1-X_0^2)\kappa_1 + \kappa_2)}{3\pi(1+X_0^2)(2\kappa_1+\kappa_2)}\mathcal{V}\left(h_{0i,j}^{(m)}h_{0j,i}^{(m)}\right)$$
$$- \frac{4(8\kappa_1 + (3-X_0^2)\kappa_2)}{3\pi l^2 X_0^2}\mathcal{V}\left(h_{0i}^{(m)2}\right), \tag{52}$$

and where we have used $3M^2\mathcal{W}_M(A) = 12\pi A - 32\pi G(m+nX_0^2)(m-n)\mathcal{W}_M(\rho)$. From the above it can be seen that even for large M the massive modes are not short-ranged:

$$h_{00}^{(m)} \doteq -\frac{2G}{3}(m+nX_0^2)(m-n)U. \tag{53}$$

Other theories

For all other theories the solution to equations (40) and (42) is

$$h_{00}^{(m)} = \frac{8G}{3}(m+nX_0^2)(m-n)\mathcal{W}_M(\rho) - \frac{\kappa_0(1+X_0^2)}{3l^2N^2}A$$
$$+ \frac{(1-X_0^2)}{4\pi(1+X_0^2)}\left[\mathcal{W}_M\left(h_{0i,j}^{(m)2}\right) - \mathcal{W}_M\left(h_{0i,j}^{(m)}h_{0j,i}^{(m)}\right)\right]$$
$$- \frac{(X_0^4\kappa_0 - 8(1-X_0^2)\kappa_1 - 2(1-3X_0^2)\kappa_2)}{4\pi l^2 X_0^2}\mathcal{W}_M\left(h_{0i}^{(m)2}\right), \tag{54}$$

where

$$A = \frac{2GX_0^2M^2l^2(m+nX_0^2)(m-n)}{(X_0^2\kappa_0 + 12\kappa_1 + 6\kappa_2)(1+X_0^2)}\mathcal{W}_N(\rho)$$
$$- \frac{(X_0^2(1-X_0^2)\kappa_0 + 8(3-X_0^2)\kappa_1 + 8(1-X_0^2)\kappa_2)}{4\pi(X_0^2\kappa_0 + 12\kappa_1 + 6\kappa_2)(1+X_0^2)}\mathcal{W}_N\left(h_{0i,j}^{(m)2}\right)$$
$$- \frac{(X_0^2(1-X_0^2)\kappa_0 + 8(1-X_0^2)\kappa_1 + 8\kappa_2)}{4\pi(1+X_0^2)(X_0^2\kappa_0 + 12\kappa_1 + 6\kappa_2)}\mathcal{W}_N\left(h_{0i,j}^{(m)}h_{0j,i}^{(m)}\right)$$
$$- \frac{M^2(8\kappa_1 + (3-X_0^2)\kappa_2)}{\pi(1+X_0^2)(X_0^2\kappa_0 + 12\kappa_1 + 6\kappa_2)}\mathcal{W}_N\left(h_{0i}^{(m)2}\right), \tag{55}$$

and where we have defined

$$N^2 \equiv \frac{2\kappa_0(1+X_0^2)(X_0^2\kappa_0 + 8\kappa_1 + 4\kappa_2)}{l^2(X_0^2\kappa_0 + 12\kappa_1 + 6\kappa_2)}, \tag{56}$$

and used

$$\frac{2\kappa_0(1+X_0^2)}{l^2N^2}A = 4G(m+nX_0^2)(m-n)\mathcal{W}_M(\rho) + \frac{3(2\kappa_1+\kappa_2)(1+X_0^2)}{\pi l^2 X_0^2}\mathcal{W}_M(A). \tag{57}$$

Now, if both M and N are large enough then there are no long-ranged terms in the massive mode:

$$h_{00}^{(m)} \doteq 0. \tag{58}$$

5. Post-Newtonian Perturbations to $O(2)$ and $O(3)$

5.1. The g_{ij} and q_{ij} terms, to $O(2)$

Let us now consider the g_{ij} and q_{ij} modes to $O(2)$, which we call 'post-Newtonian tensor modes', as they are tensors under transformations of the spatial coordinate system. From now on we will not consider further the contribution of the $h^{(m)}_{0i} \sim O(1)$ terms to the metric perturbations. If they do not have an effect at the Newtonian level, then it seems unlikely they will be required at higher orders.

Now, if we apply the gauge condition

$$h^{(0)\mu}{}_{i,\mu} - \frac{1}{2}h^{(0)\mu}{}_{\mu,i} = 0 \tag{59}$$

then the massless i-j equation, to $O(2)$, becomes

$$\nabla^2 h^{(0)}_{ij} = -8\pi G(m + nX_0^2)^2\delta_{ij}\rho, \tag{60}$$

which has the solution

$$h^{(0)}_{ij} = 2G(m + nX_0^2)^2\delta_{ij}U. \tag{61}$$

The massive modes, however, are once again dependent on the theory that is being considered. Now, the i-j equations, to $O(2)$, are

$$\begin{aligned} &\nabla^2 h^{(m)}_{ij} - M^2 h^{(m)}_{ij} + 8\pi G(m + nX_0^2)(m - n)\delta_{ij}\rho \\ &= -\frac{2(2\kappa_1 + \kappa_2)}{(X_0^2\kappa_0 + 8\kappa_1 + 4\kappa_2)}A_{,ij} - \frac{2(2\kappa_1 + \kappa_2)(1 + X_0^2)}{l^2X_0^2}\delta_{ij}A, \end{aligned} \tag{62}$$

where we have used the Bianchi identities to $O(2)$, Equation (43), and where A is again given by the solution of Equation (42).

ISS theory

In the ISS theory the solution to (42) and (62) is given by

$$h^{(m)}_{ij} = -\frac{4G}{3M^2}(m+n\sigma)(m-n)\mathcal{W}_M(\rho_{,ij}) + \frac{4G}{3}(m+n\sigma)(m-n)\delta_{ij}\mathcal{W}_M(\rho). \tag{63}$$

The term involving $\mathcal{W}_M(\rho_{,ij})$ is not in keeping with the usual PPN philosophy of avoiding terms with derivatives of ρ involved. Normally, one could perform a gauge transformation to remove such terms. Here, the massive modes are gauge invariant, but in subsequent sections we will find it possible to remove these terms from the combination of metrics that couples to the matter fields. For large M we have

$$h^{(m)}_{ij} \doteq 0. \tag{64}$$

$\kappa_0 = 0$ theory

When $\kappa_0 = 0$ the solution to (42) and (62) is

$$h^{(m)}_{ij} = \frac{2G}{3}(m+nX_0^2)(m-n)\delta_{ij}U + \frac{4G}{3}(m+nX_0^2)(m-n)\delta_{ij}\mathcal{W}_M(\rho) + \frac{G}{3\pi}(m+nX_0^2)(m-n)\mathcal{W}_M(U_{,ij}). \tag{65}$$

There is again an unsightly term containing derivatives of its integrand, which we will gauge transform away later on when considering the combined metric $\hat{g}_{ij}$. The long-range component of $h^{(m)}_{ij}$ is now given by

$$h^{(m)}_{ij} \doteq \frac{2G}{3}(m+nX_0^2)(m-n)\delta_{ij}U. \tag{66}$$

Other theories

For all other theories the solutions to (42) and (62) are given by

$$h^{(m)}_{ij} = \frac{4G}{3}(m+nX_0^2)(m-n)\delta_{ij}\mathcal{W}_M(\rho) + \frac{2G}{3}(m+nX_0^2)(m-n)\delta_{ij}\mathcal{W}_N(\rho) + \frac{(2\kappa_1+\kappa_2)}{2\pi(X_0^2\kappa_0+8\kappa_1+4\kappa_2)}\mathcal{W}_M(A_{,ij}), \tag{67}$$

where A is given by (55). Once again it can be seen that in the general case there are two different massive modes, with masses M and N. Once again, there is also a term containing derivatives. If both M and N are large enough then all of the terms in this massive mode are suppressed:

$$h^{(m)}_{ij} \doteq 0. \tag{68}$$

5.2. The g_{0i} and q_{0i} terms, to $O(3)$

Now consider the g_{0i} and q_{0i} modes to $O(3)$, which we call 'post-Newtonian vector modes', due to their properties under spatial coordinate transformations. If we again apply the gauge condition (59), together with the new condition

$$h^{(0)\mu}{}_{0,\mu} - \frac{1}{2}h^{(0)\mu}{}_{\mu,0} = -\frac{1}{2}h^{(0)}_{00,0}, \tag{69}$$

then we can write the massless combination of the field equations (5) and (6) as

$$\nabla^2 h^{(0)}_{0i} = \frac{G}{2}(m+nX_0^2)^2\nabla^2(V_i - W_i) + 16\pi G(m+nX_0^2)^2\rho v_i, \tag{70}$$

where we have used $U_{,0i} = \frac{1}{2}\nabla^2(W_i - V_i)$, as well as the previously found solution for $h^{(0)}_{00}$. It can then be seen that equation (70) has the solution

$$h^{(0)}_{0i} = -\frac{7G}{2}(m+nX_0^2)^2 V_i - \frac{G}{2}(m+nX_0^2)^2 W_i, \tag{71}$$

where V_i and W_i are the usual vector post-Newtonian potentials, as defined in [6].

If we now use the $O(2)$ Bianchi identities, (43), as well as the Bianchi identities to $O(3)$,

$$2(X_0^2\kappa_0+8\kappa_1+4\kappa_2)h^{(m)}_{0i,i} = (X_0^2\kappa_0+4\kappa_1+2\kappa_2)h^{(m)}_{ii,0}+(X_0^2\kappa_0+12\kappa_1+6\kappa_2)h^{(m)}_{00,0}, \quad (72)$$

then we can write the relevant combination of field equations from (5) and (6) as

$$\nabla^2 h^{(m)}_{0i} - M^2 h^{(m)}_{0i} = -\frac{2(2\kappa_1+\kappa_2)}{(X_0^2\kappa_0+8\kappa_1+4\kappa_2)}A_{,0i}+16\pi G(m+nX_0^2)(m-n)\rho v_i. \quad (73)$$

As above, the form of the massive modes in $\hat{g}_{0i}$ depend on the theory being considered.

ISS theory

The solution to Equation (73) in the ISS theory is given by

$$h^{(m)}_{0i} = -\frac{Gl^2(m+n\sigma)(m-n)}{3(1+\sigma)}\mathcal{W}_M(\rho_{,0i}) - 4G(m+n\sigma)(m-n)\mathcal{W}_M(\rho v_i). \quad (74)$$

As with the $O(2)$ components, there is still only one mass, M. If M is large enough then

$$h^{(m)}_{0i} \doteq 0. \quad (75)$$

$\kappa_0 = 0$ theory

When $\kappa_0 = 0$ the solution to Equation (73) is given by

$$\begin{aligned} h^{(m)}_{0i} =& \frac{Gl^2X_0^2(m+nX_0^2)(m-n)}{6(1+X_0^2)(2\kappa_1+\kappa_2)}U_{,0i} - 4G(m+nX_0^2)(m-n)\mathcal{W}_M(\rho v_i) \\ & - \frac{Gl^2X_0^2(m+nX_0^2)(m-n)}{6(1+X_0^2)(2\kappa_1+\kappa_2)}\mathcal{W}_M(\rho_{,0i}). \end{aligned} \quad (76)$$

Possible long-ranged forces are now given by

$$h^{(m)}_{0i} \doteq \frac{Gl^2X_0^2(m+nX_0^2)(m-n)}{6(1+X_0^2)(2\kappa_1+\kappa_2)}U_{,0i}. \quad (77)$$

Other theories

For all other theories we have that the solution to (73) is given by

$$\begin{aligned} h^{(m)}_{0i} =& \frac{2\kappa_0(1+X_0^2)}{3l^2N^2M^2}A_{,0i} - \frac{4G(m+nX_0^2)(m-n)}{3M^2}\mathcal{W}_M(\rho_{,0i}) \\ & - 4G(m+nX_0^2)(m-n)\mathcal{W}_M(\rho v_i). \end{aligned} \quad (78)$$

To $O(3)$ we have no long-ranged terms in $h^{(m)}_{0i}$, if M and N are both sufficiently large:

$$h^{(m)}_{0i} \doteq 0. \quad (79)$$

6. Post-Newtonian Perturbations to $O(4)$

We will now consider the g_{00} and q_{00} component of the metric to $O(4)$.

6.1. The $g^{(0)}_{00}$ and $q^{(0)}_{00}$ terms, to $O(4)$

Using known solutions and identities we can write the massless combination of the field equations (5) and (6) as

$$\begin{aligned}\nabla^2 h^{(0)}_{00} =& -8\pi G(m+nX_0^2)^2\rho\left[1+\Pi+2v^2+3\frac{p}{\rho}\right]+\frac{8\pi G X_0^2(m+nX_0^2)(m-n)\rho h^{(m)}_{00}}{(1+X_0^2)}\\ &-\frac{h^{(0)2}_{00,i}}{(1+X_0^2)}-\frac{X_0^2(h^{(m)2}_{00,i}-h^{(m)}_{00,ij}h^{(m)}_{ij})}{(1+X_0^2)}-\frac{2(2\kappa_1+\kappa_2)}{M^2l^2}h^{(m)}_{00,i}A_{,i}\\ &-\frac{X_0^2\kappa_0}{2l^2}h^{(m)2}_{00}-\frac{1}{4l^2}(X_0^2\kappa_0+12\kappa_1+6\kappa_2)A^2+\frac{M^2X_0^2}{4(1+X_0^2)}h^{(m)2}_{ij}. \end{aligned}\tag{80}$$

This equation can be integrated to give

$$\begin{aligned}h^{(0)}_{00} =& 2(m+nX_0^2)^2G\left[U-\frac{(m+nX_0^2)^2}{(1+X_0^2)}GU^2+2\Phi_1+2\frac{(m+nX_0^2)^2}{(1+X_0^2)}G\Phi_2+\Phi_3+3\Phi_4\right]\\ &-\frac{2GX_0^2(m+nX_0^2)(m-n)}{(1+X_0^2)}\mathcal{V}(\rho h^{(m)}_{00})+\frac{X_0^2\kappa_0}{8\pi l^2}\mathcal{V}(h^{(m)2}_{00})\\ &+\frac{X_0^2\left(\mathcal{V}(h^{(m)2}_{00,i})-\mathcal{V}(h^{(m)}_{00,ij}h^{(m)}_{ij})\right)}{4\pi(1+X_0^2)}+\frac{(2\kappa_1+\kappa_2)}{2\pi M^2l^2}\mathcal{V}(h^{(m)}_{00,i}A_{,i})\\ &+\frac{(X_0^2\kappa_0+12\kappa_1+6\kappa_2)}{16\pi l^2}\mathcal{V}(A^2)-\frac{M^2X_0^2}{16\pi(1+X_0^2)}\mathcal{V}(h^{(m)2}_{ij}), \end{aligned}\tag{81}$$

where the $h^{(m)}_{00}$, $h^{(m)}_{ij}$ and A to $O(2)$ are taken to be given by the expressions found in previous sections. The long-ranged modes in $h^{(0)}_{00}$ are then given, for our various different theories by the following:

ISS theory

$$\begin{aligned}h^{(0)}_{00}\doteq& 2(m+n\sigma)^2G\left[U-\frac{(m+n\sigma)^2}{(1+\sigma)}GU^2+2\Phi_1+2\frac{(m+n\sigma)^2}{(1+\sigma)}G\Phi_2+\Phi_3+3\Phi_4\right]\\ &+\frac{4\pi\sigma G^2l^2(m+n\sigma)^2(m-n)^2}{3(1+\sigma)^2}\mathcal{V}(\rho^2). \end{aligned}\tag{82}$$

$\kappa_0 = 0$ theory

$$
\begin{aligned}
h^{(0)}_{00} \doteq & 2(m+nX_0^2)^2 G\left[U - \frac{(m+nX_0^2)^2}{(1+X_0^2)}GU^2 + 2\Phi_1 + 2\frac{(m+nX_0^2)^2}{(1+X_0^2)}G\Phi_2 + \Phi_3 + 3\Phi_4\right] \\
& + \frac{8X_0^2G^2(m+nX_0^2)^2(m-n)^2}{3(1+X_0^2)}\Phi_2 + \frac{M^2X_0^2G^2(m+nX_0^2)^2(m-n)^2}{4\pi(1+X_0^2)}\mathcal{V}(U^2) \\
& - \frac{2X_0^2G^2(m+nX_0^2)^2(m-n)^2}{9(1+X_0^2)M^2}|\nabla U|^2 - \frac{4X_0^2G^2(m+nX_0^2)^2(m-n)^2}{3M^2(1+X_0^2)}\mathcal{V}(\rho_{,i}U_{,i}) \\
& + \frac{4X_0^2G^2(m+nX_0^2)^2(m-n)^2}{3M^2(1+X_0^2)}\mathcal{V}(\rho_{,i}\mathcal{W}_M(\rho)_{,i}).
\end{aligned}
\tag{83}
$$

Other theories

$$
h^{(0)}_{00} \doteq 2(m+nX_0^2)^2 G\Big[U - \frac{(m+nX_0^2)^2}{(1+X_0^2)}GU^2 + 2\Phi_1 + 2\frac{(m+nX_0^2)^2}{(1+X_0^2)}G\Phi_2 + \Phi_3 + 3\Phi_4\Big]. \tag{84}
$$

In deriving these expressions we have used $|\nabla U|^2 = \frac{1}{2}\nabla^2 U^2 - \nabla^2\Phi_2$, as well as

$$
A_{,i}B_{,i} = \frac{1}{2}\nabla^2(AB) + 2\pi\sigma A + 2\pi\rho B - \frac{1}{2}(M^2+N^2)AB \tag{85}
$$

$$
\begin{aligned}
A_{,ij}B_{,ij} = & \frac{1}{2}\nabla^2(A_{,i}B_{,i}) + 2\pi\sigma_{,i}A_{,i} + 2\pi\rho_{,i}B_{,i} - \frac{1}{4}(M^2+N^2)\nabla^2(AB) \\
& + \frac{1}{4}(M^2+N^2)^2AB - \pi(M^2+N^2)(A\sigma + B\rho)
\end{aligned}
\tag{86}
$$

where $A = \mathcal{W}_M(\rho)$ and $B = \mathcal{W}_N(\sigma)$. Further manipulation has also been done by noting that, for example, $\mathcal{V}(\rho_{,i}\mathcal{W}_M(\rho)_{,i}) \doteq 4\pi\mathcal{V}(\rho^2)$. These manipulations are performed so that the potentials are written without derivatives acting on the matter fields, as is usual in the PPN formalism.

6.2. The $g^{(m)}_{00}$ and $q^{(m)}_{00}$ terms, to $O(4)$

The equations involved in finding $g^{(m)}_{00}$ and $q^{(m)}_{00}$ to $O(4)$ are very lengthy, and we therefore choose to present them in Appendix B. Here we only state the form of the equations, and give their long-ranged components when M and N are large. We find the equation for $h^{(m)}_{00}$ looks like

$$
\nabla^2 h^{(m)}_{00} - M^2 h^{(m)}_{00} = ... \tag{87}
$$

The Green's function for this equation is therefore that of Helmholtz's equation, with the same mass term, M, as at $O(2)$. The right-hand side of this equation has terms containing A to $O(4)$. This quantity is given in Appendix B by the solution to an equation of the form

$$
(X_0^2\kappa_0 + 12\kappa_1 + 6\kappa_2)\nabla^2 A - M^2X_0^2\kappa_0 A = ... \tag{88}
$$

The Green's function for this equation can also be seen to be the same as the corresponding $O(2)$ case. The long-ranged components of the massive modes of h_{00} to $O(4)$ are given below.

ISS theory

$$h_{00}^{(m)} \doteq \frac{\pi^2 l^4 G^2}{18(1+\sigma)^4}(m+n\sigma)^2(m-n)^2\rho^2. \tag{89}$$

$\kappa_0 = 0$ theory

In this case the massive modes $h_{00}^{(m)}$ and $h_{ij}^{(m)}$ to $O(2)$ both have long-ranged components, and so the full expressions do not simplify greatly when the exponentially suppressed modes are neglected. The reader is therefore referred to equations (135) and (139) in Appendix B, where $h_{00}^{(m)}$ to $O(4)$ is given explicitly.

Other theories

$$h_{00}^{(m)} \doteq 0. \tag{90}$$

To find these expressions we have made use of (85) and (86), as well as the following relations:

$$\mathcal{W}_M(\mathcal{W}_N(X)) = \frac{4\pi}{(M^2 - N^2)}\left(\mathcal{W}_N(X) - \mathcal{W}_M(X)\right) \tag{91}$$

$$\mathcal{W}_M(\mathcal{V}(X)) \doteq \frac{4\pi}{M^2}\mathcal{V}(X). \tag{92}$$

7. Perturbations Coupled to Matter Fields

We have so far calculated the form of the massless and massive combinations of the perturbations to $g_{\mu\nu}$ and $q_{\mu\nu}$. We now want to know what these tell us about the perturbations to the metric $\hat{g}_{\mu\nu}$. Using previous relations we can write $\hat{g}_{\mu\nu}$ as

$$\hat{g}_{\mu\nu} = \hat{\eta}_{\mu\nu} + \hat{h}_{\mu\nu} \tag{93}$$

where

$$\hat{h}_{\mu\nu} = \frac{(m+nX_0^2)}{(1+X_0^2)} h_{\mu\nu}^{(0)} + \frac{X_0^2(m-n)}{(1+X_0^2)} h_{\mu\nu}^{(m)} \tag{94}$$

and $\hat{\eta}_{\mu\nu} = \eta_{\mu\nu} + \bar{\eta}_{\mu\nu}$. It is now straightforward to write down the components of $\hat{h}_{\mu\nu}$. To proceed in doing this it is useful to define two new constants:

$$G_N \equiv \frac{G(m+nX_0^2)^3}{(1+X_0^2)} \tag{95}$$

$$\mathcal{G}_N \equiv \frac{G(m+nX_0^2)}{3(1+X_0^2)}\left[3(m+nX_0^2)^2 - X_0^2(m-n)^2\right]. \tag{96}$$

The reason for doing this is that although the constant G appears in the gravitational action (2) in the same way as it does in general relativity, it is *not* necessarily the value of Newton's constant that is determined by Cavendish type experiments. For the $\kappa_0 = 0$ theories this value of Newton's constant is given by $\mathcal{G}_N$ (when M is large), while for the

ISS theory and all other theories it is given by G_N (when M is large for the ISS theory, and when both M and N are large for all other theories).

It is also useful at this point to start using infinitesimal gauge transformations to put our results in the 'standard post-Newtonian gauge'. In the preceding section we used the gauge specified by conditions (59) and (69). This has been a convenient choice, and has allowed integration of the field equations to post-Newtonian accuracy. It has, however, resulted in the metrics in question having off-diagonal components in the spatial part of the metric, as well as non-zero $O(1)$ terms in the 0-i components of the metric. These can be removed by making an infinitesimal coordinate transformation of the form $x^\mu \to x^\mu + \xi^\mu$. The metric that couples to matter, $\hat{g}_{\mu\nu}$, is then transformed in such a way that¶

$$\hat{h}_{\mu\nu} \to \hat{h}_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu} + O(\xi^2), \tag{97}$$

where covariant derivatives are with respect to $\hat{g}_{\mu\nu}$, and where indices have been lowered with $\hat{g}_{\mu\nu}$. By making coordinate transformations such that $\xi_0 \sim O(1)$ or $O(3)$ and $\xi_i \sim O(2)$ the metric perturbations then transform as

$$\hat{h}_{ij} \to \hat{h}_{ij} - 2\xi_{(i,j)} \tag{98}$$

$$\hat{h}_{0i} \to \hat{h}_{0i} - \xi_{0,i} - \xi_{i,0}. \tag{99}$$

Transformations of this kind will be used in what follows to remove all contributions from the $O(1)$ massive modes in $\hat{h}_{0i}$, as well as to diagonalise $\hat{h}_{ij}$, and to remove the potentials found above that contain derivatives on the functions in their integrands.

As in the previous section, many of the expressions involved here are quite lengthy. We will therefore once again present the full equation in Appendix C, and quote here only the terms that are long-ranged when M and N are large (after the gauge transformations discussed above have been performed+). To $O(4)$ in $\hat{h}_{00}$, $O(3)$ in $\hat{h}_{0i}$ and $O(2)$ in $\hat{h}_{ij}$ we find the results below.

ISS theory

$$\begin{aligned}\hat{h}_{00} \doteq\, & 2G_N\left[U - \frac{G_N}{(m+n\sigma)}U^2 + 2\Phi_1 + \frac{2G_N}{(m+n\sigma)}\Phi_2 + \Phi_3 + 3\Phi_4\right] \\ & + \frac{4\pi\sigma G_N^2 l^2 (m-n)^2}{3(1+\sigma)(m+n\sigma)^3}\mathcal{V}(\rho^2) + \frac{\pi^2 l^4 G_N^2 \sigma (m-n)^3}{18(1+\sigma)^3(m+n\sigma)}\rho^2\end{aligned} \tag{100}$$

$$\hat{h}_{0i} \doteq -\frac{7G_N}{2}V_i - \frac{G_N}{2}W_i \tag{101}$$

$$\hat{h}_{ij} \doteq 2G_N \delta_{ij} U. \tag{102}$$

¶ This is true as long the matter does not couple to the massive combination of metrics only, i.e. as long as $m \neq -nX_0^2$.

+ These transformations are given explicitly in Appendix C.

$\kappa_0 = 0$ theory

$$\hat{h}_{00} \doteq 2\mathcal{G}_N U + O(4) \tag{103}$$

$$\hat{h}_{0i} \doteq -\frac{7\mathcal{G}_N}{2}\frac{3(m+nX_0^2)^2}{(3(m+nX_0^2)^2 - X_0^2(m-n)^2)}V_i - \frac{\mathcal{G}_N}{2}\frac{3(m+nX_0^2)^2}{(3(m+nX_0^2)^2 - X_0^2(m-n)^2)}W_i \tag{104}$$

$$\hat{h}_{ij} \doteq 2\mathcal{G}_N \frac{(3(m+nX_0^2)^2 + X_0^2(m-n)^2)}{(3(m+nX_0^2)^2 - X_0^2(m-n)^2)}\delta_{ij}U. \tag{105}$$

Other theories

$$\hat{h}_{00} \doteq 2G_N\left[U - \frac{G_N}{(m+nX_0^2)}U^2 + 2\Phi_1 + \frac{2G_N}{(m+nX_0^2)}\Phi_2 + \Phi_3 + 3\Phi_4\right] \tag{106}$$

$$\hat{h}_{0i} \doteq -\frac{7G_N}{2}V_i - \frac{G_N}{2}W_i \tag{107}$$

$$\hat{h}_{ij} \doteq 2G_N\delta_{ij}U. \tag{108}$$

In the $\kappa_0 = 0$ theory we have not written the $O(4)$ part of $\hat{h}_{00}$ explicitly, because, as mentioned before, it contains a large number of long-ranged potentials. Appendices B and C contain enough information for the reader to calculate these terms explicitly, if they are required.

8. Post-Newtonian Parameters, and Observational Constraints

It is convenient at this point to choose units so that $m + nX_0^2 = 1$. The unperturbed metric $\hat{\eta}_{\mu\nu}$ then takes its usual form. Using the results of the previous section we can now obtain the PPN parameters of the theories we are considering, and determine the extent to which they can be observationally constrained.

ISS theory

It can be seen by comparing (100), (101) and (102) with (21), (22) and (23) that, when M is large, all PPN parameters take the same value as in General Relativity. We do, however, have two new potentials in the $\hat{h}_{00}$ term at $O(4)$. These are

$$+\frac{4\pi\sigma G_N^2 l^2(1-n(1+\sigma))^2}{3(1+\sigma)}\mathcal{V}(\rho^2) \qquad \text{and} \qquad +\frac{\pi^2 l^4 G_N^2\sigma(1-n(1+\sigma))^3}{18(1+\sigma)^3}\rho^2. \tag{109}$$

No such terms appear in the usual PPN prescription. The second of these is particularly unfamiliar as it involves ρ directly, and not ρ integrated over any volume. Such a term, however, can be expected to be small in comparison to $\mathcal{V}(\rho^2)$ so long as $l \ll d$, where d is the length scale over which the observational phenomena are being measured. Outside of any matter distribution, of course, this term will vanish entirely.

In the absence of any observational constraints on these new potentials, the only constraint that can currently be placed on this particular theory is therefore $Md \gg 1$, or equivalently

$$4(1+\sigma) \gg \frac{l^2}{d^2}. \tag{110}$$

No other constraints are available from standard PPN formalism.

$\kappa_0 = 0$ theory

In the $\kappa_0 = 0$ theory it can be seen by comparing (103), (104) and (105) with (21), (22) and (23) that, among other deviations from general relativity, we have

$$\gamma = \frac{(3 + X_0^2(1 - n(1 - X_0^2))^2)}{(3 - X_0^2(1 - n(1 - X_0^2))^2)}. \tag{111}$$

As in the ISS theory, the $\kappa_0 = 0$ theory must also obey $Md \gg 1$ in order for the massive modes to be exponentially suppressed. For this theory, this corresponds to

$$\frac{8(1 + X_0^2)}{X_0^2}(2\kappa_1 + \kappa_2) \gg \frac{l^2}{d^2}. \tag{112}$$

Unlike the ISS theory, however, we can also use the PPN constraint on γ from observations of the Cassini spacecraft to constrain the theory. To be compatible with observations γ must satisfy equation (24). We then have the constraint

$$X_0^2(1 - n(1 - X_0^2))^2 = (3.1 \pm 3.3) \times 10^{-5}. \tag{113}$$

While tight, it is worth noting that this constraint can always be evaded by an appropriate choice of X_0. Unless X_0 is known from other considerations (cosmology, for example) it is not possible to constrain the theory with this information only. It is also worth noting that this is a constraint on n only, and not the parameters determining the interactions in the theory (i.e. κ_1 and κ_2). Further constraints are also available on this theory from the $\hat{g}_{00}$ metric component at $O(4)$.

Other theories

Comparing (106), (107) and (108) with (21), (22) and (23) shows, as long as both $Md \gg 1$ and $Nd \gg 1$, that all of the PPN parameters in this general class of theories are the same as in general relativity. The only constraints that need to be satisfied by these theories are therefore

$$\frac{2(1 + X_0^2)}{X_0^2}(X_0^2\kappa_0 + 8\kappa_1 + 4\kappa_2) \gg \frac{l^2}{d^2} \tag{114}$$

and

$$2\kappa_0(1 + X_0^2)\frac{(X_0^2\kappa_0 + 8\kappa_1 + 4\kappa_2)}{(X_0^2\kappa_0 + 12\kappa_1 + 6\kappa_2)} \gg \frac{l^2}{d^2}. \tag{115}$$

If these conditions are met then all the post-Newtonian predictions of these theories are indistinguishable from those of general relativity. This shows the importance of the κ_0 term in the interaction Lagrangian.

9. Discussion

We have investigated here the weak field limit of a general class of bimetric theories, generated from the gravitational action (2), and coupled to matter via a linear combination of the two metrics involved. We have calculated the post-Newtonian limit of these theories, up to $O(4)$ in g_{00}, $O(3)$ in g_{0i} and $O(2)$ in g_{ij}, for a perfect fluid matter content distributed without any symmetries. We find that, to all orders considered, there is a natural decomposition of the perturbations to the two metrics into modes we have called 'massive' and 'massless'. These modes are defined through the combinations

$$h^{(m)}_{\mu\nu} = h_{\mu\nu} - \frac{\hbar_{\mu\nu}}{X_0^2} \qquad \text{and} \qquad h^{(0)}_{\mu\nu} = h_{\mu\nu} + \hbar_{\mu\nu}, \tag{116}$$

where $h_{\mu\nu}$ and $\hbar_{\mu\nu}$ denote the perturbations to the two metrics involved, and X_0 is a constant associated with the unperturbed background space-times.

We find that, in general, the $O(1)$ contributions to the massive modes $h^{(m)}_{0i}$ are non-zero, and cannot be made to vanish by a coordinate transformation. These quantities satisfy a Helmholtz equation with mass M (as given by Eq. (35)), and while the 0-i component of the combined metric that couples to matter can always be set to zero by a gauge transformation (as long as the matter does not couple exclusively to the massive modes), these $O(1)$ terms can in principle make a non-negligible contribution to the other components of the metric. We therefore include the effects of these terms while calculating the 0-0 component of the combined metric to $O(2)$, but neglect them to higher orders, as we see no reason to expect them to contribute substantially to post-Newtonian accuracy if they are negligible at the Newtonian level.

To Newtonian order, we find that in general the relevant metric perturbations have massless modes (that are always long-ranged), as well as massive modes with *two* different masses: M and N (where N is given by Eq. (56)). We find two special cases in which the second mass, N, does not appear. These are the ISS theory, as specified in Section 2, and theories with $\kappa_0 = 0$, in which the interaction term $\sqrt{-q}(q^{-1})^{\mu\nu}g_{\mu\nu}$ is missing from the action. In the ISS theory this occurs because the $\nabla h^{(m)}_{00} = ...$ equation can be decoupled from the $h^{(m)}_{ij}$ terms without having to solve any other differential equations. In all other cases this is not possible, and a second equation of the form $\nabla^2 A - N^2 A = ...$ must be solved, before one can integrate the $h^{(m)}_{00}$ equation. In the second anomalous case, when $\kappa_0 \to 0$ it is found that the mass parameter $N \to 0$. Theories of this kind therefore have extra massless modes, beyond the usual Newtonian term that occurs from integrating the $\nabla h^{(0)}_{00} = ...$ equation.

We then proceed to calculate the metric perturbations to post-Newtonian accuracy. These terms all involve the same two masses, M and N, that are required to Newtonian order (except for the ISS and $\kappa_0 = 0$ cases, which only require M). By expressing these perturbations in the standard post-Newtonian gauge we find that the predictions of these theories are, in general, very similar to those of general relativity. In particular, if M and N are both large enough then they can evade all preferred frame experiments, and are indistinguishable from general relativity by any experiments that constrain the

PPN parameters β and γ (including the stringent tests available from Shapiro time-delay observations [39], and lunar laser ranging measurements [40]). The ISS theory also does a remarkably good job of satisfying weak field tests of gravity when M is large, with PPN parameters that are again the same as in general relativity. In this case, however, we do find a couple of anomalous terms appearing at $O(4)$ in the 0-0 component of the metric that couples to matter. These are given in Eq. (109), and could potentially be used to observationally distinguish between general relativity and ISS theory. The $\kappa_0 = 0$ theory does not fare quite so well, due to its extra long-ranged modes. In this case observations of γ can be used to place constraints on a particular combination of the constant X_0, and the mixing angle between the two fundamental spin-2 fields that couples to matter, even when M is large.

As well as the large mass limit, one can also consider the small mass limit. In this case a viable Newtonian term can again appear in the 0-0 component of the metric, and we must then consider the problem of the vDVZ discontinuity that was discussed in Section 1. To do this we note that as $M \to 0$ and $N \to 0$ we get $\mathcal{W}_M(\rho) \to U^*$ and $\mathcal{W}_N(\rho) \to U^\dagger$, respectively (we have used the superscripts $*$ and $\dagger$ here, even though $U^* = U^\dagger = U$, so that we can keep track of where the potential has come from). Neglecting contributions from $h_{0i}^{(m)}$ at $O(1)$, the full expressions in Appendix C then give, for the most general theories,

$$\hat{h}_{00} = 2G_N U + \frac{8G_N X_0^2(1-n(1+X_0^2))^2 U^*}{3} - \frac{2G_N X_0^2(1-n(1+X_0^2))^2 U^\dagger}{3} \tag{117}$$

$$\hat{h}_{ij} = 2G_N U + \frac{4G_N X_0^2(1-n(1+X_0^2))^2 U^*}{3} + \frac{2G_N X_0^2(1-n(1+X_0^2))^2 U^\dagger}{3}. \tag{118}$$

The vDVZ discontinuity can be seen to be manifest in the U^* terms of these two equations. If these were the only terms in these expressions then we would indeed get the familiar $\gamma = 1/2$ that is anticipated from massive gravity. Here, however, we have two additional terms in both $\hat{h}_{00}$ and $\hat{h}_{ij}$ – one from the terms that are long-ranged even when M and N are large, and another from the $\mathcal{W}_N(\rho)$ terms. The cumulative effect of all three of these contributions added together shows that there is no vDVZ discontinuity in the γ parameter of the most general class of these theories, as in this case we end up with $\hat{h}_{ij} = \delta_{ij}\hat{h}_{00}$, and hence $\gamma = 1$. The same is true in the special case in which $\kappa_0 = 0$, as should be expected, as these modes already correspond to the $N \to 0$ limit of the more general theories. The ISS theory, however, does not have $\gamma = 1$ as $M \to 0$. Instead we find

$$\gamma \to \frac{3 + 2(1-n(1+\sigma))^2\sigma}{3 + 4(1-n(1+\sigma))^2\sigma}. \tag{119}$$

This is essentially because the $\mathcal{W}_N(\rho)$ terms that cancel the discontinuity from the $\mathcal{W}_M(\rho)$ terms are absent in the ISS theory.

It should be noted, however, that although the general relativistic value of γ is recovered in the most general theories as M and $N \to 0$, this does not mean that all PPN parameters will be. For example, for the parameters α_1 and α_2 that parameterise deviations from general relativity in the 0-i components of the field equations, we find that while $\alpha_1 \rightarrow 0$ (as in general relativity), we also have

$$\alpha_2 \rightarrow -\frac{(1-n(1+X_0^2))^2 X_0^2}{(2-n(1+X_0^2))^2 X_0^2}. \tag{120}$$

In terms of this expression, equation (25) then gives the constraint

$$X_0^2(1-n(1+X_0^2))^2 \lesssim 1.2 \times 10^{-7}. \tag{121}$$

This is the same combination of quantities that was constrained in the large M limit of the $\kappa_0 = 0$ theory using the parameter γ, but the constraint here is from an entirely different gravitational effect, and is even tighter. On should also note that in the limit $M \rightarrow 0$ the $O(1)$ terms in $h_{0i}^{(m)}$ are no longer short-ranged. These terms are, however, set by initial conditions, rather than by the matter content of the space-time. One may then be able to limit their magnitude by applying suitable observational constraints, but this will place constraints on the initial conditions only, and not on the gravitational theory itself.

Finally, we note that when the matter fields are coupled to the massless combination of metrics only (so that $m = n$) then many of the constraints found above are automatically satisfied. These observations can therefore be though of as constraining the amount of massive mode that is allowed to be mixed with the massless modes.

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Appendix A: Perturbed Metric Quantities

The perturbed form of the metric that we will be using is specified in (10) and (11). In order to solve (5) and (6), however, we also need to know the perturbed Ricci tensor to $O(3)$ in its off-diagonal components, and $O(4)$ in all of its diagonal components*, as well as the relevant expressions for the perturbed metric determinants, and inverse metrics. The perturbed Ricci tensor components take on their usual lengthy form, and we will not reproduce them explicitly here.

The perturbed metric determinants can be calculated using the equation $det(g) = e^{tr(ln(g))}$. To $O(2)$ in perturbations this gives

$$det(g) = -\left(1 - h_{00} + h_{ii} + h_{0i}^2\right) + O(4) \tag{122}$$

$$det(q) = -X_0^8\left(1 - \frac{\hbar_{00}}{X_0^2} + \frac{\hbar_{ii}}{X_0^2} + \frac{\hbar_{0i}^2}{X_0^4}\right) + O(4) \tag{123}$$

$$\begin{aligned} det(\hat{g}) = -\Big[&(m+nX_0^2)^2 + (h_{ii} - h_{00} + h_{0i}^2)m^2 \\ &+ (X_0^2 h_{ii} + \hbar_{ii} - X_0^2 h_{00} - \hbar_{00} + 2h_{0i}\hbar_{0i})mn \\ &+ (X_0^2\hbar_{ii} - X_0^2\hbar_{00} + \hbar_{0i}^2)n^2\Big](m+nX_0^2)^2 + O(4), \end{aligned} \tag{124}$$

and, if we ignoring $O(1)$ contributions to h_{0i} and $\hbar_{0i}$, then to higher order we have

$$det(g) = -\left(1 - h_{00} + h_{ii} - h_{00}h_{ii} + \frac{1}{2}h_{ii}^2 - \frac{1}{2}h_{ij}^2\right) + O(6) \tag{125}$$

$$det(q) = -X_0^8\left(1 - \frac{\hbar_{00}}{X_0^2} + \frac{\hbar_{ii}}{X_0^2} - \frac{\hbar_{00}\hbar_{ii}}{X_0^4} + \frac{\hbar_{ii}^2}{2X_0^4} - \frac{\hbar_{ij}^2}{2X_0^4}\right) + O(6). \tag{126}$$

In order to compute the interaction terms in the field equations (5) and (6) we also need to know the inverse metric fluctuations $(q^{-1})^{\mu\nu} = (\bar{\eta}^{-1})^{\mu\nu} + (\delta q^{-1})^{\mu\nu}$. These can be found from the definition $(q^{-1})^{\mu\alpha}q_{\alpha\nu} = \delta^\mu_\nu$, which gives

$$(\delta q^{-1})^{\mu\nu} = -(\bar{\eta}^{-1})^{\mu\alpha}(\bar{\eta}^{-1})^{\nu\beta}\hbar_{\alpha\beta} + (\bar{\eta}^{-1})^{\mu\alpha}(\bar{\eta}^{-1})^{\nu\delta}(\bar{\eta}^{-1})^{\beta\gamma}\hbar_{\alpha\beta}\hbar_{\gamma\delta} + O(h^3). \tag{127}$$

The components of $(\delta q^{-1})^{\mu\nu}$ are then given by

$$(\delta q^{-1})^{00} = -\frac{\hbar_{00}}{X_0^4} + \frac{\hbar_{0i}^2}{X_0^6} + O(4) \tag{128}$$

$$(\delta q^{-1})^{0i} = \frac{\hbar_{0i}}{X_0^4} + O(3) \tag{129}$$

$$(\delta q^{-1})^{ij} = -\frac{\hbar_{ij}}{X_0^4} - \frac{\hbar_{0i}\hbar_{0j}}{X_0^6} + O(4). \tag{130}$$

Ignoring $O(1)$ contributions to h_{0i} and $\hbar_{0i}$, the higher order components of $(\delta q^{-1})^{\mu\nu}$ are

$$(\delta q^{-1})^{00} = -\frac{\hbar_{00}}{X_0^4} - \frac{\hbar_{00}^2}{X_0^6} + O(6) \tag{131}$$

$$(\delta q^{-1})^{ij} = -\frac{\hbar_{ij}}{X_0^4} + \frac{\hbar_{ik}\hbar_{jk}}{X_0^6} + O(6). \tag{132}$$

* The i-j components are usually only required to $O(2)$, but here we also need higher order terms in order to calculate the $O(4)$ part of the 0-0 components.

Appendix B: The $g_{00}^{(m)}$ and $q_{00}^{(m)}$ Equations, to $O(4)$

Using this expression, together with other known identities and lower order solutions, the field equations (5) and (6) give

$$
\begin{aligned}
&\nabla^2 h_{00}^{(m)} - M^2 h_{00}^{(m)} \\
= &\frac{2(1+X_0^2)(2\kappa_1+\kappa_2)}{l^2X_0^2}A - \frac{4(1+X_0^2)(2\kappa_1+\kappa_2)}{M^2l^2X_0^2}A_{,00} + h_{00,00}^{(m)} \\
&-8\pi G(m+nX_0^2)(m-n)\rho\left[1+\Pi+2v^2+3\frac{p}{\rho}\right] - \frac{2(2\kappa_1+\kappa_2)}{M^2l^2X_0^2}h_{00,i}^{(0)}A_{,i} \\
&+\frac{8\pi G(m+nX_0^2)(mX_0^2+n)}{(1+X_0^2)}\rho h_{00}^{(m)} + 4\pi Gmn(1+X_0^2)\rho\left(h_{00}^{(m)}+h_{ii}^{(m)}\right) \\
&+\frac{1}{(1+X_0^2)}\left[h_{00,ij}^{(0)}h_{ij}^{(m)} - 2h_{00,i}^{(0)}h_{00,i}^{(m)}\right] + \frac{2(X_0^2\kappa_0+4\kappa_1+2\kappa_2)}{l^2X_0^2}h_{00}^{(0)}h_{00}^{(m)} \\
&+\frac{(1-X_0^2)}{2(1+X_0^2)}\left[h_{00,i}^{(m)2} - 2h_{00,ij}^{(m)}h_{ij}^{(m)} - 2h_{00,i}^{(m)}h_{ij,j}^{(m)} + h_{00,i}^{(m)}h_{jj,i}^{(m)}\right] + \frac{\kappa_1}{l^2X_0^2}h_{ii}^{(m)^2} \\
&-\frac{[X_0^2\kappa_0+8\kappa_1+6\kappa_2]}{4l^2}\left(3h_{00}^{(m)2}+2h_{00}^{(m)}h_{ii}^{(m)}+h_{ii}^{(m)2}-2h_{ij}^{(m)2}\right) \\
&-\frac{3(\kappa_1+\kappa_2)}{l^2X_0^2}h_{00}^{(m)2} + \frac{2(3\kappa_1+\kappa_2)}{l^2X_0^2}h_{00}^{(m)}h_{ii}^{(m)} + \frac{(4\kappa_1+3\kappa_2)}{l^2X_0^2}h_{ij}^{(m)2}.
\end{aligned}
\tag{133}
$$

The solutions to this equation are given below.

ISS theory

$$
\begin{aligned}
h_{00}^{(m)} = &\frac{(1+\sigma)}{2\pi l^2}\mathcal{W}_M(A) - \frac{\mathcal{W}_M(A_{,00})}{4\pi} - \frac{2G(m+n\sigma)(m\sigma+n)}{(1+\sigma)}\mathcal{W}_M(\rho h_{00}^{(m)}) \\
&-Gmn(1+\sigma)\left[\mathcal{W}_M(\rho h_{00}^{(m)}) + \mathcal{W}_M(\rho h_{ii}^{(m)})\right] - \frac{\mathcal{W}_M(h_{00,i}^{(0)}A_{,i})}{8\pi(1+\sigma)} \\
&+2G(m+n\sigma)(m-n)\left[\mathcal{W}_M(\rho)+\mathcal{W}_M(\rho\Pi)+2\mathcal{W}_M(\rho v^2)+3\mathcal{W}_M(p)\right] \\
&-\frac{(\mathcal{W}_M(h_{00,ij}^{(0)}h_{ij}^{(m)}) - 2\mathcal{W}_M(h_{00,i}^{(0)}h_{00,i}^{(m)}))}{4\pi(1+\sigma)} - \frac{\mathcal{W}_M(h_{00,00}^{(m)})}{4\pi} \\
&-\frac{\left[8\mathcal{W}_M(h_{00}^{(0)}h_{00}^{(m)}) - 4\mathcal{W}_M(h_{00}^{(m)}h_{ii}^{(m)}) - \mathcal{W}_M(h_{ii}^{(m)^2}) - \mathcal{W}_M(h_{ij}^{(m)2})\right]}{4\pi l^2} \\
&+\frac{\sigma}{4\pi l^2}\left[3\mathcal{W}_M(h_{00}^{(m)2}) + 2\mathcal{W}_M(h_{00}^{(m)}h_{ii}^{(m)}) + \mathcal{W}_M(h_{ii}^{(m)2}) - 2\mathcal{W}_M(h_{ij}^{(m)2})\right] \\
&-\frac{(1-\sigma)}{8\pi(1+\sigma)}\Big[\mathcal{W}_M(h_{00,i}^{(m)2}) - 2\mathcal{W}_M(h_{00,ij}^{(m)}h_{ij}^{(m)}) \\
&\qquad -2\mathcal{W}_M(h_{00,i}^{(m)}h_{ij,j}^{(m)}) + \mathcal{W}_M(h_{00,i}^{(m)}h_{jj,i}^{(m)})\Big].
\end{aligned}
\tag{134}
$$

$\kappa_0 = 0$ theory

$$
\begin{aligned}
h_{00}^{(m)} = &-\frac{(1+X_0^2)(2\kappa_1+\kappa_2)}{2\pi l^2 X_0^2}\mathcal{W}_M(A) + \frac{\mathcal{W}_M(A_{,00})}{8\pi} + \frac{\mathcal{W}_M(h_{00,i}^{(0)}A_{,i})}{16\pi(1+X_0^2)} \\
&+2G(m+nX_0^2)(m-n)\left[\mathcal{W}_M(\rho) + \mathcal{W}_M(\rho\Pi) + 2\mathcal{W}_M(\rho v^2) + 3\mathcal{W}_M(p)\right] \\
&-\frac{2(m+nX_0^2)(mX_0^2+n)}{(1+X_0^2)}\mathcal{W}_M(\rho h_{00}^{(m)}) - \frac{\mathcal{W}_M(h_{00,00}^{(m)})}{4\pi} \\
&-Gmn(1+X_0^2)\left[\mathcal{W}_M(\rho h_{00}^{(m)}) + \mathcal{W}_M(\rho h_{ii}^{(m)})\right] \\
&-\frac{(\mathcal{W}_M(h_{00,ij}^{(0)}h_{ij}^{(m)}) - 2\mathcal{W}_M(h_{00,i}^{(0)}h_{00,i}^{(m)}))}{4\pi(1+X_0^2)} - \frac{(2\kappa_1+\kappa_2)}{\pi l^2 X_0^2}\mathcal{W}_M(h_{00}^{(0)}h_{00}^{(m)}) \\
&+\frac{(4\kappa_1+3\kappa_2)}{8\pi l^2}\left[3\mathcal{W}_M(h_{00}^{(m)2}) + 2\mathcal{W}_M(h_{00}^{(m)}h_{ii}^{(m)}) + \mathcal{W}_M(h_{ii}^{(m)2}) - 2\mathcal{W}_M(h_{ij}^{(m)2})\right] \\
&+\frac{3(\kappa_1+\kappa_2)}{4\pi l^2 X_0^2}\mathcal{W}_M(h_{00}^{(m)2}) - \frac{(3\kappa_1+\kappa_2)}{2\pi l^2 X_0^2}\mathcal{W}_M(h_{00}^{(m)}h_{ii}^{(m)}) \\
&-\frac{\kappa_1}{4\pi l^2 X_0^2}\mathcal{W}_M(h_{ii}^{(m)^2}) - \frac{(4\kappa_1+3\kappa_2)}{4\pi l^2 X_0^2}\mathcal{W}_M(h_{ij}^{(m)2}) \\
&-\frac{(1-X_0^2)}{8\pi(1+X_0^2)}\Big[\mathcal{W}_M(h_{00,i}^{(m)2}) - 2\mathcal{W}_M(h_{00,ij}^{(m)}h_{ij}^{(m)}) \\
&\qquad -2\mathcal{W}_M(h_{00,i}^{(m)}h_{ij,j}^{(m)}) + \mathcal{W}_M(h_{00,i}^{(m)}h_{jj,i}^{(m)})\Big].
\end{aligned}
\tag{135}
$$

Other theories

$$
\begin{aligned}
h_{00}^{(m)} = &-\frac{(1+X_0^2)(2\kappa_1+\kappa_2)}{2\pi l^2 X_0^2}\mathcal{W}_M(A) + \frac{(1+X_0^2)(2\kappa_1+\kappa_2)}{\pi M^2 l^2 X_0^2}\mathcal{W}_M(A_{,00}) \\
&-\frac{\mathcal{W}_M(h_{00,00}^{(m)})}{4\pi} + \frac{(2\kappa_1+\kappa_2)}{2\pi M^2 l^2 X_0^2}\mathcal{W}_M(h_{00,i}^{(0)}A_{,i}) \\
&+2G(m+nX_0^2)(m-n)\left[\mathcal{W}_M(\rho) + \mathcal{W}_M(\rho\Pi) + 2\mathcal{W}_M(\rho v^2) + 3\mathcal{W}_M(p)\right] \\
&-\frac{2G(m+nX_0^2)(mX_0^2+n)}{(1+X_0^2)}\mathcal{W}_M(\rho h_{00}^{(m)}) - \frac{(4\kappa_1+3\kappa_2)}{4\pi l^2 X_0^2}\mathcal{W}_M(h_{ij}^{(m)2}) \\
&-Gmn(1+X_0^2)\left[\mathcal{W}_M(\rho h_{00}^{(m)}) + \mathcal{W}_M(\rho h_{ii}^{(m)})\right] - \frac{\kappa_1}{4\pi l^2 X_0^2}\mathcal{W}_M(h_{ii}^{(m)^2}) \\
&-\frac{(\mathcal{W}_M(h_{00,ij}^{(0)}h_{ij}^{(m)}) - 2\mathcal{W}_M(h_{00,i}^{(0)}h_{00,i}^{(m)}))}{4\pi(1+X_0^2)} - \frac{(3\kappa_1+\kappa_2)}{2\pi l^2 X_0^2}\mathcal{W}_M(h_{00}^{(m)}h_{ii}^{(m)}) \\
&-\frac{(X_0^2\kappa_0+4\kappa_1+2\kappa_2)}{2\pi l^2 X_0^2}\mathcal{W}_M(h_{00}^{(0)}h_{00}^{(m)}) + \frac{3(\kappa_1+\kappa_2)}{4\pi l^2 X_0^2}\mathcal{W}_M(h_{00}^{(m)2}) \\
&-\frac{(1-X_0^2)}{8\pi(1+X_0^2)}\Big[\mathcal{W}_M(h_{00,i}^{(m)2}) - 2\mathcal{W}_M(h_{00,ij}^{(m)}h_{ij}^{(m)}) \\
&\qquad -2\mathcal{W}_M(h_{00,i}^{(m)}h_{ij,j}^{(m)}) + \mathcal{W}_M(h_{00,i}^{(m)}h_{jj,i}^{(m)})\Big] \\
&+\frac{[X_0^2\kappa_0+8\kappa_1+6\kappa_2]}{16\pi l^2}\Big[3\mathcal{W}_M(h_{00}^{(m)2}) + 2\mathcal{W}_M(h_{00}^{(m)}h_{ii}^{(m)}) \\
&\qquad +\mathcal{W}_M(h_{ii}^{(m)2}) - 2\mathcal{W}_M(h_{ij}^{(m)2})\Big].
\end{aligned}
\tag{136}
$$

The expressions above should be understood to have $h^{(0)}_{00}$, $h^{(m)}_{00}$, $h^{(m)}_{ij}$ and A given to $O(2)$ by the lower order solutions found in previous sections. It now remains to find A to $O(4)$. To do this we use the field equations (5) and (6), as well as known identities and lower order solutions, to find

$$
\begin{aligned}
&(X_0^2\kappa_0 + 12\kappa_1 + 6\kappa_2)\nabla^2 A - M^2 X_0^2 \kappa_0 A \\
= &-\frac{3M^2 l^2 X_0^2}{2(1+X_0^2)} h^{(m)}_{00,00} - \frac{4\pi G M^2 l^2 X_0^2}{(1+X_0^2)}\left[3mn(1-X_0^2) + \frac{(m-n)^2 X_0^2}{(1+X_0^2)}\right]\rho h^{(m)}_{00} \\
&+6(2\kappa_1+\kappa_2)A_{,00} + \frac{4\pi G M^2 l^2 X_0^2}{3(1+X_0^2)}\left[2mn(1+X_0^2) - \frac{(m-n)^2 X_0^2}{(1+X_0^2)}\right]\rho h^{(m)}_{ii} \\
&-\frac{8\pi G M^2 l^2 X_0^2 (m+nX_0^2)(m-n)}{(1+X_0^2)}\rho\left[1 + \Pi - 3\frac{p}{\rho} + \frac{2h^{(0)}_{00}}{(1+X_0^2)}\right] \\
&+\frac{M^2(2\kappa_1+\kappa_2)}{(1+X_0^2)}\left[2h^{(0)}_{00}A + 8h^{(0)}_{00}h^{(m)}_{00} - 4h^{(m)2}_{00} + A^2 - 4h^{(m)2}_{ij}\right] \\
&+\frac{(X_0^2\kappa_0 + 12\kappa_2 + 6\kappa_2)}{(1+X_0^2)}\left[h^{(0)}_{00}\nabla^2 h^{(m)}_{ii} + h^{(m)}_{ii}\nabla^2 h^{(0)}_{00} + 2h^{(0)}_{00,i}h^{(m)}_{jj,i}\right] \\
&-\frac{(X_0^2\kappa_0 + 4\kappa_1 + 2\kappa_2)}{(1+X_0^2)}\left[h^{(0)}_{00}\nabla^2 h^{(m)}_{00} + h^{(m)}_{00}\nabla^2 h^{(0)}_{00}\right] \\
&-\frac{M^2 l^2 X_0^2}{2(1+X_0^2)^2}\left[2h^{(m)}_{ij}h^{(0)}_{00,ij} + 5h^{(0)}_{00,i}h^{(m)}_{ij,j}\right] + \frac{4(2\kappa_1+\kappa_2)}{(1+X_0^2)}h^{(0)}_{00,i}h^{(m)}_{00,i} \\
&+\frac{(X_0^2\kappa_0 + 2(1+X_0^2)\kappa_2)}{(1+X_0^2)}\left[h^{(m)}_{ij,k}h^{(m)}_{ij,k} + h^{(m)}_{ij}\nabla^2 h^{(m)}_{ij}\right] \\
&+\frac{(X_0^2\kappa_0 + 4(1-X_0^2)\kappa_1 + 4\kappa_2)}{(1+X_0^2)}\left[h^{(m)}_{ij,i}A_{,j} + h^{(m)}_{ij}A_{,ij}\right] \\
&-\frac{M^2 l^2 X_0^4}{(1+X_0^2)^2}\left[h^{(m)}_{ij,i}h^{(m)}_{00,j} + h^{(m)}_{ij}h^{(m)}_{00,ij}\right] + (X_0^2\kappa_0 + 4\kappa_1 + 4\kappa_2)\left[A_{,i}h^{(m)}_{ij,j} + Ah^{(m)}_{ij,ij}\right] \\
&-\frac{(X_0^2\kappa_0 + 2\kappa_2)}{2}\left[A_{,i}^2 + A\nabla^2 A - 2h^{(m)2}_{00,i} - 2h^{(m)}_{00}\nabla^2 h^{(m)}_{00}\right] \\
&-\frac{X_0^2(3X_0^2\kappa_0 + 16\kappa_1 + 8\kappa_2)}{(1+X_0^2)}\left[h^{(m)2}_{00,i} + h^{(m)}_{00}\nabla^2 h^{(m)}_{00}\right] \\
&-\frac{2(X_0^2\kappa_0 + 8\kappa_1 + 2(3+X_0^2)\kappa_2)}{(1+X_0^2)}\left[h^{(m)2}_{ij,j} + 2h^{(m)}_{ij}h^{(m)}_{jk,ki} + h^{(m)}_{ij,k}h^{(m)}_{jk,i}\right] \\
&-\frac{M^2 l^2 X_0^2(1-X_0^2)}{4(1+X_0^2)^2}\Big[3h^{(m)2}_{00,i} + 2h^{(m)}_{00}\nabla^2 h^{(m)}_{00} - A_{,i}^2 - 2h^{(m)}_{ij,k}h^{(m)}_{ik,j} + 2h^{(m)}_{jk}\nabla^2 h^{(m)}_{jk} \\
&\qquad +3h^{(m)2}_{ij,k} - 4h^{(m)2}_{ij,j} + 4h^{(m)}_{ij,j}A_{,i} + 2h^{(m)}_{ij}A_{,ij} - 4h^{(m)}_{jk}h^{(m)}_{ij,ik}\Big]. \qquad (137)
\end{aligned}
$$

The solutions to this equation are given below.

ISS theory

$$\begin{aligned}
\frac{24(1+\sigma)}{l^2}A &= 16\pi G\left[3mn(1-\sigma)+\frac{(m-n)^2\sigma}{(1+\sigma)}-\frac{2(m+n\sigma)^2}{(1+\sigma)}+\frac{4(m+n\sigma)(m-n)}{(1+\sigma)}\right]\rho h^{(m)}_{00}\\
&-\frac{16\pi G}{3}\left[2mn(1+\sigma)-\frac{(m-n)^2\sigma}{(1+\sigma)}-\frac{3(3+2\sigma)(m+n\sigma)(m-n)}{(1+\sigma)}\right]\rho h^{(m)}_{ii}\\
&+32\pi G(m+n\sigma)(m-n)\rho\left[1+\Pi-3\frac{p}{\rho}+\frac{h^{(0)}_{00}}{(1+\sigma)}-\frac{(3+2\sigma)h^{(m)}_{00}}{6(1+\sigma)}-\frac{A}{3}\right]\\
&+\frac{4}{l^2}\left[12h^{(0)}_{00}h^{(m)}_{00}-10h^{(m)2}_{00}-2(5+2\sigma)h^{(m)2}_{ij}\right]-A^2_{,i}-2A\nabla^2A+6A_{,00}+6h^{(m)}_{00,00}\\
&+\frac{2}{(1+\sigma)}\left[2h^{(0)}_{00,i}h^{(m)}_{00,i}+2h^{(m)}_{ij}h^{(0)}_{00,ij}+5h^{(0)}_{00,i}A_{,i}\right]-5h^{(m)2}_{ij,k}+2h^{(m)}_{ij}A_{,ij}\\
&+\frac{4\sigma}{(1+\sigma)}\left[A_{,i}h^{(m)}_{00,i}+h^{(m)}_{ij}h^{(m)}_{00,ij}\right]-\frac{(5+\sigma)}{(1+\sigma)}h^{(m)2}_{00,i}+6h^{(m)}_{ij,k}h^{(m)}_{jk,i}. \qquad (138)
\end{aligned}$$

$\kappa_0 = 0$ theory

$$\begin{aligned}
A &= \frac{4G}{3}\left[3mn(1-X_0^2)+\frac{(m-n)^2X_0^2}{(1+X_0^2)}\right]\mathcal{V}(\rho h^{(m)}_{00})\\
&-\frac{4G}{9}\left[2mn(1+X_0^2)-\frac{(m-n)^2X_0^2}{(1+X_0^2)}\right]\mathcal{V}(\rho h^{(m)}_{ii})\\
&+\frac{8G(m+nX_0^2)(m-n)}{3}\left[U+\Phi_3-3\Phi_4+\frac{2}{(1+X_0^2)}\mathcal{V}(\rho h^{(0)}_{00})\right]\\
&-\frac{(2\kappa_1+\kappa_2)}{3\pi X_0^2l^2}\left[2\mathcal{V}(h^{(0)}_{00}A)+8\mathcal{V}(h^{(0)}_{00}h^{(m)}_{00})-4\mathcal{V}(h^{(m)2}_{00})+\mathcal{V}(A^2)-4\mathcal{V}(h^{(m)2}_{ij})\right]\\
&-\frac{\left[\mathcal{V}(h^{(0)}_{00}\nabla^2h^{(m)}_{ii})+\mathcal{V}(h^{(m)}_{ii}\nabla^2h^{(0)}_{00})+2\mathcal{V}(h^{(0)}_{00,i}h^{(m)}_{jj,i})\right]}{4\pi(1+X_0^2)}+\frac{\mathcal{V}(h^{(m)}_{00,00})}{2\pi}\\
&-\frac{\mathcal{V}(h^{(0)}_{00,i}h^{(m)}_{00,i})}{6\pi(1+X_0^2)}+\frac{\left[8\mathcal{V}(h^{(m)}_{ij}h^{(0)}_{00,ij})+5\mathcal{V}(h^{(0)}_{00,i}A_{,i})\right]}{24\pi(1+X_0^2)}-\frac{\mathcal{V}(A_{,00})}{4\pi}\\
&+\frac{\left[\mathcal{V}(h^{(0)}_{00}\nabla^2h^{(m)}_{00})+\mathcal{V}(h^{(m)}_{00}\nabla^2h^{(0)}_{00})\right]}{12\pi(1+X_0^2)}-\frac{(4\kappa_1+3(1-X_0^2)\kappa_2)}{4\pi M^2l^2X_0^2}\mathcal{V}(A^2_{,i})\\
&+\frac{(3(1-X_0^2)\kappa_1+(1-2X_0^2)\kappa_2)}{6\pi(1+X_0^2)(2\kappa_1+\kappa_2)}\mathcal{V}(h^{(m)2}_{ij,k})+\frac{\mathcal{V}(h^{(m)}_{ij}A_{,ij})}{6\pi(1+X_0^2)}\\
&+\frac{(4(1-X_0^2)\kappa_1+(1-3X_0^2)\kappa_2)}{12\pi(1+X_0^2)(2\kappa_1+\kappa_2)}\mathcal{V}(h^{(m)}_{ij}\nabla^2h^{(m)}_{ij})-\frac{\kappa_1\mathcal{V}(A\nabla^2A)}{12\pi(2\kappa_1+\kappa_2)}\\
&+\frac{X_0^2}{12\pi(1+X_0^2)}\left[\mathcal{V}(A_{,i}h^{(m)}_{00,i})+4\mathcal{V}(h^{(m)}_{ij}h^{(m)}_{00,ij})\right]+\frac{(4\kappa_1+\kappa_2)}{12\pi(2\kappa_1+\kappa_2)}\mathcal{V}(h^{(m)}_{00}\nabla^2h^{(m)}_{00})\\
&+\frac{((3+X_0^2)\kappa_1+(1+3X_0^2)\kappa_2)}{6\pi(1+X_0^2)(2\kappa_1+\kappa_2)}\mathcal{V}(h^{(m)2}_{00,i})+\frac{(\kappa_1+\kappa_2)}{3\pi(2\kappa_1+\kappa_2)}\mathcal{V}(h^{(m)}_{ij,k}h^{(m)}_{jk,i}). \qquad (139)
\end{aligned}$$

Other theories

$$
\begin{aligned}
&(X_0^2\kappa_0+12\kappa_1+6\kappa_2)A \\
&= \frac{3M^2l^2X_0^2}{8\pi(1+X_0^2)}\mathcal{W}_N(h^{(m)}_{00,00}) + \frac{GM^2l^2X_0^2}{(1+X_0^2)}\left[3mn(1-X_0^2)+\frac{(m-n)^2X_0^2}{(1+X_0^2)}\right]\mathcal{W}_N(\rho h^{(m)}_{00}) \\
&\quad -\frac{3(2\kappa_1+\kappa_2)}{2\pi}\mathcal{W}_N(A_{,00}) - \frac{GM^2l^2X_0^2}{3(1+X_0^2)}\left[2mn(1+X_0^2)-\frac{(m-n)^2X_0^2}{(1+X_0^2)}\right]\mathcal{W}_N(\rho h^{(m)}_{ii}) \\
&\quad +\frac{2GM^2l^2X_0^2(m+nX_0^2)(m-n)}{(1+X_0^2)}\Big[\mathcal{W}_N(\rho)+\mathcal{W}_N(\rho\Pi)-3\mathcal{W}_N(p) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\frac{2}{(1+X_0^2)}\mathcal{W}_N(\rho h^{(0)}_{00})\Big] \\
&\quad -\frac{M^2(2\kappa_1+\kappa_2)}{4\pi(1+X_0^2)}\Big[2\mathcal{W}_N(h^{(0)}_{00}A)+8\mathcal{W}_N(h^{(0)}_{00}h^{(m)}_{00}) \\
&\qquad\qquad\qquad -4\mathcal{W}_N(h^{(m)2}_{00})+\mathcal{W}_N(A^2)-4\mathcal{W}_N(h^{(m)2}_{ij})\Big] \\
&\quad -\frac{(X_0^2\kappa_0+12\kappa_2+6\kappa_2)}{4\pi(1+X_0^2)}\left[\mathcal{W}_N(h^{(0)}_{00}\nabla^2h^{(m)}_{ii})+\mathcal{W}_N(h^{(m)}_{ii}\nabla^2h^{(0)}_{00})+2\mathcal{W}_N(h^{(0)}_{00,i}h^{(m)}_{jj,i})\right] \\
&\quad -\frac{(2\kappa_1+\kappa_2)}{\pi(1+X_0^2)}\mathcal{W}_N(h^{(0)}_{00,i}h^{(m)}_{00,i})+\frac{M^2l^2X_0^2}{8\pi(1+X_0^2)^2}\left[2\mathcal{W}_N(h^{(m)}_{ij}h^{(0)}_{00,ij})+5\mathcal{W}_N(h^{(0)}_{00,i}h^{(m)}_{ij,j})\right] \\
&\quad +\frac{(X_0^2\kappa_0+4\kappa_1+2\kappa_2)}{4\pi(1+X_0^2)}\left[\mathcal{W}_N(h^{(0)}_{00}\nabla^2h^{(m)}_{00})+\mathcal{W}_N(h^{(m)}_{00}\nabla^2h^{(0)}_{00})\right] \\
&\quad -\frac{(X_0^2\kappa_0+2(1+X_0^2)\kappa_2)}{4\pi(1+X_0^2)}\left[\mathcal{W}_N(h^{(m)}_{ij,k}h^{(m)}_{ij,k})+\mathcal{W}_N(h^{(m)}_{ij}\nabla^2h^{(m)}_{ij})\right] \\
&\quad -\frac{(X_0^2\kappa_0+4(1-X_0^2)\kappa_1+4\kappa_2)}{4\pi(1+X_0^2)}\left[\mathcal{W}_N(h^{(m)}_{ij,i}A_{,j})+\mathcal{W}_N(h^{(m)}_{ij}A_{,ij})\right] \\
&\quad +\frac{M^2l^2X_0^4}{4\pi(1+X_0^2)^2}\left[\mathcal{W}_N(h^{(m)}_{ij,i}h^{(m)}_{00,j})+\mathcal{W}_N(h^{(m)}_{ij}h^{(m)}_{00,ij})\right] \\
&\quad +\frac{(X_0^2\kappa_0+2\kappa_2)}{8\pi}\left[\mathcal{W}_N(A^2_{,i})+\mathcal{W}_N(A\nabla^2A)-2\mathcal{W}_N(h^{(m)2}_{00,i})-2\mathcal{W}_N(h^{(m)}_{00}\nabla^2h^{(m)}_{00})\right] \\
&\quad -\frac{(X_0^2\kappa_0+4\kappa_1+4\kappa_2)}{4\pi}\left[\mathcal{W}_N(A_{,i}h^{(m)}_{ij,j})+\mathcal{W}_N(Ah^{(m)}_{ij,ij})\right] \\
&\quad +\frac{X_0^2(3X_0^2\kappa_0+16\kappa_1+8\kappa_2)}{4\pi(1+X_0^2)}\left[\mathcal{W}_N(h^{(m)2}_{00,i})+\mathcal{W}_N(h^{(m)}_{00}\nabla^2h^{(m)}_{00})\right] \\
&\quad +\frac{(X_0^2\kappa_0+8\kappa_1+2(3+X_0^2)\kappa_2)}{2\pi(1+X_0^2)}\left[\mathcal{W}_N(h^{(m)2}_{ij,j})+2\mathcal{W}_N(h^{(m)}_{ij}h^{(m)}_{jk,ki})+\mathcal{W}_N(h^{(m)}_{ij,k}h^{(m)}_{jk,i})\right] \\
&\quad +\frac{M^2l^2X_0^2(1-X_0^2)}{16\pi(1+X_0^2)^2}\Big[3\mathcal{W}_N(h^{(m)2}_{00,i})+2\mathcal{W}_N(h^{(m)}_{00}\nabla^2h^{(m)}_{00})-\mathcal{W}_N(A^2_{,i})-2\mathcal{W}_N(h^{(m)}_{ij,k}h^{(m)}_{ik,j}) \\
&\qquad\qquad +2\mathcal{W}_N(h^{(m)}_{jk}\nabla^2h^{(m)}_{jk})+3\mathcal{W}_N(h^{(m)2}_{ij,k})-4\mathcal{W}_N(h^{(m)2}_{ij,j}) \\
&\qquad\qquad +4\mathcal{W}_N(h^{(m)}_{ij,j}A_{,i})+2\mathcal{W}_N(h^{(m)}_{ij}A_{,ij})-4\mathcal{W}_N(h^{(m)}_{jk}h^{(m)}_{ij,ik})\Big].
\end{aligned} \tag{140}
$$

In the above we have made use of the Bianchi identities to $O(4)$, which are given by:

$$
\begin{aligned}
&(X_0^2\kappa_0+8\kappa_1+4\kappa_2)h^{(m)}_{ij,ij}\\
&=\frac{1}{2}(X_0^2\kappa_0+4\kappa_1+2\kappa_2)\nabla^2 A+\frac{(X_0^2\kappa_0+4\kappa_1+4\kappa_2)}{2}\left[A_{,i}h^{(m)}_{ij,j}+Ah^{(m)}_{ij,ij}\right]\\
&\quad+\frac{(X_0^2\kappa_0+2(1+X_0^2)\kappa_2)}{2(1+X_0^2)}\left[h^{(m)}_{ij,k}h^{(m)}_{ij,k}+h^{(m)}_{ij}\nabla^2 h^{(m)}_{ij}\right]\\
&\quad-\frac{(X_0^2\kappa_0+2\kappa_2)}{4}\left[A_i h^{(m)}_{jj,i}+A\nabla^2 h^{(m)}_{ii}-h^{(m)}_{jj,i}h^{(m)}_{00,i}-h^{(m)}_{ii}\nabla^2 h^{(m)}_{00}\right]\\
&\quad-\frac{(X_0^2\kappa_0+4\kappa_1+2\kappa_2)}{2(1+X_0^2)}\left[h^{(0)}_{00,i}h^{(m)}_{jj,i}+h^{(0)}_{00}\nabla^2 h^{(m)}_{ii}+h^{(0)}_{00,i}h^{(m)}_{00,i}+h^{(0)}_{00}\nabla^2 h^{(m)}_{00}\right]\\
&\quad+\frac{(X_0^2\kappa_0+4(1-X_0^2)\kappa_1+4\kappa_2)}{2(1+X_0^2)}\left[h^{(m)}_{ij,i}h^{(m)}_{kk,j}+h^{(m)}_{ij}h^{(m)}_{kk,ij}\right]\\
&\quad-\frac{(X_0^2\kappa_0+6\kappa_1+3\kappa_2)}{(1+X_0^2)}\left[h^{(m)}_{00,i}h^{(0)}_{00,i}+h^{(m)}_{00}\nabla^2 h^{(0)}_{00}\right]\\
&\quad+\frac{(2\kappa_1+\kappa_2)}{(1+X_0^2)}\left[h^{(m)}_{jj,i}h^{(0)}_{00,i}+h^{(m)}_{ii}\nabla^2 h^{(0)}_{00}\right]-(X_0^2\kappa_0+8\kappa_1+4\kappa_2)h^{(m)}_{0i,0i}\\
&\quad+\frac{(X_0^2(1-5X_0^2)\kappa_0-32X_0^2\kappa_1+2(1-7X_0^2)\kappa_2)}{4(1+X_0^2)}\left[h^{(m)2}_{00,i}+h^{(m)}_{00}\nabla^2 h^{(m)}_{00}\right]\\
&\quad-\frac{(X_0^2\kappa_0+8\kappa_1+2(3+X_0^2)\kappa_2)}{(1+X_0^2)}\left[h^{(m)}_{ij,i}h^{(m)}_{jk,k}+2h^{(m)}_{ij}h^{(m)}_{jk,ki}+h^{(m)}_{ij,k}h^{(m)}_{jk,i}\right]\\
&\quad+\frac{(X_0^2\kappa_0+8\kappa_1+4\kappa_2)}{(1+X_0^2)}\left[h^{(0)}_{00}h^{(m)}_{ij,ij}-h^{(m)}_{ij}h^{(0)}_{00,ij}\right]\\
&\quad-\frac{((1+2X_0^2)(X_0^2\kappa_0+4\kappa_2)+4(1+3X_0^2)\kappa_1)}{2(1+X_0^2)}\left[h^{(m)}_{ij,i}h^{(m)}_{00,j}+h^{(m)}_{ij}h^{(m)}_{00,ij}\right].
\end{aligned}
\qquad (141)
$$

Appendix C: The $\hat{h}_{\mu\nu}$ Equations, and Gauge Transformations

As long as $m \neq -nX_0^2$ we can always use a coordinate transformation to remove all contributions from the $O(1)$ massive modes to $\hat{h}_{0i}$, so that $\hat{h}_{0i} \to 0$. We will therefore consider here only the higher order contributions to $\hat{h}_{\mu\nu}$.

Newtonian perturbations to $O(2)$

To $O(2)$ in $\hat{h}_{00}$ we find the results below.

ISS theory

$$\begin{aligned}\hat{h}_{00} = 2G_N U &- \frac{\sigma(m+n\sigma)}{4\pi(1+\sigma)^2}\left[\mathcal{V}\left(h^{(m)2}_{0i,j}\right) - \mathcal{V}\left(h^{(m)}_{0i,j}h^{(m)}_{0j,i}\right)\right] \\ &+ \frac{(11-9\sigma)(m-n)\sigma}{24\pi(1+\sigma)^2}\mathcal{W}_M\left(h^{(m)2}_{0i,j}\right) + \frac{8G_N(m-n)^2\sigma}{3(m+n\sigma)^2}\mathcal{W}_M\left(\rho\right) \\ &- \frac{(1-3\sigma)(m-n)\sigma}{24\pi(1+\sigma)^2}\mathcal{W}_M\left(h^{(m)}_{0i,j}h^{(m)}_{0j,i}\right) - \frac{\sigma(m+n\sigma)}{2\pi l^2(1+\sigma)}\mathcal{V}\left(h^{(m)2}_{0i}\right) \\ &+ \frac{(1-4\sigma)(m-n)\sigma}{6\pi l^2(1+\sigma)}\mathcal{W}_M\left(h^{(m)2}_{0i}\right).\end{aligned} \tag{142}$$

$\kappa_0 = 0$ theory

$$\begin{aligned}\hat{h}_{00} = 2\mathcal{G}_N U &+ \frac{8(m-n)^2X_0^2\mathcal{G}_N}{[3(m+nX_0^2)^2 - X_0^2(m-n)^2]}\mathcal{W}_M\left(\rho\right) \\ &- \frac{X_0^2[(m-n)((X_0^2-3)\kappa_1 + (X_0^2-1)\kappa_2) + 3(m+nX_0^2)(2\kappa_1+\kappa_2)]}{12\pi(1+X_0^2)^2(2\kappa_1+\kappa_2)}\mathcal{V}\left(h^{(m)2}_{0i,j}\right) \\ &+ \frac{X_0^2[(m-n)((1-X_0^2)\kappa_1 + \kappa_2) + 3(m+nX_0^2)(2\kappa_1+\kappa_2)]}{12\pi(1+X_0^2)^2(2\kappa_1+\kappa_2)}\mathcal{V}\left(h^{(m)}_{0i,j}h^{(m)}_{0j,i}\right) \\ &+ \frac{[(m-n)(8\kappa_1 + (3-X_0^2)\kappa_2) - 3(m+nX_0^2)(2\kappa_1+\kappa_2)]}{3\pi l^2(1+X_0^2)}\mathcal{V}\left(h^{(m)2}_{0i}\right) \\ &+ \frac{(1-X_0^2)(m-n)X_0^2}{4\pi(1+X_0^2)^2}\left[\mathcal{W}_M\left(h^{(m)2}_{0i,j}\right) - \mathcal{W}_M\left(h^{(m)}_{0i,j}h^{(m)}_{0j,i}\right)\right] \\ &+ \frac{(4(1-X_0^2)\kappa_1 + (1-3X_0^2)\kappa_2)(m-n)}{2\pi l^2(1+X_0^2)}\mathcal{W}_M\left(h^{(m)2}_{0i}\right).\end{aligned} \tag{143}$$

Other theories

$$\begin{aligned}\hat{h}_{00} = 2G_N U &- \frac{X_0^2(m+nX_0^2)}{4\pi(1+X_0^2)^2}\left[\mathcal{V}\left(h^{(m)2}_{0i,j}\right) - \mathcal{V}\left(h^{(m)}_{0i,j}h^{(m)}_{0j,i}\right)\right] \\ &- \frac{M^2X_0^2(m+nX_0^2)}{8\pi(1+X_0^2)^2}\mathcal{V}\left(h^{(m)2}_{0i}\right) + \frac{(m-n)(8\kappa_1 + (3-X_0^2)\kappa_2)}{3\pi l^2(1+X_0^2)}\mathcal{W}_N\left(h^{(m)2}_{0i}\right) \\ &+ \frac{8G_NX_0^2(m-n)^2}{3(m+nX_0^2)^2}\mathcal{W}_M\left(\rho\right) - \frac{2G_NX_0^2(m-n)^2}{3(m+nX_0^2)^2}\mathcal{W}_N\left(\rho\right) \\ &+ \frac{(1-X_0^2)(m-n)X_0^2}{4\pi(1+X_0^2)^2}\left[\mathcal{W}_M\left(h^{(m)2}_{0i,j}\right) - \mathcal{W}_M\left(h^{(m)}_{0i,j}h^{(m)}_{0j,i}\right)\right]\end{aligned}$$

$$-\frac{(m-n)(X_0^4\kappa_0-8(1-X_0^2)\kappa_1-2(1-3X_0^2)\kappa_2)}{4\pi l^2(1+X_0^2)}\mathcal{W}_M\left(h_{0i}^{(m)2}\right)$$
$$+\frac{(m-n)(X_0^2(1-X_0^2)\kappa_0+8(3-X_0^2)\kappa_1+8(1-X_0^2)\kappa_2)}{12\pi M^2l^2(1+X_0^2)}\mathcal{W}_N\left(h_{0i,j}^{(m)2}\right)$$
$$+\frac{(m-n)(X_0^2(1-X_0^2)\kappa_0+8(1-X_0^2)\kappa_1+8\kappa_2)}{12\pi M^2l^2(1+X_0^2)}\mathcal{W}_N\left(h_{0i,j}^{(m)}h_{0j,i}^{(m)}\right). \quad (144)$$

Post-Newtonian perturbations to $O(2)$ and $O(3)$

To $O(2)$ in $\hat{h}_{ij}$ and $O(3)$ in $\hat{h}_{0i}$ we find the results below.

ISS theory

$$\hat{h}_{ij}=2G_N\delta_{ij}U-\frac{l^2G_N\sigma(m-n)^2}{3(1+\sigma)(m+n\sigma)^2}\mathcal{W}_M(\rho_{,ij})+\frac{4G_N\sigma(m-n)^2}{3(m+n\sigma)^2}\delta_{ij}\mathcal{W}_M(\rho) \quad (145)$$
$$\to 2G_N\delta_{ij}U+\frac{4G_N\sigma(m-n)^2}{3(m+n\sigma)^2}\delta_{ij}\mathcal{W}_M(\rho) \quad (146)$$

and

$$\hat{h}_{0i}=-\frac{7G_N}{2}V_i-\frac{G_N}{2}W_i-\frac{4G_N(m-n)^2\sigma}{(m+n\sigma)^2}\mathcal{W}_M(\rho v_i)$$
$$-\frac{G_Nl^2\sigma(m-n)^2}{3(1+\sigma)(m+n\sigma)^2}\mathcal{W}_M(\rho_{,0i}) \quad (147)$$
$$\to -\frac{7G_N}{2}V_i-\frac{G_N}{2}W_i-\frac{4G_N(m-n)^2\sigma}{(m+n\sigma)^2}\mathcal{W}_M(\rho v_i). \quad (148)$$

The arrows in the expressions indicate the infinitesimal coordinate transformations

$$\xi_i=-\frac{G_Nl^2\sigma(m-n)^2}{6(1+\sigma)(m+n\sigma)^2}\mathcal{W}_M(\rho)_{,i} \quad (149)$$
$$\xi_0=-\frac{G_Nl^2\sigma(m-n)^2}{6(1+\sigma)(m+n\sigma)^2}\mathcal{W}_M(\rho)_{,0}. \quad (150)$$

$\kappa_0=0$ theory

$$\hat{h}_{ij}=2\mathcal{G}_N\frac{(3(m+nX_0^2)^2+X_0^2(m-n)^2)}{(3(m+nX_0^2)^2-X_0^2(m-n)^2)}\delta_{ij}U$$
$$+\frac{4\mathcal{G}_NX_0^2(m-n)^2}{(3(m+nX_0^2)^2-X_0^2(m-n)^2)}\delta_{ij}\mathcal{W}_M(\rho)$$
$$+\frac{\mathcal{G}_NX_0^2(m-n)^2}{\pi\,(3(m+nX_0^2)^2-X_0^2(m-n)^2)}\mathcal{W}_M(U_{,ij}) \quad (151)$$
$$\to 2\mathcal{G}_N\frac{(3(m+nX_0^2)^2+X_0^2(m-n)^2)}{(3(m+nX_0^2)^2-X_0^2(m-n)^2)}\delta_{ij}U$$
$$+\frac{4\mathcal{G}_NX_0^2(m-n)^2\delta_{ij}}{(3(m+nX_0^2)^2-X_0^2(m-n)^2)}\mathcal{W}_M(\rho) \quad (152)$$

and

$$\hat{h}_{0i} = -\frac{4GX_0^2(m+nX_0^2)(m-n)^2}{(1+X_0^2)}\mathcal{W}_M(\rho v_i) - \frac{7G(m+nX_0^2)^3}{2(1+X_0^2)}V_i$$
$$+\frac{Gl^2X_0^4(m+nX_0^2)(m-n)^2}{6(1+X_0^2)^2(2\kappa_1+\kappa_2)}U_{,0i} - \frac{G(m+nX_0^2)^3}{2(1+X_0^2)}W_i$$
$$-\frac{Gl^2X_0^4(m+nX_0^2)(m-n)^2}{6(1+X_0^2)^2(2\kappa_1+\kappa_2)}\mathcal{W}_M(\rho_{,0i}) \tag{153}$$
$$\to -\frac{7G(m+nX_0^2)^3}{2(1+X_0^2)}V_i - \frac{G(m+nX_0^2)^3}{2(1+X_0^2)}W_i$$
$$-\frac{4GX_0^2(m+nX_0^2)(m-n)^2}{(1+X_0^2)}\mathcal{W}_M(\rho v_i), \tag{154}$$

where the arrows indicate the coordinate transformations

$$\xi_i = \frac{\mathcal{G}_N l^2 X_0^4(m-n)^2}{4(1+X_0^2)^2(2\kappa_1+\kappa_2)(3(m+nX_0^2)^2 - X_0^2(m-n)^2)}\mathcal{W}_M(\rho)_{,i} \tag{155}$$
$$\xi_0 = \frac{\mathcal{G}_N l^2 X_0^4(m-n)^2}{4(1+X_0^2)^2(2\kappa_1+\kappa_2)(3(m+nX_0^2)^2 - X_0^2(m-n)^2)}(2U_{,0} - 3\mathcal{W}_M(\rho)_{,0}). \tag{156}$$

Other theories

$$\hat{h}_{ij} = 2G_N\delta_{ij}U + \frac{4G_NX_0^2(m-n)^2}{3(m+nX_0^2)^2}\delta_{ij}\mathcal{W}_M(\rho) + \frac{2G_NX_0^2(m-n)^2}{3(m+nX_0^2)^2}\delta_{ij}\mathcal{W}_N(\rho)$$
$$+\frac{4G_NX_0^2(m-n)^2}{3M^2(m+nX_0^2)^2}(\mathcal{W}_N(\rho_{,ij}) - \mathcal{W}_M(\rho_{,ij})) \tag{157}$$
$$\to 2G_N\delta_{ij}U + \frac{4G_NX_0^2(m-n)^2}{3(m+nX_0^2)^2}\delta_{ij}\mathcal{W}_M(\rho) + \frac{2G_NX_0^2(m-n)^2}{3(m+nX_0^2)^2}\delta_{ij}\mathcal{W}_N(\rho) \tag{158}$$

and

$$\hat{h}_{0i} = -\frac{7G_N}{2}V_i - \frac{G_N}{2}W_i - \frac{4G_NX_0^2(m-n)^2}{(m+nX_0^2)^2}\mathcal{W}_M(\rho v_i)$$
$$+\frac{4G_NX_0^2(m-n)^2}{3M^2(m+nX_0^2)^2}(\mathcal{W}_N(\rho_{,0i}) - \mathcal{W}_M(\rho_{,0i})) \tag{159}$$
$$\to -\frac{7G_N}{2}V_i - \frac{G_N}{2}W_i - \frac{4G_NX_0^2(m-n)^2}{(m+nX_0^2)^2}\mathcal{W}_M(\rho v_i) \tag{160}$$

where the arrows indicate the coordinate transformations

$$\xi_i = \frac{2G_NX_0^2(m-n)^2}{3M^2(m+nX_0^2)^2}(\mathcal{W}_N(\rho) - \mathcal{W}_M(\rho))_{,i} \tag{161}$$
$$\xi_0 = \frac{2G_NX_0^2(m-n)^2}{3M^2(m+nX_0^2)^2}(\mathcal{W}_N(\rho) - \mathcal{W}_M(\rho))_{,0}. \tag{162}$$

We will not show the full expressions for $\hat{h}_{00}$ to $O(4)$ here as they are quite lengthy, and we do not feel that writing them out explicitly will add sufficient extra insight to justify their inclusion. The interested reader can calculate these quantities straightforwardly using the relevant expressions in Appendix B, and the coordinate transformations given above.