Asymptotic normality of quadratic forms of martingale differences

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Abstract

We establish the asymptotic normality of a quadratic form Q_n in martingale difference random variables η_t when the weight matrix A of the quadratic form has an asymptotically vanishing diagonal. Such a result has numerous potential applications in time series analysis. While for i.i.d. random variables η_t, asymptotic normality holds under condition $||A||_{sp} = o(||A||)$, where $||A||_{sp}$ and $||A||$ are the spectral and Euclidean norms of the matrix A, respectively, finding corresponding sufficient conditions in the case of martingale differences η_t has been an important open problem. We provide such sufficient conditions in this paper.

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1 Main results

We study here quadratic forms

$$Q_n = \sum_{t,k=1}^{n} a_{n;tk}\eta_t\eta_k \tag{1.1}$$

where $\{\eta_k\}$ is a stationary ergodic martingale difference (m.d.) sequence with respect to some natural filtration $\mathcal{F}_t$, with moments

$$E\eta_k = 0, \quad E\eta_k^2 = 1 \quad \text{and} \quad E\eta_k^4 < \infty.$$

The real-valued coefficients $a_{n;tk}$ in (1.1) are entries of a symmetric matrix $A_n = (a_{n;tk})_{t,k=1,\dots,n}$. We denote by

$$||A_n|| = (\sum_{t,k=1}^{n} a_{n;tk}^2)^{1/2}$$

the Euclidean norm and by

$$||A_n||_{sp} = \max_{||x||=1} ||A_n x||$$

the spectral norm of the matrix A_n. For convenience, we set $a_{n;tk} = 0$ for $t \le 0$, $t > n$ or $k \le 0$, $k > n$.

The asymptotic normality property of the quadratic form Q_n has been well investigated when the random variables η_j are i.i.d. If A_n has vanishing diagonal: $a_{n;tt} = 0$ for all t, then asymptotic normality is implied by the condition

$$||A_n||_{sp} = o(||A_n||), \tag{1.2}$$

see Rotar (1973), De Jong (1987), Guttorp and Lockhart (1988), Mikosch (1991) and Bhansali, Giraitis and Kokoszka (2007a).

The aim of this paper is to extend these results to the m.d. noise η_j. We will need the following additional assumptions on the m.d. noise η_t:

$$E(\eta_j^2 | \mathcal{F}_{j-1}) \ge c > 0, \qquad (\exists c > 0). \tag{1.3}$$

The assumption (1.3) bounds the conditional variance of η_j away from zero. We also assume that A_n has an asymptotically "vanishing" diagonal in the sense:

$$\sum_{t=1}^{n} |a_{n;tt}| = o(||A_n||), \quad n \to \infty. \tag{1.4}$$

Relation (1.4) implies

$$\sum_{t=1}^{n} a_{n;tt}^2 = o(||A_n||^2), \quad n \to \infty. \tag{1.5}$$

The following theorem shows that in case of m.d. noise $\{\eta_k\}$, the condition

$$||A_n||_{sp}/||A_n|| \to 0$$

above needs to be strengthened by including the assumptions (1.8) and (1.9) on the weights $a_{n;ts}$. Its proof is based on the martingale central limit theorem.

THEOREM 1.1. *Let Q_n be as in (1.1), where $\{\eta_j\}$ is a stationary ergodic m.d. noise such that $E\eta_j^4 < \infty$ and (1.3) hold. Suppose that the $a_{n;ts}$'s are such that, as $n \to \infty$,*

$$||A_n||_{sp}/||A_n|| \to 0. \tag{1.6}$$

Then there exist $c_1, c_2 > 0$ such that

$$c_1 ||A_n||^2 \le \mathrm{Var}(Q_n) \le c_2 ||A_n||^2, \quad n \ge 1. \tag{1.7}$$

If in addition,

$$\textstyle\sum_{t,s=1:\,|t-s|\ge L}^{n} a_{n;ts}^2 = o(||A_n||^2), \quad n \to \infty, \;\; L \to \infty, \tag{1.8}$$

and

$$\sum_{t=k+2}^{n}(a_{n;t,t-k} - a_{n;t-1,t-1-k})^2 = o(||A_n||^2), \qquad \forall k \geq 1 \tag{1.9}$$

then the following normal convergence holds:

$$(\mathrm{Var}(Q_n))^{-1/2}(Q_n - EQ_n) \xrightarrow{d} N(0,1). \tag{1.10}$$

As usual, "$\xrightarrow{d} N(0,1)$" denotes convergence in distribution to a normal random variable with mean zero and variance one.

Theorem 1.1 plays an important instrumental role in establishing asymptotic properties of various estimation and testing procedures in parametric and non-parametric time series analysis where the object of interest can be written as a quadratic form

$$Q_{n,X} = \sum_{t,s=1}^{n} e_n(t-s)X_tX_s$$

of a linear (moving-average) process

$$X_t = \sum_{j=0}^{\infty} a_j\eta_{t-j}$$

of uncorrelated noise η_t and the weights $e_n(s)$ may depend on n. In the case of i.i.d. noise η_t, the asymptotic normality for $Q_{n,X}$ is established by approximating it by a simpler quadratic form

$$Q_{n,\eta} = \sum_{t,s=1}^{n} b_n(t-s)\eta_t\eta_s$$

with some different weights $b_n(t)$ and then deriving the asymptotic normality for $Q_{n,\eta}$, as in Bhansali, Giraitis and Kokoszka (2007b). For example, one sets

$$b_n(t) = \int_{-\pi}^{\pi} u_n(x)f(x)e^{itx}dx$$

where $f(x)$ is the spectral density of the sequence X_t, and where $u_n(x)$ is some convenient function related to $e_n(t)$, typically such that

$$e_n(t) = \int_{-\pi}^{\pi} u_n(x)e^{itx}dx.$$

In general, obtaining simple asymptotic normality conditions for $Q_{n,X}$ is a hard theoretical problem but of great practical importance, which for an i.i.d. noise η_t was solved in Bhansali, Giraitis and Kokoszka (2007b). In addition, in Section 6.2 in Giraitis, Koul and Surgailis (2012) one considers discreet frequencies and shows that a sum

$$S_n = \sum_{j=1}^{n/2} b_{nj}I(u_j)$$

of weighted periodograms

$$I(u_j) = (2\pi n)^{-1} \Big| \sum_{k=1}^{n} e^{iku_j} X_k \Big|^2$$

of the sequence X_t at Fourier frequencies u_j can be also effectively approximated by a quadratic form $Q_{n,\eta}$. This allows, by theorem like Theorem 1.1, to establish the asymptotic normality for such sums S_n. However, assumption of i.i.d. noise is restrictive and may be not satisfied in practical applications and in some theoretical, i.e. ARCH, settings. In a subsequent paper we will derive corresponding normal approximation results for $Q_{n,X}$ and S_n when η_t is a martingale difference process.

The following Corollary 1.1 displays situations where the conditions of Theorem 1.1 are easily satisfied. For a Toeplitz matrix A_n, that is with entries

$$a_{n;ts} = b_n(t-s),$$

the assumption (1.9) is clearly satisfied, since

$$a_{n;t,t-k} - a_{n;t-1,t-1-k} = b_n(k) - b_n(k) = 0.$$

The following lemma provides a useful bound that can be used to prove (1.6).

Lemma 1.1. *Suppose that*

$$b_n(t) = \int_{-\pi}^{\pi} e^{itx} g_n(x) dx, \quad t = 0, 1, ...,$$

where $g_n(x)$, $|x| \le \pi$ is an even real function. If there exists

$$0 \le \alpha < 1/2$$

and a sequence of constants $k_n > 0$ such that

$$|g_n(x)| \le k_n |x|^{-\alpha}, \quad |x| \le \pi,$$

then

$$||A_n||_{sp} \le C k_n n^{\alpha}, \quad n \ge 1. \tag{1.11}$$

For the proof see Theorem 2.2(i) in Bhansali *et al.* (2007a).

Suppose now, in addition, that $g_n(x) \equiv g(x)$, $n \ge 1$ and $|g(x)| \le C|x|^{-\alpha}$, $|x| \le \pi$. Then

$$\int_{-\pi}^{\pi} g^2(x) dx < \infty, \quad b_n(t) = b(t) \quad \text{and} \quad \sum_{t=-\infty}^{\infty} b^2(t) < \infty$$

and, in addition, $k_n = 1$ in (1.11). Hence

$$||A||^2 = \textstyle\sum_{t,s=1}^{n} b^2(t-s) = \sum_{k=-n}^{n} b^2(k)(n - |k|) \sim n \sum_{t=-\infty}^{\infty} b^2(t) \text{ as } n \to \infty$$

and

$$||A_n||_{sp} \le Cn^{\alpha} = o(n^{1/2}) = o(||A||)$$

which implies (1.6). Moreover,

$$\sum_{t,s=1:\,|t-s|\ge L}^{n} a_{n;ts}^2 = \sum_{t,s=1:\,|t-s|\ge L}^{n} b^2(t-s) \le n \sum_{|k|\ge L} b^2(|k|).$$

Since $\sum_{|k|\ge L} b^2(|k|) \to 0$ as $L \to \infty$, we obtain (1.8). This together with Theorem 1.1 implies the following corollary.

Corollary 1.1. *Let*

$$Q_n = \sum_{t,k=1}^{n} b(t-k)\eta_t\eta_k,$$

where $b(t) = b(-t)$, $b(0) = 0$ are real weights and $\{\eta_j\}$ is a stationary ergodic m.d. noise such that $E\eta_j^4 < \infty$ and (1.3) hold.

(i) *If $\sum_{t=0}^{\infty} |b(t)| < \infty$, then*

(1.12) $$\exists c_1, c_2 > 0: \ c_1 n \le \mathrm{Var}(Q_n) \le c_2 n, \quad n \ge 1,$$

(1.13) $$(\mathrm{Var}(Q_n))^{-1/2}(Q_n - EQ_n) \xrightarrow{d} N(0,1).$$

(ii) *If $b(t) = \int_{-\pi}^{\pi} e^{itx} g(x)dx$, $t = 0, 1, ...$, where $g(x)$, $|x| \le \pi$ is an even real function such that for some $0 \le \alpha < 1/2$ and $C > 0$,*

(1.14) $$|g(x)| \le C|x|^{-\alpha}, \qquad |x| \le \pi$$

then (1.12) and (1.13) hold.

Next we consider two quadratic forms

(1.15) $$Q_n^{(1)} = \sum_{t,s=1}^{n} a_{n;ts}^{(1)}\eta_t\eta_s, \quad \textit{and} \quad Q_n^{(2)} = \sum_{t,s=1}^{n} a_{n;ts}^{(2)}\eta_t\eta_s,$$

with corresponding matrices $A_n^{(1)}$, $A_n^{(2)}$ and a m.d. sequence η_t which satisfy the assumptions of Theorem 1.1, so that

$$(\mathrm{Var}(Q_n^{(i)}))^{-1/2}(Q_n^{(i)} - EQ_n^{(i)}) \xrightarrow{d} N(0,1), \quad i = 1, 2.$$

The next corollary provides additional sufficient condition that implies asymptotic normality of their sum.

Corollary 1.2. *Suppose that the quadratic forms $Q_n^{(1)}$, $Q_n^{(2)}$ in (1.15) satisfy the assumptions of Theorem 1.1. Set*

$$A_n = A_n^{(1)} + A_n^{(2)}.$$

If in addition

$$\lim_{n\to\infty} ||A_n^{(1)}||^{-1}||A_n^{(2)}||^{-1}\mathrm{tr}(A_n^{(1)}A_n^{(2)}) = 0 \tag{1.16}$$

then the quadratic form $Q_n := Q_n^{(1)} + Q_n^{(2)}$ *satisfies*

$$\exists c_1, c_2 > 0: \quad c_1(||A_n^{(1)}|| + ||A_n^{(2)}||) \le \mathrm{Var}(Q_n) \le c_2(||A_n^{(1)}|| + ||A_n^{(2)}||), \quad n \ge 1,$$

and

$$(\mathrm{Var}(Q_n))^{-1/2}(Q_n - EQ_n) \xrightarrow{d} N(0,1).$$

Proof. We have $Q_n = \sum_{t,s=1}^n a_{n;ts}\eta_t\eta_s$ where $a_{n;ts} = a_{n;ts}^{(1)} + a_{n;ts}^{(2)}$. Thus, to prove the corollary, it suffices to show that A_n satisfies assumptions of Theorem 1.1. This easily follows from the fact that both $A_n^{(1)}$ and $A_n^{(2)}$ satisfy assumptions of Theorem 1.1, and the property

$$||A_n||^2 = (||A_n^{(1)}||^2 + ||A_n^{(2)}||^2)(1 + o(1)).$$

The latter follows from

$$||A_n||^2 = ||A_n^{(1)}||^2 + ||A_n^{(2)}||^2 + 2\mathrm{tr}(A_n^{(1)}A_n^{(2)})$$

because the matrices $A_n^{(1)}$ and $A_n^{(2)}$ are symmetric so the cross term

$$2\sum_{t,s} a_{n;ts}^{(1)}a_{n;ts}^{(2)} = 2\sum_{t,s} a_{n;ts}^{(1)}a_{n;st}^{(2)} = 2\mathrm{tr}(A_n^{(1)}A_n^{(2)}).$$

Hence

$$||A_n||^2 = (||A_n^{(1)}||^2 + ||A_n^{(2)}||^2)(1 + r_n)$$

where

$$r_n = 2\mathrm{tr}(A_n^{(1)}A_n^{(2)})/(||A_n^{(1)}||^2 + ||A_n^{(2)}||^2).$$

Since $||A_n^{(1)}||^2 + ||A_n^{(2)}||^2 \ge 2||A_n^{(1)}||\,||A_n^{(2)}||$ we get $r_n = o(1)$ by (1.16). □

Corollary 1.2 indicates that we need the additional condition (1.16) in order to obtain the asymptotic normality of Q_n. It does not imply that in this case the components $Q_n^{(1)}$ and $Q_n^{(2)}$ are asymptotically uncorrelated and hence asymptotically independent. We conjecture that $Q_n^{(1)}$ and $Q_n^{(2)}$ will be asymptotically independent in the case when η_t is an i.i.d. noise.

2 Proof of Theorem 1.1

In the proof of Theorem 1.1 we shall use the following result.

Lemma 2.1. *(Dalla, Giraitis and Koul (2014), Lemma 10).*
(i) *Let*

$$T_n = \sum_{j\in Z} c_{nj}V_j,$$

where $\{V_j\}$, $j \in Z = \{\cdots, -1, 0, 1, \cdots\}$ *is a stationary ergodic sequence,* $E|V_1| < \infty$*, and* c_{nj} *are real numbers such that for some* $0 < \alpha_n < \infty$, $n \geq 1$,

$$\sum_{j \in Z} |c_{nj}| = O(\alpha_n), \quad \sum_{j \in Z} |c_{nj} - c_{n,j-1}| = o(\alpha_n). \tag{2.17}$$

Then

$$E|T_n - ET_n| = o(\alpha_n).$$

In particular, if $\alpha_n = 1$*, then*

$$T_n = ET_n + o_p(1).$$

(ii) *If the m.d. sequence* η_t *satisfies* $\max_t E|\eta_t|^p < \infty$*, for some* $p \geq 2$*, then*

$$E\big|\sum_{j \in Z} d_j \eta_j\big|^p \leq C\big(\sum_{j \in Z} d_j^2\big)^{p/2}, \tag{2.18}$$

for any d_j*'s such that* $\sum_{j \in Z} d_j^2 < \infty$*, where* $C < \infty$ *does not depend on* d_j*'s.*

For the convenience of the reader we provide the proof of the following lemma.

Lemma 2.2. *One has*

$$\max_{t=1,\dots,n} \sum_{s=1}^n a_{n;ts}^2 \leq ||A_n||_{sp}^2, \qquad \max_{t,s=1,\dots,n} |a_{n;ts}| \leq ||A_n||_{sp}. \tag{2.19}$$

Proof. We drop the index n and let $A = (a_{ts})$. The second inequality $|a_{ts}| \leq ||A_n||_{sp}$ follows from the first since

$$\max_{t,s} a_{ts}^2 \leq \max_t \sum_{s=1}^n a_{ts}^2 \leq ||A_n||_{sp}^2.$$

Turning to the first inequality, we have $||A_n||_{sp}^2 = \sup_{||x||=1} ||Ax||^2$ where $x = (x_1, ..., x_n)'$ and

$$||Ax||^2 = ||\sum_{s=1}^n a_{1s}x_s, ..., \sum_{s=1}^n a_{ns}x_s||^2 = (\sum_{s=1}^n a_{1s}x_s)^2 + ... + (\sum_{s=1}^n a_{ns}x_s)^2.$$

Set $y = (0, ..., 0, 1, 0, ..., 0)'$ where 1 is at the t_0 position. Note that $||y|| = 1$. Then

$$||A_n||_{sp}^2 \geq ||Ay||^2 = a_{1t_0}^2 + ... + a_{nt_0}^2 = \sum_{s=1}^n a_{st_0}^2 = \sum_{s=1}^n a_{t_0 s}^2$$

since A is symmetric. Hence

$$||A_n||_{sp}^2 \geq \max_{t_0=1,\dots,n} \sum_{s=1}^n a_{t_0 s}^2.$$

□

Proof of Theorem 1.1.

Using (1.6), the second claim of (2.19) implies

$$\max_{1 \leq k,u \leq L} |a_{n;ku}| = o(||A||), \qquad \forall L \geq 1 \text{ fixed}. \tag{2.20}$$

Relation (2.20) implies that no single $a_{n;ku}$ dominates.

• *Proof of (1.7).* Below we write a_{ts} instead of $a_{n;ts}$. Let

$$z_{nt} = 2\eta_t \sum_{s=1}^{t-1} a_{ts}\eta_s \quad \text{and} \quad z'_t = a_{tt}(\eta_t^2 - E\eta_t^2). \tag{2.21}$$

Then

$$Q_n - EQ_n = \sum_{t=2}^{n} z_{nt} + \sum_{t=1}^{n} z'_{nt} = S_n + S'_n. \tag{2.22}$$

Observe that $E\eta_t\eta_s = 0$ for $t > s$ and hence $ES_n = 0$ since η_s is a m.d. sequence. In addition,

$$ES_n^2 = 4\sum_{t=2}^{n} E\big[\eta_t^2(\sum_{s=1}^{t-1} a_{ts}\eta_s)^2\big]. \tag{2.23}$$

Using $E\eta_t^4 \le C$ and (1.4),

$$E|S'_n| \le C\sum_{t=1}^{n} |a_{tt}| = o(||A_n||), \quad ES_n'^2 \le C(\sum_{t=1}^{n} |a_{tt}|)^2 = o(||A_n||^2). \tag{2.24}$$

Now we show that

$$c_1||A_n||^2 \le ES_n^2 \le c_2||A_n||^2.$$

The lower bound follows by using (1.3) and (1.5) because of the fact that $c > 0$:

$$\begin{aligned} ES_n^2 &= 4\sum_{t=2}^{n} E\big[\eta_t^2(\sum_{s=1}^{t-1} a_{ts}\eta_s)^2\big] = 4\sum_{t=2}^{n} E\big[E[\eta_t^2|\mathcal{F}_{t-1}](\sum_{s=1}^{t-1} a_{ts}\eta_s)^2\big] \\ &\ge 4c\sum_{t=2}^{n} E(\sum_{s=1}^{t-1} a_{ts}\eta_s)^2 = 4c\sum_{t=2}^{n}\sum_{s=1}^{t-1} a_{ts}^2 \\ &= 2c\sum_{t,s=1}^{n} a_{ts}^2 - 2c\sum_{t=1}^{n} a_{tt}^2 = 2||A||^2 - o(||A||^2) \ge ||A||^2, \end{aligned} \tag{2.25}$$

for large n.

To prove the upper bound, notice that

$$\begin{aligned} ES_n^2 &= 4\sum_{t=2}^{n} E\big[\eta_t^2(\sum_{s=1}^{t-1} a_{ts}\eta_s)^2\big] \\ &\le 4\sum_{t=2}^{n} (E\eta_t^4)^{1/2}(E(\sum_{s=1}^{t-1} a_{ts}\eta_s)^4)^{1/2}\big] \le C\sum_{t,s=1}^{n} a_{ts}^2 = C||A||^2 \end{aligned} \tag{2.26}$$

by (2.18) and assumption $E\eta_t^4 = E\eta_1^4 < \infty$. To obtain (1.7), note that

$$\mathrm{Var}(Q_n) \le 2ES_n^2 + 2E{S'_n}^2 \le C||A||^2 + o(||A||^2) \le 2C||A||^2$$

by (2.24) and (2.26). In addition, (2.22)-(2.26) imply

$$\text{Var}(Q_n) = (ES_n^2)(1 + o(1)), \quad n \to \infty. \tag{2.27}$$

Indeed, by (2.22),

$$\begin{aligned} |\text{Var}(Q_n) - \text{Var}(S_n)| &= |\text{Var}(S_n') + 2\text{Cov}(S_n, S_n')| \le \text{Var}(S_n') + 2\big(\text{Var}(S_n)\text{Var}(S_n')\big)^{1/2} \\ &= o(||A||^2) + \big(O(||A||^2)o(||A||^2)\big)^{1/2} = o(||A||^2) \end{aligned}$$

so that $\text{Var}(Q_n) = \text{Var}(S_n) + o(||A||^2)$ and by (2.25) we have $ES_n^2 \ge ||A||^2$, which leads to (2.27).

• *Proof of (1.10).* We now prove the asymptotic normality of Q_n. Let $B_n^2 = \text{Var}(Q_n)$, $X_{nt} = B_n^{-1} z_{nt}$ and $X_t' = B_n^{-1} z_{nt}'$. Then, by (2.22)

$$B_n^{-1}(Q_n - EQ_n) = \sum_{t=2}^n X_{nt} + \sum_{t=1}^n X_{nt}'. \tag{2.28}$$

Observe that by (1.7) and (2.24), $E|\sum_{t=1}^n X_t'| = B_n^{-1} E|\sum_{s=1}^n z_{nt}'| \le C||A_n||^{-1} \sum_{t=1}^n |a_{tt}| = o(1)$. Therefore, to prove (1.10) it remains to show that

$$\sum_{t=2}^n X_{nt} \stackrel{d}{\to} N(0,1). \tag{2.29}$$

Since X_{nt} is a m.d. sequence, then by Theorem 3.2 of Hall and Heyde (1980), it suffices to show

$$(a)\, E\max_{1\le j\le n} X_{nj}^2 \to 0, \ \ (b)\, \max_{1\le j\le n} |X_{nj}| \to_p 0, \ \ (c)\, \textstyle\sum_{j=1}^n X_{nj}^2 \to_p 1. \tag{2.30}$$

•• To verify (a) and (b), it suffices to show that for any $\varepsilon > 0$,

$$\sum_{j=1}^n EX_{nj}^2 I(|X_{nj}| \ge \varepsilon) \to 0, \tag{2.31}$$

which clearly implies (a), while (b) follows from (2.31) noting that

$$P\big(\max_{1\le j\le n} |X_{nj}| \ge \varepsilon\big) \le \varepsilon^{-2} \sum_{j=1}^n EX_{nj}^2 I(|X_{nj}| \ge \varepsilon) \to 0.$$

To prove (2.31), let $K > 0$ be large. We consider two cases: $\eta_t^2 \le K$ and $\eta_t^2 > K$. Then,

$$\begin{aligned} EX_{nt}^2 I(X_{nt}^2 \ge \varepsilon) I(\eta_t^2 \le K) &\le \varepsilon^{-1} EX_{nj}^4 I(\eta_t^2 \le K) \le \varepsilon^{-1} 2^4 K^2 B_n^{-4} E\Big(\sum_{s=1}^{t-1} a_{ts}\eta_s\Big)^4 \\ &\le C\varepsilon^{-1} K^2 B_n^{-4} \Big(\sum_{s=1}^{t-1} a_{ts}^2\Big)^2 \le C\varepsilon^{-1} K^2 B_n^{-4} ||A||_{sp}^2 \sum_{s=1}^{t-1} a_{ts}^2 \end{aligned}$$

by (2.18) and (2.19). Recall that by (1.7), $B_n^{-2} \le C||A||^{-2}$. Thus, for any $\varepsilon > 0$ and $K > 0$,

$$\begin{aligned} \sum_{t=2}^n EX_{nt}^2 I(X_{nt}^2 \ge \varepsilon) I(\eta_t^2 \le K) &\le C\varepsilon^{-1} K^2 B_n^{-4} ||A||_{sp}^2 \sum_{t=2}^n \sum_{s=1}^{t-1} a_{ts}^2 \\ &\le C\varepsilon^{-1} K^2 (||A||_{sp}/||A||)^2 \to 0 \end{aligned} \tag{2.32}$$

by (1.6) as $n \to \infty$ for any finite K.

We now focus on the case $\eta_t^2 \geq K$. Since $E\eta_t^4 < \infty$ and, by stationarity of η_t, $\delta_K := E\eta_1^4 I(\eta_1^2 > K) \to 0$ as $K \to \infty$, this implies

$$EX_{nt}^2 I(X_{nt}^2 \geq \varepsilon) I(\eta_t^2 > K) \leq EX_{nt}^2 I(\eta_t^2 > K) \leq B_n^{-2} 2^2 E\Big[\eta_t^2 I(\eta_t^2 > K)\big(\sum_{s=1}^{t-1} a_{ts}\eta_s\big)^2\Big]$$
$$\leq C||A||^{-2}\delta_K^{1/2}(E\big(\sum_{s=1}^{t-1} a_{ts}\eta_s\big)^4)^{1/2} \leq C||A||^{-2}\delta_K^{1/2}\sum_{s=1}^{t-1} a_{ts}^2$$

by (2.18). Hence,

$$\sum_{t=2}^{n} EX_{nt}^2 I(X_{nt}^2 \geq \varepsilon) I(\eta_t^2 > K) \leq C\delta_K^{1/2}||A||^{-2}\sum_{t=2}^{n}\sum_{s=1}^{t-1} a_{ts}^2 \tag{2.33}$$
$$\leq C\delta_K^{1/2} \to 0, \qquad K \to \infty.$$

Since (2.32) holds for any fixed K as $n \to \infty$, and since (2.33) holds as $K \to \infty$ uniformly in n, we get (2.31).

•• The verification of (c) in (2.30) is particularly delicate. We want to show that $s_n \to_p 1$. Recall that $x_{nt} = B^{-1}z_{nt}$ where z_{nt} is defined in (2.21). We shall decompose $s_n = \sum_{s=1}^n X_{ns}^2$ into two parts involving $L > 1$. Write

$$s_n = 4B_n^{-2}\sum_{t=1}^{n}\eta_t^2\big(\sum_{s=1}^{t-1} a_{ts}\eta_s\big)^2 = s_{n,1} + s_{n,2}, \tag{2.34}$$

where

$$s_{n,1} := 4B_n^{-2}\sum_{t=1}^{n}\eta_t^2\big(\sum_{s=t-L}^{t-1} a_{ts}\eta_s\big)^2, \quad s_{n,2} := s_n - s_{n,1}.$$

Then,

$$s_n = Es_n + (s_{n,1} - Es_{n,1}) + (s_{n,2} - Es_{n,2}).$$

We show that as $n \to \infty$,

$$(i)\ \ Es_n \to 1; \qquad (ii)\ \ s_{n,1} - Es_{n,1} \to_p 0, \quad \forall L \geq 1; \tag{2.35}$$
$$(iii)\ \ E|s_{n,2}| \to 0, \quad n \to \infty, \quad L \to \infty$$

which proves (2.30)(c) since $E|s_n| \to 0$ implies $s_n \to_P 0$ as $n \to \infty$ and $L \to \infty$.

••• The claim (2.35)(i) follows from (2.27),

$$(ES_n^2)/\mathrm{Var}(Q_n) = B_n^{-2}ES_n^2 \to 1,$$

noting that $B_n^{-2}ES_n^2 = Es_n$, which holds by definition of s_n and (2.23).

• • • To show (2.35)(ii), open up the squares, set $s = t - k$ and $s' = t - u$, to get

$$s_{n,1} - Es_{n,1} = 4 \sum_{k,u=1}^{L} \Big\{ B_n^{-2} \sum_{t=1}^{n} a_{t,t-k} a_{t,t-u} [\eta_t^2 \eta_{t-k} \eta_{t-u} - E\eta_t^2 \eta_{t-k} \eta_{t-u}] \Big\} = 4 \sum_{k,u=1}^{L} g_{n,ku}.$$

It suffices to verify that for any fixed k and u, $g_{n,ku} = o_p(1)$. Setting

$$c_{nt} := B_n^{-2} a_{t,t-k} a_{t,t-u}$$

and

$$V_t := \eta_t^2 \eta_{t-k} \eta_{t-u} - E\eta_t^2 \eta_{t-k} \eta_{t-u},$$

write

$$g_{n,ku} = \sum_{t=1}^{n} c_{nt} V_t.$$

Since the noise $\{\eta_t\}$ is stationary ergodic and such that $E\eta_1^4 < \infty$, by Theorem 3.5.8 in Stout (1974), the process $\{V_j\}$ is stationary and ergodic, and $E|V_1| < \infty$. Because of the centering, $Eg_{n,ku} = 0$. Thus, by Lemma 2.1(i), to prove $g_{n,ku} = o_p(1)$, it remains to show that c_{nt}'s satisfy (2.17) with $\alpha_n = 1$. Observe that

$$\sum_{t \in Z} |c_{nt}| = B_n^{-2} \sum_{t=1}^{n} |a_{t,t-k} a_{t,t-u}| \le 2B_n^{-2} \sum_{t,s=1}^{n} a_{t,s}^2 = 2B_n^{-2} ||A||^2 \le C, \quad n \to \infty$$

by (1.7). On the other hand,

$$\begin{aligned}
\sum_{t \in Z} |c_{nt} - c_{n,t-1}| &= B_n^{-2} \sum_{t=1}^{n+1} |a_{t,t-k} a_{t,t-u} - a_{t-1,t-1-k} a_{t-1,t-1-u}| \\
&\le B_n^{-2} \sum_{t=1}^{n+1} \{ |a_{t,t-k} - a_{t-1,t-1-k}| |a_{t,t-u}| + |a_{t-1,t-1-k}| |a_{t,t-u} - a_{t-1,t-1-u}| \} \\
&\le B_n^{-2} \{ (\sum_{t=1}^{n+1} (a_{t,t-k} - a_{t-1,t-1-k})^2)^{1/2} + \sum_{t=1}^{n+1} (a_{t,t-u} - a_{t-1,t-1-u})^2)^{1/2} \} (\sum_{t,s=1}^{n} a_{t,s}^2)^{1/2} \\
&= o(B_n^{-2} ||A||^2) = o(1),
\end{aligned}$$

by (1.9), (2.19) and (1.7). Hence (2.17) holds. By Lemma 2.1(i) we conclude that $g_{n,ku} = o_p(1)$ and, thus, $s_{n,1} - Es_{n,1} = o_p(1)$. Hence (2.35)(ii) holds.

• • • To verify $E|s_{n,2}| \to 0$ in (2.35)(iii), write

$$s_{n,2} = s_n - s_{n,1} = 4B_n^{-2} \sum_{t=1}^{n} \eta_t^2 \big[\big(\sum_{s=1}^{t-1} a_{ts} \eta_s \big)^2 - \big(\sum_{s=t-L}^{t-1} a_{ts} \eta_s \big)^2 \big].$$

We use the identity $a^2 - b^2 = (a-b)^2 + 2(a-b)b$, to obtain

$$\begin{aligned}
|s_{n,2}| &= 4B_n^{-2}\Big|\sum_{t=1}^{n}\eta_t^2\{\big(\sum_{s=1}^{t-1}a_{ts}\eta_s\big)^2 - \big(\sum_{s=t-L}^{t-1}a_{ts}\eta_s\big)^2\}\Big| \\
&= 4B_n^{-2}\Big|\sum_{t=1}^{n}\eta_t^2\{\big(\sum_{s=1}^{t-L-1}a_{ts}\eta_s\big)^2 + 2\big(\sum_{s=1}^{t-L-1}a_{ts}\eta_s\big)\big(\sum_{s=t-L}^{t-1}a_{ts}\eta_s\big)\}\Big| \\
&\le 4q_{n,1} + 4\Big(B_n^{-2}\sum_{t=1}^{n}\eta_t^2\big(\sum_{s=1}^{t-L-1}a_{ts}\eta_s\big)^2\Big)^{1/2}\Big(4B_n^{-2}\sum_{t=1}^{n}\eta_t^2\big(\sum_{s=t-L}^{t-1}a_{ts}\eta_s\big)^2\Big)^{1/2} \\
&\le 4(q_{n,1} + q_{n,1}^{1/2}s_{n,1}^{1/2}),
\end{aligned}$$

where

$$q_{n,1} := B_n^{-2}\sum_{t=1}^{n}\eta_t^2\big(\sum_{s=1}^{t-L-1}a_{ts}\eta_s\big)^2.$$

Hence, $E|s_{n,2}| \le 4Eq_{n,1} + 4(Eq_{n,1}Es_{n,1})^{1/2}$. To bound $Eq_{n,1}$, we argue partly as in (2.26):

$$Eq_{n,1} \le C||A_n||^{-2}\sum_{t=1}^{n}\sum_{s=1}^{t-L-1}a_{ts}^2 \to 0, \quad n\to\infty,\ L\to\infty$$

by (1.8). We also have

$$Es_{n,1} \le C||A_n||^{-2}\sum_{t=1}^{n}\sum_{s=t-L}^{t-1}a_{ts}^2 \le C.$$

Hence $E|s_{n,2}| \to 0$ as $n\to\infty$ and $L\to\infty$. This completes the proof of (2.35)(iii) and the theorem. □

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