

The squeezed limit of the bispectrum in multi-field inflation

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Abstract. We calculate the squeezed limit of the bispectrum produced by inflation with multiple light fields. To achieve this we allow for different horizon exit times for each mode and calculate the intrinsic field-space three-point function in the squeezed limit using soft-limit techniques. We then use the δN formalism from the time the last mode exits the horizon to calculate the bispectrum of the primordial curvature perturbation. We apply our results to calculate the spectral index of the halo bias, $n_{\delta b}$, an important observational probe of the squeezed limit of the primordial bispectrum and compare our results with previous formulae. We give an example of a curvaton model with $n_{\delta b} \sim \mathcal{O}(n_s - 1)$ for which we find a 20% correction to observable parameters for squeezings relevant to future experiments. For completeness, we also calculate the squeezed limit of three-point correlation functions involving gravitons for multiple field models.

Keywords: Inflation, Non-Gaussianity, Bispectrum, Squeezed Limits, Soft Limits

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1 Introduction

Reliable calculations of the N-point correlation functions of the primordial curvature perturbation, ζ , are essential to confront models of inflation with present and future observational constraints. Since inflation can occur at energies as high as 10^{14}GeV , these correlation functions provide an unparalleled observational window into high energy physics, providing information about the fields and their interactions active in the early universe. In this paper we consider the three-point correlation function, and discuss how it can be calculated in a particular limit, known as the ‘squeezed limit’. Our primary aim is to provide clarity on how to accurately confront models of inflation with more than one light field against observations sensitive to this limit. This includes models in which more than one field supports inflation,

as well as spectator models such as the curvaton scenario and modulated reheating models – we refer to such models as multi-field models, and exclude cases in which additional heavy fields play a role.

The three-point function, parametrised by the bispectrum, is a function of three wave vectors which sum to zero as a result of momentum conservation, forming a triangle in momentum space. The squeezed limit refers to the case where one of the associated wave numbers is much smaller than the other two, such that the triangle looks ‘squeezed’. Calculations of the three-point function are now extremely mature, yet for technical reasons previous multiple field calculations have only been performed explicitly for the case of a mild hierarchy between wave numbers, as we will see. The highly squeezed limit is, however, very important both from an observational and a theoretical point of view.

The squeezed limit of the the bispectrum is the simplest possible example of a more general class of limits of correlation functions, referred to as soft limits. Soft limits occur when there exists a separation of scales in a physical problem. In the inflationary context, soft limits of correlation functions of the primordial curvature perturbation offer an exciting opportunity to confront theory with observations. For example, in the case of a single slow-roll field with canonical kinetic terms, Maldacena [1] found that the bispectrum of the curvature perturbation in the squeezed limit is purely determined by the tilt of the power spectrum, with the assumption of a Bunch-Davies initial state. The relation is: $12f_{\text{NL}}^{\text{sq}} = -5(n_s - 1)$, where $f_{\text{NL}}^{\text{sq}}$ is the squeezed limit of the reduced bispectrum, and $n_s - 1 = -0.032 \pm 0.006$ [2] is the spectral tilt. Creminelli & Zaldarriaga [3] (see also Ref. [4]), showed that Maldacena’s result holds even without the assumption of slow-roll¹ in all models with a Bunch-Davies initial state and where the classical solution is a dynamical attractor – the proof of which was later formalized by Cheung *et al.* [7]. Thus a detection of $f_{\text{NL}}^{\text{sq}} \gtrsim \mathcal{O}(0.01)$ would rule out all single field models with a Bunch-Davies initial state and where the classical solution is a dynamical attractor². For single field inflation, considerable work has also gone into studying more general soft limit results and providing further consistency relations amongst correlation functions [8–30]. Moreover, soft limits can be used to provide information about other fields present during inflation [31–37].

In the case of inflation with multiple light fields, in contrast to the single field case, *model independent* results such as the Maldacena consistency relation are not possible. However, many observables which constrain non-Gaussianity produced by multiple field inflation are particularly sensitive to soft limits. Examples include³ the spectral index of the halo bias [43] and CMB μ -distortions [44]. This is why explicit calculations of soft limits in the multiple field context are important when comparing *model dependent* predictions against observation.

In this paper, as a starting point to more general studies of soft limits in multiple field inflation, we explore the simplest case of the squeezed limit of the bispectrum. In most previous studies, the path to the bispectrum for multiple field models has been to first use the in-in formalism to calculate the three-point function of scalar field perturbations at a time soon after all modes have left the horizon, as was first done by Seery & Lidsey [45]. Next the δN formalism [46, 47] is applied to convert the field-space correlations to correlations

¹See, for example, [5, 6] for single field, but non-slow-roll models which obey Maldacena’s relation.

²See [8] where more general initial states are considered, and see [9] (and references therein) where non-attractor models are considered.

³One might naively think that the long-wavelength fluctuations used to model the observed CMB power asymmetry (see e.g. [38–42]) may also be an observational probe of soft limits – however, to describe the asymmetry, the long wavelength mode is required to be superhorizon, and so won’t be suitable as a soft momentum in the correlation functions considered in this paper.

of ζ [48]. The result of Seery & Lidsey for the three-point function, however, requires that there is not a large hierarchy between the three wavenumbers involved in the bispectrum, and thus that the modes of the bispectrum must cross the horizon during inflation at roughly the same time. As is clearly stated in their paper, therefore, their result is not valid in the highly squeezed limit where there is an appreciable difference in the exit times of different modes. Moreover, on using δN to convert from field-space fluctuations to ζ , one finds that the three-point function of the curvature perturbation involves copies of the two-point function of field fluctuations evaluated after all modes have exited the horizon. At this point in the procedure, previous explicit calculations have considered at most a mild hierarchy between the scales at which these two-point functions are evaluated [49, 50], with $|\log(k_1/k_3)| \sim \mathcal{O}(1)$, where k_1 is the long-wavelength mode, and k_3 is the short-wavelength mode.

Future experiments, together with the expected amount of squeezing they will be sensitive to, are shown in Table 1 [50]. These experiments will probe a hierarchy much larger than that allowed by previous theoretical calculations. Therefore, it is very important to have theoretical predictions for multiple field models valid in the highly squeezed limit, to be able to compare with observations.

Experiment	Dark Energy Survey	Euclid	μ -distortions
Squeezing	$\log(k_1/k_3) \sim -2$	$\log(k_1/k_3) \sim -8$	$\log(k_1/k_3) \sim -19$

Table 1: Future experiments and their observable range of scales.

In our study, therefore, we wish to relax the requirement of a mild hierarchy, and study models explicitly in the highly squeezed limit. To do so we will use a similar soft limit argument to Cheung *et al.* [7], but applied to calculate the three-point function of the scalar field perturbations. This result reduces to that of Seery & Lidsey for mild squeezing, but is valid in the highly squeezed limit and does not rely on slow-roll. In analogy with the single field case, where the three-point function depends on the tilt of the power spectrum, we find that the three-point function of the scalar field perturbations depends on derivatives of the field space two-point function with respect to the background value of the fields. Armed with this result, we further relax the usual assumption that the copies of the two-point function of the field perturbations, which appear in the three-point function for ζ , involve only a mild hierarchy of scales. This can easily be achieved by accounting for the evolution between horizon crossing times. We find that for any model in which previous applications of the δN formalism give rise to analytic results, analytic expressions for the highly squeezed limit are also possible. We finish by considering a specific curvaton model and compare our new squeezed limit formulae with previous expressions. We find significant differences for cases in which non-Gaussianity depends on scale.

The outline of this paper is as follows: in §2 we review the previous applications of the δN formalism, taking a pedagogical approach in this section which will be helpful when we come to extend previous work. In §3 we conveniently parametrise the evolution of the superhorizon field perturbations between exit times in terms of a ‘ Γ -matrix’, and calculate the three-point function of the scalar field perturbations at the time the last mode exits the horizon using soft-limit arguments. Putting these elements into the δN formalism we calculate the highly squeezed limit of the bispectrum of the curvature perturbation. In §4 we calculate the scale dependence of the squeezed limit of the reduced bispectrum, focussing on the spectral index of the halo bias. In §5 we provide explicit formulae for the Γ matrix, and investigate the concrete example of the mixed inflaton-curvaton scenario [51] with self-interactions [41].

Throughout this paper we work in units where $\hbar = c = 1$ and we set the reduced Planck mass $M_p = 1$.

2 Review of δN

In this section we review previous applications of the δN formalism to calculate the bispectrum of ζ . In doing so, we present key definitions and formulae essential for the subsequent sections.

2.1 Defining ζ

We will be considering Fourier modes which make up a given triangle of the bispectrum, labelled by three wave vectors \mathbf{k}_1 , \mathbf{k}_2 and \mathbf{k}_3 . These scales satisfy the condition $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ because of momentum conservation, and without loss of generality we order them such that $k_1 \leq k_2 \leq k_3$, where $k = |\mathbf{k}|$. We denote the times these modes cross the horizon by $\{t_1, t_2, t_3\}$, defined by the condition $k_1 = a(t_1)H(t_1)$ etc. and so we have $t_1 \leq t_2 \leq t_3$. For the bispectrum there are only two limiting possibilities for the relative magnitudes of wave numbers. The close to the equilateral case, with $k_1 \approx k_2 \approx k_3$, and the squeezed shape, with $k_1 \ll k_2 \approx k_3$. The amount of squeezing can be measured by the value of $\log(k_1/k_3)$, a negative number with greater absolute value indicating greater squeezing.

In the context of multiple field inflation, the cosmological perturbation of most interest is the uniform-density curvature perturbation, which coincides with the comoving curvature perturbation on superhorizon scales, where it is conserved if the system reaches an adiabatic limit [52, 53]. In general the curvature perturbation can be defined as a scalar perturbation to the spatial metric for a given foliation of spacetime, and moreover can be written as a local perturbation to the scale factor as [52, 54, 55]

$$a(t, \mathbf{x}) = a(t)e^{\psi(t, \mathbf{x})}. \quad (2.1.1)$$

Choosing the t -slicing to be such that the spatial hyper surfaces have uniform-density (UD) leads to $\psi_{\text{UD}}(t, \mathbf{x}) \equiv \zeta(t, \mathbf{x})$, while a flat t -slice is defined by $\psi_{\text{flat}}(t, \mathbf{x}) = 0$. The power spectrum, P_ζ , and bispectrum, B_ζ , of the curvature perturbation ζ are then defined by the two and three-point correlation functions

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle = P_\zeta(k_1)(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) \quad (2.1.2)$$

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = B_\zeta(k_1, k_2, k_3)(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3). \quad (2.1.3)$$

2.2 The δN formalism

Using the spatially dependent definition of the scale factor, the number of e-folds which occurs between two time slices of the perturbed spacetime, labeled by T and t_u respectively, is a function of position and is given by

$$N(T, t_u, \mathbf{x}) \equiv \int_T^{t_u} \frac{\dot{a}(t, \mathbf{x})}{a(t, \mathbf{x})} dt = \int_T^{t_u} H(t) dt + \psi(t_u, \mathbf{x}) - \psi(T, \mathbf{x}) \quad (2.2.1)$$

while the unperturbed number of e-folds is given by $N_0(T, t_u) \equiv \int_T^{t_u} H(t) dt$. Taking the T -slices to be flat, and the t_u -slices to be uniform density gives

$$\zeta(t_u, \mathbf{x}) = N(T, t_u, \mathbf{x}) - N_0(T, t_u) \equiv \delta N(t_u, \mathbf{x}) \quad (2.2.2)$$

which is the celebrated δN formula [46]. We note that δN doesn't depend on the initial time T [52]. Typically T is taken to be some time after all the modes involved in a given correlation function have exited the horizon (for the bispectrum this means $T > t_3$). If the system becomes adiabatic ζ is also independent of t_u and so the later time t_u should be the time at which adiabaticity is reached. If it doesn't become adiabatic, then t_u can be taken to be the time at which the correlations are required (see for example the discussion in Ref. [56]).

To use this formalism in practice, we must employ the separate universe approximation [47, 57] to cosmological perturbation theory. This states that on super-horizon scales positions in the perturbed universe evolve independently of one another, and do so according to the same equations as the unperturbed cosmology, so that every position can be treated as a 'separate universe'. The number of e-folds which occur at every position can therefore be calculated using the local conservation and Friedmann equations in that separate universe.

For an inflationary model with n scalar fields, ϕ_i , where i runs from one to n , we can split the field values on any flat slice into background and perturbed parts $\phi_i(t, \mathbf{x}) = \phi_i(t) + \delta\phi_i(t, \mathbf{x})$. Further demanding that the slow-roll equations of motion are satisfied at time T , such that $3H^2 = V(\phi_i)$ and $3H\dot{\phi}_i = -V_{,i}$, the initial conditions at time T for the perturbed cosmology become dependent only on $\phi_i(T, \mathbf{x})$. Consequently, the number of e-folds to any subsequent time slice becomes a function only of the initial field values, even if the cosmology evolves away from slow-roll. In particular, we can write

$$N(T, t_u, \mathbf{x}) = N((\phi_i(T, \mathbf{x})), t_u) = N((\phi_i(T) + \delta\phi_i(T, \mathbf{x})), t_u) \quad (2.2.3)$$

which gives

$$\zeta(\vec{x}) = N((\phi_i(T) + \delta\phi_i(T, \mathbf{x})), t_u) - N_0(\phi_i(T), t_u). \quad (2.2.4)$$

One can then Taylor expand in the initial flat slicing field perturbations, which in Fourier space leads to [46, 48]

$$\zeta_{\mathbf{k}_1} = N_i^{(T)} \delta\phi_{i, \mathbf{k}_1}^{(T)} + \frac{1}{2} N_{ij}^{(T)} (\delta\phi_i^{(T)} \star \delta\phi_j^{(T)})_{\mathbf{k}_1} + \dots \quad (2.2.5)$$

$$\text{where } N_i^{(T)} \equiv \frac{\partial N_0}{\partial \phi_i^{(T)}}, \quad (2.2.6)$$

and where \star denotes convolution. The additional vector (boldface) subscript indicates wavevector and the superscript in brackets is a shorthand indicating evaluation time

$$\phi_i^{(T)} \equiv \phi_i(T) \quad (2.2.7)$$

which we use from now on.

The correlation functions of the field perturbations are defined as

$$\langle \delta\phi_{i, \mathbf{k}_1}^{(T)} \delta\phi_{j, \mathbf{k}_2}^{(T)} \rangle = \Sigma_{ij}^{(T)}(k_1) (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) \quad (2.2.8)$$

$$\langle \delta\phi_{i, \mathbf{k}_1}^{(T)} \delta\phi_{j, \mathbf{k}_2}^{(T)} \delta\phi_{k, \mathbf{k}_3}^{(T)} \rangle = \alpha_{ijk}^{(T)}(k_1, k_2, k_3) (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (2.2.9)$$

such that the power spectrum [46] and bispectrum [48] of ζ are given by

$$P_\zeta(k_a) = N_i^{(T)} N_j^{(T)} \Sigma_{ij}^{(T)}(k_a) \quad (2.2.10)$$

$$B_\zeta(k_1, k_2, k_3) = N_i^{(T)} N_j^{(T)} N_k^{(T)} \alpha_{ijk}^{(T)}(k_1, k_2, k_3) \\ + N_i^{(T)} N_{jk}^{(T)} N_l^{(T)} [\Sigma_{ij}^{(T)}(k_1) \Sigma_{kl}^{(T)}(k_2) + (k_1 \rightarrow k_2 \rightarrow k_3)] \quad (2.2.11)$$

where the arrows indicate there are two additional terms formed by cyclic permutations.

One can then define the reduced bispectrum, conventionally denoted by $f_{\text{NL}}(k_1, k_2, k_3)$, by comparing the amplitude of the bispectrum to the square of the power spectrum as follows

$$f_{\text{NL}}(k_1, k_2, k_3) \equiv \frac{5}{6} \frac{B_\zeta(k_1, k_2, k_3)}{[P_\zeta(k_1)P_\zeta(k_2) + (k_1 \rightarrow k_2 \rightarrow k_3)]}. \quad (2.2.12)$$

2.3 δN for $k_1 \approx k_2 \approx k_3$

For light, canonically normalized fields, the field-space correlation function for a given wavenumber takes a very simple form at the time the wavenumber crosses the horizon⁴ (and from which the δN formalism can be employed), one finds [58, 59]

$$\Sigma_{ij}^{(1)}(k_1) = \frac{H^{(1)2}}{2k_1^3} \delta_{ij}. \quad (2.3.1)$$

In the most common application of δN , one assumes a near-equilateral momentum regime, where all the wavenumbers are approximately equal, $k_1 \approx k_2 \approx k_3$ and thus the horizon crossing times of the three wavenumbers involved in the bispectrum can be identified with a single time, t_* , such that $t_1 \approx t_2 \approx t_3 \approx t_*$. In this regime, it is then common to make the simple choice $T = t_*$.

For canonical slow-roll inflation, one can take⁵

$$\Sigma_{ij}^{(*)}(k_1) \approx \frac{H^{(*)2}}{2k_1^3} \delta_{ij}, \quad (2.3.2)$$

and similarly for the other wavenumbers, k_2 and k_3 . Moreover, Seery & Lidsey [45] used the in-in formalism to calculate $\alpha_{ijk}^{(*)}(k_1, k_2, k_3)$ in the close to equilateral momentum configuration at the time t_* . They found

$$\alpha_{ijk}^{(*)}(k_1, k_2, k_3) = \frac{4\pi^4}{k_1^3 k_2^3 k_3^3} \left(\frac{H^{(*)}}{2\pi} \right)^4 \sum_{6 \text{ perms}} \frac{\dot{\phi}_i^{(*)} \delta_{jk}}{4H^{(*)}} \left(-3 \frac{k_2^2 k_3^2}{k_t} - \frac{k_2^2 k_3^2}{k_t^2} (k_1 + 2k_3) + \frac{1}{2} k_1^2 - k_1 k_2^2 \right) \quad (2.3.3)$$

where the sum is over the six permutations of (ijk) while simultaneously rearranging the momenta k_1, k_2, k_3 such that the relative positioning of the k 's is respected. We emphasize that this result assumed that the three k modes crossed the horizon at roughly the same time and cannot be trusted when the crossing times are too different⁶, as in the case of the highly squeezed limit, which we will consider in this paper.

⁴ Strictly speaking this result is the form the two-point function takes once the decaying mode present at horizon crossing has been lost, written in terms of horizon crossing parameters.

⁵The leading order in slow-roll correction to this [58–60] is to replace $\delta_{ij} \mapsto \delta_{ij} + 2cu_{ij}$, where $c \equiv 2 - \log 2 - \gamma$, with γ the Euler-Mascheroni constant, and $u_{ij} = -(\log V)_{,ij}$, but we will not use this correction in this work.

⁶This is because in evaluating the time integrals of the in-in calculation, the time-dependent coefficients of the field perturbations were all evaluated at the common horizon crossing time, t_* , of the near-equilateral modes. This can't be done if the modes exit at largely different times, as is the case in the highly squeezed limit.

Sticking with the near-equilateral configuration, with $k_1 \approx k_2 \approx k_3$, and taking the δN initial time $T = t_*$, the results (2.3.1) and (2.3.3) can then be used together with Eqs. (2.2.10)-(2.2.12) to give the well-known expression [48], valid when $f_{\text{NL}}(k_*) \gg \mathcal{O}(\epsilon)$,

$$f_{\text{NL}} \approx \frac{5}{6} \frac{N_i^{(*)} N_{ij}^{(*)} N_j^{(*)}}{(N_l^{(*)} N_l^{(*)})^2}, \quad (2.3.4)$$

where (2.3.3) has been used to demonstrate that the second term in Eq. (2.2.11) must dominate over the first term if f_{NL} is to be large – i.e. the contribution from α is neglected [61]. Eq. (2.3.4) retains a dependence on k_* through the horizon crossing time t_* , and the relation that $k_* = a(t_*)H(t_*)$.

Given that it retains this dependence on the crossing scale, which is a measure of the overall scale of the bispectrum in the near-equilateral configuration, the bispectrum calculated above is sometimes referred to as quasi-local (to be contrasted with the local shape, in which f_{NL} is independent of all three k 's). Differentiating (2.3.4), Byrnes *et al.* [49] gave an expression for the tilt of f_{NL} , denoted $n_{f_{\text{NL}}}$, for equilateral triangles

$$n_{f_{\text{NL}}}^{(*)} \equiv \frac{d \log |f_{\text{NL}}|}{d \log k} = \frac{(n_i^{(*)} + n_j^{(*)} + n_{ij}^{(*)}) N_i^{(*)} N_{ij}^{(*)} N_j^{(*)}}{N_l^{(*)} N_{lm}^{(*)} N_m^{(*)}} - 4\hat{n} \quad (2.3.5)$$

$$\text{where } n_i^{(*)} \equiv \frac{1}{H} \frac{d \log N_i^{(*)}}{dt_*}, \quad n_{ij}^{(*)} \equiv \frac{1}{H} \frac{d \log N_{ij}^{(*)}}{dt_*} \quad \text{and } \hat{n} \equiv \frac{n_i^{(*)} N_i^{(*)} N_i^{(*)}}{N_j^{(*)} N_j^{(*)}}. \quad (2.3.6)$$

2.4 Beyond the near-equilateral configurations

In the present paper we aim to go beyond the case where $k_1 \approx k_2 \approx k_3$, to study the highly squeezed limit. Before we do, we must stress that although most previous studies use the formulae presented in the previous sub-section, some attempts have already been made to relax this near-equilateral assumption, which we briefly review now. Full details will appear when we compare our expressions to earlier ones in later sections.

In Ref. [49], in addition to the consideration of scale dependence discussed above, Byrnes *et al.* considered the more general dependence of the reduced bispectrum on k_1 , k_2 and k_3 , using the δN framework. They did so by solving the slow-roll field equations for scalar fluctuations to first order in the quantity $|\log(k_1/k_3)|$ about the time t_* . This was used to allow a mild hierarchy of scales. The result becomes inaccurate as this quantity grows with increased squeezing. Moreover, they did not calculate the squeezed limit of α_{ijk} , assuming it to contribute negligibly for an observable bispectrum of ζ – as is the case for near-equilateral configurations. We give more details on their approach and when it is accurate in §4.2 and Appendix D.

In Ref. [50] Dias *et al.* took a different approach. They used a next to leading order expression in slow-roll for the bispectrum at the time t_* , calculated by the same authors in Ref. [60]. Such next to leading order expressions automatically provides information on scale dependence, and allows for a mild hierarchy of scales. They went on to use their expressions for an investigation of the spectral index of the halo bias. The Dias *et al.* result includes the form of α_{ijk} in the case of a mild hierarchy, but also takes the form of an expansion in $|\log(k_1/k_3)|$ with only the first term calculated, and so again becomes inaccurate as the squeezing is increased. Their result is given explicitly in Appendix E.

A major aim of this paper is to determine when previous expressions can be trusted, and when the squeezing becomes large enough that the expressions we present are necessary.

3 The squeezed limit of the bispectrum with δN

In this paper, therefore, we wish to extend previous work to explicitly calculate the bispectrum at time t_u in the case of a truly squeezed momentum configuration, $k_1 \ll k_2 \approx k_3$, which will involve (perhaps very) different horizon exit times $t_1 \ll t_2 \approx t_3$. To do so we will employ expression (2.2.11), setting $T = t_3$, but will fully account for the different horizon crossing times, and also show how to calculate $\alpha_{ijk}^{(3)}$ for highly squeezed configurations. In contrast to the previous section, the new objects we need to calculate are $\Sigma_{ij}^{(3)}(k_1)$, the field perturbation two-point function for modes which cross the horizon at time t_1 evaluated at the later time t_3 , and $\alpha_{ijk}^{(3)}(k_1, k_2, k_3)$ for the $k_1 \ll k_2 \approx k_3$ configuration. Many authors have discussed how to propagate the field-space two-point correlation function past horizon crossing [58–60], but to the best of our knowledge no one has then employed these techniques to explicitly investigate the bispectrum of ζ beyond cases of mild squeezing. Moreover, to the best of our knowledge we are the first to attempt a calculation of $\alpha_{ijk}^{(3)}(k_1, k_2, k_3)$ in the squeezed limit⁷.

We proceed to discuss how to calculate each of these objects now, and then combine them to compute the bispectrum of ζ in the squeezed limit for multiple field models.

3.1 Field evolution between different crossing times

The first task is to calculate $\Sigma_{ij}^{(3)}(k_1)$. To do this we need to account for the evolution of the field-space perturbations on a flat hypersurface with wavenumber k_1 between the time this wavenumber crosses the horizon and the later time t_3 . We choose to approach this problem in a manner closely connected with the δN framework, and which therefore provides a unified treatment of the overall problem. It will also make transparent the cases in which analytic progress can be made.

Returning to the separate universe picture, perturbations at time t_3 are defined by

$$\delta\phi_i^{(3)}(\mathbf{x}) \equiv \phi_i^{(3)}(\mathbf{x}) - \phi_i^{(3)} \quad (3.1.1)$$

where $\phi_i^{(3)}(\mathbf{x})$ is the true value, and $\phi_i^{(3)}$ without an \mathbf{x} argument is the homogeneous background value. In general, for a given i , $\phi_i^{(3)}(\mathbf{x})$ will depend on the value of all the fields and field velocities at the earlier time t_1 . Assuming, however, slow-roll between t_1 and t_3 , the fields velocities become functions of the fields. One can write therefore

$$\delta\phi_i^{(3)}(\mathbf{x}) = \phi_i^{(3)}(\phi_j^{(1)}(\mathbf{x})) - \phi_i^{(3)} \quad (3.1.2)$$

$$= \phi_i^{(3)}(\phi_j^{(1)} + \delta\phi_j^{(1)}(\mathbf{x})) - \phi_i^{(3)}. \quad (3.1.3)$$

In analogy with the δN expression, one can Taylor expand this in the perturbation $\delta\phi_j^{(1)}(\mathbf{x})$ to give

$$\delta\phi_i^{(3)}(\mathbf{x}) = \delta\phi_j^{(1)}(\mathbf{x}) \frac{\partial\phi_i^{(3)}}{\partial\phi_j^{(1)}} + \dots \quad (3.1.4)$$

In this paper we won't need the higher order terms. We use the shorthand

$$\Gamma_{ij}^{(3,1)} \equiv \frac{\partial\phi_i^{(3)}}{\partial\phi_j^{(1)}} \quad (3.1.5)$$

⁷Although α in the squeezed limit was given in [62, 63] in terms of OPE coefficients.

which we will often refer to as the ‘ Γ -matrix’. We would like to emphasise that the ‘ Γ -matrix’ used here is constructed using the background cosmology and is analogous to $N_i^{(T)}$ in (2.2.6). In Fourier space we have

$$\delta\phi_{i,\mathbf{k}}^{(3)} = \Gamma_{ij}^{(3,1)}\delta\phi_{j,\mathbf{k}}^{(1)} + \dots \quad (3.1.6)$$

Objects like the Γ matrices have been used by a number of authors in the past⁸ [56, 65–71], and we note the properties: $\Gamma_{ij}^{(a,a)} = \delta_{ij}$, $\Gamma_{ki}^{(c,b)}\Gamma_{ij}^{(b,a)} = \Gamma_{kj}^{(c,a)}$ and so $\Gamma_{ki}^{(c,a)}\Gamma_{ij}^{(a,c)} = \delta_{kj}$ for arbitrary times t_a, t_b, t_c . Moreover since ζ in (2.2.5) is independent of T , at first order we find that

$$N_i^{(b)}\delta\phi_{i,\mathbf{k}}^{(b)} = N_i^{(a)}\delta\phi_{i,\mathbf{k}}^{(a)} \quad (3.1.7)$$

and so we must have

$$N_i^{(b)} = N_j^{(a)}\Gamma_{ji}^{(a,b)}. \quad (3.1.8)$$

We can now use the Γ matrices to relate correlation functions at different times, and in particular using Eq. (2.3.1) we find

$$\boxed{\Sigma_{ij}^{(3)}(k_1) = \Gamma_{ik}^{(3,1)}\Gamma_{jk}^{(3,1)}\Sigma_{ij}^{(1)}(k_1) = \Gamma_{ik}^{(3,1)}\Gamma_{jk}^{(3,1)}\frac{H^{(1)2}}{2k_1^3}} \quad (3.1.9)$$

While at one level Eq. (3.1.9) has simply swapped the unknown correlation matrix $\Sigma_{ij}^{(3)}(k_1)$ for the unknown $\Gamma_{ik}^{(3,1)}$ matrix, the point is that the problem has been reduced to a matter of solving the background cosmology, which must be done anyway to apply the δN formalism. Moreover, in all models where the derivatives of N can be calculated analytically, analytic expressions should also exist for the required Γ matrix, as we will see in an explicit example below. On the other hand, if numerical tools are needed to calculate the derivatives of N , similar tools can be applied to calculate $\Gamma_{ik}^{(3,1)}$. The formulation in terms of Γ is therefore very convenient for explicitly studying the squeezed limit.

3.2 The field-space bispectrum in the squeezed limit

The next step is to calculate $\alpha_{ijk}^{(3)}(k_1, k_2, k_3)$ in the squeezed limit. To do so we can adapt an idea used by Maldacena [1] and subsequently other authors [3, 7] in the single field context. The calculation relies on the following simplifying assumption: the long wavelength mode k_1 which exits the horizon at the first exit time, t_1 , can only affect the much shorter wavelength modes, which at that time are still deep inside the horizon, through its effect on the background cosmology. The k_1 mode shifts the background field configuration, and hence changes the background cosmology which is felt by the two modes which exit later – producing a correlation between the k_1 mode and the two-point function of the k_2, k_3 modes in the shifted background.

In the single field case the calculation is performed directly with the modes of ζ in the comoving gauge, and the shift in background cosmology induced by the long wavelength mode exiting the horizon can be thought of as a *shift in the time* at which the short modes exit.

⁸ Γ can also be generalised to be k -dependent, as in [64].

This leads to the three-point correlation function of ζ being related to the tilt of the two-point function of the short modes, and to the famous consistency relation of Maldacena⁹.

In the multiple field case we are calculating the three-point function of field perturbations in the flat gauge, and the shift in the background cosmology induced by the long wavelength mode exiting the horizon can be thought of as a *shift in the background field values*. This won't result in a simple consistency relation for ζ , but does allow analytic progress to be made in calculating α in the highly squeezed limit.

To calculate α we follow a similar calculation to that in Cheung *et al.* [7] for $\langle\zeta\zeta\zeta\rangle$ in the single field case. We show a derivation of the squeezed limit of α which relies on slow-roll - though this derivation can be easily generalised to relax the assumption of slow-roll. In the end the final result we get for the squeezed limit of $\langle\zeta\zeta\zeta\rangle$ must rely on a slow-roll approximation at the time of the last horizon crossing, t_3 , because we will be using the $\delta\mathcal{N}$ formalism at that time.

We begin our calculation in position space, denoting short-wavelength perturbations with a superscript S , and long-wavelength perturbations with a superscript L . We ask how a short-wavelength two-point function $\langle\delta\phi_j^{S(3)}(\mathbf{x}_2)\delta\phi_k^{S(3)}(\mathbf{x}_3)\rangle$ at t_3 is affected by a further long-wavelength perturbation $\delta\phi_i^{L(3)}(\mathbf{x})$. To proceed, we evaluate the long-wavelength fluctuation $\delta\phi_i^{L(3)}(\mathbf{x})$ at the midpoint $\mathbf{x} = \mathbf{x}_+ \equiv (\mathbf{x}_2 + \mathbf{x}_3)/2$. Then we Taylor expand the short-wavelength two-point function around the value it would have in the absence of the long-wavelength perturbation, denoting this value by a subscript 0. One finds

$$\begin{aligned} \langle\delta\phi_j^{S(3)}(\mathbf{x}_2)\delta\phi_k^{S(3)}(\mathbf{x}_3)\rangle\Big|_{\delta\phi_m^{L(3)}(\mathbf{x}_+)} &= \langle\delta\phi_j^{S(3)}(\mathbf{x}_2)\delta\phi_k^{S(3)}(\mathbf{x}_3)\rangle\Big|_0 \\ &+ \delta\phi_m^{L(3)}(\mathbf{x}_+)\langle\delta\phi_j^{S(3)}(\mathbf{x}_2)\delta\phi_k^{S(3)}(\mathbf{x}_3)\rangle_{,m}\Big|_0 + \dots \end{aligned} \quad (3.2.1)$$

where the subscript $,m$ denotes the partial derivative with respect to the background field $\phi_m^{(3)}$. At this stage we employ the simple soft limit argument discussed above and assume that in the squeezed limit the three-point function in momentum space will receive its largest contribution from the correlation between the long-wavelength mode, which effectively shifts the background cosmology, and the short-wavelength two-point function in the shifted background. This leads to

$$\langle\delta\phi_i^{L(3)}(\mathbf{x}_1)\delta\phi_j^{S(3)}(\mathbf{x}_2)\delta\phi_k^{S(3)}(\mathbf{x}_3)\rangle \approx \langle\delta\phi_i^{L(3)}(\mathbf{x}_1)\langle\delta\phi_j^{S(3)}(\mathbf{x}_2)\delta\phi_k^{S(3)}(\mathbf{x}_3)\rangle\Big|_{\delta\phi_m^{L(3)}(\mathbf{x}_+)} \quad (3.2.2)$$

$$\approx \langle\delta\phi_i^{L(3)}(\mathbf{x}_1)\delta\phi_m^{L(3)}(\mathbf{x}_+)\langle\delta\phi_j^{S(3)}(\mathbf{x}_2)\delta\phi_k^{S(3)}(\mathbf{x}_3)\rangle_{,m}\Big|_0 \quad (3.2.3)$$

$$\approx \langle\delta\phi_i^{L(3)}(\mathbf{x}_1)\delta\phi_m^{L(3)}(\mathbf{x}_+)\rangle\langle\delta\phi_j^{S(3)}(\mathbf{x}_2)\delta\phi_k^{S(3)}(\mathbf{x}_3)\rangle_{,m}\Big|_0 \quad (3.2.4)$$

$$\approx \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_+)+i\mathbf{q}\cdot(\mathbf{x}_2-\mathbf{x}_3)} \Sigma_{im}^{(3)}(p)\Sigma_{jk,m}^{(3)}(q)\Big|_0. \quad (3.2.5)$$

⁹Note that the consistency relation holds for all single field models with a Bunch-Davies initial state and where the classical solution is a dynamical attractor.

In what follows we will drop the subscript $|_0$ for notational ease. Now we insert $1 = \int d^3\mathbf{k}_1 \delta(\mathbf{k}_1 + \mathbf{p})$ to get

$$\begin{aligned} & \langle \delta\phi_i^{L(3)}(\mathbf{x}_1) \delta\phi_j^{S(3)}(\mathbf{x}_2) \delta\phi_k^{S(3)}(\mathbf{x}_3) \rangle \\ & \approx \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{q}}{(2\pi)^3} e^{-i\mathbf{k}_1 \cdot \mathbf{x}_1 - i\mathbf{p} \cdot \mathbf{x}_2 + i\mathbf{q} \cdot (\mathbf{x}_2 - \mathbf{x}_3)} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{p}) \Sigma_{im}^{(3)}(p) \Sigma_{jk,m}^{(3)}(q). \end{aligned} \quad (3.2.6)$$

Changing the integration variables from \mathbf{p}, \mathbf{q} to $\mathbf{k}_2 = \frac{1}{2}\mathbf{p} - \mathbf{q}$ and $\mathbf{k}_3 = \frac{1}{2}\mathbf{p} + \mathbf{q}$ we find

$$\begin{aligned} & \langle \delta\phi_i^{L(3)}(\mathbf{x}_1) \delta\phi_j^{S(3)}(\mathbf{x}_2) \delta\phi_k^{S(3)}(\mathbf{x}_3) \rangle \\ & \approx \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{d^3\mathbf{k}_3}{(2\pi)^3} e^{-i\mathbf{k}_1 \cdot \mathbf{x}_1 - i\mathbf{k}_2 \cdot \mathbf{x}_2 - i\mathbf{k}_3 \cdot \mathbf{x}_3} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \Sigma_{im}^{(3)}(k_1) \Sigma_{jk,m}^{(3)}(k_3), \end{aligned} \quad (3.2.7)$$

where we have used the relation $\mathbf{q} = \mathbf{k}_3(1 + \mathcal{O}(k_1/k_3))$. From (3.2.7) we can now read off the squeezed limit of the momentum space three-point function of the field perturbations

$$\lim_{k_1 \ll k_3, k_2} \langle \delta\phi_{i,\mathbf{k}_1}^{(3)} \delta\phi_{j,\mathbf{k}_2}^{(3)} \delta\phi_{k,\mathbf{k}_3}^{(3)} \rangle \approx (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \Sigma_{im}^{(3)}(k_1) \Sigma_{jk,m}^{(3)}(k_3) \quad (3.2.8)$$

and so

$$\boxed{\lim_{k_1 \ll k_3, k_2} \alpha_{ijk}^{(3)}(k_1, k_2, k_3) \approx \Sigma_{im}^{(3)}(k_1) \Sigma_{jk,m}^{(3)}(k_3)} \quad (3.2.9)$$

This is a very general expression for the squeezed limit of α , independent of the multiple field model, relating the squeezed limit of the three-point function of the field perturbations to the two-point function of the field perturbations and its derivatives with respect to the background fields. It is one of the principal results of this paper. It only relied on slow-roll at time t_3 in (3.2.1) where the Taylor expansion was done only in terms of the field, rather than both the field and its velocity. If one promoted the indices i to run over both fields and field velocities, then (3.2.9) with this more general index notation would still hold, independent of whether slow-roll was valid at time t_3 .

At this stage we can make our result more explicit in the case where slow-roll is valid between the horizon exit times t_1 and t_3 . In this case, we can use the Γ evolution between crossing times, which by (3.1.9) and (2.3.1) gives

$$\begin{aligned} \alpha_{ijk}^{(3)}(k_1, k_2, k_3) & \approx \Gamma_{il}^{(3,1)} \Gamma_{mn}^{(3,1)} \Sigma_{ln}^{(1)}(k_1) \Sigma_{jk,m}^{(3)}(k_3) \approx \Gamma_{il}^{(3,1)} \Gamma_{ml}^{(3,1)} \delta_{jk} \frac{H^{(1)2}}{2k_1^3} \frac{[H^{(3)2}]_{,m}}{2k_3^3} \\ & \approx -\Gamma_{il}^{(3,1)} \Gamma_{ml}^{(3,1)} \delta_{jk} \frac{H^{(1)2}}{2k_1^3} \frac{H^{(3)2}}{2k_3^3} \frac{d\phi_m^{(3)}}{dN}. \end{aligned} \quad (3.2.10)$$

where the final line uses the slow-roll equations of motion for the background fields, and for clarity we note that $dN \equiv d \log a$ is the local measure of e-folding time (and is not related to the δN formula).

3.3 The squeezed limit of the bispectrum of ζ

We now have all the ingredients we need to calculate the bispectrum of ζ (2.2.11) in the squeezed limit for multiple field inflation, and since we now use the δN formalism, the results

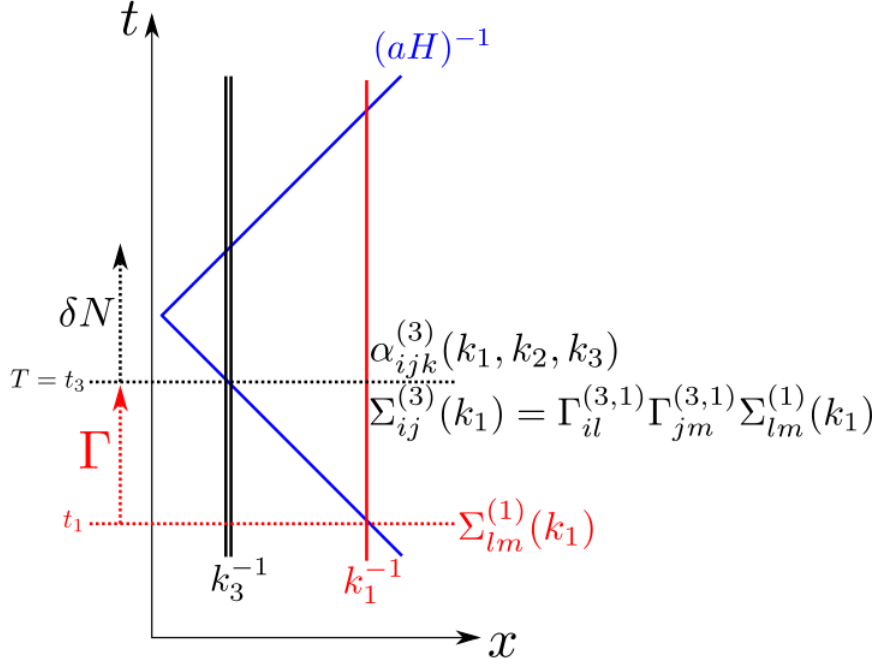


Figure 1: Schematic picture of δN evolution from time $T = t_3$ onwards, using $\Sigma_{ij}^{(3)}(k_1)$ from Subsection 3.1 and $\alpha_{ijk}^{(3)}(k_1, k_2, k_3)$ from Subsection 3.2. The Γ -evolution of field perturbations occurs between t_1 and t_3 . The blue line is the comoving Hubble radius, the solid red line is the inverse of the squeezed wavenumber, k_1 , while the solid black lines are the inverses of the other wavenumbers, k_2 and k_3 .

which follow in this and subsequent sections are only valid in slow-roll models. The schematic picture of our approach is shown in Figure 1.

Putting $\alpha_{ijk}^{(3)}(k_1, k_2, k_3)$ from (3.2.10) and $\Sigma_{ij}^{(3)}(k_1)$ from (3.1.9) into (2.2.11), with $T \mapsto t_3$, gives

$$\begin{aligned} \lim_{k_1 \ll k_2, k_3} B_\zeta(k_1, k_2, k_3) \approx & -N_i^{(3)} N_j^{(3)} N_k^{(3)} \Gamma_{ik}^{(3,1)} \Gamma_{mk}^{(3,1)} \frac{H^{(1)2}}{2k_1^3} \frac{H^{(3)2}}{2k_3^3} \frac{d\phi_m^{(3)}}{dN} \\ & + N_i^{(3)} N_{jk}^{(3)} N_k^{(3)} \left[2\Gamma_{im}^{(3,1)} \Gamma_{jm}^{(3,1)} \frac{H^{(1)2}}{2k_1^3} \frac{H^{(3)2}}{2k_3^3} + \delta^{ij} \left(\frac{H^{(3)2}}{2k_3^3} \right)^2 \right] \end{aligned} \quad (3.3.1)$$

and since $k_1 \ll k_3$ this can be simplified to

$$\begin{aligned} \lim_{k_1 \ll k_2, k_3} B_\zeta(k_1, k_2, k_3) \approx & -N_i^{(3)} N_j^{(3)} N_k^{(3)} \Gamma_{ik}^{(3,1)} \Gamma_{mk}^{(3,1)} \frac{H^{(1)2}}{2k_1^3} \frac{H^{(3)2}}{2k_3^3} \frac{d\phi_m^{(3)}}{dN} \\ & + 2N_i^{(3)} N_{jk}^{(3)} N_k^{(3)} \Gamma_{im}^{(3,1)} \Gamma_{jm}^{(3,1)} \frac{H^{(1)2}}{2k_1^3} \frac{H^{(3)2}}{2k_3^3} \end{aligned} \quad (3.3.2)$$

which is one of the main results of this paper. In Appendix A we check that it reduces to the Maldacena result [1] in the single field limit. In Appendix B we check that our result for α in (3.2.10) agrees with the result of Seery & Lidsey [45] if we take a near-equilateral limit.

We can then form the reduced bispectrum in the squeezed limit

$$\lim_{k_1 \ll k_2, k_3} \frac{6}{5} f_{\text{NL}}(k_1, k_2, k_3) \approx - \frac{N_i^{(3)} \Gamma_{ik}^{(3,1)} \Gamma_{mk}^{(3,1)}}{2N_l^{(3)} N_n^{(3)} \Gamma_{lq}^{(3,1)} \Gamma_{nq}^{(3,1)}} \frac{d\phi_m^{(3)}}{dN} + \frac{N_i^{(3)} N_{jk}^{(3)} N_k^{(3)} \Gamma_{im}^{(3,1)} \Gamma_{jm}^{(3,1)}}{N_l^{(3)} N_n^{(3)} N_p^{(3)} N_p^{(3)} \Gamma_{lq}^{(3,1)} \Gamma_{nq}^{(3,1)}}.$$
(3.3.3)

For convenience we define

$$B^{[\alpha]}(k_1, k_2, k_3) \equiv -N_i^{(3)} N_j^{(3)} N_j^{(3)} \Gamma_{ik}^{(3,1)} \Gamma_{mk}^{(3,1)} \frac{H^{(1)2}}{2k_1^3} \frac{H^{(3)2}}{2k_3^3} \frac{d\phi_m^{(3)}}{dN}$$
(3.3.4)

$$\frac{6}{5} f_{\text{NL}}^{[\alpha]}(k_1, k_2, k_3) \equiv - \frac{N_i^{(3)} \Gamma_{ik}^{(3,1)} \Gamma_{mk}^{(3,1)}}{2N_l^{(3)} N_n^{(3)} \Gamma_{lq}^{(3,1)} \Gamma_{nq}^{(3,1)}} \frac{d\phi_m^{(3)}}{dN}$$
(3.3.5)

so that superscript $[\alpha]$ labels the contribution from the term involving α .

As we noted in §2.3 for the near-equilateral configuration, the term coming from α can only be of order slow-roll [45], and so if the reduced bispectrum of ζ is to be sufficiently large to be observable by present or next generation experiments, it will be dominated by the second line of Eq. (2.2.11) [61].

For our highly squeezed case it's not immediately clear whether the contribution of $f_{\text{NL}}^{[\alpha]}(k_1, k_2, k_3)$ must be of order ϵ . One might think that because at least one component of the vector N_i is of order $\epsilon^{-1/2}$, and because as many Γ matrices appear in the numerator as in the denominator, that the likely order is indeed ϵ . However, while this is likely the most common outcome, because $\Gamma_{ij}^{(3,1)}$ is a matrix and appears with different contractions in the numerator compared with the denominator, it is not impossible that the contractions may conspire to make $f_{\text{NL}}^{[\alpha]}(k_1, k_2, k_3)$ larger than $\mathcal{O}(\epsilon)$. In this paper we do not perform a general study, but armed with this explicit expression for $f_{\text{NL}}^{[\alpha]}(k_1, k_2, k_3)$ we can consider the amplitude of this term on a case by case basis.

Finally, for completeness, we note that in addition to purely scalar correlations, it is possible to calculate the three-point functions involving both the scalar curvature perturbation and gravitons in the squeezed limit. In Appendix C we use similar techniques to those employed above to find the scalar-graviton three-point functions for multiple field models, noting that the graviton-only three-point function will be the same as for the single field case, as given in [1].

4 Scale dependence

4.1 Spectral index of the halo bias

An important quantity for large scale structure surveys is the *scale-dependent halo bias*, $\delta b(k_1)$ [43], which is sensitive to how the ratio of the bispectrum to the power spectrum, $B_\zeta(k_1, k_2, k_3)/P_\zeta(k_1)$, scales with the squeezed momentum k_1 , where $k_1 \ll k_2 \approx k_3$. This is

captured by the *spectral index of the halo bias*, $n_{\delta b} \equiv n_{\text{sq}} - (n_s - 1)$, where n_{sq} is the tilt of the squeezed limit of the bispectrum with respect to its squeezed momentum k_1 :

$$\lim_{k_1 \ll k_2, k_3} B_\zeta(k_1, k_2, k_3) \sim \frac{\mathcal{B}_\zeta}{k_1^3 k_3^3} \left(\frac{k_1}{k_s}\right)^{n_{\text{sq}}} \quad (4.1.1)$$

with k_s some arbitrary scale and \mathcal{B}_ζ roughly constant. Dias et al. [50] investigated $n_{\delta b}$ in multiple field inflation in the case where $|\log(k_1/k_3)|$ is of order a few, as discussed further in Appendix E. Here we would like to explore the highly squeezed case, and find if there are significant differences.

Our results of §3 for the bispectrum in the highly squeezed limit can be applied to calculate $n_{\delta b}$ for large values of $|\log(k_1/k_3)|$. Differentiating $\log(k_1^3 B_\zeta)$ in (3.3.2) with respect to $\log k_1$ we find

$$n_{\text{sq}} = - \frac{\left[N_i^{(3)} N_q^{(3)} (N_q^{(3)} V_{,j}^{(3)} + 6 N_{qj}^{(3)} H^{(3)2}) \right] \left[2\epsilon^{(1)} L_{ij}^{(3,1)} - P_{ij,1}^{(3,1)} \right]}{\left[N_m^{(3)} N_r^{(3)} (N_r^{(3)} V_{,n}^{(3)} + 6 N_{rn}^{(3)} H^{(3)2}) \right] L_{mn}^{(3,1)}} \quad (4.1.2)$$

where we have defined

$$L_{ij}^{(3,1)} \equiv \Gamma_{im}^{(3,1)} \Gamma_{jm}^{(3,1)} \quad (4.1.3)$$

$$P_{ij,1}^{(3,1)} \equiv \frac{dL_{ij}^{(3,1)}}{d \log k_1} = - \frac{V_{,l}^{(1)}}{V^{(1)}} \left(\Gamma_{ik,l}^{(3,1)} \Gamma_{jk}^{(3,1)} + \Gamma_{ik}^{(3,1)} \Gamma_{jk,l}^{(3,1)} \right) \quad (4.1.4)$$

$$\Gamma_{ik,l}^{(3,1)} \equiv \frac{\partial}{\partial \phi_l^{(1)}} \Gamma_{ik}^{(3,1)} \quad (4.1.5)$$

and we have used the slow-roll equations to relate $\dot{\phi}_i$ to $V_{,i}$ in (4.1.4).

The spectral index of the halo bias requires also the tilt of the power spectrum at k_1

$$n_s - 1 = \frac{d \log(k_1^3 P_\zeta(k_1))}{d \log k_1} = -2 \frac{N_i^{(3)} N_j^{(3)} \Gamma_{ik}^{(3,1)} \Gamma_{jl}^{(3,1)} M_{kl}^{(1)}}{N_m^{(3)} N_n^{(3)} L_{mp}^{(3,1)}} \quad (4.1.6)$$

where

$$M_{ij}^{(1)} \equiv \epsilon^{(1)} \delta_{ij} + u_{ij}^{(1)} \quad (4.1.7)$$

$$\text{and } u_{ij}^{(1)} \equiv \frac{V_{,i}^{(1)} V_{,j}^{(1)}}{V^{(1)2}} - \frac{V_{,ij}^{(1)}}{V^{(1)}}. \quad (4.1.8)$$

This leads to the expression for the spectral index of the halo bias

$$\boxed{n_{\delta b} = - \frac{\left[N_i^{(3)} N_q^{(3)} (N_q^{(3)} V_{,j}^{(3)} + 6 N_{qj}^{(3)} H^{(3)2}) \right] \left[2\epsilon^{(1)} L_{ij}^{(3,1)} - P_{ij,1}^{(3,1)} \right]}{\left[N_m^{(3)} N_r^{(3)} (N_r^{(3)} V_{,n}^{(3)} + 6 N_{rn}^{(3)} H^{(3)2}) \right] L_{mn}^{(3,1)}} + 2 \frac{N_i^{(3)} N_j^{(3)} \Gamma_{ik}^{(3,1)} \Gamma_{jl}^{(3,1)} M_{kl}^{(1)}}{N_m^{(3)} N_n^{(3)} L_{mn}^{(3,1)}}} \quad (4.1.9)$$

valid for large values of $|\log(k_1/k_3)|$, and is another key result of this paper. In Appendix E we show that when we take our exit times to be roughly equal (4.1.9) reduces to the same

form of (17) of [50], given in the appendix as (E.0.2), and in §5.3 we compare approaches for a concrete model.

We note that if $f_{\text{NL}}^{[\alpha]}(k_1, k_2, k_3)$ is negligible, then our expression simplifies to

$$n_{\delta b} = -\frac{N_i^{(3)} N_q^{(3)} N_{qj}^{(3)} \left(2\epsilon^{(1)} L_{ij}^{(3,1)} - P_{ij,1}^{(3,1)} \right)}{N_m^{(3)} N_r^{(3)} N_{rn}^{(3)} L_{mn}^{(3,1)}} + 2 \frac{N_i^{(3)} N_j^{(3)} \Gamma_{ik}^{(3,1)} \Gamma_{jl}^{(3,1)} M_{kl}^{(1)}}{N_m^{(3)} N_n^{(3)} L_{mp}^{(3,1)}}. \quad (4.1.10)$$

4.2 Tilts of the reduced bispectrum in the squeezed limit

In a similar manner to Refs. [49, 72–74], one can study the scaling of the squeezed limit of the reduced bispectrum (3.3.3) with respect to k_1 and k_3 . In the squeezed limit $k_2 \approx k_3$, so one can parametrize how this depends on scale by differentiating with respect to k_1 or k_3 leading to the tilts

$$n_{f_{\text{NL}}}^X \equiv \frac{\partial \log |f_{\text{NL}}(k_1, k_2, k_3)|}{\partial \log X} \quad (4.2.1)$$

where $X = k_1$ or k_3 , with the other momenta held fixed in the derivative. Note that $n_{f_{\text{NL}}}^{k_1} = n_{\delta b}$ since the k_1 dependence of f_{NL} is captured by the scaling of the ratio $B_\zeta(k_1, k_2, k_3)/P_\zeta(k_1)$ with k_1 . We calculate $n_{f_{\text{NL}}}^{k_3}$ in (F.0.3) of Appendix F. We note that each of these results are different to the scale dependence of the equilateral configuration of Byrnes *et al.* [49] given in Eq. (2.3.5), since we are working only in the squeezed limit. These authors also considered the scale dependence of near-equilateral triangles, writing the wavenumbers as $k_a = \alpha_a \tilde{k}$, and varying with respect to \tilde{k} , keeping the α_a constant. Their near-equilateral result has the same form as (2.3.5), with all $*$'s replaced by the exit time of a pivot scale k_p not too different from the k 's. Their calculation relies on an expansion to first order in $|\log(k_1/k_3)|$ and as a result is only valid for a small squeezing. We calculate the \tilde{k} tilt using our approach in Eq (F.0.8) of Appendix F, where more details on the expansion of [49] can also be found. We emphasise again that all of our expressions can be employed for a large hierarchy of scales.

5 Employing the Γ formalism in concrete models

In this paper we have advocated the use of the Γ matrices to allow the bispectrum of ζ to be calculated in the highly squeezed limit for multiple field models. These matrices allow us to account for the evolution of the inflationary fields between horizon crossing times, and help provide compact expressions for the bispectrum and its scale dependence, including the contribution from the field-space three-point function. To be of use, however, we must be able to calculate the Γ matrices in concrete settings.

In any given model, we could solve for $\Gamma_{ij}^{(3,1)}$ numerically. Either by solving the equation of motion it satisfies [56] from the initial conditions at t_1 of $\Gamma_{ij}^{(3,1)} = \delta_{ij}$, which follow from its definition, or by solving the background equations and implying finite differences as was done for the derivatives of N in, for example, Refs. [56, 75, 76]. However, what makes this parametrisation particularly useful is that in any model for which the derivatives of N can be calculated analytically, the Γ matrices also admit analytic solutions. This allows us to compute the amplitude of the highly squeezed limit of the bispectrum in such models and to compare this against previous squeezed limit expressions, as well as against the amplitude of the bispectrum for near-equilateral triangles.

We therefore calculate Γ analytically in §5.1 for sum-separable potentials. Then we consider the importance of Γ in specific settings. In §5.2 we show that in all single-source models, where only one field (which need not be the inflaton) contributes towards the curvature perturbation, the effect of the Γ matrices is to cause the reduced bispectrum in the squeezed limit to become independent of the squeezed momentum k_1 , and that because the bispectrum scales with k_1 in exactly the same way as the power spectrum, the spectral index of the halo bias will be zero, and so not observable. Then in §5.3 we consider a specific multiple-source model where more interesting results are possible. In particular, we examine a mixed curvaton-inflaton model [41, 77–84] allowing for self-interaction terms for the curvaton. For this specific model with our given parameter choices, we find that in highly squeezed cases relevant for future observations, the bispectrum is suppressed by the Γ matrices at a level of 20% when compared to using the existing expressions of Byrnes *et al.* [49] and Dias *et al.* [50] for this model. In addition, we find that the spectral index of the halo bias is enhanced at a level of 20% in this model compared with the results that would be obtained with previous expressions which assume a mild hierarchy of scales.

5.1 Calculating Γ : sum-separable potential

In the δN framework, all models which are analytically tractable have a common feature. This is that the inflationary potentials are of separable form, either sum or product separable [75, 85, 86], or of the generalised sum-separable form of Ref [87]. This is true not only for models in which the evolution is tracked during inflation, but also for models in which the post inflationary evolution is important such as the curvaton model. In this work we will focus on sum-separable potentials and confirm that we can derive analytic formulae for the Γ matrices, using similar techniques as those originally used for derivatives of N in Ref. [75].

We will initially work with the simple case of a two field model: ϕ , χ and write the potential as $W(\phi, \chi) = U(\phi) + V(\chi)$. The slow-roll equations are then

$$3H\dot{\phi} = -U_{,\phi}, \quad 3H\dot{\chi} = -V_{,\chi}, \quad 3H^2 = W. \quad (5.1.1)$$

Using these slow-roll equations we have

$$\frac{d\phi}{d\chi} = \frac{\dot{\phi}}{\dot{\chi}} = \frac{U_{,\phi}}{V_{,\chi}}. \quad (5.1.2)$$

The number of e-folds between a flat hypersurface at time t_1 and another flat hypersurface at time t_3 is

$$\Delta N \equiv \int_{t_1}^{t_3} H dt = \int_{\phi^{(1)}}^{\phi^{(3)}} \frac{H}{\dot{\phi}} d\phi = - \int_{\phi^{(1)}}^{\phi^{(3)}} \frac{W}{U_{,\phi}} d\phi \quad (5.1.3)$$

and using (5.1.2) this gives

$$\Delta N = - \int_{\phi^{(1)}}^{\phi^{(3)}} \frac{U}{U_{,\phi}} d\phi - \int_{\chi^{(1)}}^{\chi^{(3)}} \frac{V}{V_{,\chi}} d\chi. \quad (5.1.4)$$

We also have, by (5.1.2), that

$$\int_{\phi^{(1)}}^{\phi^{(3)}} \frac{1}{U_{,\phi}} d\phi = \int_{\chi^{(1)}}^{\chi^{(3)}} \frac{1}{V_{,\chi}} d\chi. \quad (5.1.5)$$

To determine $\Gamma_{ij}^{(3,1)}$ we need to find the following four derivatives of flat hypersurface fields

$$\Gamma_{ij}^{(3,1)} = \begin{pmatrix} \frac{\partial\phi^{(3)}}{\partial\phi^{(1)}} & \frac{\partial\phi^{(3)}}{\partial\chi^{(1)}} \\ \frac{\partial\chi^{(3)}}{\partial\phi^{(1)}} & \frac{\partial\chi^{(3)}}{\partial\chi^{(1)}} \end{pmatrix}. \quad (5.1.6)$$

For flat hypersurfaces, by definition, if we vary our position on the initial slice, then ΔN does not alter. This implies that the derivative of ΔN with respect to field values on the initial flat hypersurface satisfies: $\Delta N_{,\phi^{(1)}} = 0 = \Delta N_{,\chi^{(1)}}$. Employing (5.1.4) then leads to two independent equations relating the derivatives. Moreover, differentiating (5.1.5) with respect to the field values on the initial flat hypersurface yields two further independent equations relating the derivatives. Between these four equations we can then solve for each derivative and hence determine $\Gamma_{ij}^{(3,1)}$.

We begin with $\Delta N_{,\phi^{(1)}} = 0$ and $\Delta N_{,\chi^{(1)}} = 0$ giving respectively

$$-\frac{U^{(3)}}{U_{,\phi}^{(3)}} \frac{\partial\phi^{(3)}}{\partial\phi^{(1)}} - \frac{V^{(3)}}{V_{,\chi}^{(3)}} \frac{\partial\chi^{(3)}}{\partial\phi^{(1)}} = -\frac{U^{(1)}}{U_{,\phi}^{(1)}} \quad (5.1.7)$$

$$-\frac{U^{(3)}}{U_{,\phi}^{(3)}} \frac{\partial\phi^{(3)}}{\partial\chi^{(1)}} - \frac{V^{(3)}}{V_{,\chi}^{(3)}} \frac{\partial\chi^{(3)}}{\partial\chi^{(1)}} = -\frac{V^{(1)}}{V_{,\chi}^{(1)}}. \quad (5.1.8)$$

Next differentiating (5.1.5) with respect to $\phi^{(1)}$ and $\chi^{(1)}$ gives respectively

$$\frac{1}{U_{,\phi}^{(3)}} \frac{\partial\phi^{(3)}}{\partial\phi^{(1)}} - \frac{1}{V_{,\chi}^{(3)}} \frac{\partial\chi^{(3)}}{\partial\phi^{(1)}} = \frac{1}{U_{,\phi}^{(1)}} \quad (5.1.9)$$

$$\frac{1}{U_{,\phi}^{(3)}} \frac{\partial\phi^{(3)}}{\partial\chi^{(1)}} - \frac{1}{V_{,\chi}^{(3)}} \frac{\partial\chi^{(3)}}{\partial\chi^{(1)}} = -\frac{1}{V_{,\chi}^{(1)}}. \quad (5.1.10)$$

Solving, we find

$$\Gamma_{ij}^{(3,1)} = \begin{pmatrix} \frac{\partial\phi^{(3)}}{\partial\phi^{(1)}} & \frac{\partial\phi^{(3)}}{\partial\chi^{(1)}} \\ \frac{\partial\chi^{(3)}}{\partial\phi^{(1)}} & \frac{\partial\chi^{(3)}}{\partial\chi^{(1)}} \end{pmatrix} = \begin{pmatrix} \frac{U_{,\phi}^{(3)}}{U_{,\phi}^{(1)}} \frac{(U^{(1)}+V^{(3)})}{W^{(3)}} & \frac{U_{,\phi}^{(3)}}{V_{,\chi}^{(1)}} \frac{(V^{(1)}-V^{(3)})}{W^{(3)}} \\ \frac{V_{,\chi}^{(3)}}{U_{,\phi}^{(1)}} \frac{(U^{(1)}-U^{(3)})}{W^{(3)}} & \frac{V_{,\chi}^{(3)}}{V_{,\chi}^{(1)}} \frac{(V^{(1)}+U^{(3)})}{W^{(3)}} \end{pmatrix}, \quad (5.1.11)$$

which is the analytic calculation of Γ for two-field sum-separable potentials we required. Note that i labels the rows and j labels the columns.

For a model with n fields one can calculate the Γ matrix by generalizing the two-field case. In what follows we will suspend the summation convention of repeated indices, and where a sum should be taken we explicitly state this. We take the sum-separable potential $W(\phi_1, \phi_2, \dots, \phi_n) = \sum_l W_l(\phi_l)$. We have, in analogy with (5.1.4),

$$\Delta N = - \sum_l \int_{\phi_l^{(1)}}^{\phi_l^{(3)}} \frac{W_l}{W_l'} d\phi_l \quad (5.1.12)$$

where $W_l' \equiv \partial W_l / \partial \phi_l$ for which we can write the equations $\Delta N_{,\phi_j^{(1)}} = 0$ in analogy to (5.1.7) and (5.1.8). We also have the relations, in analogy to (5.1.5), for all l, i

$$\int_{\phi_l^{(1)}}^{\phi_l^{(3)}} \frac{d\phi_l}{W_{,\phi_l}} = \int_{\phi_i^{(1)}}^{\phi_i^{(3)}} \frac{d\phi_i}{W_{i,\phi_i}} \quad (5.1.13)$$

for which we can take derivatives with respect to $\phi_j^{(1)}$ giving equations analogous to (5.1.9) and (5.1.10). Combining these equations with those from $\Delta N_{,\phi_j^{(1)}} = 0$, after some algebra we arrive at

$$\Gamma_{ij}^{(3,1)} = \frac{\partial \phi_i^{(3)}}{\partial \phi_j^{(1)}} = \frac{W_i'^{(3)} (W_j^{(1)} - W_j^{(3)})}{W_j'^{(1)} W^{(3)}} + \delta_{ij} \frac{W_i'^{(3)}}{W_i'^{(1)}} \quad (5.1.14)$$

which reduces to (5.1.11) in the two-field case.

5.2 Single-source models

In what follows we will assume for now that in the highly squeezed limit we can neglect $B_\zeta^{[\alpha]}(k_1, k_2, k_3)$, the contribution to the bispectrum from intrinsic field-space three-point function. This needs to be checked on a case by case basis.

In the single-source case where only one field, which we denote χ , contributes to ζ , the bispectrum (3.3.2) (without $B_\zeta^{[\alpha]}(k_1, k_2, k_3)$) simplifies to

$$\lim_{k_1 \ll k_2, k_3} B_\zeta(k_1, k_2, k_3) \approx 2N_\chi^{(3)} N_{\chi\chi}^{(3)} N_\chi^{(3)} \Gamma_{\chi m}^{(3,1)} \Gamma_{\chi m}^{(3,1)} \frac{H^{(1)2}}{2k_1^3} \frac{H^{(3)2}}{2k_3^3}. \quad (5.2.1)$$

Now we can use (3.1.8) to relate $N_\chi^{(3)}$ to $N_i^{(1)}$ through the Γ 's to give the bispectrum and reduced bispectrum as

$$\lim_{k_1 \ll k_2, k_3} B_\zeta(k_1, k_2, k_3) \approx 2N_m^{(1)} N_{\chi\chi}^{(3)} N_m^{(1)} \frac{H^{(1)2}}{2k_1^3} \frac{H^{(3)2}}{2k_3^3} \quad (5.2.2)$$

$$\approx 2P_\zeta(k_1) P_\zeta(k_3) \frac{N_{\chi\chi}^{(3)}}{N_\chi^{(3)2}} \quad (5.2.3)$$

$$\lim_{k_1 \ll k_2, k_3} \frac{6}{5} f_{\text{NL}}(k_1, k_2, k_3) \approx \frac{N_{\chi\chi}^{(3)}}{N_\chi^{(3)2}}. \quad (5.2.4)$$

We note that the effect of the Γ matrices has been to make the bispectrum proportional to $2P_\zeta(k_1)P_\zeta(k_3)$, so that the reduced bispectrum is independent of k_1 , and coincidentally of the same form as would be derived in the near-equilateral regime assuming $t_* \approx t_3$. Because the bispectrum scales with k_1 in exactly the same way as the power spectrum, the spectral index of the halo bias will be zero, and so will not be observable unless the intrinsic contribution we have neglected is important.

Note that the results of this subsection did not rely on assuming a sum-separable potential – all single source models satisfy (5.2.4). In the next subsection we will look at a specific multiple-source model which does rely on the assumption of a sum-separable potential.

5.3 Multiple-source models: the mixed curvaton-inflaton model

We now consider multiple-source models, and will consider the concrete example of a curvaton model for which both the inflaton, ϕ , and curvaton, χ , contribute towards ζ . In order to make analytic progress we will need expressions for the derivatives of N in this multiple-source model to combine with our analytic expression for Γ , valid for sum-separable potentials.

More detailed studies of the curvaton scenario can be found in the literature, see e.g. [77–82]. Here we just give a very brief account outlining our parameter choices and quote the results for the N derivatives. Our main focus will instead be on the differences between our new expressions compared to existing formulae in the literature. We will show these differences graphically, where the plots are produced using the expressions for the N derivatives that follow, together with the Γ matrix for sum-separable potentials.

For the inflaton and curvaton, we take the potential

$$W(\phi, \chi) = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 + \lambda\chi^n, \quad (5.3.1)$$

where the curvaton is given a self-interaction with coupling λ , and $n > 2$. If we are to see a difference between the bispectrum in the highly squeezed limit and the near-equilateral limit, it is natural to expect that we will need the field configuration to differ significantly between different exit times, otherwise the $\delta\phi_i$ would not evolve between t_1 and t_3 and we would just get $\Gamma_{ij}^{(3,1)} = \delta_{ij}$. This means that we expect to see interesting effects for models which generate a significantly scale-dependent non-Gaussianity. In the present model, this means that the curvaton field, while light, cannot be completely frozen. To achieve this we take the mass of the curvaton and inflaton to be the same $m_\chi = m_\phi$.

The self-interacting curvaton model can give significant scale dependence of f_{NL} , [41, 88], and analytic expressions for the derivatives of N are available in the limit of weak self-interaction $s_k \equiv 2\lambda\chi_k^{n-2}/m_\chi^2 \ll 1$ [41]. Here, and in what follows, a subscript k indicates the result is a function of the value of the curvaton and/or inflaton fields at the time, t_k , when a given k -mode exits.

The relevant N derivatives are given by [41, 48, 51]

$$N_\phi = \frac{1}{\sqrt{2\epsilon_\phi}|_k} \quad (5.3.2)$$

$$N_\chi = \frac{2r_{\text{dec}}}{3} \left. \frac{\sigma'_{\text{osc}}}{\sigma_{\text{osc}}} \right|_k \quad (5.3.3)$$

$$N_{\chi\chi} = \frac{r_{\text{dec}}}{3} \left[\frac{\sigma''_{\text{osc}}}{\sigma_{\text{osc}}} + \left(\frac{\sigma'_{\text{osc}}}{\sigma_{\text{osc}}} \right)^2 \right] \Big|_k \quad (5.3.4)$$

$$\text{where } \sigma_{\text{osc}}(\chi_k) \propto \chi_k \left(1 + \frac{n}{2}s_k \right)^{-1/(n-2)}. \quad (5.3.5)$$

The parameter r_{dec} denotes the value of $3\rho_\chi/(3\rho_\chi + 4\rho_\gamma)$ at the time of curvaton decay, where ρ is the energy density and the subscript labels the species, with γ denoting radiation. In the example which follows we take to be $r_{\text{dec}} = 0.02$. We have neglected $N_{\phi\phi}, N_{\phi\chi}$ which are much smaller than $N_{\chi\chi}$ for the parameter choices considered.

We take the initial condition¹⁰ $\phi_0 = 16$ which leads to 63.5 e-folds of inflation and assume that all scales which exit the horizon after this time are within the horizon today and potentially observable. We will also take $\chi_0 = 2 \times 10^{-3}$ to generate a significant non-Gaussianity. To see a significantly scale-dependent non-Gaussianity we take the self-interaction with $n = 6$ and $\lambda = 0.2$, which gives $s_k \approx 0.07$.

The power spectrum for this model is shown in Figure 2a, and the tilt of the power spectrum is shown in Figure 2b. In order to see a large scale dependence of f_{NL} for this

¹⁰Note we are still working in units where the reduced Planck mass is set to one.

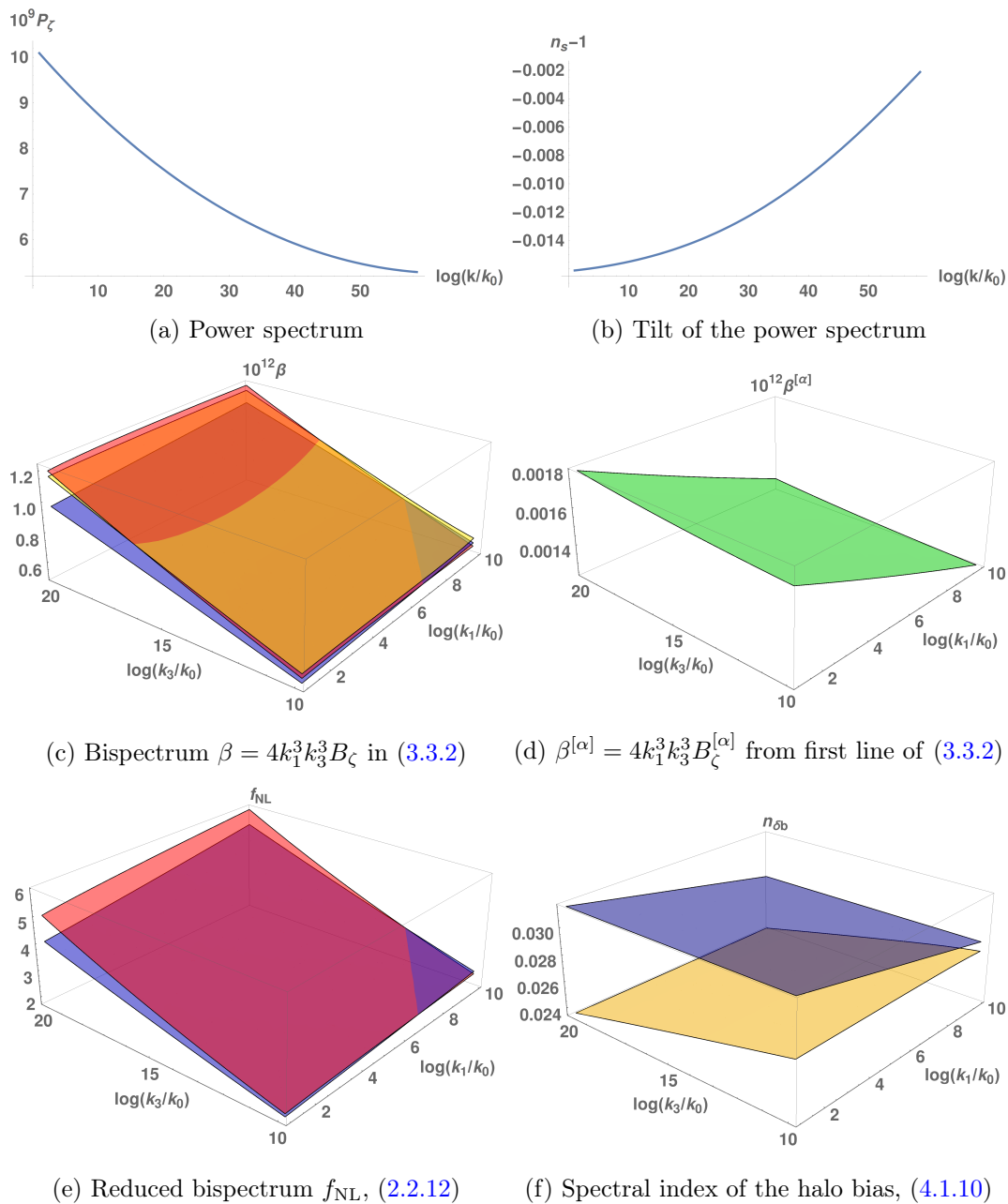


Figure 2: In the 3d plots, the blue surfaces are our expressions with Γ 's. Red surfaces are Byrnes *et al.* [49] expressions. Yellow surfaces are the Dias *et al.* [50] expressions. The near-equilateral regime is on the far right of each plot. The highly squeezed limit is on the far left of each plot. The green surface shows the $[\alpha]$ contribution is negligible for this model.

simple model, we took the curvaton mass to be as large as the inflaton mass¹¹. However this has the adverse effect of making the tilt of the power spectrum to not be as red (negative) as

¹¹ A potential issue for the mixed curvaton-inflaton model with equal masses is that since both fields begin oscillating at exactly the same time, the curvaton energy density remains subdominant for a long time – see [89] for a discussion on this.

observations suggest. This could be fixed by beginning with a more complicated model, and not requiring the curvaton mass to be as large, but this is not our focus here, and instead we focus on the results for the bispectrum.

We can now compare our new expressions with Γ 's against existing expressions in the literature. In the 3d plots, the near-equilateral limit $\log(k_1/k_3) \approx 0$ is on the far right of each plot, and the squeezing increases as you move away from this corner. The highly squeezed limit, where $\log(k_1/k_3) \sim -20$ is on the far left of each plot.

In Figure 2c we plot the bispectrum $\beta \equiv 4k_1^3 k_3^3 B_\zeta$ to show how the bispectrum scales with $\log(k_1/k_0)$ and $\log(k_3/k_0)$ where $k_0 = H^{(0)}$ at the initial time t_0 , where we have set $a_0 = 1$. The blue surface uses our new expression (3.3.2), the red surface uses (96) of Byrnes *et al.* [49] given in (D.0.1), and the yellow surface uses (14) of Dias *et al.* [50], given in (E.0.1). In the highly squeezed limit we see a percentage difference of about 20%. The near-equilateral results are within a few percent of each other for each of the three expressions – the small difference is because previous authors have included an additional constant term which appears at next order in slow-roll in their expression for Σ at horizon crossing discussed in Footnote 5 and Appendix D.

In Figure 2d we plot the contribution to the bispectrum from the field-space three-point function $\beta^{[\alpha]} = 4k_1^3 k_3^3 B_\zeta^{[\alpha]}$ where $B_\zeta^{[\alpha]}$ is the first line of (3.3.2). We see that this is a factor of 10^3 smaller than the total bispectrum for all scales considered, and so all the $[\alpha]$ terms in observables can be neglected for this model.

In Figure 2e we plot the reduced bispectrum, where the blue surface is our new expression (3.3.3) and the red surface is (96) of [49] divided by the factor $2P_\zeta(k_1)P_\zeta(k_3)$, i.e. the curly brackets of (D.0.1). Dias *et al.* don't give an explicit expression for the reduced bispectrum, so we don't plot this here. Again, in the highly squeezed limit we see a percentage difference of about 20%, and close agreement in the near-equilateral configuration.

Finally, in Figure 2f we plot the spectral index of the halo bias. Blue is our new expression (4.1.10), and the yellow surface is (17) of [50], given in (E.0.2). Byrnes *et al.* [49] don't give an expression for the spectral index of the halo bias so we don't plot this here. Similarly to the other observables we see a percentage difference of about 20% in the highly squeezed limit. We see agreement to within a few percent in the near-equilateral limit, the small discrepancy being due to the Dias *et al.* expression being evaluated at the exit time of $k_t \equiv k_1 + k_2 + k_3$ compared to ours being evaluated at t_3 .

The 20% level difference for all these observables for a squeezing of $|\log(k_1/k_3)| \sim \mathcal{O}(20)$ can be estimated heuristically as arising from a scale-dependence of the reduced bispectrum of $\mathcal{O}(0.01)$, multiplied by the squeezing. We thus expect similar levels of discrepancy for any model where the scale-dependence of the reduced bispectrum is of similar order to the scale-dependence of the power spectrum, when considering a squeezing of $|\log(k_1/k_3)| \sim \mathcal{O}(20)$, and that the percentage difference will scale linearly with the scale-dependence of the bispectrum.

The expected levels of squeezing for three future experiments were shown in Table 1 of the introduction. The results of this section show that inclusion of the effects of field evolution can be important when computing the theoretical predictions of a model for comparison against observations for large squeezing, even for this simple model.

6 Conclusion

In this work we calculated the squeezed limit of the bispectrum of the curvature perturbation for multiple field inflation. Different scales involved in one triangle of the bispectrum will

exit the horizon at different times, and previous analytic expressions have been limited to a mild squeezing where the exit times are roughly equal. Observations can at present probe only a mildly squeezed limit, but future large-scale structure surveys and observations of CMB μ -distortions will be able to probe a highly squeezed limit. It is important, therefore, to have accurate theoretical predictions for this highly squeezed limit in order to ensure the uncertainty in the prediction is less than the uncertainty in the data. For certain models, our results give a correction at a level of 20% in the highly squeezed limit compared to extrapolating existing expressions, valid only in the mildly squeezed limit, to the highly squeezed limit.

In order to study this highly squeezed limit, we suggested using the elegant Γ matrix formalism to account for the evolution on superhorizon scales of the perturbations between exit times. We also calculated the intrinsic three-point function of the field perturbations, α , in the highly squeezed limit for the first time. We did so by appealing to a soft limit argument, previously used in the single field context for the curvature perturbation. Together these elements allowed us to extend δN expressions for the bispectrum of ζ to account for multiple crossing times. From this expression, we then obtained the reduced bispectrum and the spectral index of the halo bias. Working with a specific model, the mixed inflaton-curvaton scenario with a self-interacting curvaton, we checked the difference in our theoretical prediction, valid for a large squeezing, against existing predictions, valid for a mild squeezing. As would be expected we found significant differences, especially in the highly squeezed limit for the cases in which there is significant scale dependence in the reduced bispectrum.

The overall aim of this paper was to provide clarity in how to confront models of inflation against observations sensitive to the squeezed limit of the bispectrum. Our results however, could also be useful to check numerical methods in the squeezed limit. From a theoretical and observational point of view, soft-limits – of which the squeezed limit is the simplest example – are of considerable interest, and in future work we hope to consider soft limits of higher n -point correlation functions for multiple field inflation, using a similar approach. For $n > 3$, the story can be more interesting than for the bispectrum, where the only soft limit is when a single external momenta becomes small. Firstly, one can consider multiple-soft limits, where more than one momentum becomes smaller than the others. Moreover, one can also consider the collapsed limit, when an internal momentum becomes soft. Observing these higher-point functions may be even harder than for the bispectrum, but it is still important to have theoretical predictions for multiple field inflation to constrain models using observational limits, in particular to search for deviations from single field inflation.

Finally, we mention that the intrinsic term in the bispectrum coming from α was negligible compared to the other term in the case study presented. Calculating this expression explicitly allowed us to determine this, but more work is required to investigate whether this term can ever be as large as, or even dominate over, the other contribution.

Acknowledgements

We would like to thank Chris Byrnes, Shailee Imrith, David Seery and Raquel Ribeiro for taking the time to comment extensively on a draft of this work. ZK is supported by an STFC studentship. DJM is supported by a Royal Society University Research Fellowship.

A Reduction to single field case

We should check that our expression (3.3.2) for B_ζ reduces to the Maldacena single field squeezed limit result [1] in the case of one slowly rolling scalar field, given by

$$\lim_{k_1 \ll k_2, k_3} B_\zeta(k_1, k_2, k_3) \approx -(n_s^{(3)} - 1)P_\zeta(k_1)P_\zeta(k_3) \quad (\text{A.0.1})$$

with $n_s - 1 = 2\eta_V - 6\epsilon_V$, and the superscript (3) denoting evaluation at time t_3 when k_3 exits. Here $\epsilon_V \equiv \frac{1}{2}(V'/V)^2$ and $\eta_V \equiv V''/V$ are the potential slow-roll parameters.

Thus we begin with a slowly rolling single field ϕ . In this case we have

$$N_\phi = -\frac{H}{\dot{\phi}} = \frac{1}{\sqrt{2\epsilon_V}}, \quad \frac{N_{\phi\phi}}{N_\phi^2} = 2\epsilon_V - \eta_V, \quad \frac{d}{d\phi^{(3)}} = \frac{1}{\dot{\phi}^{(3)}} \frac{d}{dt_3} \quad (\text{A.0.2})$$

and the Γ matrix is just the number 1. Then (3.3.2) simplifies to

$$\begin{aligned} & \lim_{k_1 \ll k_2, k_3} B_\zeta(k_1, k_2, k_3) \\ & \approx \frac{(N_\phi^{(1)})^2 H^{(1)2}}{2k_1^3} \frac{(N_\phi^{(3)})^2 H^{(3)2}}{2k_3^3} \left(-\frac{\dot{\phi}^{(3)}}{H^{(3)}} \right) \frac{2}{H^{(3)}} \left(\frac{1}{\dot{\phi}^{(3)}} \frac{dH^{(3)}}{dt_3} \right) \\ & + 2 \frac{(N_\phi^{(1)})^2 H^{(1)2}}{2k_1^3} \frac{(N_\phi^{(3)})^2 H^{(3)2}}{2k_3^3} \frac{N_{\phi\phi}^{(3)}}{(N_\phi^{(3)})^2} \\ & \approx [2\epsilon_V^{(3)} + 2(2\epsilon_V^{(3)} - \eta_V^{(3)})] P_\zeta(k_1) P_\zeta(k_3) \\ & \approx -(n_s^{(3)} - 1) P_\zeta(k_1) P_\zeta(k_3) \end{aligned} \quad (\text{A.0.3})$$

as required.

B Recovering Seery & Lidsey result in near-equilateral limit

A further important check is that the near-equilateral limit of our result for α , (3.2.10) goes over to the result of Seery & Lidsey [45], given earlier in (2.3.3), valid when k_1 is small, but not so small as to change the exit times appreciably. That is, we want to check that

$$\begin{aligned} \lim_{t_1 \rightarrow t_3} N_i^{(3)} N_j^{(3)} N_k^{(3)} \alpha_{ijk}^{(3)}(k_1, k_2, k_3) &= \lim_{k_1 \ll k_2, k_3} \frac{4\pi^4}{k_1^3 k_2^3 k_3^3} \left(\frac{H^{(3)}}{2\pi} \right)^4 N_i^{(3)} N_j^{(3)} N_k^{(3)} \\ &\times \sum_{6 \text{ perms}} \frac{\dot{\phi}_i^{(3)}}{4H^{(3)}} \delta_{jk} \left(-3 \frac{k_2^2 k_3^2}{k_t} - \frac{k_2^2 k_3^2}{k_t^2} (k_1 + 2k_3) + \frac{1}{2} k_1^2 - k_1 k_2^2 \right). \end{aligned} \quad (\text{B.0.1})$$

Beginning with the RHS, one can do the sum over all six permutations, then take the slightly squeezed limit to get the RHS equal to

$$-N_i^{(3)} N_j^{(3)} N_j^{(3)} \frac{H^{(3)4}}{4k_1^3 k_3^3} \frac{d\phi_i^{(3)}}{dN} \quad (\text{B.0.2})$$

which is exactly the $t_1 \rightarrow t_3$ limit of $\alpha_{ijk}^{(3)}$ in (3.2.10), in which $\Gamma_{ij}^{(3,1)} \rightarrow \delta_{ij}$, contracted with $N_i^{(3)} N_j^{(3)} N_k^{(3)}$.

C Squeezed limits of graviton correlators

Three-point functions involving gravitons (tensors) are likely significantly harder to detect observationally than those just involving scalars. Nonetheless they are interesting to calculate with a view to observations in the more distant future, and from a theoretical perspective.

Maldacena found squeezed limits of scalar-graviton and graviton-graviton three-point functions in the case of a single scalar field [1]¹². Here we use our soft-limit argument to calculate these in the multiple field case.

First order graviton perturbations, denoted by γ , are gauge invariant in contrast to scalar perturbations. They are defined as the transverse traceless perturbations of the spatial metric, h_{IJ} , such that

$$h_{IJ} = a^2(t) [(1 + 2\zeta)\delta_{IJ} + \gamma_{IJ}] \quad (\text{C.0.1})$$

where $\gamma_{II} = 0 = \partial_I \gamma_{IJ}$. We can Fourier expand γ as

$$\gamma_{IJ} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \epsilon_{IJ}^s(k) \gamma_{\mathbf{k}}^s(t) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (\text{C.0.2})$$

where s, r indices label the polarization of the graviton, and the polarization tensors $\epsilon_{IJ}^s(k)$ satisfy $\epsilon_{II}^s(k) = 0 = k_I \epsilon_{IJ}^s(k)$ and $\epsilon_{IJ}^s(k) \epsilon_{IJ}^r(k) = 2\delta_{sr}$. The two-point function of the graviton is given by

$$\langle \gamma_{\mathbf{k}_1}^s \gamma_{\mathbf{k}_2}^r \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) P_\gamma^{sr(1)}(k_1) \quad (\text{C.0.3})$$

$$P_\gamma^{sr(1)}(k_1) = \delta_{sr} \frac{2H^{(1)2}}{2k_1^3}. \quad (\text{C.0.4})$$

We now consider the squeezed limit of three-point correlation functions involving gravitons. When the soft mode k_1 is that of a ζ , there will be a correlation between $\zeta_{\mathbf{k}_1}$ and two γ 's by way of a similar soft limit argument applied now to a $\zeta\gamma\gamma$ correlator, giving the result

$$\lim_{k_1 \ll k_2, k_3} \langle \zeta_{\mathbf{k}_1} \gamma_{\mathbf{k}_2}^s \gamma_{\mathbf{k}_3}^r \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) N_i^{(3)} \Gamma_{in}^{(3,1)} \Gamma_{ml}^{(3,1)} \Sigma_{nl}^{(1)}(k_1) P_{\gamma,m}^{sr(3)}(k_3) \quad (\text{C.0.5})$$

which can be contrasted with the corresponding single field result given in [1]. In the single field case, the result is proportional to the tilt of the graviton power spectrum, providing another consistency relation between observables. Now in the multiple field case, this consistency relation no longer holds, but instead the squeezed limit three-point function is related to how the two-point γ correlator depends on the background scalar fields ϕ_i .

When the soft mode is instead a graviton, we can refer to Maldacena's argument that when the $\zeta_{\mathbf{k}_2}, \zeta_{\mathbf{k}_3}$ modes exit, the graviton with momentum k_1 exited much earlier and is already frozen, so that fluctuations of ζ at time t_3 will be those in the deformed geometry of the background $\gamma_{\mathbf{k}_1}$ mode. The main effect of the deformation of the background geometry is to change the $\delta_{IJ} k_3^I k_3^J \rightarrow \delta_{IJ} k_3^I k_3^J - \gamma_{IJ} k_3^I k_3^J$ inside the correlation function of the two ζ 's (equivalently in the second order action for ζ_{k_3}). Putting this into a soft limit type argument gives

$$\lim_{k_1 \ll k_2, k_3} \langle \gamma_{\mathbf{k}_1}^s \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle \approx -(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) P_\gamma^{sr(1)}(k_1) \epsilon_{IJ}^r(k_1) k_3^I k_3^J \frac{d}{dk_3^2} P_\zeta(k_3). \quad (\text{C.0.6})$$

¹²Maldacena and Pimentel [90] found graviton-graviton correlators for gravity theories not restricted to Einstein gravity, using a de Sitter approximation.

Similarly, the squeezed limit of the three-point correlator is exactly as given by Maldacena [1]

$$\lim_{k_1 \ll k_2, k_3} \langle \gamma_{\mathbf{k}_1}^s \gamma_{\mathbf{k}_2}^r \gamma_{\mathbf{k}_3}^t \rangle \approx -(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) P_\gamma^{sq(1)}(k_1) \epsilon_{IJ}^q(k_1) k_3^I k_3^J \frac{d}{dk_2^2} P_\gamma^{rt(3)}(k_3). \quad (\text{C.0.7})$$

We highlight that the result (C.0.5) may lead to new shape dependence compared to the single field results.

D Reduction to Byrnes *et al.*

It is also important to check our expressions match those which have previously appeared in the literature in the near-equilateral, mildly-squeezed configuration, which is the overlapping regime of validity. A result for the squeezed limit of the bispectrum was given by Byrnes *et al.* in Eq. (96) of Ref [49]

$$B_\zeta(k_1, k_2, k_3) = \left\{ \frac{N_i^{(1)} N_j^{(1)} N_{lm}^{(3)} \left[\delta_{il} + \left(2c + \log \frac{k_3}{k_1} \right) u_{il} \right] \left[\delta_{jm} + \left(2c + \log \frac{k_3}{k_2} \right) u_{jm} \right]}{N_r^{(1)} N_s^{(1)} N_t^{(3)} N_z^{(3)} (\delta_{rs} + 2cu_{rs}) (\delta_{tz} + 2cu_{tz})} \right\} \\ \times P_\zeta(k_1) P_\zeta(k_2) + 2 \text{ perms} \quad (\text{D.0.1})$$

where $c = 2 - \log 2 - \gamma$, with γ the Euler-Mascheroni constant, and u_{ij} is given by (4.1.8), and since u_{ij} is of order slow-roll, it can be evaluated at any time. Their result is valid for small $\log(k_3/k_1)u_{ij}$ with the intrinsic contribution from the three-point function of field perturbations neglected. It was derived by considering the two-point correlation at unequal times, calculated using the expression

$$\delta\phi_{i,\mathbf{k}}^{(3)} = \left[\delta_{ij} + \log \frac{k_3}{k_1} u_{ij} \right] \delta\phi_{j,\mathbf{k}}^{(1)} \quad (\text{D.0.2})$$

which is valid for small $\log(k_3/k_1)u_{ij}$. However, for a large squeezing, $\log(k_3/k_1)u_{ij}$ may not be small, even though u_{ij} is of order slow-roll, and one will instead need the full Γ expression (3.1.6) for the evolution of the field perturbations. The Γ matrix can be written formally as a time-ordered exponential [56, 65–67, 69, 70]

$$\Gamma_{ij}^{(3,1)} = T \exp \left[\int_{N_1}^{N_3} u_{ij}(N) dN \right] \quad (\text{D.0.3})$$

where N_1 is the number of e-folds corresponding to time t_1 and similarly for N_3 . Note that in the limit of small $\log(k_3/k_1)u_{ij}$, we have, at leading order in $\log(k_3/k_1)u_{ij}$

$$\Gamma_{ij}^{(3,1)} \approx \delta_{ij} + \log \left(\frac{k_3}{k_1} \right) u_{ij}^{(1)} + \dots \quad (\text{D.0.4})$$

As discussed in Footnote 5, these authors used a next-order in slow-roll expression for Σ ,

$$\Sigma_{ij}^{(*)}(k_1) \approx \frac{H^{(*)2}}{2k_1^3} (\delta_{ij} + 2cu_{ij}). \quad (\text{D.0.5})$$

Taking our expression for the squeezed limit of the bispectrum (3.3.1), and substituting the RHS of (D.0.4) for Γ , and replacing $H^{(*)2} \delta_{ij} / 2k_1^3$ with the RHS of (D.0.5), we recover (D.0.1).

The reason we didn't need to use (D.0.5) in the main part of this work was because we were throughout working to leading order in slow-roll, rather than next to leading order in slow-roll. The only time we needed to consider slow-roll terms, such as u_{ij} , are when they appear multiplied by $\log(k_3/k_1)$, which can be as large as $\mathcal{O}(20)$, in which case $|\log(k_3/k_1)u_{ij}| \sim 1$. Note that the expansion in Eq (D.0.4) cannot be truncated for $|\log(k_3/k_1)u_{ij}| \sim 1$. In the highly squeezed limit this is why the full expression for Γ given in Eq (3.1.5) needs to be used instead of Eq (D.0.4), even though we can safely neglect the slow-roll correction to the power spectrum in Eq (D.0.5).

E Reduction to Dias *et al.*

Dias *et al.* [50] used a next-to-leading order expression for the bispectrum [60]

$$\lim_{k_1 \ll k_2, k_3} B_\zeta(k_1, k_2, k_3) \approx 2N_{ij}N_lN_m [\Sigma_{il}(k_1)\Sigma_{jm}(k_2) + \Sigma_{il}(k_2)\Sigma_{jm}(k_3) + \Sigma_{il}(k_3)\Sigma_{jm}(k_1)] \quad (\text{E.0.1})$$

to calculate the spectral index of the halo bias as

$$n_{\delta b} \equiv \frac{d \log B_\zeta}{d \log k_1} = -2 \frac{N_i N_j N_k M_{im} \alpha_{mjk}^{\text{LO}} + N_{ij} N_k N_l (M_{im} \Sigma_{mk} + M_{km} \Sigma_{im}) \Sigma_{jl}}{N_n N_p N_q \alpha_{npq}^{\text{LO}} + 2N_n N_p N_{qr} \Sigma_{nq} \Sigma_{pr}} + 2 \frac{N_i N_j M_{ij}}{N_l N_l} \quad (\text{E.0.2})$$

where the right hand sides of both expressions are evaluated at the time t_t when $k_t \equiv k_1 + k_2 + k_3$ crosses the horizon, and where

$$\alpha_{ijk}^{\text{LO}} = -H^4 \frac{\dot{\phi}_i}{H} \delta_{jk} = -H^4 \frac{V_{,i}}{V} \delta_{jk} \quad (\text{E.0.3})$$

$$\Sigma_{ij}(k_a) = H^{(t)^2} \left[\delta_{ij} + 2r_{ij} - 2M_{ij} \log \left(\frac{2k_a}{k_t} \right) \right] \quad (\text{E.0.4})$$

$$\text{with } M_{ij} \equiv \epsilon \delta_{ij} + u_{ij} \quad (\text{E.0.5})$$

$$\text{and } r_{ij} \equiv \epsilon \delta_{ij} (1 - \gamma) + u_{ij} (2 - \gamma) \quad (\text{E.0.6})$$

$$\text{where } u_{ij} \equiv \frac{V_{,i} V_{,j}}{V^2} - \frac{V_{,ij}}{V} \quad (\text{E.0.7})$$

in which all quantities on the right hand side are again evaluated at t_t , and γ is the Euler-Mascheroni constant. These results can be trusted for a mild hierarchy of scales, where $|\log(k_1/k_3)|$ is of order a few. We now check that our expression for $n_{\delta b}$, (4.1.9), can recover the Dias *et al.* result, (E.0.2), in the limit where the exit times are very close. To do so we begin with the expressions

$$\frac{V_{,l}^{(1)}}{V^{(1)}} \Gamma_{ik,l}^{(3,1)} = -\frac{\dot{\phi}_l^{(1)}}{H^{(1)}} \Gamma_{ik,l}^{(3,1)} = -\frac{1}{H^{(1)}} \frac{d}{dt_1} \Gamma_{ik}^{(3,1)} = -\frac{d}{d \log k_1} \Gamma_{ik}^{(3,1)}. \quad (\text{E.0.8})$$

Now assuming we can swap the limit of differentiation with respect to k_1 and the limit of taking $t_1 \rightarrow t_3$, and using (D.0.4) we get

$$\lim_{t_1 \rightarrow t_3} \left(\frac{V_{,l}^{(1)}}{V^{(1)}} \Gamma_{ik,l}^{(3,1)} \right) = -\frac{d}{d \log k_1} \left(\delta_{ik} + \log \left(\frac{k_3}{k_1} \right) u_{ik}^{(1)} + \dots \right) = u_{ik}^{(1)}. \quad (\text{E.0.9})$$

Substituting this into $n_{\delta b}$ of (4.1.9) and setting $\Gamma_{ij}^{(3,1)} \rightarrow \delta_{ij}$ we get

$$\lim_{t_1 \rightarrow t_3} n_{\delta b} = -2 \frac{N_i^{(3)} N_q^{(3)} (N_q^{(3)} V_j^{(3)} + 6N_{qj}^{(3)} H^{(3)2}) M_{ij}^{(1)}}{N_m^{(3)} N_r^{(3)} (N_r^{(3)} V_m^{(3)} + 6N_{rm}^{(3)} H^{(3)2})} + 2 \frac{N_i^{(3)} N_j^{(3)} M_{ij}^{(1)}}{N_m^{(3)} N_m^{(3)}} \quad (\text{E.0.10})$$

which is the of the same form as (E.0.2) when (E.0.3) and (E.0.4) are substituted in. Note that in their expression everything on the RHS is instead evaluated at exit time of $k_t = k_1 + k_2 + k_3$, rather than t_3 , but in the limit where the exit times are very close, this won't affect the result significantly, and we recover their result.

F Tilt of reduced bispectrum in the squeezed limit

As discussed in §4.2 one can study the tilts of the reduced bispectrum, f_{NL} , in the squeezed configuration with respect to any combination of the k -modes which it involves. In particular, one can calculate how f_{NL} of (3.3.3) varies with respect to $k_1 \approx k_2$ or k_3 , or some combination of them. The dependence can be parametrized by

$$n_{f_{\text{NL}}}^X \equiv \frac{d \log |f_{\text{NL}}|}{d \log X} \quad (\text{F.0.1})$$

for $X = k_1, k_3$. In §4.2 we found $n_{f_{\text{NL}}}^{k_1} = n_{\delta b}$, where $n_{\delta b}$ was calculated in §4.1 in (4.1.9). To find $n_{f_{\text{NL}}}^{k_3}$ we write (3.3.3) in a form where the second square bracket contains all the k_3 dependence

$$\lim_{k_1 \ll k_2, k_3} \frac{6}{5} f_{\text{NL}}(k_1, k_2, k_3) \approx \left[\frac{1}{N_q^{(1)} N_q^{(1)}} \right] \left[L_{ij}^{(3,1)} \left(N_i^{(3)} [\log H^{(3)}]_{,j} + \frac{N_i^{(3)} N_{jk}^{(3)} N_k^{(3)}}{N_p^{(3)} N_p^{(3)}} \right) \right] \quad (\text{F.0.2})$$

so that

$$n_{f_{\text{NL}}}^{k_3} \approx \frac{1}{f_{\text{NL}}} \left[P_{ij,3}^{(3,1)} \left(N_i^{(3)} [\log H^{(3)}]_{,j} + \frac{N_i^{(3)} N_{jk}^{(3)} N_k^{(3)}}{N_p^{(3)} N_p^{(3)}} \right) + L_{ij}^{(3,1)} Q_{ij}^{(3)} \right] \quad (\text{F.0.3})$$

$$\begin{aligned} \text{where } Q_{ij}^{(3)} \equiv & \frac{1}{2} N_i^{(3)} u_{jk}^{(3)} \frac{V_{k3}}{V^{(3)}} - \frac{1}{2} N_k^{(3)} u_{ik}^{(3)} \frac{V_j}{V^{(3)}} - \frac{N_i^{(3)} N_{jk}^{(3)} N_k^{(3)} N_m^{(3)} N_n^{(3)} u_{mn}}{(N_p^{(3)} N_p^{(3)})^2} \\ & - \frac{1}{N_p^{(3)} N_p^{(3)}} \left(N_i^{(3)} N_k^{(3)} N_{jkl}^{(3)} \frac{V_l^{(3)}}{V^{(3)}} + N_k^{(3)} N_l^{(3)} N_{jk}^{(3)} u_{il}^{(3)} + N_i^{(3)} N_l^{(3)} N_{jk}^{(3)} u_{kl}^{(3)} \right) \end{aligned} \quad (\text{F.0.4})$$

$$\text{and } P_{ij,3}^{(3,1)} \equiv \frac{dL_{ij}^{(3,1)}}{d \log k_3} = -\frac{V_l^{(3)}}{V^{(3)}} \Gamma_{ml}^{(1,3)} \left(\Gamma_{ik,m}^{(3,1)} \Gamma_{jk}^{(3,1)} + \Gamma_{ik}^{(3,1)} \Gamma_{jk,m}^{(3,1)} \right). \quad (\text{F.0.5})$$

Note that we have neglected the intrinsic contribution in Eq. (3.3.3) for simplicity.

To compare with observations, one might wish to use the variables in [91], given by $\tilde{k}, \tilde{\alpha}, \tilde{\beta}$, defined as

$$\tilde{k} = \frac{1}{2} k_1 + \frac{1}{2} k_2 + \frac{1}{2} k_3, \quad \tilde{\alpha} = \frac{k_2 - k_3}{\tilde{k}}, \quad \tilde{\beta} = \frac{\tilde{k} - k_1}{\tilde{k}} \quad (\text{F.0.6})$$

which in the squeezed limit are related to k_1, k_3 by

$$k_1 = \tilde{k} \frac{1 - \tilde{\beta}}{3 - \tilde{\beta}}, \quad k_3 = \frac{\tilde{k}}{3 - \tilde{\beta}} \quad (\text{F.0.7})$$

with $\tilde{\alpha} \approx 0$. We can use the chain rule to calculate

$$n_{f_{\text{NL}}}^{\tilde{k}} = \frac{\partial \log f_{\text{NL}}}{\partial \log \tilde{k}} = \frac{\partial \log f_{\text{NL}}}{\partial \log k_1} \frac{\partial \log k_1}{\partial \log \tilde{k}} + \frac{\partial \log f_{\text{NL}}}{\partial \log k_3} \frac{\partial \log k_3}{\partial \log \tilde{k}} = n_{f_{\text{NL}}}^{k_1} + n_{f_{\text{NL}}}^{k_3} \quad (\text{F.0.8})$$

$$n_{f_{\text{NL}}}^{\tilde{\beta}} = \frac{\partial \log f_{\text{NL}}}{\partial \log \tilde{\beta}} = \frac{\partial \log f_{\text{NL}}}{\partial \log k_1} \frac{\partial \log k_1}{\partial \log \tilde{\beta}} + \frac{\partial \log f_{\text{NL}}}{\partial \log k_3} \frac{\partial \log k_3}{\partial \log \tilde{\beta}} = \frac{-2\tilde{\beta} n_{f_{\text{NL}}}^{k_1}}{(1 - \tilde{\beta})(3 - \tilde{\beta})} + \frac{\tilde{\beta} n_{f_{\text{NL}}}^{k_3}}{(3 - \tilde{\beta})}. \quad (\text{F.0.9})$$

Note that in the squeezed limit $\tilde{\beta} \approx 1$, and so if we use our expression for the squeezed limit of f_{NL} in (3.3.3), we shouldn't vary $\tilde{\beta}$ significantly away from 1.

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