FRW cosmology in Milgrom's bimetric theory of gravity

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We consider spatially homogeneous and isotropic Friedmann-Robertson-Walker (FRW) solutions of Milgrom's recently proposed class of bimetric theories of gravity. These theories have two different regimes, corresponding to high and low acceleration. We find simple power-law matter dominated solutions in both, as well as solutions with spatial curvature, and exponentially expanding solutions. In the high acceleration limit these solutions behave like the FRW solutions of General Relativity, with a cosmological constant term that is of the correct order of magnitude to explain the observed accelerating expansion of the Universe. We find that solutions that remain in the high acceleration regime for their entire history, however, require non-baryonic dark matter fields, or extra interaction terms in their gravitational Lagrangian, in order to be observationally viable. The low acceleration regime also provides some scope to account for this deficit, with solutions that differ considerably from their general relativistic counterparts.

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I. INTRODUCTION

A natural way to extend General Relativity is via the introduction of a second dynamical rank 2 metric tensor field, $\hat{g}_{ab}$. The so called 'bimetric' theories that result have a rich phenomenology (see, for example, [1–4]), and a long history [5–7]. They represent an application to gravity of the Yang-Mills approach to gauge theories, and allow for new and interesting behaviour.

A new class of these theories has recently been introduced by Milgrom [8, 9], in which the interaction term between the two metric fields is constructed from a tensor defined as the difference of the connections associated with each of them. The stated purpose of this approach is to produce a weak-field limit of the form

$$\nabla^2\Psi - \nabla\cdot\left[f\left(l|\nabla\Psi_N|\right)\nabla\Psi_N\right] = 4\pi G\rho, \qquad (1)$$

where Ψ_N obeys $\nabla^2\Psi_N = 4\pi G\rho$, and where time-like test particles obey a force law $\ddot{x}\sim -\nabla\Psi$. Here Ψ is a gravitational potential, $f(x)$ is a function to be specified, ρ is the energy density and l is a constant with units of length. The existence of gravitational fields of this type would be of considerable interest for astrophysics [10, 11], and the goal of producing a viable relativistic formulation for theories of this kind has been pursued for some time [12–20].

Let us now try to understand why a relativistic formulation of (1) is so important. If we associate an energy-momentum tensor, T_{ab}, with the right-hand side of (1), then

$$\rho = T_{ab}n^a n^b, \qquad (2)$$

where n^a is the unit vector normal to surfaces of constant t. Energy-momentum conservation then tells us that test particles follow geodesics of the metric with respect to which T_{ab} is covariantly conserved. For a perturbed metric of the form

$$ds^2 \;=\; -(1+2\Psi)dt^2 + (1-2\Phi)\delta_{ij}dx^i dx^j \qquad (3)$$

this corresponds to $\ddot{x}\sim-\nabla\Psi$, as required. In order to gain a set of field equations that are invariant with respect to coordinate transformations, and that reduce to (1) for the metric (3), we must now construct a tensor quantity that can be associated with the left-hand side of (1). We immediately encounter a problem though: The quantity $|\nabla\Psi|$ cannot be formed from any local curvature invariants of (3) alone. One may alternatively try and recover (1) from *nonlocal* functions of the space-time curvature [21]. However, such attempts have prompted a no-go theorem [22] regarding the phenomenological viability of all theories that are constructed from the space-time geometry and matter fields only. New structure is therefore required, beyond a single space-time metric.

In the absence of a relativistic theory that is well motivated by other considerations, and that reduces to (1) in the appropriate limit, it is not clear what this additional structure should be. We are then left without any way of knowing when (1) should be considered valid, or when we should prefer, for example, a Newtonian description. We can, however, use observations of relativistic effects beyond Newtonian order to gain insight. In this regard, cosmological solutions provide us with a very useful probe as they express the full non-linearity of the relativistic field equations. Motivated by this, as well as the desire to understand further the cosmological consequences of bimetric theories of gravity in general, we consider here the FRW solutions of Milgrom's class of bimetric theories.

Of course, for any theory of gravity to be considered

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viable it is now the case that it should be able to reproduce all of the major probes of observational cosmology. These include the anisotropies in the cosmic microwave background, the matter power spectrum on large scales, Hubble diagrams that extend out to $z \sim 1$, as well as the ratios of light elements from primordial nucleosynthesis. All of these require a detailed knowledge of FRW cosmology (as well as of cosmological perturbation theory, in the case of the first two). It is not our goal here to perform an exhaustive study of all of these areas, but rather to make a solid first step in understanding the underlying FRW solutions, and what they imply.

The article proceeds as follows. In Section II we describe the theory and its field equations. The general form of the theory allows considerable flexibility, and so in Section III we discuss and motivate the specific versions of the theory that we will consider. In Section IV we find the field equations in space-times with FRW symmetries, and discuss their solutions in Section V. In Section VI we discuss the degree to which current probes of cosmology can constrain these models, and in Section VII we conclude.

II. FIELD EQUATIONS

The theory we are considering has two metrics, g_{ab} and $\hat{g}_{ab}$. The action for the gravitational part of Milgrom's theory can be written in terms of these two tensor fields as

$$I = I_g + I_{\hat{g}} + I_{int}, \tag{4}$$

where

$$I_g = \beta \int \sqrt{-g} R d^4x \tag{5}$$

$$I_{\hat{g}} = \alpha \int \sqrt{-\hat{g}} \hat{R} d^4x \tag{6}$$

are the Einstein-Hilbert terms for g_{ab} and $\hat{g}_{ab}$, and I_{int} is an interaction term between them. In the equations above R is the Ricci scalar constructed from g_{ab}, and $\hat{R}$ is the Ricci scalar constructed from $\hat{g}_{ab}$.

In Milgrom's theory the interaction term, I_{int}, is specified as a function of scalars formed from the rank-3 tensor

$$C^a{}_{bc} = \Gamma^a{}_{bc} - \hat{\Gamma}^a{}_{bc}, \tag{7}$$

where $\Gamma^a{}_{bc}$ and $\hat{\Gamma}^a{}_{bc}$ are the metric connections of g_{ab} and $\hat{g}_{ab}$, respectively, such that

$$g_{ab;c} = 0 \tag{8}$$

$$\hat{g}_{ab:c} = 0, \tag{9}$$

where ; and : indicate covariant derivatives constructed from $\Gamma^a{}_{bc}$ and $\hat{\Gamma}^a{}_{bc}$. The general form of scalars that are quadratic in $C^a{}_{bc}$ can then be written as

$$Q = l^2 Q_{ad}{}^{bcef} C^a{}_{bc} C^d{}_{ef}, \tag{10}$$

where $Q_{ad}{}^{bcef}$ is built from g_{ab} and $\hat{g}_{ab}$, and l is a constant with dimensions of length. In terms of these quantities we can then write[1]

$$I_{int} = \frac{2}{l^2} \int \sqrt{-g} \sigma \mathcal{M}(Q) d^4x, \tag{11}$$

where $\mathcal{M}(Q)$ and $\sigma = \sigma(\kappa)$ are functions to be specified, and $\kappa = (g/\hat{g})^{1/4}$.

In Appendix A we find the field equations for g_{ab} can be written as

$$\beta G_{ab} + \mathcal{S}_{ab} = 8\pi G T_{ab} \tag{12}$$

where

$$\mathcal{S}_{ab} = 2\left[\sigma \mathcal{M}_Q \left(J_{(ab)}{}^c + J_{(a}{}^c{}_{b)} - J^c{}_{ab}\right)\right]_{;c} + 2\sigma \mathcal{M}_Q \frac{\delta Q_{gd}{}^{hcef}}{\delta g^{ab}} C^g{}_{hc} C^d{}_{ef} - \frac{\sigma(1+n)}{l^2} \mathcal{M} g_{ab},$$

and G_{ab} and T_{ab} are the Einstein tensor and the energy-momentum tensor, defined with respect to g_{ab} in the usual way. We have also used the notation $\mathcal{M}_Q \equiv d\mathcal{M}/dQ$ and $J_d{}^{ef} \equiv Q_{ad}{}^{bcef} C^a{}_{bc}$, and taken $\sigma = \kappa^{2n}$.

Similarly, the field equations for $\hat{g}_{ab}$ are found to be

$$\alpha \hat{G}_{ab} + \hat{\mathcal{S}}_{ab} = 8\pi G \hat{T}_{ab} \tag{13}$$

where

$$\hat{\mathcal{S}}_{ab} = -2\left[\sigma^{\frac{n+1}{n}} \mathcal{M}_Q \left(2J_{(a}{}^{(ef)} \hat{g}_{b)f} - J_d{}^{cf} (\hat{g}^{-1})^{de} \hat{g}_{ca} \hat{g}_{fb}\right)\right]_{:e} + 2\sigma^{\frac{n+1}{n}} \mathcal{M}_Q \frac{\delta Q_{yd}{}^{zcef}}{\delta \hat{g}^{ab}} C^y{}_{zc} C^d{}_{ef} + \frac{\sigma^{\frac{n+1}{n}} n}{l^2} \mathcal{M} \hat{g}_{ab},$$

and where $\hat{G}_{ab}$ and $\hat{T}_{ab}$ are the Einstein and energy-momentum tensors defined with respect to $\hat{g}_{ab}$. Here, and throughout, we write the inverse of $\hat{g}_{ab}$ as $(\hat{g}^{-1})^{ab}$ (defined such that $(\hat{g}^{-1})^{ac} \hat{g}_{bc} = \delta^a{}_b$). Any raising or lowering of indices is otherwise always done with g_{ab}.

It should be noted that in (12) and (13) above we have taken the matter fields described by T_{ab} and $\hat{T}_{ab}$ to be coupled to g_{ab} and $\hat{g}_{ab}$, respectively[2]. We presume that we (as observers) are made from matter coupled to only one of these metrics, which we take to be the former without loss of generality. We further presume that there is no interaction between T_{ab} and $\hat{T}_{ab}$, so that all observations we make will be of the other matter fields coupled to g_{ab}. Cosmological probes of the expanding Universe then give us direct information about the geometry of g_{ab} only.

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[1] Note that in terms of the formalism used by Milgrom $\sigma = f(\kappa)/\kappa$, and $Q = l^2 \Xi$.

[2] Although one could also concievably couple matter to a combination of these metrics, we prefer to restrict ourselves here to the case considered by Milgrom in [8, 9].

In the equations derived above we have not explicitly included any cosmological constant terms. However, from (12) and (13) it is clear that Λ is dynamically equivalent to a perfect fluid with $p = -\rho$. We therefore account for any possible cosmological constants by allowing for them in T_{ab} and $\hat{T}_{ab}$.

III. SPECIFICATION OF THE THEORY

In Section II we presented the field equations for the theory derived from the action specified by (5), (6) and (11). These equations provide constraints on, and specify the evolution of, the two dynamical rank two tensor fields g_{ab} and $\hat{g}_{ab}$, but they also contain considerable freedom: The tensor ${Q_{ab}}^{cdef}$ and the function $\mathcal{M}(Q)$ have yet to be specified. In order to make progress in understanding the cosmological solutions of these theories, we will therefore now restrict ourselves to specific cases.

Firstly, we will only consider ${Q_{ab}}^{(cd)(ef)} = {Q_{ba}}^{(ef)(cd)}$ that are formed from the metric tensor g_{ab}. In this case the most general ${Q_{ab}}^{cdef}$ that can be constructed is

$$\begin{aligned} {Q_{ab}}^{cdef} &= c_1 {\delta^f}_a {\delta^d}_b g^{ce} + c_2 {\delta^f}_b {\delta^e}_a g^{cd} + c_3 g^{cd} g^{ef} g_{ab} \\ &\quad + c_4 {\delta^c}_a {\delta^e}_b g^{df} + c_5 g_{ab} g^{ce} g^{df}, \end{aligned} \tag{14}$$

where c_1, c_2, c_3, c_4 and c_5 are constants, and indices are assumed to be symmetrized appropriately. In this notation, the 'concrete simple theory' of Milgrom is specified by $c_1 = 1$, $c_2 = -1$ and $c_3 = c_4 = c_5 = 0$. The form of ${Q_{ab}}^{cdef}$ given in (14) can then be seen to correspond to a generalization of this theory.

Secondly, we will consider the function $\mathcal{M}(Q)$ to have the form specified by Milgrom in the low and high acceleration limits of the theory. That is, when $|Q| \gg 1$ we take

$$\mathcal{M}_Q \simeq 0, \tag{15}$$

and when $|Q| \ll 1$ we take

$$\mathcal{M}_Q \simeq |Q|^{-1/4}. \tag{16}$$

The first of these corresponds to Newtonian gravitation in the non-relativistic limit (when $c \to \infty$), and the second corresponds to modified gravitational dynamics in the low acceleration regime of the non-relativistic limit (when $c \to \infty$ and $l \to 0$). In particular, when $\alpha + \beta = 0$ and $\hat{T}_{ab} = 0$ one recovers (1) in the weak field limit, with $f = \mathcal{M}_Q$ [8]. The weak field limit with more general α and β, and with $\hat{T}_{ab}$, has been explored in [9].

Here we will not be concerned, for the most part, with the transitional behaviour between these regimes. It is not clearly specified by the weak field limit of the theory, and is presumed to be highly sensitive to the particular form of $\mathcal{M}(Q)$ that is chosen.

IV. FRW COSMOLOGY

We will now consider the cosmological solutions of the theory discussed above. Imposing FRW symmetries on g_{ab} and $\hat{g}_{ab}$ leads to the line-elements

$$ds^2 = g_{ab} dx^a dx^b = -a(\tau)^2 d\tau^2 + a(\tau)^2 h_{ij} dx^i dx^j, \tag{17}$$

and

$$d\hat{s}^2 = \hat{g}_{ab} dx^a dx^b = -X(\tau)^2 d\tau^2 + Y(\tau)^2 \hat{h}_{ij} dx^i dx^j, \tag{18}$$

where h_{ij} and $\hat{h}_{ij}$ are the metrics of Euclidean 3-spaces of constant curvature, k and $\hat{k}$, respectively.

The reason for introducing the function $X(\tau)$ in the τ-τ component of $\hat{g}_{ab}$ is that we now only have one coordinate freedom of the form $\tau \to f(\tau)$, but two τ-τ components, in the two different metrics. It is therefore not possible to redefine τ to absorb $X(\tau)$ without introducing a second function into the τ-τ component of g_{ab}. The most general expression of FRW geometry for both metrics must therefore contain three functions of τ: $a(\tau)$, $X(\tau)$ and $Y(\tau)$.

Using the geometry specified by (17) and (18) we can now calculate the quantities that appear in the field equations (12) and (13). Of particular interest is the constraint equation, which contains only first derivatives of a, X and Y. This equation is the analogue of the Friedmann equation in general relativistic FRW cosmology, and is derived in Appendix B. As an example, for Milgrom's 'concrete simple theory' with $\hat{k} = k$ (which we will motivate later) we find

$$\beta\left(\frac{\dot{a}^2}{a^2}+k\right)+\alpha\frac{Y^3}{a^2X}\left(\frac{\dot{Y}^2}{Y^2}+k\frac{X^2}{Y^2}\right)+\frac{a^2\mathcal{M}\sigma}{3l^2}+2\sigma\mathcal{M}_Q\left(\frac{\dot{Y}}{Y}-\frac{\dot{a}}{a}\right)\left[2\frac{\dot{a}}{a}-\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}-2\frac{Y\dot{Y}}{X^2}\right]=\frac{8\pi G\rho a^2}{3}+\frac{8\pi G\hat{\rho}XY^3}{3a^2},$$

where

$$Q=\frac{3l^2}{a^2}\left[2\frac{\dot{a}}{a}\frac{\dot{Y}}{Y}-2\frac{\dot{a}^2}{a^2}+2\frac{Y\dot{Y}}{X^2}\frac{\dot{a}}{a}+\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}-\frac{Y\dot{Y}\dot{X}}{X^3}-\frac{\dot{Y}^2}{X^2}-\frac{\dot{Y}^2}{Y^2}\right], \qquad \text{and} \qquad \sigma=\left(\frac{a^4}{XY^3}\right)^n$$

and where ρ and $\hat{\rho}$ are the energy densities of the perfect fluids associated with the energy-momentum tensors T_{ab} and $\hat{T}_{ab}$, respectively. Over-dots denote differentiation with respect to τ. More general expressions for the constraint equation, and for the other quantities involved in the field equations, are given in Appendix C.

The field equations (12) and (13) also provide second-order evolution equations for the three variables a, X and Y. However, these expressions are very lengthy, and so we choose not to reproduce them explicitly here.

As usual, the energy-momentum tensors obey conservation equations. For non-interacting fluids these equations read

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{19}$$

and

$$\dot{\hat{\rho}} + 3\frac{\dot{Y}}{Y}(\hat{\rho} + \hat{p}) = 0 \tag{20}$$

where p and $\hat{p}$ are the pressures of the two ideal fluids[3]. It is convenient to treat these fluids as being polytropic, with constant equations of state w and $\hat{w}$ defined by $p = w\rho$ and $\hat{p} = \hat{w}\hat{\rho}$. Equations (19) and (20) then give $\rho \propto a^{-3(1+w)}$ and $\hat{\rho} \propto Y^{-3(1+\hat{w})}$.

V. SOLUTIONS

We will now look for solutions to the field equations in both the 'high acceleration' limit, when $|Q| \gg 1$, and the 'low acceleration' limit, when $|Q| \ll 1$.

A. High acceleration limit

For $|Q| \gg 1$ we have $\mathcal{M}_Q \simeq 0$, so the constraint and evolution equations for g_{ab} are given by

$$\frac{\dot{a}^2}{a^4} = \frac{8\pi G\rho}{3\beta} - \frac{k}{a^2} - \frac{\mathcal{M}_0(1+n)\sigma}{3\beta l^2} \tag{21}$$

$$\frac{\ddot{a}}{a^3} - \frac{\dot{a}^2}{a^4} = -\frac{4\pi G(\rho+3p)}{3\beta} - \frac{\mathcal{M}_0(1+n)\sigma}{3\beta l^2}, \tag{22}$$

while the constraint and evolution equations for $\hat{g}_{ab}$ are given by

$$\frac{\dot{Y}^2}{X^2Y^2} = \frac{8\pi G\hat{\rho}}{3\alpha} - \frac{\hat{k}}{Y^2} + \frac{\mathcal{M}_0 n\sigma^{\frac{(1+n)}{n}}}{3\alpha l^2} \tag{23}$$

$$\frac{\ddot{Y}}{X^2Y} - \frac{\dot{X}\dot{Y}}{X^3Y} = -\frac{4\pi G(\hat{\rho}+3\hat{p})}{3\alpha} + \frac{\mathcal{M}_0 n\sigma^{\frac{(1+n)}{n}}}{3\alpha l^2} \tag{24}$$

[3] It is, in principle, possible to have an interaction between the two fluids such that energy-momentum could be exchanged between them. We do not consider this possibility here, but rather treat the two fluids as being non-interacting, as is usual in cosmology.

where

$$\sigma = \left(\frac{\sqrt{1-\hat{k}r^2}}{\sqrt{1-kr^2}}\frac{a^4}{XY^3}\right)^n, \tag{25}$$

and $\mathcal{M}_0$ is the constant value of $\mathcal{M}(x)$ when $x \gg 1$. It is now clear that one or more of the following conditions must be true:

(i) The value of $\mathcal{M}_0$ is 0.

(ii) The value of n is 0 or -1.

(iii) The value of $\hat{k}$ is equal to that of k.

If none of these conditions are met then the terms containing σ have an r dependence that cannot be accommodated by any of the other terms. The first two of these are conditions on the theory. If either, or both, of these conditions are met then there can exist FRW solutions in the two metrics which have different spatial curvatures. If neither (i) nor (ii) are satisfied, then the only FRW solutions that exist are those in which the spatial curvatures of the two FRW metrics are the same.

If condition (i) is met then the Friedmann equations above can be seen to be unchanged from their general relativistic form. The two scale factors a and Y are completely decoupled, and obey constraint and evolution equations that are exactly the same as an FRW geometry in General Relativity.

If condition (ii) is true (but (i) is not) then either a or Y is driven by an additional term in its field equations that acts in exactly the same as a cosmological constant. For $n = 0$ this term can be seen to appear in the Friedmann equations for a, which are otherwise identical to their general relativistic counterparts. The additional effective cosmological constant is given by

$$\Lambda_{eff} = -\frac{\mathcal{M}_0}{\beta l^2}. \tag{26}$$

If this is the case then Y obeys field equations that are unchanged from a general relativistic FRW solution. If $n = -1$ then the situation is reversed, with Y being driven by an additional term, while the Friedmann equations for a are unchanged.

If the theory does not satisfy conditions (i) or (ii) then the only FRW solutions that exist have $\hat{k} = k$. The value of σ is then independent of r, and a obeys a set of Friedmann equations with an additional effective fluid whose energy density and pressure are given by

$$p_{eff} = -\rho_{eff} = \frac{\mathcal{M}_0(1+n)}{8\pi G l^2}\left(\frac{a^4}{XY^3}\right)^n \tag{27}$$

This effective fluid has an equation of state $w_{eff} = -1$, and so we must have ρ_{eff} =constant, and therefore $a \propto X^{1/4}Y^{3/4}$. In this case the other set of Friedmann equations also has an additional effective fluid, with

$$\hat{p}_{eff} = -\hat{\rho}_{eff} = -\frac{\mathcal{M}_0 n}{8\pi G l^2}\left(\frac{a^4}{XY^3}\right)^{1+n}. \tag{28}$$

We therefore have that one of the scale factors is driven by an additional positive effective cosmological constant, while the other is driven by a negative one (unless $0 < n < -1$, in which case both effective cosmological constant terms can have the same sign).

It has been shown above that in all possible cases the two FRW geometries effectively decouple from each other. Their evolution is then specified by a set of equations that are identical to the Friedmann equation of General Relativity, but with the possible addition of an effective cosmological constant term. Further more, unless condition (i) or (ii) is satisfied, we must also have $\hat{k} = k$ and $a \propto X^{1/4} Y^{3/4}$.

B. Low acceleration limit

The situation in the low acceleration limit is more complicated. In this case we have $|Q| \ll 1$ and

$$\mathcal{M} \simeq \text{sign}(Q) \frac{4|Q|^{3/4}}{3} + \mathcal{M}_1, \tag{29}$$

so that $\mathcal{M}_Q \simeq |Q|^{-1/4}$. Here $\mathcal{M}_1$ is an unspecified constant.

Firstly, let us consider the situation with $k = \hat{k} = 0$ and $\mathcal{M}_1 = 0$ (we will return to the situation with non-zero spatial curvature, and $\mathcal{M}_1 \neq 0$, below). In this case it can seen from Appendix C that we automatically have $Q = Q(\tau)$ and $\sigma = \sigma(\tau)$, and that all dependence on r vanishes from the field equations. Now, just as one can take $c \to \infty$ to find the non-relativistic limit of General Relativity, here we take $l \to 0$ to find the low acceleration limit of the present theory. Such a limit suppresses the contribution of G_{ab} and $\hat{G}_{ab}$ to the left hand side of the field equations, relative to S_{ab} and $\hat{S}_{ab}$, so that (12) and (13) become

$$S_{ab} \simeq 8\pi G T_{ab} \tag{30}$$

and

$$\hat{\mathcal{S}}_{ab} \simeq 8\pi G \hat{T}_{ab}. \tag{31}$$

These equations are second-order, and provide a system of constraint and evolution equations. Assuming a perfect fluid form for both energy-momentum tensors we then find the following matter dominated power-law solutions[4]:

$$a \propto \tau^p \qquad \text{and} \qquad X \propto Y \propto \tau^q, \tag{32}$$

where p and q are given by

$$p = \frac{1 - 3\hat{w}}{(1 + 2w + 8nw - 3\hat{w} - 8n\hat{w} - 6w\hat{w})} \tag{33}$$

$$q = \frac{1 - 3w}{(1 + 2w + 8nw - 3\hat{w} - 8n\hat{w} - 6w\hat{w})}, \tag{34}$$

and where w and $\hat{w}$ are the equations of state of the two fluids, as defined in Section IV. These solutions are significantly different from the corresponding solutions in General Relativity, and some examples, for cosmologically interesting equations of state, are given in Table I. In this table, and throughout, we use the time coordinates t and $\hat{t}$ to correspond to the proper time of comoving observers in each geometry, so that $t = \int a d\tau$ and $\hat{t} = \int X d\tau$.

Fluids	$a(t)$	$Y(\hat{t})$	GR
dust & dust	$t^{1/2}$	$\hat{t}^{1/2}$	$t^{2/3}$
dust & rad.	constant	$\hat{t}^{\frac{3}{(3-8n)}}$	$t^{2/3}$ & $t^{1/2}$
rad. & dust	$t^{\frac{3}{8(1+n)}}$	constant	$t^{1/2}$ & $t^{2/3}$
rad. & rad.	$t^{\frac{3+8nC-3C}{8(1+n-C)}}$	$\hat{t}^C$	$t^{1/2}$
scalar & scalar	$t^{1/4}$	$\hat{t}^{1/4}$	$t^{1/3}$
Λ & Λ	exponential	exponential	exponential

TABLE I: The functional form of matter dominated power-law solutions in the low acceleration limit of Milgrom's bimetric theory and General Relativity (GR). The fluids considered are dust ($w = 0$), radiation ($w = 1/3$), scalar fields ($w = 1$) and Λ ($w = -1$). In the left hand column, the first fluid is coupled to g_{ab}, and the second fluid to $\hat{g}_{ab}$. In the radiation & radiation case, the solutions are not specified uniquely, and hence there is an arbitrary constant C involved.

It can be seen that the functional form of these solutions is, in general, not only dependent on the fluid that is coupled to the metric in question, but also to the fluid coupled to the other metric, as well as the theory, via the parameter n. This mutual dependence should be expected as the two metrics are non-minimally coupled to each other in the action. The evolution of the two metrics can also be asymmetric under the exchange of the two fluids. This can be seen from the second and third rows of Table I. Again, this is expected, as Q (as specified in Section III) is asymmetric under the exchange of g_{ab} and $\hat{g}_{ab}$. Once these power-law solutions have been assumed, the field equations then reduce to a set of algebraic relations between the c_i's, n, α, β, l, $\rho(t_0)$ and $\hat{\rho}(t_0)$, as well as the constants of proportionality in (32), where t_0 is the present age of the Universe.

It is interesting to note that in some cases it is possible for one of the metrics to be static. This is exemplified in Table I by the two radiation & dust solutions. In fact, it can be seen from (33) and (34) that if the matter coupled to the second metric is radiation, then the first metric will be static (unless the denominator diverges, as is the case with radiation & radiation). It is also interesting

[4] By 'matter dominated' we mean that the left-hand side of the field equations scales in the same way as the right-hand side. This is in contrast to 'vacuum dominated', in which the left-hand side scales independently of the right-hand side, such that the influence of the matter fields on the space-time dynamics is negligible.

to note that in the radiation & radiation case the two scale factors are not uniquely determined, and can only be specified up to an arbitrary constant. Clearly the behaviour of these solutions is quite different to the usual general relativistic FRW solutions, which are shown in the fourth column of Table I for comparison.

A special case that should be noted is when X equals Y, rather than simply being proportional to it. In this case the solutions that are obtained are quite different to the general case, discussed above. What happens is that the contributions from any terms that involve Q vanish. The contribution of G_{ab} and $\hat{G}_{ab}$ can then no longer be considered to be suppressed, as the terms they were suppressed with respect to no longer exist. The system of equations then reduces to exactly those considered previously, in the high acceleration regime. This is, in fact, the case that was considered by Milgrom in [8].

Now let us consider non-zero spatial curvature. It can be seen from Appendix C that σ and Q are, in general, functions of both τ and r, when k and $\hat{k}$ take arbitrary values. Such a dependence on r is problematic, as the free functions in the line-elements (17) and (18) are all functions of τ only. This is a generalization of the problem involving the r dependence of σ found in the high acceleration regime, and although the dependence is more complicated, the solution is the same: σ and Q reduce to functions of τ if $\hat{k} = k$. It can then be seen that not only does all r dependence drop out of σ, Q, S_{ab} and $\hat{S}_{ab}$, but that these functions also become independent of k and $\hat{k}$ (up to possible overall multiplicative factors of $(1 - kr^2)$). The only place that non-zero k has any effect in the field equations is therefore through its contribution to G_{ab} and $\hat{G}_{ab}$, where it appears as an additional constant.

Now, although the contribution of the k-independent terms in G_{ab} are suppressed by a power of l, and can therefore be safely ignored in the low acceleration regime, the same cannot be said for k. This term is a constant, and does not have the potential to become arbitrarily small at late-times, in contrast to the other terms in G_{ab} and S_{ab}. We therefore cannot ignore it, as even if it is negligibly small at early times, due to suppression by a power of l, it can still become influential at late-times. However, as the influence of k and $\hat{k}$ is limited to its appearance in the Einstein tensor, when $\hat{k} = k$, its influence can be straightforwardly accounted for: It acts in exactly the same way as an additional fluid with $w = -1/3$ coupled to g_{ab}, and a fluid with $\hat{w} = -1/3$ coupled to $\hat{g}_{ab}$. We can therefore calculate its influence at late-times using equations (33) and (34), above. We find that the corresponding power-law solution has $a \propto X \propto Y \propto \tau^3 \propto t^{3/4}$, independent of n. This should be contrasted with the usual $a \propto t$ in General Relativity.

Next, let us consider the effect of non-zero $\mathcal{M}_1$. We find that the inclusion of such a term requires $a \propto \tau^{-1}$, if power-law solutions are to exist, which correspond to exponential evolution in the proper time of comoving observers, t. The solutions that exist for $\hat{g}_{ab}$ then depend on the matter content of the theory. If we couple a fluid with equation of state w to g_{ab} then we find the power-law solution

$$X \propto Y \propto \tau^{-\frac{(3+4n+3w)}{4n}} \propto \hat{t}^{1+\frac{4n}{3(1+w)}}, \tag{35}$$

while if we couple a fluid with equation of state $\hat{w}$ to $\hat{g}_{ab}$ we find

$$X \propto Y \propto \tau^{-\frac{4(1+n)}{(1+4n-3\hat{w})}} \propto \hat{t}^{\frac{4(1+n)}{3(1+\hat{w})}}. \tag{36}$$

Power-law solutions with fluids coupled to both metrics do not, in general, exist, unless they result in $a \propto \tau^{-1}$ anyway (such as, for example, two Λ terms).

Finally, let us consider what happens on approach to a vacuum. For monotonically expanding geometries fluids with higher equations of state give way to those with lower equations of state, such that dust domination follows radiation domination, and so forth, in the usual way. If a fluid with equation of state $w = -1$ (or less) exists then the total energy density will eventually reach a constant (or start to increase), as all other fluids become negligible in comparison. If no such fluid exists, then the total energy density will decrease monotonically forever. If this happens then the right-hand side of the field equations can eventually drop below the previously neglected terms in the Einstein tensor. For $p(1 + 3w) > 2$ this will mean that the matter becomes subdominant, and the cosmological evolution will be vacuum dominated (determined by the dynamics of the vacuum alone). In this case the only power-law solutions that exist go like τ^{-1}, which corresponds to exponential expansion. For $p(1+3w) < 2$ the right-hand side of the field equations will always be dominant over the neglected terms in the Einstein tensor, and matter dominated power-law evolution, as described above, can continue indefinitely into the future. It should be noted that if spatial curvature is non-zero, and acts like a fluid with $w = -1/3$, then the condition $p(1+3w) < 2$ is always satisfied. A curvature dominated power-law solution, as found above, can therefore always last for an indefinitely long period, provided there are no fluids with equations of state $w < -1/3$.

VI. VIABILITY AS A MODEL OF THE UNIVERSE

We will now consider the viability of the cosmological solutions found above as models of the observable Universe. Our criterion for viability will be whether or not it is possible to account for the major probes of observational cosmology, which we take to be the primordial synthesis of light elements, the position of the first acoustic peak in the CMB angular power spectrum, the growth of structure, and the late-time accelerating expansion of the Universe. For later convenience, let us now define the density fraction of a fluid i to be

$$\Omega_i \equiv \frac{8\pi G \rho_i}{3H_0^2}, \tag{37}$$

where ρ_i is the energy density of the fluid on a homogeneous hypersurface of age t_0, and H_0 is the locally measured value of the Hubble parameter (in whichever geometry the observers are taken to be coupled to).

First consider the primordial synthesis of light elements. This process requires a cosmological evolution that is close to the standard form [23], $a \sim t^{1/2}$, and is highly sensitive to the ratio of photons to baryons, η. In particular, at the end of nucleosynthesis ($t \sim 200s$) it can be shown that the observed abundances of light elements imply $\eta = (5.5 \pm 0.5) \times 10^{-10}$ [24]. If we now combine this with a CMB photon temperature of 2.725 ± 0.001K, and assume that the expansion of the Universe has been adiabatic, then we are left with the constraint

$$\Omega_b h^2 = 0.020 \pm 0.002, \tag{38}$$

to 95% confidence, where $h \equiv H_0/(100\text{kms}^{-1}\text{Mpc}^{-1})$. This is a tight constraint on the matter content of the Universe, and one that we expect to be applicable to any viable model.

Next let us consider the positions of the acoustic peaks of the CMB. In general relativistic FRW cosmology their angular extent on the sky, θ, is essentially determined by two factors: The acoustic horizon at last scattering, R_{ls}, and the angular diameter distance to the last scattering surface, d_A^*, such that

$$\theta \sim \pi \frac{d_A^*}{R_{ls}}. \tag{39}$$

Here the situation may be somewhat different. The formation of linear structure up to last scattering can be modified, as the weak-field limit can now be altered from its usual Newtonian form. This may affect the scale of correlations on the last scattering surface, but the projection of that surface onto our sky will be unaffected by this, and will therefore be sensitive to the intervening geometry of space-time in the usual way. We should therefore expect an equation similar in form to (39), but with R_{ls} now replaced by $\tilde{R}_{ls}$, referring to the new quantity that sets the scale of correlations on the last scattering surface (no longer necessarily the acoustic horizon).

A reliable calculation of $\tilde{R}_{ls}$ will require a detailed analysis of linear perturbation theory, and pre-recombination physics, within the frame-work of the current theory. For our current purposes, we will simply assume $\tilde{R}_{ls}$ has been calculated. Using the well known result that spatial curvature produces a shift in the angular scale of the first acoustic peak according to $(1-\Omega_k)^{-9/20}$, and given the compatibility of the observed CMB with a spatially flat general relativistic FRW cosmology, we can then speculate that, in the context of the present theory, the position of the first acoustic peak is likely to imply an effective spatial curvature of

$$\Omega_k \simeq 1 - \left(\frac{\tilde{R}_{ls}}{R_{ls}} \right)^{20/9}. \tag{40}$$

This can be seen to reduce to $\Omega_k \simeq 0$ when $\tilde{R}_{ls} \simeq R$.

Beyond the sensitivity of CMB observations to the curvature of the Universe, the growth of structure also depends on the evolution of the Universe. This is true for both the linear structure that we observe at last scattering, as well as the subsequent growth of non-linear structures such as galaxies, and clusters of galaxies. In most cosmological models, it is usually required that one should have a period of cosmological expansion where $a \sim t^{2/3}$ in order for structure to form. Here the situation may be somewhat different, as, once again, the weak-field limit is modified from its usual Newtonian form. Never the less, we will consider it preferable if the cosmological solutions we have found can be shown to exhibit a period where a or $X \sim t^{2/3}$.

Finally, let us consider supernova observations. The Hubble diagrams constructed from these events allow us to determine the late-time evolution of the Universe. In the standard cosmological model, with the assumption of spatial flatness, they provide strong, direct evidence for a cosmological constant, or vacuum energy, with $\Omega_\Lambda \sim 0.7$ [25, 26]. If one permits non-zero spatial curvature, as will be required in these models if $\tilde{R}_{ls} \neq R_{ls}$, then the resulting bounds on Ω_Λ change. However, it can still be shown that there exists good evidence for a non-zero cosmological constant, independent of the value of Ω_k. Using bounds derived from the 414 supernovae of the "Union" dataset one can make the approximate statement that [27]

$$0.3 \lesssim \Omega_\Lambda \lesssim 1.4, \tag{41}$$

to 3σ, independent of the amount of spatial curvature[5]. Any viable model of the Universe should therefore include a late-time period of accelerating expansion.

Having discussed the constraints available from observational cosmology, let us now consider their applicability to the solutions found in Section V.

A. High acceleration limit

It was shown above that when $|Q| \gg 1$ the gravitational field equations reduce to their usual general relativistic form, with a cosmological constant term. For a cosmological solution with dust and radiation we will therefore have periods of cosmological evolution with $a \sim t^{1/2}$ and $t^{2/3}$, when these two fluids dominate the cosmological dynamics, respectively. This is in good keeping with the requirements of primordial nucleosynthesis and structure formation.

If we apply the primordial nucleosynthesis bound (38) on Ω_B, and take this to make up the entire contribu-

[5] The precise bounds achievable depend on exactly which data one chooses to include, and how it is treated.

tion to the dust content of the Universe, then the supernovae results [27] imply a low value of $\Omega_\Lambda \sim 1/3$, that is only compatible with observations at around the 3σ level. What is more, without any other matter content this would imply $\Omega_k \sim 2/3$, which would only be compatible with the position of the first acoustic peak of the CMB angular power spectrum if $\tilde{R}_{ls} \sim 0.6 R_{ls}$. Such a large change of scale seems unlikely from previous studies using similar theories [28, 29]. We therefore find that some additional non-baryonic dust-like degrees of freedom are necessary, either in the form of cosmological dark matter, or from some other sector of the theory.

An effective cosmological constant term, $\mathcal{M}_0$, is also present in (21) and (22). This term has the attractive feature of being able to reproduce the value of the cosmological constant observed in supernova data with $O(1)$ values of $\mathcal{M}_0$ and n, for astrophysically interesting values of l. It also provides a lower limit for $|Q|$, as $Q \rightarrow$constant in the limit that $\mathcal{M}_0$ dominates. This constant is zero in the special case $X = Y$, and, as already discussed, in this case the dynamics in the low acceleration limit become identical to those of the high acceleration regime. More generally we find that in the limit $a \rightarrow a_0 \tau^{-1}$, $X \rightarrow X_0 \tau^{-1}$ and $Y \rightarrow Y_0 \tau^{-1}$ we have

$$Q \rightarrow -\frac{3(3c_3 + c_5) l^2 (X_0^2 - Y_0^2)^2}{a_0^2 X_0^4}. \tag{42}$$

If this value of $|Q|$ is $\gg 1$, and the Universe starts off in the high acceleration regime, then the whole of cosmological history could have taken place there. In this case, as discussed above, some non-baryonic cosmological dark matter would be required. Alternatively, the decelerating phase of cosmological expansion could push the value of Q into the low acceleration regime at some point, depending on parameters. One may then look to the low or intermediate acceleration regimes for the additional dust-like degrees of freedom.

As a further possibility, we note that one could consider including extra interaction terms in the action that would behave like an extra dust-like contribution during the matter dominated era, while staying in the high acceleration regime. Indeed, such a contribution was found by the authors of [3] when they considered a bimetric theory with the interaction term

$$I_{int} \sim \int \sqrt{-\hat{g}} (\hat{g}^{-1})^{ab} g_{ab} d^4 x. \tag{43}$$

One would expect that the addition of such a term in the present theory should have a similar effect in the $|Q| \gg 1$ regime, but it is not immediately clear what effects this would have on the weak-field limit.

B. Low acceleration limit

As discussed above, it seems entirely plausible that even if the Universe starts off in the high acceleration regime (with $|Q| \gg 1$), it could end up in the low acceleration regime (with $|Q| \ll 1$) after a suitable period of decelerating expansion. However, because Q depends on the c_i's, as well as Hubble-parameter like terms and l, the values of H_0 and l do not necessarily tell us anything about the value of Q today. At an extreme, if the c_i's are chosen appropriately, the Universe could be in the low acceleration limit even before recombination, independent of H_0 and l. Thus, one could also envisage a situation where the Universe is in the low acceleration regime throughout its entire history.

Now let us consider what the probes of observational cosmology, discussed above, can tell us about the possibility of cosmological expansion in the low acceleration limit. It can be seen from Table I that in this regime a period of cosmological expansion of the form $a \sim t^{1/2}$ can be achieved as a result of dust-like fluids being coupled to each of the two metrics. We therefore have the very unfamiliar situation of primordial nucleosynthesis being able to occur during a period of dust domination. However, it can also be seen that $a \sim t^{1/2}$ can be achieved for a variety of other fluids coupled to the two metrics, including radiation, if we allow $Y \sim \hat{t}^{\frac{(1+4n)}{(1+8n)}}$.

Now consider structure formation. It can be seen that a period of expansion of the form $a \sim t^{2/3}$ can be achieved if two fluids with equation of state $w = -1/4$ are coupled to each of the two metrics. Having to invent two exotic fluids in this way is somewhat distasteful, but again we have some freedom. If we are prepared to consider a second fluid with $\hat{w} = (3+16n)^{-1}$, then we can have $a \sim t^{2/3}$ being produced with a fluid of dust coupled to the first metric. For a theory with $n = -1/8$ this corresponds to dust and a scalar field. Various other situations can be read off from Equations (33) and (34). One particularly interesting example is for a theory with $n = -1/16$. In this case spatial curvature dominating the first metric, and dust coupled to the second metric, also gives an evolution $a \sim t^{2/3}$. Here, then, the dust coupled to the second metric acts, in a way, as if it were cosmological dark matter coupled to the first, which is itself an empty, open universe. Clearly there is considerable scope for new and interesting behaviour.

C. The intermediate regime

One may also ask whether a dust-like contribution could arise from the intermediate regime, where $|Q| \sim 1$. As only the asymptotic limits of this function are defined, there is considerable flexibility in terms of the function's transitional behaviour. For example, if we consider a transitional regime where $\mathcal{M} \sim |Q|^s$ then an analysis similar to that used in the low acceleration regime finds solutions $a \propto \tau^p$ and $X \propto Y \propto \tau^q$ where

$$p = \frac{2s(1-3\hat{w})}{(3 - 2s + 3w + 12nw - 9\hat{w} - 12n\hat{w} + 6s\hat{w} - 9w\hat{w})}$$

$$q = \frac{2s(1-3w)}{(3-2s+3w+12nw-9\hat{w}-12n\hat{w}+6s\hat{w}-9w\hat{w})}.$$

These equations reduce to (33) and (34) when $s = 3/4$. It can be seen from the above that, among very many other possibilities, we can arrange to have $a \sim t^{2/3}$ and $Y \sim \hat{t}^{2/3}$ when $w = \hat{w} = s - 1$. For a theory in which $s = 2/3$, we can therefore have the some of the effects of cosmological dark matter when both metrics are effectively empty, with negative spatial curvature. This is intended as an example only. Without any well defined functional form of $\mathcal{M}(Q)$ in the intermediate regime one clearly has considerable freedom to achieve whatever evolution is desired.

One final possibility arises if the entirety of cosmological history corresponds to values of Q with a different sign to that which is appropriate for the weak-field limit. Here we have considered the functional form of $\mathcal{M}(Q)$ to be set by the magnitude of Q only. If one allowed its form to be different for positive or negative Q, then the possibility could arise that the form of $\mathcal{M}(Q)$ that is appropriate for cosmology is not fixed at all by considerations of the weak-field limit. Cosmological solutions could then be considered entirely independently from other phenomenology. In this case one may look to the discussion of $|Q| \sim 1$, above, for what the cosmological dynamics could look like for simple power-law forms of $\mathcal{M}(Q)$.

VII. DISCUSSION

We have studied the FRW solutions of Milgrom's class of bimetric theories of gravity. These theories have different behaviours depending on the value of the scalar quantity $|Q|$, which is formed from the two metrics and various parameters of the theory. The two regimes that result are referred to as the 'high' and 'low acceleration' limits.

We find that in the high acceleration limit the cosmological dynamics of the two metrics essentially decouple from each another, and evolve in a similar fashion to the FRW solutions of General Relativity. In this regime, the theory also potentially provides an explanation of the cosmological constant problem, as it is possible to include a constant term in the Friedmann equations that is constructed from a factor of order unity divided by the square of the intrinsic length scale of the theory, l. This results in the correct order of magnitude for the cosmological constant measured in our observable Universe. Solutions that stay in the high acceleration regime for their entire history, however, appear to be in conflict with observations, unless non-baryonic dark matter fields, or extra interaction terms in the action, are also included.

In the low acceleration limit we find that simple power-law, matter dominated solutions also exist. There is considerable freedom in the form of these solutions, depending on the parameters of the theory, as well as the matter fields coupled to each of the two metrics. The solutions in this regime are, in general, quite different to their general relativistic counterparts, and display some interesting possibilities. For example, in some theories it is possible to have a metric that evolves like a dust dominated universe does in General Relativity, while itself being empty and spatially open. This allows for the possibility of accounting for some of the effects of cosmological dark matter via the fields coupled to the second metric.

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Appendix A: Derivation of the Field Equations

To derive the field equations, let us consider the variations of $\mathcal{M}$, σ, and $\sqrt{-g}$ with respect to g^{ab} separately. Firstly, varying $\mathcal{M}$ gives

$$\begin{aligned}\sigma\delta\mathcal{M} \;=\; & \sigma l^2 \mathcal{M}_Q \frac{\delta Q_{ad}{}^{bcef}}{\delta g^{rs}} C^a{}_{bc} C^d{}_{ef} \delta g^{rs} \\ & +2\sigma l^2 \mathcal{M}_Q J_d{}^{ef} \delta\Gamma^d{}_{ef}\end{aligned} \tag{44}$$

where $\mathcal{M}_Q \equiv d\mathcal{M}/dQ$, and $J_d{}^{ef} \equiv Q_{ad}{}^{bcef} C^a{}_{bc}$. The second term in the equation above can then be written

$$\begin{aligned}& 2\sigma l^2 \mathcal{M}_Q J_d{}^{ef} \delta\Gamma^d{}_{ef} \\ =\; & \sigma l^2 \mathcal{M}_Q J_d{}^{ef} g^{dz} (\delta g_{zf;e} + \delta g_{ez;f} - \delta g_{ef;z}) \\ \doteq\; & l^2 \left[\sigma \mathcal{M}_Q \left(J_{(ab)}{}^e + J_{(a}{}^e{}_{b)} - J^e{}_{ab}\right)\right]_{;e} \delta g^{ab},\end{aligned} \tag{45}$$

where $\doteq$ means equal up to total divergences, and where we have used the usual result $\delta g_{ef} = -g_{ea} g_{fb} \delta g^{ab}$. The quantities $J_{ab}{}^e$ and $J^e{}_{ab}$, in the equation above, have had their indices lowered with g_{ab}.

We also have

$$\sigma\mathcal{M}\frac{\delta(\sqrt{-g})}{\sqrt{-g}} = -\frac{\sigma}{2}\mathcal{M} g_{ab}\delta g^{ab}, \tag{46}$$

and

$$\mathcal{M}\delta\sigma = \frac{\delta\sigma}{\delta g^{ab}}\mathcal{M}\delta g^{ab}. \tag{47}$$

If $\sigma = \kappa^{2n}$ then $\delta\sigma = -\frac{n}{2}\sigma g_{ab}\delta g^{ab}$, and the last equation can then be written

$$\mathcal{M}\delta\sigma = -\frac{n}{2}\sigma g_{ab}\mathcal{M}\delta g^{ab}. \tag{48}$$

Summing (44), (46) and (48) gives the field equations (12) and (13) shown in Section II.

Appendix B: The Constraint Equations

Given the complicated form of the full second-order field equations, it is useful to look for the simpler, first-order constraint equation. To find this, consider general variations of the action of the form

$$\delta I \;=\; \int d^4x\sqrt{-g}(E_{ab}\delta g^{ab}+\hat{E}_{ab}\delta\hat{g}^{ab}), \qquad (49)$$

where

$$E_{ab}=\beta G_{ab}-S_{ab}-\frac{1}{2}T_{ab}=0 \qquad (50)$$

and

$$\hat{E}_{ab}=\sqrt{\frac{\hat{g}}{g}}(\alpha\hat{G}_{ab}-\hat{S}_{ab}-\frac{1}{2}\hat{T}_{ab})=0. \qquad (51)$$

If we assume that T_{ab} and $\hat{T}_{ab}$ contain at most first-order derivatives then we do not need to consider any compensatory contributions from their evolution equations in order to find the desired constraint equations.

Now consider the variation of g^{ab} and $\hat{g}^{ab}$ due to diffeomorphisms generated by the 'infinitesimal' vector field ξ^a. Firstly, let us consider the variation of δg^{ab} due to ξ^a, which gives

$$\delta I_{\xi,\delta g} \;=\; -\int E_{ab}(g^{ac}\nabla_c\xi^b+g^{cb}\nabla_c\xi^a). \qquad (52)$$

We can integrate this by parts to yield a boundary term, that we neglect, the Bianchi identity, $\nabla_a G^{ab}=0$, and

$$\delta I_{\xi,\delta g} \;=\; 2\int\nabla_c(E^{ca})\xi_a. \qquad (53)$$

Now consider variations of $\hat{g}^{ab}$, such that

$$\begin{aligned}\delta I_{\xi,\delta\hat{g}} \;&=\; \int \hat{E}_{ab}(\xi^c\nabla_c\hat{g}^{ab}-\hat{g}^{ac}\nabla_c\xi^b-\hat{g}^{cb}\nabla_c\xi^a) \qquad (54)\\ &=\int(\xi_c(\nabla^c\hat{g}^{ab})\hat{E}_{ab}+\xi_a\nabla_c(\hat{E}_b{}^a\hat{g}^{bc}+\hat{E}^a{}_b\hat{g}^{cb})).\end{aligned}$$

Collecting terms then gives

$$\begin{aligned}\delta I_\xi \;&=\; I_{\xi,\delta g}+I_{\xi,\delta\hat{g}} \qquad (55)\\ &=\int\xi_a((\nabla^a\hat{g}^{cb})\hat{E}_{cb}+\nabla_c(2E^{ca}+\hat{E}_b{}^a\hat{g}^{bc}+\hat{E}^a{}_b\hat{g}^{cb})).\end{aligned}$$

Finally, let us define the tensor $\mathcal{J}^{ca}\equiv 2E^{ca}+\hat{E}_b{}^a\hat{g}^{bc}+\hat{E}^a{}_b\hat{g}^{cb}$. If $\mathcal{J}^{0a}$ had second-order derivatives, then $\nabla_c\mathcal{J}^{ca}$ would contain third-order derivatives. However, we know from the vanishing of the variation of the action under diffeomorphisms, $\delta I_\xi=0$, that $\nabla_c\mathcal{J}^{ca}=-(\nabla^a\hat{g}^{cb})\hat{E}_{cb}$. Now, due to the structure of the theory we can see that $\hat{E}_{cb}$ contains at most second-order derivatives, so the right-hand side of this equation can contain at most second-order derivatives. We can therefore conclude that $\mathcal{J}^{0a}$ contains up to first-order derivatives only.

For a given foliation of hypersurfaces with normal n^a, and metric $h_{ab}=g_{ab}+n_an_b$, we then find that the constraint equations are given by

$$\mathcal{J}_{ab}n^ah^b{}_c=0. \qquad (56)$$

and

$$\mathcal{J}_{ab}n^an^b=0 \qquad (57)$$

The first of these is trivially satisfied by FRW geometry, where the hypersurfaces are taken to be surfaces of constant t, while the second gives the constraint equations displayed in Appendix C, and Section IV.

Appendix C: The Field Equations with FRW Geometry

Here we will write explicit expressions for some of the quantities that appear in the field equations (12) and (13), for the FRW geometries (17) and (18). Firstly, we can immediately write

$$\sigma=\left(\frac{\sqrt{1-\hat{k}r^2}}{\sqrt{1-kr^2}}\frac{a^4}{XY^3}\right)^n. \qquad (58)$$

We can also write a relatively simple expression for Q. To do this we first define five new quantities via

$$Q=c_1Q^{(1)}+c_2Q^{(2)}+c_3Q^{(3)}+c_4Q^{(4)}+c_5Q^{(5)}, \qquad (59)$$

such that $Q^{(1)}=l^2\delta^f{}_a\delta^d{}_bg^{ce}C^a{}_{cd}C^b{}_{ef}$, with the other four defined *mutatis mutandis*. We can then write

$$\begin{aligned}Q^{(1)} \;&=\; \frac{l^2}{a^2}\left[2\frac{\dot{a}^2}{a^2}+2\frac{\dot{a}}{a}\frac{\dot{X}}{X}-\frac{\dot{X}^2}{X^2}-3\frac{\dot{Y}^2}{Y^2}+\frac{(k-\hat{k})^2r^2}{(1-kr^2)(1-\hat{k}r^2)^2}+\frac{2((1-kr^2)+2(1-\hat{k}r^2))}{(1-\hat{k}r^2)}\frac{Y\dot{Y}}{X^2}\left(\frac{\dot{Y}}{Y}-\frac{\dot{a}}{a}\right)\right]\\ Q^{(2)} \;&=\; \frac{l^2}{a^2}\left[8\frac{\dot{a}^2}{a^2}+2\frac{\dot{a}}{a}\frac{\dot{X}}{X}-6\frac{\dot{a}}{a}\frac{\dot{Y}}{Y}-\frac{\dot{X}^2}{X^2}-3\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}+\frac{(k-\hat{k})^2r^2(3-2\hat{k}r^2)}{(1-kr^2)(1-\hat{k}r^2)^2}\right.\end{aligned}$$

$$+\frac{((1-kr^2)+2(1-\hat{k}r^2))}{(1-\hat{k}r^2)}\frac{Y\dot{Y}}{X^2}\left(\frac{\dot{X}}{X}+3\frac{\dot{Y}}{Y}-4\frac{\dot{a}}{a}\right)\Bigg]$$

$$Q^{(3)} = \frac{l^2}{a^2}\Bigg[-4\frac{\dot{a}^2}{a^2}-4\frac{\dot{a}}{a}\frac{\dot{X}}{X}-\frac{\dot{X}^2}{X^2}+\frac{(k-\hat{k})^2r^2(3-2\hat{k}r^2)^2}{(1-kr^2)(1-\hat{k}r^2)^2}$$
$$+\frac{((1-kr^2)+2(1-\hat{k}r^2))}{(1-\hat{k}r^2)}\frac{Y\dot{Y}}{X^2}\left(4\frac{\dot{a}}{a}+2\frac{\dot{X}}{X}-\frac{((1-kr^2)+2(1-\hat{k}r^2))}{(1-\hat{k}r^2)}\frac{Y\dot{Y}}{X^2}\right)\Bigg]$$

$$Q^{(4)} = \frac{l^2}{a^2}\Bigg[-16\frac{\dot{a}^2}{a^2}+8\frac{\dot{a}}{a}\frac{\dot{X}}{X}+24\frac{\dot{a}}{a}\frac{\dot{Y}}{Y}-\frac{\dot{X}^2}{X^2}-6\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}-9\frac{\dot{Y}^2}{Y^2}+\frac{(k-\hat{k})^2r^2}{(1-kr^2)(1-\hat{k}r^2)^2}\Bigg]$$

$$Q^{(5)} = \frac{l^2}{a^2}\Bigg[-10\frac{\dot{a}^2}{a^2}+2\frac{\dot{a}}{a}\frac{\dot{X}}{X}+12\frac{\dot{a}}{a}\frac{\dot{Y}}{Y}-\frac{\dot{X}^2}{X^2}-6\frac{\dot{Y}^2}{Y^2}+\frac{(k-\hat{k})^2r^2(1+2(1-\hat{k}r^2)^2)}{(1-kr^2)(1-\hat{k}r^2)^2}$$
$$+2\frac{((1-kr^2)+2(1-\hat{k}r^2))}{(1-\hat{k}r^2)}\frac{Y\dot{Y}\dot{a}}{X^2a}-\frac{((1-kr^2)^2+2(1-\hat{k}r^2)^2)}{(1-\hat{k}r^2)^2}\frac{Y^2\dot{Y}^2}{X^2}\Bigg].$$

When $\hat{k}=k$ (motivated in Section V), the constraint equation can then be written for arbitrary c_1, c_2, c_3, c_4 and c_5 in relatively simple form:

$$\beta\left(\frac{\dot{a}^2}{a^2}+k\right)+\alpha\frac{Y^3}{a^2X}\left(\frac{\dot{Y}^2}{Y^2}+k\frac{X^2}{Y^2}\right)+\frac{a^2\mathcal{M}\sigma}{3l^2}-\frac{8\pi G\rho a^2}{3}-\frac{8\pi G\hat{\rho}XY^3}{3a^2}$$
$$=\frac{2\sigma\mathcal{M}_Q}{3}\Bigg[2(c_1-2c_2-2c_3-8c_4-5c_5)\frac{\dot{a}^2}{a^2}+(2c_1+5c_2-4c_3+8c_4+2c_5)\frac{\dot{a}}{a}\frac{\dot{X}}{X}$$
$$-(c_1+c_2+c_3+c_4+c_5)\frac{\dot{X}^2}{X^2}+\frac{3((c_2+8c_4+4c_5)X^2+2(-c_1+2c_3+c_5)Y^2)}{X^2}\frac{\dot{a}}{a}\frac{\dot{Y}}{Y}$$
$$-\frac{3((c_2+2c_4)X^2-2c_3Y^2)}{X^2}\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}-\frac{3((c_1+3c_4+2c_5)X^4-2c_1X^2Y^2+(3c_3+c_5)Y^4)}{X^4}\frac{\dot{Y}^2}{Y^2}\Bigg],$$

where σ and Q are given by (58) and (59), above. The special case of the 'concrete simple theory' of Milgrom is given in Section IV. Due to their length, we choose not to display the evolution equations here, which make up the remaining part of the field equations.

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