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SIMPLICIAL METHODS AND THE HOMOLOGY OF GROUPS

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This thesis generalizes the simplicial methods of defining derived functors and proves characterization theorems about them. In particular, we show that our methods also generalize an alternative way of defining derived functors. We use our definitions to define the higher Baer invariants of a group relative to a variety of groups and compute them in special cases. Finally, we deduce a long exact sequence in the homology of groups.
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CONTENTS

ABSTRACT 2

ACKNOWLEDGEMENTS 3

CONTENTS 4

INTRODUCTION 5

CHAPTER I: PRELIMINARIES 7

CHAPTER II: SEMI-SIMPLICIAL METHODS IN BASE CATEGORIES 16
  2.1 Derived Functors 17
  2.2 Base Categories 31
  2.3 Semi-simplicial methods in base categories 46

CHAPTER III: SIMPLICIAL METHODS IN RINEHART CATEGORIES 60
  3.1 Simplicial Sets 61
  3.2 Simplicial methods and derived functors 64
  3.3 Simplicial methods in Rinehart categories 91

CHAPTER IV: BAER INVARIANTS AS DERIVED FUNCTORS 100
  4.1 Definitions and the first Baer invariant 101
  4.2 The second Baer invariant 113

CHAPTER V: HOMOLOGY AND BAER INVARIANTS 124
  5.1 The classical and the simplicial theories 124
  5.2 Homology and the second Baer invariant 131
  5.3 The Spectral Sequence argument 138

REFERENCES 146
INTRODUCTION

This thesis brings together four approaches to non-abelian cohomology. First, there is the theory of derived functors defined in terms of triples, developed first in Beck's thesis and later by Barr and Beck, and others; much of the work in this area is collected in volume 80 of the Springer Lecture notes in Mathematics, 1969.

A second approach is to use the theory of simplicial resolutions (with degeneracies) or semi-simplicial resolutions (without). This method is used by André in (1), Tierney and Vögel in (12), and Keune in (6). This theory bears a relationship to the triple theory similar to that born by the Eilenberg-Maclane "bar-resolution" definition of the classical cohomology of groups to the definition in terms of an arbitrary projective resolution. A key departure in Keune's work is that he considers functors taking values in a non-abelian category.

A third approach is used by Rinehart in (11). He reduces the non-abelian theory to the theory of the derived functors of an additive functor between two abelian categories by going to functor categories. This theory agrees with those considered above (in situations where the theories are all defined). It has the advantage of dealing efficiently with exact sequences arising from an epimorphism in the domain category. Its disadvantages are that the method does not lead to easy calculation and that the range category is required to be abelian.

A fourth approach is due to Fröhlich (2), who constructs a
theory of Baer-invariants. If $V \subseteq A$ are varieties of algebras, 
and if $V: A \to A$ and $U: A \to V$ are the corresponding verbal 
sub-algebra and quotient-algebra functors respectively, he considers 
$D_0V: A \to V$ and $D_1U: A \to A$ (the latter taking values in null 
algebras), though he does not produce a theory of derived functors 
to justify the notation.

The purpose of this thesis is to produce a theory of derived 
functors combining the main features of simplicial theory and 
Rinehart's theory, and which gives Fröhlich's Baer-invariants as 
derived functors, justifying his notation. This allows one to 
define higher Baer-invariants and leads ultimately to an exact se-
queness in the classical homology of groups which, while suggested by 
the above methods, is obtained classically via the Hochschild-Serre 
spectral sequence.
CHAPTER I

PRELIMINARIES

We shall use the following notation throughout this thesis.
Let \( \mathcal{C} \) be a category. \( |\mathcal{C}| \) denotes the class of objects of and if \( A, B, \xi \in |\mathcal{C}| \), then \( \mathcal{C}(A, B) \) denotes the set of \( \mathcal{C} \)-morphisms from \( A \) to \( B \). If \( f \in \mathcal{C}(A, B) \) and \( g \in \mathcal{C}(B, C) \), then we denote by \( gf \) the composite of \( f \) and \( g \).

**Definition 1.1**

A semi-simplicial \( \mathcal{C} \)-object \((X, d)\), (or \( X \) for short) is a graded \( \mathcal{C} \)-object \( \{X_n\} \ n \geq 0 \) together with \( \mathcal{C} \)-morphisms
\[
d_i^n : X_n \rightarrow X_{n-1}, \quad 0 \leq i \leq n, \quad \text{such that}
\]
\[
d_i^i d_j^j = d_j^j d_i^i d_j^j d_{j-1}^j d_i^i \quad \text{for} \quad 0 \leq i < j \leq n+1.
\]

A simplicial \( \mathcal{C} \)-object \((X, d, s)\), (or \( X \) for short), is a semi-simplicial \( \mathcal{C} \)-object \((X, d)\) together with \( \mathcal{C} \)-morphisms
\[
s_i^n : X_n \rightarrow X_{n+1}, \quad 0 \leq i \leq n, \quad \text{satisfying}
\]
\[
s_i^{i+1} s_j^{j+1} = s_j^{j+1} s_i^{i+1} s_j^{j+1} s_{j+1}^{j+1} s_i^{i+1} \quad 0 \leq i \leq j \leq n,
\]
\[
d_i^{i+1} s_j^{j+1} = s_j^{j+1} d_i^{i+1} s_j^{j+1} d_j^{j+1} d_{j+1}^{j+1} d_i^{i+1} d_{j+1}^{j+1} \quad 0 \leq i < j \leq n,
\]
\[
d_j^{j+1} s_j^{j+1} = d_j^{j+1} s_j^{j+1} = i_{X_n}, \quad 0 \leq j \leq n, \quad \text{and}
\]
\[
d_i^{i+1} s_j^{j+1} = s_j^{j+1} d_i^{i+1} \quad i > j + 1, \quad i \leq n+1.
\]
We call $d^i_n$ the face maps and $s^i_n$ the degeneracy maps.

**Definition 1.2**

If $X$ and $Y$ are semi-simplicial $C$-objects, then a semi-simplicial morphism $f$ from $X$ to $Y$ consists of $C$-morphisms $f^n : X_n \rightarrow Y_n$, $n \geq 0$, such that $f^n$ commutes with the face maps.

If $X$ and $Y$ are simplicial $C$-objects, then a simplicial morphism $g$ from $X$ to $Y$ consists of $C$-morphisms $f^n : X_n \rightarrow Y_n$, $n \geq 0$, such that $f^n$ commutes with both the face and the degeneracy maps.

**Definition 1.3**

Let $A \xrightarrow{f^0} B$ be a sequence of $n+1$ $C$-morphisms, $n \geq 0$. A simplicial kernel of $(f^0, \ldots, f^n)$ is a sequence $K \xrightarrow{k^0} A$ of $n+2$ $C$-morphisms such that $f^i k^j = f^{i-1} k^j$ for $0 \leq i < j \leq n+1$, and universal with respect to this property.

That is, if $Z \xrightarrow{h^0} A$ is any other sequence of $n+2$ $C$-morphisms satisfying $f^i h^j = f^{i-1} h^j$ for $0 \leq i < j \leq n+1$, then there exists a unique $C$-morphism $h : Z \rightarrow K$ with $k^i h = h^i$ for $0 \leq i \leq n+1$.

**Definition 1.4**

If $A \in \mathcal{C}$, an augmented (semi-) simplicial object of $A$ is a (semi-) simplicial $C$-object $X$ together with a $C$-morphism
\[ d_0^0: X_0 \to A \text{ such that } d_0^0 d_0^1 = d_0^0 d_1^1. \]

**Definition 1.5**

Let \( \mathcal{E} \) be a class of epimorphisms in a category \( \mathcal{C} \).

We say that \( P \in \mathcal{C} \) is \( \mathcal{E} \)-projective if for every \( f \in \mathcal{E} \), say \( f: A \to B \),

\[ \mathcal{C}(P,f): \mathcal{C}(P,A) \to \mathcal{C}(P,B) \text{ is surjective.} \]

If \( \mathcal{P} \) is the class of all \( \mathcal{E} \)-projectives, then we write \( \mathcal{E} \to \mathcal{P} \).

If \( \mathcal{E} \) is the class of all epimorphisms \( g \) in \( \mathcal{C} \) such that \( \mathcal{C}(P,g) \) is surjective for every \( P \in \mathcal{P} \), then we write \( \mathcal{P} \to \mathcal{E} \).

Clearly, \( \mathcal{E} \subseteq \mathcal{E} \).

We say that \( \mathcal{E} \) is closed if \( \mathcal{E} = \mathcal{E} \).

We say that \( \mathcal{E} \) is a projective class if it has "enough projectives"; i.e., if for every \( A \in \mathcal{C} \), there is a morphism \( P \to A \in \mathcal{E} \) with \( P \in \mathcal{P} \).

**Definition 1.6**

Let \( \mathcal{E} \) be a class of epimorphisms in a category \( \mathcal{C} \) and let \( \mathcal{E} \to \mathcal{P} \). An augmented (semi-)simplicial object \( X \) of \( A \in \mathcal{C} \) is called \( \mathcal{E} \)-projective if \( X_n \in \mathcal{P} \) for \( n > 0 \). Given an augmented (semi-)simplicial object \( X \) of \( A \in \mathcal{C} \), if \( K_n \xrightarrow{k_n^0} X_{n-1} \xrightarrow{k_n} X_n \) is the simplicial kernel of \( X_n \xrightarrow{d_n^1} X_{n-2} \)

\( n \geq 1 \) and putting \( X_{-1} = A \), then there exists a unique \( \mathcal{E} \)-morphism \( e_n: X_n \to K_n \) such that \( k_n e_n = d_n^i \), \( 0 \leq i \leq n \), \( n \geq 1 \). Thus, we
Definition 1.7

Let $\mathcal{C}$ be a class of epimorphisms in a category $\mathcal{C}$. An augmented semi-simplicial object $X$ of $A \in \mathcal{C}$ is called $\mathcal{C}$-exact if $d_0$ and $e_n$ are in $\mathcal{C}$, $n \geq 1$, where $e_n$ is defined as above.

$X$ is called a semi-simplicial $\mathcal{C}$-resolution of $A$ if it is both $\mathcal{C}$-projective and $\mathcal{C}$-exact.

We want to define the notion of an algebraic category. To do this we need the definitions of adjoint functors and triples. The standard reference for the rest of the chapter is MacLane (8).

Definition 1.8

Let $F : A \to B$ and $U : B \to A$ be functors. Then we say $F$ is left adjoint to $U$ (and $U$ is right adjoint to $F$) if for all $A \in |A|$, $B \in |B|$, there exists a natural bijection.

$$n_{A,B} : \mathcal{B}(FA,B) \cong \mathcal{A}(A,UB).$$

If we choose $B = FA$, we get

$$n_{A,FA} : \mathcal{B}(B,B) \cong \mathcal{A}(A,UF).$$
Put $\eta_A = n_{A,B}(1_B)$, then by naturality we can deduce that $\eta : 1_A \to UF$ is a natural transformation from the identity functor on $A$ to the functor $UF$. We call $\eta$ the unit of adjunction. If we choose $A = UB$, we get

$$n_{UB,B} : B(FUB,B) \cong A(A,B).$$

Put $\rho_B = n_{UB,B}^{-1}(1_A)$, then we can show that $\rho : FU + 1_B$ is a natural transformation of functors. We call $\rho$ the counit of adjunction.

**Definition 1.9**

A triple $\Pi = (T, \eta, \rho)$ in a category $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$, a natural transformation $\eta : 1_\mathcal{C} \to T$ and a natural transformation $\rho : T^2 \to T$ such that the following three diagrams are commutative.

\[
\begin{array}{ccc}
T & \xrightarrow{T\eta} & T^2 \\
\downarrow{1_T} & & \downarrow{\rho} \\
T & & T
\end{array}
\quad
\begin{array}{ccc}
T & \xrightarrow{\eta T} & T^2 \\
\downarrow{1_T} & & \downarrow{\rho} \\
T & & T
\end{array}
\quad
\begin{array}{ccc}
T & \xrightarrow{\rho T} & T^2 \\
\downarrow{T\rho} & & \downarrow{\rho} \\
T & & T
\end{array}
\]

$\eta$ is called the two sided unit of $\Pi$ and $\rho$ the associative composition of $\Pi$.

**Proposition 1.10**

If $F : A \to \mathcal{B}$ is left adjoint to $U : \mathcal{B} \to A$, then there exists an induced triple $\Pi$ in $A$ given by, $\Pi = (UF, \eta, \rho_U)$, where
\( n \) is the unit and \( u \) the counit of adjunction.

**Proof:** Straightforward, see Maclane (8).

**Proposition 1.11 (See Keune (6), page 25)**

Let \( \Pi = (T, n, u) \) be a triple in a category \( \mathcal{A} \). Then there exists a "universal" pair of adjoint functors \( F^\Pi: \mathcal{A} \to \mathcal{A}^\Pi \) and \( U^\Pi: \mathcal{A}^\Pi \to \mathcal{A} \), inducing the triple \( \Pi \) (in the sense of 1.10), and universal with respect to this property; i.e. if \( F, U \) is any pair of adjoint functors inducing \( \Pi \), \( F: \mathcal{A} \to \mathcal{B}, U: \mathcal{B} \to \mathcal{A} \); then there exists a unique functor \( \phi: \mathcal{B} \to \mathcal{A}^\Pi \) such that

\[
\begin{array}{c}
\mathcal{A} \\
\downarrow F \\
\mathcal{A}^\Pi
\end{array}
\begin{array}{c}
\mathcal{A}^\Pi \\
\downarrow U \\
\mathcal{A}
\end{array}
\begin{array}{c}
\mathcal{B} \\
\downarrow \phi \\
\mathcal{A}^\Pi
\end{array}
\begin{array}{c}
\mathcal{A} \\
\downarrow U^\Pi \\
\mathcal{A}^\Pi
\end{array}
\begin{array}{c}
\mathcal{A}^\Pi \\
\downarrow \phi \\
\mathcal{A}
\end{array}
\begin{array}{c}
\mathcal{B} \\
\downarrow \phi
\end{array}
\]

commute, and \( \phi e = e^\Pi \phi \), where \( e \) and \( e^\Pi \) are the counits of adjunction for \( (F, U) \) and \( (F^\Pi, U^\Pi) \) respectively.

**Proof:** We define the category \( \mathcal{A}^\Pi \) as follows.

\[
|\mathcal{A}^\Pi| = \{(A, \phi): A \in |\mathcal{A}|, \phi: TA \to A \text{ with } \phi n_A = 1_A \text{ and } \phi(T\zeta) = \zeta u_A\}.
\]

An \( \mathcal{A}^\Pi \)-morphism \( (A, \phi) \to (A', \phi') \) is a commutative diagram.

\[
\begin{array}{ccc}
TA & \xrightarrow{T\phi} & TA' \\
\downarrow \phi & & \downarrow \phi' \\
A & \xrightarrow{f} & A'
\end{array}
\]

Composition is defined by: \((g)(f) = (gf)\). Then an easy check shows that \( \mathcal{A}^\Pi \) is a category.

Define \( U^\Pi: \mathcal{A}^\Pi \to \mathcal{A} \) as the "forgetful" functor, i.e.
Define $F^\Pi : \mathcal{A} \to \mathcal{A}^\Pi$ by:

$$F^\Pi (A) = (TA, uA) \quad \text{and} \quad F^\Pi f = (f).$$

Then it is easy to see that $F^\Pi$ and $U^\Pi$ are functors and also that $F^\Pi$ is left adjoint to $U^\Pi$ with $n^\Pi = n$ the unit of adjunction, and $\epsilon^\Pi (A, \phi) = (\phi)$ the action of the counit $\epsilon^\Pi$ of adjunction. Then it is clear that $F^\Pi$, $U^\Pi$, $n^\Pi$ and $\epsilon^\Pi$ induce the triple $\Pi$.

Now given a pair $F, U$ of adjoints inducing $\Pi$, define $\circlearrowright : \mathcal{B} \to \mathcal{A}^\Pi$ by:

$$\circlearrowright B = (UB, U_\circlearrowright B), \quad \text{for} \quad B \in \mathcal{B}, \quad \text{and}$$

$$\circlearrowright f = (Uf), \quad \text{where} \quad f \quad \text{is a morphism in} \quad \mathcal{B}.$$

Then it is easy to see that $\circlearrowright$ is unique and that it satisfies the required properties.

**Definition 1.12**

A category $\mathcal{C}$ is called algebraic (or tripleable) over a category $\mathcal{A}$ if $\mathcal{C}$ is equivalent to a category $\mathcal{A}^\Pi$ for some triple $\Pi$ in $\mathcal{A}$.

We shall call a category algebraic if it is algebraic over the category $\mathcal{S}$ of sets.

**Example 1.13**

The category $\mathcal{G}$ of groups is algebraic.

**Proof:** Let $f : \mathcal{S} \to \mathcal{G}$ be the free functor left adjoint to the
forgetful functor $U: \mathcal{C} \to \mathcal{S}$. Define $T: \mathcal{S} \to \mathcal{S}$ as follows:

If $X \subseteq \mathcal{G}$, $T_X = \{\text{all reduced words in the set } \langle X \rangle\}$, where

$\langle X \rangle = \{\langle x \rangle : x \in X\}$. More precisely, $T_X$ is given by:

$$T_X = \left\{ \langle x_1 \rangle^1 \langle x_2 \rangle^2 \cdots \langle x_n \rangle^n : x_i \in X; \ i=1,\ldots,n; \ \varepsilon_i = \pm 1; \right\} \text{ if } x_i = x_{i+1}, \text{ then } \varepsilon_i + \varepsilon_{i+1} \neq 0 \right\}.$$

Define $n: 1_\mathcal{S} \to T$ by:

$$n_X(x) = \langle x \rangle, \quad x \in X \subseteq \mathcal{G}.$$ Then $n$ is a natural transformation.

Also, define $u: T^2 \to T$ by:

$$u_X(\langle w \rangle) = \text{the reduced word defined by } \langle w \rangle, \quad \text{where } \langle w \rangle \text{ is an element in } T^2_X \text{ and } \langle w \rangle \text{ is an element in } T_X.$$

Then it is easy to show that $\Pi = (T, n, u)$ is a triple in $\mathcal{S}$. But $T \cong UF$ and it can be shown that $F$ and $U$ are a universal pair of adjoint functors for the triple $\Pi$. Thus, $\mathcal{C}$ is equivalent to the category $\mathcal{S}^\Pi$, and so it is an algebraic category.

**Note:** It is well known that any class of algebras defined by universal operations and relations is an algebraic category.

We gather together some useful, though trivial, facts about algebraic categories.

**Proposition 1.14** (See, for example (8)).

Let $\mathcal{C}$ be an algebraic category. Then, (i) there exists a faithful functor $U: \mathcal{C} \to \mathcal{S}$, so that we can consider an object
of \mathcal{C} as a set endowed with some structure;

(ii) there exists a functor \( F: \mathcal{S} \to \mathcal{C} \) with \( F \) left adjoint to \( U \);

(iii) products and pullbacks exist in \( \mathcal{C} \);

(iv) \( \mathcal{C} \) has a terminal object \( * \) whose underlying set is \( \{ * \} \).

The above facts enable us to make the following definition.

**Definition 1.15**

Let \( \mathcal{C} \) be an algebraic category. Let \( U: \mathcal{C} \to \mathcal{S} \) be the faithful functor as in 1.14. We say \( f: A \to B \) in \( \mathcal{C} \) is surjective if \( Uf \) is surjective in \( \mathcal{S} \).

A subset \( X \subseteq UB \) generates \( B \) is the map \( FX \to B \), corresponding to the inclusion map \( X \subseteq UB \) via the adjoint, is surjective.
CHAPTER II

SEMI-SIMPLICIAL METHODS IN BASE CATEGORIES

In this chapter, we apply semi-simplicial methods as developed in Tierney and Vögel (12) to base categories as defined by Rinehart (11). We obtain the same result as Rinehart but feel that our methods are direct, make the theory easier to understand and, as we shall see later, can be generalized.

In §1, we give details of the semi-simplicial methods used to define derived functors of functors taking values in an abelian category. In §2, we deal with base categories. Since Rinehart does not give all the details, we have dealt with the topic in detail. In §3, we consider the semi-simplicial theory over a base category and prove that it coincides with Rinehart's theory.
§1 Derived Functors

Let $\mathcal{C}$ be a category with finite limits and $\mathcal{A}$ an abelian category. Given any functor $F: \mathcal{C} \to \mathcal{A}$, we shall define the derived functors of $F$. But first, we mention some results which will ensure the validity of the definition.

Theorem 1.1 (Comparison theorem, (12))

Let $\mathcal{C}$ be a category and $\mathcal{E}$ a projective class of epimorphisms in $\mathcal{C}$. For $A, A^1 \in |\mathcal{C}|$, let $X \xrightarrow{\hat{\epsilon}} A$ be an $\mathcal{E}$-projective augmented semi-simplicial object of $A$ and $X^1 \xrightarrow{\epsilon} A^1$ be an $\mathcal{E}$-exact augmented semi-simplicial object of $A^1$. Then any $\mathcal{C}$-morphism $f: A \to A^1$ can be extended to a semi-simplicial morphism $\bar{f} = \{\bar{f}_n\}: X \to X^1$.

Also, given any two such extensions, say $\bar{f}, \bar{g}: X \to X^1$, then there exist $\mathcal{C}$-morphisms $\bar{h}^i_n: X^i_n \to X^i_{n+1}$, $0 \leq i \leq n$, such that

\[
\begin{align*}
\varepsilon^0_n h^0_n &= \bar{f}_n; & \varepsilon^{n+1}_n h^n_n &= \bar{g}_n; \\
\varepsilon^i_n h^j_n &= \begin{cases} 
h^j_{n-1} \varepsilon^i_n & \text{if } i < j; \\
h^j_{n-1} \varepsilon^{i-1}_n & \text{if } i > j+1;
\end{cases}
\end{align*}
\]

and $\varepsilon^{j+1}_n h^{j+1}_n = \varepsilon^{j+1}_n h^j_n$.

We obtain the following diagram
Proof: We shall show the existence of \( \{ \bar{f}_n \} \) by induction on \( n \).

For \( n = 0 \), since \( \varepsilon_0^0 \in \mathcal{E} \) and \( X_0 \in \mathcal{O} \), where \( \mathcal{E} \Rightarrow \mathcal{O} \), there exists \( \bar{f}_0 : X_0 \to X_0^1 \) such that \( \varepsilon_0^0 \bar{f}_0 = \varepsilon_0^0 \). Suppose that \( \bar{f}_0, \ldots, \bar{f}_n \) have been defined for \( n > 0 \) such that \( \bar{f}_k \varepsilon_k = \varepsilon_k \bar{f}_k \) for all \( i = 0, \ldots, k \) and \( 0 < k < n \). Consider the following diagram,

\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{\varepsilon_{n+1}} & X_n \\
\downarrow{\bar{f}} & & \downarrow{\varepsilon}
\end{array}
\]

where \( (K_{n+1}, k_{n+1}) \) is the simplicial kernel of \( \varepsilon_n^1 \).

Now
\[
\varepsilon_n^i (\bar{f}_n \varepsilon_{n+1}^j) = \bar{f}_{n-1} \varepsilon_n^i \varepsilon_{n+1}^j
\]
\[
= \bar{f}_{n-1} \varepsilon_n^{j-1} \varepsilon_n^i
\]
\[
= \varepsilon_n^{j-1} (\bar{f}_n \varepsilon_{n+1}^i), \text{ for } 0 \leq i < j \leq n+1.
\]
Thus, by the universal property of simplicial kernels, there exists a unique \( C \)-morphism \( l_{n+1} : X_{n+1} \longrightarrow K_{n+1} \) such that \( k_{n+1}^j l_{n+1} = \tilde{f}_j \partial_{n+1}^j \), for \( 0 \leq j \leq n+1 \). But \( e_{n+1} \in E \) and \( X_{n+1} \in \mathcal{F} \), and so there exists a \( \mathcal{C} \)-morphism \( \tilde{f}_{n+1} : X_{n+1} \longrightarrow X_{n+1}^1 \) such that \( e_{n+1} \tilde{f}_{n+1} = l_{n+1} \).

It is easy to see that \( e_{n+1} \tilde{f}_{n+1} = \tilde{f}_n o_{n+1} \), for \( 0 \leq j \leq n+1 \). Thus, by induction, we have the existence of a semi-simplicial morphism \( \tilde{f} : X \longrightarrow X^1 \) extending \( f \).

Suppose \( \tilde{g} \) is another such semi-simplicial morphism. Then we shall define the \( h_n \)'s by induction on \( n \).

For \( n = 0 \), consider the following diagram,

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\epsilon_0} & A \\
\downarrow{\Phi_0} & & \downarrow{\Psi} \\
K_0 & \xrightarrow{\tilde{f}_0} & X_0^1 \\
\end{array}
\]

where \((K_1, k_0^0, k_1^0)\) is the simplicial kernel of \( X_0 \) and \( \tilde{g}_0 \) is a \( \mathcal{F} \)-morphism. Since \( \epsilon_0 \tilde{g}_0 = \epsilon_0 \tilde{g}_0 \), there exists a unique \( \mathcal{C} \)-morphism \( \Phi_0 : X_0 \longrightarrow K_1 \) such that \( k_0^0 \Phi_0 = \tilde{f}_0 \) and \( k_1^0 \Phi_0 = \tilde{g}_0 \).

Then, since \( \epsilon, \in \mathcal{E} \) and \( X_0 \in \mathcal{F} \), there exists a \( \mathcal{C} \)-morphism \( h_0 : X_0 \longrightarrow X_0^1 \) such that \( \epsilon_0 h_0 = \Phi_0 \). This \( h_0 \) satisfies the required conditions.

Suppose \( h_0, \ldots, h_{n-1} \) have been defined for \( n \geq 2 \), and that they satisfy the required conditions. Then we define \( h_n \) as follows.

Consider the following diagram,
Firstly, we want to define $\varphi_n^0, \ldots, \varphi_n^n : X_n \rightarrow K_{n+1}$ such that

$$k_{n+1}^0 \varphi_n^0 = \bar{f}_n; \quad k_{n+1}^{n+1} \varphi_n^n = \bar{g}_n;$$

$$k_{n+1}^i \varphi_n^j = \begin{cases} h_{n-1}^{i-1} \alpha_n^i & i < j; \\ h_{n-1}^j \alpha_n^{i-1} & i > j+1; \\ \end{cases}$$

and

$$k_{n+1}^{j+1} \varphi_n^{j+1} = k_{n+1}^{j+1} \varphi_n^j.$$

To begin with, define $\mathcal{C}$-morphisms

$$\alpha_{n-1}^j : X_{n-1} \rightarrow X_{n-1}^1, \quad 0 \leq j \leq n, \text{ by }$$

$$\alpha_{n-1}^0 = \bar{f}_{n-1},$$

$$\alpha_{n-1}^j = \varepsilon_n^{j+1} h_{n-1}^{j+1} = \varepsilon_n^{j+1} h_{n-1}^j, \quad 0 \leq j \leq n-2$$

and

$$\alpha_{n-1}^n = \bar{g}_{n-1}.$$

Now put, for $0 \leq i \leq n, \quad 0 \leq j \leq n-1,$
\[ \psi_{j+1}^{i} = \begin{cases} 
\alpha_{j}^{n-1} \circ_{n}^{i}, & 0 \leq i < j+1; \\
\alpha_{j+1}^{n-1} \circ_{n}^{i}, & i \geq j+1 
\end{cases} \]

Then, for \(0 \leq L < i \leq n\),

\[ \varepsilon_{n-1}^{L} \psi_{j+1}^{i} = \begin{cases} 
\varepsilon_{n-1}^{L} \alpha_{j}^{n-1} \circ_{n}^{i}, & i < j+1, \\
\varepsilon_{n-1}^{L} \alpha_{j+1}^{n-1} \circ_{n}^{i}, & i \geq j+1. 
\end{cases} \]

Now, for \(0 \leq L < i < j+1\),

\[ \varepsilon_{n+1}^{L} \psi_{j+1}^{i} = \varepsilon_{n-1}^{L} \alpha_{j}^{n-1} \circ_{n}^{i} \]

\[ = \varepsilon_{n-1}^{L} \varepsilon_{n}^{j} \circ_{n}^{j} \circ_{n}^{i} \]

\[ = \varepsilon_{n-1}^{j} \varepsilon_{n-1}^{L} \circ_{n}^{j} \circ_{n}^{i} \]

\[ = \varepsilon_{n-1}^{j} \varepsilon_{n-1}^{j} \circ_{n-2}^{j} \circ_{n-1}^{i} \]

\[ = \varepsilon_{n-1}^{j} \varepsilon_{n-1}^{i-1} \circ_{n-1}^{i-1} \circ_{n}^{L} \]

\[ = \varepsilon_{n-1}^{j} \varepsilon_{n}^{i-1} \circ_{n-1}^{i-1} \circ_{n}^{L} \]

\[ = \varepsilon_{n-1}^{i-1} \circ_{n-1}^{i-1} \circ_{n}^{L} \]

\[ = \varepsilon_{n-1}^{i-1} \varepsilon_{n-1}^{i} \circ_{n}^{L} \]

\[ = \varepsilon_{n-1}^{i-1} \varepsilon_{n}^{j} \circ_{n-1}^{i} \circ_{n}^{L} \]

\[ = \varepsilon_{n-1}^{i-1} \varepsilon_{n}^{j} \circ_{n-1}^{i} \circ_{n}^{L} \]

If \(0 \leq L < i\) and \(i \geq j+1\), we consider two cases.

Case (i), \(L \leq j\).
Then,
\[
\varepsilon_{n-1}^L \psi_{j+1}^i = \varepsilon_{n-1}^L \alpha_{n-1}^{i} \alpha_n^i \\
= \varepsilon_{n-1}^L \alpha_{n-1}^{i+1} h_{n-1}^i \alpha_n^i, \text{ if } j \neq n-1 \\
= \varepsilon_{n-1}^j \varepsilon_n^L h_{n-1}^i \alpha_n^i \\
= \varepsilon_{n-1}^j h_{n-2}^i \alpha_{n-1}^L \alpha_n^i \\
= \varepsilon_{n-1}^j h_{n-2}^{i-1} \alpha_n^i \\
= \varepsilon_{n-1}^j \varepsilon_n^i h_{n-1}^{j-1} \alpha_n^L \\
= \varepsilon_{n-1}^j \varepsilon_n^i h_{n-1}^{j-1} \alpha_n^L \\
= \varepsilon_{n-1}^i \alpha_n^{n-1} \alpha_n^L \\
= \varepsilon_{n-1}^i \psi_n^L \\
= \varepsilon_{n-1}^L \psi_{j+1}^i.
\]

Also,
\[
\varepsilon_{n-1}^L \psi_n^i = \varepsilon_{n-1}^L \alpha_{n-1}^{i} \alpha_n^i \\
= \varepsilon_{n-1}^L g_{n-1}^{i} \alpha_n^i \\
= \varepsilon_{n-1}^L \varepsilon_n^i g_n^i \\
= \varepsilon_{n-1}^i \varepsilon_n^{L-i} g_n^i \\
= \varepsilon_{n-1}^i \varepsilon_n^{L-i} g_n^i \\
= \varepsilon_{n-1}^i \psi_n^L \\
= \varepsilon_{n-1}^L \psi_n^i.
\]
Case (ii), \( L > j \).

\[
\varepsilon_{n-1}^{L} \psi_{j+1}^{i} = \varepsilon_{n-1}^{L} \alpha_{j+1}^{i} \delta_{n}^{i} \\
= \varepsilon_{n-1}^{L} \varepsilon_{n}^{j+1} h_{n-1}^{j} \delta_{n}^{i} \\
= \varepsilon_{n-1}^{j+1} \varepsilon_{n}^{L+1} h_{n-1}^{j} \delta_{n}^{i} \\
= \varepsilon_{n-1}^{j+1} h_{n-2}^{j} h_{n-1}^{L} \delta_{n}^{i} \\
= \varepsilon_{n-1}^{j+1} h_{n-2}^{j} \delta_{n-1}^{i-1} \delta_{n}^{L} \\
= \varepsilon_{n-1}^{j+1} \alpha_{n-1}^{i-1} \delta_{n}^{L} \\
= \varepsilon_{n-1}^{i-1} \delta_{n}^{L} \\
= \varepsilon_{n-1}^{i-1} \psi_{j+1}^{L}.
\]

Thus, \( \varepsilon_{n-1}^{L} \psi_{j+1}^{i} = \varepsilon_{n-1}^{i-1} \psi_{j+1}^{L} \) for \( 0 \leq L < i \), and so by the universal property of simplicial kernels, there exists a \( \mathcal{E} \)-morphism.

\[
\psi_{j+1}^{n} : X_{n} \longrightarrow X_{1}^{n} \quad \text{such that} \quad k_{n}^{i} \psi_{j+1}^{n} = \psi_{j+1}^{L}, \quad \text{for all} \quad i = 0, \ldots, n \quad \text{and} \quad 0 \leq j \leq n-1.
\]

Since \( \varepsilon \in \mathcal{E} \) and \( X_{n} \in \mathcal{G} \), there exists \( \alpha_{j+1}^{n} : X_{n} \longrightarrow X_{1}^{n} \), \( 0 \leq j \leq n-1 \), such that

\[
\varepsilon_{n}^{i} \alpha_{j+1}^{n} = \psi_{j+1}^{n}.
\]

Now, put \( \alpha_{0}^{n} = \bar{f}_{n} \) and \( \alpha_{n+1}^{n} = \bar{g}_{n} \) and define \( \phi_{j}^{i} : X_{n} \longrightarrow X_{1}^{n} \).
Then, as above, we can verify that $E_n^L \phi_j^i = E_n^{i-1} \phi_j^L$ for $L > i$, and hence there exists $q_j^i: X_n \to K_n^{i+1}$, $0 \leq j \leq n$, such that $k_{n+1}^i q_n^j = \phi_j^i$, $0 \leq i \leq n+1$. Now it is easy to verify that $q_{n+1}^0, \ldots, q_n^n$ satisfy the required conditions. Since $E_n^{i+1} \in \mathcal{E}$ and $X_n \in \mathcal{O}$, we obtain $h_n^j: X_n \to X_{n+1}^1$, $0 \leq j \leq n$, such that $E_n^{i+1} h_n^i = q_n^j$.

Thus, the $h_0^n, \ldots, h_n^n$ also satisfy the required conditions. By induction, now all the $h_n$'s are defined. This completes the proof.

**Definition 1.2**

Let $\mathcal{D}$ be an abelian category and $X$ a (semi-) simplicial object in $\mathcal{D}$. We define the **derived complex** $NX$, of $X$, as follows.

$$(NX)_n = X_n^i, \quad \text{for } n \geq 0 \quad \text{and the boundary } \partial_m: (NX)_m \to (NX)_{m-1}$$

is defined by

$$\partial_m = \sum_{i=0}^{m} (-1)^i d_m^i,$$ for $m \geq 1$, where $d$ is the face map of $X$. 

$$0 \leq j \leq n, \quad 0 \leq i \leq n+1, \quad \text{by}$$ 

$$\phi_j^i = \begin{cases} 
\begin{align*}
\begin{align*}
h_{n-1}^{j-1} & a_n^i, & 0 \leq i < j; \\
a_n^i & i = j;
\end{align*}
\end{align*}
\end{cases}$$

$$\begin{align*}
\begin{align*}
a_{n+1}^i & i = j+1; \\
h_n^{j-1} & a_n^{i-1}, \quad n+1 \geq i > j+1
\end{align*}
\end{align*}$$
Lemma 1.3

The derived complex of a (semi-) simplicial object in an abelian category is a chain complex.

Proof: Using the same notation as in 1.2, we have to show that
\[ \partial_n \partial_{n+1} = 0 \text{ for all } n \geq 1. \]

Now, by definition
\[ \partial_n \partial_{n+1} = \left( \sum_{i=0}^{n} (-1)^i d_i^n \right) \left( \sum_{j=0}^{n+1} (-1)^j d_{n+1}^j \right) \]
\[ = \sum_{i,j} (-1)^{i+j} d_i^n d_{n+1}^j \]
\[ = \sum_{i,j} (-1)^{i+j} d_i^{i-1} d_{n+1}^i + \sum_{i,j} (-1)^{i+j} d_i^j d_{n+1}^i. \]

Thus, for any \( i,j \), the sign of \( d_i^n d_{n+1}^j \) will be \((-1)^{i+j+1}\) from the first summation and \((-1)^{i+j+1}\) from the second. Therefore
\[ \partial_n \partial_{n+1} = 0, \] as required.

Definition 1.4

Let \( \mathcal{C} \) be a category and \( \mathcal{E} \) be a class of epimorphisms in \( \mathcal{C} \) such that semi-simplicial \( \mathcal{E} \)-resolutions exist for every \( A \in |\mathcal{C}| \). Let \( \mathcal{A} \) be an abelian category and \( F : \mathcal{C} \rightarrow \mathcal{A} \) a functor. Construct a semi-simplicial \( \mathcal{E} \)-resolution \( X \xrightarrow{d} A \) of \( A \) and apply \( F \) to get a semi-simplicial object \( F(X) \) in \( \mathcal{A} \). Form the derived complex \( \mathcal{N} F(X) \). We define the \( n \)th derived functor of \( F \) relative to \( \mathcal{E} \), \( L_n F \), by
Theorem 1.5

With the same notation as in 1.4,

(i) we can define \( L_n^E F \) on morphisms in \( \mathcal{C} \) such that \( L_n^E F : \mathcal{C} \to \mathcal{A} \) is a functor for all \( n \geq 0 \);

(ii) \( L_n^E F(A) \) is independent of the semi-simplicial resolution (up to natural isomorphism);

(iii) \( L_n^E F(P) = 0 \), for all \( P \in \mathcal{P} \) and \( n \geq 1 \), where \( E \Rightarrow \mathcal{P} \).

Proof: (ii) If we have two augmented semi-simplicial \( \mathcal{C} \) resolutions of \( A \), then consider a semi-simplicial morphism inducing the identity morphism on \( A \) to get the independence of \( L_n^E F(A) \) of any particular resolution. This proves that \( L_n^E F(A) \) is well defined for all \( A \in \{ \mathcal{C} \}, \ n \geq 0 \).

(i) Let \( f : A \rightarrow A^1 \) be a morphism in \( \mathcal{C} \). Let \( X \xrightarrow{\alpha} A \) and \( X^1 \xrightarrow{\epsilon} A^1 \) be augmented semi-simplicial \( \mathcal{C} \) resolutions of \( A \) and \( A^1 \) respectively. By 1.1, there exists a semi-simplicial \( \overline{F} = \{ \overline{f}_n \} : X \rightarrow X^1 \) extending \( f \). Applying \( F \) and forming the derived complexes, we get the following diagram,
\[ \bar{\alpha}_{n+1} = \sum_{i=0}^{n+1} (-1)^i F(\bar{\alpha}_{n+1}) \quad \text{and} \quad \bar{\varepsilon}_{n+1} = \sum_{i=0}^{n+1} (-1)^i F(\varepsilon_{n+1}), \]

\( n \geq 0. \)

Since all the squares above are commutative, \( \bar{f}_n \) induces a \( \mathcal{C} \)-morphism \( \alpha_n : \mathcal{E}_n F(A) \to \mathcal{E}_n F(A^1) \), for all \( n \geq 0. \) Define \( \mathcal{E}_n F(f) = \alpha_n, \quad n \geq 0. \) To show that \( \mathcal{E}_n F(f) \) is independent of the choice of \( \bar{f}_n \), let \( \bar{g}_n \) be another semi-simplicial morphism inducing \( f. \) Then, as above, \( \bar{g}_n \) induces a \( \mathcal{C} \)-morphism \( \beta_n : \mathcal{E}_n F(A) \to \mathcal{E}_n F(A^1), \) for all \( n \geq 0. \) We have to show that \( \alpha_n = \beta_n, \quad n \geq 0. \)

By 1.1 there exist \( \mathcal{C} \)-morphisms \( h_n^i : X_n \to X_{n+1} \) for \( 0 \leq i \leq n, \) with certain properties. Define \( \tau_n : X_n \to X_{n+1}, \quad n \geq 0, \) by

\[ \tau_n = \sum_{i=0}^{n} (-1)^i F(h_n^i). \]

Then, using the various relations involving \( h_n^i, \) as in 1.1, an easy computation shows that

\[ \bar{\varepsilon}_{n+1} \tau_n + \tau_{n-1} \bar{\alpha}_n = F(\bar{f}_n) - F(\bar{g}_n) \quad \text{for all} \quad n \geq 1, \, \text{i.e.} \]

\( F(\bar{f}_n) \) and \( F(\bar{g}_n) \) are homotopic in the usual sense via the homotopy \( \{ \tau_n \}, \quad n \geq 0. \) Hence the induced morphisms are equal and so \( \alpha_n = \beta_n, \quad n \geq 0. \) Now it is easy to see that \( \mathcal{E}_n F \) is a functor for all \( n \geq 0. \)

(iii) If \( P \in \mathcal{O}, \) then trivially

\[ \ldots P \xrightarrow{1_P} P \quad \ldots \quad P \xrightarrow{1_P} P \quad \xrightarrow{1_P} P \]
is an augmented semi-simplicial $\mathcal{E}$-resolution of $P$. Applying $F$ and forming the derived complex, we get

$$\cdots F(P) \to F(P) \to F(P) \to F(P) \to 0$$

Thus, $L_n^\mathcal{E} F(P) = \begin{cases} F(P), & \text{if } n = 0 \\ 0, & \text{if } n > 1. \end{cases}$

**Lemma 1.6 (Snake Lemma)**

Let $\mathcal{A}$ be an abelian category and suppose we have the following commutative diagram with exact rows in $\mathcal{A}$.

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & 0 \\
\downarrow{i} & & \downarrow{j} & & \downarrow{k} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C'
\end{array}
$$

Then there exists an $\mathcal{A}$-morphism $\delta : \ker h \longrightarrow \text{Coker } f$ such that the following sequence is exact,

$$
\ker f \xrightarrow{\alpha_1} \ker g \xrightarrow{\beta_1} \ker h \xrightarrow{\delta} \text{Coker } f \xrightarrow{\alpha_2^*} \text{Coker } g \xrightarrow{\beta_2^*} \text{Coker } h
$$

where $\alpha_2^*$ and $\beta_2^*$ are the induced morphisms.

Moreover, if $\alpha_1$ is a monomorphism then so is $\alpha_1$, and if $\beta_2$ is an epimorphism then so is $\beta_2^*$.

**Proof:** This is standard, see MacLane (8).

**Proposition 1.7**

Let $\mathcal{C}$ be a category and $\mathcal{E}$ a class of epimorphisms in $\mathcal{C}$ such that semi-simplicial $\mathcal{E}$-resolutions exist for every object in $\mathcal{C}$. Let $\mathcal{A}$ be an abelian category.

Let
$F_i : \mathcal{C} \xrightarrow{\cdot} \mathcal{C}, \; i = 1,2,3;$ be three functors such that for any object $P \in \mathcal{C}$, where $\mathcal{C} \xrightarrow{\cdot} \mathcal{C}$,

$$0 \rightarrow F_1(P) \rightarrow F_2(P) \rightarrow F_3(P) \rightarrow 0$$

is exact.

Then for any object $A \in \mathcal{C}$, there is a long exact sequence

$$\ldots \rightarrow L_n F_1(A) \rightarrow L_n F_2(A) \rightarrow L_n F_3(A) \rightarrow L_{n-1} F_1(A) \rightarrow \ldots$$

$$\rightarrow L_0 F_3(A) \rightarrow 0.$$

**Proof:** Let $X \rightarrow A$ be an augmented semi-simplicial $C$-resolution of $A$. Apply $F_i$, $i = 1,2,3$; and form the derived complexes.

Since $(NF_i(X))_n = F_i(X)_n$, $n > 0$, and $X_n \in \mathcal{C}$ by definition, we have the following commutative diagram with exact rows,

$$\xymatrix{ 0 \rightarrow F_1(X_n) \ar[r] & F_2(X_n) \ar[r] & F_3(X_n) \ar[r] & 0 \ar[d]^{F_1 \partial_n} \ar[d]^{F_2 \partial_n} \ar[d]^{F_3 \partial_n} \\
0 \rightarrow F_1(X_{n-1}) \ar[r] & F_2(X_{n-1}) \ar[r] & F_3(X_{n-1}) \ar[r] & 0 }$$

where $\partial_n = \sum_{i=0}^{n} (-1)^{i} d_i^2$, $n \geq 1$.

By the snake lemma, we have exact sequences

$$0 \rightarrow \text{Ker} F_1 \partial_n \rightarrow \text{Ker} F_2 \partial_n \rightarrow \text{Ker} F_3 \partial_n$$

and

$$\text{Coker} F_1 \partial_n \rightarrow \text{Coker} F_2 \partial_n \rightarrow \text{Coker} F_3 \partial_n \rightarrow 0$$

for all $n \geq 1$. 
Now, $F_{i+1}$ induces a morphism
\[ \text{Coker } F_i \rightarrow \text{Ker } F_{i+1} \quad i=1,2,3, \quad n \geq 3. \]

Thus, we have the following commutative diagram with exact rows.

\[
\begin{array}{cccccc}
\text{Coker } F_i & \rightarrow & \text{Coker } F_{i+1} & \rightarrow & \text{Coker } F_{i+2} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
O & \rightarrow & \text{Ker } F_{i+1} & \rightarrow & \text{Ker } F_{i+2} & \rightarrow & \text{Ker } F_{i+3} \\
\end{array}
\]

By the definition of $L_n F_i$ and using the snake lemma, we obtain the required long exact sequence.
§ 2 Base Categories

Definition 2.1

Let \( \mathcal{C} \) be a category and \( A \xrightarrow{\alpha} B \) and \( C \xrightarrow{\beta} B \) be morphisms in \( \mathcal{C} \). Then the \textit{fibre product} (or pullback) of \( \alpha \) and \( \beta \) is the limit of the diagram \( A \xrightarrow{\alpha} B \xleftarrow{\beta} C \). Denote the fibre product by \( (A \times_C B, \pi_1, \pi_2) \), if it exists, where \( \pi_1 \) and \( \pi_2 \) are the canonical morphisms \( A \times C \xrightarrow{\pi_1} A \) and \( A \times C \xrightarrow{\pi_2} C \) respectively.

Definition 2.2

Let \( \mathcal{C} \) be a category and \( \mathcal{E} \) a projective class of epi-morphisms in \( \mathcal{C} \). The pair \( (\mathcal{C}, \mathcal{E}) \) is called a \textit{base category} if the following conditions are satisfied.

(B1) If \( A \xrightarrow{\alpha} B \in \mathcal{E} \) and \( C \xrightarrow{\beta} B \) is any morphism in \( \mathcal{C} \), then \( \mathcal{C} \) contains the fibre product \( A \times_C B \).

(B2) \( \mathcal{E} \) is closed under composition.

(B3) If the composite \( A \xrightarrow{\alpha} B \xrightarrow{\beta} C \in \mathcal{E} \), then so is \( B \xrightarrow{\beta} C \).

(B4) If \( A^1 \xrightarrow{\alpha} A \), \( A \xrightarrow{\beta} B \) and \( C \xrightarrow{\gamma} B \) are morphisms in \( \mathcal{C} \) with \( A^1 \xrightarrow{\alpha} A \xrightarrow{\beta} B \in \mathcal{E} \), then \( A^1 \times_C B \xrightarrow{\alpha \times \gamma} A \times C \in \mathcal{E} \) iff \( A^1 \xrightarrow{\alpha} A \in \mathcal{E} \).
Example 2.3

Let $\mathcal{C}$ be an algebraic category and $\mathcal{E}$ the class of all surjections in $\mathcal{C}$. Every surjection in $\mathcal{C}$ is an epimorphism, since if $\alpha : A \rightarrow B$ is a surjection then $U\alpha : UA \rightarrow UB$ is a surjection in $\mathcal{S}$, using I.1.5. Thus, if $A \rightarrow B \xrightarrow{\beta_1} C \xrightarrow{\beta_2}$ are morphisms in $\mathcal{C}$ with $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, then $(U\beta_1)(U\alpha) = (U\beta_2)(U\alpha)$. But $U\alpha$ is an epimorphism in $\mathcal{S}$ and so $U\beta_1 = U\beta_2$. But $U$ is faithful and so $\beta_1 = \beta_2$. Thus, $\mathcal{E}$ is a class of epimorphisms in $\mathcal{C}$. Now, it is easy to see that since (B1), (B2), (B3), and (B4) are satisfied by the category of sets and surjections in sets, they are also satisfied in $(\mathcal{C}, \mathcal{E})$.

In particular, $(\mathcal{G}, \mathcal{E})$ is a base category, where $\mathcal{G}$ is the category of groups and $\mathcal{E}$ is the class of all surjections in $\mathcal{G}$.

**Definition 2.4**

Let $(\mathcal{C}, \mathcal{E})$ be a base category. Define a category $\mathcal{C}_1$ as follows. $|\mathcal{C}_1| = \mathcal{E}$, and given $A \xrightarrow{\alpha} B$ and $C \xrightarrow{\beta} D$ any two objects in $\mathcal{C}_1$, then a $\mathcal{C}_1$-morphism $h : \alpha \rightarrow \beta$ consists of a pair of $\mathcal{C}$-morphisms $h_0, h_1$ such that the diagram is commutative. Then, with the obvious composition of morphisms, it is easy to see that $\mathcal{C}_1$ is a category.

Define a class of morphisms $\mathcal{E}_1$ in $\mathcal{C}_1$ by:

$$\mathcal{E}_1 = \left\{ h = (h_0, h_1) \in \mathcal{C}_1(\alpha, \beta) : (\alpha, h_0) : A \rightarrow B \times C \in \mathcal{E} \text{ and } h_1 \in \mathcal{E}, \text{ where } \alpha, \beta \in |\mathcal{C}_1| \right\}.$$
Thus, an element \((h_0, h_1)\) of \(E_1\) yields the following commutative diagram, where \(\rightarrow\) denotes an element of \(E\).

![Diagram]

**Proposition 2.5**

Let \((E, E)\) be a base category and \((E_1, E_1)\) be defined as above. Then \((E_1, E_1)\) is a base category.

We shall prove some basic lemmas before proving 2.5.

**Lemma 2.6**

In any category \(C\), the following isomorphisms hold, where isomorphism of two objects is taken to mean that if either object exists then so does the other, and they are isomorphic.

(i) If \(A \rightarrow B\) and \(C \rightarrow B\) are morphisms in \(C\), then \(A \times C = C \times A\).

(ii) If \(A \rightarrow B\), \(C \rightarrow B\) and \(D \rightarrow A\) are morphisms in \(C\) and if \(A \times C\) exists, then \(D \times (A \times C) = D \times C\).

(iii) If \(A \rightarrow B\) and \(1_B: B \rightarrow B\) are morphisms in \(C\), then \(A \times B = A\).

(iv) Given the following diagram in \(C\),
and if $A \times C$ and $C \times E$ exist, then

$$(A \times C) \times E \cong A \times (C \times E).$$

(v) Given the following diagram in $\mathcal{C}$,

where $\theta_1$ and $\theta_2$ are the fibre product morphisms,

we have $(A_1 \times C_1) \times T \cong A_1 \times T$.

Proof: (i) This follows easily from the observation that $A \times C$ and $C \times A$ are both limits of the same diagram.

(ii) By the universal property of fibre products, we have the following diagram,

where $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ and $\pi_1$, $\pi_2$ are the obvious fibre product morphisms, $\alpha$, $\beta$, $\gamma$ are unique with properties $\alpha_1 \alpha = \pi_1$, $\alpha_2 \alpha = \beta_2 \pi_2$; $\pi_1 \beta = \alpha_1$, $\pi_2 \beta = \gamma$; and $\beta_1 \gamma = \delta \alpha_1$, $\beta_2 \gamma = \alpha_2$ respectively.
In particular, $\pi_1 \beta a = \alpha_1 a = \pi_1$ and $\pi_2 \beta a = \gamma a = \pi_2$ by the uniqueness of $\alpha$ and $\gamma$.

Thus, $\beta a = l_D \times (A \times C)$ by the uniqueness of $\beta a$.

Also, $\alpha_1 \alpha \beta = \pi_1 \beta = \alpha_1$ and $\alpha_2 \alpha \beta = \beta_2 \pi_2 \beta = \beta_2 \gamma = \alpha_2$ and so

$\alpha \beta = l_D \times C$ by the uniqueness $\alpha \beta$.

Thus, $\alpha : D \times (A \times C) \rightarrow D \times C$, and $\alpha$ is the unique isomorphism.

(iii) Let $\phi : A \rightarrow B$ be a morphism in $\mathcal{C}$. Then $l_B \phi = \phi l_A$ and so $(A, l_A, \phi)$ is the fibre product of $\phi$ and $l_B$ if $(A, l_A, \phi)$ satisfies the universal property, i.e. if $(E, \chi_1, \chi_2)$ satisfies $\phi \chi_1 = l_B \chi_2$ then there exists a unique $\mathcal{C}$-morphism $\theta : E \rightarrow A$ with $l_A \theta = \chi_1$ and $\phi \theta = \chi_2$.

But putting $\theta = \chi_1$ the universal property is satisfied. Thus,

$A \times B \not\rightarrow \ A$.

(iv) This follows from the observation that $(A \times C) \times E$ and $A \times (C \times E)$ are both limits of the diagram in the statement of the lemma.
(v) \( A_1^1 \times T \sim (A_1^1 \times B_1) \times (A_1 \times C_1) \times T \), using (i) and (ii),
\( \sim (A_1^1 \times (A_1 \times C_1)) \times T \), using (iv),
\( \sim (A_1^1 \times C_1) \times T \), using (ii).

Note: In fact (iii) & (iv) imply (ii).

Lemma 2.7

Let \( (\mathcal{C}, E) \) be a base category and \( (\mathcal{C}_1, E_1) \) be defined as in 2.4. Then,

(i) \( E \) is a closed class;

(ii) isomorphisms in \( \mathcal{C} \) are in \( E \);

(iii) if \( B \xrightarrow{h_1} D \) and \( C \xrightarrow{\beta} D \) are morphisms in \( \mathcal{C} \) and \( (B \times C, \pi_1, \pi_2) \) is their fibre product (if it exists), then \( h_1 \in E \) implies \( \pi_2 \in E \), and \( \beta \in E \) implies \( \pi_1 \in E \);

(iv) if \( \alpha \) and \( \beta \) are objects in \( \mathcal{C}_1 \) and \( h = (h_0, h_1) : \alpha \rightarrow \beta \) is a \( \mathcal{C}_1 \)-morphism in \( E_1 \), then \( h_0 \in E \). In particular, \( E_1 \) is a class of epimorphisms in \( \mathcal{C}_1 \).

Proof: (i) As in 1.1.5, let \( E \Rightarrow P \) and \( P \Rightarrow \overline{E} \). Then \( E \subseteq \overline{E} \) and we have to show \( E = \overline{E} \). Let \( \theta : A \rightarrow B \in \overline{E} \), i.e. every object in \( P \) is projective relative to \( \theta \). Now, \( E \) is a projective class and so for \( B \in |C| \), there exists some \( P \in \mathcal{P} \) with
Then, as \( \theta \in \mathcal{E} \), there exists \( \phi : P \to A \) such that \( \theta \phi = (P \to B) \). In particular, \( \theta \phi \in \mathcal{E} \).

But \((\mathcal{C}, \mathcal{E})\) is a base category and so by (B3) in 2.2, \( \theta \in \mathcal{E} \). Thus, \( \mathcal{E} = \overline{\mathcal{E}} \) and so \( \mathcal{E} \) is a closed class.

(ii) Let \( \mathcal{E} \to \mathcal{P} \) and \( \mathcal{P} \to \overline{\mathcal{E}} \). Then it is easy to see that isomorphisms are in \( \overline{\mathcal{E}} \). Thus, by (i), isomorphisms are in \( \mathcal{E} \).

(iii) We are given the following commutative diagram:

\[
\begin{array}{ccc}
B \times C & \xrightarrow{\pi_2} & C \\
\downarrow \pi_1 & & \downarrow \beta \\
B & \xrightarrow{h_1} & D
\end{array}
\]

Suppose \( h_1 \in \mathcal{E} \), then to show that \( \pi_2 \in \mathcal{E} \) consider the following diagram:

\[
\begin{array}{ccc}
B \times C & \xrightarrow{\pi_2} & C \\
\downarrow \pi_1 & & \downarrow \beta \\
B & \xrightarrow{h_1} & D
\end{array}
\]

where we have identified \( D \times C \) with \( C \) by using 2.6 (iii), condition (B3) for \((\mathcal{C}, \mathcal{E})\) and part (ii) above. Now, using condition (B4) for \((\mathcal{C}, \mathcal{E})\) we deduce that \( \pi_2 \in \mathcal{E} \).

A similar argument shows that if \( \beta \in \mathcal{E} \), then \( \pi_1 \in \mathcal{E} \).

(iv) Let \( \alpha : A \to B \) and \( \beta : C \to D \) be objects of \( \mathcal{C}_1 \). Then we obtain the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{h_0} & C \\
\downarrow \alpha & & \downarrow \beta \\
B \times C & \xrightarrow{\pi_2} & C \\
\downarrow \pi_1 & & \downarrow \beta \\
B & \xrightarrow{h_1} & D
\end{array}
\]

Then by part (iii) above, \( \pi_1 \) and \( \pi_2 \) are in \( \mathcal{E} \).

Since the composite of two elements in \( \mathcal{C} \) is in \( \mathcal{E} \), we deduce that \( h_0 \in \mathcal{E} \). Now it is easy to see that \( \mathcal{C}_1 \) is a class of epimorphisms in \( \mathcal{C} \).
Lemma 2.8

Let \((\mathcal{C}, \mathcal{E})\) be a base category and \((\mathcal{C}_1, \mathcal{E}_1)\) be defined as in 2.4. If \(A \rightarrow B \in \mathcal{E}\) and \(C \rightarrow B\) is any morphism in \(\mathcal{E}_1\), then the natural morphism \(A_0 \times C_0 \rightarrow A_1 \times C_1 \in \mathcal{E}\), where \(A = A_0 \rightarrow A_1\), \(B = B_0 \rightarrow B_1\), and \(C = C_0 \rightarrow C_1\) with \(A_0, B_0, C_0, A_1, B_1, C_1 \in |\mathcal{C}|\).

Proof: We are given the following diagram,

\[
\begin{array}{ccc}
A_0 \times C_0 & \xrightarrow{\pi_1} & A_1 \times C_1 \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
A_0 & \rightarrow & A_1 \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
A & \rightarrow & B \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
B_0 & \rightarrow & B_1 \\
\end{array}
\]

where \(\pi_1, \pi_2, \pi_1', \pi_2'\) are the fibre product morphisms and \(\alpha\) the unique morphism factoring through the fibre product.

As \(C_0 \rightarrow C_1 \in \mathcal{E}\), by 2.7 (iii) we deduce that

\[\beta_1: (A_1 \times C_1) \times C_0 \rightarrow A_1 \times C_1 \in \mathcal{E}\cdot\]

Now consider the following diagram,

\[
\begin{array}{ccc}
A_0 \times C_0 & \xrightarrow{\beta} & (A_1 \times B_0) \times C_0 \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
A_0 \times C_0 & \rightarrow & C_0 \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
A_0 & \rightarrow & B_0 \\
\end{array}
\]

where \(A_0 \rightarrow A_1 \times B_0 \in \mathcal{E}\), \(A_0 \rightarrow B_0 \in \mathcal{E}\), and

\[\beta: A_0 \times C_0 \rightarrow (A_1 \times B_0) \times C_0 \] is the unique morphism. Applying
condition (B4) we deduce that $\beta \in \mathcal{E}$. But

$$(A_1 \times B_0) \times C_0 \cong (A_1 \times C_1) \times C_0 \text{, using } 2.6 \,(i) \text{ and } (ii), \text{ and an easy check shows that } \alpha \text{ and } \beta \text{ are coherent under this isomorphism. Thus } \alpha \in \mathcal{E}. \text{ But } A_0 \times C_0 \longrightarrow A_1 \times C_1 \text{ is the composite of } \alpha \text{ and } p_1 \text{ which are both in } \mathcal{E}. \text{ Thus, by condition (B2), }$$

$$A_0 \times C_0 \longrightarrow A_1 \times C_1 \in \mathcal{E}.$$  

Note: This Lemma not only proves that if $A \longrightarrow B \in \mathcal{E}_1$ and $C \longrightarrow B$ is any morphism in $\mathcal{C}_1$, then $A \times C \in \mathcal{C}_1$, i.e. that $(\mathcal{C}_1, \mathcal{E}_1)$ satisfies condition (B1) of a base category, but also shows what $A \times C$ is, namely $A_0 \times C_0 \longrightarrow A_1 \times C_1$.  

Now we are in a position to prove proposition 2.5. We have to show that $\mathcal{E}_1$ is a projective class and that $(\mathcal{C}_1, \mathcal{E}_1)$ satisfies the conditions (B1) - (B4).

(a) $\mathcal{E}_1$ is a projective class.

Proof: By 2.7 (iv) we know that $\mathcal{E}_1$ is a class of epimorphisms in $\mathcal{C}_1$. So let $A = (A_0 \longrightarrow A_1)$ be any object in $\mathcal{C}_1$. Since $\mathcal{E}$ is a projective class, there exists $P_1 \in \mathcal{P}$, where $\mathcal{C} \Rightarrow \mathcal{P}$, with $P_1 \longrightarrow A_1 \in \mathcal{E}$. Using condition (B1) for $(\mathcal{C}, \mathcal{E})$, form

$$P_1 \times A_0 \in \mathcal{C}_1.$$
Again, we can find $P_0 \in \mathcal{P}$ with $P_0 \to P_1 \times A_0 \in \mathcal{E}$. Then as $\pi_1 \in \mathcal{E}$ by 2.7 (iii), we can deduce that $P_0 \to P_1 \in \mathcal{E}$, using condition (B2). Thus, $P = (P_0 \to P_1) \in \mathcal{P}_1$. It is easy to see that $P \to A \in \mathcal{E}_i$ and so it remains to prove that $(P_0 \to P_1) \in \mathcal{P}_1$, where $\mathcal{E}_i \to \mathcal{P}_1$.

To prove this let $B = (B_0 \to B_1)$ be any object in and let $\alpha = (\alpha_0, \alpha_1): B \to A$ be a morphism in $\mathcal{C}_i$.

\[ \text{For } B \text{ read } P \text{ and for } P \text{ read } B. \]

From the diagram above, since $P_0, P_1 \in \mathcal{P}$ and using the universal property of fibre products, we obtain the existence of three $\mathcal{C}$-morphisms $\beta_1, \gamma$ and $\beta_0$, as shown. Then, an easy check shows that $\beta = (\beta_0, \beta_1): B \to P$ is a $\mathcal{C}_i$-morphism and so $(P_0 \to P_1) \in \mathcal{P}_1$.

(b) $(\mathcal{C}_i, \mathcal{E}_i)$ satisfies condition (B1).
Proof: Follows immediately from 2.8.

(c) \((c, c')\) satisfies condition \((B2)\).

Proof: We have to show that \(c\) is closed under composition. Let \(A \to B\) and \(B \to C\) be elements of \(\mathcal{E}\), where \(A = (A_0 \to A_1)\), \(B = (B_0 \to B_1)\), \(C = (C_0 \to C_1)\) and \(A_0, B_0, C_0, A_1, B_1, C_1 \in \mathcal{E}\).

Since \(A_1 \to B_1\) and \(B_1 \to C_1\) are in \(\mathcal{E}\), by condition \((B2)\) for \((\mathcal{C}, \mathcal{E})\), we get that \(A_1 \to C_1 \in \mathcal{E}\). Thus it remains to show that \(A_0 \to A_1 \times C_0 \in \mathcal{E}\). By the universal property of fibre products, there exists a unique morphism

\[ \delta: A_0 \times B_0 \to A_1 \times C_0 \]

such that the following diagram is commutative,

\[
\begin{array}{ccc}
A_0 & \xrightarrow{\delta} & B_0 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A_1 \times C_0 & \xrightarrow{\kappa} & B_1 \times C_1 \\
\downarrow{\delta} & & \downarrow{\beta} \\
A_1 & \xrightarrow{\kappa} & B_1 \\
\end{array}
\]

In particular, \(\alpha = \delta \gamma\). By hypothesis, \(\gamma \in \mathcal{E}\), and so it is enough to show that \(\delta \in \mathcal{E}\). Consider the following diagram,

\[
\begin{array}{ccc}
B_0 \times A_1 & \xrightarrow{\Theta} & (B_1 \times C_0) \times A_1 \\
\downarrow{\beta} & & \downarrow{\beta} \\
B_0 & \xrightarrow{\beta} & B_1 \times C_0 \\
\downarrow{\beta} & & \downarrow{\beta} \\
B_1 & \xrightarrow{\alpha} & B_1 \\
\end{array}
\]

where \(\delta: B_0 \to B_1 \in \mathcal{E}\) and \(\beta \in \mathcal{C}\) by hypothesis. By applying condition \((B4)\) for \((\mathcal{C}, \mathcal{E})\), we deduce that
\[ \Theta : B_0 \times A_1 \to (B_1 \times C_0) \times A_1 \in \mathcal{E} \]. But
\[ (B_1 \times C_0) \times A_1 \cong A_1 \times C_0 \] by using 2.6 (i) and (ii),
and \( \delta \) and \( \theta \) are coherent under this isomorphism. Thus \( \delta \in \mathcal{E} \)
and so \( \alpha \in \mathcal{E} \). This shows that \((\mathcal{C}, \mathcal{E})\) satisfies condition (B2).

(d) \((\mathcal{C}, \mathcal{E})\) satisfies condition (B3).

Proof: We have to show that if \( \alpha, \beta \in \mathcal{E} \), then \( \alpha, \beta \in \mathcal{E} \). Using the same notation as in (c) above,
\[ A_1 \to B_1 \to C_1 \in \mathcal{E} \] implies \( B_1 \to C_1 \in \mathcal{E} \) by condition (B3)
for \((\mathcal{C}, \mathcal{E})\). Thus, it remains to show that, assuming \( \alpha \in \mathcal{E} \), \( \beta \)
is also in \( \mathcal{E} \).

Now, as above, \( \alpha = \delta \gamma \) and so \( \delta \in \mathcal{E} \) by using condition
(B3) for \((\mathcal{C}, \mathcal{E})\). Thus, \( \delta \in \mathcal{E} \) and so by using condition (B4) for
\((\mathcal{C}, \mathcal{E})\) and the same diagram as above, we deduce that \( \beta \in \mathcal{E} \), as re-
quired. This shows that \((\mathcal{C}, \mathcal{E})\) satisfies condition (B3).

(e) \((\mathcal{C}, \mathcal{E})\) satisfies condition (B4).

Proof: Let \( A^1 \to A \to B \in \mathcal{E} \) and \( C \to B \) be any
morphism in \( \mathcal{C} \). We have to prove that \( A^1 \times C \to A \times C \in \mathcal{E} \),
iff \( A^1 \to A \in \mathcal{E} \). With the usual notation, we get the
following diagram,
By using condition (B4) for \((\mathcal{E}, \mathcal{E})\) it is clear that \(A_1^1 \times C_1 \rightarrow A_1 \times C_1 \in \mathcal{E}\)

iff \(A_1^1 \rightarrow A_1 \in \mathcal{E}\). Thus, it remains to show that \(\delta \in \mathcal{E}\) iff \(\beta \in \mathcal{E}\). For this consider the following diagram,

\[
\begin{array}{c}
Z \xrightarrow{\mu} X \xrightarrow{\chi} (A_1^1 \times B_0) \times C_0 \\
\downarrow \quad \downarrow \\
A_1^0 \xrightarrow{\beta} A_1^1 \times A_0 \xrightarrow{\delta} A_1^1 \times B_0
\end{array}
\]

where \(X = (A_1^1 \times A_0) \times ((A_1^1 \times B_0) \times C_0) \cong (A_1^1 \times A_0) \times C_0\) by 2.6 (ii) and

\[
Z = A_0^1 \times ((A_1^1 \times B_0) \times C_0) \cong A_0^1 \times C_0, \text{ by } 2.6(ii).
\]
As $\delta \beta = \alpha \in \mathcal{E}$, by condition (B4) for $(\mathcal{C}, \mathcal{E})$ we deduce that $\mu \in \mathcal{E}$ iff $\beta \in \mathcal{E}$. Now, by 2.6 (iv) and (v), we deduce that

$$Y \sim A_1 \times (A_0 \times C_0) \sim (A_1 \times A_0) \times C_0 \sim X.$$

Thus, since it is easy to check that the various isomorphisms are compatible, we deduce that $\mu \in \mathcal{E}$ iff $\gamma \in \mathcal{E}$. This implies that $\gamma \in \mathcal{E}$ iff $\beta \in \mathcal{E}$, as required, and so $(\mathcal{C}, \mathcal{E})$ satisfies condition (B4).

The parts (a), (b), (c), (d) and (e) above complete the proof of 2.5.

**Definition 2.9**

Let $(\mathcal{C}, \mathcal{E})$ be a base category. Define $(\mathcal{C}_1, \mathcal{E}_1)$ as in 2.4. Then by 2.5, $(\mathcal{C}_1, \mathcal{E}_1)$ is also a base category and so we can define, by using 2.4 and 2.5, a base category $(\mathcal{C}_2, \mathcal{E}_2)$. Thus, inductively, we have a base category $(\mathcal{C}_n, \mathcal{E}_n)$, $n \geq 0$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{E}_0 = \mathcal{E}$. We call $(\mathcal{C}_n, \mathcal{E}_n)$ the $n$th based category on $(\mathcal{C}, \mathcal{E})$.

**Lemma 2.10**

Let $(\mathcal{C}, \mathcal{E})$ be a base category and define base categories $(\mathcal{E}_n, \mathcal{E}_n)$ as above. Let $\mathcal{P}_n$ be the class of $\mathcal{E}_n$-projectives in $\mathcal{E}_n$, $n \geq 0$, i.e., $\mathcal{E}_n \Rightarrow \mathcal{P}_n$. Then,

$$\mathcal{P}_n = \left\{ p_0 \rightarrow p_1 \in \mathcal{E}_{n-1} : p_o, p_1 \in \mathcal{P}_{n-1} \right\}, \text{ for } n \geq 1.$$

**Proof:** By induction, it is enough to prove that

$$\mathcal{P}_1 = \left\{ p_0 \rightarrow p_1 \in \mathcal{E}_0 : p_o, p_1 \in \mathcal{P}_0 \right\}.$$
By part (a) of the proof for 2.5, the right hand side is contained in $\mathcal{E}_1$. So let $Q_0 \xrightarrow{(a)} Q_1 \in \mathcal{E}_1$, $Q_0, Q_1 \in \mathcal{E}_0$.

By using part (a) of the proof for 2.5, we can find $(P_0 \xrightarrow{(a)} P_1) \in \mathcal{E}_0$ such that $(P_0 \xrightarrow{(a)} P_1) \xrightarrow{(\alpha, \beta)} (Q_0 \xrightarrow{(a)} Q_1) \in \mathcal{E}_1$ and $P_0, P_1 \in \mathcal{E}_0$.

But as $(Q_0 \xrightarrow{(a)} Q_1) \in \mathcal{E}_1$, we have a splitting, i.e. there exist $u: Q_0 \xrightarrow{} P_0$ and $\eta: Q_1 \xrightarrow{} P_1$ such that $\alpha u = 1_{Q_1}$, $\beta \eta = 1_{Q_1}$.

Then $Q_0$ and $Q_1$ are split factors in $P_0$ and $P_1$ respectively.

But $P_0, P_1 \in \mathcal{E}_0$ and it can be easily shown that $\mathcal{E}_0$ is closed under split factors. Thus, $Q_0, Q_1 \in \mathcal{E}_0$, as required.
§ 3 Semi-simplicial methods in base categories

Definition 3.1

Let $\mathcal{C}$ be a category with zero object, $\mathbb{1}_\mathcal{C}$. Let $\alpha: A \rightarrow B$ be a morphism in $\mathcal{C}$. Then a $\mathcal{C}$-morphism $k: K \rightarrow A$ is called the kernel of $\alpha$ if whenever $\alpha k = 0$, the zero morphism from $K$ to $B$, and if $\alpha l = 0$ for some $\mathcal{C}$-morphism $l: L \rightarrow A$, then there exists a unique $\mathcal{C}$-morphism $h: L \rightarrow K$ with $kh = l$, i.e. the following diagram is commutative.

Dually, we define the cokernel.

Definition 3.2

Let $(\mathcal{C}, \mathcal{E})$ be a base category and define $(\mathcal{C}_n, \mathcal{E}_n)$, $n \geq 0$, as in § 2. Let $\mathcal{D}$ be a category with kernels and $F: \mathcal{C} \rightarrow \mathcal{D}$ a functor. Then we define associated functors $F_n: \mathcal{C}_n \rightarrow \mathcal{D}$ as follows.

For $n = 0$, $F_0 = F$.

For $n = 1$, if $\alpha \in |\mathcal{C}_1|$, define

$$F_1(\alpha) = \text{Ker}(F_0\alpha)$$

and if $h = (h_0, h_1): \alpha \rightarrow \beta$ is a morphism in $\mathcal{C}_1$, then define $F_1(h) = F_0(h_0)\left|_{F_1(\alpha)}\right.$, i.e. the unique morphism making the following diagram commutative.
For $n > 2$, suppose that $F_1, \ldots, F_{n-1}$ have been defined. Let $\alpha \in |C_n|$ and define

$$F_n(\alpha) = \text{Ker} (F_{n-1}(\alpha)) \text{ and if } h = (h_0, h_1): \alpha \to \beta$$

is a morphism in $C_n$, then define

$$F_n(h) = F_{n-1}(h_0)|_{F_n(\alpha)}.$$

Thus, inductively, all $F_n$ are defined and it is easy to see that they are all functors.

**Definition 3.3**

If $X \xrightarrow{\epsilon} A$ is an augmented semi-simplicial $C$ object, then $\text{strip } X \xrightarrow{\epsilon} A$ is the augmented semi-simplicial object $Y \xrightarrow{\delta} Y_{-1}$ where $Y_n = X_{n+1}$ for $n \geq -1$ and $\delta^i_n = \epsilon^{i+1}_{n+1}$ for $0 \leq i \leq n$. Similarly, we define strip on a finite augmented semi-simplicial $C$-object.

**Definition 3.4**

Let $C$ be a category and $\mathcal{E}$ a class of epimorphisms in $C$. Let $X \xrightarrow{\epsilon} A$ be a (semi-)simplicial $\mathcal{E}$-resolution of $A$ in $C$. Then
Proposition 3.5

Let $X \xrightarrow{\varepsilon} A$ be an augmented semi-simplicial $C$-object. If the $(m-1)^{\text{st}}$ simplicial kernel of strip $X \xrightarrow{\varepsilon} A$ exists, say $(L_{m-1}, L_i)$, $m \geq 2$, $0 \leq i < m-1$, and if the $(m-1)^{\text{st}}$ simplicial kernel of $X \xrightarrow{\varepsilon} A$ exists, say $(K_{m-1}, k_{m-1}^i)$, then there exist unique $C$-morphisms $E_{m-1} : X_{m-1} \xrightarrow{\varepsilon_{m-1}} K_{m-1}$ and $g_{m-1} : L_{m-1} \xrightarrow{\varepsilon_{m-1}} K_{m-1}$, and $X_{m-1} \times L_{m-1}$ is the $m^{\text{th}}$ simplicial kernel of $X \xrightarrow{\varepsilon} A$, provided the fibre product exists.

Proof: We have the following diagram.

Since $\varepsilon_{m-2}^i \varepsilon_{m-1}^j = \varepsilon_{m-2}^i \varepsilon_{m-1}^{i-1} \varepsilon_{m-1}^i$ for $0 \leq i < j \leq m-1$, by the universal property of $K_{m-1}$, there exists a unique morphism $E_{m-1}$ such that $k_{m-1}^i \varepsilon_{m-1}^j = \varepsilon_{m-1}^i$. By definition, we have $E_{m-1}^{i+1} L_j = E_{m-1}^j L_i$ for $0 \leq i < j \leq m$. Put $d_j = E_{m-1}^0 L_j$, $j = 0, \ldots, m$; then $E_{m-2}^i d_j = E_{m-2}^i E_{m-1}^0 L_j = E_{m-2}^0 E_{m-1}^i L_j = E_{m-2}^0 E_{m-1}^j L_i = E_{m-2}^j L_i$, for $0 \leq i < j \leq m$. 

\[ X_n \xrightarrow{\varepsilon_n^0} X_{n-1} \quad \cdots \quad X_1 \xrightarrow{\varepsilon_1^0} X_0 \xrightarrow{\varepsilon_0^0} A \] is called a truncated (semi-)simplicial $C$-resolution of length $n > 0$. 

Since $E_{m-2}^i \varepsilon_{m-1}^j = \varepsilon_{m-2}^i \varepsilon_{m-1}^{i-1} \varepsilon_{m-1}^i$ for $0 \leq i < j \leq m-1$, by the universal property of $K_{m-1}$, there exists a unique morphism $E_{m-1}$ such that $k_{m-1}^i \varepsilon_{m-1}^j = \varepsilon_{m-1}^i$. By definition, we have $E_{m-1}^{i+1} L_j = E_{m-1}^j L_i$ for $0 \leq i < j \leq m$. Put $d_j = E_{m-1}^0 L_j$, $j = 0, \ldots, m$; then $E_{m-2}^i d_j = E_{m-2}^i E_{m-1}^0 L_j = E_{m-2}^0 E_{m-1}^i L_j = E_{m-2}^0 E_{m-1}^j L_i = E_{m-2}^j L_i$, for $0 \leq i < j \leq m$. 

\[ \cdots \]
Thus $d^i$'s satisfy the semi-simplicial identities and so there exists a unique morphism $g_{m-1}$ such that $k^j_{m-1} g_{m-1} = d^j$.

Now, suppose that we can form the fibre product $(X_{m-1} \times L_{m-1}, P_j)$ and consider $(X_{m-1} \times L_{m-1}, P_j)$

where $P_j = \begin{cases} P_0 & \text{if } j = 0; \\ L^j \times L & \text{if } 1 \leq j \leq m. \end{cases}$

For $1 \leq i < j \leq m$, $\varepsilon^i_{m-1} P_j = \varepsilon^i_{m-1} L^j \times L = \varepsilon^i_{m-1} L^j \times L = \varepsilon^i_{m-1} P_i$,

and for $0 < j$, $\varepsilon^0_{m-1} P_j = \varepsilon^0_{m-1} L^j \times L = d^{j-1} L = \varepsilon^j_{m-1} g_{m-1} L$

Thus the semi-simplicial identities are satisfied by $P_j$.

Now, suppose that $H \xrightarrow{h} X_{m-1}$ also satisfies the semi-simplicial identities. Then, put $t^i = h^{i+1}$, $i = 0, \ldots, m-1$ and so $\varepsilon^i_{m-1} t^j = \varepsilon^i_{m-1} t^i$ for $0 \leq i < j \leq m$, and so there exists a unique morphism $h: H \xrightarrow{h} L_{m-1}$ such that $L^i h = t^i = h^{i+1}$, $i = 0, \ldots, m-1$.

Now it is easy to see that $\varepsilon^i_{m-1} h^o = \varepsilon^i_{m-1} h$ and so there exists a unique morphism $H \xrightarrow{h} X_{m-1} \times L_{m-1}$ as required. Hence, $(X_{m-1} \times L_{m-1}, P_j)$ is the $m$th simplicial kernel of $X \xrightarrow{\varepsilon} A$.

**Theorem 3.6**

Let $(\mathcal{C}, \mathcal{E})$ be a base category. Then every object in $\mathcal{C}$ has a semi-simplicial $\mathcal{E}$-resolution.
Proof: Let $A$ be an object in $C$. We shall construct a resolution $X \xrightarrow{\epsilon} A$ by induction on the length $n$ of a truncated resolution.

For $n = 0$, choose $X_0 \in \mathcal{P}$ with $X_0 \xrightarrow{\epsilon_0} A \in \mathcal{E}$, where $\mathcal{E} \Rightarrow \mathcal{P}$.

For $n = 1$, we first form the fibre product $(X_0 \times X_0, k^0, k^1)$ which exists by condition (B1) of a base category, and then choose $X_1 \in \mathcal{P}$ with $X_1 \xrightarrow{\epsilon_1} X_0 \times X_0 \in \mathcal{E}$. We define $\epsilon_1^0, \epsilon_1^1$ by $\epsilon_1^0 = k^0 \epsilon_1$ and $\epsilon_1^1 = k^1 \epsilon_1$ to obtain the following diagram.

For $n > 1$, assume by an inductive hypothesis that a truncated semi-simplicial $\mathcal{E}$-resolution of $A$ of length $n-1$ has been constructed. We first claim that strip $(X_{n-1} \xrightarrow{\epsilon_{n-1}} \ldots X_0 \xrightarrow{\epsilon} A) = Y \xrightarrow{\delta} Y_{-1}$, say, is a truncated semi-simplicial $\mathcal{E}$-resolution of $Y_{-1}$ of length $n-2$. Clearly we have all the semi-simplicial identities being satisfied and all the $Y_i$'s are $\mathcal{E}$-projective. Thus, it remains to show that the morphisms $\alpha_t$ as shown below are in $\mathcal{E}$, for $0 < t < n-1$.

\[ \cdots \xrightarrow{\delta_t} Y_t \xrightarrow{\alpha_t} Y_{t-1} \xrightarrow{\delta_{t-1}} \cdots \]

where $L_t$ is the simplicial kernel as in 3.5.
Now, by 3.5, \( K_{t+1} = L_t \times Y_{t-1} \) and we have
\[
X_{t+1} = Y_t \xrightarrow{e_{t+1}} K_{t+1} \in \mathcal{E}.
\]
But \( K_{t+1} \to L_t \) is also in \( \mathcal{E} \) since we can take \( \alpha_{t-1} \in \mathcal{E} \) by induction and then use 2.7 (iii). Hence, if \( \alpha_1 \in \mathcal{E} \), we can deduce that \( \alpha_t : Y_t \to L_t \in \mathcal{E} \).
But it is easy to check that \( \alpha_1 \in \mathcal{E} \). Therefore
\[
\text{strip} \ (X_{n-1} \to \ldots \to X_1 \to A) \text{ is a truncated semi-simplicial } \mathcal{E} \text{-resolution of length } n-2.
\]
Hence, we can construct the \( n^{th} \) simplicial kernel
\[
K_n = L_{n-1} \times X_{n-1} \quad \text{using 3.5, and we choose } \ X_n \in \mathcal{E}
\]
with
\[
X_n \xrightarrow{e_n} K_n \in \mathcal{E}.
\]
We define the \( e_n^i \), \( 0 \leq i \leq n \), by the usual composition and so we have a truncated semi-simplicial \( \mathcal{E} \)-resolution of \( A \) of length \( n \). Hence, by induction, \( A \) has a semi-simplicial \( \mathcal{E} \)-resolution.

Now we are in a position to apply semi-simplicial methods to define derived functors of associated functors.

**Definition 3.7**

Let \( (\mathcal{C}, \mathcal{E}) \) be a base category and let \( (\mathcal{C}_n, \mathcal{E}_n) \) be defined as before for all \( n \geq 0 \). Let \( \mathcal{A} \) be an abelian category.

Let \( F : \mathcal{C} \to \mathcal{A} \) be a functor and \( F_n : \mathcal{C}_n \to \mathcal{A} \) be the associated functors for all \( n \geq 0 \). Then for each \( m \geq 0 \) and \( A \in \mathcal{C}_m \), by 3.3, there exists a semi-simplicial \( \mathcal{E}_m \)-resolution of \( A \) in \( \mathcal{C}_m \).

Then, by 1.4, we define the associated derived functors
\[
\mathcal{L}_n^m F_m : \mathcal{C}_m \to \mathcal{A}
\]
for all \( n \geq 0, m \geq 0 \). We shall write
\[
L^m_n F_m
\]
Proposition 3.8

Let \((\mathcal{C}, \mathcal{E})\) be a base category and let \((\mathcal{E}_n, \mathcal{E}_n)\) be defined as before for all \(n \geq 0\). Let \(A \xrightarrow{\alpha} B\) be an object in \(\mathcal{C}_m\) for some \(m \geq 1\), and let \((\mathcal{O} \rightarrow \mathcal{Q}) \rightarrow (\mathcal{A} \rightarrow \mathcal{B})\) be a (augmented) semi-simplicial \(\mathcal{E}_m\)-resolution of \(\alpha\). Then \(\mathcal{O} \rightarrow A\) and \(\mathcal{Q} \rightarrow B\) are (augmented) semi-simplicial \(\mathcal{E}_{m-1}\)-resolutions of \(A\) and \(B\) respectively.

Proof: Let the following be a (augmented) semi-simplicial \(\mathcal{E}_m\)-resolution of \((X, E_{m-1}, \mathcal{E}_0: \mathcal{O} \rightarrow \mathcal{Q})\):

\[
\begin{array}{ccc}
\mathcal{O} & \xleftarrow{\alpha} & \mathcal{Q} \\
\vdots & & \vdots \\
\mathcal{P}_n & \xleftarrow{\mathcal{E}_n^i} & \mathcal{Q}_n \\
\mathcal{P}_1 & \xleftarrow{\mathcal{E}_1^o} & \mathcal{Q}_1 \\
\vdots & & \vdots \\
\mathcal{P}_0 & \xleftarrow{\mathcal{E}_0^o} & \mathcal{Q}_0 \\
\end{array}
\]

As \(\mathcal{P}_n \rightarrow \mathcal{Q}_n\) is \(\mathcal{E}_m\)-projective, \(n \geq 0\), by 2.10, \(\mathcal{P}_n\) and \(\mathcal{Q}_n\) are \(\mathcal{E}_{m-1}\)-projective. Also, since composition is componentwise, the semi-simplicial identities are satisfied by \(\mathcal{E}_n^i\) and \(\mathcal{E}_n^o\) for all \(n \geq 0\), \(0 \leq i \leq n\). Thus, \(\mathcal{O}\) and \(\mathcal{Q}\) are \(\mathcal{E}_{m-1}\)-projective semi-simplicial objects in \(\mathcal{C}_{m-1}\). But an easy check shows that the simplicial kernel of the morphisms \((\mathcal{E}_n^i, \mathcal{E}_n^o)\), \(n \geq 0\) in \(\mathcal{C}_m\) is the induced morphism between the simplicial kernels of the morphisms \(\mathcal{E}_n^i\) and \(\mathcal{E}_n^o\) in \(\mathcal{C}_{m-1}\). Thus, by the definition of \(\mathcal{E}_m\) and using 2.7 (iv), \(\mathcal{P}_n\) and \(\mathcal{Q}_n\) factor through the respective simplicial kernels via elements of \(\mathcal{E}_{m-1}\) for all \(n \geq 1\). Also, we know that \(\mathcal{E}_0^i\) and \(\mathcal{E}_0^o\) are in \(\mathcal{E}_{m-1}\), and so \(\mathcal{O} \rightarrow A\) and \(\mathcal{Q} \rightarrow B\) are (augmented) semi-simplicial \(\mathcal{E}_{m-1}\)-resolutions of \(A\) and \(B\) respectively.
Theorem 3.9

Let \((\mathcal{C}, \mathcal{E})\) be a base category and let \((\mathcal{C}_n, \mathcal{E}_n)\) be defined as before for all \(n \geq 0\). Let \(\mathcal{O}\) be an abelian category and let \(F: \mathcal{C} \to \mathcal{O}\) be a functor with associated functors \(F_m: \mathcal{C}_m \to \mathcal{O}, m \geq 0\).

If \(L_n F_m: \mathcal{C}_m \to \mathcal{O}\) is defined as in 3.7 for all \(n \geq 0, m \geq 0\), then

(i) \(L_n F_m\) vanishes in \(\mathcal{C}_m\)-projectives for all \(n \geq 1, m \geq 0\);

(ii) If \(\xymatrix{A \ar[r] & B}\) is an object of \(\mathcal{C}_m, m \geq 1\), then there exists a long exact sequence

\[
\ldots L_{n+1} F_m(A) \to L_n F_m(A) \to L_{n-1} F_m(A) \to L_{n-2} F_m(A) \to \ldots \to L_0 F_m(B) \to 0.
\]

Proof: (i) This follows immediately from 1.5 (iii).

(ii) As in 3.8, let \(\xymatrix{P \ar[r] & Q \ar[r] & (A \ar[r] & B)\)} be an augmented semi-simplicial \(\mathcal{E}_m\)-resolution of \(\alpha\). Then, using 3.8,

\[
F_m(\alpha_n) = \text{Ker}(F_{m-1}(\alpha_n)) = K_n, \text{ say, } n \geq 0.
\]

Thus, by the definition of \(L_n F_m\),

\[
L_n F_m(\alpha) = H_n(\ldots \to K_n \to K_{n-1} \to \ldots \to K_1 \to K_0 \to 0),
\]

where

\[
k_n = \sum_{i=0}^{n} (-1)^i \left(F \mathcal{E}_n^{i}\right)_{K_n}, \quad n \geq 0.
\]
Also,

\[ L_n F_{m-1}(A) = H_n(\ldots \rightarrow F_{m-1}(P_n) \rightarrow P_n \rightarrow F_{m-1}(P_{n-1}) \rightarrow \ldots) \]

\[ \rightarrow F_{m-1}(P_0) \rightarrow 0), \]

where \[ p_n = \sum_{i=0}^{n} (-1)^i F_{m-1} e_i, \quad n \geq 0, \]

and

\[ L_n F_{m-1}(B) = H_n(\ldots \rightarrow F_{m-1}(Q_n) \rightarrow \tau_n \rightarrow F_{m-1}(Q_{n-1}) \rightarrow \ldots) \]

\[ \rightarrow F_{m-1}(Q_0) \rightarrow 0), \]

where \[ \tau_n = \sum_{i=0}^{n} (-1)^i F_{m-1} \delta_i, \quad n \geq 0. \]

Now, \[ P_{n-1} \rightarrow Q_{n-1} \]

is a split epimorphism of \( C_{m-1}, \quad n \geq 1, \)

and it is easy to see that a functor takes split epimorphisms to split epimorphisms; and so, using the definition of \( K_n, \) we obtain the following commutative diagram with exact rows in \( \mathcal{O}, \)

\[ \begin{array}{ccccccc}
0 & \rightarrow & K_{n-1} & \rightarrow & F_{m-1}(P_{n-1}) & \rightarrow & F_{m-1}(Q_{n-1}) & \rightarrow & 0 \\
\downarrow & & \downarrow k_{n-1} & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K_{n-2} & \rightarrow & F_{m-1}(P_{n-2}) & \rightarrow & F_{m-1}(Q_{n-2}) & \rightarrow & 0
\end{array} \]

for all \( n \geq 2. \) Then, as usual, the "snake lemma" completes the proof.
Definition 3.10

Let $(\mathcal{C}, \mathcal{E})$ be a base category and $F: \mathcal{C} \rightarrow \mathcal{A}$ a functor, where $\mathcal{A}$ is an abelian category. We say that $F$ is simplicially right exact if for $A \xrightarrow{\alpha} B \in \mathcal{E}$, we have

$$\xymatrix{ F(A \times A) \ar@<0.5ex>[r]^-{F\pi_1} & F(A) \ar[r] & F(B) \ar[r] & 0 }$$

an exact sequence, where $(A \times A, \pi_1, \pi_2)$ is the fibre product of $\alpha$ with itself.

Proposition 3.11

Let $(\mathcal{C}, \mathcal{E})$ be a base category and $F: \mathcal{C} \rightarrow \mathcal{A}$ be a functor with $\mathcal{A}$ an abelian category. Then,

(i) $L_0 F$ is simplicially right exact;

(ii) $L_0 F \cong F$ iff $F$ is simplicially right exact.

Proof: Let $A \xrightarrow{\alpha} B \in \mathcal{E}$ and construct part of a semi-simplicial resolution of $\alpha$ as follows,

$$\xymatrix{ P_1 \ar[d] \ar[r] & Q_1 \ar[d] & \ar[d] \\
\downarrow & \downarrow & \\
P_0 \ar[r] & Q_0 \ar[d] & \\
A \ar[r] & B \ar[d] }$$

where $P_0, P_1, Q_0, Q_1$ are in $\mathcal{P}$ and $\mathcal{E} \Rightarrow \mathcal{P}$.

As $P_0 \rightarrow Q_0 \in \mathcal{E}$ we have a splitting and if $(P_0 \times P_0, \delta_1, \delta_2)$ is the fibre product of $P_0 \rightarrow Q_0$ with itself, then
is an exact sequence. But \( L_0 F(P_i) = F(P_i) \) and \( L_0 F(Q_i) = F(Q_i) \), 
\( i = 1,2 \) by \( 1.5(iii) \). Therefore \( L_0 F(P_0 \times P_0) \rightarrow F(P) \rightarrow F(Q) \rightarrow 0 \)

is an exact sequence.

Now consider the following commutative diagram,

\[
\begin{array}{ccc}
L_0 F(P_0 x P_0) & \xrightarrow{L_0 F_1 - L_0 F_2} & L_0 F(P_0) \\
& & \rightarrow L_0 F(Q_0) \\
& & \rightarrow 0
\end{array}
\]

where the columns are exact by the definition of \( L_0 F \) and the top two 
rows are exact by the remarks above. Hence, by diagram chasing, 
the bottom row is exact and so \( L_0 F \) is simplicially right exact.

(ii) If \( F \sim L_0 F \), then by (i) above, \( F \) is simplicially 
right exact.

Conversely, let \( F \) be simplicially right exact and let \( A \) 
be an object in \( \mathcal{C} \). Consider the usual part of the semi-simplicial 
\( \mathcal{C} \)-resolution for \( A \). Since \( F \) takes elements in \( \mathcal{C} \) to epimor-
isms, we obtain the following commutative diagram with exact rows.
Thus, by diagram chasing, $L_0F(A) \cong F(A)$ and so $L_0F \cong F$, as required.

**Proposition 3.12**

Let $(\mathcal{C}, \mathcal{E})$ be a base category and define $(\mathcal{C}_m, \mathcal{E}_m)$ as before for all $m \geq 0$. Let $\mathcal{A}$ be an abelian category and $F: \mathcal{C} \rightarrow \mathcal{A}$ a functor. Define the associated derived functors $L_n^\mathcal{F}$ for all $n \geq 0$, $m \geq 0$. Then,

(A1) $L_{0}^{\mathcal{F}}(P) = F_{m}(P)$ for $P \in \mathcal{P}_m$, where $\mathcal{E}_m \rightarrow \mathcal{P}_m$;

(A2) $L_{0}^{\mathcal{F}}$ is simplicially right exact;

(A3) $L_{n}^{\mathcal{F}}(P) = 0$ for $P \in \mathcal{P}_m$; $n > 0$;

(A4) if $A \rightarrow B \in \mathcal{E}_m$, $m \geq 0$, then there exists a long exact natural sequence,

$$
\cdots \rightarrow L_{n+1}^{\mathcal{F}}(B) \rightarrow L_{n+m+1}(\alpha) \rightarrow L_{n}^{\mathcal{F}}(A) \rightarrow L_{n}^{\mathcal{F}}(B) \rightarrow 0
$$

Furthermore, these four conditions characterize all the associated derived functors.
Proof: By 1.5(iii), 3.9 and 3.11 (ii), conditions (A1) to (A4) are satisfied.

Suppose we are given functors $S_{n,m}^\xi : C_m \rightarrow 0\Pi$ which also satisfy conditions (A1) to (A4). Then we shall show

$L_{n,m}^{\xi F} \sim S_{n,m}^\xi$ for all $n \geq 0$, $m > 0$, by induction on $n$.

For $n = 0$, with the usual notation and using (A1) and (A2) we obtain the following commutative diagram with exact rows.

\[
\begin{array}{cccccc}
F_m(p_0) & \rightarrow & F_m(p_0) & \rightarrow & L_o F_m(A) & \rightarrow O \\
\downarrow & & \downarrow & & \downarrow & \\
S_{0,m}^\xi (p_0 \times A) & \rightarrow & S_{0,m}^\xi (p_0) & \rightarrow & S_{0,m}^\xi (A) & \rightarrow O \\
& & \downarrow & & \downarrow & \\
& & & & & 0
\end{array}
\]

Hence, by diagram chasing, we have $L_{o,m}^{\xi F} (A) \sim S_{o,m}^\xi (A)$ and so $L_{o,m}^{\xi F} \sim S_{o,m}^\xi$.

For $n > 0$, assume by an inductive hypothesis that $L_{n-1,m}^{\xi F} \sim S_{n-1,m}^\xi$ for all $m > 0$. Let $P \xrightarrow{\alpha} A \in E_m$ with $P \in C_m$. By conditions (A1), (A3) and (A4) we obtain two long exact natural sequences, and hence $\alpha$

\[
\begin{array}{cccccc}
\cdots & 0 & \rightarrow & L_{2,m}^\xi F(A) & \rightarrow & L_{1,m}^\xi F(A) & \rightarrow 0 \\
& & & \xrightarrow{\xi L_{o,m}^{F+1} (a)} & \rightarrow & L_{o,m}^{\xi F} (A) & \rightarrow 0. \\
\cdots & 0 & \rightarrow & S_{2,m}^\xi (A) & \rightarrow & S_{1,m+1}^\xi (a) & \rightarrow 0 \\
& & & \xrightarrow{\xi S_{o,m+1}^\xi (a)} & \rightarrow & S_{o,m}^\xi (A) & \rightarrow 0.
\end{array}
\]

commutative diagram.
Thus, induction and the case $n = 0$ imply that $L_{n,m}^F(A) \cong S_{n,m}^C(A)$ for all $n \geq 0$, $m \geq 0$. This implies that conditions (A1) to (A4) characterize all the associated derived functors.

Remark Rinehart also defines the associated derived functors of a functor $F$ from a base category $(\mathcal{C}, \mathcal{E})$ to an abelian category $\mathcal{A}$. He considers the categories $(\mathcal{C}, \mathcal{A})$ of functors from $\mathcal{C}$ to $\mathcal{A}$ and $S_{\text{Rex}}(\mathcal{C}, \mathcal{A})$ of simplicially right exact functors from $\mathcal{C}$ to $\mathcal{A}$. He defines an adjoint pair of functors

$$\left(\mathcal{C}, \mathcal{A}\right) \xrightarrow{\#} S_{\text{Rex}}(\mathcal{C}, \mathcal{A}),$$

where $I$ is the inclusion functor. He proves that $I$ is an additive, right exact functor of abelian categories. Thus, using "classical" methods one can define the derived functors $D_m I$ for all $m \geq 0$. Finally, to define derived functors $S_n F$ (or satellite functors as he calls them), he puts $S_n F = D_n(F \#)$. This method also yields the long exact sequence and in fact the functors satisfy conditions (A1) to (A4). Thus, our definition agrees with Rinehart's.
CHAPTER III

SIMPLICIAL METHODS IN RINEHART CATEGORIES

In this chapter we generalize some of the results of the last chapter. We have done this to emphasise both the semi-simplicial and the simplicial theories of defining derived functors.

In §1, we state the "standard" results on simplicial sets. In §2, we use these simplicial methods to define functors from a not necessarily algebraic category to a not necessarily abelian category. Thus we have a generalization of a theory of Keune (6) to what we call Rinehart categories. At this point, we briefly compare the semi-simplicial and the simplicial theories and sketch their relative merits.

In §3, we use simplicial techniques in Rinehart categories to obtain a generalization of Rinehart's theory.
§ 1  Simplicial Sets

The standard reference for this section is May (9), and so we leave out the proofs.

**Definition 1.1**

A simplicial set \((X, d, -)\) is called a Kan complex if for every collection of \((n+1)\) elements \(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}\) in \(X_n, n \geq 0\) such that \(d^i_n x_j = d^{i-1}_n x_i\) for \(k \neq i < j \neq k\), there exists an element \(x\) in \(X_{n+1}\) such that \(d^i_{n+1} x = x_i\) for all \(i \neq k\).

**Definition 1.2**

Let \((E, d, -)\) and \((B, e, -)\) be simplicial sets and \(p: E \rightarrow B\) a simplicial map. Then \(p\) is called a fibration if for every collection of \((n+1)\) elements \(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}\) in \(E_n, n \geq 0\), such that \(d^i_n x_j = d^{i-1}_n x_i\) for \(k \neq i < j \neq k\); and for \(y \in B_{n+1}\) with \(e^{i} y = px_i\) for \(i \neq k\), there exists an element \(x\) in \(E_{n+1}\) such that \(d^i_{n+1} x = x_i\) for all \(i \neq k\) and \(px = y\).

**Definition 1.3**

Let \((X, d, S)\) be a Kan complex and let \(x\) and \(x^1\) be elements in \(X_n\) for some \(n \geq 0\). Then \(x\) and \(x^1\) are called homotopic \((x \sim x^1)\), if there exists an element \(y \in X_{n+1}\) such that \(d^i_{n+1} y = s^{n-1}_{n-1} d^i_{n-1} x = s^{n-1}_{n-1} d^i_{n-1} x^1\) for \(0 \leq i < n\), and \(d^i_{n+1} y = x, d^i_{n+1} y = x^1\).
Proposition 1.4

With the same notation as in 1.3, \( \sim \) is an equivalence relation on \( X_n \) for all \( n \geq 0 \).

Definition 1.5

Let \((X, d, -)\) be a Kan complex. Let \(*\) be an element in \( X_0 \) and denote by \(*\) all \( S_{n-1} \cdots S_1 S_0 \) for all \( n \geq 1 \). Then \((X, *)\) is called a pointed Kan complex. Also, for \( n \geq 1 \), define

\[
\tilde{X}_n = \{ x \in X : (d^0_n x, \ldots, d^n_n x) = (*, \ldots, *) \}.
\]

Put \( \Pi_n(X, *) = \begin{cases} \tilde{X}_n/\sim, & \text{if } n \geq 1, \\ X_0/\sim, & \text{if } n = 0. \end{cases} \)

We call \( \Pi_n(X, *) \) the \( n \)th homotopy group \( n \geq 0 \).

Theorem 1.6

Let \( E \) and \( B \) be simplicial sets and let \( p : E \longrightarrow B \) be a fibration. Let \(*\) be an element of \( B_0 \) and put \( F = p^{-1}(*) \).

Then, \( E \) \( F \) are Kan complexes and \( E \) is iff \( B \) is, and

(i) there exists a long exact sequence of pointed sets,

\[
\cdots \longrightarrow \Pi_{n+1}(B, *) \longrightarrow \Pi_n(F, *) \longrightarrow \Pi_n(E, *) \longrightarrow \Pi_n(B, *) \longrightarrow \cdots \longrightarrow \Pi_0(B, *) \longrightarrow *
\]

where some \( * \in F_0 \) acts as base point in \( E \) and \( F \);
(ii) we can define canonical group structures such that the following is a long exact sequence of groups,

\[ \cdots \rightarrow \Pi_{n+1}(B,*) \rightarrow \Pi_{n}(F,*) \rightarrow \Pi_{n}(E,*) \rightarrow \Pi_{n}(B,*) \rightarrow \cdots \rightarrow \Pi_{n}(B,*) ; \]

(iii) \( \Pi_{n}(-,*) \) is an abelian group for \( n \geq 2 \).
§2. Simplicial methods and derived functors.

As in chapter 1, we have a definition of an augmented simplicial $\mathcal{C}$-resolution for an object in a category $\mathcal{C}$ with epimorphisms $\mathcal{E}$. We now prove a result which states a set of conditions under which a category has augmented simplicial $\mathcal{E}$-resolutions.

Theorem 2.1 (The "step-by-step" construction)

Let $(\mathcal{C}, \mathcal{E})$ be a base category such that

(i) a finite number of elements in $\mathcal{P}$ have a coproduct in $\mathcal{C}$, where $\mathcal{E} \rightarrow \mathcal{P}$;

(ii) morphisms in $\mathcal{E}$ have kernels in $\mathcal{C}$;

(iii) for all commutative diagrams,

\[
\begin{array}{c}
K_i \Rightarrow B_i \Rightarrow A_0 \\
\downarrow \alpha \downarrow \beta \\
K_0 \Rightarrow B_0 \Rightarrow A_0
\end{array}
\]

where $K_i$ is the kernel of $B_i \rightarrow A_i$, $i = 0, 1$; $A_1 = A_0$; and $\Rightarrow$ denotes an element in $\mathcal{E}$.

Then every object in $\mathcal{C}$ has an augmented simplicial $\mathcal{E}$-resolution.

Proof: Let $A$ be an object in $\mathcal{C}$. We shall construct the resolution $\eta \Rightarrow \varepsilon A$ by induction on the length of the truncated resolution.
For \( n = 0 \), there exists \( M_0 \in \mathcal{P} \) with \( M_0 \xrightarrow{\omega_0} A \in \mathcal{E} \).

Put \( N_0 = M_0 \) and \( \varepsilon^0 = \omega_0 \) to get

\[
N_0 \xrightarrow{\varepsilon^0} A.
\]

For \( n = 1 \), let \( R_1 = \text{Ker} \varepsilon^0 \) and choose \( M_1 \in \mathcal{P} \) with

\[
M_1 \xrightarrow{\omega_1} R_1 \in \mathcal{E}.
\]

Put \( N_1 = M_0 \sqcup M_1 \), where \( \sqcup \) is the coproduct symbol and define \( \varepsilon^0_1, \varepsilon^1_1 \) and \( \eta^0_1 \) as follows

\[
\varepsilon^0_1 : \begin{cases}
M_0 \\
M_1 \xrightarrow{i \omega_1} N_0,
\end{cases}
\]

where \( i : R_1 \rightarrow N_0 \)

is the inclusion.

\[
\varepsilon^1_1 : M_1 \rightarrow 1,
\]

where \( 1 \) is the initial object which exists, being the coproduct of \( \phi \).

\[
\eta^0_1 : M_0 \xrightarrow{1} M_0.
\]

By the universal property of coproducts, we have induced \( \mathcal{E} \)-morphisms \( \varepsilon^0_1, \varepsilon^1_1 \) and \( \eta^0_1 \) which satisfy the simplicial identities. We obtain the following diagram.

\[
\begin{array}{ccc}
N_1 & \xrightarrow{\varepsilon^0_1} & N_0 \\
\varepsilon^1_1 & & \varepsilon^0_1 \\
& \varepsilon^0_1 \downarrow & \downarrow \\
& N_0 & \rightarrow A
\end{array}
\]
Now, it remains to show that $\text{NN}_1 \rightarrow \text{N}_0 \times \text{N}_0 \in \mathcal{E}$. For this, we first show that given any commutative diagram,

\[
\begin{array}{ccc}
M & \rightarrow & B_1 \\
\downarrow \gamma & & \downarrow \gamma \\
K_0 & \rightarrow & B_0 \\
\end{array}
\]

where $K_0$ is the kernel of $B_0 \rightarrow A_0$. $\gamma \in \mathcal{E} \Rightarrow \alpha \in \mathcal{E}$. For this, consider the factorization through the kernel $K_1$ of $B_1 \rightarrow A_0$ and apply condition (iii) of the hypothesis.

\[
\begin{array}{ccc}
M & \rightarrow & B_1 & \rightarrow & A_0 \\
\downarrow \gamma & & \downarrow \gamma & & \downarrow \\
K_0 & \rightarrow & B_0 & \rightarrow & A_0 \\
\end{array}
\]

Then $\alpha \in \mathcal{E}$ iff $\beta \in \mathcal{E}$. But $\gamma \in \mathcal{E}$ implies $\beta \in \mathcal{E}$ by using condition (B3) for a base category. Hence, $\gamma \in \mathcal{E}$ implies $\alpha \in \mathcal{E}$.

Now, consider the diagram,

\[
\begin{array}{ccc}
M_1 & \rightarrow & m_1 & \rightarrow & N_1 & \rightarrow & \epsilon_1 \\
\downarrow \omega_1 & & \downarrow & & \downarrow & & \downarrow \\
R_1 & \rightarrow & N_0 \times N_0 \rightarrow & \pi_2 & & \rightarrow & \rightarrow & N_0 \\
\end{array}
\]

where $m_1: M_1 \rightarrow N_1$ is the "coproduct" map $M_1 \rightarrow M_0 \sqcup M_1$ and $R_1 \sim \text{Ker} (N_0 \times N_0 \rightarrow A_0 \rightarrow N_0)$. 
Since \( \varepsilon_1^1(M_1) = 1 \), we have \( \varepsilon_1^1 M_1 (M_1) = 1 \). Also, we can assume that the diagram is commutative and so, as \( w_1 \in \mathcal{E} \), we have \( N_1 \rightarrow N_0 \times N_0 \in \mathcal{E} \), as required. By induction, suppose that we have defined \( N_0, \ldots, N_{n-1} \) and \( M_0, \ldots, M_{n-1} \); \( n \geq 2 \), together with the face and degeneracy morphisms satisfying the various simplicial identities. Also, assume that the various factorizations through the simplicial kernels are via elements in \( \mathcal{E} \). Consider the diagram,

\[
\begin{array}{ccc}
R_n & \rightarrow & K_n = N_{n-1} \times L_{n-1} \\
\downarrow & & \downarrow \\
L_{n-1} & \rightarrow & K_{n-1} \\
\end{array}
\]

where \( K_{n-1} \) is the simplicial kernel of \( (\varepsilon_{n-2}^0, \ldots, \varepsilon_{n-2}^1) \), \( L_{n-1} \) is the simplicial kernel of \( (\varepsilon_{n-1}^1, \ldots, \varepsilon_{n-1}^0) \). Then, using \( \Pi \cdot 3.5 \), \( K_n \) is the simplicial kernel of \( (\varepsilon_{n-1}^0, \ldots, \varepsilon_{n-1}^1) \). Put \( R_n = \text{Ker}(K_n \rightarrow L_{n-1}) \).

Choose \( M_n \in \mathcal{E} \) with \( M_n \rightarrow R_n \in \mathcal{E} \) and define \( N_n \) as follows.

For \( k \in \mathbb{N} \), let \( (k) \) denote the set of integers \( \{0, 1, \ldots, k\} \) and consider all order preserving surjections \( \sigma : (n) \rightarrow (m) \) for all \( m \leq n \). It is easy to see that there are \( \binom{n}{m} \) such surjections for each \( m \leq n \).

Put \( N_n = \bigoplus_{m \leq n} M_{m}^{\sigma} \), where \( M_{m}^{\sigma} \) is
an isomorphic copy of $M_m^n$, with given isomorphism $\phi_3 : M_m^n \to M_m^n$.

Thus, $N_n = \bigcup_{j=0}^{(n-2)} M_j \cup M_{n-1} \cup M_n$, where $M_j$ denotes the coproduct of $k$ distinct but isomorphic copies of $M_j$. Now, we can define the face and degeneracy maps. For $j = 0, \ldots, n$;

define $\tilde{e}^j_n : (n-1) \to (n)$ by

$$
\tilde{e}^j_n(i) = \begin{cases} 
  i & \text{if } i < j; \\
  i+1 & \text{if } i \geq j.
\end{cases}
$$

Put $e^j_n = \tilde{e}^j_n : (n-1) \to (m)$, $j = 0, \ldots, n$ and all $\sigma : (n) \to (m)$ for $m \leq n$.

Case (i). If $\sigma^k$ is a surjection for some $k \in \{0, 1, \ldots, n\}$, then define

$$
\epsilon^k_n (M^\sigma_m) = 1.
$$

Case (ii). If $\sigma^0$ is not a surjection, then we have two sub-cases.

(a) If $0 \in \text{Im} (\sigma^k)$, (for example when $k > 0$), then define

$$
\epsilon^k_n (M^\sigma_m) = 1.
$$

(b) If $0 \notin \text{Im} (\sigma^k)$, (i.e. if $k = 0$ and $\sigma^{-1}(0) = \{0\}$), then we proceed as follows.

Define $\tilde{\sigma}^k : (n-1) \to (m-1)$ by $\tilde{\sigma}^k(i) = \sigma^k(i) - 1$ for all $i$. Then $\tilde{\sigma}^k$ is a surjection. Now, given

$$
\rho : (m-1) \to (p) \quad \text{as an order preserving surjection, so is } \bar{\rho} = \rho_{\sigma^k}.
$$
Thus we can embed $N^{m-1}$ in $N^{n-1}$ by mapping each summand $M^p$ of $N^{m-1}$ to the summand $M^q$ of $N^{n-1}$ via the isomorphism defined by the $i$'s. In particular, we have the following diagram,

$$
\begin{array}{c}
M^p \\
\downarrow \iota_p \\
M^m \\
\downarrow \sigma_m \\
M^{m-1} \\
\end{array} \xrightarrow{t_m \omega_m} \begin{array}{c}
N^{m-1} \\
\downarrow \text{emb.} \\
N^{n-1} \\
\end{array}
$$

where \text{emb.} is the above embedding and $t_m : R^{m} \rightarrow N^{m-1}$ is the monomorphism induced. We define $\varepsilon^k_n$ on $M^m$ such that the above diagram is commutative.

The above cases define the action of $\varepsilon^i_n$ on all $M^m_{0 \leq i \leq n}$, $m < n$, and so this induces $\mathcal{C}$-morphisms $\varepsilon^i_n : N_n \rightarrow N^{n-1}$. A straightforward check shows that these morphisms satisfy the required simplicial identities. To define the degeneracy maps, we first define, for $k = 0, \ldots, n-1$:

$$
\tilde{\eta}^k_n : (n) \rightarrow (n-1) \quad \text{by} \quad \tilde{\eta}^k_n(i) = \begin{cases} 
 i \text{ is } i < k; \\
 i-1 \text{ if } i > k.
\end{cases}
$$

As before, to every $M^m$ in $N^{n-1}$ we can associate an order preserving surjection $\sigma : (n-1) \rightarrow (m)$. Then $\sigma \tilde{\eta}^k_n : (n) \rightarrow (m)$ is also an order preserving surjection and so we define $\eta^k_n$ on $M^m$ by the obvious composite of $i$'s followed by an inclusion in $N_n$. In particular, $\eta^k_n(M^m) = M^m$. Thus, for the same reason as above, we have a definition for the degeneracy morphisms. Now, a routine check shows that these face and degeneracy maps satisfy the various simplicial identities.
Finally, it remains to show that \( \xrightarrow[n]{N} K \in \mathcal{E} \). For this, consider the commutative diagram,

\[
\begin{array}{ccc}
M_n & \xrightarrow{m_n} & N_n \\
\downarrow \omega_n & & \downarrow \nu_n \\
R_n & \xrightarrow{m_n} & K_n \\
\end{array}
\]

where \( m_n : M_n \rightarrow N_n \) is the "coproduct" map, \( R_n = \text{Ker}(K_n \rightarrow L_{n-1}) \) and \( \nu_n \) is the unique induced map, which is in \( \mathcal{E} \) by the proof of II.3.6.

Then, as before, \( w_n \in \mathcal{E} \) implies \( \xrightarrow[n]{N} K \in \mathcal{E} \), as required. Hence, by induction, every object in \( \mathcal{E} \) has a simplicial \( \mathcal{E} \)-resolution.

Note: The above construction is a modified version of the procedure described by Andre (1). To see that our definitions agree with Andre's, observe that \( R_n = \bigcap_{i=0}^{n-1} \text{Ker} \epsilon_i \), if the right-hand side exists. We shall call a resolution obtained by the procedure in 2.1, an Andre \( \mathcal{E} \)-resolution.

Now, we want to define derived functors by using the simplicial resolutions. But before we can do that we need to have a comparison theorem and the notion of a simplicial homotopy.

**Definition 2.2**

Let \((K, \epsilon, \sigma)\) and \((L, \alpha, \mu)\) be simplicial objects in a category \( \mathcal{C} \) and let \( f \) and \( g \) be simplicial morphisms from \( K \) to \( L \). Then \( f \) and \( g \) are said to be **simplicially homotopic**
if there exist \( \tau \)-morphisms \( h^i_n: K_n \rightarrow L_{n+1}, \ i = 0, \ldots, n; \)
\( n > 0, \) such that

(a) \( \partial_{n+1}^o h^o_n = f_n; \)

(b) \( \partial_{n+1}^{n+1} h_n = g_n; \)

(c) \( \partial_{n+1}^i h_n^j = h_{n-1}^{i-1} \epsilon_n^i \) if \( i < j; \)

(d) \( \partial_{n+1}^{j+1} h_{n+1}^j = \partial_{n+1}^{j+1} h_n^j; \)

(e) \( \partial_{n+1}^{i} h_n^j = h_n^{j-1} \epsilon_n^{i-1} \) if \( i > j+1; \)

(f) \( \mu_n^i h_{n-1}^j = h_n^{j+1} \sigma_{n-1}^i \) if \( i \leq j; \)

(g) \( \mu_n^i h_n^{j-1} = h_n^j \sigma_{n-1}^{i-1} \) if \( i > j. \)

We obtain the following diagram.

\[ \begin{array}{c}
\ldots \ K_{n+1} \Longrightarrow K_n \Longrightarrow K_{n-1} \ldots \\
\downarrow \sigma_{n+1} \downarrow \varepsilon_{n+1} \downarrow \sigma_{n-1} \downarrow \varepsilon_{n-1} \\
\downarrow \sigma_n \downarrow \varepsilon_n \downarrow \sigma_{n-1} \downarrow \varepsilon_{n-1} \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
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\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\downarrow \sigma_n \downarrow \varepsilon_n \\
\end{array} \]
Theorem 2.3

The Simplicial Comparison Theorem.

Let \( N \xrightarrow{\partial} A \) be an Andre \( \mathcal{E} \)-resolution for \( A \) and \( N' \xrightarrow{\partial} A' \) be an augmented simplicial \( \mathcal{E} \)-exact object in a category \( \mathcal{C} \), where \( \mathcal{E} \) is a projective class of epimorphisms in \( \mathcal{C} \). Let \( \alpha: A \rightarrow A' \) be a \( \mathcal{C} \)-morphism. Then,

(i) there exists a simplicial \( \mathcal{C} \)-morphism \( \gamma: N \rightarrow N' \) which extends \( \alpha \); and

(ii) if \( \gamma': N \rightarrow N' \) is any other simplicial \( \mathcal{C} \)-morphism extending \( \alpha \), then \( \gamma \) and \( \gamma' \) are simplicially homotopic.

Proof: (i) We need the following diagram.

We shall define \( \{ \gamma_n \} \) by induction on \( n \). For \( n = 0 \), since \( N_0 \in \mathcal{P} \) and \( \delta_0^\circ \in \mathcal{E} \), there exists a \( \mathcal{C} \)-morphism \( \gamma_0: N_0 \rightarrow N'_0 \) such that \( \alpha \gamma_0 = \gamma_0 \delta_0^\circ \). So suppose that \( \gamma_0, \ldots, \gamma_{n-1} \) have been defined for \( n \geq 1 \), and that they all commute with the face and degeneracy maps as required.
As $N_n$ is the coproduct of $M_j^\theta$, $j \leq n$ and $\theta : (n) \to (j)$ an order preserving surjection, as in II.2.1, we shall define $\gamma_n$ on the $M_j^\theta$ separately, and hence by the universal property of coproducts on $N_n$.

Case (i) $M_j^\theta$ is non-degenerate, i.e., $M_j^\theta$ is not an image of $\sigma_{n-1}$.

By Theorem II.1.1, there exists $\gamma_n : N_n \to N_n'$ such that $\{\gamma_n\}$ is a semi-simplicial morphism from $N$ to $N'$.

Define $\gamma_n(M_j^\theta) = \gamma_n |_{M_j^\theta}$

Case (ii) $M_j$ is degenerate.

Let $M_j^\theta = \sigma_{n-1}^{-k}(M_t^\phi)$ for $j, t \leq n-1$, $\phi : (n-1) \to (t)$ an order preserving surjection.

As $\sigma_{n-1}$ is an isomorphism, we define

$$\gamma_n(M_j^\theta) = \gamma_{n-1}^{-1} k \gamma_{n-1}^{-1} (M_j^\theta);$$

i.e. such that the following diagram is commutative.

$$\begin{array}{c}
M_j^\theta \\
\downarrow \gamma_n \\
N_n' \\
\downarrow \mu_{n-1} \\
N_{n-1}
\end{array} \qquad \begin{array}{c}
\sigma_{n-1}^{-k} \\
\downarrow \\
M_t^\phi \\
\downarrow \gamma_{n-1} \\
N_{n-1}'
\end{array}$$

It is easy to verify that $\gamma_n$ is well-defined for different values of $t$ and $\phi$.

Hence, by induction, we have defined $\gamma_n$ for all $n \geq 0$.

From the definition, it is obvious that $\gamma_n$ commute with the degeneracy maps. Thus, it remains to show that $\gamma_n$ commute with the
face morphisms. Now, by definition, the non-degenerate $M^\theta_j$ commute with the face maps. So let $M^\theta_j$ be degenerate, say $M^\theta_j = \sigma_{n-1}^k (M^\phi_t)$, as above. We consider three cases.

Case (i) $i < k$.

Then

$$\left( \sigma_n^i \gamma_n \left( \sigma_{n-1}^k (M^\phi_t) \right) \right) = \sigma_n^i \gamma_{n-1} (M^\phi_t)$$

$$= u_{n-2}^{k-1} \gamma_{n-1} (M^\phi_t)$$

$$= u_{n-2}^{k-1} \gamma_{n-2} \epsilon_{n-1} (M^\phi_t)$$

$$= \gamma_{n-1} \sigma_{n-2}^k \epsilon_{n-1} (M^\phi_t)$$

$$= \gamma_{n-1} \epsilon_{n} \sigma_{n-1} (M^\phi_t)$$

$$= \gamma_{n-1} \epsilon_{n} (M^\phi_t).$$

Case (ii) $i = k$ or $i = k+1$

Then

$$\left( \sigma_n^i \gamma_n \left( \sigma_{n-1}^k (M^\phi_t) \right) \right) = \sigma_n^i \gamma_{n-1} (M^\phi_t)$$

$$= \gamma_{n-1} (M^\phi_t)$$

$$= \gamma_{n-1} \epsilon_n \sigma_{n-1} (M^\phi_t) = \gamma_{n-1} \epsilon_n (M^\phi_t).$$
Case (iii) \( i > k+1 \)

Then
\[
\Phi_n \gamma_n (\sigma_{n-1}^k (M^\phi_t)) = \Phi_n \mu_n \gamma_{n-1} (M^\phi_t)
\]
\[
= u_{n-2}^k \gamma_{n-2} \gamma_{n-1} \mu_{n-1} (M^\phi_t)
\]
\[
= u_{n-2}^k \gamma_{n-2} \gamma_{n-1} \varepsilon_{n-1} (M^\phi_t)
\]
\[
= \gamma_{n-1} \gamma_{n-2} \varepsilon_{n-1} (M^\phi_t)
\]
\[
= \gamma_{n-1} \varepsilon_n (M^\phi_t)
\]
\[
= \gamma_{n-1} \varepsilon_n (M^\phi_t).
\]

Thus, \( \{ \gamma_n \} \) is a simplicial morphism extending \( \alpha \) and the proof of (i) is complete.

(ii) We shall define a simplicial homotopy \( h : \gamma \to \gamma' \) by induction. We need the following diagram.

We define \( h_0^o \) in the same way as in Theorem II'. Suppose that
\( h_0, \ldots, h_{n-1} \) have been defined and that they satisfy the required conditions. We define \( h_n \) as follows.

Case (i) If \( M_j^\theta \) is non-degenerate in \( N_n \), then we define

\[
\bar{h}_n^i(M_j^\theta) = h_n^i(M_j^\theta),
\]

where \( \bar{h}_n^i \) is defined as in the semi-simplicial case.

Case (ii) If \( M_j^\theta \) is degenerate, say

\[
M_j^\theta = \sigma_{n-1}^k(M_l^\phi).
\]

Then define \( h_n^i(M_j^\theta) \) such that the following diagram is commutative, depending on whether \( k < i \) or \( k > i \); i.e.

\[
u_n^k h_{n-1} = h_n^i \sigma_{n-1}^k, \quad \text{if } k < i;
\]

\[
u_n^{k-1} h_i = h_n^i \sigma_{n-1}^k, \quad \text{if } k > i.
\]

Thus, by induction, we have a definition for \( h_n \) for all \( n \geq 0 \).

It remains to show that \( h_0^n, \ldots, h_n^n \) satisfy conditions (a) to (g) as in definition. By construction, the degenerate \( M_j^\theta \) satisfy conditions (f) and (g), and so conditions (f) and (g) are satisfied.
Also, by Theorem II.11, the non-degenerate $M_j^\theta$ satisfy conditions (a) to (e). Thus, it remains to show that the degenerate $M_j^\theta$ satisfy conditions (a) to (e).

Let $M_j^\theta = \sigma_{n-1}^k(M_t^\phi)$, as before.

(a)  
\[ \partial_{n+1}^o h_n^o (M_j^\theta) = \partial_{n+1}^o h_{n-1}^o \sigma_{n-1}^k (M_t^\phi) \]
\[ = \partial_{n+1}^o u_n^{k+1} h_n^o (M_t^\phi) \]
\[ = u_n^{k} \partial_n^o h_n^o (M_t^\phi) \]
\[ = u_n^{k} f_n (M_t^\phi) \]
\[ = \gamma_n \sigma_{n-1}^k (M_t^\phi) \]
\[ = \gamma_n (M_j^\theta) \]
\[ \therefore \partial_{n+1}^o h_n^o = \gamma_n. \]

(b)  
\[ \partial_{n+1}^o h_n^o (M_j^\theta) = \partial_{n+1}^o h_n^o \sigma_{n-1}^k (M_t^\phi) \]
\[ = \partial_{n+1}^o u_n^{k} h_n^o (M_t^\phi) \]
\[ = u_n^{k} \partial_n^o h_n^o (M_t^\phi) \]
\[ = u_n^{k} \gamma_n (M_t^\phi) \]
\[ = \gamma_n' \sigma_{n-1}^k (M^\phi_t) \]

\[ = \gamma_n' (M^\phi_j) \]

\[ \therefore \, a_{n+1}^n h_n^n = \gamma_n'. \]

(c) We have to show that

\[ a_{n+1}^i h_n^L = h_{n-1}^{L-1} e_n^i, \text{ if } i < L, \text{ on degenerate } M^\theta_j. \]

So let \( M^\theta_j \) = \( \sigma_{n-1}^k (M^\phi_t) \).

Case (i) \( i < k < L \), i.e. when \( L \geq 2 \).

\[ a_{n+1}^i h_n^L (M^\theta_j) = a_{n+1}^i h_n^L (\sigma_{n-1}^k (M^\phi_t)) \]

\[ = a_{n+1}^i h_n^{L-1} (M^\phi_t) \]

\[ = u_{n-1}^{k-1} h_n^{L-2} e_n^i (M^\phi_t) \]

\[ = h_{n-1}^{L-1} e_n^i (M^\phi_t) \]

\[ = h_{n-1}^{L-1} e_n^i (M^\phi_j) \]

\[ = h_{n-1}^{L-1} e_n^i (M^\phi_j). \]
Case (ii) \( k < L \), \( i = k \) or \( i = k + 1 \).

Then

\[
\begin{align*}
\partial_{n+1}^i h_n^L (M_{ij}^\phi) &= \partial_{n+1}^i h_n^L (\sigma_{n-1}^k (M_{ij}^\phi)) \\
&= \partial_{n+1}^i u_n^k h_{n-1}^L (M_{ij}^\phi) \\
&= h_{n-1}^L (M_{ij}^\phi) \\
&= h_{n-1}^L i^k \sigma_{n-1} (M_{ij}^\phi) = h_{n-1}^L i (M_{ij}^\phi).
\end{align*}
\]

Case (iii) \( k < L \) and \( i > k+1 \) (so \( L \geq 2 \) and \( k < L-2 \))

Then

\[
\begin{align*}
\partial_{n+1}^i h_n^L (M_{ij}^\phi) &= \partial_{n+1}^i h_n^L (\sigma_{n-1}^k (M_{ij}^\phi)) \\
&= \partial_{n+1}^i u_n^k h_{n-1}^L (M_{ij}^\phi) \\
&= u_{n-1}^k \partial_n^i h_{n-1}^L (M_{ij}^\phi) \\
&= u_{n-1}^k h_{n-2}^i \epsilon_{n-1} (M_{ij}^\phi) \\
&= h_{n-1}^L i^{k-1} \epsilon_{n-1} (M_{ij}^\phi) \\
&= h_{n-1}^L i^k \epsilon_{n} (M_{ij}^\phi) = h_{n-1}^L i (M_{ij}^\phi).
\end{align*}
\]
Case (iv) \( k \geq \ell \).

Then

\[
\begin{align*}
\delta_{n+1}^i h_n^L (M_j^\theta) & = \delta_{n+1}^i h_n^L (\sigma_{n-1}^k (M_t^\phi)) \\
& = \delta_{n+1}^i u_{n+1}^{k+1} h_n^{L-1} (M_t^\phi) \\
& = u_{n-1}^{k+1} h_{n-1}^L (M_t^\phi) \\
& = u_{n-1}^{k+1} h_{n-2}^{L-1} \epsilon_{n-1} (M_t^\phi) \\
& = h_{n-1}^{L-1} \epsilon_{n-2} (M_t^\phi) \\
& = h_{n-1}^{L-1} \epsilon_{n} (M_t^\phi) \\
& = h_{n-1}^{L-1} \epsilon_{n} (M_t^\phi).
\end{align*}
\]

This completes the proof of (c).

(d) We have to show that

\[
\delta_{n+1}^i h_n^L = \delta_{n+1}^i h_n^{L-1}
\]

on degenerate \( M_j^\theta \).

Case (i) \( k+1 < \ell \).

Then

\[
\begin{align*}
\delta_{n+1}^L h_n^L (M_j^\theta) & = \delta_{n+1}^L h_n^L (\sigma_{n-1}^k (M_t^\phi)) \\
& = \delta_{n+1}^L h_n^L (M_t^\phi).
\end{align*}
\]
Case (ii) \( k+1 = L \).

Then

\[
\begin{align*}
\alpha_{n+1}^L u_n^k h_n^{L-1} (M_\phi^\theta) &= \alpha_{n+1}^L h_n^L (\sigma_{n-1} (M_\phi)) \\
&= \alpha_{n+1}^L u_n^k h_n^{L-1} (M_\phi^\theta) \\
&= \alpha_{n+1}^L u_n^L h_n^{L-1} (M_\phi^\theta) \\
&= \alpha_{n+1}^L u_n^{L-1} h_n^{L-1} (M_\phi^\theta) \\
&= \alpha_{n+1}^L h_n^{L-1} (M_\phi^\theta) \\
&= \alpha_{n+1}^L h_n^{L-1} (M_\phi^\theta) \\
\end{align*}
\]
Case (iii) $k \geq L$.

Then

\[
\begin{align*}
3^L_{n+1} \cdot h^L_n \cdot (M_j^\theta) &= 3^L_{n+1} \cdot h^L_n \cdot (\sigma_{n-1}^k (M_t^\phi)) \\
&= 3^L_{n+1} \cdot u^k+1_n \cdot h^L_{n-1} \cdot (M_t^\phi) \\
&= u^k_{n-1} \cdot 3^L_n \cdot h^L_{n-1} \cdot (M_t^\phi) \\
&= u^k_{n-1} \cdot 3^L_n \cdot h^L_{n-1} \cdot (M_t^\phi) \\
&= 3^L_{n+1} \cdot u^k+1_n \cdot h^L_{n-1} \cdot (M_t^\phi) \\
&= 3^L_{n+1} \cdot h^L_{n-1} \cdot (\sigma_{n-1}^k (M_t^\phi)) \\
&= 3^L_{n+1} \cdot h^L_{n-1} \cdot (M_t^\phi).
\end{align*}
\]

This completes the proof of (d).

(e) We have to show that $3^i_{n+1} \cdot h^L_n = h^L_{n-1} \cdot \sigma^i_{n-1}$, if $i \geq L+1$, on degenerate

\[
M_j^\theta = \sigma_{n-1}^k (M_t^\phi),
\]

as before.
Case (i) \( k < L \) (and so \( k+1 < i \)).

Then

\[
\begin{align*}
\delta_{n+1}^i h_n^L (M_j^\theta) &= \delta_{n+1}^i h_n^L (\sigma_{n-1}^k (M^\phi_t)) \\
&= \delta_{n+1}^i u_n^k h_{n-1}^L (M^\phi_t) \\
&= u_{n-1}^k \delta_n^i h_{n-1}^L (M^\phi_t) \\
&= u_{n-1}^k \delta_n^i h_{n-2} h_{n-1}^L (M^\phi_t) \\
&= h_{n-1}^L \delta_n^i h_{n-2} h_{n-1}^L (M^\phi_t) \\
&= h_{n-1}^L \delta_n^i h_{n-2} (M^\phi_t) \\
&= h_{n-1}^L \delta_n^i h_{n-1} (M_j^\theta).
\end{align*}
\]

Case (ii) \( L+1 < i < k+1 \)

Then

\[
\begin{align*}
\delta_{n+1}^i h_n^L (M_j^\theta) &= \delta_{n+1}^i h_n^L (\sigma_{n-1}^k (M^\phi_t)) \\
&= \delta_{n+1}^i u_n^{k+1} h_{n-1}^L (M^\phi_t) \\
&= u_{n-1}^k \delta_n^i h_{n-1}^L (M^\phi_t) \\
&= u_{n-1}^k \delta_n^i h_{n-2} h_{n-1}^L (M^\phi_t) \\
&= u_{n-1}^k \delta_n^i h_{n-2} h_{n-1}^L (M^\phi_t) \\
&= u_{n-1}^k \delta_n^i h_{n-2} (M^\phi_t).
\end{align*}
\]
Case (iii) \( i = k+1 \) or \( k+2 \)

Then

\[
\begin{align*}
\varepsilon_{n+1}^i h_n^{L} (M_j^\alpha) &= \varepsilon_{n+1}^i h_n^{L} (\sigma_{n-1}^k (M_t^\phi)) \\
&= \varepsilon_{n+1}^{i+k+1} h_n^{L} (M_t^\phi) \\
&= h_n^{L} (M_t^\phi) \\
&= h_n^{L} \varepsilon_n^{i-1} \sigma_{n-1}^k (M_t^\phi) \\
&= h_n^{L} \varepsilon_n^{i-1} (M_j^\alpha).
\end{align*}
\]

Case (iv) \( L+1 < k+2 < i \).

Then

\[
\begin{align*}
\varepsilon_{n+1}^i h_n^{L} (M_j^\alpha) &= \varepsilon_{n+1}^i h_n^{L} (\sigma_{n-1}^k (M_t^\phi)) \\
&= \varepsilon_{n+1}^{i+k+1} h_n^{L} (M_t^\phi) \\
&= \varepsilon_{n+1}^{i+k+1} h_n^{L} (M_t^\phi)
\end{align*}
\]
This completes the proof of (e) and hence also of the comparison theorem!

**Definition 2.4**

A base category \((\mathcal{C}, \mathcal{E})\) is called a **Rinehart category** if

1. a finite number of elements in \(\mathcal{O}\) have a coproduct in \(\mathcal{C}\), where \(\varepsilon \rightarrow \mathcal{O}_j\)
2. morphisms in \(\mathcal{E}\) have kernels in \(\mathcal{C}\);
3. for all commutative diagrams,

\[
\begin{array}{ccc}
K_1 & \rightarrow & B_1 & \rightarrow & A_0 \\
\downarrow \alpha & & \downarrow \beta & & \\
K_0 & \rightarrow & B_0 & \rightarrow & A_0
\end{array}
\]

where \(K_i\) is the kernel of \(B_i \rightarrow A; i = 0, 1, A_1 = A_0;\)
and \( \rightarrow \) denotes an element of \( \mathcal{E} \),

\[
\alpha \in \mathcal{E} \text{ iff } \beta \in \mathcal{E}.
\]

**Example 2.5**

Algebraic categories are Rinehart categories.

**Proof:** It is easy to see that (sets, surjections) is a Rinehart category. If \( \mathcal{U} \) is the underlying set functor for an algebraic category, then \( \mathcal{U} \) preserves limits, and preserves and reflects surjections. Hence, all algebraic categories are Rinehart categories. Observe that by 2.1, Rinehart categories have Andre \( \mathcal{E} \)-resolutions.

Now, we are in a position of defining derived functors from a Rinehart category.

**Definition 2.6**

Let \((\mathcal{C}, \mathcal{E})\) be a Rinehart category. Let \((\mathcal{S}, *)\) be the category of pointed sets. Let \(F: \mathcal{C} \to (\mathcal{S}, *)\) be a functor such that

(i) \(F(1_\mathcal{C}) = *\); and

(ii) if \(g = \{g_n\}: K \to L\) is a simplicial \(\mathcal{C}\)-morphism between augmented simplicial \(\mathcal{E}\)-resolutions \(K\) and \(L\) such that \(g_n \in \mathcal{E}\) for all \(n > 0\), then \(F(g_n)\) is a fibration for all \(n > 0\).

Then we define the derived functors of \(F\) as follows. Let \(A \in \mathcal{C}\) and construct an Andre \(\mathcal{E}\)-resolution \(N \leftarrow \cdots \to A\).
of A.

Define $L^n_{F(A)} = \prod_n (FN)$, $n \geq 0$, see 1.5.

$(FN, \ast)$ is a pointed Kan complex by conditions (i) and (ii) above.

If $f : A \rightarrow B$ is a $\mathcal{C}$-morphism, let $N \xrightarrow{f} B$ be an Andre $\mathcal{E}$-resolution of $B$ and let $\tilde{f} : N \rightarrow N'$ be a simplicial morphism extending $f$. Then define

$$L^n_{F(f)} = \prod_n (\tilde{F})^n, \quad n \geq 0.$$ 

The comparison theorem ensures a valid definition of $L^n_{F}$ for all $n \geq 0$, see Keune (6, page 44 and 45) for details.

**Theorem 2.7**

Let $F : \mathcal{C} \rightarrow (\mathcal{S}, \ast)$ be defined as above. Then,

(i) $L^n_{E}$ vanishes on $\mathcal{E}$-projectives for all $n \geq 1$;

(ii) for all $A \xrightarrow{\alpha} B \in \mathcal{E}$, there exists a long exact sequence of pointed sets

$$\cdots \rightarrow L^n_{F(A)} \rightarrow L^n_{F(B)} \rightarrow L^n_{F(\ker F \alpha)} \rightarrow L^{n-1}_{n-1}$$

$$\xrightarrow{\sim} \rightarrow L^n_{F(A)} \rightarrow \cdots \rightarrow L^n_{F(B)},$$

where $\alpha$ is the simplicial morphism, unique up to simplicial homotopy, extending $\alpha$.

**Proof:** (i) See II.3.9. and use $1_p$ as degeneracies.

(ii) Immediate from 1.6 (i).
From 1.6 (ii), we have group structures on $L^F_n(-)$ for $n \geq 1$, and these give rise to a long exact sequence of groups. We now want to look at this group structure in some detail.

**Definition 2.8**

Let $G$ be a simplicial group with face morphism $\partial$. Then define the Moore chain complex, $MG$, of $G$ as follows.

$$(MG)_0 = G_0 \quad \text{and} \quad (MG)_n = \bigcap_{i=1}^{n} \ker \partial^i_n, \quad n \geq 1.$$  

The differential $\theta_n : (MG)_n \to (MG)_{n-1}$ is defined by,

$$\theta_n = \partial^o_n \bigg|_{(MG)_n}.$$  

**Lemma 2.9**

$(MG)_n$ with differential $\theta_n$, as above, is a chain complex of not necessarily abelian groups.

**Proof:** Firstly, we need to check that $\theta_n$ maps $(MG)_n$ into $(MG)_{n-1}$ and that $\theta_n \circ \theta_{n+1} = 0$. Let $z \in (MG)_n$, then

$$\partial^i_{n-1}(\theta_n z) = \partial^i_{n-1} \partial^o_n z = \partial^o_{n-1} \partial^i_n z = 1,$$

for $i = 1, \ldots, n-1$.  

$$\theta_n z \in (MG)_{n-1}, \quad \text{as required.}$$

Now, let $x \in (MG)_{n+1}$, then
\[ \theta_n \theta_{n+1}(x) = \theta_n^0 \theta_n^0 (x) = \theta_n^0 \theta_{n+1}^1 (x) = 1 \text{ and so} \]

\[ \theta_n \theta_{n+1} = 0. \]

Finally, we have to show that \( \text{Im} \theta_{n+1} \) is a normal subgroup of \( \text{Ker} \theta_n \). If we show that \( Z^{-1} \theta_{n+1}(x). Z \in I \theta_{n+1} \), then \( \text{Im} \theta_{n+1} \) is normal in \((MG)_n\) and so in \( \text{Ker} \theta_n \).

Let \( u = \sigma_n^0 Z^{-1} x. \sigma_n^0 Z \in G_{n+1} \), where \( \sigma \) is the degeneracy morphism. As \((MG)_{n+1}\) is trivially normal in \( G_{n+1} \), being an intersection of kernels, we have \( u \in (MG)_{n+1} \).

But \[ \theta_{n+1} u = \theta_{n+1}^0 \sigma_n^0 Z^{-1} \theta_{n+1}^0 x. \theta_{n+1}^0 \sigma_n^0 Z \]

\[ = Z^{-1} \theta_{n+1}^1(x) Z \]

\[ \therefore \quad Z^{-1} \theta_{n+1}(x). Z \in \text{Im} \theta_{n+1} \text{ and so } \text{Im} \theta_{n+1} \]

is normal in \( \text{Ker} \theta_n \).

We shall now quote an "old" result.

**Proposition 2.10** (See, for example, Keune (6)).

The homotopy groups of a simplicial group coincide with the homology groups of its Moore chain complex.

Thus, if a functor \( F \), as in 2.8, takes values in the category of groups, then to compute \( L^n_F(\_ ) \), we look at the homology groups of \( MF(\_ ) \). We shall do this for a special functor in the next chapter.

We now state the following result, for reference.
Theorem 2.11

Let \((\mathcal{C}, \mathcal{E})\) be a Rinehart category. Let \(F: \mathcal{C} \to \mathcal{G}\) be a functor such that \(F(1_\mathcal{E}) = 1_\mathcal{G}\). Then,

(i) \(L_n^\mathcal{E}F\) vanishes on \(\mathcal{E}\)-projectives for \(n \geq 1\);

(ii) for all \(A \xrightarrow{\alpha} B \in \mathcal{E}\), there exists a long exact sequence of groups,

\[
\ldots \xrightarrow{L_n^\mathcal{E}F} L_n^\mathcal{E}(B) \xrightarrow{L_n^\mathcal{E}(\text{Ker } F \alpha_{n-1})} L_{n-1}^\mathcal{E}F(A) \xrightarrow{\ldots} \]

To end this section we shall briefly compare the semi-simplicial and the simplicial methods by considering functors taking value in groups.

There are at least two places where the semi-simplicial method does not give a valid definition for functors taking value in groups. Firstly, since \(\mathcal{G}\) is not an abelian category, we cannot "measure the difference" between two group homomorphisms. Thus the semi-simplicial comparison theorem does not work. In fact it is the degeneracy maps which rectify the situation in the simplicial case.

Secondly, having defined the Moore chain complex, we need to have the image of a differential as a normal subgroup of the kernel of the next differential. If we do not have degeneracy maps, it is not clear, to me, how one can ensure this normality condition.
§3  Simplicial methods in Rinehart categories.

Lemma 3.1

Let \((\mathcal{C},\mathcal{E})\) be a Rinehart category and consider the commutative diagram in \(\mathcal{C}\),

\[
\begin{array}{ccc}
K_1 \to & A & \to & A \\downarrow & \downarrow & \downarrow \\downarrow & \downarrow & \downarrow \K_0 \to & B & \to & C
\end{array}
\]

where \(K_1 = \text{Ker} (A \times B \to A)\) and \(K_0 = \text{Ker} (B \to C)\).

Then \(\alpha\) is an isomorphism.

Proof: Consider the morphisms \(K_0 \to B \to C\) and the zero morphism \(K_0 \to A\). Hence, by the universal property of the fibre product \(A \times B\), there exists a unique morphism \(\gamma : K_0 \to A \times B\) satisfying the usual conditions. In particular, \(K_0 \to A \times B \to A\) is the zero morphism and so by the universal property of the kernel \(K_1\) we have a unique morphism \(\beta : K_0 \to K_1\) satisfying \((K_0 \to K_1 \to A \times B) = \gamma\).

Then, an easy check shows that \(\alpha \beta = \mathbf{1}_{K_0}\) and \(\beta \alpha = \mathbf{1}_{K_1}\). Thus, \(\alpha\) is an isomorphism.

Proposition 3.2

Let \((\mathcal{C},\mathcal{E})\) be a Rinehart category. Then for every commutative diagram,
Proof: We shall consider two cases.

Case (i) $\gamma = 1_{A_0}$.

Then, by condition (iii) for a Rinehart category, $\alpha \in \mathcal{E}$ iff $\delta \in \mathcal{E}$.

But then $A_0 \times B_0 = B_0$ and so $\delta \in \mathcal{E}$ iff $\beta \in \mathcal{E}$.

Thus, $\alpha \in \mathcal{E}$ iff $\beta \in \mathcal{E}$, as required.

Case (ii) Arbitrary $\gamma$.

Consider the following commutative diagram,

\[
\begin{array}{ccc}
K_1 & \xrightarrow{\alpha} & B_1 \\
\downarrow & & \downarrow \beta \\
K & \xrightarrow{\Theta} & A_1 \\
\downarrow & & \downarrow \gamma \\
K_0 & \xrightarrow{\delta} & A_0 \\
\end{array}
\]

where $K = \text{Ker} (A_1 \times B_0 \rightarrow A_1)$ and $\alpha = \delta \alpha$.

By 3.1, $\Theta$ is an isomorphism and so $(\alpha \in \mathcal{E}$ iff $\beta \in \mathcal{E})$ is equivalent to $(\alpha \in \mathcal{E}$ iff $\beta \in \mathcal{E})$. In particular, we can thus reduce this case to when $\gamma = 1_{A_0}$; i.e. to case (i).

Hence, $\alpha \in \mathcal{E}$ iff $\beta \in \mathcal{E}$ for arbitrary $\gamma$. 

where $K_i = \text{Ker} (B_i \rightarrow A_i)$, $i = 0, 1; \alpha \in \mathcal{E}$ iff $\beta \in \mathcal{E}$. 

Proposition 3.3.

If \((\mathcal{C}, \mathcal{E})\) is a Rinehart category then so is \((\mathcal{C}_n, \mathcal{E}_n)\) for all \(n \geq 0\).

Proof: By induction, it is enough to prove that \((\mathcal{C}_1, \mathcal{E}_1)\) is a Rinehart category. Now, \((\mathcal{C}_1, \mathcal{E}_1)\) is a base category and so we need to check the three conditions in the definition of a Rinehart category.

(i) Let \(P\) and \(Q\) be in \(\mathcal{P}_1\), where \(\mathcal{E}_1 \Rightarrow \mathcal{P}_1\). Then

\[
P = P_0 \rightarrow P_1\text{ and } Q = Q_0 \rightarrow Q_1
\]

for some \(P_0, P_1, Q_0, Q_1\) in \(\mathcal{P}\) and \(\alpha\) and \(\beta\) in \(\mathcal{E}\). Thus we have splittings

\[
\alpha_0 : P_1 \rightarrow P_0 \text{ and } \beta_0 : Q_1 \rightarrow Q_0
\]

such that \(\alpha_0 \alpha = 1_{P_1}\) and \(\beta_0 \beta = 1_{Q_1}\). Now, consider the following diagram,

\[
\begin{array}{c}
P_0 \downarrow \beta_0 \quad \delta \quad \gamma \\
\downarrow \alpha_0 \quad \delta \quad \gamma \\
\end{array}
\]

where \(\gamma\) and \(\delta\) are the canonical morphisms. By their uniqueness, we have \(\gamma \delta = 1_{P_1 \sqcup Q_1}\) and so, by condition (B3) of a base category, \(\gamma \in \mathcal{E}\). Thus, \(P_0 \sqcup Q_0 \rightarrow P_1 \sqcup Q_1\) is an object in \(\mathcal{C}_1\) and it is easy to see that it is the coproduct of \(P\) and \(Q\). This shows that the coproduct of two elements in \(\mathcal{P}_1\) exists in \(\mathcal{C}_1\) and so, by induction, does the coproduct of a finite number of elements in \(\mathcal{P}_1\).

(ii) If \(B = (B_1 \rightarrow B_0)\) and \(A = (A_1 \rightarrow A_0)\) with
B \rightarrow A \in \mathcal{E}_1$, then by 3.2, $K = (K_1 \rightarrow K_0)$ is the kernel of $B \rightarrow A$. Thus, morphisms in $\mathcal{E}_1$ have kernels in $\mathcal{E}_1$.

(iii) Consider the following commutative diagram in $\mathcal{E}_1$,

\[
\begin{array}{ccc}
K & \xrightarrow{\alpha} & A \\
\downarrow & & \downarrow \beta \\
K' & \xrightarrow{\gamma} & A'
\end{array}
\]

where $K = \text{Ker}(A \rightarrow B)$, $K' = \text{Ker}(A' \rightarrow B)$ and $\rightarrow$ denotes an element in $\mathcal{E}_1$.

Also, let $A = (A_0 \rightarrow A_1)$, $B = (B_0 \rightarrow B_0')$, $A' = (A_1 \rightarrow A_1')$, $K = (K_0 \rightarrow K_0')$ and $K' = (K_1 \rightarrow K_1')$, where $\rightarrow$ denotes an element of $\mathcal{E}$. Then we can construct the following diagram in $\mathcal{C}$,
where \( \alpha = (\alpha_0, \alpha_1), \beta = (\beta_0, \beta_1), K_2 \) is the kernel of \( \beta_0, K'_2 \) is the kernel of \( \beta_1, K_3 \) is the kernel of \( K_2 \rightarrow \mathbb{1} \), \( K'_3 \) is the kernel of \( K'_2 \rightarrow \mathbb{1} \), \( \beta_0, \beta_1 \in \mathcal{C} \) since \( A \rightarrow B \in \mathcal{C} \), and \( \alpha_0, \alpha_1 \in \mathcal{C} \) since \( (\mathcal{C}, \mathcal{E}) \) is a Rinehart category.

Now, \( \beta \in \mathcal{C} \) iff \( \bar{\beta} \in \mathcal{C} \) iff \( \bar{\beta} \in \mathcal{C} \), using 3.2; iff \( \bar{k}_2 \in \mathcal{C} \); iff \( \bar{\alpha} \in \mathcal{C} \), using 3.2; iff \( \alpha \in \mathcal{C} \).

This completes the proof of the proposition.

**Proposition 3.4**

Let \((\mathcal{C}, \mathcal{E})\) be a Rinehart category and define \((\mathcal{C}_n, \mathcal{E}_n)\) as before for all \( n \geq 0 \). Let \( A \xrightarrow{\alpha} B \) be an object in \( \mathcal{C}_m \) for some \( m \geq 1 \), and let \( (\mathcal{P} \xrightarrow{\alpha} \mathcal{Q}) \rightarrow (A \xrightarrow{\alpha} B) \) be an Andre \( \mathcal{E}_m \) resolution of \( \alpha \). Then \( \mathcal{P} \rightarrow A \) and \( \mathcal{Q} \rightarrow B \) are Andre \( \mathcal{E}_{m-1} \) resolutions for \( A \) and \( B \) respectively.

**Proof:** This is obvious, especially since we already have such a result in the semi-simplicial case.

**Theorem 3.5**

Let \((\mathcal{C}, \mathcal{E})\) be a Rinehart category and define \((\mathcal{C}_n, \mathcal{E}_n)\) as before for all \( n \geq 0 \).

(a) Let \( F : \mathcal{C} \rightarrow (\mathcal{G}, \ast) \) be a functor as in 2.6 and
define $F_n : E_n \rightarrow S$ as before for all $n \geq 0$. Then,

(i) $L_n F_m$ vanishes on $E_m$ projectives for all $n \geq 1, m \geq 0$;

(ii) if $A \xrightarrow{\alpha} B \in E_m$, $m \geq 1$, then there exists a long exact sequence of pointed sets,

$$\ldots L_{n-1} F_1(A) \rightarrow L_{n-1} F_1(B) \rightarrow L_{n-1} F_1(\alpha) \rightarrow L_{n-1} F_1(A) \rightarrow \ldots$$

\[ \xrightarrow{\ldots} L_{o-1} F_1(A) \rightarrow \ast. \]

(b) Let $F : E \rightarrow E$ be a functor as in 2.13, and define

$F_n : E_n \rightarrow E_n$ as before for all $n \geq 0$. Then,

(i) $L_n F_m$ vanishes on $E_m$ projectives for all $n \geq 1, m \geq 0$;

(ii) if $A \xrightarrow{\alpha} B \in E_m$, $m \geq 1$, then there exists a long exact sequence of groups,

$$\ldots L_{n-1} F_1(A) \rightarrow L_{n-1} F_1(B) \rightarrow L_{n-1} F_1(\alpha) \rightarrow L_{n-1} F_1(A) \rightarrow \ldots$$

\[ \xrightarrow{\ldots} L_{o-1} F_1(B). \]

**Proof:**

(a) Put together 2.7, 3.3, and 3.4.

(b) Put together 2.11, 3.3 and 3.4.

**Definition 3.6**

Let $F : E \rightarrow (S, \ast)$ be a functor defined as in 2.6.

Then $F$ is called simplicially exact if $L_0 F = F$. (or simplicially right exact)
Proposition 3.7

Let $F: \mathcal{C} \to (\mathcal{F}, \ast)$ be a functor defined as in 2.6.

Then,

(i) $L_0 F(P) = F(P)$ and $L_n F(P) = \ast$ for all $P \in \mathcal{F}$, $n \geq 1$;

(ii) $L_0 F$ is simplicially exact.

Proof: (i) We consider the simplicial $\mathcal{E}$-resolution of $P$ consisting of $P$ in every dimension and the identity as the face and the degeneracy maps.

Then, by definition

$$L_0 F(P) = \frac{F(P)}{\ast} = F(P)/\ast = F(P),$$

and

$$L_n F(P) = \frac{F(P)}{\ast} = \ast/\ast = \ast$$ for $n \geq 1$,

as required.

(ii) By (i), since $L_0 F$ and $F$ agree on $\mathcal{E}$-projectives, we have

$$L_0 (L_0 F) = L_0 F$$ and so $L_0 F$ is simplicially exact.

Proposition 3.8

Let $F: \mathcal{C} \to (\mathcal{F}, \ast)$ be defined as in 2.6 and define

$F_m: \mathcal{C}_m \to (\mathcal{F}, \ast)$ for all $m \geq 0$.

Then,

(A1) $L_0 F_m(P) = F_m(P)$ for all $P \in \mathcal{F}_m$ and $m \geq 0$;
(A2) $L_{n_0}F_m$ is simplicially exact for all $m \geq 0$;

(A3) $L_{n,m}F(P) = \ast$ for all $P \in \mathcal{C}_m$ and $m \geq 0$, $n \geq 1$;

(A4) if $A \xrightarrow{\alpha} B \in \mathcal{C}_m$, $m \geq 0$, then there exists a long sequence of pointed sets,

$$\cdots \xrightarrow{L_{n+1,m}F(B)} L_{n,m+1}F(\alpha) \xrightarrow{L_{n,m}F(A)} L_{n,m}F(B) \xrightarrow{L_{n,m}F(B)} \cdots$$

Furthermore, these four conditions characterize the functors $L_{n,m}F$ for all $n \geq 0$, $m \geq 0$.

**Proof:** By 3.5 and 3.7 we know that $L_{n,m}F$ satisfy conditions (A1) to (A4). Suppose we are given functors $S_{n,m} : \mathcal{C}_m \rightarrow (S, \ast)$ which also satisfy the conditions (A1) to (A4). Then we shall show, by induction on $n$, that $L_{n,m}F = S_{n,m}$.

For $n = 0$, since $L_{0,m}F$ is simplicially exact, we have $L_{0,m}F = F_m$. But $S_{0,m}$ is also simplicially exact and it agrees with $F_m$ on $\mathcal{C}_m$-projectives. Hence, $S_{0,m} = F_m$.

For $n > 0$, assume by an inductive hypothesis that $L_{n-1,m}F = S_{n-1,m}$ for all $m \geq 0$. Let $A \in \mathcal{C}_m$ and $P \xrightarrow{\alpha} A \in \mathcal{C}_m$ with $P \in \mathcal{C}_m$. By conditions (A1) (A3) and (A4) and the case $n = 0$, we obtain two long exact natural sequences in pointed sets.
Thus, \( L_n F_m (A) = S_{n,m} (A) \) for all \( m \geq 0 \). Hence, by induction we have \( L_n F_m = S_{n,m} \) for all \( n \geq 0, m \geq 0 \).

**Corollary 3.9**

The simplicial and the semi-simplicial methods of defining derived functors agree whenever both theories apply.

**Proof:** Using II.3.12 and 3.8 it is enough to check that the two conditions (A2) agree. Now, \( F \) is simplicially right exact iff \( L_0 F = F \) iff \( F \) is simplicially exact. Thus, the two conditions (A2) agree, as required.

**Note:** The above theory generalizes Rinehart's theory and, by our characterization, agrees with Rinehart's whenever both theories apply.
CHAPTER IV

BAER INVARIANTS AS DERIVED FUNCTORS

In this chapter, we define the higher Baer invariants of a group relative to a variety of groups, although the definition applies more generally, to \( \Omega \)-groups for instance.

In §1, we give some basic group theoretic definitions and define the higher Baer invariants. We compute the first Baer invariant and show that our definition agrees with Frohlich's definition.

In §2, we consider the second Baer invariant. We prove that the second Baer invariant of a group is a homomorphic image of the first Baer invariant of a certain fibre product. Finally, we obtain an expression for the second Baer invariant of a group when the variety is the variety of abelian groups; i.e. a kind of "Hopf formula" for \( H_3(G, \mathbb{Z}) \) in terms of presentations

\[
1 \longrightarrow R_0 \longrightarrow F_0 \longrightarrow G \longrightarrow 1 \quad \text{and} \quad 1 \longrightarrow R_1 \longrightarrow F_1 \longrightarrow F_0 \otimes_{\mathbb{Z} G} F_0 \longrightarrow 1.
\]
§1 Definitions and the first Baer invariant.

We shall state some basic definitions by using notations as given in Hanna Neumann (10). A variety is a class of groups closed under the formation of subgroups, quotient groups and Cartesian products. Let $X_\infty$ denote the group freely generated by \{ $x_1, x_2, \ldots$ \}. If $V$ is a variety, $V(X_\infty)$ denotes the intersection of the kernels of all homomorphisms of $X_\infty$ into all groups in $V$. If $G$ is any group, then $V(G)$ denotes the union of all images of $V(X_\infty)$ under all homomorphisms of $X_\infty$ into $G$.

Then it can be easily shown that $V(G)$ is a fully invariant subgroup of $G$ and that $G \in V$ iff $V(G) = 1_G$. We call $V(G)$ the verbal subgroup of $G$ (with respect to the variety $V$). If $n \geq 1$, the subgroup of $X_\infty$ generated by \{ $x_1, \ldots, x_n$ \} is denoted by $X_n$; any element of $X_\infty$ is called a word and an element of $X_n$ is an $n$-letter word. Any element of $V(X_\infty)$ is called a law of $V$, and a law which is also an $n$-letter word is an $n$-letter law of $V$.

If $\mathcal{G}$ denotes the category of groups, then $V: \mathcal{G} \rightarrow \mathcal{G}$ defined by $V: G \rightarrow V(G)$ for all $G \in \mathcal{G}$, is an endofunctor on $\mathcal{G}$ and is called the verbal subgroup functor (with respect to the variety $V$). If we define $U: \mathcal{G} \rightarrow \mathcal{G}$ by $U: G \rightarrow G/V(G)$ for all $G \in \mathcal{G}$, then we get another endofunctor, called the verbal quotient group functor.
Definition 1.1

Let $G$ be a group and $V$ a variety. If $U: G \rightarrow G$ is the verbal quotient group functor, then the $n$th Baer invariant of $G$ with respect to $V$ is defined to be $\lambda_n U(G)$ for all $n \geq 1$.

Since the functor $U$ may not take values in an abelian category, we have to use the simplicial method to define derived functors, as in Chapter III. Also, this definition immediately gives us the fact that our definition of the Baer invariant of a group is independent of the presentation of the group. For the rest of this section, we shall work towards the computation of the first Baer invariant of a group.

Proposition 1.2 (André (1))

Let $\mathcal{E}$ be the class of all epimorphisms in the category of groups. Then for a group $G$ we can construct a simplicial $\mathcal{E}$-resolution $F \xrightarrow{\sigma} G$ which begins as follows.

$$
\cdots \xrightarrow{} F_2 = F_1 \ast \widehat{R}_0 \ast \overline{R}_1 \xrightarrow{\sigma} F_1 \xrightarrow{} F_0 \xrightarrow{} G \xrightarrow{} \cdots
$$

where $1 \rightarrow R_0^F \rightarrow F_0 \rightarrow G \rightarrow 1$ is a free presentation for $G$, $1 \rightarrow R_1^F \rightarrow F_1 \rightarrow F_0 \times_{G} F_0 \rightarrow 1$ is a free presentation for the fibre product $F_0 \times_{G} F_0$; $\widehat{R}_0$ and $\overline{R}_1$ are isomorphic copies of $R_0$ and $R_1$ is an isomorphic copy of $R_1$ and the face maps are given as follows.
Proof: We use the "step-by-step" method to obtain the above resolution. We can also describe how the degeneracy maps act, but they are not needed for our calculations. For explicit details of the above resolution, see, for example, Johnson (5).

We shall now prove a statement, the first two parts of which have been proved by Fröhlich (2) in greater generality.

Proposition 1.3

Let \( V \) be a variety and \( V \) and \( U \) the verbal subgroup and the verbal quotient group functors respectively. Then, for any group \( G \),

(i) \( L_0 U(G) = U(G) \);

(ii) there exists a natural exact sequence,

\[
1 \longrightarrow L_1 U(G) \longrightarrow L_0 V(G) \longrightarrow V(G) \longrightarrow 1
\]
(iii) $L_{n+1}U = L_nV$ as functors for all $n > 1$.

Proof: (i) We can prove this by a "general nonsense" proof, but we shall do it by an explicit method to get some insight into the detailed explanation. We shall first show that $U$ is a right exact functor. Let $1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$ be an exact sequence of groups, then we have to show that $U(A) \rightarrow U(B) \rightarrow U(C) \rightarrow 1$ is also an exact sequence.

As $\beta : B \rightarrow C$ is a surjection by the definition of $U$, $\beta^* = U(\beta)$ is also a surjection. Thus, the sequence is exact at $U(C)$. Now, for $\overline{a} = a \cdot V(A) \in U(A)$,

$$(\beta^* a^*)(\overline{a}) = (\beta a)(a) \cdot V(C) = V(C) = 1,$$ where $a^* = U(a)$.

$\therefore$ Im $a^* \subseteq$ Ker $\beta^*$.

Conversely, let $\overline{b} = b \cdot V(B) \in \ker \beta^*$ and so $\beta(b) \in V(C)$. As $\beta$ is a surjection, so is $\beta \mid_{V(B)} : V(B) \rightarrow V(C)$.

$\therefore \exists b_1 \in V(B)$ with $\beta(b_1) = \beta(b)$.

$\therefore b_1^{-1} \in \ker \beta = \text{Im } a$, by hypothesis.

$\therefore \exists a_1 \in A$ with $a(a_1) = b_1^{-1}$.

Finally, $a^*(a_1 \cdot V(A)) = b_1^{-1} V(B) = b \cdot V(B) = \overline{b}$

and so Ker $\beta^* \subseteq$ Im $a^*$. 
Thus the sequence is also exact at $U(B)$ and so $U$ is a right exact functor. Now, by definition of $L_0 U$ and 1.2,

$$L_0 U(G) = \frac{U(F_0)}{(U_{\mathcal{E}_1})(\text{Ker } (U_{\mathcal{E}_1}))}$$

$$= \frac{U(F_0)}{(U_{\mathcal{E}_1})(U(R_{\mathcal{O}}^1))}$$

But, as $1 \longrightarrow R_{\mathcal{O}}^0 \longrightarrow F_0 \longrightarrow G \longrightarrow 1$ is an exact sequence and $U$ is right exact, we deduce that $L_0 U(G) = U(G)$.

(ii) We have an exact sequence of functors, $* \longrightarrow V \longrightarrow I \longrightarrow U \longrightarrow *$, where $I$ is the identity functor and $*$ is the zero functor. This yields a long exact sequence of derived functors,

$$\cdots L_{n+1}I \longrightarrow L_{n+1}U \longrightarrow L_n V \longrightarrow L_n I \longrightarrow \cdots$$

$$\cdots L_1I \longrightarrow L_1U \longrightarrow L_0V \longrightarrow L_0I \longrightarrow L_0U \longrightarrow *.$$
But $L_0 I = I$ and $L_1 I = \ast$, and so

$$
\ast \rightarrow L_1 U \rightarrow L_0 V \rightarrow I \rightarrow L_0 U \rightarrow \ast
$$

is exact.

But $L_0 U = U$ by part (i), and so

$$
\ast \rightarrow L_1 U \rightarrow L_0 V \rightarrow V \rightarrow \ast
$$

is exact, i.e.

$$
1 \rightarrow L_1 U(G) \rightarrow L_0 V(G) \rightarrow V(G) \rightarrow 1
$$

is an exact sequence of groups.

(iii) From the long exact sequence in (ii), since $L_n I = \ast$ for all $n > 1$, we deduce that

$$
L_{n+1} U = L_n V, \quad n \geq 1.
$$

**Definition 1.4**

Let $V$ be a variety with laws $V$. Then for any group $G$, the $V$-marginal subgroup of $G$ is the subset $V^*(G)$ of $G$ forming the subgroup

$$
V^*(G) = \left\{ g \in G : v(g_1, \ldots, g_s, \ldots, g_s) = v(g_1, \ldots, g_s), \quad \text{for all integers } S; \quad g_j \in G, \ j = 1, \ldots, s; \quad i = 1, \ldots, s \quad \text{and} \quad v \text{ an } S\text{-letter law.} \right\}
$$

With abuse of terminology, we shall call $V^*(G)$ the marginal subgroup of $G$ (with respect to the variety $V$).
Definition 1.5

Let \( V \) be a variety with laws \( V \). Let \( N \) be a normal subgroup of a group \( G \). Then the \( V \)-marginalizer subgroup of \( N \) in \( G \), \( V(N, G) \), is the subgroup of \( G \) generated by the set

\[
\left\{ v(g_1, \ldots, g_1^n, \ldots, g_s) v(g_1, \ldots, g_s)^{-1} : n \in N; \ g_i \in G; \right. \\
l, j = 1, \ldots, s; \ v \in V \text{ and } v \text{ an } s\text{-letter law for } \ S \in N \right\}.
\]

We shall now prove a standard result about \( V(N, G) \).

Proposition 1.6

Let \( V(N, G) \) be defined as above. Then,

(i) \( V(N, G) \) is the largest normal subgroup of \( G \) such that for every homomorphism \( \phi : G \rightarrow H \) with \( \phi(N) \subseteq V^*(H) \), \( V(N, G) \subseteq \text{Ker } \phi \);

(ii) \( V(N, G) \) is the least normal subgroup of \( G \) such that

\[
\frac{N \cdot V(N, G)}{V(N, G)} \subseteq V^* \left( \frac{G}{V(N, G)} \right);
\]

(iii) \( [N, V(G)] \subseteq V(N, G) \subseteq N \cap V(G) \), where \( [X, Y] \) denotes the commutator subgroup of the groups \( X \) and \( Y \).

Proof: (i) To see that \( V(N, G) \) is a normal subgroup of \( G \), observe that for \( g \in G \), \( n \in N \) and \( v \in V \),

\[
g^{-1} (v(g_1, \ldots, g_1^n, \ldots, g_s) v(g_1, \ldots, g_s)^{-1}) g \quad \text{equals}
\]

\[
v(g_1^g, \ldots, g_1^{n^g}, \ldots, g_s^g) v (g_1^g, \ldots, g_s^g)^{-1},
\]
and as \( N \) is normal in \( G \), we can deduce that \( V(N, G) \) is also normal in \( G \). Now, it is easy to see that \( V(N, G) \) satisfies the required property. Suppose a normal subgroup \( K \) of \( G \) also satisfies the same property. Then consider the natural homomorphism
\[
\pi: G \longrightarrow \frac{G}{V(N, G)}.
\]
Then, \( \pi(N) = \frac{N \cdot V(N, G)}{V(N, G)} \leq V^* \left( \frac{G}{V(N, G)} \right) \) and so \( K \leq \text{Ker } \pi = V(N, G). \therefore V(N, G) \) is the largest normal subgroup of \( G \) satisfying the required property.

(ii) From (i) we have \( \pi(N) \leq V^* \left( \frac{G}{V(N, G)} \right) \). Suppose \( L \) is a normal subgroup of \( G \) with \( \frac{NL}{L} \leq V^* \left( \frac{G}{L} \right) \).

Then, \( v(g_1, \ldots, g_n, \ldots, g_s)L = v(g_1, \ldots, g_s)L \) for all \( v \in V \), \( n \in N \), \( g_1, \ldots, g_s \in G \).

\( \therefore V(N, G) \leq L \) and so \( V(N, G) \) is the least normal subgroup of \( G \) with the given property.

(iii) Since \( N \) is a normal subgroup of \( G \) and \( N/N \leq V^* (G/N) \), by (ii) we can deduce that \( V(N, G) \leq N \). Also, it is obvious from the definition that \( V(N, G) \leq V(G) \). \( \therefore V(N, G) \leq N \cap V(G) \).

For the first inclusion, since
\[
\frac{N}{V(N, G)} \leq V^* \left( \frac{G}{V(N, G)} \right)
\]
and using the easy fact that
\[
[V^*(H), V(H)] = 1 \text{ for any group } H, \text{ see P.Hall (13)},
\]
we deduce that
\[
\left[ \frac{N}{V(N,G)} , \ V\left( \frac{G}{V(N,G)} \right) \right] = 1.
\]

But \[ V\left( \frac{G}{V(N,G)} \right) = \frac{V(G)}{V(N,G)} \] since \( V(N,G) \subseteq V(G) \).

\[
\therefore \ [N, V(G)] \subseteq V(N,G), \text{ as required.}
\]

**Examples 1.7**

(i) If \( V \) is the variety of abelian groups and \( N \) is a normal subgroup of a group \( G \), then \( V(N,G) = \left[ N; G \right] \).

(ii) If \( V \) is the variety of all nilpotent groups of class at most \( c \), then \( V(N,G) = \left[ N, G, \ldots, G \right] \) \( c \) times.

**Proof:** Routine, see Leedham-Green and McKay (7).

We now prove two technical lemmas which we need to find an expression for the first Baer invariant.

**Lemma 1.8**

Let \( R \) and \( S \) be free groups and \( F \) their free product.

Then, for any variety \( V \),

\[ R^F \cap V(F) = V(R^F,F). \]

**Proof:** By 1.6 (iii), \( V(R^F,F) \subseteq R^F \cap V(F) \).

Conversely, let \( y \in R^F \cap V(F) \).
As \( y \in V(F) \), we can express \( y \) as a product,

\[
y = \prod_{s=1}^{n} v_s (f_{s_1}, \ldots, f_{s_t})
\]

\( v_s \in V \),

\( f_{s_k} \in F \)

As \( F = R \ast S \), we can also write \( y \) as a word, \( y = w(r_{j_i}, s_{i_j}) \), \( r_j \in R \), \( s_i \in S \), \( i_j \geq 0 \), and where

\[
w(x_{11}, x_{12}, \ldots, x_{it}, x_{21}, x_{31}, \ldots, x_{nt})
\]

\[
= \prod_{s=1}^{n} v_s (x_{s_1}, \ldots, x_{s_t})
\]

\( v_s \in V \).

Since \( v_s \) is a law for \( V \) for \( s = 1, \ldots, n \); so is \( w \).

Define a homomorphism \( \beta : F \longrightarrow F \) by

\[
\beta | R = 1 \quad \text{and} \quad \beta | S = 1_S, \quad \text{the identity morphism on } S.
\]

Then, using the above expression for \( y \),

\[
y \in R^F \cap V(F) \iff y = w(r_{j_i}, s_{i_j}) \quad \text{and} \quad w(1, \ldots, 1, s_i) = 1
\]

Thus we can write

\[
y = w(r_{j_i}, s_{i_j}) w(1, \ldots, 1, s_i)^{-1} \in V(R^F) \quad \text{as}
\]
w is a law for \( \mathcal{V} \).

\[ \therefore R^F \cap V(F) = V(R^F, F), \text{ as required.} \]

**Lemma 1.9**

Let \( R \) be a subgroup of a free group \( F \) and let \( \overline{R} \) be an isomorphic copy of \( R \). Let \( Y \) be a free group and put \( X = F \ast \overline{R} \ast Y \). Let \( \theta : X \to F \) be a surjection such that \( \theta(\overline{R}) = R \). Then,

\[ \theta(R^X \cap V(X)) = V(R^F, F) \text{ for any variety } \mathcal{V}. \]

**Proof:** \[ \theta(R^X \cap V(X)) = \theta(V(R^X, X)), \text{ using 1.8.} \]

\[ \quad = V(\theta(R^X), \theta(X)) \]

\[ \quad = V(R^F, F), \text{ as required.} \]

**Theorem 1.10**

Let \( G \) be a group and \( 1 \to R_0 \to F_0 \to G \to 1 \) a free presentation of \( G \). Then the first Baer invariant of \( G \) with respect to a variety \( \mathcal{V} \), is given by

\[ \frac{R_0 \cap V(F_0)}{V(R_0, F_0)}. \]

**Proof:** By definition, if \( U \) is the verbal quotient group functor associated to \( \mathcal{V} \), then the first Baer invariant of \( G \) is given by \( L_1 U(G) \).
Using 1.3 (ii), we get $L_1 U(G) = \ker(L_0 V(G) \to V(G))$.

We shall compute $L_0 V(G)$ by using the explicit resolution given in 1.2.

By definition, $L_0 V(G)$ equals

$$\frac{V(F_0)}{\text{Im}(V_{\varepsilon_1}) \cap \ker(V_{\varepsilon_1})}.$$ 

Now, if $V$ is the verbal subgroup functor, $V_{\varepsilon_1} = \varepsilon_1|_{V(F_1)}$ and $\ker V_{\varepsilon_1} = \ker \varepsilon_1 \cap V(F_1)$

$$= R_0 \cap V(F_1).$$

\[\therefore \text{Im}(V_{\varepsilon_1} \cap \ker(V_{\varepsilon_1})) = \varepsilon_1(R_0 \cap V(F_1))
= V(R_0^{o}, F_0), \text{ using 1.9.}\]

\[\therefore L_0 V(G) = \frac{V(F_0)}{V(R_0^{o}, F_0)}\]

\[\therefore L_1 U(G) = \ker \left( \frac{V(F_0)}{V(R_0^{o}, F_0)} \to V(G) \right)
= \frac{R_0 \cap V(F_0)}{V(R_0^{o}, F_0)}, \text{ as required.}\]

**Note:** Lemma 1.8 is now just the easy fact that the first Baer invariant of the free group $S$ is trivial. We deduce two corollaries, one of which is easy to prove by a direct method and the other is well known.
Corollary 1.11

\[
\frac{R \cap V(F)}{V(R,F)}
\]

is independent of the presentation for any variety \( V \), and any presentation \( 1 \to R \to F \to G \to 1 \) of a group \( G \).

Corollary 1.12

The Schur multiplier \( \frac{R \cap [F,F]}{[R,F]} \) of a group \( G \), where \( 1 \to R \to F \to G \to 1 \) is a free presentation of \( G \), is independent of the presentation.

Proof: Choose \( V \) to be the variety of abelian groups. Then \( V(F) = [F,F] \) and \( V(R,F) = [R,F] \).

Note: From 1.11 we can deduce that our definition of the first Baer invariant agrees with Frohlich's definition. See Leedham-Green and McKay (7) for a group theoretic version of Frohlich's definition.

§2. The Second Baer Invariant

In this section we shall try to compute the second Baer invariant of a group. I have not been able to compute it in general but we prove a result which links the second Baer invariant of a group to the first Baer invariant of a certain fibre product.

Now, by definition, the second Baer invariant of a group...
G, with respect to a variety $V$, is given by $L_2 U(G)$, where $U$ is the verbal quotient group functor associated to $V$.

By 1.3 (iii), $L_2 U(G) \cong L_1 V(G)$, where $V$ is the verbal subgroup functor associated to $V$. Thus, we shall try to compute $L_1 V(G)$.

**Theorem 2.1**

Let $G$ be a group and $1 \rightarrow R_0 \rightarrow F_0 \rightarrow G \rightarrow 1$ a free presentation of $G$. Then, for any variety $V$, the second Baer invariant of $G$ is a homomorphic image of the first Baer invariant of the fibre product $F_0 \times F_0$.

**Proof:** By the remarks above, we shall consider $L_1 V(G)$.

Using the resolution in 1.2,

$$L_1 V(G) = \frac{\text{Ker} \left( V_{\varepsilon_1}^0 \right)}{\text{Ker} \left( V_{\varepsilon_1}^1 \right)} \bigg/ \frac{\text{Im} \left( V_{\varepsilon_2}^0 \right)}{\text{Ker} \left( V_{\varepsilon_2}^1 \right) \cap \text{Ker} \left( V_{\varepsilon_2}^2 \right)}$$

$$= \frac{\text{Ker} \ v_{\varepsilon_1}^0 \cap \text{Ker} \ v_{\varepsilon_1}^1 \cap V(F_1)}{v_{\varepsilon_2}^0 \ (\text{Ker} v_{\varepsilon_2}^1 \cap \text{Ker} v_{\varepsilon_2}^2 \cap V(F_2))}$$

$$= \frac{R_1 \cap V(F_1)}{D}, \quad \text{say, where}$$

$$1 \rightarrow R_1 \rightarrow F_1 \rightarrow F_0 \rightarrow 1$$

is an exact sequence, for
then \( \text{Ker} \varepsilon_1^0 \cap \text{Ker} \varepsilon_1^1 = F_1 \).

Now, \( D = \varepsilon_2^0(\text{Ker} \varepsilon_2^1 \cap \text{Ker} \varepsilon_2^2 \cap V(F_2)) \)

\[ \supseteq \varepsilon_2^0 \left( R_1^2 \cap V(F_2) \right), \text{ by the definitions of } \varepsilon_2^1 \text{ and } \varepsilon_2^2. \]

\[ = V \left( R_1^1, F_1 \right), \text{ by lemma 1.9} \]

\[ \therefore \frac{F_1^1 \cap V(F_1)}{V(R_1^1, F_1)} \text{ maps homomorphically onto } L_1V(G) \]

But as \( 1 \rightarrow R_1^1 \rightarrow F_1 \rightarrow F_0 \times F_0 \rightarrow G \rightarrow L \rightarrow 1 \) is a free presentation of the fibre product,

\[ L_1U(F_0 \times F_0) \rightarrow L_2U(G), \text{ as required.} \]

We now want to compute the kernel of the surjection obtained in 2.1. To do this we need some lemmas.

**Lemma 2.2**

Let \( w \) be an \( n \) letter word in the alphabet \( \{ x_1, x_2, \ldots \} \), say \( w(x_1, \ldots, x_n) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_t}^{\varepsilon_t} \), where \( \varepsilon_j = \pm 1, \)

\( j = 1, \ldots, t \) and \( x_{ij} \in \{ x_1, \ldots, x_n \} \).

Define an \( n \) letter word \( w' \) by;

\[ w'(x_1, \ldots, x_n) = x_{i_t}^{\varepsilon_t} x_{i_{t-1}}^{\varepsilon_{t-1}} \ldots x_{i_1}^{\varepsilon_1}. \] Then
(i) \((w(x_1^{-1}, \ldots, x_n^{-1}))^{-1} = w'(x_1, \ldots, x_n)\);

(ii) If \(G\) is a group in which \([r_k s_k, t_t] = 1\) for some set of elements \(\{r_k, s_k\}\) in \(G\), \(k, t_t = 1, \ldots, n\); then

\[w(r_1 s_1, r_2 s_2, \ldots, r_n s_n) = w'(r_1 s_1, r_2 s_2, \ldots, r_n s_n)w(s_1, s_2, \ldots, s_n)\]

Proof: (i) By definition,

\[
(w(x_1^{-1}, \ldots, x_n^{-1}))^{-1} = (x_1^{-e_1} x_2^{-e_2} \ldots x_t^{-e_t})^{-1} = x_t^{-e_t} \ldots x_2^{-e_2} x_1^{-e_1} = w'(x_1, \ldots, x_n).
\]

(ii) \[w(r_1 s_1, r_2 s_2, \ldots, r_n s_n) = (r_1 s_1)^{e_1} (r_2 s_2)^{e_2} \ldots (r_t s_t)^{e_t} = r_t^{e_t} (r_1 s_1)^{e_1} (r_2 s_2)^{e_2} \ldots (r_t s_t)^{e_t} t_t \]

using the results \([x^{-1}, y] \equiv [y, x]^{-1}\) and \([x^{-1}, y^{-1}] \equiv [x, y]^{(xy)^{-1}}\) for the hypothesis in (ii).
Let $\varepsilon_1$ and $\varepsilon_2$ be defined as in 1.2 and let

$$M = \left\{ \left( \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \end{array} \right) \mid 1 \neq \varepsilon \in \hat{R}_0, \quad \varepsilon \mapsto \gamma \right\}$$

under the natural isomorphism between $\hat{R}_0$ and $\hat{R}_0^\ast$, $f, g \in F_0$, $s \in R_0^\ast$.

then $\text{Ker} \varepsilon_1 \cap \text{Ker} \varepsilon_2 = M$.

**Proof:** By the definition of $\varepsilon_2$, we deduce that

$$\text{Ker} \varepsilon_2^1 = \left\{ \left( \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \end{array} \right) \mid 1 \neq \varepsilon \in \hat{R}_0, \quad \varepsilon \mapsto \gamma \right\}$$

under the natural isomorphism between $\hat{R}_0$ and $\hat{R}_0^\ast$, $f \in F_2$.

Now, $F_2 = F_0 \ast \hat{R}_0 \ast \hat{R}_0^\ast \ast \hat{R}_1$ and we claim that we do not need conjugation of $(\varepsilon \gamma^{-1})$ by the whole of $F_2$ and it is sufficient to allow conjugation by the elements in $F_0 \ast \hat{R}_1$.

To see this, let
Then, \((\bar{\tau}^{-1})\bar{\tau} \cdot \bar{\tau}^{-1}\) = \((\bar{\tau}^{-1})\bar{\tau} \cdot \bar{\tau}^{-1}\) which is a product of generators already included.

Similarly, \((\bar{\tau}^{-1})\bar{\tau} \cdot \bar{\tau}^{-1}\) = \((\bar{\tau}^{-1})\bar{\tau} \cdot \bar{\tau}^{-1}\)

which is also a product of generators already included.

By the definition of \(\varepsilon_2\), we deduce that \(\text{Ker } \varepsilon_2 = \frac{F_2}{R_1}\), \((\bar{\tau}^{-1})\bar{\tau} \cdot \bar{\tau}^{-1}\) for \(\bar{\tau} \in R_0\).

Then it is easy to see that \(M \subseteq \text{Ker } \varepsilon_2 \cap \text{Ker } \varepsilon_2\).

Conversely, let \(x \in \text{Ker } \varepsilon_2 \cap \text{Ker } \varepsilon_2\).

As \(x \in \text{Ker } \varepsilon_2\) we can write

\[x = w(\bar{\tau}_{11}, \bar{\tau}_{12}, \ldots, \bar{\tau}_{1m}, \bar{\tau}_{01}, \bar{\tau}_{02}, \ldots, \bar{\tau}_{0n})\]

where \(w\) is some \((m+n)\) letter word, \(\bar{\tau}_{1k} \in R_1\), \(k = 1, \ldots, m;\)

\(h_k \in F_2, g_i \in F_0 \cdot R_1\), \(i = 1, \ldots, n;\) \(m\) and \(n\) are positive integers.
As \( x \in \text{Ker } \epsilon_2 \), we must have

\[
\begin{align*}
w(1,1,\ldots,1, \ (\tau_{01})^{-1} \epsilon_2(g_1), (\tau_{02})^{-1} \epsilon_2(g_2), \ldots, (\tau_{0n})^{-1} \epsilon_2(g_n)) \ & \ m \text{ times} \\
\ & = 1 
\end{align*}
\]

Define a homomorphism \( \theta: \mathbb{F}_2 \longrightarrow \mathbb{F}_2 \) by

\[
\begin{align*}
F_0 & \longrightarrow F_0 \\
R_0 & \longrightarrow 1 \\
\theta: & \ \\
R_0 & \sim R_0 \\
R_1 & \longrightarrow 1 
\end{align*}
\]

Thus, \( x \in \text{Ker } \epsilon_2 \) also implies that

\[
w(1,1,\ldots,1, \ (\tau_{01})^{-1} \theta(g_1), (\tau_{02})^{-1} \theta(g_2), \ldots, (\tau_{0n})^{-1} \theta(g_n)) = 1.
\]

But \( g_1, \ldots, g_n \in F_0 \ast \bar{R}_1 \) and so, regarding \( F_1 \) as a subgroup of \( F_2 \), we have

\[
\theta\bigg|_{F_1} = \epsilon_2\bigg|_{F_1}, \ i.e. \ \theta(g_i) = \epsilon_2^2(g_i), i = 1, \ldots, n.
\]

\[
\therefore \ w(1,\ldots,1, \ (\tau_{01})^{-1} \epsilon_2^2(g_1), (\tau_{02})^{-1} \epsilon_2^2(g_2), \ldots, (\tau_{0n})^{-1} \epsilon_2^2(g_n)) = 1 \quad (2)
\]
Now consider $x \text{ mod } M$.

$$x \equiv w(1, \ldots, 1, (\tau_{01}^{-1} g_1), \ldots, (\tau_{on}^{-1} g_n) \text{ mod } M,$$

as $R_1 \subseteq M$.

$$x \equiv w(1, \ldots, 1, (\tau_{01}^{-1} g_1^2), \ldots, (\tau_{on}^{-1} g_n^2) \text{ mod } M,$$

since $g_i \equiv g_i^2 \text{ mod } M$, $g_i \in F_0 * R_1$, $i = 1, \ldots, n$.

If we define an $n$-letter word $w_i$ by

$$w_i(x_1, \ldots, x_n) = w(1, \ldots, 1, x_1, \ldots, x_n),$$

then by

$$x \equiv w_i((\tau_{01}^{-1} g_1^2), \ldots, (\tau_{on}^{-1} g_n^2) \text{ mod } M.$$

Now, as in 2.2, we can define an $n$-letter word $w_i'$ and then

$$x \equiv w_i'(\tau_{01}^{-1} g_1^2, \ldots, \tau_{on}^{-1} g_n^2) \text{ mod } M.$$

by using 2.2 (ii) since

$$\left[ \begin{array}{ccc} \varepsilon_2(g_k) & -\varepsilon_2(g_k) & -\varepsilon_2(g_1) \\ \tau_{ok} & \tau_{ok} & \tau_{01} \end{array} \right] \in M$$

for $k, \ell = 1, \ldots, n$ and $\varepsilon_2(g_k) \in F_0$. 
Now,
\[ \omega_{1}(\tau_{01}^{-1}c_{2}(g_{1}), \ldots, \tau_{0n}^{-1}c_{2}(g_{n})) \]

\[ = \underbrace{\omega(1,1,\ldots,1,}_{m \text{ times}} (\tau_{01}^{-1}c_{2}(g_{1}), \ldots, (\tau_{0n}^{-1}c_{2}(g_{n})) = 1 \]

by (2).

Also,
\[ \omega'_{1}(\tau_{01}^{-1}c_{2}(g_{1}), \ldots, \tau_{0n}^{-1}c_{2}(g_{n})) \]

\[ = \left( \omega_{1}(\tau_{01}^{-1}c_{2}(g_{1}), \ldots, \tau_{0n}^{-1}c_{2}(g_{n})) \right)^{-1} \text{ by 2.2.(i),} \]

and so it equals
\[ \underbrace{\omega(1,1,\ldots,1,}_{m \text{ times}} (\tau_{01}^{-1}c_{2}(g_{1}), \ldots, (\tau_{0n}^{-1}c_{2}(g_{n}))^{-1} \]

\[ = 1 \text{ by using (1).} \]

\[ \therefore x \equiv 1 \mod M \text{ and so we have the equality,} \]

\[ \ker \epsilon_{2}^{1} \cap \text{Ker } \epsilon_{2}^{2} = M. \]

**Proposition 2.4**

With the same notation as before, if \( \mathcal{V} \) is the variety of abelian groups, then the second Baer invariant of a group \( G \) is the quotient of the first Baer invariant of the fibre product \( F_{0} \times F_{0} \) by the normal subgroup \( N \) given by \( G \).
\[ N = \left[ \frac{F_1}{R_1}, F_1 \right] \left\langle (\bar{\tau}^{-1}f, s^g) : 1 \neq \bar{\tau} \in \bar{R}_0, \bar{\tau} \mapsto \tau, s \in S, f, g \in F_0 \right\rangle \]

Proof: By definition, the second Baer invariant

\[ L_2U(G) = L_1V(G) = \frac{F_1 \cap V(F_1)}{D} \]

where \( D = \epsilon_2^0(M \cap \left[ F_2, F_2 \right]) \). If \( V \) is the variety of abelian groups, \( V(F_2) = \left[ F_2, F_2 \right] \), and using 2.3, we have

\[ D = \epsilon_2^0(M \cap \left[ F_2, F_2 \right]) \]

\[ \therefore D = \epsilon_2^0 \left( \left[ \frac{F_2}{F_2}, F_2 \right], \left[ (\bar{\tau}^{-1}f, s^g) : 1 \neq \bar{\tau} \in \bar{R}_0, \bar{\tau} \mapsto \tau, f, g \in F_0 \right\rangle \right) \]

using 1.8.

\[ \therefore D = \epsilon_2^0 \left[ \frac{F_2}{F_2}, F_2 \right] \epsilon_2^0 \left( \left[ (\bar{\tau}^{-1}f, s^g) : 1 \neq \bar{\tau} \in \bar{R}_0, \bar{\tau} \mapsto \tau, f, g \in F_0 \right\rangle \right) \]

\[ = \left[ \frac{F_1}{R_1}, F_1 \right] \left\langle (\bar{\tau}^{-1}f, s^g) : 1 \neq \bar{\tau} \in \bar{R}_0, \bar{\tau} \mapsto \tau, f, g \in F_0 \right\rangle \].
\[ L_{2}U(G) = \frac{\left( \frac{F_{1} \cap V(F_{1})}{[F_{1} : F_{1}]} \right)}{[F_{1} : F_{1}]} \]

\[ = \frac{\left[ \frac{F_{1}}{F_{1}} \right]}{[F_{1} : F_{1}]} \left\langle \left( \tau \tau^{-1}, g \right) : 1 \neq \tau \in \bar{F}_{0}, \tau, g, \in F_{0} \right\rangle \]

\[ = \frac{L_{1}U(F_{0} \times F_{0})}{G} \]

, as required.

\[ \text{Note: Even in the variety of abelian groups, we have a rather complicated expression for } N. \text{ We shall get a neater expression for } N \text{ in the next chapter by using classical homology theory.} \]
CHAPTER V

HOMOLOGY AND BAER INVARIANTS

In this chapter we link up the homology groups of a group with the Baer invariants of a group and use this connection to obtain a result relating \( H_3(G, \mathbb{Z}) \) and \( H_2(F \times F, \mathbb{Z}) \).

In §1, we prove that the simplicial theory, under suitable conditions, gives us the classical theory. In §2, we use results of chapters III and IV to obtain a result which involves a second Baer invariant of a group. In §3, we use a spectral sequence argument to obtain a short exact sequence in homology.

§1. The classical and the simplicial theories.

Definition 1.1

Let \( G \) be a group and \( \mathbb{Z}G \) its integral group ring. Let \( \epsilon : \mathbb{Z}G \to \mathbb{Z} \) be the ring homomorphism given by \( \epsilon(g) = 1 \) for all \( g \in G \). Then \( \text{Ker} (\epsilon) \) is called the augmentation ideal of \( G \), denoted by \( \text{IG} \).

Let \( A \) be a left \( \mathbb{Z}G \)-module. We define a functor \( \text{Diff}(G, -) \) from the category of left \( \mathbb{Z}G \)-modules to abelian groups by \( \text{Diff}(G, A) = \text{IG} \otimes \mathbb{Z}G \).

We also define a functor \( \text{Der}(G, -) \) from the category of right \( \mathbb{Z}G \)-modules to abelian groups by \( \text{Der}(G, B) = \text{Hom}_{\mathbb{Z}G}(\text{IG}, B) \), for
all right \( \mathbb{Z}G \)-modules \( B \). We call \( \text{Der}(G, B) \) the derivations of \( G \) into \( B \).

**Definition 1.2**

Let \( \mathcal{C} \) be a category and let \( A \in \mathcal{C} \). We define a category \( (\mathcal{C}, A) \) of objects over \( A \) as follows.

\[
|\mathcal{C} \downarrow A| = \{ C \rightarrow A : \alpha \in \mathcal{C}(C, A), C \in |\mathcal{C}| \} \quad \text{and morphisms are evident compositions.}
\]

We call \( (\mathcal{C}, A) \) the **comma category** over \( A \).

**Definition 1.3**

A positive chain complex \( C = \{ C_n \}_{n \geq 0} \) over an abelian category is called **acyclic** if \( \text{H}_n(C) = 0 \) for \( n \geq 1 \).

**Definition 1.4**

Let \( G \) be a group. Let \( \overline{B}_n = \overline{B}_n(G), n \geq 0 \), be the free abelian group on the set of all \((n+1)\) tuples \( (g_0, \ldots, g_n) \) of elements of \( G \). Define a right \( G \)-module structure on \( \overline{B}_n \) by

\[
(g_0, \ldots, g_n) \cdot g = (g_0 g, \ldots, g_n g) \quad \text{for all } g \in G.
\]

Define a differential \( \partial_n : \overline{B}_n \rightarrow \overline{B}_{n-1} \) by

\[
\partial_n (g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n).
\]
Then, \[ \cdots \to \overline{B}_n \xrightarrow{d_n} \overline{B}_{n-1} \to \cdots \to \overline{B}_1 \xrightarrow{d_1} \overline{B}_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0 \]
is called the non-normalized, homogeneous bar resolution of \( \mathbb{Z} \), where \( \epsilon(g) = 1 \), for all \( g \in \overline{B}_o \). It is easy to show that \( \{ \overline{B}_n \} \) is in fact an acyclic chain complex and that \( \overline{B}_n \) is a free \( \mathbb{Z} \)-module for all \( n \geq 0 \).

**Definition 1.5**

Let \( G \) be a group and let \( A \) be a left \( \mathbb{Z} \)-module. Let \( \{ \overline{B}_n \} \) be the bar resolution of \( \mathbb{Z} \). Define a functor \( B_n(-,A) \) from the comma category \( (\mathcal{G} \downarrow G) \) to abelian groups by

\[
B_n(H,A) = \overline{B}_n(H) \otimes A, \quad \text{for all } H \to G \in (\mathcal{G} \downarrow G). 
\]

Thus, we can obtain a chain complex of abelian groups;

\[
\bigtriangleup = \{ \cdots \to B_n(G,A) \to B_{n-1}(G,A) \to \cdots \to B_1(G,A) \to B_0(G,A) \to 0 \}.
\]

As usual, we define the \( n \)-th homology group of \( G \) with coefficient in \( A \), \( H_n(G,A) \), to be \( H_n(\bigtriangleup) \) for all \( n \geq 0 \). For the cohomology groups, if \( B \) is a right \( \mathbb{Z} \)-module and \( B^n(H,B) = \text{Hom}_{\mathbb{Z} \text{-} \text{mod}}(B,B^n) \) defines functors from \( (\mathcal{G} \downarrow G) \) to abelian groups, then first we can form a cochain complex

\[
\bigtriangledown = \{ \cdots \to B^n(G,B) \to B^{n-1}(G,B) \to \cdots \to B^0(G,B) \to 0 \}.
\]
and then we define \( H^n(G, B) = H^n(\mathcal{D}) \) for all \( n \geq 0 \).

Now, \( \text{Diff} (G, A) = IG \otimes A \) and \( B_0(G, A) = \mathcal{B}_0 \otimes A \) and so \( B_0(G, A) \twoheadrightarrow \text{Diff} (G, A) \).

Thus we can form an augmented chain complex of functors,

\[
\cdots \rightarrow B_n(\_, A) \rightarrow B_{n-1}(\_, A) \rightarrow \cdots \rightarrow B_1(\_, A) \rightarrow B_0(\_, A) \twoheadrightarrow \text{Diff}(\_, A) \rightarrow \ast
\]

by regarding \( \text{Diff} (\_, A): (\mathcal{C}, G) \rightarrow \text{Ab} \) as a functor.

**Definition 1.6**

Let \( \mathcal{C} \) be a category and \( \mathcal{E} \) a class of epimorphisms in \( \mathcal{C} \). Let \( F: \mathcal{C} \rightarrow \mathcal{D} \) be a functor to a category \( \mathcal{D} \) such that the derived functors \( L_n^F \) are defined for all \( n \geq 0 \). Then we say that \( F \) is \( \mathcal{E} \)-representable if

\[
L_n^F = \begin{cases} 
\ast & \text{for } n > 0; \\
F & \text{for } n = 0.
\end{cases}
\]

**Proposition 1.7**

Let the functors \( B_n(\_, A) \) be defined as above for all \( n \geq 0 \). Let \( \mathcal{E} \) be the class of all epimorphisms in \( (\mathcal{C}, G) \). Then \( B_n(\_, A) \) is \( \mathcal{E} \)-representable for all \( n \geq 0 \).

**Proof:** Let \( H \rightarrow G \in (\mathcal{C}, G) \). It is easy to see that all group homomorphisms from free groups to \( G \) are \( \mathcal{E} \)-projective
objects in \((G \downarrow G)\). Thus we can use a simplicial resolution \(N\) of \(H\) by free groups as a simplicial resolution \(N \rightarrow G\) of \(H \rightarrow G\) by \(E\)-projectives in \((G \downarrow G)\). Now, to compute \(L^E_m B_n (H, A)\) we apply \(B_n\) to \(N\) to get \(B_n (N)\), form the derived complex as in \(\Pi X\), and take homology.

But we can forget that \(H\) is anything but a set. Then we can compute \(L^E_m B^* (H, A)\) where \(B^* (-, A): (\mathcal{C} \downarrow G) \rightarrow \mathcal{C}\) is the functor we obtain by forgetting the group structures. We can use the same construction of \(N\) as before and we shall get the same set theoretic result.

But as a set, \(H\) is projective relative to surjections in \(\mathcal{C}\) and so

\[
L^E_m B^*_n (H, A) = \begin{cases} 
\{*\} & \text{if } m > 0; \\
B^*_n (H, A) & \text{if } m = 0.
\end{cases}
\]

Thus, by the remarks above, we deduce that \(B_n (-, A)\) is \(E\)-representable for all \(n > 0\).

**Proposition 1.8**

Let \(\{\ldots \rightarrow K_m \rightarrow K_{m-1} \rightarrow \ldots \rightarrow K_0 \rightarrow K_{-1}\}\) be a chain complex of functors from a category \(\mathcal{C}\) to an abelian category \(\mathcal{A}\). Let \(\mathcal{E}\) be a projective class of epimorphisms in \(\mathcal{C}\) such that

(i) \(K_i\) is \(\mathcal{E}\)-representable for all \(i \geq 0\);
(ii) for all $E$-projective objects $P$ in $C$, the sequence.

$$\ldots K_m(P) \longrightarrow K_{m-1}(P) \longrightarrow \ldots \longrightarrow K_0(P) \longrightarrow K_{-1}(P) \longrightarrow 0$$

is acyclic.

Then $L_n(K_{-1}) = H_n(K)$ for all $n \geq 0$.

**Proof:** Define a functor $T: C \to \mathcal{A}$ by $T(A) = \text{Ker}(K_0(A) \to K_{-1}(A))$ for all $A \in C$ and put $R_1 = L_0 T$.

Thus, $*$ $R_1$ $K_0$ $K_{-1}$ $*$ is an exact sequence on $E$-projectives.

Now, consider the commutative diagram,

$$\begin{array}{ccc}
K_2 & \longrightarrow & K_1 \\
& \downarrow & \downarrow \\
& R_1 & \longrightarrow K_0 \\
& & \downarrow \\
& & K_{-1}
\end{array}$$

Then, since $L_0(K_i) = K_i$, $i = 0,1,2$ and $L_0(R_1) = L_0(L_0 T) = L_0 T = R_1$;

we have exact sequences,

1. $K_2 \longrightarrow K_1 \longrightarrow R_1 \longrightarrow *$ and
2. $R_1 \longrightarrow K_0 \longrightarrow L_0 K_{-1} \longrightarrow *$.

But, as $K_1 \longrightarrow R_1$ is an epimorphism, we have $R_1 \longrightarrow K_0 \longrightarrow H_0(K) \longrightarrow *$ is also exact.
\[ L_0(K_{-1}) = H_0(K), \text{ using } (2). \]

Also, the long exact sequence of derived functors for \( K \) yields

\[ * \rightarrow L_nK_{-1} \rightarrow L_{n-1}R_1 \rightarrow * \quad \text{for } n > 1, \text{ and} \]

\[ * \rightarrow L_nK_{-1} \rightarrow R_1 \rightarrow K_0 \rightarrow L_0K_{-1} \rightarrow *, \text{ since } K_0 \]

is representable.

But then, \( L_nK_{-1} = L_{n-1}R_1 \) and \( L_1(K_{-1}) = H_1(K) \).

Considering the sequence

\[ \ldots \rightarrow K_2 \rightarrow K_1 \rightarrow R_1 \rightarrow *, \text{ and applying induction,} \]

we have

\[ L_{n-1}(R_1) = H_n(K), \quad n \geq 1. \]

Thus \( L_n(K_{-1}) = H_n(K), \) for all \( n > 0. \)

**Theorem 1.9**

Let \( G \) be a group and let \( A \) be a left \( \mathbb{Z}G \)-module.

Then, \( L_n(\text{Diff}(G,A)) = H_{n+1}(G,A) \) for all \( n > 0. \)

**Proof:** By the definition of homology groups, the sequence (1) of functors in 1.5, 1.7 and 1.8 we deduce the result.

**Corollary 1.10**

Let the functor \( u: \mathcal{C} \rightarrow \mathcal{O} \) be defined by \( u(G) = G/[G,G] \).

Then,

\[ L_nu(G) = H_{n+1}(G, \mathbb{Z}) \quad \text{for all } n > 0, \text{ and where } \mathcal{C} \text{ is the} \]
class of all surjections in \( \mathcal{G} \).

**Proof:** \( \text{Diff} (G, \mathbb{Z}) = \frac{\text{IG} \otimes \mathbb{Z}}{\mathcal{ZG}} \).

\[ \approx \frac{\text{IG}}{(\text{IG})^2} \]

\[ \approx \frac{G}{[G,G]} \text{, see Hilton and Stammbach (4)} \]

\[ \therefore \text{Diff} (\_ , \mathbb{Z}) = U(\_ ) \text{ and so the result follows.} \]

We can also prove a similar theorem for cohomology groups. Here we just state the result.

**Theorem 1.11**

Let \( G \) be a group and \( B \) a right \( \mathbb{Z}G \)-module. Then,

\[ L_n^\mathcal{E} (\text{Der}(G,B)) \approx \begin{cases} n^{n+1} (G,B), & \text{if } n > 0; \\ \text{Der} (G,B), & \text{if } n = 0. \end{cases} \]

**§2 Homology and the second Baer invariant**

Let \( \mathcal{G} \) be the category of groups and \( \mathcal{E} \) the class of all surjections in \( \mathcal{G} \). Then, by \( \mathbb{3}.2.5 \), \( (\mathcal{G}, \mathcal{E}) \) is a Rinehart category and so we can form Rinehart categories \( (\mathcal{G}_n, \mathcal{E}_n) \) for all \( n \geq 0 \).

Let \( V \) be a variety of groups and \( V \) and \( U \) be the associated verbal subgroup and verbal quotient group functors respectively. As in \( \mathbb{3}.3.2 \), we can define the associated functors \( V_n, U_n : \mathcal{G}_n \rightarrow \mathcal{G} \) for all \( n \geq 0 \). We want to compute
$L_2 U(G) \cong L_1 V(G)$ for an arbitrary group $G$. We shall use our version of Rinehart's result to compute $L_2 U(G)$ by considering the functors $V_2$ and $U_2$.

**Lemma 2.1**

Let $V$ and $U$ be the verbal subgroup and the quotient group functors respectively, associated to a variety $V$. If $(\mathcal{G}, \mathcal{E})$ is the Rinehart category defined as above, then

$$\ast \rightarrow V_2 \rightarrow I_2 \rightarrow U_2 \rightarrow \ast$$

is an exact sequence of functors on $\mathcal{G}_2$, where $\ast$ denotes the trivial functor and $I$ the identity functor.

**Proof:** Let $\vec{A} = \begin{array}{ccc} A & \xrightarrow{g_1} & C \\ \downarrow f_1 & & \downarrow f_2 \\ B & \xrightarrow{g_2} & D \end{array}$ be an object of $\mathcal{G}_2$.

Then, by definition

$$V_2(\vec{A}) = \ker V(f_1) \cap \ker V(g_1) = \ker f_1 \cap \ker g_1 \cap V(A),$$

since $V$ is a subfunctor of $I$.

Also, $I_2(\vec{A}) = \ker f_1 \cap \ker g_1$, and by definition
Let $\mathcal{C}$ be a base category and let $F$ be an $\mathcal{E}$-representable functor defined on $\mathcal{C}$. Then $F_m$ is $\mathcal{E}_m$-representable for all $m > 0$.

**Proof:** We induct on $m$.

For $m = 0$, $F_0 = F$ and the result is obvious. So assume $F_k$ is $\mathcal{E}_k$-representable for some $k > 0$. 

Thus, $V_2(\bar{A}) \longrightarrow I_2(\bar{A}) \longrightarrow U_2(\bar{A})$ is an exact sequence of groups and so 

$$* \longrightarrow V_2 \longrightarrow I_2 \longrightarrow U_2 \longrightarrow *$$

is an exact sequence of functors.

**Lemma 2.2**

Let $(\mathcal{C}, \mathcal{E})$ be a base category and let $F$ be an $\mathcal{E}$-representable functor defined on $\mathcal{C}$. Then $F_m$ is $\mathcal{E}_m$-representable for all $m > 0$. 

**Proof:** We induct on $m$.

For $m = 0$, $F_0 = F$ and the result is obvious. So assume $F_k$ is $\mathcal{E}_k$-representable for some $k > 0$. 

Thus, $V_2(\bar{A}) \longrightarrow I_2(\bar{A}) \longrightarrow U_2(\bar{A})$ is an exact sequence of groups and so 

$$* \longrightarrow V_2 \longrightarrow I_2 \longrightarrow U_2 \longrightarrow *$$

is an exact sequence of functors.

**Lemma 2.2**

Let $(\mathcal{C}, \mathcal{E})$ be a base category and let $F$ be an $\mathcal{E}$-representable functor defined on $\mathcal{C}$. Then $F_m$ is $\mathcal{E}_m$-representable for all $m > 0$. 

**Proof:** We induct on $m$.

For $m = 0$, $F_0 = F$ and the result is obvious. So assume $F_k$ is $\mathcal{E}_k$-representable for some $k > 0$. 

Thus, $V_2(\bar{A}) \longrightarrow I_2(\bar{A}) \longrightarrow U_2(\bar{A})$ is an exact sequence of groups and so 

$$* \longrightarrow V_2 \longrightarrow I_2 \longrightarrow U_2 \longrightarrow *$$

is an exact sequence of functors.

**Lemma 2.2**

Let $(\mathcal{C}, \mathcal{E})$ be a base category and let $F$ be an $\mathcal{E}$-representable functor defined on $\mathcal{C}$. Then $F_m$ is $\mathcal{E}_m$-representable for all $m > 0$. 

**Proof:** We induct on $m$.

For $m = 0$, $F_0 = F$ and the result is obvious. So assume $F_k$ is $\mathcal{E}_k$-representable for some $k > 0$. 

Thus, $V_2(\bar{A}) \longrightarrow I_2(\bar{A}) \longrightarrow U_2(\bar{A})$ is an exact sequence of groups and so 

$$* \longrightarrow V_2 \longrightarrow I_2 \longrightarrow U_2 \longrightarrow *$$

is an exact sequence of functors.
Let \( A \rightarrow B \in \mathcal{E}_{k+1} \), then we can get a long exact sequence,

\[
\ldots \longrightarrow L_n F_k(A) \longrightarrow L_n F_k(B) \longrightarrow L_{n-1} F_k(A) \rightarrow L_{n-1} F_k(B) \longrightarrow L_{n-1} F_k(A) \rightarrow \ldots
\]

But since \( F_k \) is \( \mathcal{E}_k \)-representable,

\[ L_n F_k = \begin{cases} * & \text{if } n > 0; \\ F_k & \text{if } n = 0. \end{cases} \]

\[ L_n F_{k+1}(\alpha) = 1 \text{ for } n > 0, \text{ and} \]

\[ L_0 F_{k+1}(\alpha) = \ker (F_k(A) \xrightarrow{F_k(\alpha)} F_k(B)) \]

\[ = F_{k+1}(\alpha), \text{ by definition}. \]

\( F_{k+1} \) is \( \mathcal{E}_{k+1} \)-representable and so, by induction, the result holds.

**Lemma 2.3**

Let \( G \) be a group and \( F \) a free group such that \( F \) maps onto \( G \), i.e.

\[ F \xrightarrow{\alpha} G \in \mathcal{E} \text{ where } (G, \mathcal{E}) \text{ is the Rinehart category considered before.} \]

Form the fibre product square
Then, $L_n^\mathcal{E} U_2(X) = L_{n-1}^\mathcal{E} V_2(X)$ for $n \geq 1$ and $L_0^\mathcal{E} U_2(X) = 1$, where $V$ and $U$ are the verbal subgroup and the quotient group functors associated to a variety $\mathcal{V}$, respectively.

**Proof:** From the short exact sequence of functors in 2.1 we obtain a long exact sequence of derived functors. As $I$ is an $\mathcal{E}$-representable functor, by using 2.2 we deduce that $I_2$ is $\mathcal{E}_2$-representable. Hence,

$$L_n^\mathcal{E} U_2 = L_{n-1}^\mathcal{E} V_2 \text{ for } n \geq 2$$

and

$$L_n^\mathcal{E} U_2(X) = L_{n-1}^\mathcal{E} V_2(X) \text{ for } n \geq 2.$$ 

But $L_0^\mathcal{E} I_2(X) = I_2(X) = \ker \pi_1 \cap \ker \pi_2 = 1$ and so from the long exact sequence we further deduce that $L_1^\mathcal{E} U_2(X) = L_0^\mathcal{E} V_2(X)$ and $L_0^\mathcal{E} U_2(X) = 1$, as required.

Now, we shall apply the above result in the special case where $V$ is the commutator subgroup functor and $U$ is the commutator quotient group functor. Then, from 5.1, we know that for any group $G$,

$$L_n^\mathcal{E} U(G) = H_{n+1}(G, \mathbb{Z}) \text{ for } n \geq 0.$$
We shall write \( H_{n+1}(G) \) for \( H_{n+1}(G, \mathbb{Z}) \) and also write \( H_{n+1}(\alpha) \) for \( L^p_n(\alpha), \alpha \in [e]_m \). Then using Rinehart's result, for any surjection \( E \xrightarrow{\alpha} G \in \mathcal{C} \), where \((\mathcal{G}, \mathcal{C})\) is the Rinehart category as defined before, we obtain a long exact sequence,

\[
\begin{array}{ccccccccc}
\cdots & H_n(E) & \xrightarrow{} & H_n(G) & \xrightarrow{} & H_{n-1}(\alpha) & \xrightarrow{} & H_{n-1}(E) & \xrightarrow{} & \cdots \\
\xrightarrow{} & H_1(\alpha) & \xrightarrow{} & H_1(E) & \xrightarrow{} & H_1(G) & \xrightarrow{} & 1.
\end{array}
\]

Theorem 2.4

Let \( X \) be defined as in 2.3. Then with the same notation as above, we have a natural short exact sequence,

\[
1 \xrightarrow{} H_4(G) \xrightarrow{} H_2(X) \xrightarrow{} H_2(F \times F) \xrightarrow{G} H_3(G) \xrightarrow{} 1.
\]

Proof: Consider the element \( F \xrightarrow{\alpha} G \in \mathcal{C} \) and use the sequence \( 1 \) above. Since \( F \) is a free group, it is \( \mathcal{C} \)-projective and so

\[
L^p_n(F) = 1 \text{ for } n \geq 1, \text{ i.e. } H^p_n(F) = 1 \text{ for } n \geq 1.
\]

\( 1 \) \( : \ H_n(G) = H^p_{n-1}(\alpha) \text{ for } n \geq 3. \)

Now consider the element \( F \times F \xrightarrow{\pi_1} F \in \mathcal{C} \) and use the sequence \( 1 \) above to get

\( 2 \) \( H^p_{n-1}(F \times F) \xrightarrow{G} H^p_{n-1}(\pi_1) \text{ for } n \geq 3. \)
Finally, consider the element $X = (\pi_2^\alpha): \pi_1 \to \alpha \in \mathcal{L}_2$ and apply Rinehart's result to get a long exact sequence,

$$
\cdots \to H_n(\pi_1) \to H_n(\alpha) \to H_{n-1}(X) \to H_{n-1}(\pi_1) \to \cdots
$$

$$
\to H_1(X) \to H_1(\pi_1) \to H_1(\alpha) \to 1.
$$

Using (1) and (2), we obtain

$$
\cdots \to H_3(F \times F)_G \to H_4(G) \to H_2(X) \to H_2(F \times F)_G
$$

$$
\to H_3(G) \to H_1(X) \to H_1(\pi_1) \to H_1(\alpha) \to 1.
$$

But $H_1(X) = L_0U_2(X) = 1$ by 2.3, and since $F \times F$ is a subgroup of $F \times F$, we have $H_3(F \times F)_G = 1$ by considering an easy "homology dimension" argument, see, for example (3).

Thus, our exact sequence above yields,

$$
1 \to H_4(G) \to H_2(X) \to H_2(F \times F)_G \to H_3(G) \to 1.
$$

**Note:** The above result tells us what the kernel of the surjection

$$
L_1^G(U(F \times F)) \to L_2^G(U(G), \text{ as obtained in IV.2.1}, is
$$

when $U$ is the commutator quotient group functor. The required kernel turns out to be $H_2(X)$ and to get a "better" expression $\frac{H_2(X)}{H_4(G)}$

we need to compute $H_2(X)$. I tried to compute $H_2(X)$ by using a simplicial $\mathcal{L}$-resolution for $X$ but did not get a "neat" expression
for the kernel required. Thus, unfortunately, we shall use a spectral sequence argument to complete our computation.

§3 The Spectral Sequence Argument

We shall use the standard notation for spectral sequences as given in Maclane's "Homology" and for homology we shall use the notation as given in Hilton and Stammbach (4). We shall first put together some results in homology which we need.

Lemma 3.1

Let G be a group and \( \mathbb{R} \xrightarrow{\alpha} \mathbb{F} \xrightarrow{\beta} \mathbb{G} \) be a free presentation of G. Then,

(i) \( H_n(\mathbb{R} \times \mathbb{R}, \mathbb{Z}) = H_n(\mathbb{F} \times \mathbb{F}, \mathbb{Z}) \) for \( n \geq 3 \), where \( \mathbb{R} \times \mathbb{R} \) is the direct product of \( \mathbb{R} \) with itself and \( \mathbb{F} \times \mathbb{F} \) is the fibre product of \( \alpha \) with itself;

(ii) \( H_1(\mathbb{R} \times \mathbb{R}, \mathbb{Z}) = \mathbb{R}^{ab} \oplus \mathbb{R}^{ab} \), where \( \mathbb{R}^{ab} \) stands for \( \mathbb{R} \) factored out by the commutator subgroup of \( \mathbb{R} \);

(iii) \( H_2(\mathbb{R} \times \mathbb{R}, \mathbb{Z}) = \mathbb{R}^{ab} \otimes \mathbb{R}^{ab} \), where \( \otimes \) denotes \( \otimes \mathbb{Z} \);

(iv) \( H_{i+2}(\mathbb{G}, \mathbb{Z}) = H_i(\mathbb{G}, \mathbb{R}^{ab}) \) for \( i > 0 \) and where \( \mathbb{R}^{ab} \) is made into a \( \mathbb{Z} \mathbb{G} \)-module via \( \alpha \);

(v) \( H_{i+4}(\mathbb{G}, \mathbb{Z}) = H_i(\mathbb{G}, \mathbb{R}^{ab} \otimes \mathbb{R}^{ab}) \) for \( i > 0 \) and where \( \mathbb{R}^{ab} \otimes \mathbb{R}^{ab} \) is the \( \mathbb{Z} \mathbb{G} \)-module with diagonal action;

(vi) \( H_n(\mathbb{G}, \mathbb{R}^{ab} \oplus \mathbb{R}^{ab}) = H_n(\mathbb{G}, \mathbb{R}^{ab}) \oplus H_n(\mathbb{G}, \mathbb{R}^{ab}) \) for \( n \geq 0 \) and where \( \mathbb{R}^{ab} \oplus \mathbb{R}^{ab} \) is the \( \mathbb{Z} \mathbb{G} \)-module with diagonal action.
Proof: (i) Follows from the fact that $R$ is a free group and that $F \times F$ is a subgroup of the direct product of two free groups.

(ii) Follows from $\frac{R \times R}{[R \times R, R \times R]} \cong \frac{R}{[R, R]} \oplus \frac{R}{[R, R]}$.

(iii) Follows from the Künneth formula and the fact that $H_2(R, \mathbb{Z}) = 1$.

(iv) This is a standard reduction theorem.

(v) Follows from the beginning of the Gruenberg resolution,

\[ 0 \longrightarrow R^{ab} \bigotimes R^{ab} \longrightarrow (\mathbb{Z}G \bigotimes \mathbb{F}) \bigotimes R^{ab} \longrightarrow \mathbb{Z}G \bigotimes R^{ab} \]

see Gruenberg (3) for details.

(vi) If \( P \longrightarrow R^{ab} \) is a \( \mathbb{Z}G \)-projective resolution of \( R^{ab} \),
then \( P \bigoplus P \longrightarrow R^{ab} \bigoplus R^{ab} \) is a \( \mathbb{Z}G \)-projective resolution of \( R^{ab} \bigoplus R^{ab} \).

Lemma 3.2

With the same notation as above,

(i) $R \times R \xrightarrow{u} F \times F \xrightarrow{\alpha \pi_1} G$ is an exact sequence of groups, where $u$ is inclusion and $\pi_1: F \times F \longrightarrow F$ is the projection onto the first component;

(ii) the spectral sequence for the short exact sequence in (i) yields

\[ H_p(G, H_\mathbb{F}(R \times R, \mathbb{Z})) \longrightarrow H_{p+q}(F \times F, \mathbb{Z}) \quad \text{whose } E_2^{p,q} \]
term is given as follows,

\[ \begin{array}{cccc}
& H_5(G) & H_c(G) \\
R^a \otimes R^b & d_{21} & d_{41} \\
(R^a \oplus R^b) & d_{31} & \ & \ \\
& H_3(G) \otimes H_3(G) & H_6(G) \otimes H_6(G) & H_5(G) \otimes H_5(G) \\
& & & H_c(G) \otimes H_c(G) \\
& \mathbb{Z} & H_1(G) & H_2(G) & H_3(G) & H_4(G) & H_5(G) \\
\end{array} \]

where \( H_n(G) \) stands for \( H_n(G, \mathbb{Z}) \), \( n \geq 1 \).

**Proof:**
(i) This is obvious.

(ii) We use the Lyndon-Hochschild-Serre spectral sequence and lemma 3.1 to get the result.

**Lemma 3.3**

With the same notation as above,

(i) \( d_{20} \) is an injection;

(ii) \( \text{Im} \: d_{p0} = H_p(G, \mathbb{Z}) \) for \( p \geq 3 \), where \( d_{p0} \) is the diagonal map, and are injections.
Proof: Consider the following commutative diagram with exact rows,

\[ \begin{array}{cccccc}
R \times R & \xrightarrow{u} & F \times F & \xrightarrow{\alpha \pi_1} & G \\
\Delta_0 & \downarrow{\pi_1} & \Delta_1 & \downarrow{\pi_1} & 1_G \\
R & \xrightarrow{u_0} & F & \xrightarrow{\alpha} & G
\end{array} \]

where \( u_0, u \) are inclusions; \( \Delta_0, \Delta_1 \) are diagonal morphisms and \( \pi_1 \) is the projection onto the first component. In particular, 
\[
(\pi_1 | \Delta_0) = 1_R \quad \text{and} \quad \pi_1 \Delta_1 = 1_F.
\]

We shall consider spectral sequences for the exact sequences \( \text{(A)} \) and \( \text{(B)} \), and so will obtain an induced morphism between the spectral sequences. If \( E_p^r \) denotes an arbitrary term of the spectral sequence associated to \( B \), then \( E_2^{p,r} = H_p(G, H_r(R, \mathbb{Z})) \) is given by the following diagram.

Let \( f: \mathbb{C} \to \mathbb{C} \) be the induced morphisms of the spectral sequences.

\[ \begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{array} \]

(i) We have \( f_{20} : H_2(G) \to H_2(G) \) induced by \( 1_G \) and so 
\[
f_{20} = 1_{H_2(G)}.
\]

We also have \( f_{01} : (R^{ab})^G \to H_0(G, H_1(R)) \),
\[ f_{01}: H_0(G, H_1(R \oplus R)) \rightarrow H_0(G, H_1(R)) \] which is induced by the projection \( \pi_1 \) and so \( f_{01} \) is the projection map onto the first component \( \text{proj}_1 \). This yields the following commutative diagram:

Thus \( d_{20} = (d_{20}', -) \). But replacing \( \pi_1 \) by \( \pi_2 \) we also get \( d_{20} = (-, d_{20}') \) and so \( d_{20} = (d_{20}', d_{20}) \). As \( H_2(F, \mathbb{Z}) = 1 \), its filtration will be trivial and so \( E_{20}' = E_{20}'' = 1 \).

\[ \therefore \text{Ker } d_{20}' = 1 \quad \text{and so Ker } d_{20} = 1, \text{ i.e.} \]

\( d_{20} \) is an injection.

(ii) For \( p = 3 \) and \( 4 \), we obtain the following commutative diagram,

where \( f_{p,0} = 1_H^p(G) \) as it is induced by \( 1_G \) and \( f_{p-2,1} = \text{proj}_1 \).
as it is induced by \(\pi_1\). Thus, by the same argument as in (i), we deduce that 
\[
d_{p,0} = (d'_p, d'_p), \quad p = 3, 4.
\]
As \(H_2(F, \mathbb{Z}) = 1\), considering its filtration we deduce that 
\[
E'_\infty = 1.
\]
But 
\[
E_1^\infty = E_3^8 = \frac{H_1(G, H_1(R))}{\text{Im} \, d_{30}^1} = 1.
\]
\[
\therefore \text{Im} \, d_{30}^1 = H_1(G, H_1(R)) = H_3(G, \mathbb{Z}) \quad \text{and so}
\]
\[
\text{Im} \, d_{30}^1 = H_3(G, \mathbb{Z}).
\]
Now, \(H_0(G, H_2(R, \mathbb{Z})) = 1\) and so \(\ker d_{21}^1 = H_2(G, H_1(R))\).
As \(H_3(F, \mathbb{Z}) = 1\), its filtration will be trivial and so 
\[
E_2^\infty = 1.
\]
But, 
\[
E_2^\infty = E_2^3 = \frac{\ker d_{21}^1}{\text{Im} \, d_{40}^1} = 1.
\]
\[
\therefore \text{Im} \, d_{40}^1 = H_2(G, H_1(R)), \quad \text{by above}
\]
\[
= H_4(G, \mathbb{Z})
\]
\[
\therefore \text{Im} \, d_{40}^1 = H_4(G, \mathbb{Z}) \quad \text{as required.}
\]
Finally, as \(d_{p,0}^1\) is a natural isomorphism for \(p = 3, 4\), it is easy to see that \(d_{p,0}^1\) is the diagonal map.

**Theorem 3.4**

Let \(G\) be a group and \(R \twoheadrightarrow F \xrightarrow{\alpha} G\) be a free presentation of \(G\). Let \(F \times F\) be the fibre product of \(\alpha\) with itself.

Then there exists a natural exact sequence
Proof: From the spectral sequence considered in 3.2 we also get a filtration for \( H_2(F \times F, \mathbb{Z}) \) as follows

\[
1 \subseteq F_0 \subseteq F_1 \subseteq F_2 = H_2(F \times F, \mathbb{Z})
\]

where

\[
F_0 = E_0, \quad \frac{F_1}{F_0} = E_1 \quad \text{and} \quad \frac{F_2}{F_1} = E_2
\]

Now, \( E_2 = E_3 = \ker d_2 = 1 \) by 3.3(i). \( \therefore F_1 = F_2. \)

Also, \( E_3 = E_4 = \frac{H_3(G) \otimes H_3(G)}{\text{Im } d_3} = H_3(G), \) using 3.3(ii).

\( \therefore E_{02} \rightarrow H_2(F \times F, \mathbb{Z}) \rightarrow H_3(G) \) is exact.

Now, \( E_{02} = E_{02} = E_{02} = \frac{R^{ab} \otimes_G R^{ab}}{\text{Im } d_3}, \) as \( d_3 \) is an injection.

But \( H_4(G) = \text{Im } d_{40} \subseteq \ker d_{21}, \) by using 3.3(ii).

Consider the filtration \( 1 \subseteq T_0 \subseteq T_1 \subseteq T_2 \subseteq T_3 = H_3(F \times F, \mathbb{Z}). \)

As \( H_3(F \times F, \mathbb{Z}) = 1, \ T_j = 1 \) for \( j = 0, 1, 2, 3. \)

\( \therefore \) In particular \( E_{21} = 1. \)
But \( E_{21}^\infty = E_{21}^3 = \frac{\text{Ker } d_{21}}{\text{Im } d_{40}} = 1 \)

\[ \therefore \text{Ker } d_{21} = \text{Im } d_{40} = H_4(G), \text{ using 3.3(ii)}. \]

Thus, \( \text{Im } d_{21} \cong \frac{H_4(G) \oplus H_4(G)}{\text{Ker } d_{21}} = H_4(G) \).

\[ \therefore E_{02} = \frac{R^{ab} \otimes R^{ab}}{H_4(G, G)}. \]

Thus, using (1), we obtain the exact sequence

\[ 1 \rightarrow H_4(G, \mathbb{Z}) \rightarrow R^{ab} \otimes R^{ab} \rightarrow H_2(F \times F, \mathbb{Z}) \rightarrow H_3(G, \mathbb{Z}) \rightarrow 1, \text{ as required.} \]

**Note:** This, in particular, implies that \( H_2(X) = R^{ab} \otimes R^{ab}, \)

the term we were trying to compute in section two of this chapter.
REFERENCES


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