Unique Low Rank Completability of Partially Filled Matrices

Bill Jackson* Tibor Jordán† Shin-ichi Tanigawa‡

Revised 1 August, 2016

Abstract

We consider the problems of completing a low-rank positive semidefinite square matrix M or a low-rank rectangular matrix N from a given subset of their entries. We study the local and global uniqueness of such completions by analysing the structure of the graphs determined by the positions of the known entries of M or N.

We show that, in the generic setting, the unique completability testing of rectangular matrices is a special case of the unique completability testing of positive semidefinite matrices. We prove that a generic partially filled positive semidefinite $n \times n$ matrix is globally uniquely ranks d completable if any principal minor of size $n-1$ is locally uniquely ranks d completable. These results are based on new geometric observations that extend similar results of the theory of rigid frameworks. We also give an example showing that global completability is not a generic property in $\mathbb{R}^2$.

We provide sufficient conditions for local and global unique completability of a partially filled matrix in terms of either the minimum number of known entries per row or the total number of known entries.

1 Introduction

We consider the problem of determining the uniqueness of a low-rank positive semidefinite completion of a partially filled matrix. This completion problem and its variants arise in various practical problems, such as computer vision, machine learning and control, and several completion algorithms have been developed and implemented in recent decades,

*School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, England. email: B.Jackson@qmul.ac.uk

†Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary, and MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Budapest, Hungary. e-mail: jordan@cs.elte.hu

‡Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan, and Centrum Wiskunde & Informatica (CWI), Postbus 94079, 1090 GB Amsterdam, The Netherlands. e-mail: tanigawa@kurims.kyoto-u.ac.jp

see for example [24, 21, 18, 16]. It is also related to the fundamental problem of Euclidean distance geometry and has been investigated from several different viewpoints, see for example [9, 17].

Singer and Cucuringu [20] initiated an analysis of this problem using techniques from graph rigidity theory. They defined the underlying graph of a partially filled positive semidefinite matrix $M = (m_{ij})$ of size n as the graph G with vertex set $V = \{1, \dots, n\}$, in which ij is an edge if and only if the (i, j)-th entry (or (j, i)-th entry) is known. Note that G is *semisimple*, meaning that it has no parallel edges but may have loops.

Recall that a positive semidefinite matrix of size n and rank d can be written as $P^\top P$ for some $d \times n$ matrix P. Hence, finding a completion of M corresponds to finding a map $p : V \to \mathbb{R}^d$ such that

$$\langle p_i, p_j \rangle = m_{ij} \qquad \text{for all } ij \in E$$

where $p_i = p(i)$. Therefore, assuming that a completion is known in advance, the unique completability problem can be restated as follows. We are given a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$. We need to decide whether there exists a $q : V \to \mathbb{R}^d$ such that $\langle p_i, p_j \rangle = \langle q_i, q_j \rangle$ for all $ij \in E$ and $\langle p_k, p_l \rangle \neq \langle q_k, q_l \rangle$ for some $k, l \in V$.

We will adopt the terminology from rigidity theory and refer to a pair (G, p) as a *(d-dimensional) framework*. Two maps $p : V \to \mathbb{R}^d$ and $q : V \to \mathbb{R}^d$ are said to be *congruent* if

$$\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \qquad \text{for all } i, j \in V \tag{1}$$

and we say that (G, q) is *equivalent* to (G, p) if

$$\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \qquad \text{for all } ij \in E. \tag{2}$$

A d-dimensional framework (G, p) is called *globally uniquely completable* (or, simply, *globally completable*) in $\mathbb{R}^d$ if for every d-dimensional framework (G, q) which is equivalent to (G, p) we have that p and q are congruent. A recent result of E.-Nagy, Laurent, and Varvitsiotis [5] shows that the decision problem of asking whether a partially filled matrix can be completed to a positive semidefinite matrix of rank at most d is NP-hard for any fixed integer $d \geq 2$ (even if it has an all-ones diagonal). The proof of this result, combined with ideas from Saxe [19], can be used to show that testing the global completability of d-dimensional frameworks is also hard for all $d \geq 2$.

The local version of the uniqueness of the completion can also be defined by using terminology inspired by rigidity theory. A framework (G, p) is *locally uniquely completable* (or, simply, *locally completable*) in $\mathbb{R}^d$ if there exists an open neighborhood $N(p)$ of p in $\mathbb{R}^{d|V|}$ (regarding a map p as a point in $\mathbb{R}^{d|V|}$) such that for any $q \in N(p)$ the equivalence of (G, q) to (G, p) implies that p and q are congruent.[1]

[1] It can be seen that, as in the case of rigidity, the local completability of (G, p) is equivalent to the fact that every continuous motion of the vertices of (G, p) in $\mathbb{R}^d$ which preserves equivalence must also preserve congruence.

The local or global rigidity of a framework is defined by replacing each inner product with the squared distances in (1) and (2), respectively. An important fact in rigidity theory is that local and global rigidity are both generic properties, meaning that, if a framework (G, p) is locally (globally) rigid for a generic p, then (G, q) is locally (globally) rigid for all generic q. (We say that p is *generic* if the set of the coordinates in $p(V)$ is algebraically independent over $\mathbb{Q}$.) This was first pointed out by Gluck [7] and Asimov and Roth [1] for local rigidity. For global rigidity, the generic property was first conjectured by Connelly [3] and was recently confirmed by Gortler, Healy, and Thurston [8]. This leads to a polynomial-time randomized algorithm for checking local or global rigidity of generic frameworks. Singer and Cucuringu [20] showed that several concepts in rigidity theory can be naturally extended to the completability setting and gave a randomized algorithm for checking local completability as well as a heuristic algorithm for global completability in the generic case. An advantage of this approach is that the algorithms use only the underlying graphs of the frameworks.

There is a direct connection between rigidity and completability. For a framework (G, p) with a simple graph G and a map $p : V \to \mathbb{S}^d$, the rigidity of (G, p) on the d-dimensional sphere $\mathbb{S}^d$ is equivalent to the completability of (G°, p) in $\mathbb{R}^{d+1}$, where G° denotes the graph obtained from G by adding a loop at each vertex. The rigidity of frameworks on the sphere is a classical concept and is closely related to the rigidity in Euclidean space via the so-called coning technique (see, e.g., [4, 27]). In fact, at a generic level, the local (global) completability of G° in $\mathbb{R}^{d+1}$ is equivalent to the local (global) rigidity of G in $\mathbb{R}^d$ (see [13, Corollary 2.6 and Corollary 2.7]).

In [13] we began a more detailed analysis of the relationship between rigidity and completability. This paper is a sequel to [13]. We will show that two-dimensional global completability is not a generic property, suggesting a difficulty for checking global completability in the existing theory. On the positive side, we show that a generic framework (G, p) is globally completable if $(G - v, p)$ is locally completable for every $v \in V$. This in turn implies that a generic partially filled semidefinite $n \times n$ matrix is globally uniquely rank d completable if any principal minor of size $n - 1$ is locally uniquely rank d completable.

The paper is organized as follows. In Section 2, we give preliminary results that will be used throughout the paper. In Section 2.3 we discuss the unique completability problem for low rank rectangular matrices (which was also introduced by Singer and Cucuringu [20]). In Section 3.1 and Section 3.2, we shall introduce the concept of canonical positions and standard positions, which are adaptations of concepts from rigidity theory. In Section 3.3, we show that the unique completability testing of rectangular matrices is a special case of the unique completability testing of positive semidefinite matrices. In Section 4, we give some geometric observations on completability. These observations are used in Section 5 to show that a generic partially filled matrix is globally completable if any principal minor of size $n - 1$ is locally completable. In Section 6, we give three examples that indicate a difficulty in characterizing 2-dimensional generic global completability by using existing techniques from rigidity theory. In particular we give an example showing that global completability is not a generic property in $\mathbb{R}^2$. In Section 7 we provide sufficient conditions for local and global unique rank d completability of a partially filled $n \times n$ matrix in terms of either the minimum number of known entries per row, or the total number of known entries, as functions of n and d.

2 Preliminaries

2.1 Infinitesimal completability and the completability matroid

One may define the infinitesimal version of local completability based on an analogy with infinitesimal rigidity. A map $\dot{p} : V \to \mathbb{R}^d$ is called an *infinitesimal c-motion* of (G, p) if

$$\langle p_i, \dot{p}_j \rangle + \langle p_j, \dot{p}_i \rangle = 0 \qquad (ij \in E). \tag{3}$$

The $|E| \times d|V|$-matrix representing this system of linear equations with variables $\dot{p}$ is the *completability matrix* of (G, p), denoted by $C(G, p)$. (Thus the entries of $C(G, p)$ in the d-tuples of positions i and j of row $e = ij$ are p_j and p_i, respectively, and all other entries are zeros.)

For any $d \times d$ skew-symmetric matrix S, the map $\dot{p} : V \to \mathbb{R}^d$ defined by $\dot{p}_i = Sp_i$ for $i \in V$ is an infinitesimal c-motion. (The infinitesimal c-motions of this kind are called *trivial*.) Therefore, if $|V| \geq d$, then

$$\operatorname{rank} C(G, p) \leq dn - \binom{d}{2}. \tag{4}$$

Clearly the rank of $C(G, p)$ is also bounded above by the number of edges in the complete semisimple graph on n vertices. A framework (G, p) is said to be *infinitesimally completable* if $\operatorname{rank} C(G, p) = dn - \binom{d}{2}$ when $n \geq d$ or $\operatorname{rank} C(G, p) = \binom{n+1}{2}$ when $n \leq d$. It is *c-independent* if $\operatorname{rank} C(G, p) = |E|$. Note that the rank of $C(G, p)$ will be the same for all generic realizations of G. Singer and Cucuringu [20] showed that infinitesimal completability is a sufficient condition for local completability, and that the two properties are equivalent when (G, p) is generic. Hence we say that the graph G is *locally completable* or *c-independent* in $\mathbb{R}^d$ if some (or equivalently, every) generic realization of G in $\mathbb{R}^d$ is locally completable or c-independent. It follows that in the generic case, the local uniqueness of a completion of a partial positive semi-definite matrix depends only on the underlying graph G, which is determined by the positions of the known entries.

The d-dimensional *completability matroid* $\mathcal{C}_d(G)$ of G is the matroid on E in which a set of edges is independent if and only if the corresponding set of rows in $C(G, p)$ is linearly independent, for some generic $p : V \to \mathbb{R}^d$. We say that G is *c-independent* if E is independent in $\mathcal{C}_d(G)$. The following necessary condition for c-independence was observed in [20]. We use $i_G(X)$ to denote the number of edges induced by a set X of vertices in graph G.

Lemma 1 ([20]). *Let $G = (V, E)$ be c-independent in $\mathbb{R}^d$. Then*
(i) $i_G(X) \leq d|X| - \binom{d}{2}$ for all $X \subseteq V$ with $|X| \geq d$, and
(ii) for each bipartite subgraph $H = (V_1, V_2; F)$ on vertex set $X = V_1 \cup V_2$ with $|V_i| \geq d$, $i = 1, 2$ we have $i_H(X) \leq d|X| - d^2$.

We say that a graph G is *globally completable* in $\mathbb{R}^d$ if *every* generic realization of G in $\mathbb{R}^d$ is globally completable. In Section 6 we show that global completability is not a generic property in general, unlike in the case of global rigidity.

2.2 The rectangular matrix model

Singer and Cucuringu [20] also considered the unique completability of *low rank rectangular matrices*, i.e. rectangular matrices of the form $P^\top Q$ for some $d \times n$ matrix P and $d \times m$ matrix Q. In this case the known entries of the rectangular matrix define a bipartite graph $G = (V, E)$ with bipartition (U, W) in which $|U| = n$, $|W| = m$, and an edge ij corresponds to the known scalar product of row i in $P^\top$ and column j in Q.

We say that two bipartite frameworks (G, p) and (G, q) are *bicongruent* if $\langle p_i, p_j \rangle = \langle q_i, q_j \rangle$ holds for every pair $i \in U$ and $j \in W$. This is equivalent to saying that there exists an invertible matrix A such that $q_i = A^T p_i$ and $q_j = A^{-1} p_j$ for all $i \in U$ and $j \in W$. The framework (G, p) is *globally bicompletable* if every framework which is equivalent to (G, p), is bicongruent to (G, p). Similarly, (G, p) is said to be *locally bicompletable* if there exists an open neighborhood $N(p)$ of p such that for any $q \in N(p)$ the equivalence of (G, q) to (G, p) implies that the two frameworks are bicongruent.

For any $d \times d$ matrix A, the map $\dot{p} : \mathbb{R}^d \to \mathbb{R}^d$ by $\dot{p}_i = A p_i$ for $i \in U$ and $\dot{p}_j = -A^T p_j$ for $j \in W$ is an infinitesimal c-motion of (G, p), and hence

$$\text{rank } C(G, p) \leq d|V| - d^2 \tag{5}$$

whenever $|U|, |V| \geq d$. (This gives Lemma 1(ii).) It is also bounded above by $|U|\,|W|$. We say that (G, p) is *infinitesimally bicompletable* if rank $C(G, p) = d|V| - d^2$ when $\min\{|U|\,|W|\} \geq d$ and rank $C(G, p) = |U|\,|V|$ when $\min\{|U|\,|W|\} < d$. Local bicompletability and infinitesimal bicompletability are equivalent for generic bipartite frameworks. Hence we say that a bipartite graph G is *locally bicompletable in* $\mathbb{R}^d$ if (G, p) is infinitesimally bicompletable for some (or equivalently, every) generic d-dimensional framework (G, p). We say that G is *globally bicompletable in* $\mathbb{R}^d$ if every generic d-dimensional framework (G, p) is globally bicompletable.

Király et al. [16] also considered the uniqueness of matrix completion in the rectangular matrix model over the complex field. They discussed combinatorial characterizations of 1-dimensional bicompletability and corank-1-dimensional bicompletability, a sufficient condition for global bicompletability, and bicompletability of random graphs. Infinitesimal bicompletablity was also analyzed (as a special case) in Kalai et al. [15] in a different context.

2.3 Graph operations

We introduce several graph operations which were shown to preserve local (global) completability in [13]. Let $G = (V, E)$ be a semisimple graph. The (d-dimensional) *0-extension* operation adds a new vertex v to G and d new edges $vu_1, \ldots, vu_d$ for distinct vertices $u_1, \ldots, u_d \in V + v$. If we only add less than d new edges, the operation is called a *partial 0-extension*. Note that we allow one of the new edges to be a loop by taking $u_i = v$. If necessary, we will specify whether or not a loop is added by referring to the operation as a *looped extension* or a *simple extension*.

Lemma 2. *[13, Lemma 2.3] Suppose that G is obtained from G' by a 0-extension operation. Then G' is c-independent (resp. locally completable) in $\mathbb{R}^d$ if and only if G is c-independent (resp. locally completable) in $\mathbb{R}^d$.*

Lemma 3. *[13, Theorem 6.7] Let G be a globally completable graph in $\mathbb{R}^d$, and let G' be a graph obtained from G by a simple 0-extension. Then G' is globally completable in $\mathbb{R}^d$.*

Let $G = (V, E)$ be a semisimple graph. The (d-dimensional) *double 1-extension* operation removes an existing edge $e = ab$ from G and inserts two new vertices v_1 and v_2 with new edges av_1, v_1v_2, v_2b and $v_1u_1^1, v_1u_1^2, \ldots, v_1u_1^{d-1}$ and $v_2u_2^1, v_2u_2^2, \ldots, v_2u_2^{d-1}$, where $\{u_1^1, u_1^2, \ldots, u_1^{d-1}\}$ and $\{u_2^1, u_2^2, \ldots, u_2^{d-1}\}$ are $d-1$ distinct vertices in $(V + v_1) \setminus \{a\}$ and $(V + v_2) \setminus \{b\}$, respectively. We allow the possibility that e is a loop (in which case $a = b$).

Lemma 4. *[13, Lemma 4.1] Let $G = (V, E)$ be a graph and $G' = (V', E')$ be the graph obtained from G by a double 1-extension. If G is c-independent (resp. locally completable) in $\mathbb{R}^d$ then G' is also c-independent (resp. locally completable) in $\mathbb{R}^d$.*

For a vertex v_1 in a semisimple graph G, $N_G(v_1)$ denotes the set of vertices adjacent to v_1 in G, taking $v_1 \in N_G(v_1)$ when v_1 is incident to a loop. The *d-dimensional vertex-splitting* (or simply *vertex-d-splitting*) operation at v_1 (with respect to some fixed partition $\{U_0, U^*, U_1\}$ of $N(v_1)$ with $|U^*| = d$) removes the edges between v_1 and the vertices in U_0, inserts a new vertex v_0, and inserts new edges v_0u for $u \in U_0 \cup U^*$. Note that v_0 and v_1 are adjacent in the resulting graph if and only if there is a loop incident with v_1 in G and $v_1 \in U_0 \cup U^*$.

Lemma 5. *[13, Lemma 4.3] Let $G = (V, E)$ be a graph and $G' = (V', E')$ be the graph obtained from G by a vertex-d-splitting at vertex v_1. If G is c-independent in $\mathbb{R}^d$ then G' is also c-independent in $\mathbb{R}^d$.*

Let $G = (V, E)$ be a semisimple graph. The *looped cone extension* $G \circ v$ of G is obtained by adding a new vertex v and all edges uv for $u \in V + v$.

Lemma 6 ([13])**.** *Let $G = (V, E)$ be a graph and $G \circ v$ be its looped cone extension. Then G is locally completable in $\mathbb{R}^d$ if and only if $G \circ v$ is locally completable in $\mathbb{R}^{d+1}$.*

2.4 Complete graphs

Recall that for a loopless graph G we use G° to denote the graph obtained from G by adding a loop incident with each vertex. Lemmas 1 and 2 imply the following.

Lemma 7. *The looped complete graph K_n° is c-independent if and only if $n \leq d$.*

By Lemmas 1 and 2 we also have the following.

Lemma 8. *The complete bipartite graph $K_{n,m}$ is c-independent in $\mathbb{R}^d$ if and only if $n \leq d$ or $m \leq d$. In particular, the edge set of $K_{d+1,d+1}$ is a circuit in $\mathcal{C}_d(K_{d+1,d+1})$.*

Proof. Observe that, for any edge e, $K_{d+1,d+1} - e$ can be constructed from a graph with d vertices and no edges by a sequence of partial 0-extensions. Hence $K_{d+1,d+1} - e$ is c-independent. On the other hand $K_{d+1,d+1}$ is c-dependent by Lemma 1(ii). Thus $K_{d+1,d+1}$ is a circuit.

This also implies that, if $K_{n,m}$ is c-independent, then $n \leq d$ or $m \leq d$ holds. Conversely, if $n \leq d$, then $K_{n,m}$ can be constructed from a graph with d vertices and no edges by a sequence of partial 0-extensions, so is c-independent. $\square$

We will need the following characterisation of global (local) completability of complete tripartite graphs.

Lemma 9. *The complete tripartite graph $K_{a,b,c}$ is globally (or locally) completable in $\mathbb{R}^d$ if and only if $\min\{a,b,c\} \geq d$.*

Proof. Let $G = K_{a,b,c}$ and let A, B, C be the sets in the tripartition. Suppose $|C| < d$. Since $G - C$ is bipartite, it is not locally completable in $\mathbb{R}^1$ by (5). Lemma 6 now implies that G is not locally completable in $\mathbb{R}^t$ for any $t > |C|$.

Suppose on the other hand that $\min\{a,b,c\} \geq d$. Let (G,p) be a generic realization and take any equivalent realization (G,q) to (G,p). Since $K_{a,b}$ is globally bicompletable, there exists a $d \times d$ matrix M such that $q_i = M^T p_i$ and $q_j = M^{-1} p_j$ for all $i \in A$ and $j \in B$. Similarly, since $K_{a,c}$ is globally bicompletable, $q_i = N^T p_i$ and $q_k = N^{-1} p_k$ for all $i \in A$ and $k \in C$ for some $d \times d$ matrix N. Hence $(M^T - N^T)p_i = 0$ for all $i \in A$. Since p is generic and $|A| \geq d$, we have $M = N$. The same argument for $K_{b,c}$ now gives $q_k = M^T p_k = M^{-1} p_k$ for all $k \in C$. Hence $(M^T - M^{-1})p_k = 0$ for all $k \in C$. Since p is generic and $|C| \geq d$ this implies that $MM^T = I_d$. Hence M is orthogonal and $q_i = M^T p_i$ for all vertices i of G. This gives $\langle p_i, p_j \rangle = \langle q_i, q_j \rangle$ for all pairs i, j of vertices of G, so (G,q) is congruent to (G,p). $\square$

3 Canonical Positions

When analyzing the rigidity of frameworks, pinning down some points to factor out trivial motions is a useful tool. We will introduce a corresponding technique for completability in this section and use it frequently in the rest of this paper. In particular we will use it to show that testing bicompletability of bipartite graphs can be reduced to completability testing in Subsection 3.3.

For a vector p, let $\mathbb{Q}(p)$ denote the field extension of the rationals by the coordinates of p. Let $\overline{\mathbb{Q}(p)}$ denote the algebraic closure of $\mathbb{Q}(p)$ in $\mathbb{C}$.

3.1 Completability

Let $G=(V,E)$ be a semisimple graph. We define the completability function $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ by

$$f_G(p) = (\dots, \langle p_i, p_j \rangle, \dots) \qquad (p \in \mathbb{R}^{d|V|} \text{ and } ij \in E).$$

Notice that the completion matrix $C(G,p)$ is the Jacobian of f_G at p.

For a finite set V with $|V| \geq d$ and a sequence $S=(k_1,\dots,k_d)$ of d elements in V, let

$$W_S = \left\{ p \in \mathbb{R}^{d|V|} \;\middle|\; \begin{array}{l} \langle p(k_i), \boldsymbol{e}_j \rangle = 0 \ (\forall i = 1,\dots,d-1, \forall j = i+1,\dots,d) \\ \langle p(k_i), \boldsymbol{e}_i \rangle \geq 0 \ (\forall i = 1,\dots,d-1) \end{array} \right\},$$

where $\boldsymbol{e}_j$ be the j-th vector of the standard basis in $\mathbb{R}^d$. For a point $p \in W_S$, a coordinate of p that is set to zero is called a *fixed coordinate*. We say that p is in *canonical position (with respect to S)* if $p \in W_S$. Notice that for any $p \in \mathbb{R}^{dV}$ there is a $\hat{p} \in W_S$ that is congruent to p, and that $\hat{p}$ is unique when $p(S)$ is linearly independent. A point $p \in W_S$ is called *semi-generic* if the set of non-fixed coordinates of p is algebraically independent over $\mathbb{Q}$, or equivalently, if the transcendence degree of $\mathbb{Q}(p)/\mathbb{Q}$ is $d|V| - \binom{d}{2}$. Our next result shows that the completability matroid of G is determined by any semi-generic realisation of G.

Lemma 10. *Let (G,p) be a semi-generic framework in canonical position with respect to S. Then* $\operatorname{rank} C(G,p) = \operatorname{rank} \mathcal{C}_d(G)$.

Proof. Take any generic $q : V \to \mathbb{R}^d$. Then there is an orthogonal matrix A such that $A \cdot q \in W_S$. Then $\dot{q} \in \ker C(G,q)$ if and only if $A \cdot \dot{q} \in \ker C(G, A\cdot p)$, which means

$$\operatorname{rank} C(G,p) \leq \operatorname{rank} \mathcal{C}_d(G) = \operatorname{rank} C(G,q) = \operatorname{rank} C(G, A\cdot q) \leq \operatorname{rank} C(G,p),$$

where the last inequality follows since both p and $A \cdot q$ are in W_S and p is semi-generic. □

The next four propositions are analogous to results from [13].

Proposition 11. *Suppose that p is (semi)generic and G is c-independent. Then $f_G(p)$ is generic.*

Proof. This is a direct application of [12, Lemma 3.1]. □

Proposition 12. *Suppose that p is semi-generic and G is locally completable. Then $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_G(p))}$.*

Proof. Let G' be a spanning c-independent and locally completable subgraph of G. Since f_G is a polynomial map, we have $\overline{\mathbb{Q}(f_{G'}(p))} \subseteq \overline{\mathbb{Q}(f_G(p))} \subseteq \overline{\mathbb{Q}(p)}$. The point $f_{G'}(p)$ is generic by Proposition 11, and hence the transcendence degree of $\overline{\mathbb{Q}(f_{G'}(p))}/\mathbb{Q}$ is $d|V(G)| - \binom{d}{2}$, which is equal to the transcendence degree of $\overline{\mathbb{Q}(p)}/\mathbb{Q}$. We thus have $\overline{\mathbb{Q}(f_{G'}(p))} = \overline{\mathbb{Q}(f_G(p))} = \overline{\mathbb{Q}(p)}$. □

Proposition 13. *Suppose that G is locally completable, and p and q are in canonical positions with $f_G(p) = f_G(q)$. Suppose that p is semi-generic. Then q is semi-generic and $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(q)}$.*

Proof. By Propositions 11 and 12 we have $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_G(p))} = \overline{\mathbb{Q}(f_G(q))} \subseteq \overline{\mathbb{Q}(q)}$. Since p is semi-generic, q is semi-generic and $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(q)}$ follows. □

Proposition 14. *Let V be a finite set, S be a sequence of d distinct elements in V, p be a generic point in $\mathbb{R}^{d|V|}$, and p' be a point in W_S which is congruent to p. Then p' is semi-generic.*

Proof. Let G be a c-independent and locally completable graph on V. Then by Proposition 11, $f_G(p)$ is generic. Now we have $\overline{\mathbb{Q}(f_G(p))} = \overline{\mathbb{Q}(f_G(p'))} \subseteq \overline{\mathbb{Q}(p')}$. Since G is c-independent and locally completable, $|E| = d|V| - \binom{d}{2}$, and hence the transcendence degree of $\overline{\mathbb{Q}(p')}/\mathbb{Q}$ is at least $d|V| - \binom{d}{2}$. In other words, p' is semi-generic. □

3.2 Bicompletability

Suppose $G = (U, W; E)$ is a bipartite graph and $S = (u_1, \ldots, u_d)$ is a sequence of distinct elements in U. Define $\tilde{W}_S$ by

$$\tilde{W}_S = \left\{ p \in \mathbb{R}^{d|U \cup W|} \mid p(u_i) = \boldsymbol{e}_i \ (\forall i = 1, \ldots, d) \right\}.$$

We say that (G, p) is in *standard position with respect to S* if $p \in \tilde{W}_S$. It is easy to see that (G, p) is bicongruent to a unique framework $(G, \tilde{p})$ in standard position with respect to S whenever $p(S)$ is linearly independent (where S is regarded as a set). In particular two generic realisations of G are bicongruent if and only if they are both bicongruent to the same realisation in standard position with respect to S. Since bicongruence is an equivalence relation, we have the following.

Lemma 15. *Let (G, p) be a realization of a bipartite graph $G = (U, V; E)$ in $\mathbb{R}^d$, S be a sequence of d distinct elements in U such that $p(S)$ is linearly independent, and $\tilde{p}$ be the configuration bicongruent to p which is in standard position with respect to S. Then (G, p) is globally bicompletable if and only if $(G, \tilde{p})$ is globally bicompletable.*

A point $p \in \tilde{W}_S$ is called *semi-generic* if the set of coordinates in $p(V \setminus S)$ is algebraically independent over $\mathbb{Q}$. It is straightforward to check that the counterparts of the propositions given in the last subsection obtained by replacing "completability" with "bicompletability" and "canonical position" with "standard position" all hold.

3.3 From Bicompletability to Completability

In this subsection we establish a relation between bicompletability and completability, and show that bicompletability testing of bipartite graphs can be reduced to completability testing.

For a finite set X, let $K^\circ(X)$ be the graph on X whose edge set is $\{ij \mid i, j \in X\}$ (including loops). We begin with infinitesimal completability.

Lemma 16. *Let (G, p) be a realisation of a bipartite graph $G = (U, W; E)$ in $\mathbb{R}^d$ and $S = \{u_1, \dots, u_d\}$ be a set of d distinct vertices in U such that $p(S)$ is linearly independent. Let $G^+ = G \cup K^\circ(S)$. Then*

$$\operatorname{rank} C(G, p) = \operatorname{rank} C(G^+, p) - \binom{d+1}{2}.$$

Hence (G, p) is infinitesimally bicompletable if and only if (G^+, p) is infinitesimally completable.

Proof. Let $C'(G, p)$ be the matrix obtained from $C(G, p)$ by deleting the columns indexed by $u_1, u_2, \dots, u_d$. Let T be the set of all infinitesimal c-motions $\dot{p}$ of (G, p) such that, for some fixed $d \times d$ matrix A, $\dot{p}_i = Ap_i$ and $\dot{p}_j = -A^T p_j$ for all $i \in U$ and $j \in W$. (So T is the space of all 'trivial' infinitesimal c-motions of (G, p).) Let F be the space of all infinitesimal c-motions $\dot{p}$ of (G, p) which keep $u_1, u_2, \dots, u_d$ fixed i.e. $\dot{p}(u_i) = 0$ for all $1 \le i \le d$. Then $\ker C(G, p) = T \oplus F$ and $\dim T = d^2$. The fact that F is isomorphic to $\ker C'(G, p)$ now gives $\operatorname{rank} C'(G, p) = \operatorname{rank} C(G, p)$.

We have $C(G^+, p) = \begin{pmatrix} C(K^\circ(S), p) & 0 \\ * & C'(G, p) \end{pmatrix}$. Hence

$$\operatorname{rank} C(G^+, p) = \operatorname{rank} C'(G, p) + \operatorname{rank} C(K^\circ(S), p) = \operatorname{rank} C(G, p) + \binom{d+1}{2}.$$

□

As a corollary we obtain the following result for graphs.

Theorem 17. *Suppose that $G = (U, W; E)$ is a bipartite graph with $|U|, |W| \ge d$ and $S = \{u_1, \dots, u_d\}$ is a set of d distinct vertices in U. Then G is locally bicompletable in $\mathbb{R}^d$ if and only if $G^+ = G \cup K^\circ(S)$ is locally completable in $\mathbb{R}^d$.*

We next give the global completability counterpart to this result. We need the following technical lemma.

Lemma 18. *Let $(G = (U, W; E), p)$ be a locally bicompletable framework with $|U|, |W| \geq d$, and (G, q) be a framework equivalent to (G, p). Suppose p is generic. Then any d points in $q(U)$ are linearly independent.*

Proof. We first prove that $q(U)$ spans $\mathbb{R}^d$. Suppose not. Let $q(u_1), q(u_2), \ldots q(u_t)$ be a basis for the subspace of $\mathbb{R}^d$ spanned by $q(U)$ with $t < d$. By applying a suitable congruence to (G, q) we may assume that $q(u_1), q(u_2), \ldots q(u_t)$ are the first t vectors in a standard basis for $\mathbb{R}^d$. Let (G, q') be the projection of (G, q) onto $\mathbb{R}^t$. Since the last $(d-t)$ coordinates of $q(u)$ are zero for all $u \in U$, we have $\langle q'(u), q'(w)\rangle = \langle q(u), q(w)\rangle$ for any $u \in U$ and $w \in W$. Therefore $\mathbb{Q}(f_G(q)) = \mathbb{Q}(f_G(q'))$. Since the transcendence degree of $\mathbb{Q}(f_G(q'))/\mathbb{Q}$ can be at most $t|U \cup W| - t^2$, the transcendence degree of $\mathbb{Q}(f_G(q))/\mathbb{Q}$ is at most $t|U \cup W| - t^2$.

On the other hand, since G is locally bicompletable and p is generic, (the bicompletability version of) Propositions 12 and 14 imply that the transcendence degree of $\mathbb{Q}(f_G(p))/\mathbb{Q}$ is equal to $d|U \cup W| - d^2$. Since $f_G(q) = f_G(p)$, the transcendence degree of $\mathbb{Q}(f_G(q))/\mathbb{Q}$ is equal to $d|U \cup W| - d^2$, a contradiction.

Therefore $q(U)$ spans $\mathbb{R}^d$ and we may choose $X \subseteq U$ such that $q(X)$ is a basis for $\mathbb{R}^d$. Then there is the unique $\bar{q}$ such that $\bar{q}$ is bicongruent to q and is in standard position with respect to X (assuming any order on the elements of X). Then by (the bicompletability version of) Propositions 13 and 14, $\bar{q}$ is semi-generic. This in turn implies the statement since $\bar{q}(U)$ is an image of $q(U)$ by a nonsingular linear map. □

Theorem 19. *Suppose that $G = (U, W; E)$ is a bipartite graph with $|U|, |W| \geq d$ and $S = \{u_1, \ldots, u_d\}$ is a set of d distinct vertices in U. Then G is globally bicompletable in $\mathbb{R}^d$ if and only if $G^+ = G \cup K^\circ(S)$ is globally completable in $\mathbb{R}^d$.*

Proof. Let $p : U \cup W \to \mathbb{R}^d$ be generic.

Suppose that (G, p) is globally bicompletable. Let $\bar{G}^+$ be the graph obtained from G^+ by adding all edges from U to W. Let (G^+, q) be a framework equivalent to (G^+, p). Since (G, p) is globally bicompletable, $(\bar{G}^+, p)$ and $(\bar{G}^+, q)$ are equivalent. Since $(\bar{G}^+, p)$ can be obtained from K_d° by simple 0-extensions and edge-additions, $(\bar{G}^+, p)$ is globally completable by Lemma 3. Hence p and q are congruent.

Suppose that (G, p) is not globally bicompletable in $\mathbb{R}^d$. Then there exists an equivalent framework (G, q) such that $\langle p(a), p(b)\rangle \neq \langle q(a), q(b)\rangle$ for some pair $a \in U$ and $b \in W$. By Lemma 18, $q(S)$ is linearly independent. Let P and Q be the $d \times d$ matrices whose i-th columns are $p(u_i)$ and $q(u_i)$, respectively. Define

$$q'(v) = \begin{cases} PQ^{-1}q(v) & \text{if } v \in U \\ (P^{-1})^\top Q^\top q(v) & \text{if } v \in W \end{cases}$$

We claim that (G^+, q') is equivalent, but not congruent, to (G^+, p) . To see this observe that $q'(u_i) = p(u_i)$ for $1 \leq i \leq d$. Hence $\langle q'(u_i), q'(u_j)\rangle = \langle p(u_i), p(u_j)\rangle$ for any $1 \leq i, j \leq d$. Also for any $u \in U$ and $w \in W$ we have $\langle q'(u), q'(w)\rangle = \langle PQ^{-1}q(u), (P^{-1})^\top Q^\top q(w)\rangle = \langle q(u), q(w)\rangle$. Hence $\langle q'(u), q'(v)\rangle = \langle p(u), p(v)\rangle$ for all $uv \in E(G^+)$, and $\langle q'(a), q'(b)\rangle \neq \langle p(a), p(b)\rangle$. This implies that (G^+, q') is equivalent, but not congruent, to (G^+, p). □

We do not know whether a similar relation between bicompletability and completability holds at the level of frameworks, i.e., whether it is true that (G, p) is globally bicompletable if and only if (G^+, p) is globally completable even for non-generic p.

4 Geometric Observations

In this section we shall provide several geometric tools for constructing globally completable graphs. Our proof strategy using algebraic independence is inspired by [12, 22], but extends and clarifies the existing theory.

Proposition 20. *Let (G, p) and (G, q) be d-dimensional frameworks and let v be a vertex in G with $\{1, 2, \ldots, d+1\} \subseteq N_G(v)$. Suppose that $\langle p_v, p_i\rangle = \langle q_v, q_i\rangle$ for all $1 \leq i \leq d$. If*

$$\langle p_v, p_{d+1}\rangle = \langle q_v, q_{d+1}\rangle, \tag{6}$$

then

$$\det \begin{pmatrix} q_1 & q_2 & \cdots & q_{d+1} \\ \langle p_v, p_1\rangle & \langle p_v, p_2\rangle & \ldots & \langle p_v, p_{d+1}\rangle \end{pmatrix} = 0. \tag{7}$$

Conversely, if (7) holds and $q_1, \ldots, q_d$ are linearly independent, then (6) holds.

Proof. If (6) holds, we have

$$\begin{pmatrix} q_v \\ -1 \end{pmatrix}^\top \begin{pmatrix} q_1 & q_2 & \cdots & q_{d+1} \\ \langle p_v, p_1\rangle & \langle p_v, p_2\rangle & \ldots & \langle p_v, p_{d+1}\rangle \end{pmatrix} = 0.$$

This implies (7).

Conversely suppose that (7) holds and $q_1, \ldots, q_d$ are linearly independent. Then (7) implies

$$\langle p_v, \sum_{i=1}^{d+1} (-1)^i (\det Q_i) p_i\rangle = 0 \tag{8}$$

where Q_i is a $d \times d$-matrix whose columns are $\{q_1, q_2, \ldots, q_{d+1}\} \setminus \{q_i\}$ in this order. Since $\langle p_v, p_i\rangle = \langle q_v, q_i\rangle$ for $1 \leq i \leq d$, we have

$$\langle q_v, \sum_{i=1}^{d} (-1)^i (\det Q_i) q_i\rangle + (-1)^{d+1} (\det Q_{d+1}) \langle p_v, p_{d+1}\rangle = 0\,. \tag{9}$$

Also we have

$$\sum_{i=1}^{d+1}(-1)^i(\det\, Q_i)q_i = 0 \tag{10}$$

since each coordinate of the left vector is the determinant of a $(d+1)\times(d+1)$-submatrix of the following matrix of rank d:

$$\begin{pmatrix} q_1 & \cdots & q_{d+1} \\ q_1 & \cdots & q_{d+1} \end{pmatrix}.$$

Combining (9) and (10) we get $(-1)^{d+1}(\det\, Q_{d+1})\langle p_v, p_{d+1}\rangle = (-1)^{d+1}(\det\, Q_{d+1})\langle q_v, q_{d+1}\rangle$. Since $\det\, Q_{d+1} \neq 0$, this gives $\langle p_v, p_{d+1}\rangle = \langle q_v, q_{d+1}\rangle$ as required. $\square$

Proposition 21. *Let (G,p) be a generic framework and v be a vertex with $\{1,\ldots,d+1\} \subseteq N_G(v)\setminus\{v\}$. Suppose $(G-v,p)$ is locally completable. Then for any (G,q) equivalent to (G,p), $\{q_1,\ldots,q_{d+1}\}$ is a linear image of $\{p_1,\ldots,p_{d+1}\}$ (i.e., there is a $d\times d$-matrix A such that $q_i = Ap_i$ for all $i = 1,\ldots,d+1$).*

Proof. Let S be a sequence of d distinct vertices in $V\setminus\{v\}$. By Proposition 14 it suffices to show the statement for a semi-generic (G,p) and for any equivalent (G,q) both in canonical position with respect to S. By Proposition 20 we have (7). In particular, we have (8).

Since $(G-v,p)$ is locally completable, $\overline{\mathbb{Q}(q|_{V-v})} = \overline{\mathbb{Q}(p|_{V-v})}$ by Proposition 13. Hence, the left hand side of (8) is a linear combination of the components of p_v with coefficients in $\overline{\mathbb{Q}(p|_{V-v})}$. Since p is generic, each coefficient is zero and hence

$$\sum_{i=1}^{d+1}(-1)^i(\det Q_i)p_i = 0. \tag{11}$$

This in turn implies

$$\operatorname{rank}\begin{pmatrix} p_1 & \cdots & p_{d+1} \\ q_1 & \cdots & q_{d+1} \end{pmatrix} = d,$$

which means that there is a $d\times d$-matrix A such that $q_i = Ap_i$ for $i = 1,\ldots,d+1$. $\square$

For $i,j \in V$, we say that i and j are *globally c-linked* in G (in $\mathbb{R}^d$) if $\langle p_i, p_j\rangle = \langle q_i, q_j\rangle$ for all generic realizations (G,p) in $\mathbb{R}^d$ and all equivalent realizations (G,q). We use this term even when $i = j$.

Theorem 22. *Let G be a graph and uv be a non-loop edge in G with $|N_G(u)\setminus\{u\}| > d$ and $|N_G(v)\setminus\{v\}| > d$. Suppose that $G-u$ and $G-v$ are locally completable in $\mathbb{R}^d$. Then i and j are globally c-linked in G for any $i \in N_G(u)\setminus\{u\}$ and $j \in N_G(v)\setminus\{v\}$.*

Proof. By Proposition 14, we may focus on a semi-generic (G, p) in canonical position with respect to S with $S \cap \{u, v\} = \emptyset$ and any framework (G, q) that is equivalent to (G, p) and is in canonical position with respect to S. By Proposition 21, there are linear maps A and B such that

$$q_i = Ap_i \text{ for all } i \in N_G(u) \setminus \{u\} \qquad \text{and} \qquad q_j = Bp_j \text{ for all } j \in N_G(v) \setminus \{v\}. \tag{12}$$

Since $G - u$ is locally completable and $G - u$ is a subgraph of G, $\overline{\mathbb{Q}(p|_{V-u})} = \overline{\mathbb{Q}(q|_{V-u})}$ by Proposition 13. In particular, the set of entries of p_u is algebraically independent over $\mathbb{Q}(p|_{V-u}, q|_{V-u})$. Symmetrically the set of entries of p_v is algebraically independent over $\mathbb{Q}(p|_{V-v}, q|_{V-v})$. Therefore, the entries of q_i are algebraic over $\overline{\mathbb{Q}(p|_{V-u-v})}$ for all $i \in V - u - v$. (Otherwise the transcendence degree of $\mathbb{Q}(p)/\mathbb{Q}$ becomes more than $d|V| - \binom{d}{2}$ by $\overline{\mathbb{Q}(q)} = \overline{\mathbb{Q}(p)}$.)

Since $|N_G(u) \setminus \{u, v\}| \geq d$ and $|N_G(v) \setminus \{u, v\}| \geq d$, A and B are determined by the following equations,

$$q_i = Ap_i \text{ for all } i \in N_G(u) \setminus \{u, v\} \quad \text{and} \quad q_j = Bp_j \text{ for all } j \in N_G(v) \setminus \{u, v\}. \tag{13}$$

(Recall that a linear map is determined by d linearly independent vectors.) This in particular implies that the entries of A and B are algebraic over $\overline{\mathbb{Q}(p|_{V-u-v})}$.

Since G contains edge uv, we have

$$0 = \langle p_u, p_v \rangle - \langle q_u, q_v \rangle = p_u^\top (I_d - A^\top B) p_v.$$

Since the entries of p_u and p_v are algebraically independent over $\overline{\mathbb{Q}(p|_{V-u-v})}$, we have $A^\top B = I_d$.

To complete the proof, consider any $i \in N_G(u) \setminus \{u\}$ and $j \in N_G(v) \setminus \{v\}$. Then we have $\langle p_i, p_j \rangle - \langle q_i, q_j \rangle = p_i^\top (I_d - A^\top B) p_j = 0$ as required. □

Corollary 23. *Let G be a graph and let uv be a non-loop edge in G with $|N_G(u) \setminus \{u\}| > d$ and $|N_G(v) \setminus \{v\}| > d$. Suppose that*

- *$G - u$ and $G - v$ are locally completable in $\mathbb{R}^d$, and*
- *the graph obtained from $G - u - v$ by adding all edges of the form $\{ij \mid i \in N_G(u) \setminus \{u\}, j \in N_G(v) \setminus \{j\}\}$ (including loops at vertices in $(N_G(u) \setminus \{u\} \cap N_G(v) \setminus \{v\})$) is globally completable in $\mathbb{R}^d$.*

Then, G is globally completable in $\mathbb{R}^d$.

Proof. By Theorem 22, G is globally comletable if and only if the graph G' obtained from G by adding all edges between $N_G(u)$ and $N_G(v)$ is globally completable. By assumption, $G' - u - v$ is globally completable. Lemma 3 now implies that G' is globally completable since we can obtain a spanning subgraph of G' from $G' - u - v$ by simple 0-extensions. □

It is straightforward to check that an analogous result to Corollary 23 holds for bicompletability.

We next derive a similar statement to Corollary 23 for a vertex incident with a loop.

Theorem 24. *Let G be a graph with $|V| \geq d+1$, and let v be a vertex in G having a loop with $|N(v) \setminus \{v\}| \geq d$. Suppose that $G - v$ is locally completable in $\mathbb{R}^d$. Then i and j are globally c-linked in G for all $i, j \in N_G(v)$.*

Proof. Take any semi-generic (G, p) in canonical position with respect to S with $v \notin S$, and consider any (G, q) that is equivalent to (G, p) and is in canonical position with respect to S. Since G and $G - v$ are locally completable, q is semi-generic and $\overline{\mathbb{Q}(q|_{V-v})} = \overline{\mathbb{Q}(p|_{V-v})}$ by Proposition 13.

Take any d vertices from $N(v) \setminus \{v\}$ and, without loss of generality, denote them by $\{1, \ldots, d\}$. Since $\{p_1, \ldots, p_d\}$ and $\{q_1, \ldots, q_d\}$ are linearly independent, there is a $d \times d$ nonsingular linear map A such that

$$q_i = Ap_i \text{ for all } 1 \leq i \leq d. \tag{14}$$

More specifically A can be expressed as $A = QP^{-1}$, where P is the $d \times d$ matrix whose i-th column is p_i and Q is the $d \times d$ matrix whose i-th column is q_i. Hence the entries of A are contained in $\overline{\mathbb{Q}(p|V - v)}$.

Since G has edge vi, we have

$$\langle p_v, p_i \rangle = \langle q_v, q_i \rangle \qquad \text{for all } 1 \leq i \leq d$$

which can be written as

$$p_v^\top P = q_v^\top Q.$$

Hence we have $q_v = (Q^{-1})^\top P^\top p_v = (A^{-1})^\top p_v$. Since G has a loop at v, we also have

$$0 = \langle p_v, p_v \rangle - \langle q_v, q_v \rangle = p_v^\top (I_d - A^{-1}(A^{-1})^\top) p_v.$$

Since the entries of p_v are algebraically independent over $\overline{\mathbb{Q}(p|V - v)}$ and $I_d - A^{-1}(A^{-1})^\top$ is symmetric, we have $A^{-1}(A^{-1})^\top = I_d$, implying that A is orthogonal. Therefore, for any $1 \leq i, j \leq d$, we have $\langle p_i, p_j \rangle - \langle q_i, q_j \rangle = p_i^\top (I_d - A^\top A) p_j = 0$ as required. □

Corollary 25. *Let G be a graph and let v be a vertex in G having a loop with $|N(v) \setminus \{v\}| \geq d$. Suppose that*

- *$G - v$ is locally completable in $\mathbb{R}^d$, and*
- *the graph obtained from $G - v$ by adding all edges of the form $\{ij \mid i, j \in N_G(v)\}$ (including loops) is globally completable in $\mathbb{R}^d$.*

Then, G is globally completable in $\mathbb{R}^d$.

The following connection between rigidity and completability is a corollary of the coning arguments given in [4, 27], which will enable us to use the results of this section to obtain new results on global rigidity.

Proposition 26. *([13]) Let G be a simple graph. Then G° is globally/locally completable in $\mathbb{R}^d$ if and only if G is globally/locally rigid in $\mathbb{R}^{d-1}$.*

Corollary 25 and Proposition 26 now give the following, which was implicit in [22] and was shown to be a powerful tool for analyzing the global rigidity of graphs in [23].

Corollary 27. *Let G be a simple graph and let v be a vertex in G with $|N_G(v)| \geq d+1$. Suppose that*

- *$G - v$ is rigid in $\mathbb{R}^d$, and*
- *the graph obtained from $G - v$ by adding all non-loop edges of the form $\{ij \mid i, j \in N_G(u)\}$ is globally rigid in $\mathbb{R}^d$.*

Then, G is globally rigid in $\mathbb{R}^d$.

There is a corresponding concept to global c-linkedness in rigidity theory, and the main theorem of [12] is the rigidity counterpart of Theorem 24. We note that this would also follow from Theorem 24 if we knew that coning preserves global c-linkedness. However it is not straightforward to extend Proposition 26 to global linkedness since the proof of Proposition 26 is based on stress matrices.

We can use similar geometric arguments to derive results on infinitesimal completability. Theorem 29 below is an infinitesimal counterpart of Theorem 24 and Corollary 30 is a completability analogue of a well-known result on infinitesimal rigidity. We first need to establish one technical lemma.

Lemma 28. *Let (G, p) be a generic framework such that $\operatorname{rank} \mathcal{C}_d(G) = d|V| - \binom{d}{2} - 1$. Then there is a nontrivial infinitesimal c-motion $\dot{p}$ of (G, p) such that $\mathbb{Q}(\dot{p}) \subseteq \mathbb{Q}(p)$.*

Proof. Let C' be the matrix obtained from $C(G, p)$ by deleting the first $d-i$ columns from the d columns indexed by i, for all $1 \leq i \leq d-1$. Then $\operatorname{rank} C' = \operatorname{rank} C(G, p)$. More specifically, an infinitesimal c-motion $\dot{p}$ of (G, p) is nontrivial if $\dot{p}$ is obtained by extending a nonzero $\dot{p}' \in \ker C'$ by adding zero components in positions corresponding to the columns we deleted from $C(G, p)$. We take $\dot{p}' \in \ker C'$ such that one specific nonzero entry is equal to one. Then $\mathbb{Q}(\dot{p}') \subseteq \mathbb{Q}(p)$ since $\dot{p}'$ is the unique solution to a system of $d|V| - \binom{d}{2} - 1$ linear equations in $d|V| - \binom{d}{2} - 1$ unknowns with coefficients in $\mathbb{Q}(p)$. Thus $\mathbb{Q}(\dot{p}) \subseteq \mathbb{Q}(p)$ follows. $\square$

We say that an edge ij is *implied* in the completability matroid $\mathcal{C}_d(G)$ if the rank remains unchanged after adding ij.

Theorem 29. *Let $G = (V, E)$ be a graph, $v \in V$, and $N(v) = \{v, 1, \dots, d\}$. Suppose that the rank of $\mathcal{C}_d(G - v)$ is $d(|V| - 1) - \binom{d}{2} - 1$. Then ij is implied in $\mathcal{C}_d(G)$ for all $1 \leq i, j \leq d$.*

Proof. Take any generic framework (G, p). By Lemma 28, $(G - v, p|_{V-v})$ has a nontrivial infinitesimal c-motion $\dot{p}$ such that $\mathbb{Q}(\dot{p}) \subseteq \mathbb{Q}(p|_{V-v})$. Let ℓ be the loop at v. Also let P and $\dot{P}$ be the matrices whose columns are $p_1, \dots, p_d$ and $\dot{p}_1, \dots, \dot{p}_d$, respectively. Since $G - \ell$ is a 0-extension of $G - v$, $\dot{p}$ can be extended to a nontrivial infinitesimal c-motion of $G - \ell$, which we will again denote by $\dot{p}$. More specifically, since $\langle p_v, \dot{p}_j \rangle + \langle p_j, \dot{p}_v \rangle = 0$ for $1 \leq i \leq d$, we have $\dot{p}_v = (P^\top)^{-1} \dot{P} p_v$.

Since G has loop ℓ at v, we have $\langle p_v, \dot{p}_v \rangle = 0$, implying $p_v^\top (P^\top)^{-1} \dot{P} p_v = 0$. The facts that the entries of $(P^\top)^{-1} \dot{P}$ are algebraic over $\mathbb{Q}(p|V - v)$ and p_v is algebraically independent over $\mathbb{Q}(p|V - v)$ now imply that $(P^\top)^{-1} \dot{P}$ is skew-symmetric. This in turn implies that there is a skew-symmetric matrix S such that $\dot{p}_i = S p_i$ for all $1 \leq i \leq d$. Thus $\langle p_i, \dot{p}_j \rangle + \langle p_j, \dot{p}_j \rangle = 0$ for any $1 \leq i, j \leq d$, so ij is implied. $\square$

Given a graph $G = (V, E)$ and distinct vertices $v_1, v_2, \dots, v_d \in V$ with $v_i v_j \in E$ for some $1 \leq i, j \leq d$, the (d-dimensional) *looped* 1*-extension operation* constructs a new graph H by deleting the edge $v_i v_j$ and then adding a new vertex v_0 and edges $v_0 v_0, v_0 v_1, \dots, v_0 v_d$.

Corollary 30. *Suppose H is locally completable in $\mathbb{R}^d$ and G is a looped* 1*-extension of H. Then G is locally completable in $\mathbb{R}^d$.*

Proof. Since H is locally completable, Theorem 29 implies that $v_i v_j$ is an implied edge in G. Since $G + v_i v_j$ can be obtained from H by a 0-extension and an edge addition, G is locally completable. $\square$

Notice that, for a simple graph G, a looped 1-extension of G° can be expressed as H° for some simple graph H. The operation for constructing H from G is known as *(($d - 1$)-dimensional) 1-extension* in rigidity theory, and is widely used for analyzing global/local rigidity (see, e.g., [11, 26]). By Proposition 26, Corollary 30 extends the well-known fact that 1-extension preserves local rigidity.

5 Vertex Redundancy Implies Global Completability

A graph is said to be *vertex redundantly completable* if $G - v$ is locally completable for all $v \in V$. In this section we shall prove the following theorem, which implies that a partially filled positive semidefinite matrix of order n is globally rank d completable if every $n - 1$ principal submatrix is locally rank d completable.

Theorem 31. *Let $G = (V, E)$ be a vertex redundantly completable graph in $\mathbb{R}^d$ with $|V| \geq d + 1$ for some $d \geq 2$. Then G is globally completable in $\mathbb{R}^d$.*

For the proof, we need the geometric results from the previous section as well as the following combinatorial lemmas.

For finite sets X and Y that may be intersecting, let $K^\circ(X,Y)$ be the graph on $X \cup Y$ whose edge set is $\{ij \mid i \in X, j \in Y\}$ (including loops at vertices in $X \cap Y$).

Lemma 32. *Let X and Y be sets with $|X| = |Y| = d+1$. Then $K^\circ(X,Y)$ is not c-independent in $\mathbb{R}^d$. Moreover, for any edge ij with $i \in X \setminus Y$ and $j \in Y \setminus X$, $K^\circ(X,Y) - ij$ is c-independent in $\mathbb{R}^d$.*

Proof. Let $k = |X \cap Y|$. Then $K^\circ(X \setminus Y, Y \setminus X)$ is isomorphic to $K_{d+1-k,d+1-k}$, whose edge set is a circuit in $\mathcal{C}_{d-k}(K^\circ(X \setminus Y, Y \setminus X))$ by Lemma 8. Observe that $K^\circ(X,Y)$ can be obtained from $K^\circ(X \setminus Y, Y \setminus X)$ by a sequence of k looped cone extensions. Lemma 6 now implies the claim. $\square$

Given a semisimple graph $G = (V,E)$ with $i \in V$ and $F \subseteq E$, let $d_F(i)$ be the number of edges in F incident with i.

Our next result follows from repeated applications of Lemma 32.

Lemma 33. *Let X and Y be sets with $|X| \geq d+1$ and $|Y| \geq d+1$. Then for any $i \in X \setminus Y$ and $j \in Y \setminus X$ there is a base B of $\mathcal{C}_d(K^\circ(X,Y))$ such that $ij \notin B$ and $d_B(i) = d_B(j) = d$.*

Proof. We first suppose that $|X \cap Y| \geq d$. Choose $Z \subseteq X \cap Y$ with $|Z| = d$. Then Z induces $K^\circ(Z)$ in $K^\circ(X,Y)$, and $K^\circ(Z)$ is c-independent and locally completable in $\mathbb{R}^d$ by Lemma 7. Hence the edge set of $K^\circ(Z)$ can be extended to the desired base of $\mathcal{C}_d(K^\circ(X,Y))$ by 0-extension operations.

Next suppose that $|X \cap Y| < d$. Take $X' \subseteq X \setminus \{i\}$ and $Y' \subseteq Y \setminus \{j\}$ with $|X'| = |Y'| = d$ and $X \cap Y = X' \cap Y'$. The edge set of $K^\circ(X',Y')$ is c-independent by Lemma 32. We may extend it to a spanning edge subset F of $K^\circ(X,Y)$ by 0-extension operations in such a way that each new vertex in X is connected to all vertices in Y' and each new vertex in Y is connected to all vertices in X'. We claim that F spans $\mathcal{C}_d(K^\circ(X,Y))$. To see this, take any $k \in X \setminus X'$ and $l \in Y \setminus Y'$. Then F contains all edges of $K^\circ(X'+k, Y'+l)$ except kl, and kl is spanned by F by Lemma 32. Hence F spans $\mathcal{C}_d(K^\circ(X,Y))$, and F is a base satisfying the degree condition. $\square$

Lemma 34. *Let X be a finite set with $|X| \geq d$. Then for any $i \in X$, $\mathcal{C}_d(K^\circ(X))$ has a base B such that $d_B(i) = d$.*

Proof. Take any $X' \subseteq X - i$ such that $|X'| = d-1$. Then $K^\circ(X'+i)$ is c-independent and locally completable in $\mathbb{R}^d$. By adding vertices of $X \setminus (X'+i)$ by 0-extension, one can obtain a desired base of $\mathcal{C}_d(K^\circ(X))$. $\square$

Lemma 35. *Let $G = (V,E)$ be a vertex redundantly completable graph in $\mathbb{R}^d$ with $|V| \geq d \geq 2$. Then for any $v \in V$, $|N_G(v)| \geq d+1$.*

Proof. If $|N_G(v)| < d+1$, then $|N_{G-w}(v)| < d$ for any $w \in N_G(v) - v$. Then $G - w$ is not locally completable since every vertex has at least d neighbors in a locally completable graph with at least d vertices. This contradicts the hypothesis that G is vertex redundantly completable. □

We are now ready to prove Theorem 31.

Proof of Theorem 31. We use induction on $|V|$. Note that K_d° is c-independent and locally completable in $\mathbb{R}^d$. Hence K_{d+1}° is the only vertex redundantly completable graph with $|V| = d+1$. Since K_{d+1}° is clearly globally completable in $\mathbb{R}^d$, we may assume that $|V| > d+1$. We split the remainder of the proof into two cases depending on whether or not G has a loop.

Suppose G has a vertex v incident with a loop. Let G_1 be the graph obtained from G by adding all edges $\{ij \mid i,j \in N(v)\}$ and let $G_1' = G_1 - v$. By the induction hypothesis and Corollary 25, it suffices to show that G_1' is vertex redundantly completable in $\mathbb{R}^d$. To this end, take any vertex w in G_1'. Since G is vertex redundantly completable, $G_1 - w$ is locally completable. Notice that $G_1 - w$ contains a subgraph which is isomorphic to K_d° and contains v by Lemma 35. Hence by Lemma 34 there is a base B of $\mathcal{C}_d(G_1 - w)$ such that $d_B(v) = d$. Note that B induces a locally completable subgraph in $G_1 - w$ since $G_1 - w$ is locally completable. Therefore $\mathcal{C}_d(G_1 - w - v)$ contains a locally completable spanning subgraph by Lemma 2, which in turn implies that $G_1 - w - v = G_1' - w$ is locally completable. In other words G_1' is vertex redundantly completable in $\mathbb{R}^d$.

Suppose G has no loop. We say that an edge is *bad* if exactly one of its endvertices has degree $d+1$. If every edge is bad then exactly one of the endvertices of each edge has degree $d+1$, and hence G is bipartite. This gives a contradiction since no bipartite graph can be locally completable by Lemma 1.

Thus there is an edge uv that is not bad. Let G_2 be the graph obtained from G by adding any edges between $N_G(u)$ and $N_G(v)$ including loops at $N_G(u) \cap N_G(v)$, and let $G_2' = G_2 - u - v$. By the induction hypothesis and Corollary 23, it suffices to show that G_2' is again vertex redundantly completable. Since uv is not bad, we have the following two cases according to Lemma 35.

Case 1: Suppose that $|N_G(u)| = d+1$ and $|N_G(v)| = d+1$. We claim that $N_G(v) \cap N_G(u) = \emptyset$. Suppose not, and let $w \in N_G(v) \cap N_G(u)$. Then $G - w$ is locally completable since G is vertex redundantly completable. Moreover $G - w - v$ is also locally completable since v has degree d in $G - w$. However $|N_{G-w-v}(u)| = d-1$, which means that $G - w - v$ is not locally completable, a contradiction. Thus $N(u) \cap N(v) = \emptyset$ and G_2 contains a subgraph isomorphic to $K_{d+1,d+1}$ and covering uv.

Let us take any vertex w in G_2'. Since G is vertex redundantly completable, $G_2 - w$ is locally completable. If $w \notin N(u) \cup N(v)$ then $G_2 - w - uv$ is locally completable since uv is covered by a subgraph isomorphic to $K_{d+1,d+1}$ whose edge set is a circuit in $\mathcal{C}_d(G_2 - w)$ by Lemma 8. Since u and v have degree d in $G_2 - w - uv$, $G_2' - w$ is locally completable.

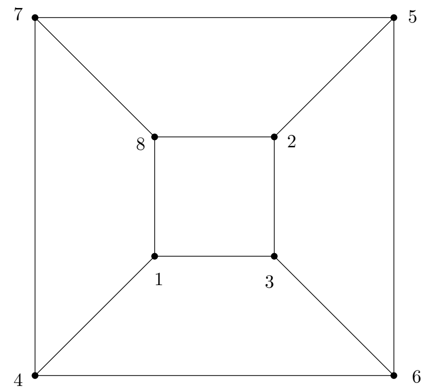

Figure 1: The graph of Example 1.

On the other hand, if $w \in N(u) \cup N(v)$, then $G_2' - w$ can be obtained from $G_2 - w$ by the inverse operations to 0-extension, which implies that $G_2' - w$ is locally completable.

Case 2: Suppose that $|N(u)| \geq d+2$ and $|N(v)| \geq d+2$. Take any vertex w in G_2'. Observe that $G_2 - w$ contains $K^\circ(N_G(u) \setminus \{w\}, N_G(v) \setminus \{w\})$ as an induced subgraph. Since $|N_G(u) \setminus \{w\}| \geq d+1$ and $|N_G(v) \setminus \{w\}| \geq d+1$, Lemma 33 implies that $\mathcal{C}_d(G_2 - w)$ has a base B such that $uv \notin B$ and $d_B(u) = d_B(v) = d$. Let H be the subgraph of $G_2 - w$ induced by B. Then H is locally completable since $G_2 - w$ is locally completable. Since $H - u - v$ can be obtained from H by the inverse operations to 0-extension, $H - u - v$ is locally completable and hence $G_2' - w$ is locally completable. $\square$

6 Three Examples

In this section we shall present three examples that indicate it will be difficult to characterize 2-dimensional generic global completability by using existing techniques from rigidity theory.

Example 1. Suppose G is the graph of the cube labeled as in Figure 1. Since G is a maximal planar bipartite graph, G is locally bicompletable in $\mathbb{R}^2$ by [13, 15]. We will show that (G, p) is not globally bicompletable when p is generic. By Lemma 15 and (the bicompletability version of) Proposition 14, it will suffice to consider a semi-generic framework (G, p), in standard position with respect to $(1, 2)$. We will compute all possible realizations (G, q) which are equivalent to (G, p). Since G is locally bicompletable, we may assume that q is semi-generic and in standard position with respect to $(1, 2)$ by (the bicompletability version of) Proposition 13. Since $G[1, 2, 3, 8]$ is globally bicompletable, we have $p_i = q_i$ for $i \in \{1, 2, 3, 8\}$.

For each p_i, let $p_i^\perp$ be the vector obtained by rotating p_i by $\pi/2$. Since G has edges

14, 25, we have

$$q_4 = p_4 + t_4 p_1^{\perp} \tag{15}$$

$$q_5 = p_5 + t_5 p_2^{\perp}, \tag{16}$$

for some $t_4, t_5 \in \mathbb{R}$.

By Proposition 20 and the semi-genericity of q, the constraints by edges $36, 46, 56$ are equivalent to

$$\det \begin{pmatrix} q_3 & q_4 & q_5 \\ \langle p_6, p_3 \rangle & \langle p_6, p_4 \rangle & \langle p_6, p_5 \rangle \end{pmatrix} = 0. \tag{17}$$

Thus we get

$$0 = \det \begin{pmatrix} p_3 & p_4 + t_4 p_1^{\perp} & p_5 + t_5 p_2^{\perp} \\ \langle p_6, p_3 \rangle & \langle p_6, p_4 \rangle & \langle p_6, p_5 \rangle \end{pmatrix} \tag{18}$$

$$= \det \begin{pmatrix} p_3 & p_4 & p_5 \\ \langle p_6, p_3 \rangle & \langle p_6, p_4 \rangle & \langle p_6, p_5 \rangle \end{pmatrix} + \det \begin{pmatrix} p_3 & t_4 p_1^{\perp} & p_5 \\ \langle p_6, p_3 \rangle & 0 & \langle p_6, p_5 \rangle \end{pmatrix}$$

$$+ \det \begin{pmatrix} p_3 & p_4 & t_5 p_2^{\perp} \\ \langle p_6, p_3 \rangle & \langle p_6, p_4 \rangle & 0 \end{pmatrix} + \det \begin{pmatrix} p_3 & t_4 p_1^{\perp} & t_5 p_2^{\perp} \\ \langle p_6, p_3 \rangle & 0 & 0 \end{pmatrix} \tag{19}$$

Since $\det \begin{pmatrix} p_3 & p_4 & p_5 \\ \langle p_6, p_3 \rangle & \langle p_6, p_4 \rangle & \langle p_6, p_5 \rangle \end{pmatrix} = 0$, this gives

$$A t_4 t_5 + B t_4 + C t_5 = 0, \tag{20}$$

where $A = \begin{pmatrix} p_3 & p_1^{\perp} & p_2^{\perp} \\ \langle p_6, p_3 \rangle & 0 & 0 \end{pmatrix}$, $B = \det \begin{pmatrix} p_3 & p_1^{\perp} & p_5 \\ \langle p_6, p_3 \rangle & 0 & \langle p_6, p_5 \rangle \end{pmatrix}$, and $C = \det \begin{pmatrix} p_3 & p_4 & p_2^{\perp} \\ \langle p_6, p_3 \rangle & \langle p_6, p_4 \rangle & 0 \end{pmatrix}$.

A similar calculation for the constraints represented by the edges $87, 47, 57$ gives

$$a t_4 t_5 + b t_4 + c t_5 = 0, \tag{21}$$

where $a = \begin{pmatrix} p_8 & p_1^{\perp} & p_2^{\perp} \\ \langle p_7, p_8 \rangle & 0 & 0 \end{pmatrix}$, $b = \det \begin{pmatrix} p_8 & p_1^{\perp} & p_5 \\ \langle p_7, p_8 \rangle & 0 & \langle p_7, p_5 \rangle \end{pmatrix}$, and $c = \det \begin{pmatrix} p_8 & p_4 & p_2^{\perp} \\ \langle p_7, p_8 \rangle & \langle p_7, p_4 \rangle & 0 \end{pmatrix}$.

Equations (20) and (21) imply that $t_5 = k t_4$ for some constant k depending only on p. Substitution back into equation (20) gives a quadratic equation for t_4 with two distinct real roots, one of which is $t_4 = 0$. The other root gives us a realisation (G, q) which is equivalent but not congruent to (G, p). Hence (G, p) is not globally bicompletable in $\mathbb{R}^2$.

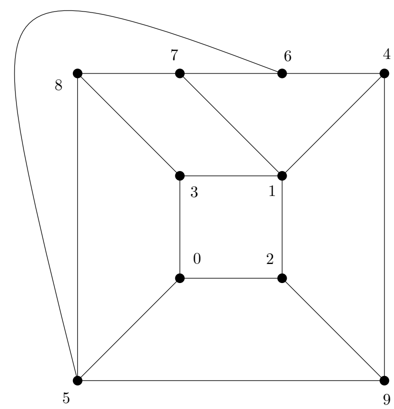

Figure 2: The graph of Example 2.

Theorem 19 now implies that the graph G^+ obtained from G by adding the edges $\{11, 12, 22\}$ is not globally completable in $\mathbb{R}^2$.

It is known that the 1-extension operation introduced in Section 4 preserves the global rigidity of graphs. We showed in [13] that the double 1-extension operation (defined in Section 2.3), which is a natural analogue of 1-extension in the completability setting, preserves global completability *if* the initial graph satisfies the completability-stress rank condition given in [20]. Since the graph G^+ can be constructed from the globally completable graph K_2° by 0-extension and double 1-extension operations, and since 0-extension preserves global completability, we may conclude that double 1-extension does not preserve global completability in general.

Example 2. We next investigate the configuration space of a semi-generic realization of the graph G given in Figure 2 in $\mathbb{R}^2$. Since G is a maximal planar bipartite graph, G is locally bicompletable in $\mathbb{R}^2$ by [15, 13]. We will show that (G, p) is globally bicompletable for some but not all generic p.

By Lemma 15 and (the bicompletability version of) Proposition 14, it will suffice to consider a semi-generic framework (G, p), in standard position with respect to $(0, 1)$. We will compute all possible realizations (G, q) which are equivalent to (G, p). Since G is locally bicompletable, we may assume that q is semi-generic and in standard position with respect to $(0, 1)$ by (the bicompletability version of) Proposition 13. Since $G[0, 1, 2, 3]$ is globally bicompletable, we have $p_i = q_i$ for $i \in \{0, 1, 2, 3\}$. Since G has edges $14, 17, 05$, we also have

$$q_4 = p_4 + t_4 p_1^\perp \tag{22}$$

$$q_5 = p_5 + t_5 p_0^\perp \tag{23}$$

$$q_7 = p_7 + t_7 p_1^\perp \tag{24}$$

for some $t_4, t_5, t_7 \in \mathbb{R}$.

Proposition 20 implies that the constraints represented by the three sets of edges $\{38, 58, 78\}$, $\{29, 59, 49\}$ and $\{46, 56, 76\}$ are equivalent to the system of three quadratic equations

$$a_1t_5t_7 + a_2t_5 + a_3t_7 = 0, \tag{25}$$

$$b_1t_4t_5 + b_2t_5 + b_3t_4 = 0, \tag{26}$$

$$c_1t_4t_5 + c_2t_4t_7 + c_3t_5t_7 + c_4t_4 + c_5t_5 + c_6t_7 = 0 \tag{27}$$

where:

$a_1 = \det\begin{pmatrix} p_3 & p_0^\perp & p_1^\perp \\ \langle p_8, p_3\rangle & 0 & 0 \end{pmatrix}$, $a_2 = \det\begin{pmatrix} p_3 & p_0^\perp & p_7 \\ \langle p_8, p_3\rangle & 0 & \langle p_8, p_7\rangle \end{pmatrix}$, $a_3 = \det\begin{pmatrix} p_3 & p_5 & p_1^\perp \\ \langle p_8, p_3\rangle & \langle p_8, p_5\rangle & 0 \end{pmatrix}$;

$b_1 = \det\begin{pmatrix} p_2 & p_0^\perp & p_1^\perp \\ \langle p_9, p_2\rangle & 0 & 0 \end{pmatrix}$, $b_2 = \det\begin{pmatrix} p_2 & p_0^\perp & p_4 \\ \langle p_9, p_2\rangle & 0 & \langle p_9, p_4\rangle \end{pmatrix}$, $b_3 = \det\begin{pmatrix} p_2 & p_5 & p_1^\perp \\ \langle p_9, p_2\rangle & \langle p_9, p_5\rangle & 0 \end{pmatrix}$;

$c_1 = \det\begin{pmatrix} p_1^\perp & p_0^\perp & p_7 \\ 0 & 0 & \langle p_6, p_7\rangle \end{pmatrix}$, $c_2 = \det\begin{pmatrix} p_1^\perp & p_5 & p_1^\perp \\ 0 & \langle p_6, p_5\rangle & 0 \end{pmatrix}$, $c_3 = \det\begin{pmatrix} p_4 & p_0^\perp & p_1^\perp \\ \langle p_6, p_4\rangle & 0 & 0 \end{pmatrix}$,

$c_4 = \det\begin{pmatrix} p_1^\perp & p_5 & p_7 \\ 0 & \langle p_6, p_5\rangle & \langle p_6, p_7\rangle \end{pmatrix}$, $c_5 = \det\begin{pmatrix} p_4 & p_0^\perp & p_7 \\ \langle p_6, p_4\rangle & 0 & \langle p_6, p_7\rangle \end{pmatrix}$,

$c_6 = \det\begin{pmatrix} p_4 & p_5 & p_1^\perp \\ \langle p_6, p_4\rangle & \langle p_6, p_5\rangle & 0 \end{pmatrix}$.

Clearly $c_2 = 0$. We can rewrite (26) and (27) as

$$t_4(b_1t_5 + b_3) + b_2t_5 = 0 \tag{28}$$

$$t_4(c_1t_5 + c_4) = -(c_3t_5t_7 + c_5t_5 + c_6t_7) \tag{29}$$

We can now substitute (29) into the equation we get by multiplying (28) by $(c_1t_5 + c_4)$ to obtain

$$-(c_3t_5t_7 + c_5t_5 + c_6t_7)(b_1t_5 + b_3) + b_2t_5(c_1t_5 + c_4) = 0. \tag{30}$$

We next rewrite (25) and (30) as

$$t_7(a_1t_5 + a_3) + a_2t_5 = 0 \tag{31}$$

$$t_7(c_3t_5 + c_6)(b_1t_5 + b_3) = t_5[b_2(c_1t_5 + c_4) - c_5(b_1t_5 + b_3)]. \tag{32}$$

We then substitute (32) into the equation we get by multiplying (31) by $(c_3t_5+c_6)(b_1t_5+b_3)$ to obtain

$$t_5(d_1t_5^2 + d_2t_5 + d_3) = 0 \tag{33}$$

where $d_1 = a_1(b_2c_1 - b_1c_5) + a_2b_1c_3$, $d_2 = a_1(b_2c_4 - b_3c_5) + a_3(b_2c_1 - b_1c_5) + a_2(b_3c_3 + b_1c_6)$, and $d_3 = a_3(b_2c_4 - b_3c_5) + a_2b_3c_6$. This cubic equation will have either one or three real roots

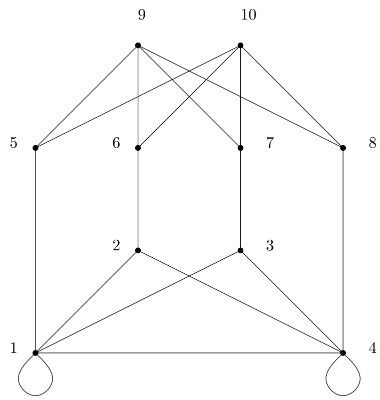

Figure 3: The graph of Example 3.

depending on the sign of the discriminant $D = d_2^2 - 4d_1d_3$. It follows that the framework (G,p) will be globally bicompletable when $D < 0$, and will not be globally bicompletable when $D > 0$. It remains to show that both alternatives are possible.

If we take p such that $p_0^\perp = \begin{pmatrix}1\\0\end{pmatrix}, p_1^\perp = \begin{pmatrix}0\\1\end{pmatrix}, p_3 = \begin{pmatrix}1\\-1\end{pmatrix}, p_4 = \begin{pmatrix}1\\1\end{pmatrix}, p_5 = \begin{pmatrix}1\\1\end{pmatrix}, p_6 = \begin{pmatrix}0\\1\end{pmatrix}, p_7 = \begin{pmatrix}1\\0\end{pmatrix}, p_8 = \begin{pmatrix}1\\1\end{pmatrix}, p_9 = \begin{pmatrix}1\\0\end{pmatrix}$, then $a_1 = 0, b_1 = p_{2,x}, c_1 = 0, c_3 = 1, a_2 = 1, a_3 = -2, b_2 = p_{2,x} - p_{2,y}, b_3 = 0, c_4 = 1, c_5 = 0, c_6 = 0$. Hence $d_1 = p_{2,x}$, $d_2 = 0$, $d_3 = -2(p_{2,x} - p_{2,y})$ and $D = 8p_{2,x}(p_{2,x} - p_{2,y})$, which can be both positive and negative depending on the entries of p_2.

This example shows that global bicompletability is not a generic property. We can now apply Theorem 19 to also deduce that global completablity is not a generic property.

Example 3.

Theorems of Connelly [3], and Jackson and Jordán [11] for $d = 2$, or Gortler, Healy, and Thurston [8] for general dimension, imply that the global rigidity of graphs can be characterized by a rank condition on stress matrices. An analogous condition, which we have referred to as the completability-stress rank condition, was shown to be sufficient to imply global completability in [13]. This rank condition is not necessary in general, however, since any graph which can be constructed from a globally completable graph by a simple 0-extension is globally completable by Lemma 3, but cannot satisfy the rank condition. It is perhaps plausible that all gobally completable graphs can be constructed from graphs satisfying the completability-stress rank condition by a sequence of 0-extensions. In Figure 3 we give an example which shows that this is not the case.

Let G be the graph in Figure 3, and let (G,p) be a generic realization in $\mathbb{R}^2$. Consider any realization (G,q) equivalent to (G,p). Note that $\{1,2,3,4\}$ induces a globally completable subgraph since it can be constructed from K_2° by 0-extension. Hence we may assume $p(i) = q(i)$ for $i = 1, \ldots, 4$. Also $\{5, \ldots, 10\}$ induces $K_{2,4}$ which is globally bicompletable. Hence there is a 2-by-2 nonsingular matrix A such that $q(i) = Ap(i)$ for $i = 5, \ldots, 8$, and $q(i) = (A^{-1})^\top p(i)$ for $i = 9, 10$. Due to the existence of the four bridging edges between $\{1, 2, 3, 4\}$ and $\{5, \ldots, 10\}$, we have $p(i)^\top (I - A)p(i+4) = 0$ for $i = 1, \ldots, 4$, which implies that $A = I$ since p is generic. Thus (G, q) is congruent to (G, p), and G is globally completable in $\mathbb{R}^2$.

Since G is globally completable in $\mathbb{R}^2$, it is locally completable in $\mathbb{R}^2$, and hence is c-independent since $|E| = 2|V| - 1$. It follows that the only completability-stress of a generic realization of G is the zero stress and hence the rank condition cannot be satisfied. Since G has no vertex of degree two, it cannot be constructed by a 0-extension.

A similar construction can be used to give a globally completable graph from any pair consisting of a globally completable graph and a globally bicompletable graph with sufficiently many vertices.

7 Combinatorial sufficient conditions for completability

In this section our goal is to show that if the minimum degree of an n-vertex graph G is sufficiently large, or the number of pairs of non-adjacent vertices is sufficiently small, compared to n and d, then G is locally (resp. globally) completable in $\mathbb{R}^d$. Our bounds are essentially tight in most cases.

These results will imply that if sufficiently many entries are known in each row/column of the given partially filled matrix (or if the number of unknown entries is sufficiently small) then - in the generic case - the completion is locally (resp. globally) unique.

We shall frequently use the fact that the graph operations introduced in Section 2.3 preserve local (or global) completability in $\mathbb{R}^d$.

7.1 Minimum degree bounds

The *degree* of a vertex v in a graph G is the number of edges incident to v, counting loops once. Let $\delta(G)$ denote the minimum degree of G. We will use the fact that $K_4 + e$ (the graph obtained from K_4 by adding a loop) and $K_5 - e$ (the graph obtained from K_5 by deleting an edge) are locally completable in $\mathbb{R}^2$, since they can be obtained from K_2° by a 0-extension and one or two vertex splits, respectively.

Since the complete tripartite graph $K_{m,m,1}$ is not locally completable in $\mathbb{R}^2$ by Lemma 9, the bound in the next result is almost tight.

Theorem 36. *Let $G = (V, E)$ be a semisimple graph on n vertices with $\delta(G) \geq \lceil n/2 \rceil + 2$. Then G is locally completable in $\mathbb{R}^2$.*

Proof. We use induction on n. Note that the minimum degree condition implies that $n \geq 4$.

We first show that G has a locally completable subgraph on at least four vertices. Suppose not. Then $K_5 - e \not\subseteq G$.

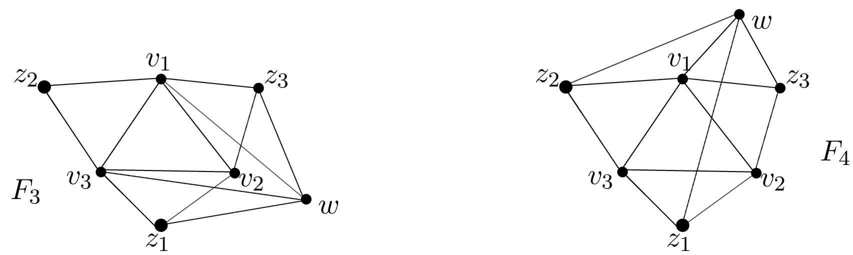

Figure 4: The subgraphs F_3 and F_4.

Suppose that G has a subgraph H isomorphic to K_4. Since $K_5 - e \nsubseteq G$, each vertex of $G - H$ is adjacent to at most two vertices of H. We can now apply induction to $G - H$ to deduce that $G - H$ is a locally completable graph on at least four vertices. Hence we may assume that $K_4 \not\subset G$.

The minimum degree condition implies that there exists a subgraph F_1 of G which is isomorphic to K_3. Let $V(F_1) = \{v_1, v_2, v_3\}$. The minimum degree condition also implies that v_i and v_{i+1} have a common neighbour z_{i+2} in $G - F_1$, reading subscripts modulo three, and the fact that G has no K_4 implies that z_1, z_2, z_3 are distinct. Let $F_2 = G[v_1, v_2, v_3, z_1, z_2, z_3]$.

If each vertex of $G - F_2$ is adjacent to at most three vertices of F_2 then we may apply induction to deduce that $G - F_2$ is a locally completable graph on at least four vertices. Hence some vertex w of $G - F_2$ is adjacent to four vertices of F_2. Since $K_4 \not\subset G$, $G[V(F_2) \cup \{w\}]$ contains one of the two graphs F_3, F_4 shown in Figure 4.

We can reduce F_3 to the locally completable graph $K_5 - e$ by deleting z_2 and then contracting (i.e. applying the inverse of vertex-splitting to) the pair z_1, z_3. Hence F_3 is locally completable. Similarly, we can reduce F_4 to a $K_4 + e$ by contracting the pair z_1, z_3 and then applying the inverse of double 1-extension. Thus F_4 is locally completable. It follows that G has a locally completable subgraph on at least four vertices.

We may now choose a maximal locally completable subgraph H of G. Let $|V(H)| = t$. Suppose that $H \neq G$. If $t < \lceil n/2 \rceil$ then the minimum degree condition implies that there are at least $t(\lceil n/2 \rceil + 2 - t) > n - t$ edges from H to $G - H$, and hence some vertex of $G - H$ is adjacent to two vertices of H. On the other hand, if $t \geq \lceil n/2 \rceil$ then each vertex of $G - H$ is adjacent to at least $\lceil n/2 \rceil + 2 - n + t \geq 2$ vertices of H. In both alternatives we may construct a larger locally completable subgraph by performing a 0-extension on H. $\square$

By using Theorem 31 we can deduce the following sufficient condition for global completability.

Theorem 37. *Let $G = (V, E)$ be a semisimple graph on n vertices. Suppose that $\delta(G) \geq \lceil n/2 \rceil + 3$. Then G is globally completable in $\mathbb{R}^2$.*

We close this subsection by considering a possible extension to $\mathbb{R}^d$. Lemma 9 implies that $K_{m,m,d-1}$ is not locally completable in $\mathbb{R}^d$ for all m, and hence that the degree bound in the following conjecture would be best possible.

Conjecture 38. *For every $d \geq 1$ there is an integer c_d such that every semisimple graph G on $n \geq c_d$ vertices with $\delta(G) \geq (n+d)/2$ is locally completable in $\mathbb{R}^d$.*

Some evidence in favour of this conjecture can be deduced from our next result.

Theorem 39. *For all $d \geq 1$ and $\epsilon > 0$ there exists an integer $N = N_{d,\epsilon}$ such that every semisimple graph G on $n > N$ vertices with $\delta(G) \geq n(1+\epsilon)/2$ is locally completable in $\mathbb{R}^d$.*

Proof. The Erdős-Stone Theorem [6] tells us that there exists an N such that every semisimple graph G on $n > N$ vertices with $\delta(G) \geq n(1+\epsilon)/2$ has a subgraph F isomorphic to $K_{d,d,d}$. Lemma 9 implies that F is a locally completable graph on $3d$ vertices. We can now choose a maximal locally completable subgraph H of G, and use (the d-dimensional version of) the argument given in the last paragraph of the proof of Theorem 36 to deduce that $H = G$. □

7.2 Bounds on the number of missing edges

Let $G = (V, E)$ be a simple graph. We say that a pair $u, v \in V$ of non-adjacent vertices with $u \neq v$ is a *missing edge* of G. If G is locally (or globally) completable in $\mathbb{R}^d$ on at least $d+1$ vertices then each vertex must be incident with at least d edges. This implies that there exist simple graphs on n vertices with $n-d$ missing edges which are not locally (or globally) completable in $\mathbb{R}^d$.

The number of edges from a vertex v to a set X of vertices is denoted by $d_G(v, X)$.

Theorem 40. *Let $G = (V, E)$ be a simple graph on $n \geq 3(2d+1)$ vertices. Suppose that G has at most $n-d-1$ missing edges. Then G is globally completable in $\mathbb{R}^d$.*

Proof. We first remark that K_{2d+1} is locally completable in $\mathbb{R}^d$. This can be checked by first observing that a simple 0-extension G' of K_d° is locally completable in $\mathbb{R}^d$ and that a spanning subgraph of K_{2d+1} can be obtained from G' by vertex-splitting operations by eliminating a loop at each step. Hence K_{2d+2} is globally completable in $\mathbb{R}^d$ by Theorem 31.

Now the number of edges in G is at least $\binom{n}{2} - (n-d-1)$, which is larger than $(1 - \frac{1}{2d+1})\frac{n^2}{2}$ since $n \geq 3(2d+1)$. Therefore by Turán's theorem [25] G contains a subgraph H which is isomorphic to K_{2d+2}.

To conclude the proof we show that a spanning subgraph of G can be obtained from H by a sequence of simple 0-extensions. Let $\{v_1, \ldots, v_{2d+2}\}$ be the vertices of H, and consider an ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertices which starts with the vertices of H and satisfies

$$d(v_i, \{v_1, v_2, \ldots, v_{i-1}\}) \geq d(v_j, \{v_1, v_2, \ldots, v_{i-1}\})$$

for all $2d+3 \leq i < j \leq n$. Such an ordering can be found greedily.

We claim that for all $2d+3 \leq i \leq n$ we have $d(v_i, \{v_1, v_2, ..., v_{i-1}\}) \geq d$ (which implies the statement of the theorem). Indeed, by assuming that the inequality fails for v_i we can deduce that all vertices after v_i send at most $d-1$ edges back to the set $\{v_1, v_2, ..., v_{i-1}\}$, which means that the number of missing edges is at least $(n-i+1)(i-d) \geq n-d$. This contradicts the fact that G has at most $n-d-1$ missing edges. □

By a more detailed analysis it is possible to reduce the lower bound for n. We shall demonstrate this for local completability in $\mathbb{R}^2$. First we use an observation of Berger, Kleinberg, and Leighton [2]. For completeness we give (a slightly simplified) proof of their result.

The *degree-k extension* operation adds a new vertex v to a graph and at least k new edges incident with v.

Lemma 41. *[2] Let $G = (V, E)$ be a simple graph on n vertices. Suppose that G has at most $n-5$ missing edges. Then G can be obtained from K_5 by a sequence of degree-4 extensions.*

Proof. Let H be the complement of G. Since H has n vertices and at most $n-5$ edges, it has at least five connected components. By choosing vertices from five different components we can find five pairwise non-adjacent vertices v_1, v_2, v_3, v_4, v_5 in H. Consider an ordering $v_1, v_2, ..., v_n$ of the vertices of G which starts with the five chosen vertices and satisfies

$$d_G(v_i, \{v_1, v_2, ..., v_{i-1}\}) \leq d_G(v_j, \{v_1, v_2, ..., v_{i-1}\})$$

for all $6 \leq i < j \leq n$. Then we can use the argument given in the last paragraph of the proof of Theorem 40 to conclude that $d_G(v_i, \{v_1, v_2, ..., v_{i-1}\}) \geq 4$ for $6 \leq i \leq n$. □

Theorem 42. *Let $G = (V, E)$ be a simple graph on $n \geq 6$ vertices. Suppose that G has at most $n-3$ missing edges. Then G is locally completable in $\mathbb{R}^2$.*

Proof. It follows from Lemma 41 that G can be obtained either from K_5 or $K_5 - e$ by a sequence of degree-2 extensions, or from K_5 minus two edges by a sequence of degree-4 extensions.

In the first case we can deduce that G is locally completable in $\mathbb{R}^2$ by observing that a degree-2 extension corresponds to applying a 0-extension and possibly adding some new edges, and using the fact that $K_5 - e$ is locally completable.

Consider the second case. The graph obtained from K_5 minus two edges by a degree-4 extension operation is a graph H on six vertices with at most three missing edges. It is easy to see that H can be obtained from $K_5 - e$ by a degree-3 extension (and hence it is locally completable in $\mathbb{R}^2$) or H can be obtained from K_6 by deleting three disjoint edges (and hence is isomorphic to $K_{2,2,2}$ which is locally completable by Lemma 9). We can now deduce that G is locally completable in $\mathbb{R}^2$ as in the first case. □

The preceding results in this subsection have been restricted to simple graphs. It is also natural to consider semisimple graphs on n vertices and to compare the number of edges to that of the complete semisimple graph K_n°. We close this section with two results of this type, which are valid in all dimensions. Henceforth, we will also consider a pair u, u to be a missing edge of G if there is no loop incident with u in G.

Theorem 43. *Let $G = (V, E)$ be a semisimple graph on $n \geq d$ vertices. Suppose that G has at most $n - d$ missing edges. Then G is locally completable in $\mathbb{R}^d$.*

Proof. Note that the hypotheses imply that each vertex has at least d incident edges. We use induction on the number of vertices. We may assume that $G \neq K_n^\circ$ and hence that $n > d$.

Choose a vertex v of G with at most $n - 1$ incident edges. Then $G - v$ has at most $n - d - 1 = (n - 1) - d$ missing edges and hence is locally completable by induction. Since G can be obtained from $G - v$ by a (possibly non-simple) 0-extension and edge additions, G is also locally completable in $\mathbb{R}^d$. $\square$

The bound $n - d$ is best possible. To see this consider the graph obtained from a complete semisimple graph by attaching a vertex v of degree $d - 1$. Theorems 43 and 31 imply a similar bound for global completability.

Theorem 44. *Let $G = (V, E)$ be a semisimple graph on $n \geq d$ vertices. Suppose that G has at most $n - d - 1$ missing edges. Then G is globally completable in $\mathbb{R}^d$.*

The graph obtained from a complete semisimple graph by attaching a vertex v of degree d which has a loop on it shows that this bound is also best possible.

8 Concluding remarks

We conclude the paper with some open questions. As we noted earlier, the complexity of deciding whether a given graph is locally (or globally) completable in $\mathbb{R}^d$ remains open for all $d \geq 2$.

One may also consider the completion problem of matrices with complex entries and search for a charaterization of those partially filled Hermitian matrices which have a unique complex completion. We note that, in this case, global completability is known to be a generic property by [14, Lemma 4.4].

Motivated by corresponding results for rigidity and global rigidity, we also ask the following questions. Is it true that, if a graph is redundantly locally completable in $\mathbb{R}^d$ (i.e., it is locally completable after removing any edge), then G is globally completable if and only if the completability-stress condition holds (c.f. [13, Theorem 6.2])? In particular, is global completability a generic property of redundantly locally completable graphs?

The *configuration space* of a framework (G, p) is the set of all $q \in \mathbb{R}^{d|V|}$ for which (G, q) is equivalent to (G, p). It seems likely that the proof technique used by Hendrickson [10] can be used to show that, if (G, p) is generic and globally completable, then for each $e \in E$

the framework $(G-e,p)$ is either locally completable or has an unbounded configuration space. Hence it would be useful to determine when the configuration space of a (generic) framework in $\mathbb{R}^d$ is bounded.

Acknowledgement

This work was supported by the Hungarian Scientific Research Fund grant no. K81472, CK80124, K109240 and K115483. The third author was supported by JSPS Postdoctoral Fellowships for Research Abroad and JSPS Grant-in-Aid for Young Scientists (B) (No. 15K15942).

References

[1] L. Asimov and B. Roth, The rigidity of graphs. *Tran. Amer. Math. Soc.* 245 (1978) 279–289.

[2] B. Berger, J. Kleinberg, T. Leighton, Reconstructing a three-dimensional model with arbitrary errors, *J. ACM* 46, (1999) 212–235.

[3] R. Connelly, Generic global rigidity, *Discrete Comput. Geom.* 33 (2005) 549–563.

[4] R. Connelly and W. Whiteley, Global rigidity: The effect of coning. *Discrete Comput. Geom.* 43 (2010) 717–735.

[5] M. E.-Nagy, M. Laurent, and A. Varvitsiotis, Complexity of the positive semidefinite matrix completion problem with a rank constraint, in *Discrete Geometry and Optimization*: K. Bezdek et al. (eds.), Fields Institute Communications 69, 2014 105–120.

[6] P. Erdős and A.H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* 52 (1946) 1087–1091.

[7] H. Gluck, Almost all simply connected closed surfaces are rigid, in *Geometric topology*, Springer, Lecture Notes in Mathematics 438 1975 225–240.

[8] S. Gortler, A. Healy, and D. Thurston, Characterizing generic global rigidity, *American J. Math.*, 132 (2010) 897–939.

[9] D. Hadwin, K.J. Harrison, and J.A. Ward, Rank-one completions of partial matrices and completely rank-nonincreasing linear functionals, *Proc. Amer. Math. Soc.* 134, (2006) 2169-2178.

[10] B. Hendrickson, Conditions for unique graph realizations, *SIAM J. Comput.*, 12 (1992) 65–84.

[11] B. Jackson and T. Jordán, Connected rigidity matroids and unique realizations of graphs. *J. Combin. Theory Ser. B*, 94 (2005) 1–29.

[12] B. Jackson, T. Jordán and Z. Szabadka, Globally linked pairs of vertices in equivalent realizations of graphs, *Discrete Comput. Geom.*, 35 (2006) 493–512.

[13] B. Jackson, T. Jordán and S. Tanigawa, Combinatorial conditions for the unique completability of low rank matrices, *SIAM J. Discrete Math.* 28 (2014) 1797–1819.

[14] B. Jackson and J. C. Owen, The number of equivalent realisations of a rigid graph, arXiv:1204.1228, 2012.

[15] G. Kalai, E. Nevo, and I. Novik, Bipartite rigidity, *Tran. Amer. Math. Soc.* 368 (2016) 5515–5545.

[16] F. Király, L. Theran and R. Tomioka, The algebraic combinatorial approach for low-rank matrix completion, *J. Machine Learning Research* 16 (2015) 1391–1436.

[17] L. Liberti, C. Lavor, N. Maculan, and A. Mucherino, Euclidean distance geometry and applications, *SIAM Rev.* 56 (2014) 3–69.

[18] M. Mesbahi and G.P. Papavassilopoulos, On the rank minimization problem over a positive semidefinite matrix inequality, *IEEE Trans. Automat. Control* 42 (1997) 239–243.

[19] J.B. Saxe, Embeddability of weighted graphs in k-space is strongly NP-hard, Tech. Report, Computer Science Department, Carnegie-Mellon University, Pittsburgh, PA, 1979.

[20] A. Singer and M. Cucuringu, Uniqueness of low rank matrix completion by rigidity theory, *SIAM J. Matrix Anal. Appl.* 31 (2010) 1621–1641.

[21] N. Srebro, Learning with matrix factorizations, PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 2004.

[22] Z. Szabadka, Globally rigid frameworks and rigid tensegrity graphs in the plane, PhD. Thesis, Department of Operations Research, Eötvös Loránd University, 2010.

[23] S. Tanigawa, Sufficient conditions for the global rigidity of graphs, *J. Comb. Theory, Series B* 113 (2015) 123–140.

[24] C. Tomasi and T. Kanade, Shape and motion from image streams under orthography: A factorization method, *Int. J. Comput. Vision* 9 (1992), 137–154.

[25] P. Turán, On an extremal problem in graph theory (in Hungarian), *Mat. Fiz. Lapok* 48 (1941) 436–452.

[26] W. Whiteley, Some matroids from discrete applied geometry, in *Matroid Theorey*, J.E. Bonin et al eds., Contemporary Mathematics 197 1997 171–312.

[27] W. Whiteley, Cones, infinity and one-story buildings, *Structural Topology* 8 (1983) 53–70.