THE MEASUREMENT AND CALCULATION OF ACOUSTIC NOISE
RADIATED BY SMALL ELECTRICAL MACHINES

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Thesis presented in the University of London for the degree of Doctor of Philosophy in Engineering
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ABSTRACT

With the increasing noise level present in a mechanised society today, it is necessary to reduce unwanted sound in any economic way possible. The work done for this thesis is part of a research project at Queen Mary College into the noise produced by electric machines. By gaining a thorough knowledge of the way in which noise is produced in machines it will be possible to calculate and minimise the noise output of machines at the design stage.

The first requirement of the project is that the noise produced by a machine should be measured accurately. Methods are developed for measuring, in an anechoic chamber, the acoustic power radiated by a machine. Also studies of the vibration and acoustic radiation characteristics of several machines are made.

The second requirement is to identify the sources of the noise components in a machine. This is done by calculating the resonant frequencies of the mechanical parts of the machine and then analysing the noise with the machine running at several different speeds. Examples of the application of the methods to the identification of noise components in small induction machines are given.

The third requirement is the accurate calculation of components of the noise and comparison of the magnitudes with measured values. A method for calculating the noise produced by electromagnetic sources is given. The method is basically applicable to all types of machine although certain parts of the calculation must be modified in some cases. The calculation is divided into three parts: the calculation of the forces at the iron surfaces of the air gap, the mechanical response
and the acoustic radiation characteristics. Each part is considered separately. The mechanical response, radiation characteristics and overall calculation are compared with measured results and good agreement is obtained.
ACKNOWLEDGEMENTS

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LIST OF SYMBOLS

Where appropriate, the location of the definition of the symbol is shown in parenthesis.

\( a \) Area

\( A_1, A_2 \ldots \) Arbitrary constants

\( B \) Flux density

\( c \) Speed of propagation of sound

\( C \) Capacitance (Fig. 2.11)

\( d \) Variable distance along axis of machine (Sect. 5.5.1)

\( D_1 \) Rotor core length (Sect. 5.5.1)

\( D_2 \) Length of shaft between bearings (Sect. 5.5.1)

\( D_3 \) Length of stator core (Appendix B)

\( e \) Eccentricity (Sect. 4.3.3)

\( E \) Modulus of elasticity

\( E' \) Modified modulus of elasticity (Eqn. B.17)

\( f \) Cyclic frequency

\( F \) Magneto-motive force

\( g \) Air-gap length

\( g' \) Modified air-gap length (Sect. 4.3)

\( g \) Acceleration due to gravity

\( G \) Bending moment (Fig. B.1)

\( h \) Integer

\( h^{(m)}, h^{(n)} \) Spherical Hankel functions (Sect. 6.3.4)

\( H \) Non-dimensional variable (Eqn. B.27)

\( i \) Integer

\( i_1, i_2 \ldots \) Instantaneous currents (Fig. 2.11)
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* Unless otherwise stated, force levels are at a base of $10^{-4}$ N/m² and pressure levels to a base of $2 \times 10^{-5}$ N/m².
\( u \)  
Particle velocity

\( v_1, v_2 \)  
Instantaneous voltages (Fig. 2.11)

\( w = w + jw \)  
Complex plane (Appendix F)

\( w \)  
Loading forces (Eqns. 5.19 and 5.27)

\( x \)  
Apparent distance for radiation (Eqn. 6.12)

\( X \)  
Co-ordinate in stator response derivation (Fig. B.2)

\( y \)  
\[ \cos \theta \] (Sect. 6.3.3)

\( Y \)  
Co-ordinate in stator response derivation (Fig. B.2)

\( j = j' + j'' \)  
Complex plane (Appendix F)

\( z \)  
Co-ordinate in stator response derivation (Fig. B.2)

\( \alpha \)  
Acoustic absorption coefficient

\( \beta \)  
Periodic term (Sect. 6.4)

\( \gamma \)  
Function of integers (Eqn. C.1)

\( \Gamma, \Gamma', \Gamma'' \)  
Constants in pressure equation (Sect. 6.4 and Eqn. 6.42)

\( \epsilon \)  
Strain (Eqn. B.8)

\( \chi = \chi + j\chi'' \)  
\{Complex planes (Appendix F)\}

\( \eta, \eta', \eta'' \)  
Polynomials in \( \chi \) (Eqns. 6.14, 6.15 and 6.23)

\( \theta \)  
Polar angle

\( \Theta \)  
Bending angle of stator element (Fig. B.2)

\( k_1, k_2, \ldots \)  
Constants (Eqn. 5.2)

\( \lambda \)  
Wave length or tooth pitch

\( \Lambda \)  
Permeance

\( \mu_0 \)  
Permeability of free space

\( \delta \)  
Displacement

\( \rho \)  
Density

\( \sigma \)  
Force per unit area
$\sum$ Summation

$\Sigma$ Force per unit axial length of tooth tip (Eqns. 4.39 and 4.40)

c Tooth width (Sect. 4.3)

$\gamma$ Definite integral (Appendix C)

$\hat{\theta}$, $\tilde{\theta}$ Equatorial angles

$\hat{\theta}$ Function

$\psi$ Arbitrary angle

$\xi$ Angular function of $x$ (Eqns. 6.25 and 6.33)

$\omega$ Angular frequency

$\omega_0$ Angular frequency of power supply

Subscripts

Numerical subscripts, when applied to variables denote special values of the variables (E.g. $t_1$ indicates a particular time) and when applied to integers denote particular members of a set of integers. The following subscripts denote special values of variables. Where no such subscript is present, the instantaneous value is indicated:

- $r.m.s.$ Root mean square value
- $p_p$ Per unit value, based on specified value
- $p_t$ Peak of sinusoidal variable
- $m_{ax}$ Maximum value of variable
- $a_{av}$ Average value

The following subscripts indicate quantities associated with parts of the machine, other variables etc.:

- $St.$ Stator
- $Rr.$ Rotor
- $e$ Eccentricity
- $g$ Air-gap
\( sh \)  Shaft

\( su \)  Stator surface

\( c \)  Stator core

\( ne \)  Neutral axis

\( w \)  Winding

\( re \)  Resonant

\( a \)  Air

\( ir \)  Iron

\( u \)  Velocity

\( p \)  Pressure

\( P \)  Power

\( T \)  Tension

\( sh \)  Shear

\( r, \phi, \theta \)  Indicate components in the directions of \( r, \phi \) and \( \theta \), except when used with \( z \) when functions of \( r, \phi \) and \( \theta \) are indicated.

\( w, 3, \nu \)  In pairs in Appendix F denote transformations from the former to the latter

\( h, i, j \)  Denote components with these harmonic numbers or frequency
CHAPTER I

INTRODUCTION

1.1) General.

Noise has been defined by the British Standards Institution as 'sound which is undesired by the recipient.' (9) As processes in the home, transport and industry become more mechanised, the sounds, often inherent in the mechanisation, tend to increase. Most of these sounds do not contribute directly to the efficient action of such mechanisms although in some cases they may be regarded as a warning that the mechanism is operating. To people not concerned directly with the operation of the mechanism these sounds are a nuisance and must be classed as noise.

Noise produced by mechanisation is perhaps the most extensive source of annoyance to people living in a mechanised society. Devices as diverse as jet aircraft and electric typewriter motors produce annoyance. The increasing level of noise has caused so much concern that a government committee was appointed in 1960 to study the problem. Their final report (23) was published in 1963.

The annoyance caused by a sound is not merely a function of its physical intensity. Monotonous high pitched noises or slowly but regularly changing noises can produce more annoyance than music or speech at a much higher level. The harm that such noise may do to the recipient is difficult to assess. In extreme cases of subjection to very high levels of noise physical damage to the ear may occur. More often the effects of the noise are psychological. In either case, as stated...
in the government report (paragraph 550):

Since health is defined as "a state of complete physical, mental and social well-being and not merely an absence of disease and infirmity", there is no doubt that noise affects health.

Another conclusion of the report is given in paragraph 549:

The price of sweeping measures to bring about large reductions of noise quickly would not be acceptable to the community, but, in our judgment, the present level of noise is such that some additional cost in money and in restriction of liberty to make noise is justifiable to prevent further increase and in time to achieve some reduction.

There are three main ways of reducing the noise received by the listener. First, the noise producing objects may be placed further away from the listener. This is usually not a practicable solution as the presence of the noise producing objects is often essential in inhabited areas. However, for some noisy industrial processes, this is a possible solution and then the minimum number of people are exposed to the noise.

Secondly, a sound insulating enclosure may be placed round the noise source or the listener so that the level of the noise is reduced. In the case of structure borne noise resilient mountings may be used to reduce the noise. Where existing noise has to be reduced, this is usually the only method, but it is usually expensive, as enclosures have to be specially designed to produce sufficient attenuation at all frequencies for a particular source.

The third method is to reduce the noise at source. This involves studying the ways in which noise is produced and identifying the
surfaces from which it is radiated. The apparatus is then redesigned to reduce the various components of noise. This may sometimes be done only in ways which involve lowering its efficiency or changing its operation in some way. However, if a sufficient understanding is gained of all the processes involved in its operation it may often be redesigned to produce a compromise for a given situation or even to reduce the noise without changing the other functions. Sometimes the possible reduction by this method is small and the second method of noise reduction must be used.

The noise produced by electric plant is not regarded by most people as especially potent. It does not produce the public outcry that motor vehicle and aircraft noise produces. This is because the larger machines and transformers which produce the most noise are usually confined to power stations away from built-up areas and factories where noise is expected. The noise from these machines does however annoy people who have to work near them.

More and more electric machines are being used in appliances in offices and in the home. It is probably in this field that electric machine noise affects most people today. Thus, it is not only necessary to make the very noisy machines quiet, but also to make the reasonably quiet, but still annoying, machines quieter. For this reason a detailed study of all components of the noise should be made.

As larger numbers and larger sizes of electric machines are used in ships, so the level of the noise caused by them increases. In addition to being annoying, this noise is radiated well in the water so that the ship is easily detected. This is another field where there is an interest in the reduction of machine noise.
1.2) Aims of Research Project.

The work described in this thesis is the initial part of a long-term investigation, at Queen Mary College, of the noise produced by electric machines. The aim of the project is to obtain such an understanding of the noise produced by machines that the noise characteristics may be accurately predicted when a machine is designed.

The final criterion for the noisiness of a machine must always be subjective. Since different people react in different ways to noise, this often leads to anomalous assessment of noisiness. In fact, different reactions to the same noise may occur according to their mental attitude at the time. It is therefore important that physical measurements should be made, which may be related to the average subjective assessment. Also, the physical noise criterion should not vary as the acoustic surroundings are varied but the noise present under any acoustic conditions should be obtainable from it. Reasons are given in Chapters 2 and 3 for the acceptance of total acoustic power and the forces at the mounting points in the required frequency band as criteria.

Since these measured values are physical, it is possible to calculate values for them from a knowledge of the physical structure of the machine. If a physical specification of the maximum permissible noise can be given, it may be possible to design a machine to give the required characteristics. This physical specification may also be related to the acoustic surroundings and the subjective requirements of the user.

The first step, therefore, is to develop methods for measuring these physical quantities so that any experimenter will obtain the same result and the result will be meaningful in terms of subjective requirements.
Thus, the measurement and presentation of the measurements of total power must be standardised for all field conditions. The most fundamental measurements may be made under free-radiating conditions. Once the power can be measured accurately under these conditions, comparison may be made with measurements under less ideal conditions. When sufficient accuracy is obtained from simple measurements under less ideal conditions, the method may then be standardised. Standards for the measurement of electric machine noise are being drafted by the International Organisation for Standardisation and the British Standards Institution. The first part of this research project is to investigate the acoustic fields of electric machines and determine the accuracy of power measurements made under free conditions in the anechoic chamber.

It is of little use knowing the total acoustic power radiated by a machine over its whole frequency range as this does not indicate the sources of the noise components and the subjective effect may not be obtained from it. There are three main types of analysis which are used to analyse noise. First, narrow band analysis may be used to measure the magnitude of pure tones and is therefore of use in investigating the sources of components of the noise. Secondly, octave, 1/2-octave or 1/3-octave band analyses may be used to give the distribution of energy over the frequency range. Such analyses may be used in the design of insulating enclosures or resilient mountings and may also be used to calculate the response of the human ear to the noise. Thirdly, overall weighting networks may be used. The responses of these correspond roughly to the response of the human ear to pure tones at various levels and may therefore be used to give a rough assessment of
the subjective loudness of noise.

Vibration, transmitted to the surrounding objects by the mountings of a machine, is also important. Methods of measuring this and from it assessing the secondary radiation should also be developed if all of the noise produced by a machine is to be assessed.

Once measuring techniques have been developed, the reasons for the production of noise may be investigated. It is important to know the fundamental causes of all components of the noise and also determine the way in which the mechanical and acoustic properties of a machine affect the production of noise. Once the causes of each component are found, techniques can be developed for calculating the power and frequency of the noise produced in each way. Means of reducing the magnitudes of the components may then be proposed and tested.

Noise is growing in importance as a factor in the design of electric machines. It is one of many factors which must be considered at the design stage. Cost, efficiency, torque characteristics, voltage ripples and temperature rises are among the factors which must be considered in addition to the basic requirements of the design. For a particular job, a machine must be designed to make as many of these factors 'good' as is possible. The relative importance of each factor depends on the purpose for which the machine is intended. It may be possible to decrease the noise produced by a machine by increasing the thickness of the frame so that the forces produce less movement. However, the cost will then increase. So far as is possible, means must be found for improving all such parameters without adversely affecting the others. It is therefore necessary to study all these factors in a fundamental way. For instance, noise producing forces, parasitic torques, voltage ripples and losses may all be investigated by more accurate studies.
of the magnetic field produced by a machine.

Although for several years methods for calculating the noise components produced by synchronous machines\(^{(1,53)}\) and induction machines\(^{(1,2)}\) have been available they have been little used in design. Quiet machines have been built from the experience of the designers and by using such rules for the numbers of slots as those given by Kron\(^{(33)}\).

With the advent of the high-speed digital computer, complex calculations may be made quickly. Some manufacturers now use such computers in the design of electric machines. A logical extension of this trend is to programme the computer with the expected noise output in addition to the other design factors. This has already been done for particular components of the noise as explained by Ridley\(^{(43)}\). Eventually all possible components could be investigated by a computer and facilities provided to minimise particular design factors.

Thus, methods of calculating the magnitudes of the noise components produced by a machine and which are suitable for programming for a computer, should be developed. Theoretical work has been done for particular types of machine\(^{(2,21,53)}\), but no theory which may be applied to any type of machine has been developed. Noise is produced fundamentally by all machines in similar ways. Therefore, similar paths may be followed in the calculation for each type of machine. For different machines, parts of the calculation may have to be treated differently. For example, the mechanical response of a stator depends on its construction but the electromagnetic forces may be calculated in a similar way if the air-gap configurations are similar. In a computer calculation, different routines could be called in for particular cases.
Dissymmetries in the magnetic field or mechanical construction of a machine are a major source of noise especially in small mass produced machines \((35)\). Dissymmetries may arise directly from the design of a machine, as when it has an odd number of rotor or stator slots. They may also be produced by manufacturing methods. Eccentric air-gaps, faulty rotor bars, burred laminations and bad contact between the stator laminations and the stator shell are among the faults which may produce dissymmetries. They may cause basically similar machines to radiate different noise power magnitudes. The effect of such factors on the noise must be investigated so that manufacturing standards may be set.

Induction machines produce noise which is modulated at a frequency proportional to that of the slip currents. Noises of this type can be more annoying than steady noises at a lower level. The loudness calculated from the total powers radiated by these machines does not give an accurate assessment of the subjective effect of this noise. Means must be provided for checking characteristics of this type which might affect the subjective assessment of the noise.

The eventual aim is therefore to understand, evaluate and minimise the noise produced by any type of electric machine at the design stage.

1.3) The Production of Noise by Electric Machines.

The causes of most of the noise produced by electric machines may be classified into three types according to the source of the noise. These types are aerodynamic, mechanical and electromagnetic in origin. Each type of noise is present in all types of machines, but in some machines one particular type of noise predominates, depending on the size and type of machine.
Aerodynamic noise is produced by the cooling air in a machine. There are three main ways in which noise may be produced by moving air. First, general turbulence is caused by the air moving quickly over uneven surfaces, producing wide band noise. Secondly, a siren effect may be caused by air moving past fan blades with stationary objects close to them. Thirdly, noise is produced as air passes an obstacle at high speed producing aeolian tones even if laminar flow is maintained in general. Most of this noise is radiated by the air stream as it leaves the machine. Ducts may be used to take the air, and hence the noise, away from the machine but care must be taken to prevent the ducts from resonating. Alternatively, closed cooling systems or silencers may be used. Aerodynamic noise may also cause vibration in the mechanical parts of the machine and this may be radiated from the surface or transmitted to the supports. Aerodynamic noise is of most importance when high air speeds are involved.

The main mechanical sources of noise are mechanical unbalance and vibration produced in the bearings. Unbalance produces radial forces at the bearings rotating at the speed of the rotor. These forces produce movement of the stator from which the noise is radiated. Since these forces occur at low frequencies, their main effect is to produce vibration at the supports of the machine and, hence, structure borne noise. The amount of noise produced by a bearing varies considerably with the type. Sleive bearings, which produce wide band noise are the quietest. Ball and roller bearings are often noisy. The production of such noise is not thoroughly understood but it is thought that some components may be due to resonances of the bearing housings excited by ball and bearing surface irregularities.
Electromagnetic forces are produced by the flux in the air-gap of the machine. The forces produced by the flux are inherent in the operation of the machine since it is these forces which also produce the torque. In most machines it is not the main flux waves which produce the most noise but the harmonic components produced by irregularities in the field. These produce forces of higher frequency and the sound is then easily radiated from the surface. In large machines, however, the twice-line-frequency component produced by the fundamental flux often predominates. The forces on the teeth and poles of a machine cause vibration in the iron parts and the noise is radiated from the surface and transmitted by the supports to other apparatus.

The work presented in this thesis is mainly concerned with the noise produced by electromagnetic sources. These components predominate in the induction machines tested. The measuring techniques and some of the theory are also applicable to other sources of noise.

1.4) The Calculation of Electromagnetically Produced Noise.

The determination, from its design data, of the noise produced in a machine by electromagnetic sources may conveniently be considered in four parts. Each part of the calculation gives a value for a physical quantity at some position in the machine. The solution for each section may be considered separately and checked by physical measurement. After the analysis of each section has been tested, it may be joined with that of other sections to calculate the radiated noise.

The first section deals with the determination of the currents
flowing in the windings from the design data, the shaft load and the terminal voltage. These currents consist not only of supply frequency components but also harmonic components caused by slot permeance, space harmonics, and saturation. For a complete analysis of the noise, these components must also be calculated. This section is not considered in detail in this thesis as measurements were confined to no load conditions and only simple cases of noise analysed. For the future calculation of on-load noise and the effect of harmonics the work done in other research projects at Queen Mary College by Nagappan and Randell, using the generalised theory of electric machines taking into account harmonics, should be useful. The currents may easily be measured using a harmonic analyser so that this section of the calculation may be checked.

The second section, which is considered in Chapter 4, deals with the calculation of the electromagnetic forces on the teeth and poles of a machine from currents flowing in the windings. To calculate these forces it is necessary to know the flux distribution on the iron surfaces. This is a complicated problem even in the simple unsaturated case. It is solved most accurately by using conformal transforms, as shown by Carter. However, for a doubly slotted air-gap surface this becomes extremely complicated. A method for solving the field problem has been developed by Hinns but it has not yet been applied to calculating the forces. Another simpler method of solving the field problem is to assume that the flux is radial and calculate an equivalent permeance distribution for the slots. Using either method it is very difficult to take into account saturation in the tooth tips and other parts of the magnetic circuit. This occurs under normal
conditions in most machines.

The third section, which is dealt with in Chapter 5, concerns the mechanical response of the rotor and stator to the forces acting on the ends of the teeth and poles. The result of this calculation is the magnitude and distribution of the vibration displacement on the surface of the machine. If the stator can be considered as a single cylinder, the response is relatively simple. If the only contact between the laminations and the outer casing is key bars or ribs, the response of two linked cylinders must be considered. This has been done by Erdelyi(122). Whichever method is used considerable approximations must be made. For example, the resonant frequencies of the stator may be found by assuming that there is no damping in the iron, but if this is done, the magnitude of the response near resonance cannot be calculated.

The final section deals with the calculation of the radiation characteristics and the total power from the displacement at the surface of the machine. For this calculation, a simplified surface shape and displacement variation must be considered. Usually the surface is assumed to be cylindrical or spherical. Both methods involve approximations and the method chosen depends on the properties of the field which must be calculated in a particular situation.

In calculating some components of the noise it is not possible to separate these sections of the calculation completely. For example, if a one-sided magnetic pull acts on the rotor due to eccentricity, elastic movement of the rotor may occur changing the amount of eccentricity. In this way an oscillating system may be set up as described by Robinson(44). In this case the second and third sections must be taken together.
2.1) Choice of Parameter for Comparison of Noise Radiated by Machines

When a value, or series of values for different frequency bands, is given to represent the noise radiated by a certain electric machine, it is important that the measurement should be repeatable under various field constraints. Also, an assessment of the subjective effect of the noise under any chosen condition must be obtainable from the parameter. Fundamentally, an electric machine is a constant velocity acoustic source, since the source of the acoustic wave is a periodic displacement of its surface. The velocity is not affected by surrounding objects to a measurable extent so long as they do not touch the machine. However, the velocity varies considerably with position on the surface of the machine. It is therefore necessary to make measurements at many places on the surface of the machine and integrate the velocity over the surface to find the volume velocity which is known as the strength of the source. This is a value which is constant for a machine under most conditions and is often used to represent the sound produced by less complicated sources.

The strength of a source does not, however, give a reliable measurement of the conditions in the field of a machine where the noise is heard. The sound pressure at a large distance from the source,
for a particular source strength, varies considerably with the ratio of the source circumference to the wavelength of the sound. This ratio will be called the effective size of the source. The smaller the effective size of the source the more difficult it is for it to radiate acoustic energy.

So far only the sound radiated directly by the machine has been considered. Often much of the noise is transmitted mechanically to associated equipment and surrounding objects and is then radiated by these. It then becomes necessary to include the effect of the secondary sources in the total noise figures. In this case the source becomes more complicated and it is then difficult to define the total source strength. Also, some of the acoustic energy in the field of a machine, especially a large open-ventilated machine, is carried from the machine by the cooling air or is caused by turbulence in the air stream \( (18) \). The additional source strength due to these sources is difficult to measure.

It is therefore desirable that measurements of radiated noise should be made in the field of the source rather than be calculated from the surface velocity since then the ventilating noise, the effective size of the source and the sound radiated by secondary sources is automatically taken into account. Most microphones produce a voltage which is proportional to the sound pressure. It is therefore easy and convenient to measure the sound pressure distribution in the field of a machine. Pressure measurements are convenient for another reason: with the use of weighting networks or filters quasi-subjective measurements may be made. However, the sound pressure varies considerably with position in the field of a machine and it
is impossible to define, for measurements on all kinds of apparatus, a standard position which would give a general picture of the noise. Thus the single-point pressure measurement is only of use in comparing the noise produced by similar machines, e.g. as a production line noise test.

A much more meaningful measure of noise is the total acoustic power radiated by a machine as this takes into account the noise radiated in all directions. The power may be calculated from pressure measurements made under several different conditions and is usually expressed as a power level referred to $10^{-12}$ watts. Under the semi-reverberant conditions in which machines usually work the sound pressure at distances greater than a few feet is proportional to the power and inversely proportional to the absorption coefficient of the room. The sound pressure levels and therefore the quasi-subjective levels to be expected under particular working conditions may therefore be simply assessed. Also, it is possible to make power measurements with and without auxiliary apparatus so that the effect on the radiated noise of the extra radiating surfaces and different mechanical linkage between items of apparatus may be assessed.

2.2) The Measurement of Total Acoustic Power.

The total acoustic power radiated by a source may be measured in several different ways depending on the environment of the source. The most fundamental way is the free field method. If a source is
allowed to radiate sound freely in all directions as it would at the
centre of a large volume of air with no reflecting surfaces near the
source, then the acoustic power crossing any closed surface enclosing
it is a constant equal to the total power radiated. Thus, if the
power crossing any surface enclosing the source can be measured the
total source power is known.

The power intensity at a point in a sound wave is equal to the
product of the r.m.s. particle velocity, the r.m.s. pressure and the
cosine of the phase angle between them and is in the direction of
the particle velocity. In the far field of a source the velocity
is in phase with the pressure and also directly proportional to it.
Thus the Power Intensity,

\[ I = \frac{p^2}{\rho^2 c_a} \]

where \( p \) is the r.m.s. pressure, \( \rho_a \) is the mean air density, \( c_a \) is the
velocity of sound waves in air and \( \rho_a c_a \) is known as the characteristic
impedance of the air \( (9) \).

The total power radiated by a source may be found by integrating
the intensity over a surface enclosing the source. Thus the total
power,

\[ P = \int I \, da \]

\[ = \frac{I}{\rho_a c_a} \int p^2 \, da \]

The integrating surface may be any shape as long as it encloses
the source and the intensity normal to the surface is used.

These conditions may be artificially attained by suspending
the source in the centre of an anechoic chamber. The power may then
be calculated by integrating the intensity over a sphere or cylinder
enclosing the source, usually by taking equally spaced points and
averaging the squares of the pressure measurements.

Alternatively, the source may be placed close to a reflecting surface such as the floor of a room, the walls of which are acoustically non-reflecting, and the power measured by integrating over a hemisphere. Similarly, the power could also be measured by placing the source near the junction of two or three orthogonal reflecting surfaces. If the surfaces are perfectly reflecting the power measured under these conditions would be the same as that under entirely free conditions. Therefore, the values measured in practice will generally be smaller, although at some frequencies and measuring distances they may be larger due to interference between direct and reflected waves.

The second method of measuring the total power radiated by a source is the reverberant field method. In order to use this method a large amount of reflecting material is placed in the field of the source, care being taken not to produce standing waves by appropriately inclining the reflecting surfaces. In this way the sound pressure in the room builds up when a source is present and if the surfaces were perfectly reflecting would continue to build up as long as the source is radiating power. However, the surfaces are never perfectly reflecting, a certain fraction (α) of the incident energy being absorbed. Owing to the multiple reflections the sound pressure in the room away from the source is a constant and therefore the power absorbed by the surfaces,

\[ P = \frac{\alpha \beta^2 a}{\rho c \alpha} \]

where \( a \) is the total area of the reflecting surfaces. Thus if \( (\alpha \beta) \), for a reverberant room at a particular frequency, is known the energy
absorbed by the walls at this frequency may be calculated from the constant sound pressure. In the equilibrium condition the power absorbed must be equal to the power radiated by the source. Thus, the power radiated by the source at this frequency is also known.

Since the surfaces often absorb different proportions of energy at different frequencies this method is only accurate when filters are used to analyse the sound and cannot be used for overall measurements unless special weighting networks to match the room are used. The free-field method can, however, be used for overall measurements. Another advantage of the free-field method is that investigation of the source radiation patterns may also be made. The advantage of the reverberant field method is the speed with which measurements may be made once a room is calibrated as only one measurement or the average of two or three are required to measure the power.

Generally, the conditions under which power measurements are required are neither free-field nor reverberent-field. In a machine test shop conditions are usually semi-reverberent. There are no large reflecting surfaces close to the source but large complex surfaces further away. It may be shown from considerations of the conservation of radiated power that the pressure drop in the field of a simple source radiating freely is 6 dB when the distance of the measuring point from the centre is doubled. Thus, the pressure / distance curve for a simple source radiating freely is shown by curve A in Fig. 2.1.

As reflecting materials are placed in the field the reverberant field builds up giving pressure / distance curves like B and C in Fig. 2.1. Since the reverberant level is low the field close to
the source is not greatly changed. In order to measure the power radiated under such conditions, it is usual to assume that the reverberant field close to the source is negligible and use the method described for free-field conditions \((30, 39)\). Alternatively, measurements may be made in this way and then corrected for the reverberant field according to the room constant for the particular frequency and position of measurement. Wells \(^{(54)}\) suggests using a machine to calibrate the room, the sound power radiated by the machine in each octave band having been measured in an anechoic chamber, while Baron \(^{(3)}\) uses a small loudspeaker to calibrate the room. The former method is probably the more accurate if a machine of the type to be tested is used for calibration. However, if a machine of a different size or with a different type of cooling is used the loudspeaker method has probably equal accuracy.
Corrected semi-reverberant field measurements can be sufficiently accurate for most purposes, especially if the room is large enough for the measuring points to be outside the near field of the source without being in the reverberant field.

2.3) Choice of Frequency Bands.

The noise produced by electric machines has components at frequencies covering the whole audible range. The main noise components are pure tones, although turbulent air streams and bearings often produce wide band noise. The simplest method of measuring the noise is to measure the r.m.s. value of the total sound pressure. However, this does not give a value indicative of the subjective value of the noise. The response of the ear to pure tones is less at high and low frequencies than it is at the centre of the audible range and the effective attenuation at any frequency varies with the loudness of the noise.

A tone with a subjective loudness equal to that of a 1000 c/s tone has a loudness level, in phons, equal to the sound pressure level \(20 \log_{10} \left(\frac{\rho}{0.000\,02}\right)\), where \(\rho\) is the r.m.s. sound pressure in N/m\(^2\) of the 1000 c/s tone. Thus the value in phons depends on the subjective effect of the noise but is not directly proportional to the loudness. A second subjective scale has been standardised giving the sone \(\frac{2(\text{phon} - 40)}{20}\) which is roughly proportional to the loudness. The equal loudness curves apply accurately only to pure tones and therefore observers are required
when an accurate assessment of a complex noise is required.

Three standard sound level meter weighting networks\(^{(12)}\) are often used to give a rough assessment of the subjective value of the noise. These have frequency responses similar to those of the human ear to pure tones at 40, 60 and 80 phons. They are termed A, B and C weightings respectively and the values so measured are often designated dBA, dBB and dBC.

The response of the human ear to a complex noise is not in general the same as the response of a meter weighted in accordance with the pure tone equal loudness curves, since masking often occurs between components and also wide band noise may be present. It is therefore necessary to split the frequency range into bands and measure the sound pressure level in each band. By modifying and summing the values for each band, a close approximation to the subjective loudness level may be obtained for most types of noise. The three most common methods in use are those due to Stevens, Zwicker and Kryter. These methods are discussed by King\(^{(28,29)}\)

So far only the subjective effect of the noise has been considered. Often it is necessary to design acoustic insulation for a machine and the attenuation provided by a particular enclosure depends on the frequency of the sound. If an enclosure is to be effective, it must provide sufficient attenuation over the whole frequency range. It is therefore necessary to measure the components of machine noise over this range in order to design such an enclosure. For this purpose, contiguous bands one-octave, \(1/2\)-octave or \(1/3\)-octave wide with fixed centres at standard frequencies\(^{(27)}\) are used. The attenuation of the enclosure for each frequency band may be found from the properties of
its materials and the noise to be expected outside the enclosure calculated. If necessary the calculated loudness may then be found. Octave and 1/3-octave analyses of the same noise together with the overall weighted levels are shown in Fig. E.13.

Measurements of this type are of little use to the machine designer or for research into the causes of noise where a more detailed analysis of the noise is required. The frequency and magnitude of each pure tone must be found accurately and, therefore, the tone must be separated from the noise at other frequencies. Narrow band analysers having pass bands with continuously variable centre frequencies are used in this case. There are two main types of narrow band analyser: the heterodyne type has a constant band-width and the feedback type has a band-width which is a constant percentage of its centre frequency. The latter is probably the more convenient to use since, at high frequencies, it does not require such accurate tuning as the former. Often an analyser may be used in conjunction with a level recorder to plot automatically the frequency spectrum as in Fig. E.7. This is an analysis of the same noise as the octave and 1/3-Octave analyses in Fig. E.13.

2.4) Extraneous Noise.

When measuring the noise produced by a machine, three general types of extraneous noise occur. First, there is the general background noise produced by other plant, etc., adjacent to the test area. This adds to the noise measured by the instruments and, if the difference between the background noise with the machine switched off and the
total level in any frequency band is less than about 10 dB, a correction should be made by subtracting the background noise on an energy basis. Ideally measurements should be made in a soundproof room and for measurements on small machines this becomes essential. Also, when low levels of noise have to be measured, the background electrical noise of the measuring apparatus is often in evidence and care must be taken to detect and eliminate such noise. The electronic noise level in most measuring apparatus is below the threshold of hearing and so the machine noise modified by this is at a very low level and therefore relatively unimportant.

Secondly, extraneous noise may be produced by secondary radiation. Vibrations travel from the machine via its mountings to surrounding apparatus and can cause considerable error in the noise measurements, especially if mechanical resonances occur in any part. It is, therefore, always advisable resiliently to mount the plant on which noise tests are made. This problem is prevalent in single-phase induction machines where pulsating torques, which produce little direct radiation, from the surface of a machine, produce large components of vibration on the mounting flanges and other surfaces which are not concentric with the stator core. Care must also be taken in the placing of the microphone as vibration in its supports can produce an appreciable error in the measurements. It is, therefore, best to use resilient mountings for the microphone also.

Thirdly, extraneous noise can be produced if the microphone is placed in the cooling-air stream of the machine. The air passing across the microphone grill produces low frequency noise. For this reason, microphones are often shielded by thin mesh wind screens when they have to be placed in or near an air stream.
2.5) Design of Microphone and Machine Support Apparatus.

Since an anechoic chamber was available for measuring the noise radiated by machines, the free-field method could be used to obtain the total radiated power by integrating over a complete sphere. To do this it is necessary to support the machine at the centre of the chamber and to be able to place the microphone accurately at any chosen position in the field. The machine is therefore suspended from an adjustable frame on thin steel wires with rubber inserts to reduce the transmission of vibration. The support system is shown in Fig. 2.2. The motor may be adjusted to any height and may also be adjusted a few inches horizontally. An alternative mounting arrangement using the same principle but capable of supporting larger machines has also been built and is shown in Fig. 2.3.

The microphone is held in a one-inch diameter tube attached to a six-foot diameter steel ring as shown in Fig. 2.3. The ring is mounted so that it may be rotated about a vertical diameter passing through the centre of the machine. The machine support is fixed to the stationary part of the top bearing of the ring. The ring may be set to any position, its angle to a reference (taken through the shaft of the machine which is lined up with a diagonal of the room) being measured on a scale at the top of the ring. By using extension tubes, the microphone may be set at any radius inside the ring, pointing towards the centre and may be located in notches at five degree intervals round the periphery of the ring.

The top bearing of the ring is held by four steel rods screwed into the roof of the chamber and supported laterally by tensioned steel wires fixed to the wall at the level of the roof wedge tips.
Fig. 2.2. Support Apparatus for Small Machines.
Fig. 2.3. Microphone ring and support for large machines.
Fig. 2.5. Microphone ring drive motor and gear boxes.
The bottom bearing is supported by a three-inch diameter steam pipe bolted firmly to the floor of the chamber, its top being just below the wire mesh floor. At this bearing there is a height adjustment so that the ring may be adjusted to a circular shape. This adjustment may be seen in Fig. 2.4.

As well as being adjustable to any position, the microphone may also be rotated continuously about the axis of the ring via a drive shaft, passing through the steam pipe to the space under the chamber. The shaft is driven through a three-speed gear box, two worm gear boxes and two antivibration couplings by a small synchronous motor as is shown in Fig. 2.5. A tapered wooden pin links the final gear box to the drive shaft so that the gear boxes are protected against impulsive torques which would be produced if the ring were knocked.

Owing to the stiffness of the microphone cable and the machine power supply wires, it is possible for the ring to revolve only once automatically before it must be unclamped from the drive shaft and returned to its starting position. To prevent damage, an arm, attached to the ring, automatically switches off the driving motor after about one and a quarter revolutions. The same arm also operates two other contacts, one closing and the other opening as the microphone passes the reference position. After one revolution of the ring the contact positions are reversed again. The contacts are shown in Fig. 2.4. They are used in the remote control circuits of the tape recorder and level recorder, so that recordings may be made over exactly one revolution of the ring. The speed of the drive is such that the microphone ring makes one revolution as the polar paper on the level recorder revolves once, so that complete radiation patterns may be
plotted. Three different speeds, each synchronised with a recorder speed, are available.

Alternatively, the microphone may be attached to a carriage at any required height above the floor and moved automatically at constant speed along a rail on a horizontal diagonal of the chamber. The carriage is actuated by allowing a weight, connected to it by a cord, to fall under gravity, the speed being kept constant by a governor. The carriage may be started remotely by means of an electromagnet. A contact, attached to the carriage and rubbing on the rail, operates the remote control marking circuit of the level recorder. The surface of the rail consists of alternate inch lengths of insulating and conducting materials so that a mark is made on the level recordings every inch of the microphone's movement and the position of the microphone at any point on the recording is known. This apparatus may be used for plotting the variation of pressure with distance from the source as shown in Figs. E.1 to E.3.

2.6) Measuring Apparatus.

A block diagram and a photograph of the apparatus used to measure machine noise are shown in Figs. 2.6 and 2.8 respectively. Two condenser microphones (B & K, type 4131), either of which may be used inside or outside the chamber, are connected through a two way microphone switch (B & K, type 4408) to the microphone amplifier and analyser (B & K type 2107). The analyser is of the feedback type with variable bandwidth and a frequency rejection section. The selective section may be switched out so that the analyser may be used as a wide band
Fig. 2.6. ARRANGEMENT OF APPARATUS
FOR MACHINE NOISE MEASUREMENT.
amplifier. Also sound level meter weighting networks or external filters may be switched in. A 1/3-octave/one-octave filter (B & K type 6111) is available and may be switched by hand or by pulses in a remote control circuit. The peak, r.m.s. or mean value of the resulting signal may be read on the output meter of the analyser. The output signal may also be fed to the tape recorder (Philips Pro 20), the level recorder (B & K, type 2305), the oscilloscope and the long-time-constant mean-square calculating circuit. Usually only one or two of these output devices are required at any one time.

The level recorder produces a pen deflection proportional to the logarithm of the peak, r.m.s., mean or direct value of the signal. The scale of the movement may be changed by interchanging the scaling potentiometers, a linear potentiometer also being obtainable. There are wide ranges of recording paper and pen speeds and the recorder will take two different widths of roll paper and also polar charts.

An external drive from the recorder may be used to change the selected frequency of the analyser or a beat frequency oscillator (B & K, type 1014) which is used for calibration and for driving a vibration generator. The recorder also produces pulses which may be used in the remote control circuit of the 1/3-octave filter. For each external drive, frequency calibrated paper may be used to produce frequency spectra automatically.

The tape recorder has two channels and two tape speeds (7½ and 15 inches per second). At the highest of these speeds, the frequency response is flat within ±2dB from 60 c/s to 10 kc/s although by careful adjustment much greater accuracy than this claimed figure was obtained. A monitoring meter and monitoring output are provided in
each channel. At low frequencies, noise from underground trains passing under the building produces a sound level in the chamber as high as that produced by test machines at higher frequencies. This noise was recorded with the motor noise, producing distortion in the tape amplifier. A high-pass filter, having a critical frequency a little under 100 c/s, which is the lowest frequency of interest in the acoustic field of most machines, was therefore built and inserted in the analyser external filter circuit.

When a machine is available for only a short time, or a large amount of analysis has to be carried out, the noise is recorded on the tape recorder and then analysed by connecting the apparatus as shown in Fig. 2.7.

Several different types of measurement may be made in the field of a source using this apparatus. First, with the microphone stationary, octave, 1/3-octave and narrow band analyses may be made and the overall values measured in any position. Secondly, the variation of noise with time or change of voltage, etc., in any frequency band may be investigated. Thirdly, the microphone ring may be revolved with the microphone set to the required distances and angles and the polar diagrams plotted. Fourthly, as the ring revolves, the average sound intensity in the path of the microphone may be measured and, using several positions of the microphone on the ring, the total power may be calculated. Fifthly, the waveform of the noise may be studied and photographed on the oscilloscope.

The noise produced by a machine varies with the voltage applied to its terminals. It was found that the normal mains fluctuations of the three-phase supply to the small induction machines tested affected
their noise outputs and so a three-phase voltage stabiliser was fitted and is used to supply the machines for noise tests. For each set of noise measurements the input voltage, current and power in each phase and also the speed and slip for induction motors are noted.

A good indication of the source of machine noise may be obtained by changing the supply frequency. A large generator, driven from the laboratory d.c. supply, was installed for this purpose and is also used for tests on machines rated at frequencies other than 50 c/s. Remote controls and indication for frequency and voltage are installed close to the noise measuring apparatus so that the voltage and frequency may be quickly changed or corrected. The measuring apparatus and the machine control board are shown in Fig. 2.8.


For the first method used to calculate the total power radiated freely by an electric machine, the spherical measuring surface, obtained by keeping the microphone at a constant distance from the centre of the machine, was divided into a large number of approximately square areas and the sound pressure measured at the centre of each. To find the power, the square of the sound pressure was then calculated, multiplied by a factor proportional to the area represented by each measurement and the resulting values summed. Since the microphone could be easily moved in the horizontal plane, it was decided that the surface of the sphere should be split into horizontal bands by equally spaced lines of latitude. Each band was split into a number of equal sections so that the area for each section was approximately the same for each band.
The value of the sound pressure level at the centre of each section was taken from polar diagrams of the noise radiated by the source at each main frequency of its spectrum. A computer programme was written to calculate the power from the sound pressure level values, the radius of the measuring sphere, the number of bands and the number of sections in each band. A block diagram of the programme is shown in Fig. 2.9.

In addition to calculating the total power, the computer was also programmed to find the maximum and minimum sound pressure levels and the apparent sound power level found by averaging the sound pressure level values and correcting for the radius of the measuring sphere. Also facilities were provided for calculating the apparent power using only selected points from the data in the summation. This involves feeding steering data into the computer before the main data. This indicates the points in the main data which are not required. The calculations are then based on the assumption that each point represents a section of equal area. It was found that this assumption was completely justifiable from results using all data and comparing the results with those obtained by calculating the individual area for each band. If all sections have equal areas, then for spherical radiation, the sound power level,

\[ L_p = 10 \log_{10} 4\pi r^2 + 10 \log_{10} \frac{j}{i} \leq 10 \frac{L_{ph}}{i} \]  

(2.3)

where \( L_{ph} \) is the sound pressure level in the \( h \)th out of \( i \) sections.

The details of the ways in which the surface of the sphere was divided to obtain various numbers of points are given in Appendix A. The basic division involves splitting the surface into 272 sections. It was assumed and subsequently shown (see Sect. 7.5) that this was
**Fig 2.9.**

**Block Diagram of Computer**

**Programme to Calculate Total Power From SPL Readings.**
a sufficient number to give very small errors in the calculated power levels due to the non-uniform distribution of the sound at any frequency.

2.8) Analogue Method for Calculating Acoustic Power.

For the second method used for calculating the total power radiated freely by a machine, the surface of the measuring sphere is split into latitudinal bands as for the first method. However, instead of dividing the bands into small sections and measuring the pressure at the centre of each, the average of the square of the sound pressure along the centre line of each of the bands is measured. The voltage at the output of the frequency analyser is proportional to the sound pressure in the frequency band required. Thus, if the voltage is squared and integrated as the microphone is moved at constant speed along the centre of a latitudinal band, a voltage proportional to the average of the square of the pressure along the centre line of the band is obtained. The power radiated through this band is equal to the area of the band multiplied by \( \frac{\langle p^2 \rangle_\omega}{\rho c} \), where \( \langle p^2 \rangle_\omega \) is the average of the square of the pressure and \( \rho c \) is the characteristic impedance of the air.

Let the radius of the measuring sphere be \( r \) and the angle measured from the pole be \( \Theta \). The area of the incremental band in Fig. 2.10 is

\[
2 \pi r^2 \sin \Theta \, d\Theta
\]

and the area of the band between the angles \( \Theta \) and \( \Theta_a \) is

\[
\int_{\Theta}^{\Theta_a} 2 \pi r^2 \sin \Theta \, d\Theta
\]

\[
= 2 \pi r^2 (\cos \Theta - \cos \Theta_a)
\]

Thus, the power radiated through the band
and the total power radiated by the source is

\[ P = \frac{2\pi r^2}{\rho C} \sum_{i} (\cos \theta_i - \cos \theta_{i+1}) \text{ (2.4)} \]

where the summation is made over \( i \) bands.

A similar method has been used by Baron\(^3\) to find the average intensity over a large semi-cylindrical measuring surface enclosing a large electric machine in a test shop. In this case the microphone was moved backwards and forwards in an axial direction on the surface of the cylinder and the average intensity calculated from the rise in temperature of a block of copper heated by an element carrying a current proportional to the sound pressure. This method is inconvenient as the copper block must be allowed to cool between measurements to prevent excessive heat loss to the atmosphere during the measurement. The B.B.C.\(^4\) have used a converted kilowatt-hour meter to square and integrate the voltage and obtain the power. However, considerable modification is required to obtain accurate results. Cheissa\(^5\) describes a method for finding the average or r.m.s. value of a signal over a long period of time using a diode squaring circuit and an electronic integrator. It was decided that this method was easiest to use and so an instrument based on this principle was designed and built.

In this instrument, the signal is squared using a Hall-effect multiplier connected so that the same current passes through its plate...
and the coil producing the magnetic field. The output current is proportional to the square of the current and, therefore, from Fig. 2.11, \( v_1 \propto i^2 \). An A.E.I. type C multiplier with approximately the same current rating in the plate and coil and a sensitivity of 1.62 \( \mu V/ma^2 \) was chosen.

A power amplifier was therefore built having a flat current response when feeding the input of the multiplier and a variable gain for calibration. The current fed to the multiplier is proportional to the signal fed to the amplifier from the analyser which is proportional to the sound pressure. Thus,

\[
v_1 = k' \cdot i^2
\]  

(2.5)

In fact, there is a slight excess amplification at frequencies greater than about 5kc/s and so a calibration curve for the amplifier was obtained so that corrections could be applied at these frequencies. The output of the amplifier was made free from earth so that a potentiometer could be connected as shown in Fig. 2.11, to correct for the voltage at the output terminals produced by the driving current.
due to slight misalignment of the output terminals. In fact, this effect produces only a small error as the output current due to it is alternating and only the direct component of the output produces a significant value when integrated. The alternating output does, however, produce unwanted pick-up.

The squared signal is then integrated using a high gain d.c. amplifier (taken from an analogue computer) with capacitive feedback as shown in Fig. 2.11. The input of the amplifier has a very high impedance so that the input current can be assumed to flow into the capacitor. Thus

$$V_i C = - \int i_s dt \quad \text{and} \quad V_i = i_s R$$

The gain of the amplifier is very high and so the potential of the amplifier input is near earth potential. Thus, $V_i \approx V_s$ and

$$V_s = \frac{-1}{RC} \int V_i dt = \frac{k}{RC} \int p^2 dt \quad (2.6)$$

The total power radiated by the source

$$\frac{2\pi r^4}{k} \cdot \frac{R}{C} \sum \leq V_s \cdot \langle C, \theta_i + C, \theta_i \rangle \quad (2.7)$$

by substituting in Eqn. 2.4.

Several different values of capacitance (0.01 μF to 1 μF) and resistance (100 kΩ to 1 MΩ) are available to change the scale of the integration although for maximum accuracy the values should be as high as possible. The interchangeable analogue computer amplifier and component box units were plugged into a specially made frame which also holds a switch having discharge, earth inputs, compute and freeze positions. The output of the amplifier is measured on a 0 to 100 V d.c. voltmeter and the h.t. and heater supplies for the integrator are supplied by a separate power pack.

To measure the total power radiated by a source, the integrator
is first discharged and then switched to compute. The reading of the voltmeter is taken after the microphone ring has completed one revolution. This reading is then multiplied by \([C_a\delta - C_d\delta]\) for the latitudinal band and the sum of these products over all bands calculated. To ten times the common logarithm of the sum is added a calibration figure and corrections for the distance of measurement, the gain of the microphone amplifier, the frequency response, the time of integration and the integrator scale factor to give the sound power level of the source. The correction factors were tabulated for ease of computation.

To save time in resetting the microphone for each reading when the sound power levels at several frequencies have to be calculated, the total noise in each latitudinal band is recorded on the tape recorder. The sound power level in each frequency band is then found by replaying the tape through the analyser and power calculator. Polar diagrams of the noise in each frequency band may be obtained at the same time if required.
CHAPTER 3

INVESTIGATION OF MECHANICAL VIBRATION.

3.1) General.

As was mentioned in Chapter 1, measurements of vibration on the surface of a machine are useful to check intermediate stages in the calculation of the total noise radiated by it as a result of electromagnetic forces. Since these components of the noise are radiated by movement of the surface of the machine, the acoustic source is this entire surface. If the variation of the vibration on the surface is measured in both magnitude and phase at each main noise frequency, a suitable simplified mathematical model of the surface and its vibration may be constructed and used to predict the radiation characteristics and the total acoustic power.

If the absolute magnitude of the surface vibration at a particular point is also measured the total radiated power may be calculated using the theory developed in Chapter 6. The result of this calculation may then be compared with the measured total power and the accuracy of the calculation checked. This has been done for one machine using the techniques described in this chapter and the results are discussed in Chapter 8.

This thesis is mainly concerned with noise radiated directly by the surfaces of machines. However, in many cases this is
insignificant compared with radiation from objects mechanically coupled to the machine caused by vibration transmitted from the machine via the supports. A knowledge of the frequencies and magnitudes of the components of vibration at the mounting points must therefore be obtained so that mountings, suitable to prevent excessive mechanical vibration transmission, may be designed.

This vibration could be measured by mounting the machine on a rigid base with force transducers at each fixing point. The effective system of forces and moments could then be obtained at each frequency and the result used to calculate the response required for the mountings for a particular purpose.

In addition to its use in checking the acoustic radiation stage of the noise calculation, the measurement of vibration may also be used to check the mechanical response stage. This has been done by exciting the machines externally using a vibration generator. Vibration is produced at a known frequency and by varying this frequency the resonant frequencies of the machines may be measured and compared with the calculated resonances. If a force transducer was inserted between the vibrator and the machine, it would then be possible to check the magnitude of the response also.

This chapter describes the measurement of self-excited surface vibration made on several of the test machines and also the investigation of the resonances of the machines.

The transducers used to give voltages, for analysis, proportional to the vibration acceleration are piezoelectric vibration pick-ups (B&K type 4131). These are firmly attached to the positions on the machines where the acceleration is required. Each accelerometer is connected to a pre-amplifier which has a variable gain for calibration and also a calibrating device. The output of the pre-amplifier may be integrated once or twice to give voltages proportional to the velocity or displacement of the vibration.

Any of these three voltages may be analysed using the equipment which is used to analyse the acoustic noise. This is shown in Fig. 2.8 and described in Sect. 2.6. Thus, frequency analyses of the acceleration, velocity and displacement may be obtained. A block diagram of the apparatus used to investigate machine vibration is shown in Fig. 3.1.

For vibration measurements the machines are resiliently supported on a frame using a similar supporting arrangement to that used for acoustic noise measurements. The presence of reflecting surfaces has negligible effect on the surface vibration unless they are touching the machine. Vibration tests were therefore made in the laboratory outside the anechoic chamber rather than inside it so that the machines were near the measuring apparatus.

The measurement of surface vibration may conveniently be considered in three parts. First, the measurement of the absolute value at a representative position. Secondly, the variation of magnitude over the surface and thirdly, the variation of phase angle over the
Fig. 3.1. ARRANGEMENT OF APPARATUS

for MOTOR VIBRATION INVESTIGATIONS.
surface. These three sections will be considered separately. Although exploratory measurements of velocity and displacement were made, the main measurements were limited to acceleration. Since frequency analyses of the acceleration were made, the velocity and displacement could easily be found by division by the angular frequency.

3.2.1) The Absolute Measurement of Acceleration.

Before measurements of the actual magnitude of the acceleration on the surface of a machine were made the accelerometer was first calibrated. For this it is firmly attached to a small electromagnetically excited vibrating table and the magnitude of the vibration varied until a peak acceleration equal to the acceleration due to gravity is attained. This is judged by the movement of a small ball in a tube attached to the table. At this acceleration the amplification of the pre-amplifier is adjusted to give a known output voltage so that for subsequent output voltages the acceleration is known.

The accelerometer was then attached to the surface of the machine at the position where the acceleration was required. The reference point on most machines was taken at the side of the machine, as it is normally mounted, and near the axial centre of the stator. Since the accelerometer surface is flat and the machine surface is cylindrical, a thin packing piece was made and inserted between them. The packing piece is flat on one side and concave cylindrical on the other to fit both surfaces. The accelerometer was fixed to the surface either by a screw or by holding it pressed to the surface with a rubber strap.

Frequency analyses, similar to those shown in Figs. E.11 and E.12 were then taken using the level recorder.
3.2.2) Variation of Magnitude.

The vibration pick-up has a large contact surface and is therefore inconvenient for measuring the variation of the magnitude of the vibration on the surfaces of small machines where the variation can be very rapid. Also the accelerometer is difficult to mount on small surfaces such as bearing housings. A probe, about two inches long, was therefore attached to the accelerometer and held on the surface of the machine by hand or by using rubber straps. However, the response of the probe is not the same at all frequencies as is the response of the accelerometer. Therefore, the magnitudes of the measurements at different frequencies may not be directly compared with each other when the probe is used and the calibration is no longer valid.

If the absolute values of the components of the acceleration at a reference point have been obtained as described in the previous section, the magnitude of the acceleration at any position on the surface may be found without measuring the frequency response of the probe. Frequency analyses of the voltages produced by the accelerometer-probe combination at each position, including the reference position, may be obtained. The variation at a particular frequency may then be plotted from these analyses and referred to the absolute measurements since the absolute value at the reference point is known.

Initially, narrow band frequency analyses were made at 15 intervals round the machine and the variation at each frequency plotted. Such a diagram for one component is shown in Fig. E.4.
Secondly, the probe was moved to several other places on the machine and frequency analyses again taken. The acceleration at each position and at each frequency was then compared with that measured on the cylindrical part of the stator. From these measurements an indication of the resonances and the comparative effectiveness in the radiation of noise of the end shields, mounting flanges, starting capacitor etc. were obtained.

3.2.3) Variation of Phase.

Usually all points on the surface of a machine do not vibrate in phase with each other. In order to investigate the change of phase over a surface two vibration measuring systems are required as shown in Fig. 3.1. The output from each accelerometer is amplified and analysed so that the component at the same frequency is selected on each channel. The outputs of the two channels are connected to the 'X' and 'Y' plates of a cathode-ray oscilloscope so that the phase angle between the two signals may be observed.

The two accelerometers, with probes attached, are first held close together on the surface of the machine and the trace formed on

![Oscilloscope Traces Obtained for Various Phase Angles Between Signals](image)

**Fig. 3.2** Oscilloscope Traces Obtained for Various Phase Angles Between Signals.
the oscilloscope screen noted. This was usually a straight line as shown for zero degrees in Fig. 3.2. By moving one probe away from the other, traces similar to those shown on the other parts of Fig. 3.2 are obtained and thus the variation of phase angle round the machine may be noted.

3.2.4) Interpretation of Measurements.

From measurements it was shown that the feet on the f.h.p. machines had little effect on most components of the noise and vibration and therefore, for vibration measurements, they were removed to simplify the surfaces. It was also found that, at most frequencies, the part of the stator surface parallel to the shaft is the main radiating surface and along axial lines on this surface there is little variation in phase angle. Therefore, for most components, a two dimensional vibration distribution could be assumed.

It is shown in Sect. 5.3 that all displacement waves on the surface of a machine may be mathematically reduced to combinations of sinusoidal travelling waves. However, when they are measured, they exhibit different characteristics since all traveling wave components at a particular frequency are measured whatever the number of poles or the direction of rotation.

When standing waves are present at a particular frequency, the magnitude of the surface acceleration falls to a very small value at each node. From the number of nodes the mode number may be found by dividing by two. Fig. 3.3 shows the first three modes of vibration of a ring or infinitely long cylinder. When standing waves are present,
all points between two adjacent nodes move in phase with each other; there is a rapid change of phase at the nodes where the magnitude theoretically becomes zero. If two unequal traveling waves are present, with one traveling in each direction, then a similar pattern is obtained. However, the movement never falls to zero and the phase angle, although varying continuously over the whole periphery, changes most rapidly near the nodes.

![Diagram showing surface configurations for various modes of vibration.](image)

**Fig. 3.3.** SURFACE CONFIGURATIONS FOR VARIOUS MODES OF VIBRATION.

When only traveling waves are present the surface configurations shown in Fig. 3.3 appear to rotate. Therefore, the magnitude is constant round the machine but the phase angle varies at a constant rate. The number of times round the circumference at which the vibration is in phase with the reference is equal to the mode number. If a modulated travelling wave, caused by varying stiffness of the cylinder, is present then the phase angle will vary uniformly but the magnitude will vary according to the modulation.

In practice the measurements may not be as simple as those described here since variations of the oscilloscope trace may occur at speeds proportional to the slip frequency or the speed of rotation. Thus at some frequencies, patterns similar to that shown in Fig. 3.4a
are obtained. The number of ellipses is equal to the frequency of the vibration divided by the cyclic speed of the rotor. At some frequencies the whole pattern rotates at a speed proportional to the slip frequency. However at the in-phase and anti-phase points a straight line is usually obtained. Another difficulty occurs when the component at one frequency is large compared with that at another frequency close to it. The analyser then allows through part of the first component when tuned to the frequency of the second component. In this case patterns similar to those in Fig. 3.4 are obtained, the actual pattern depending on whether the larger component is at a higher or lower frequency than the smaller component.

Finally, on single-phase induction machines there are large tangential components of vibration at several frequencies. Thus, unless the probe is kept normal to the surface, these components interfere with the measurement of radial vibration. At twice the line frequency the tangential component is so large that it is impossible to investigate the radial component using the method described here.

3.3) Investigation of Resonances.

Many of the components of noise produced by a machine are
amplified by mechanical resonances of various parts of the machine. It is useful to investigate the resonances which may occur in a machine so that the components of the noise which are due mainly to resonances may be identified and the calculation of resonant frequencies checked. A vibration exciter, shown in Fig. 3.5, has been used to apply external vibration to machines. A block diagram of the vibrator and vibration measuring apparatus is shown in Fig. 3.6.

A coil, connected through a power amplifier to a beat-frequency oscillator, is attached to the moving table of the vibrator. The coil is placed in the field of a strong permanent magnet so that the force produced on the vibrating table is proportional to the current flowing in the coil. Thus, by varying the output of the oscillator, the frequency and magnitude of the vibration may be changed. Cooling air for the coil is supplied by a blower which is shown on the right of Fig. 3.5.

The vibrator is connected to the test motor by a wedge shaped connector which is bolted to both the motor and the vibrator. The vibration is therefore applied in a radial direction along an axial line as it would be if the machine was excited by the air-gap forces acting on the teeth. The motor is mounted on rubber blocks and its position is accurately adjusted so that only small lateral forces are transmitted by the coupling.

It is hoped that eventually a signal, proportional to the force transmitted by the coupling, will be fed back to the oscillator compression circuit as shown by the broken line in Fig. 3.6. In this way the response of the vibrator and amplifier may be made linear so that a constant force is applied to a test object at all frequencies and the response of the driving system eliminated.
**Fig. 3.6.** ARRANGEMENT OF APPARATUS

FOR RESONANCE INVESTIGATION.
The response of the machine to the vibration is measured using an accelerometer as described in Sect. 3.2. At resonances, standing waves, with a number of nodes depending on the mode of vibration, are formed. In order to pick up all resonances the accelerometer must be connected near an anti-node of each mode of vibration. It was therefore usually fixed on the same axial line as the exciter as this must be an anti-node for all modes. The output of the accelerometer is connected through an amplifier to the level recorder which is mechanically coupled to the oscillator as shown in Fig. 3.6. Frequency responses may then be plotted automatically as shown in Figs. E.15 and E.16.

In addition to knowing the resonant frequencies, it is useful to know the mode of vibration at each resonance. This may be done in a similar way to that described in Sect. 3.2. However, in this case no travelling waves are present and so the mode, at a particular resonant frequency, may be determined from the total number of stationary points. In some cases it was found to be difficult to identify the nodes as they were not equally spaced due to the one-sided excitation. Thus the oscilloscope was used: one pair of plates was connected to the oscillator output and the other pair connected to the accelerometer output. The nodes were then identified approximately, by finding the positions at which the phase of the vibration was reversed, and then identified more accurately by finding where the vibration was reduced to zero.

When the resonances of the complete machine had been investigated, they were gradually dismantled, the rotor being first removed, then the end shields, starting capacitor etc. At each stage the response was obtained so that the effect of these parts on the resonances could be
investigated. In addition to the testing of machines, the resonances of a six-inch diameter pipe coupling, shown on the left of Fig. 3.5, were investigated. The coupling has the form of a hollow cylinder and the results of this test were compared with those given by the theory for the vibration of cylinders developed in Chapter 5.
CHAPTER 4

CALCULATION OF ELECTROMAGNETIC FORCES

4.1) Introduction.

Much of the noise and vibration produced by electric machines is caused by electromagnetic forces acting on the air-gap surfaces. The production of electromagnetic forces is inherent in the operation of all electric machines. A force is produced whenever magnetic flux crosses the boundary between two media which have different values of permeability. If flux crosses normally a surface between a medium of infinite permeability and free space, then the force on the surface per unit area, as shown by Kraus (32) on p. 265, is given by:

\[ \sigma = \frac{B^2}{2\mu_0} \]

The total force on the surface may be calculated by integrating this stress over the surface. Often, unsaturated iron may be considered as an infinitely permeable medium. If this assumption is made, the flux must enter the surface normally, otherwise, owing to defraction at the surface, it would never penetrate the surface. Therefore, only normal flux need be considered if the iron has infinite permeability.

Most of the flux in electric machines crosses the air-gap in a nearly radial direction and enters the iron surfaces of the air-gap normally. This produces radial forces on the two iron members. The flux in the air gap of a machine is not uniform but varies considerably
with position and time. Thus, the radial forces also vary. The magnitudes of the flux and forces vary as the currents in the windings and the relative position of the rotor and stator vary.

The windings of many machines are situated in open or semi-closed slots and so some flux, which crosses the air-gap opposite a slot, does not enter the top of a tooth but enters the tooth side normal to its surface, most of it being concentrated near the tooth tip. This flux produces forces which act on the tooth tips in a direction tangential to the air-gap. These forces also vary considerably with time and position as do the radial forces. However, the main components, when integrated round the air-gap, produce a total force which is independent of time and supplies the shaft torque of the machine. The tangential forces can also produce pulsating torques and varying tangential forces which, when integrated round the air-gap, produce no resultant torque but do, however, produce vibration in the teeth and cores.

In machines with closed slots, infinite permeability of the iron can no longer be assumed since the slot bridges become saturated. Tangential components of force are produced on the surface, since the flux does not enter the surface normally. The main components of the tangential force again supply the shaft torque. More complicated surface integral methods, as explained by Carpenter\(^{13}\), must be used to calculate the forces in this case.

Saturation also commonly occurs at the tips of semi-closed slots and to some extent throughout the iron members, causing redistribution of the flux depending on the flux density at a particular operating condition. An accurate analysis of this complex condition is extremely
difficult and in practice it is only possible to take account of 
saturation empirically as an overall effect. Among other things which 
affect the forces produced on the teeth are air-gap eccentricity, slot 
skewing and herringboning and the exact shape of the slots.

All force variations on the iron surfaces may be analysed into 
components which vary sinusoidally in time and space. As is shown in 
 Chapters 5 and 6, some of these components are more effective in 
producing noise than others.

It is generally thought that the change in phase of the force 
along a tooth due to skewing the slots on one member reduces the 
magnitude of the vibration. This is usually a very small effect, since 
the amount of skew is often small compared with the wavelengths of the 
main force waves. An exception to this is the large water-wheel 
alternator where force waves with large numbers of poles are important. 
This is the case considered by Walker and Kerruish\(^{53}\). However, skew 
probably has little effect on the noise produced by most other types 
of machines. The main use of skewing is to reduce voltage ripples. 
Since the skew has little effect on the forces, a two-dimensional model 
of the air-gap will be considered and the force per unit length 
calculated as if the machine was infinitely long.

Whatever method is used to calculate the force on the teeth, the 
currents flowing in the stator and rotor windings must be known, either 
as harmonic slot currents or as harmonic current sheets. The harmonic 
currents due to supply, saturation, space or permeance harmonics may 
be obtained from generalised theory\(^{42}\). The resulting force system 
consists of radial and tangential forces acting at the centres of the 
tips of the stator and rotor teeth.
4.2) **Calculation of Force on Teeth Assuming Radial Flux**

A method of representing the air gap of an electric machine which is often used in machine analysis is to assume that the flux in the air-gap is radial. The currents in the slots are represented by harmonic current sheets on each member and the variation in the flux due to slots and eccentricity is represented by a variable permeance between the two current sheets as shown in Fig. 4.1b for slotted surfaces.

![Diagram](Image)

**Fig. 4.1. Models of Air-gap of Machine.**

The current distribution in the model is the apparent current distribution as viewed from the air-gap. This is expressed as a Fourier series as shown in Eqs. 4.25 and 4.27. The permeance variation is found by assuming that one member is at a constant unit magnetic potential with respect to the other and calculating the flux distribution in the system. To do this it is usual to assume that the magnetic
potential is applied to the tooth roots and calculate the apparent
path length between two equivalent points, one on the stator and the
other on the rotor. If the flux is assumed to be radial, this variation
may also be given in harmonic form. If the flux may not be assumed
radial, the flux density distribution and the effective permeance
vary across the air-gap.

In order to find the resulting flux distribution the harmonic
expression for the m.m.f. distribution, calculated from the current
distribution, is multiplied by the expression for the permeance. This
assumes that the permeance remains the same for a sinusoidally varying
m.m.f. as it is for a constant m.m.f. This approximation, while not
strictly true, has been shown to give reasonable results.

The force acting on unit area of the iron behind the current
sheets is proportional to the square of the flux density as mentioned
in Sect. 4.1. Since the teeth are no longer present in the representation,
it is assumed that the force on the actual teeth is the same as the
force per unit area calculated in this way and integrated over a tooth
pitch.

Similarly, there are no longer tooth sides on which the tangential
forces may act. However, in this air-gap model, forces equivalent to
the tangential forces act on the current sheets. The force per unit
area acting on a small section of a current sheet due to the flux
produced by all other parts of the current sheets is equal to the flux
density multiplied by the current density. The current flows axially,
the flux is radial and, therefore, the force must act tangentially.

Thus, an expression for the distribution of tangential force may be
found by multiplying the expression for the current density distribution
by the expression for the total flux density distribution.

This method may be applied to any type of electric machine. However it is probably most accurate when used for synchronous machines with lumped windings on one member. Since only one air gap surface is slotted, the magnetic field is easier to calculate in this case.

4.3) Representation of Permeance.

4.3.1) Singly Slotted Air-gap.

Consider an air gap with one iron member smooth, the other having $S$ teeth of width $\tau$ and pitch $\lambda$. It is shown by Coe and Taylor\(^{(17)}\) that for most tooth shapes used in practice, the slots may be assumed infinitely deep. The error involved in this assumption is negligible if the slot depth is greater than 1.5 times the slot width and so infinitely deep slots will be considered here. It is also assumed that the tooth widths have been adjusted to take into account fringing and saturation in the tips. The former may be corrected for by using Carter's coefficients (E.g. as given by Gibbs\(^{(22)}\) in Eqn. 9.72). The latter may be corrected for by reducing the actual tooth width slightly.

The variation of permeance due to slotting is shown in Fig. 4.2. The permeance is equal to $\frac{\mu_s}{g}$ where a tooth is present and zero elsewhere.

Carrying out a Fourier analysis

$$\Lambda_0 = \frac{1}{i\pi} \int \frac{\mu_s}{\lambda \phi} \text{d}\phi$$

$$= \frac{\mu_s \tau}{\lambda g}$$

$$= \frac{\mu_s}{\mu_g}$$

(4.1)

where $\mu_g$ is the gap coefficient.
The total permeance is given by the series

\[ \Lambda_s = \sum_{j=0}^{j=\infty} \Lambda_{h,j} \cos(j \phi) \]

Basing the harmonic values on the average permeance this becomes

\[ \Lambda = \Lambda_o \left[ \sum_{j=0}^{j=\infty} \Lambda_{h,j} \cos(j \phi) \right] \]  \hspace{1cm} (4.3)

where

\[ \Lambda_{h,j} = \frac{2 \mu_0}{2 \pi j n} \sin \left( \frac{\pi j \pi}{\lambda} \right) \]

for \( j \neq 0 \)  \hspace{1cm} (4.4)

and

\[ \Lambda_{h,0} = 1 \]

These values are independent of the air-gap length. The values of these per unit permeance harmonics are plotted in Fig. 4.3 against \( \gamma/\lambda \) for values of \( j \) from one to six.

If the average permeance of the air-gap, \( \Lambda_o \), can be calculated either from Eqn. 4.1 or by using Carter's coefficient, the harmonic permeances may be found by multiplying \( \Lambda_o \) by the values obtained from Fig. 4.3. This method does not take into account the redistribution of flux due to fringing and therefore will not produce accurate results especially for the higher harmonics.
The complete permeance variation for a slotted stator

\[ \Lambda_{\text{st}} = \Lambda_{\text{st}}^{\text{cos}} \left( \sum_{j=0}^{\infty} \Lambda_{\text{ps},j} \cos j \omega t \right) \]  

(4.5)

and for a slotted rotor

\[ \Lambda_{\text{rt}} = \Lambda_{\text{rt}}^{\text{cos}} \left( \sum_{j=0}^{\infty} \Lambda_{\text{ps},j} \cos j \omega t \right) \]  

(4.6)

4.3.2) Doubly Slotted Air-gap.

There are two main ways of combining the permeances of two singly slotted air-gaps to give the permeance of a doubly slotted air-gap. Both give results of the same form but with different magnitudes. The form is given by

\[ \Lambda = \Lambda_{\text{st}}^{\text{cos}} \left( \sum_{j=0}^{\infty} \Lambda_{\text{ps},j} \cos j \omega t \right) \]  

(4.7)

where \( \Lambda_{\text{ps},0} = 1 \) and \( \Lambda_{\text{st}} = \frac{\mu_0}{j} = \frac{\mu_0}{j} k_j \)  

(4.8)

and \( k_j \) is the gap coefficient for the two slotted surfaces taken together. The different methods give different values for \( \Lambda_{\text{st}} \).

In the first method, used by Harris (24), the air gap is divided into two equal parts by a surface which is assumed to be an equipotential. The permeances between the two iron surfaces and the equipotential are given by Eqns 4.5 and 4.6 where

\[ \Lambda_{\text{st}} = \frac{2 \mu_0 \tau_{\text{st}}}{j} \]  

and \( \Lambda_{\text{rt}} = \frac{2 \mu_0 \tau_{\text{rt}}}{j} \)  

The total permeance between the two iron surfaces at any position is obtained by summing the permeances in the same way as series conductances. Thus, the total permeance,  

\[ \frac{\Lambda_{\text{rt}} \cdot \Lambda_{\text{st}}}{\Lambda_{\text{rt}} + \Lambda_{\text{st}}} \]  

If the fluctuations are small, the variable components in the denominator may be neglected and the permeance becomes

\[ \frac{2 \mu_0}{j} \Lambda_{\text{st}}^{\text{cos}} \left( \sum_{j=0}^{\infty} \Lambda_{\text{ps},j} \cos j \omega t \right) \]
Therefore  \[ K_s = \frac{1}{2} \left( K_{g,r} + K_{g,t} \right) \]  

The second method is obtained from the form given by Alger and Kron. The per unit permeances developed in Sect. 4.3.1 are the ratios of the magnitudes of the harmonic permeances to the average permeance. The permeance at a position in the air-gap when only one slotted surface is present is the average permeance multiplied by the series for the harmonics. If this permeance is taken as the basis for the series giving the harmonics of the other slotted surface, the total permeance is the average value multiplied by the two series. 

\[
\Lambda = \Lambda_o \left[ \sum \sum \frac{i}{j} \Lambda_{p_k j_k} \Lambda_{p_j j_t} \cos \left( \frac{\pi}{s} \right) S_{m_j j_t} r - S_{m_j j_t} \cos \theta \right]
\]

**Fig. 4.4. Diagram Defining Equivalent Air-Gap Lengths.**

To obtain an expression for \( \Lambda_o \) consider the rotor and stator shown in Fig. 4.4. The two broken lines define surfaces behind the teeth such that the distance between them is the apparent air-gap length. Consider each set of teeth opposite a plane surface at the position of the broken line on the other member. The effective permeance of the air-gap is given by

\[
\frac{\mu_o}{\gamma'_{g,r}} = \frac{\mu_o}{\gamma_{g,r}} = \frac{\mu_o}{J_{g,s}} = \frac{\mu_o}{\gamma'_{g,s}}
\]
From Fig. 4.4

\[ g' = g_{sr} + j_{rr} - g \]

Therefore eliminating \( g_{sr}' \) and \( g_{rr}' \)

\[
\kappa_3 = \frac{1}{\left(\frac{1}{\kappa_{sr}'} + \frac{1}{\kappa_{rr}'} - 1\right)} \quad (4.10)
\]

Other methods have been used to find \( \Lambda \) for a doubly slotted air-gap. Say\(^{(45)}\) uses the gap coefficients for the two slotted surfaces multiplied together so that

\[
k_3 = k_{3sr} \cdot k_{3rr} \quad (4.11)
\]

Carter also states that this form is used in design and also gives the form

\[
k_3 = (k_{1sr} + k_{1rr} - 1) \quad (4.12)
\]

Comparing the values given by these two forms with the value obtained using conformal transforms to solve the field between two thin parallel plates with slots in them, Carter prefers the latter approximation.

Binns\(^{(7)}\) suggests the empirical form

\[
k_3 = \frac{1}{2} \left( k_{3sr} + k_{3rr} - 1 \right) \quad (4.13)
\]

as this gives a value within 0.1% of the value obtained using conformal transforms to solve the field between two iron surfaces with infinitely deep slots in them. This form for \( k_3 \) will therefore be used to find \( \Lambda \) for a doubly slotted air-gap.

4.3.3) Permeance Harmonics Due to Air-gap Eccentricity,

To calculate the effect of eccentricity on the variation of permeance round the air-gap it is first necessary to obtain an expression for the variation of the gap length with position in the air-gap. This may be found in terms of the gap length when the rotor and stator are concentric, \( g \), and the eccentricity, \( e \). Either the stator or the
rotor may be eccentric. In the former case the centre of the rotor air-gap surface is at the axis of rotation but the centre of the stator air-gap surface is not. In the latter case the centre of the rotor air gap surface is not at the centre of rotation. The two cases produce similar permeance variations except for the time variation which appears in the expression for the rotor.

Consider the stator and rotor surfaces as two smooth iron surfaces with different centres as in Fig. 4.5. If the eccentricity is small, the centre of rotation may be assumed to be half-way between the two centres. The analysis may then be applied to both stator and rotor eccentricity. Using a modification of the cosine rule on the two triangles in Fig. 4.5,

\[ AB = \frac{e_i}{2} \cos \phi + \sqrt{\frac{e_i^2 \cos \phi}{4} - \left( \frac{e_i}{2} - r_{slr} \right)^2} \]

\[ AC = \frac{e_i}{2} \cos \phi + \sqrt{\frac{e_i^2 \cos \phi}{4} - \left( \frac{e_i}{2} - r_{slr} \right)^2} \]

Thus,
Since \( e \) is always much less than \( \xi_{r} \) or \( \xi_{s} \), this may be simplified to

\[
8 \epsilon = (\xi_{r} - \xi_{s}) + A \cos \phi
\]

Thus assuming radial flux, the general expression for the permeance between the two smooth surfaces is

\[
\frac{\mu_{0}}{g + e \cos \phi}
\]

(4.15)

Swann (50) shows that an expression similar to this, when multiplied by the fundamental m.m.f. wave, produces results similar to those produced by using conformal transforms to solve the field problem.

Although the denominator of this expression contains only the fundamental term in \( \phi \), the permeance will contain harmonics and so a harmonic analysis must be carried out. The average component is given by

\[
\Lambda_{e,0} = \frac{1}{2\pi} \int_{0}^{\pi} \frac{\mu_{0}}{g + e \cos \phi} \, d\phi
\]

(4.16)

and the harmonic components by

\[
\Lambda_{e,k} = \frac{1}{\pi} \int_{0}^{\pi} \frac{\mu_{0} \cos k\phi}{g + e \cos \phi} \, d\phi
\]

(4.17)

From Dwight (18) 858.536

\[
\int_{0}^{\pi} \frac{\cos k\phi}{1 - c_{2}\cos \phi} \, d\phi = \frac{\pi}{(\xi_{2})^{k}} \left\{ \sqrt{1 - (\xi_{2})^{2}} - 1 \right\}
\]

Thus

\[
\Lambda_{e,0} = \frac{\mu_{0}}{\frac{1}{3} \sqrt{1 - (\xi_{2})^{2}}}
\]

(4.18)

and

\[
\Lambda_{e,k} = \frac{2\mu_{0} (g)^{k}}{\frac{3}{3} \left\{ \sqrt{1 - (\xi_{2})^{2}} - 1 \right\}}
\]

(4.19)

In per unit terms as for the slot permeance, the permeance due to rotor eccentricity is

\[
\Lambda_{e,0,\epsilon_{r}} \leq \Lambda_{e,0,\epsilon_{r}} \cos \epsilon_{r}(\phi - \epsilon_{r})
\]

(4.20)

and for the stator eccentricity

\[
\Lambda_{e,0,\epsilon_{s}} \leq \Lambda_{e,0,\epsilon_{s}} \cos \epsilon_{s}(\phi)
\]

(4.21)

where

\[
\Lambda_{e,0,\epsilon_{r}} = \frac{1}{\sqrt{1 - (\xi_{2})^{2}}}
\]

and

\[
\Lambda_{e,0,\epsilon_{s}} = \frac{1}{\sqrt{1 - (\xi_{2})^{2}}}
\]

(4.22)
and $\Lambda$ is the average permeance of the air-gap if the iron members were concentric. When teeth are present the values of $\eta$ should be replaced by $\eta \kappa_\eta$, where $\kappa_\eta$ is the gap coefficient due to the teeth.

As for the slot permeance, the eccentricity permeance due to eccentricity of both members may be obtained by multiplying the two per unit permeances together if the eccentricity is small. Thus,

$$
\Lambda = \frac{\Lambda_s}{2} \sum \sum \Lambda_{p_s,t_s} \Lambda_{p_r,t_r} \cos \left[ (C_{s} + x_{s}) \phi - C_{r} \omega_{r} t \right]
$$

(4.23)

4.4) **Representation of M.M.F. Waves.**

The total m.m.f. acting across the air-gap is the sum of the individual m.m.f.s due to the rotor and the stator. Assuming that the iron surfaces of the air-gap form equipotential surfaces, there is a change of m.m.f. only where a slot is crossed. The total change in m.m.f. across a slot is equal to the current flowing in the slot as shown in Fig. 4.6.
If the diagram is continued for all slots, the form of the m.m.f. wave may be found and a harmonic analysis of it carried out (40). The m.m.f. distribution then consists of waves travelling in positive and negative directions in the air-gap. For the stator, the general form of the m.m.f. is given by

\[ F_{st} = \sum \sum F_{p,k,q,s,t} \cos (k_{st} \phi - q_{st} \omega_s t) \quad (4.24) \]

where \( k \) is the number of pole pairs of the space distribution for a particular component and \( q \) is the order of the harmonic current flowing in the slot and may take positive or negative values according to the direction of rotation of the wave. If \( q \) is zero the expression gives the m.m.f of a d.c. winding. In a.c. windings the space harmonics may be divided into two types; slotting harmonics where \( k \) is a multiple of the number of slots, and phase belt harmonics where \( k \) is the harmonic of the phase belt polygon.

The differential of the m.m.f. with respect to distance round the air-gap gives the current per unit peripheral length in the equivalent current sheet.

\[ = \sum \sum \frac{-k_{st}}{i_3} F_{p,k,q,s,t} \sin (k_{st} \phi - q_{st} \omega_s t) \quad (4.25) \]

Similar forms apply for the rotor where \( \phi \) is replaced by \((\phi - \omega_s t)\). However, in induction machines, the fundamental rotor currents are at slip frequency and the m.m.f. is given by

\[ F_{rt} = \sum \sum F_{p,r,k,q,s,t} C_{r,s} [k_{rt}(\phi - \omega_r t) - q_{rt} \omega_r s t] \]

\[ = \sum \sum \frac{-k_{rt}}{i_3} F_{p,r,k,q,s,t} C_{r,s} [k_{rt} \phi - (k_{rt} \omega_r t + q_{rt} \omega_r s t)] \quad (4.26) \]

and the current in the current sheet

\[ = \sum \sum \frac{-k_{rt}}{i_3} F_{p,r,k,q,s,t} \sin [k_{rt} \phi - (k_{rt} \omega_r t + q_{rt} \omega_r s t)] \quad (4.27) \]

This does not include all induction machine rotor components as components induced by stator harmonics are not included. However, this does give
4.5) **Calculation of Force Waves.**

Expressions have been obtained in Sect. 4.3 for the per unit change of permeance at a position in the air-gap due to eccentricity and slotting separately. The actual permeance variation round the gap due to eccentricity is obtained by multiplying the per unit value by the average permeance of the gap. This permeance may then be taken as the basis for the per unit slotting permeance and the total permeance found by multiplying the two series together giving

\[
\Delta \omega \sum_{p, j, m, r} \Lambda_{p, j, m, r} \Lambda_{p, j, m, r} \Lambda_{p, j, m, r} A \left[ (j \mu S + j \mu z \omega S + j \mu \tau + j \mu \omega \tau \omega) - (j \mu S + j \mu z \omega S + j \mu \tau + j \mu \omega \tau \omega) \right]
\]

This expression would give the variation of flux density round the air-gap when a constant m.m.f. is applied across it. In fact the m.m.f. is not constant but also varies with position and time. The total variation of flux density round the air-gap is the product of the permeance and the m.m.f. if the redistribution of flux due to the changing m.m.f along the air-gap is neglected. The error involved here will probably not be greater than the error involved in calculating the permeance variation. Thus, the flux density becomes

\[
B = \sum_{k, h, r, t, \omega} \cos \left[ (k + j \mu S + j \mu z \omega S + j \mu \tau + j \mu \omega \tau \omega) \phi - (k + j \mu S + j \mu z \omega S + j \mu \tau + j \mu \omega \tau \omega) \right] + \sum_{k, h, r, t, \omega} \cos \left[ (k + j \mu S + j \mu z \omega S + j \mu \tau + j \mu \omega \tau \omega) \phi - (k + j \mu S + j \mu z \omega S + j \mu \tau + j \mu \omega \tau \omega) \right]
\]

Any phase angles will be ignored in this expression since it may be assumed that when \( t = 0 \), \( \phi = 0 \) at a position where a rotor tooth is opposite a stator tooth and maximum eccentricity occurs. Thus, only the phase angle between the rotor and stator m.m.f. need be considered.
This must be inserted when the force waves resulting from the interaction of m.m.f. waves produced by both the rotor and the stator are considered. \( j_{st}, j_{sr}, \ell_{s}, \) and \( \ell_{tr} \) may equally well take positive and negative values for each permeance harmonic. The values of the amplitudes are given by

\[
B_{j,k,l,q,r,m} = \frac{\Delta_{j}}{16} \text{Fr}_{k,q,r,m} \Lambda_{p_{tr},c_{tr}} \Lambda_{p_{sr},c_{sr}} \Lambda_{p_{st},c_{st}}, \nabla_{p_{st},r_{st}} \nabla_{p_{sr},r_{sr}}
\]

The radial force per unit area

\[
\sigma_{r} = \frac{B_{r}}{2\mu_{0}}
\]

\[
= \sum_{\omega k} \sigma_{r,k,l,q,m} \cos \left[ (k' + j_{st} s_{tr} + \ell_{s} + j_{sr} s_{sr} + \ell_{tr}) + q' \omega_{0} + j_{st} s_{tr} + \ell_{s} + j_{sr} s_{sr} + \ell_{tr} \right]
\]

where

\[
\sigma_{r,j,k,l,q,m} = \frac{1}{4\mu_{0}} \sum_{k} \sum_{l} B_{j,k,l,q,m} \Lambda_{p_{tr},c_{tr}} \Lambda_{p_{sr},c_{sr}} \Lambda_{p_{st},c_{st}} \Lambda_{p_{st},r_{st}} \nabla_{p_{st},r_{st}}
\]

for values of \( i \) and \( k \) which satisfy the following conditions

\[
k' = \left( k_{st} \pm k_{sr} \right) \quad \text{or} \quad \left( k_{sr} \pm k_{tr} \right) \quad \text{or} \quad \left( k_{tr} \pm k_{tr} \right)
\]

\[
q' = \left( q_{st} \pm q_{sr} \right) \quad \left[ \left( q_{st} \pm q_{st} \right) \right] \left( q_{tr} \pm q_{tr} \right) \right)
\]

\[
J' = j_{st} \pm j_{sr}
\]

\[
\ell' = \ell_{s} \pm \ell_{tr}
\]

The \( \pm \) signs must be the same in each expression. \( k \) takes only positive values, \( q \) has a sign depending on the direction of rotation of the m.m.f. wave and \( j \) and \( \ell \) may take positive or negative signs even when they are zero.

This may be reduced to

\[
\sigma_{r} = \sum_{m} \sigma_{r,m} \cos (m \phi + \omega t)
\]

where

\[
\sigma_{r,m} = \sum_{k' l' q'} \sigma_{r,k',l',q',m}
\]

and the summation is made over all components such that

\[
\left( k' + j_{st} s_{tr} + \ell_{s} + j_{sr} s_{sr} + \ell_{tr} \right) = m
\]

and

\[
\left( q' \omega_{0} + j_{st} s_{tr} + \ell_{s} + j_{sr} s_{sr} + \ell_{tr} \right) = \omega
\]

Similarly, as explained in Sect. 4.2, the tangential force per unit area, \( \sigma_{\theta} = 8 i \)
from Eqs. 4.25, 4.27 and 4.28 where $B$ and $i$ are 90° out of phase.
The amplitude is given by

$$
\sigma_{\phi,i,j,k_1,k_2,m} = \sum_{k_1} \sum_{k_2} \sum_{m} \frac{k_i}{k_j} \tilde{E}_{n,m} \cdot B_{k_1 k_2} \cdot \sin(n \phi + \omega_t \cdot t)
$$

(4.36)

summed over all values of $k_1, k_2, k_2$ and $k_2$ as defined in Eqn. 4.32.

The total tangential force

$$
\sigma_{\phi} = \sum_{\phi, m, \omega} \sigma_{\phi, m, \omega} \cos(m \phi \pm \omega t)
$$

(4.37)

where

$$
m = \left( k_1 \cdot j \cdot \omega_t + \omega_r \right)
$$

and

$$
\omega = \left( \gamma_1 \omega_0 + \omega_r \cdot \omega_t \right)
$$

(4.38)

and $\gamma_0 + \gamma_0$ is the phase angle between $B$ and $i$ and also the angle between the radial and tangential forces of the same frequency.

Consider a tooth with its centre at a position $\phi$, in the air-gap.

Half the angular pitch of the tooth is $\frac{\lambda}{2 \phi}$ and the total radial force on the tooth per unit axial length,

$$
\Sigma_r = \tau_3 \int_{\phi - \frac{\lambda}{2 \phi}}^{\phi + \frac{\lambda}{2 \phi}} \sum_{m} \sigma_{\phi, m, \omega} \cos(m \phi \pm \omega t) \, d\phi
$$

$$
= \tau_3 \sum_{m} \frac{1}{m} \sin \left( \frac{m \phi + \lambda}{2 \phi} \right) \cdot \left( \cos \left( \frac{m \phi - \lambda}{2 \phi} \right) - \cos \left( \frac{m \phi + \lambda}{2 \phi} \right) \right) \pm \omega t
$$

$$
= \tau_3 \sum_{m} \frac{1}{m} \sin \left( \frac{m \phi + \lambda}{2 \phi} \right) \cdot \cos \left( m \phi \pm \omega t \right)
$$

(4.39)

In a similar way the tangential force per unit length of tooth

$$
\Sigma_{\phi} = \tau_3 \sum_{m} \frac{1}{m} \sin \left( \frac{m \phi}{2 \phi} \right) \cdot \cos \left( m \phi \pm \omega t \pm \omega t \right)
$$

$$
= \tau_3 \sum_{m} \sum_{j, i} \sum_{k_1, k_2, m} \cos \left( m \phi \pm \omega t + \omega t \right)
$$

(4.40)

There are many approximations involved in calculating the forces on the teeth by this method. A method using conformal transforms would probably produce more accurate results. An outline of this method
is given in Appendix F. However, much calculation is involved in
applying this method to a doubly slotted air gap. The method given
in this chapter is therefore used to determine which force waves are
present and roughly to estimate their magnitudes. It is, therefore, not
expected that this method will give accurate results.

The components of the forces on the teeth which need to be
considered are determined by the mechanical response of the stator
and rotor as discussed in Chapter 5 and the radiating properties of the
surface as discussed in Chapter 6. The frequencies of the components
which are most likely to be mechanically transmitted depend on the
resonant frequencies for the particular mode of vibration. The components
most likely to radiate noise from the surface are those with small
values of $m$ and high values of $\omega$. In larger machines, components
with appropriately larger values of $m$ may produce potent noise sources.

Once the range of important values of $m$ and $\omega$ have been
ascertained for a particular machine, Eqs 4.34 or 4.38 may be inspected,
bearing in mind the machine data, and any force waves in this range
investigated. When the components have been selected, the summations
of Eqs. 4.31, 4.33 and 4.36 may be made and the magnitude of the
force wave calculated. From this the radiated noise may be calculated.

This method of calculating the force wave magnitude may be
applied to most types of machine although it is shown here in a form
suitable for the induction machine. By making $k_r=\gamma_r$ and $\gamma_r=0$, the
force waves in a salient pole synchronous machine may be found. This
is of a similar form to that used by Walker and Kerruish. Saturation harmonics may also be simulated by making $k_r=\gamma_r$ for particular
components of the m.m.f. wave.
Forms for both the radial and tangential forces are given here. Although both must be borne in mind, it is shown in Sect. 8.4 that the radial component is generally of more importance.
5.1) **Introduction.**

In Chapter 4, expressions were obtained for the radial and tangential magnetic forces acting on the tips of the teeth of a machine. These forces are transmitted to the rotor and stator via the teeth. The vibration is then transmitted to the bearings, end shields, mounting brackets, terminal boxes etc. If the resonant frequencies of any of these parts are near frequencies of force waves on the teeth then large displacements may occur producing noise.

The main radiating surface of a machine is the frame enclosing the stator laminations and most of the published work analysing the noise produced by machines deals with the frequency response of this part of the machine. This is actually a problem involving the responses of short cylinders \(^{(52)}\). However, it is usually assumed that, since the teeth on which the forces are produced are axial and even if skewed produce little force variation in the axial direction, the stator may be treated as a section of an infinitely long cylinder so that the axial modes of vibration may be neglected. In some machines, however, clamping bolts etc. may induce axial modes.

In small machines the stator shell is often formed of sheet steel which is wrapped round the stator laminations. In this case the laminations and stator shell may be considered as one thick cylinder. In larger machines the stator laminations form a separate cylinder
from a fabricated or cast shell, the two cylinders being separated by stiffeners and key bars. In this case the machine must be considered as two mechanically linked cylinders.

Alger\(^{(2)}\) and Walker and Kerruish\(^{(53)}\) assume that the stator may be represented by a single thin ring. The ring is then split peripherally into a number, equal to the number of force wave nodes, of short, simply supported beams. The beams are then assumed to be straight and the simple theory for their static displacement applied. Alger then states a more accurate form for cylinders but with no explanation. Walker and Kerruish use the straight beam theory directly. This is probably justifiable since the only force waves which are considered have a large number of nodes so that the short beams may be assumed straight. In each case the effect of resonance is applied as a multiplying factor after the static displacement has been calculated.

Carter\(^{(14)}\) develops an expression for the response of a ring having both radial and tangential forces applied at its inner radius. This expression takes into account both tension and bending in the ring. Using this method the dynamic response of the ring may be calculated directly and two resonances for each mode predicted.

Erdelyi\(^{(21)}\), using an energy method, considers the combined movement of two coupled concentric rings. When a double stator construction is present, the simple methods cannot be used to represent the movement of the outer surface from which most of the sound is radiated and so a procedure of this type must be used.

In this chapter the responses of the teeth, stator core, and rotor will be investigated. The iron will be assumed to be perfectly elastic with no losses. The losses will actually change the resonant
frequencies slightly but the expressions developed give approximate values for the actual resonant frequencies. No attempt will be made to take into account the losses in calculating the displacement for a given force. The calculated displacements are therefore only applicable at frequencies away from resonance. At frequencies close to resonance the calculated values will be considerably higher than the measured values.

5.2) Vibration of Rotor and Stator Teeth.

Since the electromagnetic forces act primarily on the ends of the teeth, the reaction of the teeth must first be considered. For simplicity, each tooth will be considered as a short cantilever, held firmly at one end by the rotor or stator core. Longitudinal and lateral forces act on the other end as shown in Fig. 5.1. The extent of the cantilever in the direction perpendicular to the forces will be considered to be infinite so that axial modes of vibration may be neglected. In fact the core does not hold the teeth firmly but tends to move with the teeth and so vibration is produced in the core. However, the results obtained using these simplifications will give a

![Diagram of electromagnetic forces acting on a tooth.](image)
guide to the actual results which may be expected.

In the following sections the frequencies of the resonances induced by the radial and tangential forces, acting on the ends of the teeth, will be found together with expressions for the forces transmitted to the core when the teeth are not resonating.

5.2.1) Resonances Excited by Radial Forces.

Radial force waves, acting on the ends of the teeth, produce longitudinal motion of the teeth. If the teeth were sufficiently long, alternate areas of compression and rarefaction would be produced along the teeth forming longitudinal waves similar to sound waves in air. For longitudinal waves in an elastic medium the velocity of propagation, as given by Kinsler and Frey on p.58, is

$$c = \sqrt{\frac{E}{\rho}}$$ (5.1)

For iron, this velocity is 5300 m/s. Thus, at 20 kc/c, the upper limit of the audible frequency range, the wavelength of the longitudinal waves in iron is 0.27 m.

The longitudinal resonances of a tooth occur when waves, reflected from the inner or outer surfaces of the core, reinforce the original waves. The first resonance occurs when the tooth length is equal to half the wavelength of the vibration. Thus, the shortest tooth in which a resonance in the audible range can occur, is 0.13 m long. It is therefore apparent that longitudinal tooth resonances will only be found in large machines and at high frequencies.

Reflection may also occur at the outer surface of the stator core. However, resonances would again only occur in large machines and at high frequencies. It may therefore be assumed that, in small
machines, the radial forces acting at the tooth tips may be considered
to act at the tooth roots without any change due to the dynamic response
of the teeth.

5.2.2) Resonances Excited by Tangential Forces.

Although, in many machines, the tangential force components are
small compared with the radial components, they act in such a direction
as to cause bending moments to act on the teeth. The resonant frequencies
of the bending modes are lower than the resonant frequencies of the
longitudinal modes and so large displacements may occur at frequencies
within the audible range. The resonance is caused when the transfer
of energy between the energy stored in bending and the kinetic energy
of the moving tooth occurs at an optimal rate. Fig.5.2 shows the
configurations of a cantilever for the first three modes of bending.

It is shown by Morse (37) that the resonant frequencies of a
uniform rectangular cantilever,

\[ f_{r,i} = \frac{2 \pi k_i^2}{(\eta - \rho^2)^{1/2}} \]  

(5.2)

where the radius of gyration about the neutral axis is \( \eta / \sqrt{\pi} \), \( \eta - \rho \) is
the length and \( \tau \) is the width of the cantilever and the values of \( k_i \)
are given by

\[ \cot^2 \left( \pi k_i \right) = -\eta / k_i^2 \]

Thus, \( k_i \) may take the values 0.2985, 0.745, 1.25, ..., \( \frac{1}{2} (i + \frac{1}{2}) \) when \( i \)
is larger than three. From this, the first four overtones of the
vibration are 6.27, 17.5, 34.4 and 53.4 times the frequency of the
fundamental mode which for an iron cantilever is

\[ f_r = \frac{870 \tau}{(\eta - \rho)^2} \]  

(5.3)

where all lengths are in metres.

This formula applies to parallel teeth and may also be used to
find the resonant frequencies of tapered teeth if the taper is not too steep and the average width of the tooth is substituted for $\gamma$. Alternatively, if the teeth have steep tapers, the resonant frequencies may be found using energy methods $^{(36)}$. It is apparent from the above analysis that only in very large machines will any resonance except that of the fundamental mode of vibration be important.

In fact, the teeth are not equivalent to cantilevers as they are linked to the elastic core. Therefore, when teeth resonate displacement of the core also occurs. This will cause noise to be radiated from the surface of the machine. Thus, the main importance of these resonances is not the noise radiated by the resonating parts but the vibration transmitted to other parts.

The windings in the slots of a machine impose forces on the teeth resisting the vibration. These will damp the vibration of the teeth, the amount of damping depending on the type of winding. In addition to reducing the movement these forces may also change the resonant frequencies slightly.

![Diagram](image)

Fig. 5.2. **First 3 Lateral Modes of Vibration of Cantilever.**
5.2.3) Calculation of Forces Transmitted by Teeth.

If the frequencies of the forces acting on the teeth of a machine are not near the resonant frequencies of the teeth, the transmitted forces may be found by equating the forces acting on the tips of the teeth to the restraining forces applied by the core and neglecting the forces required to accelerate the teeth. The system of forces is shown in Fig. 5.3. $\Sigma_r$ and $\Sigma_\phi$ are forces per unit axial length of the tooth and $\sigma_r$ and $\sigma_\phi$ are forces per unit area of the tooth root.

\[ \sigma_{r,c} = \frac{\Sigma_r}{T_r} \]

Similarly, the tangential stress at the tooth roots,

\[ \sigma_{\phi,c} = \kappa \cdot \frac{\Sigma_\phi}{T_r} \quad (5.4) \]

where $\kappa$ is a constant to take into account the forces applied to the sides of the teeth by the windings in the slots. This will depend on the type of winding, type of insulation etc. The radial stress shown in the above equation in not the total radial stress but is
a stress equivalent to the broken line in Fig. 5.3b. There will also be a stress which varies over the tooth root and opposes the moment at the tooth root due to $\Sigma_\phi$. This stress is proportional to the distance from the centre line of the tooth and is equal to

$$\left(\sigma_{r_c,\infty} - \sigma_{r_c,\infty}\right) \frac{2t_c}{\Sigma_\phi} \phi'$$

where $\phi'$ is the angle measured from the centre of the tooth. Thus, where $k_1$ takes into account the moment applied by the windings, the magnitude of this extra stress may be found by equating the moments of the forces about the centre of the tooth root

$$k_1 (r_c - r_i) \Sigma_\phi = \frac{2\pi^2 r_c^2}{\Sigma_\phi} \int (\sigma_{r_c,\infty} - \sigma_{r_c,\infty}) \phi' \phi' d\phi'$$

Thus,

$$\left(\sigma_{r_c,\infty} - \sigma_{r_c,\infty}\right) = \frac{6k_1 (r_c - r_i) \Sigma_\phi}{t_c}$$

and the total radial stress,

$$\sigma_{r_c} = \frac{\Sigma_{r_c}}{t_c} + \frac{12k_1 (r_c - r_i) \Sigma_\phi}{t_c} \phi'$$

(5.5)

Both $\Sigma_r$ and $\Sigma_\phi$ vary considerably with the position of the tooth and with time. However, the time variation of $\sigma_{r_c}$ and $\sigma_{\phi_c}$ will be the same as the time variation of the tooth tip forces causing them. Only the spatial distribution of the forces will be changed. The spatial variation of $\sigma_{r_c}$ and $\sigma_{\phi_c}$ over a tooth is shown in Figs. 5.3b and 5.3c.

The magnitudes of these pulses vary with the tooth tip forces and are given by Eqns. 5.4 and 5.5. If, with time kept constant, Fourier analyses of these expressions are obtained, the magnitudes of the force waves acting on the core may be obtained. The Fourier analyses are carried out by multiplying the expressions for $\sigma_{r_c}$ and $\sigma_{\phi_c}$ by $\cos m' \phi$ where $m'$ is the number of pole pairs of the required wave, and integrating over the whole core surface. Thus, the radial force
wave component,

$$\sigma_{r,c,m,\omega} = \frac{j \omega}{k} \sum_{m=1}^{\infty} \int \left[ \frac{\Sigma_r + i \frac{2 L}{r_i}}{2 \Sigma_\phi} \Sigma_\phi (r_i - r_o) \right] \cos (m \phi + \varphi_i) \, d\phi.$$  \hspace{1cm} (5.6)

The values of $\Sigma_r$ and $\Sigma_\phi$ are constant for a particular tooth at a particular time and therefore their variations do not affect the integral but only the summation. Splitting the integral into two parts, the first part becomes

$$\frac{j \omega}{k} \sum_{m=1}^{\infty} \frac{2 \Sigma_r}{m r_i} \sin \left( \frac{m \phi + \varphi_i}{2 r_i} \right) \cos (m' \varphi_i)$$

Substituting for $\Sigma_r$ from Eqn. 4.39 this becomes

$$\frac{j \omega}{k} \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} \frac{\Sigma_r m_{m'} m_{m'} \omega \sin \left( \frac{m \phi + \varphi_i}{2 r_i} \right)}{m r_i} \cos (m \varphi_i + m' \varphi_i)$$ \hspace{1cm} (5.7)

for a particular value of $\omega$ at one time. Since the values of $\varphi_i$ are multiples of $\frac{2\pi}{S}$ the outside summation gives a non-zero result only when $(m' + m)$ is zero or a multiple of $S$. The main solution occurs when $m' = m$ giving terms of the form

$$\frac{5 \omega}{k} \sum_{m=1}^{\infty} \frac{\Sigma_r m_{m'} m_{m'} \omega \sin \left( \frac{m \phi + \varphi_i}{2 r_i} \right) \cos (m \phi + \omega t)}{m r_i}$$ \hspace{1cm} (5.8)

This is the original wave multiplied by a constant. The other non-zero parts of Eqn. 5.7 are force waves with numbers of poles which are multiples of the number of slots plus or minus the number of pole pairs of the original wave. These would be expected from a study of the stress distributions in Fig. 5.3.

Integrating the second term of Eqn. 5.6 by parts gives

$$\frac{j \omega}{k} \sum_{m=1}^{\infty} \frac{2 L}{r_i} \Sigma_\phi \left[ \sin \left( \frac{m \phi + \varphi_i}{2 r_i} \right) - \frac{m \phi}{2 r_i} \cos \left( \frac{m \phi + \varphi_i}{2 r_i} \right) \right] \sin (m' \varphi_i)$$ \hspace{1cm} (5.9)

The form of $\Sigma_\phi$ is given by Eqn. 4.40. Substituting and summing as before the main terms are of the form

$$\frac{12 \omega}{k} \frac{L}{r_i} \Sigma_\phi \sum_{m=1}^{\infty} \frac{m \omega \sin \left( \frac{m \phi + \varphi_i}{2 r_i} \right) - \frac{m \phi}{2 r_i} \cos \left( \frac{m \phi + \varphi_i}{2 r_i} \right) \cos (m \phi + \omega t + \varphi_i)}{m^3 r_i}$$ \hspace{1cm} (5.10)

These components have the same number of poles and the same frequency as the tangentential components at the tooth tips but in terms of a cosine function instead of a sine function showing a phase shift of $90^\circ$. 
In addition, there are components with different values of \( m' \) where
\( m' \neq m \) is a multiple of the number of slots and \( m \) is the number of pole pairs of the original force wave.

Combining the main terms for the radial stress at the core and summing over all values of \( m \) and \( \omega \),

\[
\sigma_{r2} = \frac{S_2}{\pi m \gamma} \sin \left( \frac{m \gamma}{2 \gamma} \right) \left[ \sum_{n} \sum_{m, \omega} C_{m, \omega} \cos (\omega x + \omega t) + \frac{12k \gamma (r_i - r_f)}{\pi \gamma} \sum_{m, \omega} C_{m, \omega} \cos (\omega x + \omega t + \frac{\pi}{2}) \right] \tag{5.11}
\]

The tangential component of the stress acting on the core may be found in a similar way by carrying out a Fourier analysis of Eqn. 5.4. Thus,

\[
\sigma_{t2} = \frac{S_2}{\pi m \gamma} \sin \left( \frac{m \gamma}{2 \gamma} \right) \sum_{n} \sum_{m, \omega} C_{m, \omega} \sin (\omega x + \omega t + \frac{\pi}{2}) \tag{5.12}
\]

where the phase angle is the same as that in Eqn. 5.11.

As mentioned previously, Eqns. 5.11 and 5.12 do not give the total force variation on the core. The other components normally have high numbers of pole pairs and so may be ignored as little noise will be radiated by them. In cases where force waves have numbers of pole-pairs similar to the number of teeth these terms may become important and noise may be produced. The terms are given completely by Eqns. 5.11 and 5.12 if \( m \) is replaced by \( m' \) where

\[ m' = m + iS \]

The number of pole pairs of the original wave is \( m \) and \( i \) is any integer. In some cases such waves may reinforce waves given directly by Eqns. 5.11 and 5.12.

5.3) **Types of Wave Produced on the Core.**

It was shown in Sect. 5.2.3 that when force waves, as calculated
in Chapter 4, act on the tooth tips, waves of the same form act on the stator core. The displacement of the surface of the machine depends on the force wave applied to the core and its mechanical response. There are four main cases to be considered and in each case it may be shown that the surface displacement may be split into travelling displacement waves.

First, a travelling flux wave produces a travelling force wave plus a constant force. This constant term may be obtained from Eqn. 4.30 by making the harmonic numbers equal to zero. A similar procedure may be followed for the tangential forces. However, if the phase angle between the current and flux waves at the same frequency is 90° as assumed in Chapter 4, there is no constant force. If the machine is on load the phase angle is not 90° and the constant tangential force depends on the cosine of the phase angle. This is as would be expected since this is the component of force which produces torque. The torque is zero in the no load case analysed in Chapter 4. The general form for both the radial and the tangential forces produced by travelling flux waves is

\[ \sigma = \sigma_0 + \sum \sum B_{m,n} \cos(m \phi \omega t + \phi_{m,n}) \]  

(5.13)

Secondly, there may be standing flux waves produced by disymmetries due to single-phase windings, eccentricity, etc. These are given by expressions of the form

\[ B = \sum \sum B_{m,n} \cos(m \phi \omega t + \phi_{m,n}) \]  

Eqn. 4.30 gives the general form of the force wave obtained when the whole series of flux waves is squared. If one standing wave term is squared and expanded, the resulting force wave,

\[ \sigma_{m,n} = \sum \sum \left[ 1 + \frac{1}{2} \cos(m \phi \omega t + \phi_{m,n}) + \frac{1}{2} \cos(m \phi - \omega t + \phi_{m,n}) + \cos(2m \phi) + \cos(2 \omega t + 2 \phi_{m,n}) \right] \]  

(5.14)
This expression consists of a constant term, two travelling waves rotating in opposite directions, a space distribution, independent of time, and a component varying in time but constant in space. The latter two expressions are special cases of travelling waves with first \( \omega \) and then \( \nu \) equal to zero.

Thirdly, there may be standing waves with non-infinite standing wave ratios. In this case the wave may be split into two travelling waves of different magnitudes, rotating in opposite directions.

Fourthly, it has so far been assumed that the response of the stator core is the same at all positions round its circumference so that the surface displacement is proportional to the force. If there are welded mounting flanges etc. this may not be so and the displacement wave will be modulated. If the modulation is split into a Fourier series, the resulting displacement waves are of the form

\[
E = \sum_{m,n} \sum_{\nu} E_{m,n,\nu} \cos(mf + \nu \omega) \cos(mf + \nu \omega)
\]

\[
= \frac{1}{2} \sum_{m,n} \sum_{\nu} \left\{ \cos[(m+\nu)f + \nu \omega + k] + \cos[(m+\nu)f + \nu \omega - k] \right\}
\]

This is again in the form of travelling waves.

It is therefore apparent that any surface displacement variation produced by force waves may be expressed in terms of travelling displacement waves together with their degenerate forms. If the analyses for the stator shell response and the radiation from the surface are expressed in terms of travelling waves then any conditions may be satisfied. The analysis of the stator core as a regular ring only will be developed in this chapter and so the fourth case will not be considered in detail. The radiation equations in Chapter 6 will, however, apply in this case.

The two degenerate forms of the displacement wave where there
is no time variation cannot produce noise and will, therefore, be ignored. The third form is equivalent to a pulsating hydraulic pressure acting inside the stator core and causing simple tangential extension in it. Since this requires simple tensional elongation of the iron a large force is required to produce a small movement and so it is unlikely that appreciable noise will be produced. However, if the response of the stator core is not the same at all angles, these forces may produce standing displacement waves on the surface as may be seen from Eqn. 5.15 with \( \omega \) equal to zero. Thus, if the frequency of any of these force waves is near any of the stator resonant frequencies, a potent noise source may be produced.

Forces of the third degenerate type may also be produced in a tangential direction by the interaction of standing current and flux waves. Since these forces act on all teeth in the same direction, they produce an oscillating torque. These forces are most predominant in single-phase induction machines, especially at twice the line frequency. The movement of the machine due to these forces is circumferential, the magnitude depending on the mounting. Any surface of a machine facing in a circumferential direction, such as the feet or starting capacitor, will radiate this noise. Often these forces are transmitted to other apparatus by the mountings. Such transmitted forces can cause potent noise sources.

5.4) **Response of Stator Core.**

In Sect. 5.2.3 the forces transmitted by the teeth to the core
of a machine were calculated. The expressions developed give the sinusoidally distributed components of force, with various numbers of poles, travelling round the rotor or stator core. The stator core response for a single shell type of stator construction will now be considered.

It is assumed that axial modes of vibration will not be excited so that the core may be considered as a ring, reducing the problem to two dimensions. It is also assumed that the laminated core and the stator shell vibrate as one ring and that the thickness of the ring is small compared with the radius. The forces acting on the ring produce bending, tension and shear in the ring and all of these forces are considered although shear strains are ignored, it being assumed that if a plane was perpendicular to the neutral axis before being displaced it will also be perpendicular to it after displacement. The effect of the tension in the ring on the bending moment and any damping effect in the iron are also ignored.

Carter\(^{14}\) states a solution to this problem and a derivation, bearing in mind the above assumptions, is given in Appendix B. Thus, the radial surface displacement due to radial and tangential forces acting inside the ring,

\[
\varepsilon_r = \frac{\partial \epsilon_z \tau_z^2}{J_c E'} \left\{ \sigma_{\tau_c} + \left[ H \frac{\tau_c}{\tau_o} + 1 - \frac{\tau_c}{\tau_o} \right] \frac{\partial \sigma_{\tau_c}}{\partial \phi} \right\}
\]

\[
= \frac{m^2(m^2-1)(1-H) - (1+H) M \tau_c^2 \omega^2}{E' J_c} (5.16)
\]

where

\[
H = \frac{1}{m^2 - M \tau_c^2 \omega^2 \rho_c E'}
\]

\[
(5.17)
\]

5.4.1) Static Displacement of Stator Shell.

The static displacement of the stator core when sinusoidally varying radial and tangential forces are applied is obtained by making
the frequency in Eqn. 5.16 equal to zero. Thus, the static displacement,

\[ F_r = \frac{D_0 r_c r_{ac}^3}{J_c E' (m^2-1)^2} \left\{ \sigma_{rc} + \left( \frac{r_c}{r_{ac}} - 1 - \frac{r_{ac}}{r_c} \right) \frac{d^2 \sigma_{rc}}{d \theta^2} \right\} \]  

(5.18)

If \( W_m \) is the total, sinusoidally distributed, radially applied load over one pole of the force wave, the radial stress applied to the core,

\[ \sigma_{rc} = \frac{m \omega^2 W_m}{2 \pi r_c} \]

Then the displacement due to radial forces only

\[ F_r = \frac{6m r_{ac}^3 W_m}{(r_{ac} - r_c)^2 E' (m^2-1)^2} \]

(5.19)

since, when the stator diameter is large compared with the core depth,

\[ J_c = \frac{(r_{ac} - r_c)^3 D_0}{12} \]

(5.20)

When values of \( m \) are substituted,

\[ F_r = \frac{1}{6} \frac{(2 r_c)^3 W_m}{(r_{ac} - r_c)^2 E'} \quad \text{for } m = 2 \]

(5.21)

\[ F_r = \frac{9}{16} \frac{(2 r_c)^3 W_m}{(r_{ac} - r_c)^2 E'} \quad \text{for } m = 3 \text{ etc.} \]

These are the formulae quoted by Alger in Eqn. 10.5 for bending modes in a ring, showing that at frequencies below resonance the two methods are equivalent when only radial forces are considered. The more general expression in Eqn. 5.18 also includes the effect of tangential forces acting on the stator.

5.4.2) Resonant Frequencies.

If only radial components of the force are considered, there will be resonances of the stator when the denominator of Eqn. 5.16 becomes zero.

\[ m^2 (m^2-1)(1-H) - (1+H) M r_{ac}^3 \omega^2 / E' J_c = 0 \]  

(5.22)
From this equation two resonant frequencies may be found for each value of \( m \); one is associated with the bending modes of vibration and the other with longitudinal vibration travelling circumferentially. Normally, the frequency of the former is much lower than the frequency of the latter which is associated with a discontinuity in the value of \( H \). For the bending mode resonances it may, therefore, be assumed that

\[ H \approx \frac{1}{m^2} \]

giving the bending resonant frequencies

\[ \omega_{bc} = \frac{m(m^2-1)}{\sqrt{1+m^2}} \sqrt{\frac{E}{J_c}} \frac{1}{\rho_s} \]

However,

\[ M = k_s (r_{sc} - r_c) \]

where \( k_s \) is the ratio of the mass of the teeth and core to the mass of the core. Substituting for \( M \) and for \( J_\perp \) from Eqn. 5.20,

\[ f_{bc} = \frac{m(m^2-1)(r_{sc} - r_c)}{4\pi \sqrt{3} \sqrt{1+m^2}} \frac{E'}{r_{nc}^2 \sqrt{\rho_{sc} k_s}} \]

For iron, this may be simplified to

\[ f_{bc} = \frac{248 (r_{sc} - r_c)}{r_{nc}^2 \sqrt{3 k_s}} \times \text{Factor for mode} \]

where the factor is given in Table 5.1. Eqn. 5.23 is of the same form as that quoted by Alger (2) for the resonant frequencies of a thin cylindrical shell.

When \( m \) is unity, there is a resonance at zero frequency. However, since this involves lateral movement of the whole machine, the displacement may be reduced by the supports at low frequencies. At higher frequencies the supports would not restrain the movement so much and the displacement would be near that predicted by Eqn. 5.16.

At frequencies higher than the bending resonant frequency the second term of Eqn. 5.22 predominates. Thus, at a frequency where \( (1 + H) \) is near zero, the whole expression is near zero and the second resonance occurs.
TABLE 5.1

<table>
<thead>
<tr>
<th>m</th>
<th>Transverse mode</th>
<th>Longitudinal mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1.414</td>
</tr>
<tr>
<td>2</td>
<td>2.65</td>
<td>2.23</td>
</tr>
<tr>
<td>3</td>
<td>7.6</td>
<td>3.16</td>
</tr>
<tr>
<td>4</td>
<td>14.5</td>
<td>4.12</td>
</tr>
<tr>
<td>5</td>
<td>22.5</td>
<td>5.09</td>
</tr>
<tr>
<td>6</td>
<td>34.5</td>
<td>6.07</td>
</tr>
<tr>
<td>7</td>
<td>47.5</td>
<td>7.06</td>
</tr>
</tbody>
</table>

Thus, from Eqn. 5.17, the second resonant frequency for each value of $m$,

$$f_{re} = \frac{\sqrt{m^2 + 1}}{2\pi \tau_\infty} \sqrt{\frac{\alpha}{M}}$$

$$= \frac{\sqrt{m^2 + 1}}{2\pi \tau_\infty} \sqrt{\frac{E'}{k_s}}$$

(5.24)

since $\alpha = D_3 (v_{\infty} - v_s)$.

For iron, this may be simplified to

$$f_{re} = \frac{862}{2\pi \tau_\infty} \sqrt{\frac{1}{k_s}} \times \text{factor for mode}$$

where the factor for the mode is given in Table 5.1.

If $m$ is large so that the sections of the ring could be considered as straight and if the ring had no teeth attached to it

$$f_{re} = \left(\frac{m}{2\pi \tau_\infty}\right) \sqrt{\frac{E'}{\rho_s}}$$

(5.25)

The former section of this expression is the inverse of the wavelength of the force wave and the latter part the velocity of longitudinal waves in iron. This demonstrates that these resonances
are due to longitudinal waves, repeating an exact number of times round the shell.

The same resonant frequencies apply to the tangential forces, although the second resonance at each value of \( m \) is slightly modified by the presence of \( H \) in the numerator of Eqn. 5.16. The two expressions in Eqns. 5.23 and 5.24 produce accurate results only when the two resonant frequencies are not close together. If the ring is thick or \( m \) is large or if the actual magnitude of the vibration is required, Eqn. 5.16 may not be split up and the whole expression must be computed for various values of \( \omega \) and the resonant frequencies found by plotting the results. In this way the complete frequency response may be found. A computer programme has been written to determine the response and a modified form of it used in the calculation of the total acoustic power from the air-gap forces.

5.4.3) Response of Stator Core and Teeth.

Eqn. 5.16 gives the response of the stator core represented by a thin cylinder acted on by sinusoidally distributed forces. The forces are applied by the tooth roots and the magnitudes of the forces are given by Eqns. 5.11 and 5.12. Substituting in Eqn. 5.16, the surface displacement

\[
\mathbf{E} = \frac{D_{\beta}}{\Delta m} \mathbf{S}_c \mathbf{S}_w \mathbf{S}_t \sum_{m, n, \omega} \mathbf{C}_m (m \omega s w) \left[ \frac{-2 k_m \left( r_m - r_\psi \right)}{m \Delta_{\omega}} \right] \mathbf{G}_m (r_m \Delta_{\omega}) + \mathbf{m} \left[ H \mathbf{G}_m (r_m \Delta_{\omega}) \right] \mathbf{G}_m (r_m \Delta_{\omega}) \sum_{m, n, \omega} \mathbf{C}_m (m \omega s w) \mathbf{F}_c \mathbf{F}_w \mathbf{F}_t
\]

(5.26)
5.5) **Rotor Resonances.**

In addition to stator and tooth resonances, rotor resonances associated with its critical speeds may also be excited. If a rotor runs at a speed near one of its critical speeds any slight unbalance of the rotor tends to make its centre of gravity move radially away from its centre of rotation and high rotor eccentricity may be produced. The movement of the centre of gravity round the centre of rotation produces a radial force on the bearings travelling at the speed of the rotor. This force is transmitted via the end plates to the stator surface causing noise to be radiated. Shafts are normally designed so that the first critical speed is much higher than the running speed. The main exception is the two pole turbo-alternator which is often run at speeds above the first critical speed. Considerable noise may be produced in this case as the machine is run up to speed.

It was shown in Sect. 4.5 that force waves having one pole pair but higher frequencies than the speed of rotation could be produced on the teeth of a machine. These may be caused by an odd number of rotor or stator slots or by eccentricity harmonics. These force waves produce a one sided pull on the rotor tending to make its centre of gravity move away from its centre of rotation. If the speed of any of these force waves is near the first critical speed of the rotor assembly, resonance occurs and noise is produced as explained above. If the length of the shaft between the rotor laminations and the bearings is not the same at each end, the rotor is axially unsymmetrical and resonances at the higher critical speeds may be excited.

It is important, therefore, to calculate the critical speeds of
the rotor, even though they are not near the running speed. These may then be compared with the frequencies of the single pole pair force waves produced in the air-gap. If they coincide, it is probable that there is a potent noise source at this frequency.

The accurate calculation of the resonant frequencies of a complex rotor assembly may be carried out using a computer or by a graphical method as described by Morrill (36). However, a rough calculation may be made as described in the following section.

5.5.1) Calculation of First Critical Speed of Rotor.

It may be shown (36) that the critical speeds of a rotating assembly are the same as the resonant frequencies of the assembly acting as a beam if similar constraints act on the supports of the beam and the bearings of the rotor. It will be assumed that the bearing area is small, or that the bearings are self-aligning so that the equivalent beam is simply supported. A typical rotor assembly for a small induction machine is shown in Fig. 5.4a. If the rotor is large compared with the diameter of the shaft, it may be assumed that all of the weight is concentrated at the centre of the shaft as shown.

![Diagram of Rotor Assembly](image)

**Fig. 5.4. Simplified Diagram of Rotor.**
in Fig. 5.4b. The shaft is assumed weightless but with a stiffness similar to the stiffness of the original shaft.

The static displacement of the beam with a load equal to the mass of the rotor at its centre (36)

\[ E = \frac{J_r \pi r_s^4 D \cdot d \left( \frac{3}{2} D_s^4 - d^4 \right) g}{12 E J_a} \]

where \( d \) is the distance from one end. It is shown by Morrill (Eqn. 5.16) that for a weightless beam with point loads, the natural angular frequency is given by

\[ \omega_n^2 = \frac{g \sum W_i \frac{E_i}{E}}{\sum W_i \frac{F_i}{F}} \]  

where the summation is made over all loads such as \( W_i \) and \( F_i \) is the displacement at the point of application of \( W_i \) due to all loads. For a single load at the centre of the shaft this reduces to

\[ \omega_n^2 = \frac{9g}{E} \]

\[ = \frac{48 E J_a}{J_r \pi r_s^4 D \cdot D_s^4} \]

The second moment of area of the shaft about a diameter,

\[ J_a = \frac{\pi}{4} r_a^4 \]

Therefore the the cyclic natural frequency or critical speed

\[ f_c = \frac{1}{2 \pi} \frac{\omega_n}{D_s r_s} \sqrt{\frac{12 E}{J_r D_s D_s}} \]

\[ = 0.55 \frac{r_a^4}{D_s r_s} \sqrt{\frac{E}{J_r D_s D_s}} \]

\[ = 2.970 \frac{r_a^4}{D_s r_s^3} \sqrt{\frac{1}{D_s D_s}} \]  

(5.28)

for a steel shaft and core. It is assumed that the dynamic deflection curve is the same as the static deflection curve. This is usually a reasonable approximation.

The expression given in Eqn. 5.28 gives reasonably accurate results if modifications are made to some dimensions. The value of
$D_1$ should be made less than the actual distance between the bearings to take account of the extra stiffening due to the rotor laminations. Also, an estimate of the effective diameter of the shaft must be made. This should be between the two shaft sizes shown in Fig. 5.4.
CHAPTER 6

THEORETICAL INVESTIGATION OF ACOUSTIC FIELD

6.1) Representation of Acoustic Source

In general, an electric machine is not an easy source to represent in mathematical form. First, the geometric shape is often not simple; fundamentally, most machines are short cylinders with rounded ends. To this basic shape are added such things as terminal boxes and mounting flanges. Also, vibration is often transmitted to other objects to which the machine is attached. These then radiate sound and must therefore be considered as part of the source. In order to simplify the problem, a resiliently supported machine will be considered and the surrounding objects treated as separate acoustic sources driven by the machine mounting flanges.

Secondly, machines are often mounted close to acoustically reflecting surfaces such as walls and floors. Thus, the field conditions are not simple under practical constraints. However, reasonable approximations to the sound levels obtained in the field of a machine under practical conditions may be determined from the results of calculations assuming free-field conditions. If total acoustic power is accepted as a noise criterion, this may be measured under various acoustic conditions as explained in Sect. 2.2.

Thirdly, the phase and magnitude of the vibration on the surface of a machine are not the same at all points. Most of the driving forces for the electromagnetically excited vibration are transmitted to the
casing by the stator core. The magnitude of the surface vibration will therefore be greatest at the centre of the stator and decrease to a small value at the bearings; the actual magnitude will vary with the frequency of the vibration and the resonant frequencies of the parts making up the stator casing. The magnitude of the vibration of the connecting boxes etc. will depend on where they are mounted and on the resonant frequencies of their mountings. Since the driving force wave on the stator laminations depends on the flux wave, the vibration round the stator periphery is sinusoidally distributed. As is shown in Sect. 5.3, the possible distributions may be represented by combinations of travelling waves moving round the machine. Thus, there is a variation of phase on the surface of the machine, although points on the same axial line will be in phase if the slots are not excessively skewed.

Vibration produced by mechanical and electromagnetic forces acting on the rotor is transmitted by the bearings to the stator. This vibration, together with that produced by the bearings, may also be represented by travelling waves on the stator although the variation of magnitude in the axial direction may be different.

Fourthly, the air surrounding an openly-ventilated machine is moving and much of the aerodynamic noise, together with some magnetic and mechanical noise, is carried from the machine by the cooling air. This may not be treated by simple radiation theory and so a machine with a small air flow will be considered here. The problem of the acoustic power radiated by the air-flow could be considered separately.

The problem may therefore be reduced to a source which is a short cylinder with rounded ends having traveling waves of known axial variation moving round it, radiating freely in all directions. It will
be noticed that, for this representation to be physically possible, the distribution must be such as to give stationary points at both ends of the machine.

Several investigators have developed solutions to this problem. Alger\(^2\) represents a machine by a cylinder of indefinite length with a sinusoidal distribution of vibration round its periphery. This method would probably yield reasonable results for the pressure close to the side of the machine where the ends have little effect. It is difficult, however, to calculate the total power radiated by the source as the power integrating surface passes through its surface. Erdelyi\(^{21}\), extending Alger's work, considers the source as a cylinder of indefinite length having a vibrating band with sinusoidally varying waves travelling round it. The solution developed gives the sound pressure at a point close to the stator. However, since the method involves a numerical integration at each point where the pressure is required, it is unsuitable for calculating the total power radiated by a machine.

A paper has recently been published by Williams et al.\(^{55}\) solving the field of a short cylinder with closed ends. This may be of use in solving the field of a machine since the total power may be found by integrating round the source and the pressure at the ends of the machine may be calculated.

Walker and Kerruish\(^{53}\) assume a simple source and so use the inverse square law to calculate the sound pressure at a point although they assume an infinitely long cylinder in another part by using Alger's formula for finding the noise reduction factor due to source size. This probably yields sufficient accuracy in finding the pressure at points close to the stator of large machines but for small machines where the noise reduction due to the effective size of the source is
more critical, it would probably be inadequate.

Carter\(^{(14)}\), in finding the effect of source size and mode of vibration on the intensity of the sound produced by a machine, considers the source to be a sphere with sinusoidal travelling waves round the periphery and a \((\text{sine})^4\) distribution in the axial direction for a four pole-pair vibration wave. Carter's published work is limited to a demonstration of the intensity cut-off effect as the effective size of the source varies but it is felt that this approach could be used for a general study of the sound radiated by a machine and also for calculating the total power.

The solution will therefore be considered in terms of spherical co-ordinates as defined in Fig. 6.1 where \(r\) is the radial distance from the centre of the source.

**Fig. 6.1.**

Spherical Polar Co-ordinates Used in Analysis of the Acoustic Field of a Machine.

6.2) Formulation and Separation of Wave Equation.

As was stated in Sect. 2.1, a machine is fundamentally a constant velocity acoustic source. However, the acoustic velocity field is a vector field requiring a vector wave equation and so the scalar pressure field will first be solved. The variation of instantaneous sound pressure \(p\) in a homogeneous, elastic medium may be described by the wave equation as derived by Beranek (Ref. 4, p. 22)
Since a spherical source has been assumed, the Laplacian operator is best expressed in spherical co-ordinates as shown in Fig. 6.1. Thus the wave equation becomes

\[ \frac{1}{r^2} \frac{\partial^2 p}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial p}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 p}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \]  

(6.1)

Any solution of this equation may be formed from combinations of separable solutions (Ref. 39, p. 497) of the form

\[ p = \tilde{f}_r \cdot \tilde{f}_\theta \cdot \tilde{f}_\phi \cdot \tilde{f}_t \]

where \( \tilde{f}_r, \tilde{f}_\theta, \tilde{f}_\phi \) and \( \tilde{f}_t \) are functions of \( r, \theta, \phi \) and \( t \) respectively.

Thus, the wave equation becomes

\[ \frac{\tilde{f}_r \tilde{f}_\theta \tilde{f}_\phi \frac{d}{dr} \left( r^2 \frac{d \tilde{f}_r}{dr} \right) + \frac{\tilde{f}_r \tilde{f}_\theta \tilde{f}_\phi}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d \tilde{f}_r}{d\theta} \right) + \frac{\tilde{f}_r \tilde{f}_\theta \tilde{f}_\phi \tilde{f}_t}{r^2 \sin^2 \theta} \frac{d^2 \tilde{f}_t}{d\phi^2} - \frac{\tilde{f}_r \tilde{f}_\theta \tilde{f}_\phi \tilde{f}_t}{c^2} \frac{d^2 \tilde{f}_t}{dt^2} = 0 \]

(6.2)

and, dividing through by \( p \), the right hand side of the equation contains only functions of \( t \), there being no function of \( t \) on the other side. Thus, each side of the equation must be equal to a constant which, for later convenience will be called \( -\frac{\omega^2}{c^2} \) where \( c \) is the velocity of sound.

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \tilde{f}_r}{dr} \right) \left( \frac{\sin \theta}{r^2} \frac{d}{d\theta} \left( \sin \theta \frac{d \tilde{f}_r}{d\theta} \right) \right) - \frac{1}{r^2 \sin^2 \theta} \frac{d^2 \tilde{f}_r}{d\phi^2} = \frac{1}{c^2} \frac{d^2 \tilde{f}_t}{dt^2} = -\omega^2 \]

Thus, from the right hand side

\[ \frac{d^2 \tilde{f}_r}{dr^2} + \omega^2 \tilde{f}_t = 0 \]

(6.4)

Multiplying the remaining equation by \( r^2 \) and rearranging

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \tilde{f}_r}{dr} \right) + \frac{\omega^2 r^2}{c^2} = -\frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d \tilde{f}_r}{d\theta} \right) - \frac{1}{r^2 \sin^2 \theta} \frac{d^2 \tilde{f}_r}{d\phi^2} = \eta' \]

where \( \eta' \) is another constant, since the two sides of the equation are again independent. Thus, an equation in terms of \( r \) only is obtained.

\[ \frac{d}{dr} \left( r^2 \frac{d \tilde{f}_r}{dr} \right) + \left( \frac{\omega^2 r^2}{c^2} - \eta' \right) = 0 \]

(6.5)

Multiplying the remaining equation by \( \sin^2 \theta \) and rearranging

\[ \frac{\sin \theta}{r^2} \frac{d}{d\theta} \left( \sin \theta \frac{d \tilde{f}_r}{d\theta} \right) + \eta' \sin^2 \theta = -\frac{1}{r^2 \sin \theta} \frac{d}{d\phi^2} \frac{d \tilde{f}_r}{d\phi^2} = \eta' \]

where \( \eta \) is another constant. The remaining two variables may therefore be separated to give
The resulting separated equations, 6.4 to 6.7, will now be considered separately and a general solution for each found. These will then be multiplied together and an expression for the particle velocity found and fitted to the boundary conditions on the surface of the machine.

6.3) **Solution of Separated Equations.**

6.3.1) **Time Variation.**

Eqn. 6.4, governing the time variation, is the equation for simple harmonic motion. The solution is well known and is most conveniently expressed in the following form

\[ E_t = A \, e^{i\omega t} + A_1 \, e^{-i\omega t} \]  
(6.8)

The more general solution is the sum of many such terms with different values of \( \omega \). Thus any variation with time may be expressed in terms of this solution by splitting it into its Fourier components.

6.3.2) **\( \phi \) Variation.**

The equation for \( E_\phi \) is in the same form as the equation for \( E_t \) and therefore gives a solution of the same form

\[ E_\phi = A_1 \, e^{i\omega \phi} + A_2 \, e^{-i\omega \phi} \]  
(6.9)

However, in this case the independent variable is the angle round the sphere so that the dependent variable must repeat itself if \( \phi \) continues to increase after passing round the sphere once. Thus, the value at \( \phi + 2\pi \) must be equal to the value at \( \phi \). Thus transforming to trigonometric form and substituting this condition
\[ \cos (m\phi + \psi) = \cos [m(\phi + 2n) + \psi] \]

where \( \psi \) is an angle depending on \( A_1 \) and \( A_4 \). Applying trigonometric identities this becomes

\[ \sin (m\phi + \psi + \pi n) \cdot \sin (m\pi) = 0 \]

This is only satisfied if \( \sin (m\pi) \) is zero. Therefore, \( m \) must be zero or an integer.

6.3.3) \( \Theta \) Variation.

This is governed by Eqn. 6.7. Substituting \( \gamma \) for \( \cos \Theta \) and rearranging, this becomes

\[ \frac{d}{d\gamma} \left[ (\gamma^2) \frac{d}{d\gamma} f_\theta \right] + \left[ n' + \frac{m}{(\gamma^2)} \right] f_\theta = 0 \]

This is the associated Legendre equation which may be solved by a series method. Since \( \gamma \) has limits of 1 and -1, the series must converge over this range. This limits the values of \( n' \) to \( n(n+1) \) where \( n \) is an integer. The equation then becomes

\[ \frac{d}{d\gamma} \left[ (\gamma^2) \frac{d}{d\gamma} f_\theta \right] + \left[ n(n+1) - \frac{m}{(\gamma^2)} \right] f_\theta = 0 \]

The solution is the sum of associated Legendre functions of the first and second kind. The second kind tend to infinity at the points \( \gamma = \pm 1 \) and so only the first kind is of any interest when considering fields containing these points. (Ref. 47, p. 406 and 38, p. 1264 show more detailed solutions of this equation.)

Thus,

\[ f_\theta = \Lambda_\gamma P_n^\gamma (\gamma) = \Lambda_\gamma P_\gamma^\gamma (\cos \theta) \]

which may be expressed in the following form, where \( P_\gamma^\gamma \) is the Legendre function

\[ P_n^\gamma = (\gamma^2)^{\gamma^2} \frac{d^n}{d\gamma^n} P_\gamma^\gamma (\gamma) \]

\[ = \frac{(\gamma^2)^{\gamma^2}}{2^n n!} \frac{d^n}{d\gamma^n} (\gamma^2-1)^n \]

\[ (6.10) \]

It may be shown that associated Legendre functions form an orthogonal set and that
This is proved in Ref. 51, p. 699 and will be of use later in this analysis.

6.3.4) Radial Variation.

This is governed by Eqn. 6.5. Substituting the expressions
\[
x = \omega' \frac{r}{c}
\]
\[
\Phi_r = \frac{x^4}{r^4} \Phi_r
\]
Eqn. 6.5 becomes
\[
x^4 \frac{d^4}{dx^4} + x \frac{d}{dx} + \left[ x^2 - n(n+1) - \frac{1}{4} \right] \Phi_r = 0
\]
which is Bessel's equation of order \((n+\frac{1}{2})\). The most convenient form of the solution for \(\Phi_r\) is the sum of Hankel functions of the first and second kind, being equivalent to the exponential solution of the simple harmonic motion equation.

\(\Phi_r\) may now be found from Eqn. 6.13 by substituting for \(\Phi_r\).

\[
\Phi_r = A_6 h_n^{(1)}(x) + A_7 h_n^{(2)}(x)
\]
where \(h_n^{(1)}\) and \(h_n^{(2)}\) are spherical Hankel functions of the first and second kinds, defined as ordinary Hankel functions of order \((n+\frac{1}{2})\) divided by \(\sqrt{x^2 + \frac{X^2}{W}}\). (See Refs. 37, p. 264 and 47, p. 404 for their properties)

For convenience, these functions will be modified. Stratton, in Ref. 47, p. 405, shows that they may be represented by exponential functions with power series multipliers, where the power series end after \((n+1)\) terms. Thus, from Stratton

\[
h_n^{(1)}(x) = (-j)^{n+1} \ e^{\frac{ix}{2}} \left( \eta_n'(x) + j \eta_n''(x) \right)
\]
\[
h_n^{(2)}(x) = j^{n+1} \ e^{\frac{-ix}{2}} \left( \eta_n'(x) - j \eta_n''(x) \right)
\]
where

\[
\eta_n'(x) = \frac{1}{x} - \frac{n(n+2)}{2x} + \frac{n(n+1)(n+2)}{2x^2} - \frac{n(n+1)(n+2)(n+3)}{2x^3}
\]
and

\[
\eta_n''(x) = \frac{n(n+1)}{2x^2} - \frac{n(n+1)(n+2)(n+3)}{2x^3}
\]
Thus, 
\[ \Phi = j^{m n} \left\{ A_1 (-1)^m (\eta_1^* + j \eta_2^*) e^{i \phi} + A_1 (-1)^n (\eta_1^* - j \eta_2^*) e^{-i \phi} \right\} (6.16) \]

It is convenient to continue to use \( z \) as an independent variable, it being regarded as an apparent radius depending on the frequency.

6.4) Total Solution for Pressure.

The general solution for the pressure at any point in the field is obtained by substituting the partial solutions in the separation equation. The total solution is the sum of all such solutions as \( m, n \) and \( \omega \) take all their combinations of values. Thus,

\[ \Phi = \sum \sum A_s P_{m n}^0 \left[ A_s e^{i \phi} + A_s e^{-i \phi} \right] \]

\[ = \sum \sum A_s j^{m n} \left[ A_s A_s A_s A_s (-1)^m (\eta_1^* + j \eta_2^*) e^{i \phi} + A_s A_s A_s A_s (\eta_1^* - j \eta_2^*) e^{-i \phi} \right] (6.17) \]

Examination of this equation shows that the first and third lines of this expression represent waves traveling towards the source. These are irrelevant to the present investigation and will therefore be dropped as only waves radiated directly by the source need be considered for free-field radiation. The remaining two lines represent waves travelling away from the source but moving in opposite directions round it. For simplicity, only one component will be considered and \( \phi \) will be substituted for \( (x \sin \phi - \omega t) \). Thus, a typical solution is

\[ \Phi = A_s j^{m n} P_{m n}^0 \left\{ A_1 (-1)^m (\eta_1^* + j \eta_2^*) (\cos \beta + j \sin \beta) + A_1 (-1)^n (\eta_1^* - j \eta_2^*) (\cos \beta - j \sin \beta) \right\} \]

The real and imaginary parts of the expressions are each solutions and so the most general solution is the sum of multiples of these parts.
Thus, splitting into real and imaginary parts and summing

$$p = P_n(y) \left\{ \gamma' (\eta'_n \cos \beta - \eta'_n' \sin \beta) + \gamma'' (\eta'_n \sin \beta + \eta''_n \cos \beta) \right\}$$  \hspace{1cm} (6.19)

where $\gamma'$ and $\gamma''$ are new constants. Rearranging the terms,

$$p = P_n(y) \left\{ (\gamma' \eta'_n + \gamma'' \eta''_n) \cos \beta + (\gamma'' \eta'_n - \gamma' \eta''_n) \sin \beta \right\}$$  \hspace{1cm} (6.20)

$$= P_n(y) \left\{ \gamma' \eta'_n \cos (\beta + \gamma) + \gamma'' \eta''_n \right\}$$  \hspace{1cm} (6.21)

where

$$\gamma^2 = \gamma'^2 + \gamma''^2$$  \hspace{1cm} (6.22)

$$\eta^2 = \eta'_n^2 + \eta''_n^2$$  \hspace{1cm} (6.23)

and

$$\psi = \tan^{-1} \left( \frac{\eta'_n}{\eta''_n} \right)$$  \hspace{1cm} (6.24)

where $r$ is the boundary conditions. All other terms are known functions.

This form applies to any line of Eqn. 6.18. However, since only the first and third lines are of interest in this case, the most general solution of interest is

$$p = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \gamma_{n\alpha} \gamma_{n\beta} P_n(y) \eta_{n\alpha} \cos (\kappa \alpha \pi f - \omega t + \psi_r + \theta \gamma)$$  \hspace{1cm} (6.26)

For each component there are two constants, $r$ and $\psi_r$, which are set by the boundary conditions. All other terms are known functions.

6.5) Solution for Particle Velocity.

The non-rotational particle velocity may be found from the pressure by considering Newton's force law applied to a small volume of the transmitting medium. Thus, from Ref. 4, p. 18

$$\nabla p = -\rho_u \frac{\partial u}{\partial t}$$  \hspace{1cm} (6.27)

where $\rho_u$ is the density of the air and $u$ is the particle velocity.
In this case, since the movement of the air is radial near the source, only the radial component of the velocity need be considered and Eqn. 6.27 becomes:

$$-\frac{\partial b}{\partial r} = \rho_0 \frac{\partial u}{\partial t}$$  \hspace{1cm} (6.28)

Differentiating the expression for \( b \) in Eqn. 6.20,

$$-\frac{1}{\partial r} \frac{\partial b}{\partial r} = -\frac{\rho_0}{\rho_0} \left\{ \left( \Gamma + d \right) \left( \frac{d^2 x}{d^2 x} + \Gamma^2 \right) \right\} \cos \beta - \left( \Gamma^2 + \Gamma^2 \right) \sin \beta$$

Thus,

$$\frac{d}{d\tau} \left\{ \left( \Gamma + d \right) \left( \frac{d^2 x}{d^2 x} + \Gamma^2 \right) \right\} \cos \beta - \left( \Gamma^2 + \Gamma^2 \right) \sin \beta$$

Integrating with respect to \( t \) and putting the expression in modulus form

$$u = \frac{\rho_0}{\rho_0} \Gamma \eta \cos (\beta + \psi)$$ \hspace{1cm} (6.30)

where

$$\Gamma \eta = \Gamma \eta + \Gamma \eta$$

$$\eta_\eta = \left( \frac{d^2 x}{d^2 x} + \eta \right) \left( d \eta + \eta \right)$$ \hspace{1cm} (6.31)

$$\psi = \tau \left( \frac{\eta}{\eta} \right) - \tau \left( \frac{\eta}{\eta} \right)$$ \hspace{1cm} (6.32)

$$= \frac{\eta}{\eta} - \psi$$ \hspace{1cm} (6.33)

Therefore, a general expression for the velocity is

$$u = \frac{\rho_0}{\rho_0} \eta \cos (\beta + \psi)$$ \hspace{1cm} (6.34)

This is the same form as that for the pressure variation. The constants, \( \tau \) and \( \eta \), are the same as those in Eqn. 6.26. Also, as \( r \) is increased, since the differentials are of an order in \( r \) less than \( \eta \) and \( \eta \),

$$\eta_\eta \rightarrow \eta_\eta \text{ and } \psi_\eta \rightarrow \psi_\eta$$

Thus,

$$u \rightarrow \frac{\rho}{\rho}$$ \hspace{1cm} (6.35)

as would be expected in the plane wave conditions present at a large distance.
The phase angle between the pressure and velocity waves is

$$\varphi_{m} - \varphi_{s} = \tan^{-1} \left( \frac{\eta''(\frac{d^2 \varphi}{dx^2}) - \eta''(\frac{1}{x} \frac{d \varphi}{dx})}{\eta'(\frac{d^2 \varphi}{dx^2}) + \eta''(\frac{1}{x} \frac{d \varphi}{dx})} \right)$$

$$= \tan^{-1} \left( \frac{\eta'' \frac{d^2 \varphi}{dx^2} + \eta' \frac{d \varphi}{dx}}{\eta'' \frac{d^2 \varphi}{dx^2} + \eta' \frac{d \varphi}{dx} - \eta'' \frac{d^2 \varphi}{dx^2}} \right)$$

(6.36)

6.6) Substitution of Boundary Conditions.

The only boundary conditions which are required result from the known movement of the surface of the source. The radial surface movement is assumed to be the same as the movement of the layer of air adjacent to the surface. As is shown in Chapter 5, the surface displacement may be expressed in terms of traveling waves for all common conditions. Thus, the general boundary condition for the total movement of the source may be expressed in the following form:

$$u_{s} = \frac{J_{m}}{\frac{d}{dx}} \varphi_{s} \cdot \underset{\alpha \leq \theta \leq \beta}{\text{e}} \cdot \varphi_{s}(x) \cdot C_{n}(x + x_{s} - \eta_{s} + \mu)$$

(6.37)

This expression must be equivalent to the series in Eqn. 6.34 evaluated at the surface of the source. Equating individual terms of the two series, having the same values of m and \(\omega\)

$$u_{p}(x_{s} + x - \omega t + \eta_{s} + \mu) = \frac{\Gamma_{n}}{\frac{d}{dx}} \varphi_{s}(x) \cdot C_{n}(x + x_{s} - \omega t - \eta_{s} + \mu)$$

(6.38)

giving

$$\varphi_{s}(x) = x_{s} + x_{s} \cdot \omega t - \eta_{s}$$

(6.39)

$$u_{p}(x_{s} + x - \omega t + \eta_{s} + \mu) = \frac{\Gamma_{n}}{\frac{d}{dx}} \varphi_{s}(x) \cdot C_{n}(x)$$

(6.40)

From Eqn. 6.39, since \(x_{s}\) and \(x_{s}\) are known, \(\varphi_{s}\) may be found. This is the same for both the pressure and the velocity expressions. Eqn. 6.40 gives the modulus constant, \(\Gamma_{n}\). The given \(\theta\) variation of the boundary velocity is equated to a series of associated Legendre functions modified by \(\eta_{n,m}\) which depends on \(n\) and \(x_{m}\). It is assumed that the velocity variation \(u_{n,m}\) is known, either in analytical form or as a
A simplified form for many distributions is a constant velocity over the central part of the machine adjacent to the stator core and zero elsewhere. This step or pulse distribution will be assumed in the absence of detailed knowledge of the distribution. The expression on the right of Eqn. 6.40 is an associated Legendre function series. Any function which falls to zero at the poles may be represented by such a series. As with Fourier series, the larger the number of terms the more accurate the representation.

The 'fundamental component' is the component with \( n \) equal to \( m \) and the 'harmonic components' have values of \( n \) greater than this. If the distribution is symmetrical about the equator of the source, the components with \( (m+n) \) equal to an odd integer will be zero.

It is necessary to express the \( \Theta \) distribution in this way since \( \eta_{n,n}, \eta_{n,0} \), and \( \eta_{n,0} \) vary considerably with \( n \). In general, the larger the value of \( n \) the more difficult it is for the source to radiate, although in certain circumstances the larger values may become important.

The values of \( \Gamma_{n,m} \) for the components of a wave may be found using a similar method to that used to find the constants in a Fourier series. Thus, multiplying both sides of Eqn. 6.40 by \( \frac{\partial}{\partial m} \) and integrating with respect to \( y \) between the limits of \(-1\) and \(+1\)

\[
\int u_{n,m,m,w}(\Theta) \cdot P_n^m(y) \, dy = \sum_{n} \int \frac{\partial}{\partial \Theta} P_n^m(y) \cdot P_n^m(y) \cdot \eta_{n,m} \, dy 
\]

(6.41)

Evoking the identity in Eqn. 6.11 it is apparent that all of the terms on the right hand side are zero except the one with \( n \) equal to \( n \).

Thus,

\[
\int u_{n,m,m,w}(\Theta) \cdot \phi_{n,m}(\Theta, \sigma) \, d\Theta = \frac{2}{(2\pi)^{\frac{3}{2}}} \frac{(n+m)!}{(n-m)!} \Gamma_{n,m,w} \phi_{n,m}(\Theta, \sigma)
\]

and

\[
\Gamma_{n,m,w} = \frac{2}{(2\pi)^{\frac{3}{2}}} \frac{(n-m)!}{(n+m)!} \phi_{n,m}(\Theta, \sigma) \int u_{n,m,m,w}(\Theta) \cdot P_n^m(\Theta, \sigma) \, d\Theta 
\]

(6.42)

The evaluation of the remaining integral is discussed in Appendix C.
6.7) Total Power Radiated by a Travelling Wave.

The total acoustic power radiated by a source is equal to the rate at which energy crosses any surface enclosing it if absorption is neglected. A surface may therefore be chosen where the equations for the field become simplest. A large sphere, concentric with the source, will be chosen as the integrating surface. A sphere is chosen so that the radial terms are constant over the surface and it is made large so that the waves effectively become plane waves with the pressure and velocity in phase with each other and linked by the characteristic impedance of the air as in Eqn. 6.35. At large distances

\[ \eta_{n.o} \rightarrow \eta_{n.e} \rightarrow \frac{1}{r} \]

and

\[ \bar{\eta}_{n.e} \rightarrow \bar{\eta}_{n.e} \rightarrow 0 \]

Thus, the expressions for the pressure and the velocity become

\[ p = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{P}_{n,m}(\cos \theta) \cos (n \pm m \phi - \omega t + \gamma_r) \]  \hspace{1cm} (6.43)

and

\[ u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{P}_{n,m}(\cos \theta) \cos (n \pm m \phi - \omega t + \gamma_r) \]

The instantaneous energy density in a sound wave is the sum of the kinetic and potential energies and is equal to

\[ \frac{1}{2} \rho_a \left( u^2 + p^2 / \rho_a^2 \right) = \frac{1}{2} \rho_a \left( u^2 + p^2 \right) \]  \hspace{1cm} (6.44)

for the waves considered here. Thus the average of the energy density taken over a long period of time \( t \),

\[ = \frac{1}{\rho_a^2 c_a^2} \int_0^\infty \left[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{P}_{n,m}(\cos \theta) \cos (n \pm m \phi - \omega t + \gamma_r) \right]^2 dt \]  \hspace{1cm} (6.45)

In the case where a single component is present, \( t \), may be one period of the wave.

The flow of energy is in the direction of the particle velocity and at the speed of sound in air, \( c_a \). Thus, the power intensity or rate at which energy crosses unit area of the surface is the energy
density multiplied by $\zeta_\alpha$. It is a maximum in the direction of the particle velocity which is perpendicular to the integrating surface at all points. Therefore, integrating over the whole sphere, the total power,

$$P = \frac{1}{4}c_0^2 \int \left[ \sum_{\mathcal{E}} \frac{\overline{\mathcal{E}}}{2 \pi r^2} \rho_{\alpha}(c_0) \cos (\alpha \mathcal{E} + \omega t - \gamma r) \right]^2 r^2 \sin \theta \, d\theta \, d\phi \, dr \, d\sigma$$

$$= \frac{1}{4}c_0^2 \int \left[ \sum_{\mathcal{E}} \frac{\overline{\mathcal{E}}}{2 \pi r^2} \rho_{\alpha}(c_0) \cos (\alpha \mathcal{E} + \omega t - \gamma r) \right]^2 \sin \theta \, d\theta \, d\phi \, dr \, d\sigma \tag{6.46}$$

where an element of area on the surface of the sphere is $r^2 \sin \theta \, d\theta \, d\phi$.

This integral is formed of terms of the type

$$\int \frac{1}{4}c_0^2 \int \left[ \sum_{\mathcal{E}} \frac{\overline{\mathcal{E}}}{2 \pi r^2} \rho_{\alpha}(c_0) \cos (\alpha \mathcal{E} + \omega t - \gamma r) \right]^2 \sin \theta \, d\theta \, d\phi \, dr \, d\sigma \tag{6.47}$$

This may be considered in two parts

$$\int \frac{1}{4}c_0^2 \int \frac{1}{\omega \mathcal{E}} \cos \left( \frac{\omega t - \gamma r}{2} \right) \left[ (\mathcal{E} \alpha \mathcal{E} + \omega t - \gamma r) \right] \sin \theta \, d\theta \, d\phi \, dr \, d\sigma \tag{6.48}$$

and

$$\int \rho_{\alpha}(c_0) \rho_{\alpha}(c_0) \sin \theta \, d\theta \, d\sigma \tag{6.49}$$

The former term is zero unless $\omega \mathcal{E} = \omega_\alpha$, $(\mathcal{E} \alpha \mathcal{E} + \omega t - \gamma r)$ and the negative sign is taken. The asterisks denote that the signs marked in this way must be the same as each other. Thus, there is only a power component for waves of the same frequency and pole number, traveling in the same direction and power produced by combinations of waves is zero.

Substituting the condition, $\omega = \omega_\alpha$ in Eqn. 6.49 and using Eqn. 6.11, Eqn. 6.49 is only non-zero when $\omega = \omega_\alpha$. Thus there is no power due to interaction between waves with different values of $\omega$.

Thus, multiplying together the non-zero parts of Eqns. 6.48 and 6.49 and summing for all values of $m, n$ and $\omega$, the total power becomes

$$\frac{\pi c_0}{\omega} \sum_{\mathcal{E}} \left[ \frac{(\alpha \mathcal{E} - m) (\alpha \mathcal{E} - n)}{2 \lambda \mathcal{E}} \right] \rho_{\alpha}(c_0)$$

where the summation is made for both plus and minus rotations.

Substituting for $\Gamma_{\alpha \omega}$ from Eqn. 6.42 this becomes

$$P = \pi c_0 \int \left[ \sum_{\mathcal{E}} \left( \frac{\alpha \mathcal{E}}{2 \pi r^2} \right) \rho_{\alpha}(c_0) \cos (\alpha \mathcal{E} + \omega t - \gamma r) \right]^2 \sin \theta \, d\theta \, d\phi \, dr \, d\sigma \tag{6.50}$$

$$= \pi c_0 \int \left[ \sum_{\mathcal{E}} \frac{u_{\alpha \mathcal{E} \omega} \mathcal{E} \omega}{(\mathcal{E} \alpha \mathcal{E} - \omega t - \gamma r)} \rho_{\alpha}(c_0) \cos (\alpha \mathcal{E} + \omega t - \gamma r) \right]^2 \sin \theta \, d\theta \, d\phi \, dr \, d\sigma \tag{6.51}$$

Substituting for $\Gamma_{\alpha \omega}$ from Eqn. 6.42 this becomes

$$P = \pi c_0 \int \left[ \sum_{\mathcal{E}} \frac{u_{\alpha \mathcal{E} \omega} \mathcal{E} \omega}{(\mathcal{E} \alpha \mathcal{E} - \omega t - \gamma r)} \rho_{\alpha}(c_0) \cos (\alpha \mathcal{E} + \omega t - \gamma r) \right]^2 \sin \theta \, d\theta \, d\phi \, dr \, d\sigma \tag{6.52}$$
where 
\[ Q_{n} \approx \frac{2^{n+1} (n-m)!}{(n+m)!} \left[ \int u_{n,m,0}^{2} d(\cos \theta) \right]^{2} \]  \hspace{1cm} (6.53)

\( u_{n,m,0} \) is the per unit velocity of a component of vibration at a point on the surface of the machine and is based on the maximum value of the component on the surface, \( u_{n,m,0} \). Thus the power radiated by each component of the vibration may be calculated and summed to give the total power.

6.8) Application to Field of Electric Machine.

In the preceding sections, expressions are given for the pressure, velocity and total power in the field of a spherical source on which vibrations with constant phase along lines of longitude are present. It was demonstrated that the normal mode of vibration of such a source is a sinusoidal travelling wave in the \( \phi \) direction and an associated Legendre function in the \( \theta \) direction. If the actual surface distribution is split up in this way, then the variation of pressure and velocity in the field and the total radiated power may be simply predicted for each component separately.

For each value of \( n \) in the associated Legendre function series there is a particular relationship between pressure, velocity and the distance from the centre of the source. This relationship is independent of the size of the source. The variations, with distance, of the pressure, velocity and the phase angle between them have been calculated as described in Appendix C. These variations are plotted against \( x \left( \frac{r}{c} \right) \) in Figs. C.2 to C.4. The pressure and velocity are plotted as logarithmic variations from the inverse square law (i.e. \( 10 \log_{10} [x^{2}] \), \( 20 \log_{10} [x^{2}] \)). The sound pressure level may then
be calculated assuming the inverse square law and the correction for
the actual radial variation of a particular mode added.

It may be seen that at small values of \( x \) there is a large increase
in pressure and velocity from the values given by the inverse square
law and a corresponding increase in the angle between them. Thus the
power flow remains constant at all distances. The large increase from
the inverse square law as \( x \) decreases is caused by the reactive
energy stored in the near field. At a certain value of \( x \) for each
value of \( n \) there is a rapid change from the predominantly reactive
near field to the predominantly in-phase far field. Fig. 6.2 shows
the wave fronts of travelling waves leaving a spherical source. It
can be seen that they are practically radial at the centre becoming
tangential at a large distance from the source. If there are two waves
of equal magnitude, one traveling in each direction round the sphere,
then standing waves are formed on the surface and the wave fronts are
totally tangential.

The magnitudes of the pressure
and the velocity also depend on \( \Gamma \)
in Eqns. 6.21 and 6.30. \( \Gamma \) is
given as a function of the boundary
conditions in Eqn. 6.42. From this
it may be seen that \( \Gamma \) depends on
the value of \( \eta_\omega \) on the surface of
the source. If the surface of the
source is within the reactive near
field, then most of the surface
displacement will be 'absorbed' by
the reactive near field and, as may be seen from Figs. C.2 and C.3, the magnitudes of the velocity and pressure decrease greatly as the distance from the source increases. Thus, if the surface of the source is within the reactive field, the source is not as effective as it would be if it was in the far field with the same volume velocity. It follows that the higher the value of \( n \) the less is the probability that a component will radiate sound effectively.

The smallest value of \( n \) possible for a particular travelling displacement wave is when \( n = m \). This component is the component which is most likely to radiate noise effectively. If it does not radiate effectively then components with higher values of \( n \) will not do so. It is therefore apparent that in many cases only the 'fundamental' component, with \( n = m \), need be considered as this is the only one which will produce considerable radiation.

It was found from vibration measurements that the polar distribution of vibration on a machine may often be considered to be constant over a band centred on the 'equator' of the measuring system and zero outside this band. Applying an associated Legendre function analysis of such a distribution it was found that for a band 60° to 120° wide, centred on the equator, most of the harmonic components \( (n > m) \) are very small compared with the fundamental. This is another reason why the higher values of \( n \) may often be neglected. If the distribution is symmetrical about the equator then the associated Legendre function components with \( (m+n) \) equal to an odd integer are zero as these are asymmetrical components.

From these considerations it may seem that, to a reasonable degree of approximation, the noise radiated by an electric machine
may be represented by taking only the component with \( n = m \).

In this way, approximations for the pressure, velocity, phase angle between them and the total power may be obtained from the graphs shown in Appendix C.

The sound pressure level at a point,

\[
L_p = 20 \log (\frac{P_{\text{rms}}}{10^{-10}}) = 20 \log (\frac{P_{\text{rms}}}{2 \times 10^{-10}}) + \frac{20 \log (\frac{P_{\text{rms}}}{2 \times 10^{-10}})}{10 \log (\frac{\pi}{10})} + \frac{20 \log (\frac{P_{\text{rms}}}{2 \times 10^{-10}})}{10 \log (\frac{\pi}{10})} - 20 \log (\frac{\pi}{10})
\]

where the substitution for \( P \) is made from Eqns. 6.21 and 6.42 for one component in \( m, n \) and \( \omega \). If a vibrating band, centred on the centre of the stator, is assumed, the approximations mentioned above may be made and the pressure level due to one component becomes

\[
L_p = 20 \log (2.8 \times 10^{4} \times \text{rms}) + 10 \log (\frac{2 \pi m}{2 \pi m}) + 10 \log (\frac{2 \pi m}{2 \pi m}) + 10 \log (\frac{2 \pi m}{2 \pi m}) - 20 \log (\frac{\pi}{10})
\]

since \( m \leq m \), \( \sin m \theta = 2 \pi m \), \( \rho_{m} \), and the original integral is twice the integral in this expression since the band is symmetrical.

In a similar way the level of the power radiated by one component,

\[
L_P = 10 \log \left\{ \frac{4 \pi}{10^{-12}} \frac{P_{\text{rms}}}{\text{rms}} \left( \frac{2 \pi m}{2 \pi m} \right) \left( \frac{n-m}{n+m} \right) \left( \frac{2 \pi m}{2 \pi m} \right) \left( \frac{n-m}{n+m} \right) \left( \frac{2 \pi m}{2 \pi m} \right) \left( \frac{n-m}{n+m} \right) \right\}
\]

The final three components of Eqn. 6.55 and the final two components of Eqn. 6.56 may be found from the graphs in Appendix C.

If the distribution on the surface of a machine may not be represented by a band of width between 60° and 120° centred on the equator and \( x_{m} \) is large enough for components other than the one with \( m \) to produce appreciable radiation then the power must be calculated for each component separately and the total power found.
Read \( m, f, T_{su} \) and maximum value of \( n \).

Set \( n \) starting at \( m \).

Calculate \( \frac{2n+1}{2} \frac{(m-n)!}{(m+n)!} \) and store.

Calculate coefficients of \( \eta_n, \eta_n', \eta_n'', \frac{\partial \eta_n'}{\partial x} \) and \( \frac{\partial \eta_n''}{\partial x} \).

Calculate \( x_{su} = \frac{\omega \eta_n}{C} \).

Substitute \( x_{su} \) to find \( \eta_n(u(x_{su})) \) and store.

Calculate \( \eta_n' \) for all possible values of \( i \) and store.

Select analytical or numerical Integration.

**Analytical**
- Read limits of integrating band and magnitude of velocity.
- Calculate Cosines, Sines and Tangents.
- Select \( n \).
- Find number of terms in expansion of \( \eta_n^m \) and integrate each using reduction formulae.
- Sum all terms and store.

**Numerical**
- Read width of integrating band and table of values of velocity for each band or a constant velocity and number of integrating steps.
- Select band.
- Calculate \( \cos \theta \sin \theta \) for each band angle and sum expansion of each \( \eta_n^m \).
- Store \( \sin \theta \) for all bands.
- Multiply by band width and constants.
- Select \( n \).
- Find individual powers from stored quantities.
- Print and sum individual powers.
- Calculate sound power levels.
- Print total power and power level.

**Fig. 6.3. Flow Diagram of Computer Programme**

To calculate total power from surface velocity.
before the level is calculated. A computer programme to do this for values of \( n \) up to \( m^{20} \) has been written and used. A block diagram of it is shown in Fig. 6.3. In most cases this is unnecessary as a larger error is involved in assuming that the machine may be treated as a spherical source.
7.1) Introduction.

In Chapter 2 some of the types of measurement which may be made in the acoustic field of an electric machine were described. These techniques have been applied to measurements in the acoustic fields of the five machines described in Appendix D. Measurements were made of the variation of the sound pressure level in various frequency bands in the fields of machines radiating freely. Also, many measurements were made of the total power radiated in various frequency bands using both the analogue and digital methods. Examples of the sound level recordings made in the fields of the machines together with a table showing the total power in various frequency bands for each machine are shown in Appendix E.

In this chapter the general results of these measurements are discussed. In Sects. 7.3 and 7.4 the typical sound pressure variations are discussed and in the remaining sections the limitations of the measuring techniques are investigated.

Frequency analyses of the sound pressure at one point in the field of each machine are shown in Figs. E.6 to E.10. These show that there are pure tone components over the whole audible frequency range. By using the narrow band frequency analyser to select various pure tone components the variation of the field distribution with frequency was investigated. In addition some measurements were made of the pressure
passed by sound level meter weighting networks and one-third-octave band filters.

At each frequency there is a different pattern of variation of the sound pressure level throughout the field. Fig. E.15 shows the variation, at a constant distance, of the sound pressure level in the field of Motor No. 2, analysed into one-third-octave bands. Also the sound pressure level which would be present at the same distance if the same power were radiated uniformly is plotted. The variation is even greater than this if narrow band analysis is used. This shows the need for an overall measurement of noise, such as total power in a particular frequency band, rather than single sound pressure level measurements which vary widely with position and thus give little indication of the total noise. It is also of little use defining a particular position to measure the sound pressure level, except for comparison of machines manufactured identically, since the relative magnitudes of components at various frequencies varies widely with position.

7.2) Variation of Apparent Sound Power With Distance.

When sound power measurements, as described in Chapter 2 are made it is assumed that the source may be treated as a point source and that under all conditions the power is given by

\[ \frac{1}{2\pi c_a} \int p^2 da \]

the integral being taken over a spherical surface enclosing the source.

This is true only in the far field of the source where the sound waves effectively become plane. Closer to the source, in the near-field, the particle velocity is no longer in phase with the pressure as was
shown for the simplified case of a spherical source in Chapter 6. The power then depends also on the cosine of the phase angle between them. Close to the source, in maintaining constant radiated power, the pressure and, therefore, the apparent power as calculated from the above formula, must rise. The apparent power may also vary with distance due to the complicated shape of the machine since the particle velocity may not always be perpendicular to the measuring surface.

Table 7.1 shows the variation with distance of the apparent power radiated by Motor No. 2 in various frequency bands. Fig. 7.1 shows the variation of the r.m.s. sound pressure with distance from the source in some of the same frequency bands. The variations at the other frequencies are very similar to these. Motor No. 2 is a single phase induction machine with starting and running capacitors mounted on the top, providing a complicated source. Each of the power measurements was made using 272 points in the summation as described in Chapter 2. Thus errors in reading s.p.l. values from the level recordings will tend to cancel out to produce small total errors.

These results show that except for the component of noise at 100 Hz this motor may be treated as a point source at the distances used. There are several explanations for the small errors observed in some of the results. First, the distance of the microphone from the centre of the source may not have been obtained accurately as it is difficult to measure and an error of 1% can produce a change of sound power level of 0.1 dB. This is further indicated by the fact that over half the highest sound power levels occur at a nominal measuring distance of 24° and over half the lowest values were obtained at a nominal distance of 18°.
The variation of the apparent power radiated by Motor No. 2 with radius of measuring sphere using 272 points in the summation. Frequency bands are A and B weightings and narrow bands.

<table>
<thead>
<tr>
<th>Frequency (c/s)</th>
<th>Sound power level (dB re 10⁻¹⁰ watts)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>r = 12&quot;</td>
</tr>
<tr>
<td>A</td>
<td>40.6</td>
</tr>
<tr>
<td>B</td>
<td>42.0</td>
</tr>
<tr>
<td>100</td>
<td>38.2</td>
</tr>
<tr>
<td>600</td>
<td>35.0</td>
</tr>
<tr>
<td>880</td>
<td>25.3</td>
</tr>
<tr>
<td>1100</td>
<td>27.2</td>
</tr>
<tr>
<td>1200</td>
<td>26.5</td>
</tr>
<tr>
<td>1400</td>
<td>24.2</td>
</tr>
<tr>
<td>2200</td>
<td>18.7</td>
</tr>
<tr>
<td>2800</td>
<td>17.6</td>
</tr>
<tr>
<td>3500</td>
<td>18.4</td>
</tr>
<tr>
<td>4400</td>
<td>28.8</td>
</tr>
<tr>
<td>5500</td>
<td>30.3</td>
</tr>
<tr>
<td>7700</td>
<td>25.0</td>
</tr>
</tbody>
</table>
Fig. 7.1. Variation of Level of R.M.S. Sound Pressure with Radius of Measuring Sphere for Noise Radiated by Motor No. 2 in Various Narrow Frequency Bands.
Secondly, the power levels at 2200, 2800 and 3500 c/s are up to 25 dB below the overall level and at the largest distance of measurement the sound pressure measured at these frequencies is near the background noise level. This is indicated in the results at these frequencies since the values measured at 31" tend to be higher than those measured at the other radii.

Thirdly, the walls of the chamber and the microphone ring may interfere with the measurements made at the largest radius since at this distance the microphone is only six inches from the ring and 18" from the wall treatment. This effect will mainly be confined to low frequencies.

The first and second points may be taken into account by adding 0.2 dB to the value measured at 18", subtracting 0.2 dB from the values obtained at 24" and ignoring the values involving low sound pressure levels. If this is done, the variation with distance for the 'A' and 'B' weightings and the nine frequency bands with centre frequencies above 1000 c/s is less than 0.5 dB which is approximately the error which may be involved in the pressure measurement. At 880 c/s the value obtained at 30" is 1dB too small. This is probably due to the presence of the wall treatment. At 600 c/s the value at 18" is 1dB too small. There is no obvious cause for this but it may be caused by standing waves since the microphone is one wavelength from the surface of the motor.

The 100 c/s component of noise does not appear to give constant power at all radii. This would be expected from the theoretical results shown in Fig. C.2. The values of \( z \) for 100 c/s at the appropriate distances have been calculated and the theoretical results superimposed
on the measured curve in Fig. 7.1 assuming that $m$ is equal to one for this wave. This agrees well with the measured curve except at the most distant point where the effect of the proximity of the chamber treatment is evident.

Therefore, measurements made at all frequencies except the lowest frequencies are in the far field of the source even with a measuring sphere of one foot radius and the measured power is the actual power radiated. The theoretical curves in Appendix C give a good guide for the extent of the near field even though they may not give accurate near-field correction factors since the motors are not exactly spherical. However, it is evident from the results that the complexity of the source does not seriously affect the power measurements.

The noise measurements made for these power calculations were taken over a period of several months with the motor running all the time from a stabilised power supply. Occasionally during this time measurements were made at one particular position in the field and it was found that there was no identifiable change in any component of noise during this time. This is further indicated by the constancy of the sound power measurements.

7.3) Variation of Sound Pressure With Distance.

When the distance of the measuring point from a freely radiating point source is doubled the sound pressure level decreases by 6 dB. The anechoic chamber allows sources to radiate freely at frequencies above 70 c/s and, as was demonstrated in Sect. 7.2, the measured power radiated by a machine is constant at all distances outside the near field and away from the walls. Thus, the equivalent sound pressure level if the
machine was radiating the same power uniformly decreases 6 dB when the
distance from the source is doubled.

An electric machine does not necessarily constitute a point
source of sound and, as was found by measurement, the variation of
sound pressure along radial lines from the centre of the machine does
not always vary in this way. Polar diagrams of the noise in various
frequency bands at various distances together with the variation with
distance along radial lines were plotted automatically for Motor No. 2.
The results of these measurements at three frequencies are shown in
Figs. E.1 to E.3. These frequencies are distributed over the audible
frequency range and the results are representative of the variations at
most frequencies and angles.

From these results it is apparent that although at some frequencies
there is a 6dB decrease in sound pressure level when the distance is
doubled, at other positions there are greater and smaller decreases.
Thus, although the size and shape of the source has little effect on the
total power it does affect the radiation in individual positions. Thus,
a machine may only be treated as a point source when total power
measurements are made.

The curves of pressure level against distance are taken in the
same plane as the polar plots and there should be agreement between the
two sets of measurements. They do in fact agree except in the case
of the 100 c/s component which falls more sharply on the pressure level
against distance curves than on the corresponding polar plots. This
is probably due to the fact that the apparent power close to the source
is greater than it is further away as explained in Sect. 7.2. Thus, at
distances less than about one foot from the centre of the machine there
is probably a very large rise in pressure so that the level recorder could not follow the variation as the microphone moved away, the first few inches of microphone movement occurring before the pen of the recorder starts. At a constant distance from the source as in the polar charts there is not such a rapid change in pressure. It is also apparent that at this frequency there is a high level of background noise so that at distances greater than about two feet and on one half of the polar diagram the background noise predominate.

At the other two frequencies there is a discrepancy between the two sets of measurements where the polar diagram is changing quickly with angle. This is probably due to slight misalignment of the polar diagrams or swaying of the microphone in the pressure level against distance curves. On each of the curves of pressure level against distance the theoretical curve for a point source has been drawn so that the actual curves may be compared with it.

7.4) Measured Field Distribution.

The actual sound pressure distribution in the field of an electric machine is very complicated since different parts of the machine vibrate with different phases and amplitudes. The surface vibrations depend on the sources and frequencies of the driving forces. If there are travelling waves on the stator, the vibration magnitude is reasonably constant in an 'equatorial' direction but, if standing waves are produced, there are a number of maxima and minima according to the mode of vibration. The variation in the polar direction is generally a constant over the parts of the stator enclosing the core and the parts of the
end castings parallel to this and then quickly decreases on the end castings as the measuring point is moved towards the bearings. At some frequencies resonances may occur in the end castings, connecting boxes etc., modifying the vibration pattern.

In general, the radiation pattern may not be the same as the vibration distribution on the motor surface since interference takes place between waves from different parts of the surface if the source is not exactly spherical. The variation of a component of noise at a constant distance from a motor in planes parallel to the shaft is shown in Fig. E.5. The variation in a plane perpendicular to the shaft for another machine is shown in Fig. E.4. In general, patterns with similar numbers of lobes are produced in each direction but the spacings are not regular.

To simplify the distribution round a motor so that a general comparison with the theory could be made, test motors were suspended in the anechoic chamber with their shafts vertical. The average power intensities along lines of latitude were then obtained at various angles of latitude using the analogue power calculator which was described in Chapter 2. Since the time for one revolution of the microphone ring is constant, the voltage produced by the integrator at the end of each revolution is proportional to the average power intensity along the microphone path. The results of such measurements at several of the main frequencies of the noise produced by Motor No. 2 are shown in Figs. 7.2 and 7.3. The intensity is plotted as a proportion of the intensity in the band opposite the centre of the stator. Similar results were obtained for other fractional horsepower induction machines.

In the simplified theory given in Chapter 6 it was assumed that
Fig. 7.2. Average Sound Intensity
Along Line of Constant $\theta$ Plotted
Against the Angle $\theta$. 
Fig. 7.3. **Average Sound Intensity**

Along Line of Constant $\theta$ Plotted

Against the Angle $\theta$. 

Motor No. 2.
the variation of vibration in the polar direction was a pulse
distribution. It was then shown that in most cases only the
fundamental associated Legendre function produced significant power,
the other modes often being attenuated. Even if they were radiated
the higher harmonics would represent only a small proportion of the
total power. If only the fundamental component is radiated, the
distribution in the $\theta$ direction is proportional to $\sin^m \theta$ where $m$
is the number of pole pairs of the displacement wave. $\sin^m \theta$ has
been drawn in Fig. 7.2 as an example of this distribution. This is
the form of distribution assumed by Carter (14).

From the practical results it is apparent that this theory is
approximately true for many frequencies as many of the curves are
similar to the theoretical curve. At other frequencies, usually
higher ones, other peaks are added to the curves. In some cases these
are components with relatively large vibration magnitudes on the end
castings.

A more frequent cause of the extra maxima at high frequencies is
interference between waves from different parts of the surface. When
the surface of a source completely covers a plane of the measuring
co-ordinates, as in the case of a spherical source or infinite plane
source, radiation theory predicts that no change of the radiation pattern
occurs with distance. Thus, if the source were completely spherical,
no interference would occur and the developed theory would hold.
However, machines are better represented by short cylinders with
curved ends where the surface does not correspond with a simple
co-ordinate plane.

In this case interference occurs and secondary lobes are formed.
The transition from the surface vibration distribution to the sound pressure distribution occurs in a refraction area close to the surface so that at most of the measuring distances the far-field polar distribution is formed and the pattern becomes nearly the same at all distances as described in Sect. 7.2. The uniform radiation from short cylinders has been analysed by Williams et al.\(^{(55)}\) and the published results show that such lobes are formed. For high frequency radiation it would be useful to extend this theory to apply to electric machines.

In general, these results justify the use of the simplified theory of Chapter 6 especially at low frequencies or where secondary lobes do not exist on the radiation pattern.

7.5) **Number of Points Required for Summation.**

The number of points at which sound pressure measurements should be made to determine the sound power radiated by a source depends on the accuracy required of the measurement, the spread of the sound pressure values and the number of lobes in the sound pressure distribution. Table 7.2 gives a summary of the results of using the various combinations of points described in Appendix A to calculate the power radiated by Motor No. 2 in various frequency bands which are collected into characteristic groups. The maximum variation from the value obtained using 272 points is given from the results of measurements at four different distances.

It is immediately apparent from the results that it would rarely be necessary to use more than 92 points in the power summation as the maximum difference between the power measured in this way and that obtained using three times this number of points is only 0.2 dB. This
TABLE 7.2

Error involved in reducing the number of points in the power calculation.
Maximum error from results at all distances for Motor No. 2

<table>
<thead>
<tr>
<th>Centre Frequency of Narrow Bands (c/s)</th>
<th>Number of Points around Machine</th>
<th>Maximum spread of S.P.L. Values (dB)</th>
<th>Maximum Variation from Value obtained using 272 Points (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>110 lbs</td>
</tr>
<tr>
<td>A &amp; B weightings</td>
<td>12</td>
<td>7</td>
<td>0.1</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>31</td>
<td>0.1</td>
</tr>
<tr>
<td>600,880</td>
<td>2</td>
<td>14</td>
<td>0.1</td>
</tr>
<tr>
<td>1100,1200,1400</td>
<td>30/4</td>
<td>20</td>
<td>0.1</td>
</tr>
<tr>
<td>2200,2800,3500</td>
<td>6</td>
<td>14</td>
<td>0.1</td>
</tr>
<tr>
<td>4400,5500,7700</td>
<td>12</td>
<td>20</td>
<td>0.1</td>
</tr>
</tbody>
</table>

TABLE 7.3

Power levels measured in the fields of Motor No. 3 in the anechoic chamber and in the laboratory.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Power Measured in Anechoic Chamber</th>
<th>Power Measured in Laboratory using 10 points</th>
<th>Power Measured in Laboratory using 24 points</th>
</tr>
</thead>
<tbody>
<tr>
<td>240</td>
<td>45.3</td>
<td>48.5</td>
<td>48.8</td>
</tr>
<tr>
<td>350</td>
<td>51.4</td>
<td>53.1</td>
<td>57.5</td>
</tr>
<tr>
<td>480</td>
<td>60.1</td>
<td>62.1</td>
<td>60.5</td>
</tr>
<tr>
<td>1110</td>
<td>74.5</td>
<td>69.2</td>
<td>74.0</td>
</tr>
<tr>
<td>1200</td>
<td>69.9</td>
<td>70.5</td>
<td>70.7</td>
</tr>
<tr>
<td>2000</td>
<td>52.2</td>
<td>55.4</td>
<td>54.3</td>
</tr>
<tr>
<td>3500</td>
<td>58.3</td>
<td>60.3</td>
<td></td>
</tr>
</tbody>
</table>
is less than the accuracy which can be claimed for individual pressure measurements. It may therefore be assumed that 92 points will give a sufficiently accurate power measurement for most purposes. For the overall weightings eight points are sufficient to give the sound power to an accuracy of ±0.5 dB. Usually overall levels are not required to any greater accuracy than this and so an even smaller number of equally spaced points may suffice.

When the noise is analysed into narrow frequency bands the error involved in using a small number of points is larger since the spread of the values is greater than the spread for the overall bands. In general, the larger the spread and and the greater the number of maxima in the distribution, the larger the error. The accuracy obtained with 20 points would probably be sufficient for narrow band motor noise measurement. The maximum spread of the pressure level values was obtained at a distance of one foot in most cases and may not be as great for the low frequencies at larger radii where measurements are normally made. Thus, the maximum error would probably be in the order of ±0.6 dB.

The set of 36 points was badly selected as it involved splitting the measuring surface into 18 triangles and 18 rectangles. Ideally the areas should be identical regular polygons so that the points are evenly distributed as when using 20 points. Thus, a greater error would be expected when 36 points are used.

7.6) **Number of Bands Required for Analogue Method.**

To determine the number of bands required to calculate the power using the analogue method, the power radiated by Motor No. 2 was
measured at the same frequencies as were used for the digital method. The results of measurements using five different arrangements of latitudinal bands are shown in Table 7.4 together with the power calculated by the digital method. All measurements were made at a distance of 18" from the centre of the motor. Measurements were made from tape recordings of the overall noise in each latitudinal band.

At most frequencies there is little difference between the results of the analogue method and the digital method even though the motor was running for several months between the two sets of measurements. At 100 c/s and 600 c/s there is a definite discrepancy between the two methods. At both frequencies there is considerable time modulation of the noise at multiples of the slip frequency causing variations of up to 15 dB in the sound pressure level. To obtain a reasonably steady value the damping of the level recorder pen was increased to its maximum value and the digital values taken from polar recordings made in this way. Unfortunately, this gives the mean of the level of the r.m.s. pressure and not the level of the r.m.s. pressure with the mean square taken over a long period of time. This is what is required to calculate the total power and is also the value given by the analogue method. Therefore, the value obtained using the analogue method is more likely to be the true value than that obtained from the digital method.

The maximum difference of power level between measurements made with 12 latitudinal bands and any other set of bands is less than 1dB when the motor is mounted in the same position each time. During the measurements two different channels of the tape recorder and two different tapes were used. The accuracy of the recorder is claimed as ± 2 dB from 60 c/s to 10 kc/s and the accuracy of the tape ± 0.5 dB
TABLE 7.4

Variation of effective sound power level with width of band and frequency using the analogue method, for Motor No. 2 at a distance of 18" from its centre. Narrow frequency bands and A and B weightings.

<table>
<thead>
<tr>
<th>Frequency (c/s)</th>
<th>Sound Power Level (dB re 10^-12 Watts.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Digital Method (212 pts)</td>
</tr>
<tr>
<td>A</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
</tr>
<tr>
<td>600</td>
<td></td>
</tr>
<tr>
<td>880</td>
<td></td>
</tr>
<tr>
<td>1100</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td></td>
</tr>
<tr>
<td>1400</td>
<td></td>
</tr>
<tr>
<td>2200</td>
<td></td>
</tr>
<tr>
<td>2800</td>
<td></td>
</tr>
<tr>
<td>3500</td>
<td></td>
</tr>
<tr>
<td>4400</td>
<td></td>
</tr>
<tr>
<td>5500</td>
<td></td>
</tr>
<tr>
<td>7700</td>
<td></td>
</tr>
</tbody>
</table>
within a tape and ±1.0 dB form tape to tape. By accurate adjustment the frequency response was improved. The within-tape error tends to cancel out over a whole length of tape and to eliminate tape to tape error each tape was calibrated using a 1000 c/s signal of known amplitude and the average response of the recorder found. When these precautions are taken the measuring accuracy should be of the order of ±0.5 dB. Thus, the error due to the various band configurations is not greater than about ±0.5 dB.

It is apparent that four bands are sufficient to give results of approximately the same accuracy as 20 points using the digital method. In general the analogue method is faster and more convenient than the digital method. As polar diagrams of the noise were made simultaneously with the power measurements it was decided that most measurements should be made using six bands. This is also convenient since all recordings for one machine may be made on one side of a tape.

7.7) Error Involved in Averaging S.P.L. to Find Power Intensity.

When approximate measurements of the sound power level are required, the average of the s.p.l. values is often taken as the s.p.l. which would be present at the same distance if the same power was radiated uniformly. The power level is then calculated by adding a factor to take into account the measuring distance as in Eqn. 2.3. In order to investigate the error involved in this approximation, the sound power level was calculated in this way using the computer programme on the results from Motor No.2 using 272 points, 14 frequency bands and four distances. The results were compared with the values calculated as describes in Sect. 7.2
The errors involved are shown plotted against the spread of the s.p.l. values in Fig. 7.4. The value obtained from the approximate method is always smaller than the correct value. The largest error was 4.5 dB for a spread of 31 dB. The error depends not only on the spread of the points but also on the distribution of the points within this range. The maximum error occurs when half the values are at one end of the spread and half of the values are at the other end when the error is

\[ 10 \log_{10} \left( \frac{10^{(L_P_{max} - L_P_{min})/10} + 1}{2} \right) = \frac{1}{2} (L_P_{max} - L_P_{min}) \]

This has been plotted against spread on the same diagram as the measured error. Also 0.3 times the theoretical maximum has been plotted. This second curve is near the centre of the measured points. Thus, although the distribution of the points within the spread is different for different components, by applying a correction corresponding with this curve, the results for this machine are not more than 1 dB in-error. Hence, although at large spreads the error involved in using this approximation is large, a reasonable correction may be made knowing only the spread of the points.

7.8) Measurements Under Non-free Conditions.

In factory conditions, especially where large machines are involved, it is not possible to measure the spherically radiated power and other methods must be used. Often the machine is mounted on the floor of as large a room as possible and hemi-spherical radiation assumed. The measuring distance must be great enough for the measuring point to be outside the near field of the source while not being large enough for it to be in the reverberant field. The power is then
Fig. 7.4. Error involved in averaging S.P.L. values in calculating total power plotted against the spread of the S.P.L. values.
measured either by taking s.p.l. values round the machine at the
height of the shaft\(^{(39)}\) or else at equally spaced points over the
surface of a hemi-sphere based on the centre of the base of the machine.\(^{(4)}\)
These two methods have been used to calculate the power radiated by
Motor No.3 as the fractional horsepower machines did not produce
enough noise to be measured above the background level outside the
anechoic chamber.

The machine was resiliently slung close to the floor in the
centre of a space in the machines laboratory adjoining the anechoic
chamber so that the nearest bench was about six feet from it and the
roof about 15 feet from it. Frequency analyses of the noise were then
made at ten equally spaced points round the machine about one foot
from its surface and the length of the locus of the microphone measured.
The total power was calculated assuming that the mean intensity
obtained from the s.p.l. values was equal to the average power intensity
over a hemi-sphere with a base circumference equal to the length of
the microphone locus.

A second set of measurements was taken at 24 points on the
surface of a hemi-sphere of three foot radius with the centre of its
base at the centre of the base of the machine. The points were taken
at the centres of three latitudinal bands with centres at 15°, 45°, and
75°. The appropriate number of points was taken in each band so that
the area represented by each point was constant. The power was then
calculated assuming hemi-spherical radiation.

The results of both of these sets of measurements are shown in
Table 7.3 and comparison is made with the values measured in the
anechoic chamber. It is apparent that neither method gives the same
value as that measured in the anechoic chamber although at most frequencies both values are quite close to it. The larger errors are probably due to interference between the direct waves and the waves reflected from the floor and surrounding objects.

It should be emphasised that this is the result for only a single machine. More exhaustive tests would have to be made before any firm conclusion on the accuracy of such measurements can be produced.
CHAPTER 8

CORRELATION OF THEORY AND TEST RESULTS

8.1) Introduction.

In Chapter 7 the general results of measurements in the sound fields of electric machines were discussed. In this chapter the results of measurements on particular machines are compared with those obtained from the theoretical approach developed in Chapters 4, 5 and 6. The theory given in these chapters is applicable to most types of machine, the main differences occurring in the calculation of force waves. The measurements which were made were confined to induction machines and so consequently the application of the theory is also confined to induction machines.

It is not proposed to attempt a comprehensive analysis of the noise. Only an indication of the techniques to be adopted, together with examples of the calculations involved will be given.

The first step in the correlation of the theoretical and test results is the identification and calculation of the resonant frequencies in a machine which is considered in Sect. 8.2. Once these are identified, their effect on the noise produced by a machine may be assessed by a study of frequency analyses of the noise when the machine is supplied at different frequencies. The probable sources of components may then be identified as described in Sect. 8.3. When frequencies of components are near resonances, the magnitude of the vibration is mainly determined by the mechanical damping of the iron so that the theory given in this thesis may not be used to calculate
the magnitudes of these components. If the frequency of a component is away from resonance, the magnitude of the component may be calculated. An example of such a calculation is given in Sect. 8.4.

As was stated in Chapter 1, it is useful to be able to check each stage of the theory independently. Since the magnitudes of the vibration and the sound power can be measured, the theory developed in Chapter 6 may be tested. Comparison between the measured and calculated powers is given in Sect 8.5.

8.2) Calculation of Resonant Frequencies.

The modes of mechanical resonance which may occur in a machine are very complicated due to their complex structure. In the theory developed in Chapter 5 the problem is simplified by considering the parts of the machine separately. In fact resonances occur as a result of vibration of the whole machine. The theoretical approach used in Chapter 5 is, however, useful to give an indication of the resonances which are likely to occur in the various parts, although the actual frequencies would probably be modified by the other parts of the machine. In view of this and the assumptions made in the theory, e.g. the stiffening effect of the ends of the machine is neglected and the rotor is considered as a point mass in the centre of the shaft, accurate determination of the resonant frequencies cannot be expected.

Applying Eqs. 5.23 and 5.24 directly, using the machine data in Appendix D, the stator resonant frequencies shown in Table 8.1 were obtained. These have been limited to frequencies in the audible range. Similarly, by applying Eqn. 5.28, the first rotor resonant frequencies
### TABLE 8.1
Calculated Stator Resonances of Machines

<table>
<thead>
<tr>
<th>m</th>
<th>LATERAL MODES</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Motors 1+2</td>
<td>Motor 3</td>
<td>Motors 4+5</td>
<td>CYLINDER</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.420</td>
<td>1.220</td>
<td>1.800</td>
<td>980</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4.050</td>
<td>3.430</td>
<td>5.150</td>
<td>2.820</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>7.800</td>
<td>6.550</td>
<td>10.550</td>
<td>5.400</td>
<td></td>
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<tr>
<td>5</td>
<td>12.000</td>
<td>10.200</td>
<td>15.300</td>
<td>8.350</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>18.500</td>
<td>15.600</td>
<td>23.400</td>
<td>12.800</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>21.500</td>
<td></td>
<td>17.700</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>14.200</td>
<td>7.600</td>
<td>14.600</td>
<td>16.700</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>22.000</td>
<td>12.000</td>
<td>23.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>17.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 8.2
Calculated Rotor Resonances of Machines

<table>
<thead>
<tr>
<th></th>
<th>Rotor Resonant Frequencies (c/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Motors 1+4</td>
<td>Motor 3</td>
</tr>
<tr>
<td>870</td>
<td>400</td>
</tr>
</tbody>
</table>
as shown in Table 8.2 were obtained. In order to take into account the
stiffening of the shaft due to the presence of the rotor laminations,
the actual length between the bearings was shortened by half the length
of the rotor core before substituting it in the resonance expression.
This was an arbitrary approximation but it was subsequently shown to
give reasonable results in all cases where this resonance was measured.
No tooth resonances were found to be within the audible range.

The mechanical responses of all five machines and a hollow
cylinder were obtained as explained in Sect. 3.3. These are shown in
Figs. E.15 and E.16. The response of the vibrator and coupling alone
is shown by the light trace in Fig. E.16c. It is apparent that these
resonances occur within the range of the higher resonant frequencies
of the machines. Thus, care must be taken when interpreting the results
of response measurements. Below 2000 c/s the force applied is fairly
constant. Above this frequency the force increases and decreases as the
response of the coupling changes. In future measurements more accurate
results could be obtained by inserting a force transducer in the
coupling to feed back a correcting signal to the oscillator as explained
in Sect. 3.3. In this way the force applied to the test object could
be kept constant at all frequencies and the actual response of the
test object found.

The rotor resonances of the four f.h.p. machines were identified
by removing the rotors and repeating the resonance measurements. In each
case the resonance at the lowest frequency was eliminated. Motor No.3
has grease packed ball-bearings so that little effect of the rotor
resonance is shown on the stator when the rotor is stationary. By
attaching the accelerometer to one end of the shaft, this resonance was
identified as may be seen from the light trace in Fig. E.15c. The
calculated resonances have been included on the response diagrams and reasonable agreement with the practical results was obtained for the rotor resonances.

Other resonances were identified by measuring the number of nodes of the vibration. Where this was possible the mode of resonance is shown at the top of the diagram. The second cylindrical mode for the stator was identified for each machine and there is again good agreement with the calculated frequencies. At higher frequencies resonances of the vibrator in addition to those of the test machine are present and the modes of vibration become more complex. However, unless the resonant frequencies of the two systems are coincident or very close to each other, the resonant frequencies of the machines will not be altered much by the resonances of the vibrator. Thus, some of the resonant frequencies of the system will be among the peaks shown in the resonance curves. Some of the resonances were identified as shown but in many cases this could not be done.

It was decided that many of these resonances were machine resonances as their frequencies agree with some of the peaks of the noise analyses in Figs. E.6 to E.10. All measured resonances together with the calculated values were therefore marked on these analyses. It should be stressed that these are not reliable values as in some cases they may be affected by the vibrator resonances. It is possible that some resonances with the same displacement variation but different modes may occur in a machine as explained by Den Hartog (25) since the teeth may vibrate in different ways. Thus, more resonant frequencies than those predicted by the theory in Chapter 5 would be present. Two discrete frequencies would probably only become apparent for the higher modes where the number of teeth becomes of the same order as
the mode number. Another explanation for the larger number of resonances at high frequencies is that the stator laminations and the shell may tend to act as separate cylinders at high frequencies. Such a system is analyzed by Erdelyi\(^{(21)}\) for larger machines and a larger number of resonances than the number which occur in a single cylinder were shown to exist.

As a study of a less complicated object, a hollow cylinder was vibrated and the response is shown in Fig. E.16c. Here the second mode of resonance agrees with the calculated value. However, it appears that two resonances occur for each of the third and fourth modes. This is probably because the resonant frequencies of the vibrator are coincident with the resonances of the cylinder causing the cylinder to act as a dynamic vibration absorber for the vibrator.

The first resonance of the vibrator is probably at a slightly lower frequency than that shown by the light trace in Fig. E.16c since the clamping bolts had been removed when this response was obtained. This frequency is nearly the same as the calculated third mode resonance of the cylinder. Such a system is analyzed by Den Hartog\(^{(25)}\) for the simplified case of two masses and two springs. This is equivalent to the system present here as may be seen from Fig. 8.1. Applying the
theory given by Den Hartog on p. 96 and taking into account the equivalent mass of the cylinder in the third mode of vibration, the resonance peak splits to form two resonance peaks with a shift of approximately ±30% from the original resonant frequency. The shift depends on the ratio of the equivalent masses of the two systems. Thus, in the case of the fourth mode resonance a smaller shift occurs as shown in the response curve. Since the two resonances must be at frequencies close to each other for this effect to occur, the calculated resonant frequencies must be close to the measured values.

After removing the rotors from the f.h.p. machines, the responses were obtained with the end castings also removed. The only change was found to be a slight lowering of the stator second-mode resonant frequency. The removal of the starting capacitor was found to have no effect on the resonant frequencies.

A computer programme has been prepared to evaluate Eqn. 5.26 for any mode number or frequency and this has been used to plot the response of the stators to various modes of vibration. The theoretical response to radial forces acting on the teeth of Motor No.3 is shown in Fig. 8.2. This shows that the response of the first mode is infinite at zero frequency as would be expected since it involves lateral movement of the whole machine. The other resonance peak on this curve is the longitudinal mode resonance. The responses of the other modes are constant at low frequencies and then increase to the first resonance where the phase is reversed. The magnitude then falls through zero where the phase is again reversed. It then rises to the longitudinal mode resonance and gradually decreases as the frequency is further increased.
Theoretical Response of Stator of Motor No. 3 in Various Modes of Vibration.

Fig. 8.2
The curves drawn in Fig 8.2 are for the magnitude of the displacement with constant radial forces acting on the teeth. Since the acceleration may be obtained from the displacement by multiplying by the square of the frequency, the magnitude of the acceleration for constant displacement rises 40 dB when the frequency is multiplied by ten. Thus, the sloping lines in Fig. 8.2 represent axes for acceleration. Similar axes, dropping 20 dB when the frequency is multiplied by ten could be drawn for the velocity of the vibration.

Resonances due to tangential forces on the teeth were also investigated. It was found that the resonant frequencies were same as those obtained for the radially excited modes although the shape of the resonance curve was modified at high frequencies. The resonances could be different for tangential excitation only if the dynamic response of the teeth was considered when deriving the expressions in Sect. 5.2.3.

8.3) Identification of Components of Noise.

Frequency analyses of the noise produced by the five test motors are shown in Figs. E.6 to E.10. It can be seen that most of the noise radiated by these machines is in the mid frequency range between 200 c/s and 2kc/s and is therefore in the range where the ear is most sensitive.

The feedback-type analyzer used to obtain these curves has a frequency response similar to those in Fig.8.3 with a continuously variable centre frequency. Fig. 8.3 shows the analyses of pure tones at three frequencies. In each case the response has dropped 45 dB at one octave from the frequency of the tone.

One frequency range of the analyzer is half of one chart in the
**Fig. 8.3.**

*Frequency Analyses of Pure Tones at Frequencies of 70 c/s, 100 c/s and 160 c/s on Frequency Calibrated Paper using B&K Type 2107 Analyser and B&K Type 2305 Level Recorder.*
diagrams in Appendix E. and the frequency scale is not logarithmic as may be expected. One octave change in frequency at the start of a frequency range (say 20c/s to 40c/s) is not the same linear distance on the scale as an octave at the end of a frequency range (say 30c/s to 60c/s). Thus, a pure tone appears as a narrower peak at the end of a frequency range than at the beginning as is shown in Fig. 8.3. This must be kept in mind when interpreting the analyses. Thus, in Fig. E.7 the light peak at 80c/s, the dark peak at 100c/s and the dark peak at 5500c/s all indicate pure tones with background levels much lower than the peak.

When several components of noise occur at frequencies near each other, the analyser does not register the distinct components. When tuned to a particular frequency, much of the energy from adjacent components is also allowed through the analyser. The analysis then takes the form of a smooth curve like that above 6 kc/s in Fig. E.7. In this case only the largest components may be discerned and the values obtained for these will be higher than the magnitude of the pure tone at that frequency. Between the large components it is impossible to determine what frequency components are present. A better type of analyser for these measurements would be a constant band width analyser which could identify pure tones over the whole frequency range even when they are separated by only a few cycles per second. These are, however, more difficult to use as the sweep frequency must be very slow to indicate all components.

At low sound pressure levels and low frequencies, background noise predominates. The general background noise level in the anechoic chamber is below 0 dB at frequencies above 200 c/s and is in the order
of 15 to 20 dB at lower frequencies. When underground trains pass under the chamber the noise level may rise as high as 40 dB, giving peaks similar to that at 28 c/s in Fig. E.7. However, no distinguishable noise from this source was present at frequencies above 100 c/s. This is not a serious limitation, as at all frequencies the background noise is below the threshold of hearing. However, it must be considered when interpreting the analyses.

At some frequencies there are rapid variations of sound pressure, as appear at frequencies near 600 c/s in Fig. E.7. These are variations of noise with time and not with frequency as they appear to be. The fluctuations occur at twice the frequency of the slip currents and may be explained by interference between forces travelling at the speed of the rotor and synchronous speed. Similar variations in the mains frequency and twice mains frequency components of induction machine noise are discussed by Summers(49). Similar principles may be applied to the higher frequency components. Such periodically varying components are often subjectively more annoying than steady components at higher levels. The magnitude of the variation was of the order of 15 dB at some places in the acoustic field.

Remembering these limitations of the measurements, the analyses of the noise may be studied. As explained in Chapter 1, the noise produced by a machine depends on several factors; the magnitude of the initial forces on the teeth, the mechanical response of the machine and the radiation characteristics.

There are very large variations of sound pressure in the field of an electric machine. Fig. E.14 shows the total variation round Motor No.2 of the sound pressure level of the noise analysed into 1/3-octave bands. The maxima and minima do not occur at the same position at
each frequency so that the overall level does not change by more than about 5 dB. A narrow band analysis at one point cannot, therefore, represent the total power radiated at each frequency. Thus, some components which appear small in Figs. E.6 to E.10 may represent large components of noise.

From a study of such analyses at several different points in the field, the dominant frequency components were selected for sound power measurements. The powers measured at these frequencies for the five test motors are shown in Table E.1. Although they do not give the relationship between the magnitudes of the total powers at different frequencies, the s.p.l. analyses do give an indication of the components of noise which are present.

The relationship between the vibration on the surface of the machine and the sound pressure at a point in the field depends on the position where the sound pressure is measured, the frequency, the mode of vibration and the size of the machine. In general, most of the components appearing in the analyses of the vibration also appear in the noise analyses of the noise although at low frequencies the relative magnitudes may be greatly reduced in the noise analyses. This may be seen by comparing the analyses in Figs. E.11 and E.12 with the analyses in Figs. E.7 and E.8.

It is assumed that most of the components of noise produced by these machines have electromagnetic sources. Some low frequency components may be due to mechanical unbalance. Bearing noise may be produced by Motor No. 3 but in the other machines it will be small since they have slieve bearings. Since the air flow in all of the machines was small, aerodynamic noise is unlikely to be a major source. It seems likely, therefore, that the main influences on the noise
analyses in Figs. E.6 to E.10 are the forces on the teeth and the mechanical response of the machine.

Most of the resonances found be externally exciting the machine are shown on the frequency analyses of the noise together with the calculated values. It should be noted that some of the measured resonances may be due to vibrator resonances and therefore they cannot all be considered as reliable (see Sect. 8.2).

Very large components of noise occur when the frequency of the force wave acting on the teeth is close to a resonant frequency of the mode of vibration with the same number of (poles) as the force wave. Very small force waves can produce large components of noise when resonances are excited in this way. Also, in some cases resonances may be excited by force waves with numbers of poles not corresponding to the mode of resonance. This may occur most easily when forces of zero mode with forces of the same phase and magnitude on all teeth, are present. In some machines parts of the stator are less rigid than other parts as in the f.h.p. test machines where four holes are made for clamping bolts to pass through the laminations. The deflection at these points when the machine is excited by the zero mode forces will be different than at other places and the four node resonance may be excited if the frequency is correct.

A simplified representation of this effect may be seen in Eqn. 5.15 with \( m = 0 \). This shows that standing waves of the form \( \sin \phi \cos \omega t \) are produced by zero mode forces. Forces with \( m \neq 0 \) may easily be produced as may be seen from Eqn. 4.34.

In some cases where large force components with low numbers of poles are present, large components of noise at frequencies away from the resonant frequencies may occur. There are not many such components
produced by the machines tested. They usually involve eccentricity or slot combinations which were recognised by Kron\(^{33}\) to produce noise if the frequency is near the resonant frequency of the particular mode of vibration.

To investigate the relative importance of the force magnitudes and mechanical resonances on the magnitude of the sound pressure, analyses of the noise produced by the machines were taken with the machine supplied at several different power frequencies. The analyses for two supply frequencies are shown on the same charts for each machine. In each case the voltage was adjusted to be proportional to the frequency of the supply giving the rated voltage at the rated frequency. In this way the stator currents and the fundamental flux density were kept constant.

Motors 1 and 2 are of similar construction the only difference being in the core length. This also applies to Motors 4 and 5. The causes of the noise in each set of machines will now be considered separately.

8.3.1) Motors 1 and 2.

These are single phase induction machines of identical cross section. Motor No. 2 has a slightly longer core than Motor No. 1. As would be expected, components of noise at the same frequencies are present in both machines although the power magnitudes are different. Motor No. 2 has larger power values at low frequencies and Motor No. 1 has higher power values at high frequencies.

The component at the lowest frequency in the acceleration analysis occurs at a frequency equal to the speed of rotation of the machine. This is probably caused by slight rotor unbalance. It could also be produc-
electromagnetically due to rotor eccentricity. No detectable power is radiated at this frequency as may be seen from the s.p.l. analyses.

The next component occurs at twice the frequency of the supply voltage. This is a pure tone and is caused by the fundamental flux in the machine. However, it is not mainly due to the four pole-pair radial force wave which causes the twice-mains-frequency hum in larger induction machines, but is caused mainly by the 100c/s variation of torque. This causes rotational movement of the stator and sound is radiated from the feet and starting capacitor. Power may also be radiated from the sides of the machine due to the correcting couple exerted on the feet by the mountings. This causes the motor to rock about an axis below the motor shaft and hence causes radial movement. If the motor is firmly fixed instead of being resiliently mounted, these vibrations will be transmitted to the supports causing sound to be radiated from secondary sources.

The next band of noise occurs in the region of the frequency of the rotor resonance. This consists of a fairly wide band of noise with several peaks on it. The lengths of the rotors of the two machines are slightly different so that the rotor resonant frequencies are different. For both machines, the maximum amplitude is obtained at a frequency lower than the resonant frequency measured when the rotor was stationary. This is probably because the effective bearing points move away from the centre of the machine when the rotor is rotating. This lowers the resonant frequency.

Without thoroughly investigating the magnetic fields of these machines and including the effects of saturation, slotting, eccentricity space harmonics and current harmonics, it is difficult to positively identify the forces which drive the vibration in the region of the
However, an example will be given to show how a 300 c/s single pole-pair vibration may be excited. This could be due to the interaction of saturation, space and stator-eccentricity harmonics. The values of \( k' \) and \( q' \) for the equations in Sect. 4.5 are 10 and -1 for the fifth space harmonic and 10 and 5 for the fifth saturation harmonic. Thus
\[
k' = 10 - 10 = 0, \quad q' = 5 - (-1) = 6
\]

If there is slight stator eccentricity making \( \omega = 300 \text{ c/s} \), then from Eqn. 4.34
\[
m = 1, \quad \omega = 300 \text{ c/s}
\]
Alternatively, if there is no stator eccentricity,
\[
m = 0, \quad \omega = 300 \text{ c/s}
\]

If the response of the stator is not the same at all positions, then displacement waves with different values of \( m \) may be produced.

The second resonant frequency occurs at about 1400 c/s. This is the resonant frequency of the second stator-bending mode. From the noise analyses it may be seen that this resonance produces the largest component of noise when the machine is supplied at the lower frequency. This is probably due to coincidence of the frequency of a component of the force with the resonant frequency.

Since there are 44 slots on the rotor causing a permeance variation at 1100 c/s, slotting will begin to affect the force wave in this frequency range. This permeance wave, combined with the fundamental m.m.f. wave produces a force wave with a large number of poles. However, when stator slotting and eccentricity are taken into account, force waves with smaller numbers of poles and different frequencies in the region of the rotor slotting frequency may be produced.

At higher frequencies there are several measured and calculated resonances. Many of these appear to correspond with peaks of the
noise analyses. It appears that at most of the resonances large
driving forces do not have frequencies which correspond with the
resonant frequencies and so large noise components are not produced.
However, at 4400 c/s and 5500 c/s force wave frequencies do appear to
coincide with with resonant frequencies and pure tones above the
general level of the noise are formed. With a supply frequency of
50 c/s a force wave with four pole-pairs at 4400 c/s is formed by
slotting harmonics and the fundamental m.m.f. wave. This may be seen
from Eqn. 4.34 by substituting

\[ k' = 2 - 2 = 0 \quad , \quad q' = 1 - 1 = 0 \]

\[ s_{rt} = 44 \quad , \quad s_{sr} = 36 \]

\[ j_{rt} = 4 \quad , \quad j_{sr} = 5 \]

The latter two terms may be formed from any of the slotting harmonics
such that

\[ (\pm j_{r1}) + (\pm j_{r2}) = 4 \quad , \quad (\pm j_{s1}) + (\pm j_{s2}) = 5 \]

Similarly, a four pole-pair force wave at 5500 c/s may be predicted
by making

\[ j_{rt} = 5 \quad , \quad j_{st} = 6 \]

The machine has semi-closed slots on both the rotor and the
stator and so quite large force components may be formed. It was not
possible to count the number of poles exactly as they were not evenly
spaced and the capacitor interfered with the measurements but a
number of poles of this order were identified. At a supply frequency of
40 c/s the equivalent peaks are much smaller showing that resonance
has a large effect on the magnitude.

The general rise in the background level in the mid frequency
range may be due to wide band noise produced by bearings and air
turbulence
8.3.2) Motor 3.

This machine is rated at 60 c/s and therefore the two supply frequencies used for the analyses in Fig. E.8 are 50 c/s and 60 c/s. The first component of the vibration analysis in Fig. E.12 is at a frequency equal to the speed of rotation of the rotor as with the single phase machine. However, since this machine is much larger than the others, more sound is radiated by this component as shown in Fig. E.8. In addition to the causes mentioned for the single phase machines, this component may be caused by magnetic unbalance due to an odd number of rotor teeth. This component is quite large and could cause considerable transmission of vibration to other connected objects. All harmonics of this frequency up to the eighth are also present. They are of considerably greater magnitude when the machine is supplied at the rated frequency than they are at the reduced frequency.

Near the rotor resonant frequency, there are two components of noise on each trace. The components at 350 c/s and 450 c/s on the light trace are equivalent to the components at 290 c/s and 350 c/s on the dark trace. The relative magnitudes of these components show that the rotor resonance probably has some effect on these components although the resonances would appear to be at a higher frequency than that measured with the rotor stationary.

In the frequency range between 600 c/s and 2000 c/s there appear to be several resonances. The main one is the second mode stator resonance occurring at 1200 c/s. There is also an off-resonance component of noise at 1110 c/s and 925 c/s on the light and dark traces respectively. This is due to rotor and stator slotting which produces a single pole-pair force wave when combined with the fundamental m.m.f. wave. This component is discussed more fully in Sect. 8.4.
The components at 1110 c/s and 1200 c/s appear to be radiated mainly by the drip shields over the ventilating apertures of the machine. These may be seen in Fig. 2.3. When these were removed, the overall power on the 'A' weighting decreased by 6 dB and the power radiated in the 1200 c/s band decreased by 9 dB. This change is not fully explained by the increase of the effective radius of the source and may, therefore, be due to resonances in the shields.

At higher noise frequencies there is very little difference in shape between the curves obtained with the two supply frequencies. It appears that in this frequency range there are a large number of force waves of nearly equal magnitudes. Thus, the shape of the analysis curve is determined mainly by the mechanical response of the machine. It is also possible that these resonances are excited by impact noise. This could be produced by turbulent air streams or small irregularities in the bearings. In order to confirm this a constant band width analyser could be used to identify particular components.

Some of the components in the machine which were not positively identified may be produced by the bearings since the machine has ball races. The best way to determine which components are caused in this way is to run the machine unexcited at the same speed using a drive motor outside the anechoic chamber.

8.3.3) Motors 4 and 5.

The frequency analyses of the noise produced by these machines are shown in Figs. E.9 and E.10. These machines are three-phase induction machines of similar construction to the single-phase machines. The two main mechanical differences are that the rotor slots are
closed and the stator cores are slightly deeper. It is apparent that many of the noise components produced by these machines are similar to those produced by the single-phase machines.

There is no measurable component of noise at twice the line frequency. This is because there is no twice-line-frequency torque variation and so the main force wave at this frequency has four pole-pairs. The resulting displacement on the surface is, therefore, very small. Also, even if considerable surface vibration were caused by this force wave, the machine is too small to radiate much power in this mode at this frequency.

The variation at frequencies close to the rotor resonant frequency is similar to that for the single phase machines. Again the resonant frequency appears to be lower than that measured with the rotor stationary. The second mode stator resonance is at a lower frequency than the calculated value. However, it does agree with the value measured by exciting the vibration externally. The discrepancy is probably due to the approximations made in the theory or else errors in the machine data.

These machines also produce two node components at a frequency of 1200 c/s. The total power radiated in the narrow band centred on this frequency is considerably higher for the smaller of the two machines than for the larger one. Single pole-pair force waves may be produced in a machine only when it has an asymmetrical construction as explained in Sect. 8.4. Since there are even numbers of slots both on the stator and on the rotor, this component is probably caused by air-gap eccentricity or faulty rotor bars. This would explain why different power magnitudes are radiated by the two machines.
The components at 4400 c/s and 5500 c/s are much smaller than are those produced by the single phase machines. This is because the slots on the rotor are fully closed reducing the magnitudes of the permeance harmonics. Also, since the stator has slightly different dimensions, the resonant frequencies may not be the same as those for the single-phase machines.

8.4) Calculation of Power due to Forced Vibration of Stator

If the forces applied to the stator teeth are large enough, potent vibration is produced on the stator surface even if the frequency is not near the resonant frequency of the particular mode of vibration. Such vibration is most effective for low mode numbers and was present in the first mode in the induction machines tested. First mode vibration consists of eccentric movement of the whole machine, which may be reduced when the machine is firmly mounted. However, in this case it would cause strong structure-borne vibration.

First mode vibration may only be produced by asymmetrical machine construction, as may be seen from Eqn. 4.34. This may take the form of an odd number of stator or rotor slots or eccentricity of the stator or rotor. The former is the cause of the 1110 c/s component in Motor No. 3 and the latter is probably the cause of the 1200 c/s components in Motors 4 and 5. As an example of the calculation of the magnitudes of these forced components, the power radiated by the 1110 c/s component of the noise radiated by Motor No. 3 will be calculated. It will be assumed that the machine is resiliently mounted. The design data for the machine are given in Appendix D.
The main m.m.f. wave component on no-load is the stator fundamental component. For this \( k = 2 \) since it is a four pole machine. Thus, assuming that this is the main m.m.f. contribution to the required force wave, from Eqn. 4.32

\[ k' = 2 \pm 2 \]

Taking the positive sign, all the other expressions in the group of Eqns. 4.32 must have positive signs making

\[ k' = 4 \quad q' = 2 \]

If eccentricity harmonics are ignored, the only values which \( \ell_{kr} \) and \( \ell_{r} \) may take are \( \pm 0 \). Therefore,

\[ \ell_{kr}' = \ell_{r}' = 0 \quad (8.1) \]

Taking the first slotting permeance harmonic terms in Eqn. 4.34

\[ j_{kr}' = -j_{r}' = 4 \quad (8.2) \]

Substituting all integers in Eqn. 4.34 gives

\[ m = 1 \quad \text{and} \quad \omega = 1110 \, \text{r/s} \]

on no-load, where the slip may be ignored.

The magnitude of this force wave may be found from the summations in Eqns. 4.31 and 4.34 together with the expressions for flux density in Eqn. 4.29. It will be assumed that the summation in Eqn. 4.34 may be taken as one term only, since the force waves with other values of \( j', k', \ell' \) and \( q' \), giving the same values of \( m \) and \( \omega \), will be small compared with the component derived above. This leaves the summation of Eqn. 4.31. Each term consists of the product of two flux density terms, which are given by Eqn. 4.29. For all components of the summation it is assumed that \( k' \) is formed as above, so that all the m.m.f. terms are identical and equal to the fundamental m.m.f.

The magnitude of the fundamental m.m.f. may be found from the machine data as follows:
The machine has a single-layer winding as shown for one pole-pair in Fig. 8.4. This also shows the two extremes of the phase band m.m.f. configuration. Taking the fundamental component as the average of the maxima of these two configurations,

\[ \bar{E}_{f} = \frac{1}{2} \left[ 1 + \frac{\sqrt{2}}{2} \right] I \]

where \( I \) is the maximum value of the total current in a phase band.

This machine has a no-load current of 9 amps r.m.s., 18 conductors in series per slot and three slots per pole per phase. Therefore, the fundamental component of m.m.f.,

\[ E_{f} = \frac{1}{2} \left[ 1 + \frac{\sqrt{2}}{2} \right] 9 \sqrt{2} \cdot 3 \cdot 18 = 640 \text{ At} \] (8.3)

The average permeance of the air-gap, \( \Lambda \), may be found using Eqn. 4.13 and the gap coefficients for a singly slotted air-gap as given by Gibbs(22) on p.123. Thus,

\[ \Lambda = \frac{\mu_0}{0.0006 \cdot 1.5} \]

\[ = 880 \mu_0 \text{ H/m}^2 \] (8.4)
The per unit permeance terms which are present are given by multiplying together two flux density terms such as those in Eqn. 4.29. If eccentricity is ignored, the four eccentricity terms will be unity. However, as mentioned in Sect. 4.5, both positive and negative signs for \( \ell \) must be considered even when \( \ell \) is zero. Thus, there are 16 identical terms in the summation of Eqn. 4.31 for the total force if the eccentricity is neglected, since

\[ l_{\ell} = (\pi o) + (\pi o) \quad \text{and} \quad l_{\ell}^* = (\pi o) + (\pi o) \]

There are many combinations of the slotting harmonics which can satisfy Eqn. 8.2 with the limitation of using only the positive signs in Eqn. 4.32. The first are formed by making \( j_{\phi h} \) and \( j_{\phi l} \) equal to one or \( \pi o \). There are four combinations: \( j_{\phi h} = 1, j_{\phi l} = \pi o \); \( j_{\phi h} = \pi o, j_{\phi l} = 1 \); \( j_{\phi h} = \pi o, j_{\phi l} = \pi o \). For each of these cases there is an equal number satisfying the stator condition in Eqn. 8.2 giving 16 terms in the permeance expression with a coefficient equal to \( \Lambda_{\phi h} \Lambda_{\phi l} \Lambda_{\phi l} \Lambda_{\phi h} \). This becomes \( \Lambda_{\phi h} \Lambda_{\phi l} \Lambda_{\phi l} \Lambda_{\phi h} \) since \( \Lambda_{\phi h} \) is equal to unity.

In a similar way the number of identical terms for other combinations of permeance harmonics satisfying Eqn. 8.2 may be found. The total expression for the force wave amplitude then becomes

\[
\sigma_{r_{ij},\ldots} = \frac{1}{16} \left( \frac{\pi o \mu_0 40}{16} \right)^2 \text{16.5} \quad \text{N/m}^2
\]

Thus, from Eqn. 4.39, the radial force acting on unit axial length of a tooth at a position \( \psi \),
\[ \Sigma_r = \frac{6400 \cdot 2}{7} \cdot 0.0825 \cdot 0.087 \cos(\phi - 110t) \]
\[ = 92 \cos(\phi - 110t) \quad (8.6) \]

In a similar way the tangential force on a tooth
\[ \Sigma_\phi = 4.3 \sin(\phi - 110t) \quad (8.7) \]

Thus, in this case, the tangential component is much smaller than the radial component. However, in some cases it may be important since it can excite different modes of vibration from the radial forces as explained by Den Hartog (25). If the design data and the magnitudes of these forces are directly substituted in Eqn. 5.26, the surface displacement may be found. This has been done using a digital computer and gives a maximum displacement of \(9 \times 10^{-8}\) m. The computer programme is fundamentally the same as that used to plot the mechanical responses of machines as explained in Sect. 8.2.

Using the computer programme developed for the total acoustic power calculation, this surface displacement gives a total power level of 73 dB assuming a 120° radiating band centred on the centre of the stator. A block diagram of the computer programme is shown in Fig. 6.3. The surface radius taken for this calculation was the radius of the drip shields on the ends of the machine. The measured power level was 74.5 dB. With the shields removed, the measured value was 68.0 dB and the calculated value 71.0 dB. There is good agreement between the measured and calculated results although the power change due to the removal of the end shields is not fully explained.

The main source of error in this calculation is probably involved in the calculation of the electromagnetic forces since saturation in the tooth tips would probably change their magnitudes. The effect of this saturation may be to increase or decrease the magnitude of this
component since it depends not only on the fundamental slot permeance but on all harmonics. If the machine was firmly mounted, this mode of vibration would probably be restricted. Therefore, although good agreement was obtained with the machine resiliently supported, under normal conditions the radiated power may be reduced.

8.5) Correlation Between Surface Acceleration and Total Power.

In Chapter 7 the production of sound waves by vibration of the surface of the machine was discussed qualitatively. Both the surface vibration and the total acoustic power may be measured as described in Chapters 2 and 3. From such measurements the theory for the radiation of sound from a machine as developed in Chapter 6 may be quantitatively tested by calculating the power from the surface vibration and comparing it with the measured power.

Motor No. 4 was used in these tests as it has no starting capacitor and so vibration could be measured easily over all its surface. The variation of vibration acceleration round the stator in a plane perpendicular to the shaft has a number of maxima equal to twice the mode number of the vibration since all measurable vibration waves on this machine were found to be standing waves. The variation of the component of vibration at 1400 c/s having four nodes is shown in Fig. E.4.

For ease of measurement of vibration, the machine was mounted with its shaft vertical. The powers measured with the machine mounted in this way were found to be slightly different from those measured with the shaft horizontal which are given in Table E.1. The powers measured
with the motor mounted vertically are given in Table 8.3 together with the other measurements required for this test. At each frequency, the power was calculated assuming that the peak value of the acceleration occurs at the reference position. The power so obtained was then modified by the difference between the vibration level at the reference and the level of the average of the maxima of the vibration distribution.

The power levels may be calculated from Eqn. 6.56 using values from Figs. C.1 and C.3. This calculation will be demonstrated for the 1400 c/s component. This component has a standing wave variation and may be split into two travelling waves travelling in opposite directions. From Table 8.3, the magnitude of each travelling wave of acceleration is 0.09 m/s². The magnitude of the velocity wave may be obtained by dividing the acceleration by the angular frequency and the first term of Eqn. 6.56 becomes

\[
20 \log_{10} \left[ \frac{7.13^7 \cdot 0.09 \cdot 0.091}{1400 \cdot 2\pi} \right]
\]

<table>
<thead>
<tr>
<th>Frequency (c/s)</th>
<th>Measured Acceleration m/s²</th>
<th>Mode Number</th>
<th>Meas. Level - Ref. Level (dB)</th>
<th>Power Calculated from Ref. Level (dB)</th>
<th>Thr. Calculated Power Level (dB)</th>
<th>Measured Level (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>0.023</td>
<td>1</td>
<td>9</td>
<td>11.6</td>
<td>20.6</td>
<td>17.0</td>
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<tr>
<td>400</td>
<td>0.012</td>
<td>1</td>
<td>10</td>
<td>12.0</td>
<td>22.0</td>
<td>19.0</td>
</tr>
<tr>
<td>600</td>
<td>0.040</td>
<td>1</td>
<td>8</td>
<td>21.9</td>
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<td>31.5</td>
</tr>
<tr>
<td>800</td>
<td>0.080</td>
<td>1</td>
<td>0</td>
<td>29.4</td>
<td>29.4</td>
<td>31.4</td>
</tr>
<tr>
<td>1200</td>
<td>0.230</td>
<td>1</td>
<td>2</td>
<td>38.3</td>
<td>40.3</td>
<td>40.3</td>
</tr>
<tr>
<td>1400</td>
<td>0.180</td>
<td>2</td>
<td>5</td>
<td>33.3</td>
<td>38.3</td>
<td>39.8</td>
</tr>
<tr>
<td>2300</td>
<td>0.018</td>
<td>3</td>
<td>6</td>
<td>12.6</td>
<td>18.6</td>
<td>18.4</td>
</tr>
<tr>
<td>3500</td>
<td>0.055</td>
<td>3</td>
<td>1</td>
<td>19.5</td>
<td>20.5</td>
<td>21.1</td>
</tr>
</tbody>
</table>
From Eqn. 6.12, the apparent radius of the surface of the machine,

\[ a = \left( \frac{0.081 \cdot 1400 \cdot 2\pi}{345} \right) \]

\[ = 2.07 \]

Thus, from Fig. 6.3, the second term of Eqn. 6.56 is approximately 3 dB.

The length of the cylindrical surface of the machine is equivalent to a 90° band on a spherical source. Therefore, from Eqn. 6.1, the third term of Eqn. 6.56 is -1.8 dB.

Summing all terms in Eqn. 6.56, the total power produced by one wave

\[ = 32.2 - 3 - 1.8 \]

\[ = 30.4 \]

making the total power for the two travelling waves 33.4 dB.

The calculated values shown in Table 8.3 were obtained using the full calculation for the first ten associated Legendre function components of the square wave distribution assumed for the surface vibration. A block diagram of the computer programme used for this calculation is shown in Fig. 6.3. It may be seen from the result for the 1400 c/s component that it was unnecessary to calculate the power due to any other component than the fundamental. Thus, the approximate calculation shown above is sufficient.

From the polar curve in Fig. 6.4 the correction for the reference position for the 1400 c/s component is approximately 5 dB making the total power level 38.3 dB. The measured power level is 39.8 dB.

All of the measured values are within 4 dB of the calculated values. Although this represents an error of the order of 100% in the power, this is a reasonable error for this type of calculation considering the complexity of the source. As is shown in Sect. 7.4.
secondary lobes are produced on the radiation patterns at high frequencies due to interference. Some of the errors could be due to the over-simplified distribution assumed in these cases. Also, at low frequencies, the value of $x$ is critical, as the source size correction factor changes rapidly with $x$ when $x$ is small. In view of these approximations involved in the calculation, the agreement between the two sets of results is better than was expected.
CHAPTER 9

CONCLUSIONS

The main purpose of the work described in this thesis is to provide a basis for the research project outlined in Sect. 1.2. This basis has been provided in three ways. First, techniques have been developed for the accurate measurement of the noise radiated by electric machines. Thus, the calculated values may be compared with the actual noise produced by a machine. Secondly, methods for identifying components of the noise have been demonstrated. This identification is required so that methods of calculating the magnitudes and frequencies of components may be developed. Thirdly, a basic method for calculating the noise components produced by electromagnetic sources has been developed and tested.

9.1) Measurement.

It was decided that the total power radiated in the required frequency band was the most convenient parameter for defining the noise radiated by electric machines. Apparatus was built to determine the total power radiated by a machine under anechoic conditions, by two methods. In the first method the power was calculated from s.p.l. measurements at many points on an imaginary sphere enclosing the source. In the second method, the output of a microphone was squared and integrated so that, as the microphone moved slowly round the machine, a signal proportional to the average power intensity was obtained. From this, the total power was calculated.
The number of points used in the first method and the number of times that the microphone was rotated in the second method were both varied, so that the effect on the accuracy of the results could be determined. The detailed results are discussed in Chapter 7. In general, the two methods produced results which were repeatable to within ±1 dB when the mounting and terminal voltage of the machine were carefully controlled. Similar agreement was obtained even when the noise for the second method of measurement was recorded on magnetic tape.

It was decided that the second method was the quicker method of obtaining the power and also had the advantage that the noise in the field of a machine along the microphone paths could be recorded so that the machine is required for a maximum of only about two hours. The total power in any frequency band can then be obtained in about half an hour from the tape recording. It is also convenient since polar diagrams of the noise may be made simultaneously with power measurements.

Measurements were also made to determine the variation of pressure in the acoustic fields of machines. It was found that, except at low frequencies, the machines could be treated as point sources for the measurement of power at distances from the machine as small as twelve inches for the small machines tested. However, at high frequencies interference occurs in the fields. Although the total power measured at all distances remains constant, the variation along radial lines from the centre of the machine is not in accordance with the inverse square law. At some angles the pressure drop is greater than that predicted by the inverse square law and at others it is less. Interference was also indicated by the presence of extra lobes on the polar diagrams of the noise at high frequencies.
Some comparisons were also made between the total power measured in the anechoic chamber and that measured for hemi-spherical radiation from a machine in a large room. The results were not conclusive but reasonable agreement was indicated. Further tests of this type would be useful, as this is the only type of measurement which may normally be made in industry. It is therefore important to determine the accuracy of measurements which may be made under these conditions.

Measurements could initially be made in the anechoic chamber with a large reflecting surface under the machine so that comparison could be made between ideal hemi-spherical radiation and free radiation. Measurements could then be made outside the anechoic chamber with the machine resiliently supported close to the floor and a comparison again made. The effect on the result of the number and distribution of the test points and the distance of measurement could then be found. Comparison could also be made between measurements made in the anechoic chamber and those made in accordance with standards set by the International Organisation for Standardisation when they are published.

Measurements of vibration on the surfaces of machines have also been made and close agreement obtained between this and the sound pressure close to the surface. Future work on vibration measurement should include measurement of the vibration transmitted by the supports. From this, studies of mountings and the noise radiated by other connected apparatus could be made.

9.2) Identification of Components.

An indication of the likely causes of many of the components of the noise may be obtained from frequency analyses of the noise radiated
by a machine while rotating at different speeds, together with a knowledge of the resonant frequencies. The application of this method, to the noise radiated by the induction machines tested, is explained in Sect. 8.3. It was found that many of the components were due to electromagnetic forces and machine resonances.

It may be possible to determine the causes of some of the electromagnetic components more exactly by running the machine at reduced flux to eliminate harmonics produced by saturation. A study of the effect of space harmonics could be made using a machine with each coil brought out of the machine separately. Also, a study of the effect of supply harmonics may be made using a harmonics generator. Machines suitable for both of these studies are available in the machines laboratory.

It would also be useful to study in detail the effect of load on the noise produced by machines. This could be done either using a small silent brake or else a drive shaft loaded outside the anechoic chamber. An eddy current brake with water cooling would probably be the most convenient.

9.3) Calculation of Electromagnetically Produced Components.

9.3.1) Air-gap Forces.

The forces on the teeth of a machine have been calculated using a simplified model of the air-gap assuming radial flux. It is difficult to know how accurate this method is since the forces have not been measured directly. This may be possible, however, if a force transducer could be attached to a tooth. The method has been used in the calculation of a component of the noise as shown in Sect. 8.4.
Although the total power obtained showed reasonable agreement with the measured value, this does not directly check the force calculation. It was shown, using this method, that the magnitudes of the tangential forces are small compared with the magnitudes of the radial forces. However, for other components, especially those with larger numbers of poles, the tangential forces may have more effect.

It would be useful to calculate the forces using conformal transforms and compare them with the values obtained from the method given in Chapter 4. This could be done by extending the work of Binns to the calculation of forces as explained in Appendix F. Another problem which requires investigation is the effect of tooth shape on the forces. This could be done using models of various teeth in an electrolytic tank. Also the effects of load currents and saturation should be investigated. The effect of the latter is very complex. However, it may be possible to study it by obtaining the flux distribution in a tooth by a relaxation method knowing the magnetisation characteristics of a small specimen of the iron. Most of these suggestions would be of use not only in noise calculations but also in other electric machine investigations concerned with such subjects as stray losses, parasitic torques etc.

9.3.2) Mechanical Response.

The calculated values of the resonant frequencies are compared with the measured values in Sect. 8.2. At the lower frequencies the calculation of the resonances appears to be quite accurate. However, at higher frequencies, more experimental investigation is required involving modification of the vibration generator to give a constant force at all frequencies.
There is little doubt that these resonances have a considerable effect on the noise radiated by a machine, even when there is no large force component at the resonant frequency. It would be useful to determine the exact way in which these resonances are excited and also to be able to calculate the magnitude of the response near resonance. This is limited by the mechanical damping of the iron. It was not possible to check the magnitude of the off-resonance response of the machines. However, as with the calculation of the force wave, the results of Sect. 8.4 indicate that the response is of the right order.

The application of the developed theory is limited to machines of a single cylinder construction. If the work of Erdelyi(21) was programmed and included in the power calculating programme, it would extend the work to a much larger number of machines. Energy methods of the type used by Erdelyi could also be used to determine the behaviour of stators more accurately by considering the core and teeth as a single member rather than separately as in Chapter 5.

9.3.3) Radiation of Sound.

The method for calculating the total acoustic power radiated by a source used in Chapter 6 assumes a spherical source. The results of calculations using this method are shown in Sect. 8.5 where reasonably accurate results are obtained. However, at high frequencies it is apparent that machines actually have radiation characteristics more like short cylinders than spheres, as secondary lobes caused by interference occur on the polar diagrams of the pressure. This does not seriously affect the result of the power calculation however. Also the pressure level measurements made in the part of the field near the 'equator' of the co-ordinate system would be expected to be near the
values calculated assuming a spherical source.

A more accurate result may be obtained by developing the solution for a short cylindrical source with closed ends as given by Williams et al. (55) The radiation characteristics for a cylindrical source in this paper are similar to those obtained from actual machines.

9.4) General.

The measuring techniques described here may be used to measure any component of the noise, if the microphone is not placed in a fast moving air stream. The calculation has been confined to noise produced by electromagnetic sources. The power radiated by other components of the noise is less easy to predict from the design data. Therefore, it is often unnecessary to predict the power produced electromagnetically very accurately.

However, the accurate determination of the electromagnetic noise leads to a better understanding of the way in which a machine works and contributes to the solution of other problems in machine analysis. In a similar way attempts should be made to calculate the other components of the noise more accurately. The mechanical response and radiation calculations given in this thesis may also be applied to mechanically produced noise. Ventilation noise must be considered separately.
REFERENCES


5. BERANEK, L.L.: "Acoustic measurement", (Chapman and Hall, 1959)


11. "Normal equal loudness contours for pure tones and normal threshold of hearing under free field listening conditions", British Standard 3383: 1961


15. CARTER, F.W.: "The magnetic field of the dynamo-electric machine", J. IEE, 1926, Vol. 64, p. 1115


17. COE, R.T., and TAYLOR, H.W.: "Some problems in electrical machine design involving elliptic functions", Phil. Mag., 1928, Vol. 6, p. 100


26. "Expression of the physical and subjective magnitudes of sound
or noise", International Organisation for Standardisation, 1959
27. "Preferred frequencies for acoustic measurements", International
Organisation for Standardisation, 1960
28. KING, A. J.: "The measurement and suppression of noise, with
special reference to electrical machines", (Chapman and Hall, 1965)
30. KING, A. J.: "Setting standards for machine noise", Engineering,
1964, Vol. 198, p. 93
(Wiley, 1962)
33. KRON, G.: "Slot combinations of induction motors", Electrical
Engineering, 1931, p. 937
34. LORD, P., and THOMAS, F. L.: "Noise measurement and control",
(Heywood, 1963)
35. MAGYAR, L. W.: "Sources of electro-magnetic vibration in single
phase induction motors", Trans Amer. Inst. Elect. Engrs, 1959,
Vol. 78, p. 81
37. MORSE, P. M.: "Vibration and sound", (McGraw-Hill, 1936)
38. MORSE, P. M., and FESHBACH, H.: "Methods of theoretical physics",
(McGraw-Hill, 1953)
A.C.E.C. Review, 1964, No. 1, p. 10


52. TIMOSHENKO, S.: "Theory of plates and shells", (McGraw-Hill, 1940)


* References marked in this way are not referred to in the text but were found to be of general assistance.
APPENDIX A

DETAILS OF SPLITTING OF THE SPHERICAL MEASURING SURFACE

A.1) Basic division.

For the basic division, the measuring surface is split into 13 bands by lines of latitude at the angles shown in the first column of Table A.1. Each of these bands is then split into an appropriate number of sections to make the area of each section approximately the same. The number of sections in each band and the area of each section in the band are shown in the third and ninth columns respectively.

The next two arrangements use the same latitudinal bands, but the sections are two and three times as large. For these distributions, the points used for the first distribution may be taken and every second and third point used. Where there were nine points in a band, five were taken in the lower of the two bands and four in the upper one when half the points were selected. The polar points were taken in each set.

Each of these divisions involves different areas per section in different bands. The remaining divisions involve sections of equal area.

A.2) 36 Points.

The surface is first divided into four bands by the 330°, 0°, 30° lines of latitude as shown in Fig. A.1. These are then split by lines of longitude into nine bands of equal area. Although they are not exactly the centres of the sections, points on the 15° and 60° lines
<table>
<thead>
<tr>
<th>Angle (Degrees)</th>
<th>Number of Points in Band</th>
<th>Area Represented by Each Reading (on sphere of unit radius)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 Points</td>
<td>8 Points</td>
</tr>
<tr>
<td>82.5</td>
<td>90</td>
<td>1</td>
</tr>
<tr>
<td>75</td>
<td>75</td>
<td>9</td>
</tr>
<tr>
<td>67.5</td>
<td>60</td>
<td>18</td>
</tr>
<tr>
<td>52.5</td>
<td>45</td>
<td>24</td>
</tr>
<tr>
<td>37.5</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>22.5</td>
<td>15</td>
<td>36</td>
</tr>
<tr>
<td>7.5</td>
<td>0</td>
<td>36</td>
</tr>
<tr>
<td>352.5</td>
<td>345</td>
<td>36</td>
</tr>
<tr>
<td>337.5</td>
<td>330</td>
<td>30</td>
</tr>
<tr>
<td>322.5</td>
<td>315</td>
<td>24</td>
</tr>
<tr>
<td>307.5</td>
<td>300</td>
<td>18</td>
</tr>
<tr>
<td>292.5</td>
<td>285</td>
<td>9</td>
</tr>
<tr>
<td>277.5</td>
<td>270</td>
<td>1</td>
</tr>
</tbody>
</table>

Total Number of Points in Bands: 272, 137, 92, 36, 20, 8
of latitude are taken from the basic distribution to give sound pressure levels representative of the sections.

A.3) 20 Points.

This is the distribution of points given by Beranek in Ref. 4, p. 110. The measuring surface consists of 20 equilateral triangles as shown in Fig. A.2. The angles required to give points at the centres of the triangles and the angles to give the nearest points to them are shown in Table A.2. The nearest points were used in the computer programme to calculate the total power.

![Fig. A.2.](image)

<table>
<thead>
<tr>
<th>Section Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Required ( \theta ) (degrees)</strong></td>
<td>45</td>
<td>45</td>
<td>45</td>
<td>315</td>
<td>315</td>
<td>315</td>
<td>315</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>19</td>
<td>19</td>
<td>341</td>
<td>341</td>
<td>71</td>
<td>71</td>
<td>181</td>
<td>289</td>
<td></td>
</tr>
<tr>
<td><strong>Actual ( \theta ) (degrees)</strong></td>
<td>45</td>
<td>45</td>
<td>45</td>
<td>315</td>
<td>315</td>
<td>315</td>
<td>315</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>15</td>
<td>345</td>
<td>345</td>
<td>60</td>
<td>75</td>
<td>30</td>
<td>270</td>
<td></td>
</tr>
<tr>
<td><strong>Required ( \phi ) (degrees)</strong></td>
<td>135</td>
<td>45</td>
<td>225</td>
<td>315</td>
<td>135</td>
<td>45</td>
<td>225</td>
<td>315</td>
<td>19</td>
<td>241</td>
<td>197</td>
<td>161</td>
<td>90</td>
<td>270</td>
<td>90</td>
<td>270</td>
<td>180</td>
<td>0</td>
<td>180</td>
<td>0</td>
</tr>
<tr>
<td><strong>Actual ( \phi ) (degrees)</strong></td>
<td>135</td>
<td>45</td>
<td>225</td>
<td>315</td>
<td>135</td>
<td>45</td>
<td>225</td>
<td>315</td>
<td>20</td>
<td>240</td>
<td>200</td>
<td>160</td>
<td>90</td>
<td>270</td>
<td>90</td>
<td>270</td>
<td>180</td>
<td>0</td>
<td>180</td>
<td>0</td>
</tr>
</tbody>
</table>

A.4) 8 Points.

In this case the sphere is divided into octants by three great circles including the equator. The angle of latitude at the centre of area of an octant is approximately 35°. Since there are no points at this latitude in the basic distribution, they were taken alternately at angles of 30° and 45°.
APPENDIX B

RESPONSE OF STATOR CORE

This section gives the derivation of the expression for the response of a thick cylinder to sinusoidal travelling waves stated by Carter(14). It is assumed that the problem may be reduced to the two dimensional problem of a ring, any axial variation of the forces being neglected. The force per unit area, acting on the inner surface of the ring, is split into radial and tangential components

\[
\sigma_{r,c} = \sum \sum \sigma_{r,c,m,n} \cos(m \phi + n \omega t) \tag{B.1}
\]

\[
\sigma_{\phi,c} = \sum \sum \sigma_{\phi,c,m,n} \sin(m \phi + n \omega t) \tag{B.2}
\]

The displacement and stresses in the iron will also be of sinusoidal form and the second differentials of any of the varying quantities may be simplified

\[
\frac{\partial^2 \sigma_{r,c,m,n}}{\partial t^2} = \omega^2 \sigma_{r,c,m,n}
\]

\[
\frac{\partial^2 \sigma_{r,c,m,n}}{\partial \phi^2} = m^2 \sigma_{r,c,m,n} \quad \text{etc.} \tag{B.3}
\]

**Fig. B.1.**

**Forces Acting on Element of Ring.**
Consider a small element of ring as shown in Fig. B.1. The stresses may be split up into a shear stress, an average tensional stress and a moment at each section. Equating forces radially and circumferentially on an element of the ring

\[ a \frac{\partial \sigma_r}{\partial \phi} - a_\epsilon \sigma_m + r_3 D_3 \sigma_{r_3} = M \tau_\epsilon \frac{\partial^2 \xi}{\partial \phi^2} \]  

and

\[ a \frac{\partial \sigma_m}{\partial \phi} + a_\epsilon \sigma_r - r_3 \theta_3 \sigma_{r_3} = -M r_\epsilon \frac{\partial^2 \xi}{\partial \phi^2} \tau_\epsilon \]

where the area of cross section of the ring,

\[ a_\epsilon = D_3 (r_m - r_3) \]

and \( M \) is the mass of the ring per unit length of the neutral axis.

The position of the neutral axis is given by Eqn. B.19 and is also the position of the centre of mass of the ring section. Thus,

\[ M = \frac{D_3}{2 \pi r_m} \left\{ \pi (r_m^2 - r_3^2) + \frac{r_3^2}{a} \sum r_m (r_m - r_3) \right\} + \frac{M_0}{2 \pi r_m} \]

\[ = \frac{k_3 D_3 \rho_f (r_m^2 - r_3^2)}{2 r_m^2} \]

where \( M_0 \) is the equivalent mass of the winding, depending on the contact with the teeth. \( k_3 \) is the ratio of the mass of the stator with teeth and windings to the mass of the stator core. If the ring is thin this may be simplified to

\[ M = k_3 D_3 \rho_f (r_m - r_3) \]

Equating the moments of all the forces acting on the element

\[ \frac{\partial^2 \xi}{\partial \phi^2} + a_\epsilon r_m \sigma_m - (r_m - r_3) r_3 D_3 \sigma_{r_3} = 0 \]

where the rotory inertia of the section is ignored.

Take a thin section of the ring of undeformed length \( \delta z_0 \) as shown in Fig. B.2. After deformation the length becomes

\[ \delta z = (1 + \epsilon) \delta z_0 \]

where \( \epsilon \) is the strain of the element.

\( \chi \) and \( \nu \) are co-ordinates tangential and normal to the undeformed
Fig. B.2. Displacement of Section of Ring.

Expressing the distances between the ends of the deformed element in terms of these co-ordinates:

$$\delta x = \delta x_0 - \delta r, \quad \delta x_c \delta \phi + \delta \phi_c \delta x - \delta r$$

$$\delta y = \delta y_0 + \delta \sigma, \quad \delta x_0 \delta \phi + \delta \phi_0 \delta x + \delta \phi_c \delta x - \delta r$$

Dividing both sides by $\delta z$ and substituting from Eqn. B.8

$$\frac{\delta x}{\delta z} = \frac{1}{(1 + \varepsilon)} \left\{ \frac{\delta x_0}{\delta z_0} - \frac{\delta r}{\delta z_0} + \frac{\delta \phi_c \delta x}{\delta z_0} - \frac{\delta \phi}{\delta z_0} \right\}$$  \hspace{1cm} (B.9)

$$\frac{\delta y}{\delta z} = \frac{1}{(1 + \varepsilon)} \left\{ \frac{\delta y_0}{\delta z_0} + \frac{\delta \sigma \delta x}{\delta z_0} + \frac{\delta \phi_c \delta y}{\delta z_0} - \frac{\delta \phi}{\delta z_0} \right\}$$  \hspace{1cm} (B.10)

In the limit when $\delta z$ and $\delta z_0$ are small,

$$S_{xz} \delta \phi \to \frac{\delta z_0}{\delta z}, \quad C_0 \delta \phi \to 1$$

$$\frac{\delta x}{\delta z} \to 1, \quad \frac{\delta y}{\delta z} \to S_{xz} \phi \to 0$$

Substituting these expressions in Eqn. B.9

$$(1 + \varepsilon) = \left[ 1 - \frac{\delta \phi_c}{\delta z_0} + \frac{\delta \phi_c}{\delta z} - \frac{\delta \phi}{\delta z_0} \right]$$

Giving

$$\varepsilon = \frac{1}{\varepsilon} \left[ \frac{\delta \phi_c}{\delta \phi} - \delta r \right]$$  \hspace{1cm} (B.11)

Since $\delta \phi = \frac{1}{2} \delta z_0$ and $\delta \phi_0 \to \delta r$

Substituting the limiting values in Eqn. B.10
since $\varepsilon$ is much less than unity.

In general, the strain produced by shear forces is small compared with that due to the bending moment. If it is assumed that the strain produced by shear is negligible, a plane perpendicular to the neutral axis before deformation is perpendicular to it after deformation as shown in Fig. B.3.

Thus, for one particular value of $\phi$, $\Theta$ is constant for all values of $\gamma$ and from Eqn. B.12

$$\Theta = \frac{1}{1+\varepsilon} \left\{ \frac{\varepsilon_{xx} - \varepsilon_r}{\varepsilon_{zz}} + \frac{1}{r} \varepsilon_r \right\}$$

(B.12)

For pure bending the radial and tangential movement of any section may be put in terms of the displacement of the neutral axis and $\Theta$.

$$\varepsilon_r = (1 - \cos \Theta) \times + \varepsilon_{r,nc}$$

$$\varepsilon_\phi = \varepsilon_{\phi,nc} + \times \sin \Theta$$

where $\times$ is zero at the neutral axis. Since $\Theta$ is small

$$\cos \Theta = 1 \quad \text{and} \quad \sin \Theta \approx \Theta$$

Substituting from Eqn. B.13, the radial and tangential displacements are

$$\varepsilon_r = \varepsilon_{r,nc}$$

(B.14)

$$\varepsilon_\phi = \varepsilon_{\phi,nc} + \frac{\times}{r} \left[ \frac{\partial \varepsilon_{\phi,nc}}{\partial \phi} + \varepsilon_{\phi,nc} \right]$$

(B.15)
Substituting these expressions in Eqn. B.11, the strain at any
distance \( X \) from the neutral axis

\[
\varepsilon_x = \frac{1}{E} \left\{ -\frac{F}{r_{\text{nc}}} + \frac{\partial F}{\partial \phi} + \frac{X}{r_{\text{nc}}} \left[ \frac{\partial^2 F}{\partial \phi^2} + \frac{\partial F}{\partial \phi} \right] \right\}
\]  (B.16)

In a solid the stress (see Ref. 52, p.2)

\[
\sigma_x = \frac{E}{(1-\nu)} \varepsilon_x = \sigma_x \varepsilon_x
\]  (B.17)

This is a stress acting in a circumferential direction and
varying with the distance from the neutral axis. The average tensional
stress over the section is obtained by integrating this stress over the
whole area and dividing by the area. Thus,

\[
\sigma_r = \frac{E}{\pi r_{\text{nc}}} \left( -\frac{F}{r_{\text{nc}}} + \frac{\partial F}{\partial \phi} \right)
\]  (B.18)

since, by the definition of \( r_{\text{nc}} \), the remaining integral is zero. To find
the position of the neutral axis, therefore, the following equation
must be solved

\[
r_{\text{nc}} \int_{r_{\text{nc}}}^{r_{\text{nc}}} \frac{X}{r_{\text{nc}} + X} \ D_3 \ dX = 0
\]

Integrating

\[
r_{\text{nc}} = \left( \frac{r_a - r_c}{\log r_a / r_c} \right)
\]  (B.19)

which is also the centre of mass of a small sector of the ring.

The moment, \( G \), at a section of the ring is given by multiplying
the expression for stress in Eqn. B.17 by \( X \) and integrating

\[
G = \frac{E}{\pi r_{\text{nc}}} \int_{r_{\text{nc}}}^{r_{\text{nc}}} \left[ \frac{\partial^2 F}{\partial \phi^2} + \frac{\partial F}{\partial \phi} \right] X \ D_3 \ dX
\]

\[
= \frac{J_c \ E}{\pi r_{\text{nc}}} \left[ \frac{\partial^2 F}{\partial \phi^2} + \frac{\partial F}{\partial \phi} \right]
\]  (B.20)

where

\[
J_c = \int_{r_{\text{nc}}}^{r_{\text{nc}}} \frac{X^2}{r_{\text{nc}} + X} \ D_3 \ dX
\]

\[
= \frac{D_3}{r_{\text{nc}}} \left[ \frac{r_{\text{nc}}^3 - r_c^3}{2} - 2 r_{\text{nc}} (r_{\text{nc}} - r_c) + r_{\text{nc}}^2 \log \left( \frac{r_{\text{nc}}}{r_c} \right) \right]
\]  (B.21)

If the ring is thin compared with its radius

\[
J_c = \left( r_{\text{nc}} - r_c \right)^2 \ D_3
\]  (B.22)
and \[ r_{\alpha \epsilon} = \frac{(r_{\alpha \epsilon} + r_c)}{2} \] (B.23)

In calculating the moment, the strain produced by tension alone may be ignored. By making \( \sigma_r \) equal to zero in Eqn. B.18

\[ \frac{\partial E}{\partial r_{\alpha \epsilon}} \frac{\partial E}{\partial r_{\alpha \epsilon}} \]

Substituting in Eqn. B.20 and simplifying

\[ G = \frac{T E^*}{r_{\alpha \epsilon}} (m^* - 1) F_{r_{\alpha \epsilon}} \] (B.24)

From the five equations, B.4, B.5, B.7, B.18, B.24, the variables \( G, \sigma_r, c_\omega, \) and \( F_{r_{\alpha \epsilon}} \) may be eliminated. Differentiating Eqn. B.24,

\[ \frac{\partial G}{\partial \phi} = \frac{J_c E^* (1 - m^*)}{r_{\alpha \epsilon}} \frac{\partial E}{\partial r_{\alpha \epsilon}} \]

Substituting in Eqn. B.7 and rearranging

\[ \sigma_{\omega \epsilon} = \frac{J_c E^* (m^* - 1)}{r_{\alpha \epsilon}} \frac{\partial E}{\partial \phi} + (r_{\alpha \epsilon} - r_c) \frac{r_c}{r_{\alpha \epsilon}} D_\epsilon \sigma_{\omega \epsilon} \]

differentiating this expression

\[ \frac{\partial \sigma_{\omega \epsilon}}{\partial \phi} = -\frac{J_c E^* m^*(m^* - 1)}{r_{\alpha \epsilon}} F_{r_{\alpha \epsilon}} + (r_{\alpha \epsilon} - r_c) \frac{r_c}{r_{\alpha \epsilon}} D_\epsilon \frac{\partial \sigma_{\omega \epsilon}}{\partial \phi} \] (B.25)

Substituting in Eqn. B.5 from Eqns. B.3 and B.25, and rearranging

\[ \sigma_r = \frac{1}{\epsilon_c} \left\{ \left[-M r_{\alpha \epsilon} \omega + \frac{J_c E^* m^*(m^* - 1)}{r_{\alpha \epsilon}} \right] F_{r_{\alpha \epsilon}} - 2 D_\epsilon \sigma_{\omega \epsilon} - (1 - \frac{r_c}{r_{\alpha \epsilon}}) \tau_c D_\epsilon \frac{\partial \sigma_{\omega \epsilon}}{\partial \phi} \right\} \] (B.26)

Rearranging Eqn. B.18

\[ \frac{\partial F_{r_{\alpha \epsilon}}}{\partial \phi} = \frac{r_{\alpha \epsilon} \sigma_r}{E^*} + F_{r_{\alpha \epsilon}} \]

This term may also be found be differentiating Eqn. B.4 and rearranging the terms to give

\[ \frac{\partial F_{r_{\alpha \epsilon}}}{\partial \phi} = \frac{1}{M r_{\alpha \epsilon} \omega^*} \left[ m^* a_c \sigma_r + a_c \frac{\partial \sigma_{\omega \epsilon}}{\partial \phi} - \tau_c D_\epsilon \frac{\partial \sigma_{\omega \epsilon}}{\partial \phi} \right] \]

Equating these two expressions and rearranging

\[ M r_{\alpha \epsilon} \omega^* F_{r_{\alpha \epsilon}} = \frac{\sigma_r}{H} + a_c \frac{\partial \sigma_{\omega \epsilon}}{\partial \phi} - \tau_c D_\epsilon \frac{\partial \sigma_{\omega \epsilon}}{\partial \phi} \]
where \( \frac{1}{H} = m^2 - \frac{M r_n^2}{a_n E'} \) \hspace{1cm} \( \text{(B.27)} \)

Substituting for \( \frac{\partial \sigma}{\partial \phi} \) and \( \sigma_n \) from Eqns. B.25 and B.26,

\[
M r_n \omega^2 \sigma_{\phi, n} \left\{ \frac{1}{H} \left[ -M r_n \omega^2 + \frac{J_r E'}{r_n^2} m^2 E_n \right] - \frac{J_r E'}{r_n^2} m^2 E_n \right\} \sigma_{\phi, n} \\
- \frac{D_n r_n^2}{H} \left[ \frac{1}{H} \left( 1 - \frac{r_n}{r_n} \right) r_n D_n - \left( 1 - \frac{r_n}{r_n} \right) r_n D_n + r_n D_n \right] \frac{D_n}{H} \frac{D_n}{r_n^2}
\]

Simplifying,

\[
\sigma_{\phi, n} = \frac{D_n}{H} \frac{D_n}{r_n^2} \left\{ \frac{\sigma_{\phi, n} + \left[ \frac{H r_n^2}{r_n} + 1 - \frac{r_n}{r_n} \right] \frac{D_n}{H} \frac{D_n}{r_n^2}}{m^2 E_n} \right\}
\]
\hspace{1cm} \( \text{(B.28)} \)

since, from Eqn. B.14, the surface radial displacement is approximately the same as the neutral axis radial displacement.
EVALUATION OF RADIATION FUNCTIONS

C.1) Evaluation of $Q_{m,n}$

The non-dimensional expression given in Eqn. 6.53 may be evaluated and tables or graphs formed so that the calculation does not have to be carried out each time the noise radiated by a machine is calculated. The expression given in Eqn. 6.53 may be expanded using Eqn. 6.10 to give

$$Q_{m,n} = \left( \frac{2n+1}{2} \right) \left( \frac{n-m}{n+m} \right) \left[ \int \frac{u_{m,n}(\phi)}{\sin \phi} \frac{d^{n-m} \left( y^{2n} \right)}{dy^{2n}} \right]^2$$

by the binomial expansion. Thus, differentiating, this becomes

$$Q_{m,n} = \left( \frac{2n+1}{2} \right) \left( \frac{n-m}{n+m} \right) \left[ \int \frac{u_{m,n}(\phi)}{\sin \phi} \frac{y^{2i(n-m)}}{i!} \right]^2$$

and substituting

$$i = \frac{2 \pi}{\cos \Theta}$$

and

$$\cos \Theta = y$$

$$Q_{m,n} = \left( \frac{2n+1}{2} \right) \left( \frac{n-m}{n+m} \right) \left[ \int \frac{u_{m,n}(\phi)}{(-1)^n \sin \phi} \frac{i^{n-m} \cos \phi}{\cos \phi} \right]^2$$

If $u_{m,n}$ is known, either as an analytical expression or as a numerical distribution in $\Theta$ for a particular value of $m$, this expression may be easily evaluated using a computer. A programme has been written to do this and is included in the the programme used to calculate the total power radiated by a machine.

In many cases the vibration may be assumed to be constant over a band on the surface adjacent to the stator core and zero elsewhere. This is equivalent to a pulse in the $\Theta$ direction. In this case
where \( u_{m, \theta, \phi} \) now becomes a constant equal to unity and \( \Theta \) and \( \Theta' \) are the upper and lower limits of the vibrating band. This expression may be evaluated numerically as before or analytically using combinations of the following reduction formulae

\[
\begin{align*}
T_i &= e^{\int \sin^2 \theta \cos \theta \, d\theta} \\
&= -\frac{1}{i} \left[ \sin^{i-1} \theta \cos \theta \right]_\Theta + \frac{\pi}{i} \, Y_i(\Theta) \\
Y_i &= e^{\int \sin \theta \cos \theta \, d\theta} \\
&= -\frac{1}{i} \left[ \sin^{i} \theta \cos \theta \right]_\Theta + \frac{\pi}{i} \, Y_i(\Theta) \\
Y_i &= e^{\int \cos^2 \theta \, d\theta} \\
&= \frac{1}{i} \left[ \sin^{i} \theta \cos \theta \right]_\Theta + \frac{\pi}{i} \, Y_i(\Theta)
\end{align*}
\]

By applying these, all components of Eqn. C.3 may be reduced to simple integrals. This method has been programmed for the computer and the programme is included in the programme used to calculate the total power. This as an alternative to the numerical method when the distribution may be considered constant over a band.

In addition, tables of

\[
\frac{\pi}{2} \frac{(n-m)!}{(n+m)!} \left[ \int_0^{\pi} \rho_n(\cos \theta) d(\cos \theta) \right]^2
\]

have been compiled for various values of \( \Theta \), \( n \) and \( m \). From these tables, \( Q_{m,n} \) over any angle may be calculated since

\[
Q_{m,n}(\Theta + \phi) = (-1)^{m+n} Q_{m,n}(\Theta - \phi)
\]

Therefore

\[
Q_{m,n} = \frac{(2n+1)(n-m)!}{2(n+m)!} \left[ \int_0^{\pi} \rho^m_n(\cos \theta) d(\cos \theta) + (-1)^m \int_0^{\pi} \rho^m_n(\cos \phi) d(\cos \phi) \right]^2
\]

When, as in many cases, the band is symmetrical about \( \Theta = 90^\circ \)

\[
Q_{m,n} = \left\{ \begin{array}{ll}
\frac{(2n+1)(n-m)!}{2(n+m)!} \left[ \int_0^{\pi} \rho^m_n(\cos \theta) d(\cos \theta) \right]^2 & , m+n \text{ even} \\
0 & , m+n \text{ odd}
\end{array} \right.
\]
Since the distribution may be considered symmetrical in most cases, this function, in a logarithmic form based on the value with $m = n = 0$, has been plotted in Fig. C.1 for various values of $m$ and with $m = n$. The fundamental component with $m = n$ contains most of the power and so these curves may be used for most calculations.

C.2) **Evaluation of Radial Function.**

All of the radial variations of pressure and velocity depend on the value of $n$ and may be expressed in terms of $\eta_n'$ and $\eta_n''$ as defined in Eqns. 6.14 and 6.15. The calculation of these expressions has been divided into three steps. The first step is to obtain a series in terms of powers of $x$ for each value of $n$ such that

$$\eta_n' = \frac{A_{n,1}}{x} + \frac{A_{n,3}}{x^3} + \cdots$$

$$\eta_n'' = \frac{A_{n,3}}{x^3} + \frac{A_{n,5}}{x^5} + \cdots$$

$$\frac{d\eta_n'}{dx} = -\frac{A_{n,1}}{x^2} - \frac{2A_{n,3}}{x^4} + \cdots$$

$$\frac{d^2\eta_n''}{dx^2} = -\frac{3A_{n,3}}{x^4} - \frac{4A_{n,5}}{x^6} + \cdots$$

The values of $A_{n,i}$ for these expressions have been evaluated and are shown for the first few values of $n$ in Table C.1. The second stage is to substitute values of $x$ and obtain $\eta_n', \eta_n''$, etc. which may then be substituted in Eqns. 6.23, 6.31 and 6.36 to find the values of $\eta_n', \eta_n''$ and the phase angle between the pressure and the velocity. These steps have been programmed for the computer and the variation of the phase angle with $x$ and $n$ is shown in Fig. C.4.

Instead of calculating $\eta_n'$ and $\eta_n''$ directly it was decided that the ratio of the actual variation to the variation given by the inverse square law was more often required and so this has been calculated.
and plotted in logarithmic form in Figs. C.2 and C.3. These are of
the form \(20 \log_{10} \left( k^2 r^2 \right) \) and \(20 \log_{10} \left( k r^2 \right)\) and give the departure from the
inverse square law in dB. This is similar to the form given by
Alger (2) for an indefinitely long cylindrical source.

In studying these curves it may be seen that when \(r\) is small
the pressure and velocity are both larger than the values predicted
by the inverse square law. However, the phase angle between them
increases as \(r\) decreases so that the total power passing through a
sphere at any distance is a constant. This is the near field of the
source. At a value of \(r\) dependant on \(n\) far field conditions are
attained and the pressure and velocity are nearly in phase with each
other.
**Fig. C.1.**

Graph of $10 \log_{10} \frac{(2n+1)(n-m)!}{(n+m)!} \left[ \int_{90-\theta}^{90+\theta} P_{m,n}^{(0)}(\mu \theta) \, d\theta \right]^2$

Against $\theta$, with $m=n$ for various values of $m$.

($\theta$ is half the width of the vibrating band)
### TABLE C.1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Coefficients of Radial Functions for Various Values of $n$.</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>-----</td>
<td>----</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
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<td>5</td>
<td>105</td>
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<td>7</td>
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<td>9</td>
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<td>3</td>
<td>-2</td>
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<td>5</td>
<td>60</td>
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<tr>
<td>8</td>
<td></td>
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<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
Fig. C.2.

Graphs of $\log_{10} [x \eta_{np}(x)]$ against $x$

For various values of $\eta$. 

---

$20 \log_{10} [x \eta_{np}(x)]$ dB
Fig. C.3.
Graph of $20 \log_{10} [x \cdot \eta_{\eta, \mu}(x)]$ against $x$ for various values of $\eta$. 
Fig. C.4.

Graphs of Phase Angle Between Pressure and Velocity for Various Values of \( \phi \).
## APPENDIX D

### DESIGN DATA OF TEST MACHINES

<table>
<thead>
<tr>
<th>Type</th>
<th>Motor No.1</th>
<th>Motor No.2</th>
<th>Motor No.3</th>
<th>Motor No.4</th>
<th>Motor No.5</th>
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<tbody>
<tr>
<td>Horse-power</td>
<td>1/6</td>
<td>1/3</td>
<td>10</td>
<td>1/2</td>
<td>3/4</td>
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<tr>
<td>Number of phases</td>
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<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
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<tr>
<td>Rated Frequency (mHz)</td>
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<td>50</td>
<td>50</td>
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<td>Number of Poles</td>
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<td>4</td>
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<td>4</td>
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<td>Line Voltage (V)</td>
<td>240</td>
<td>240</td>
<td>440</td>
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<td>415</td>
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<td>Line Current on No-Load (Amp)</td>
<td>1.7</td>
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<td>No-Load Power (Watt)</td>
<td>80</td>
<td>119</td>
<td>770</td>
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<td>91</td>
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<td>Air-gap Length (mm)</td>
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<td>0.305</td>
<td>0.76</td>
<td>0.305</td>
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<td>Air-gap Radius (cm)</td>
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<td>4.66</td>
<td>8.25</td>
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<td>Stator Core Radius (cm)</td>
<td>6.7</td>
<td>6.7</td>
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<td>6.4</td>
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<td>Rotor Core Radius (cm)</td>
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<td>8.1</td>
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<td>Stator Core Length (cm)</td>
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<td>19.0</td>
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<td>6.0</td>
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<tr>
<td>Rotor Core Length (cm)</td>
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<td>19.0</td>
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<td>6.0</td>
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<td>15.0</td>
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<td>15.0</td>
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<td>Equivalent Shaft Radius (cm)</td>
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<td>Type of Bearing</td>
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<td>Sleeve</td>
<td>Ball Races 7 Bear/Arm</td>
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<td>Sleeve</td>
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<tr>
<td>Mass of Stator Wires (kg)</td>
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<td>1.5</td>
<td>14.5</td>
<td>1.0</td>
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<td>Main Wdg. Turns in Series Per Coil</td>
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<td>33/51/69/85</td>
<td>18</td>
<td>96/09</td>
<td>40/15</td>
</tr>
<tr>
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<td>41/22/93/51</td>
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<td>-</td>
<td>-</td>
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<tr>
<td>Main Wdg. Turns in Series Per Phase</td>
<td>644</td>
<td>952</td>
<td>54</td>
<td>820</td>
<td>644</td>
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<td>Main Wdg. Turns in Series Per Phase</td>
<td>8.24</td>
<td>11.48</td>
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### Appendix D (continued)

<table>
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<tr>
<th></th>
<th>Motor No.1</th>
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<th>Motor No.3</th>
<th>Motor No.4</th>
<th>Motor No.5</th>
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<tr>
<td><strong>STATOR</strong></td>
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<td></td>
</tr>
<tr>
<td>Number of slots</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>Type of slot</td>
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<td>Semi-Closed</td>
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<td>Semi-Closed</td>
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<td>Skew</td>
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<td>None</td>
<td>None</td>
<td>None</td>
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<tr>
<td>Width of slot opening (mm)</td>
<td>2.3</td>
<td>2.3</td>
<td>6.3</td>
<td>2.3</td>
<td>2.3</td>
</tr>
<tr>
<td>Tooth width near gap (mm)</td>
<td>3.5</td>
<td>3.5</td>
<td>8.05</td>
<td>4.04</td>
<td>4.04</td>
</tr>
<tr>
<td>Tooth width near core (mm)</td>
<td>3.5</td>
<td>3.5</td>
<td>13.9</td>
<td>4.04</td>
<td>4.04</td>
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<td>Slot width near gap (mm)</td>
<td>4.7</td>
<td>4.7</td>
<td>6.3</td>
<td>4.16</td>
<td>4.16</td>
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<td>Slot width near core (mm)</td>
<td>8.1</td>
<td>8.1</td>
<td>6.3</td>
<td>8.0</td>
<td>8.0</td>
</tr>
<tr>
<td>Slot pitch (mm)</td>
<td>8.2</td>
<td>8.2</td>
<td>14.4</td>
<td>8.2</td>
<td>8.2</td>
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<td><strong>ROTOR</strong></td>
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<td></td>
<td></td>
<td></td>
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<td>Number of slots</td>
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<td>44</td>
<td>33</td>
<td>44</td>
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<td>Type of slot</td>
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<td>Semi-Closed</td>
<td>Semi-Closed</td>
<td>Closed</td>
<td>Closed</td>
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<tr>
<td>Skew</td>
<td>I.C.R.A.</td>
<td>I.S.S.P.</td>
<td>I.S.S.P.</td>
<td>1.53P.</td>
<td>1.53P.</td>
</tr>
<tr>
<td>Width of slot opening (mm)</td>
<td>0.76</td>
<td>0.76</td>
<td>1.0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Tooth width near gap (mm)</td>
<td>3.04</td>
<td>3.04</td>
<td>11.5</td>
<td>3.04</td>
<td>3.04</td>
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<tr>
<td>Tooth width near core (mm)</td>
<td>3.04</td>
<td>3.04</td>
<td>7.0</td>
<td>3.04</td>
<td>3.04</td>
</tr>
<tr>
<td>Slot width near gap (mm)</td>
<td>3.6</td>
<td>3.6</td>
<td>4.2</td>
<td>3.6</td>
<td>3.6</td>
</tr>
<tr>
<td>Slot width near core (mm)</td>
<td>2.2</td>
<td>2.2</td>
<td>4.2</td>
<td>2.2</td>
<td>2.2</td>
</tr>
<tr>
<td>Slot pitch (mm)</td>
<td>6.64</td>
<td>6.64</td>
<td>15.7</td>
<td>6.64</td>
<td>6.64</td>
</tr>
</tbody>
</table>
APPENDIX E
MACHINE TEST RESULTS

This appendix contains photographs of typical results obtained from the level recorder for the five machines tested, together with a table of the power levels obtained for each machine using the analogue power calculator. In all cases the machines were resiliently mounted and for all except Fig. E.4 the shaft was horizontal. The three-phase machines were so connected that the rotation was counter-clockwise looking from the drive end. It was found that the radiation characteristics and the power values were slightly different when the machines were mounted vertically or the other direction of rotation was used.

All pressure levels are referred to $2 \times 10^{-5} \text{ N/m}^2$ and all power levels are referred to $10^{-12}$ watts unless otherwise stated.

The level recorder settings for the traces in this appendix were as follows:

Polar traces
Writing speed $= 4 \text{ mm/s}$, paper speed = one revolution in 4.5 minutes

Inverse Square Law Traces
Writing speed $= 16 \text{ mm/s}$, paper speed $= 0.3 \text{ mm/s}$.

Narrow Band and 1/3-octave Analyses
Writing speed $= 80 \text{ mm/s}$, paper speed $= 1.0 \text{ mm/s}$.

Frequency Responses
Writing speed $= 125 \text{ mm/s}$, paper speed $= 1 \text{ mm/s}$.
### TABLE E.1

Levels of Power Radiated by Test Machines

<table>
<thead>
<tr>
<th>Motor No.1</th>
<th>Motor No.2</th>
<th>Motor No.3</th>
<th>Motor No.4</th>
<th>Motor No.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (c/s)</td>
<td>Sound Power Level (dB)</td>
<td>Frequency (c/s)</td>
<td>Sound Power Level (dB)</td>
<td>Frequency (c/s)</td>
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<tr>
<td>100</td>
<td>33.0</td>
<td>100</td>
<td>35.6</td>
<td>240</td>
</tr>
<tr>
<td>600</td>
<td>31.0</td>
<td>600</td>
<td>37.2</td>
<td>360</td>
</tr>
<tr>
<td>1200</td>
<td>19.7</td>
<td>1200</td>
<td>26.7</td>
<td>480</td>
</tr>
<tr>
<td>1400</td>
<td>31.8</td>
<td>1400</td>
<td>23.6</td>
<td>1110</td>
</tr>
<tr>
<td>3500</td>
<td>23.8</td>
<td>3500</td>
<td>18.1</td>
<td>1200</td>
</tr>
<tr>
<td>4400</td>
<td>29.4</td>
<td>4400</td>
<td>28.4</td>
<td>3500</td>
</tr>
<tr>
<td>5500</td>
<td>31.6</td>
<td>5500</td>
<td>29.8</td>
<td></td>
</tr>
<tr>
<td>7700</td>
<td>27.4</td>
<td>7700</td>
<td>25.5</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>42.3</td>
<td>A</td>
<td>41.7</td>
<td>A</td>
</tr>
</tbody>
</table>

'A' weighing,
FIG. E.1.

VARIATION OF SOUND PRESSURE LEVEL AT 100 C/I S IN THE HORIZONTAL PLANE THROUGH THE SHAFT OF MOTOR No. 2.

(A) POLAR DIAGRAMS AT VARIOUS DISTANCES FROM THE MOTOR.

(B) VARIATION OF SOUND PRESSURE LEVEL WITH DISTANCE AT VARIOUS ANGLES.
FIG. E.2.

VARIATION OF SOUND PRESSURE LEVEL AT 1200 C/S IN THE HORIZONTAL PLANE THROUGH THE SHAFT OF MOTOR No. 2

(a) POLAR DIAGRAMS AT VARIOUS DISTANCES FROM THE MOTOR

(b) VARIATION OF SOUND PRESSURE LEVEL WITH DISTANCE AT VARIOUS
FIG. E. 3.

VARIATION OF SOUND PRESSURE LEVEL AT 5500 C/S IN THE HORIZONTAL PLANE THROUGH THE SHAFT OF MOTOR No. 2

(a) POLAR DIAGRAMS AT VARIOUS DISTANCES FROM THE MOTOR.

(b) VARIATION OF SOUND PRESSURE LEVEL WITH DISTANCE AT VARIOUS ANGLES.
FIG. E. 4 VARIATION OF 1400 C/S COMPONENT OF SURFACE ACCELERATION AND SOUND PRESSURE IN THE ACOUSTIC FIELD WITH ANGLE ROUND MOTOR No. 4.

Sound pressure level curves obtained in a plane perpendicular to the shaft and passing through the centre of the motor. The distance of the microphone from the centre is shown on each curve. The points through which the acceleration curve is drawn were obtained from narrow band analyses of the vibration at 24 points round the stator. The reference level for each curve is arbitrary.
FIG. E.5  VARIATION OF 1200 C/S COMPONENT OF NOISE WITH ANGLE IN HORIZONTAL PLANE OF MOTOR No. 2.

MICROPHONE AT A CONSTANT DISTANCE OF 18" FROM THE CENTRE OF THE MOTOR.

TRACES OBTAINED WITH MICROPHONE AT VARIOUS ANGLES OF LATITUDE AS INDICATED.
FIG. E.6  NARROW-BAND ANALYSIS OF NOISE RADIATED BY MOTOR No. 1

MICROPHONE FACING SIDE OF MOTOR AND 18" FROM ITS CENTRE.

POWER SUPPLY := 240 V, 50 C/S (DARK TRACE); 192 V, 40 C/S (LIGHT TRACE)

MEASURED RESONANT FREQUENCIES SHOWN BY BROKEN LINES
CALCULATED RESONANT FREQUENCIES SHOWN BY FULL LINES
FIG. E.7  NARROW-BAND FREQUENCY ANALYSIS OF NOISE RADIATED BY MOTOR No. 2.

MICROPHONE FACING SIDE OF MOTOR AND 18" FROM ITS CENTRE

POWER SUPPLY: 240V, 50 C/S (DARK TRACE); 192V, 40 C/S (LIGHT TRACE)

MEASURED RESONANT FREQUENCIES SHOWN BY BROKEN LINE

CALCULATED RESONANT FREQUENCIES SHOWN BY FULL LINES
Fig. E-8. NARROW-BAND FREQUENCY ANALYSIS OF NOISE RADIATED BY MOTOR No. 3.

MICROPHONE FACING SIDE OF MOTOR AND 24" FROM ITS CENTRE.

POWER SUPPLY: 440 V LINE, 60 C/S (LIGHT TRACE); 387 V LINE, 50 C/S (DARK TRACE).

MEASURED RESONANT FREQUENCIES ARE SHOWN BY BROKEN LINES.
CALCULATED RESONANT FREQUENCIES ARE SHOWN BY FULL LINES.
FIG. E.9 NARROW-BAND FREQUENCY ANALYSIS OF NOISE RADIATED BY MOTOR No. 4.

MICROPHONE FACING SIDE OF MOTOR AND 18" FROM ITS CENTRE.

POWER SUPPLY – 415 V LINE, 50 C/S (DARK TRACÉ), 332 V LINE, 40 C/S (LIGHT TRACÉ)

MEASURED RESONANT FREQUENCIES SHOWN BY BROKEN LINES.

CALCULATED RESONANT FREQUENCIES SHOWN BY FULL LINES.
FIG. E.10. NARROW-BAND FREQUENCY ANALYSIS OF NOISE RADIATED BY MOTOR No. 5.

MICROPHONE FACING SIDE OF MOTOR AND 18" FROM ITS CENTRE.

POWER SUPPLY: 415 V LINE, 50 C/S (DARK TRACE); 332 V LINE, 40 C/S (LIGHT TRACE).

MEASURED RESONANT FREQUENCIES SHOWN BY BROKEN LINES.

CALCULATED RESONANT FREQUENCIES SHOWN BY FULL LINES.
FIG. E.11. NARROW BAND FREQUENCY ANALYSIS OF VIBRATION ON THE SURFACE OF MOTOR No. 2.

ACCELEROMETER ON SIDE OF MACHINE CLOSE TO ITS AXIAL CENTRE.

ANALYSIS OF NOISE RADIATED BY THE SAME MACHINE IS SHOWN IN FIG. E.7.
FIG. 12. NARROW BAND FREQUENCY ANALYSIS OF VIBRATION ON THE SURFACE OF MOTOR No. 3.
ACCELEROMETER ON SIDE OF MACHINE CLOSE TO ITS AXIAL CENTRE.
ANALYSIS OF NOISE RADIATED BY SAME MACHINE IS SHOWN IN FIG. E 8.
FIG. E.13  $\frac{1}{3}$-OCTAVE BAND (DARK TRACE) AND OCTAVE BAND (LIGHT TRACE) FREQUENCY ANALYSES OF THE NOISE RADIATED BY MOTOR No. 2.

Analysis of same noise as the narrow band analysis in Fig. E.7.
Bands are shown symmetrically about their centre frequencies.

FIG. E.14  DIAGRAM SHOWING THE RANGE OF SOUND PRESSURE LEVELS, ANALYSED IN $\frac{1}{3}$-OCTAVE BANDS, IN THE FIELD OF MOTOR No. 2 AT A DISTANCE OF 18$^\circ$ FROM ITS CENTRE.

The middle line in each band shows the sound pressure level which would be present at the same distance if the same power was radiated uniformly.
Figure 15. Frequency response of complete motors to externally applied vibration.

Constant voltage applied to input of vibrator amplifier.
Acceleration measured near point of application of vibration.
FIG. E.16. FREQUENCY RESPONSE OF COMPLETE MOTORS AND TEST CYLINDER TO EXTERNALLY APPLIED VIBRATION.

CONSTANT VOLTAGE APPLIED TO INPUT OF VIBRATION AMPLIFIER.
ACCELERATION MEASURED NEAR POINT OF APPLICATION OF VIBRATION.
The method used to calculate the electromagnetic forces on the teeth used in Chapter 4 assumed that the flux was radial. In order to represent the flux distribution more accurately, the actual flux distribution in the air-gap between simplified iron surfaces may be found using conformal transforms. The basis of the calculation of field distributions is given by Gibbs and the development to more complicated problems using numerical methods by Binns and Lawrenson.

First, the iron surfaces must be simplified, usually to form a polygonal shape, the angles often being confined to multiples of 90°. It is also usual to consider a small repeatable section of the air-gap and assume that the adjoining sections have no effect on the field in the section considered. The method could, however, be extended to include the effect of adjoining sections.

Secondly, the equipotential surfaces must be identified. It is usually assumed that the slots are infinitely deep and that the air-gap is infinitely long circumferentially. With normal slot depths and tooth widths this can be shown to have little effect on the flux distribution. If two slotted surfaces are considered with current flowing in slots in both members, each tooth is at a different magnetic potential from the others in the same area. Thus, three potential differences are required to describe the field between two facing slots. The most convenient way of splitting the field is that considered by Binns. The three components are the homo-polar field and the two hetero-polar fields excited from each side of the air-gap. In crossing
from one side of the air-gap to the other, there is a potential change
equal to the integral of all the currents down the air-gap, since the
currents in the other slots may be assumed to be on the air-gap surface.
Crossing either slot, there is a potential change equivalent to the
current flowing in the slot. Each of these field problems may be
solved separately and summed to find the total field distribution.
From the field distribution the force acting on a tooth may be calculated.

The method may also be used in a simpler form for synchronous
machines on no-load as investigated by Carter\(^{(14)}\) and for synchronous
machines on load by using a two component potential distribution. To
a limited extent the induction machine may be considered as a singly
excited machine while on no-load but this may not give all forces acting
on the iron.

The method may be generalised as follows:
Let \( \mathcal{J} \cdot j\mathbf{e} \mathbf{c} \) describe the plane in which the actual slots exist and let
\( \mathcal{W} \cdot \mathcal{W} + j\mathcal{W} \) describe variations in the plane in which the whole iron
surface is transformed to a straight line. \( \mathcal{X} = \mathcal{X}' + j\mathcal{X}'' \), \( \mathcal{X} = \mathcal{X}' + j\mathcal{X}'' \)
are planes in which the original equipotentials for a particular
component of the field become parallel infinite surfaces and therefore
have a uniform field between them. Using the Schwarz-Christoffel
transformation the relation between these fields may be found such that
\[
\begin{align*}
\frac{d\mathcal{J}}{d\mathcal{W}} &= \tilde{\mathcal{F}}_{\mathcal{J},\omega}(\mathcal{W}) \\
\frac{d\mathcal{X}}{d\mathcal{W}} &= \tilde{\mathcal{F}}_{\mathcal{X},\omega}(\mathcal{W}) \\
\frac{d\mathcal{X}'}{d\mathcal{W}} &= \tilde{\mathcal{F}}_{\mathcal{X}',\omega}(\mathcal{W})
\end{align*}
\] (F.1)

If required, the relation between \( \mathcal{J}, \mathcal{W} \) and \( \mathcal{X} \) may be determined
by analytical or numerical integration. This integration is usually
required to find the constants involved in the functions, \( \tilde{\mathcal{F}} \). Once the
constants have been determined they may be substituted in the expressions for \( i \). Thus, from Eqns. F.1

\[
\frac{d\chi'}{d\nu} = \frac{\chi'_{\omega}}{\chi_{\nu,\omega}} \frac{\chi_{\nu,\omega}}{d\nu} \quad \text{etc.}
\] (F.2)

In the \( \lambda \) plane there is a uniform field produced by a potential difference \( F_i \) between surfaces a distance \( \delta \lambda \) apart. The potential difference is the same as the actual component of potential difference in the original plane and the value of \( \delta \lambda \) depends on the transformation. The uniform flux density in the \( \lambda \) plane

\[
\delta \lambda = \frac{\mu_0}{\delta \lambda} F_i.
\]

The flux crossing a small area of unit axial length and width \( \delta \lambda \) on an equipotential surface in the \( \lambda \) plane is

\[
\frac{\mu_0}{\delta \lambda} F_i \delta \lambda.
\]

This flux enters an area of unit axial length and width \( \delta \lambda \) in the \( \lambda \) plane. Therefore, the flux density in the \( j \) plane is

\[
\frac{\mu_0}{\delta \lambda} F_i \delta \lambda.
\]

or in the limit becomes

\[
\frac{\mu_0}{\delta \lambda} F_i \frac{\chi'_{\omega}}{\chi_{\nu,\omega}} \frac{\chi_{\nu,\omega}}{d\nu} \quad \text{etc.}
\]

from Eqn. F.2. Since all components are normal to the iron surfaces, the total flux density at the tooth surface

\[
= \mu_0 \sum_{\nu} \frac{F_i}{\delta \lambda} \frac{\chi'_{\omega}}{\chi_{\nu,\omega}} \frac{\chi_{\nu,\omega}}{d\nu} \quad \text{(F.3)}
\]

where the summation is made for all components of the potential distribution.

The force per unit area is given by the square of the flux density multiplied by \( \frac{1}{2\mu_0} \). Integrating over the surface on which the force is required, the total force per unit axial length

\[
= \frac{\mu_0}{2} \int \frac{1}{\delta \lambda} \frac{\chi'_{\omega}}{\chi_{\nu,\omega}} \frac{\chi_{\nu,\omega}}{d\nu} \left[ \left( \frac{F_i}{\delta \lambda} \right)^2 \right] d \lambda
\]
Substituting for \( j \) from Eqn. F.1, the force per unit axial length becomes

\[
F = \frac{\mu_0}{2} \int \frac{1}{F_{j_i}} \left[ \frac{F_{j_i}}{j_i} \frac{F_{j_i}}{(w)} \right]^2 dw
\]

where the summation is made over all components of the field and the integration is made along the real axis of the \( w \) plane between the points corresponding to the limits of the required surface in the \( j \) plane.

To calculate the force, \( F \) must be known. The values vary with time and with slot position. When the variation is known analytically, a link can be obtained between the air-gap currents and the m.m.f. The force may then be obtained analytically for a simple case. For more complicated iron configurations or more complex slot currents, numerical methods would have to be used and the force on each part of the tooth calculated after each time interval and the resulting variation analysed into harmonic components. It should be noted that a considerable number of points would have to be taken as the force also varies with rotor position and the constants in \( F \) therefore vary with time. This would necessitate the use of numerical methods on a digital computer.

It is difficult to take saturation into account when using this method since in this case the iron surfaces are no longer equipotentials. Also, the flux is not necessarily normal to the surface and forces acting tangentially to the iron surfaces must be considered.