To my mother and father
INFORMATION, EXPECTATIONS AND MACROECONOMIC POLICY

by

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ABSTRACT

The thesis is motivated by some important recent developments in macroeconomic theory and the theory of macroeconomic policy. A common theme emphasized throughout is the integration of rational expectations into macroeconomic policy evaluation and the sophism of conducting evaluations predicated on alternative expectations hypotheses.

The application of rational expectations to optimal control theory inspires a game-theoretic paradigm for the derivation of optimal economic policies. This transforms fundamentally the way in which economists should address the control problem. The recent proliferation of research in the area is subject to the first systematic investigation. The necessity for assimilating model uncertainty into the problem of policy evaluation is emphasized. This is made operational with respect to a particularly topical issue concerning the optimal choice of monetary instrument.

A substantial part of the thesis is devoted to a rigorous exploration of the information structure conditioning expectations. Particular emphasis is on partial ignorance. An intelligent system can exploit statistical filtering techniques to extract the information content of certain economic variables. The thesis illustrates vividly the potentially critical dependence of the laws of motion of the system on the information structure. It calls for a detailed explication of that structure as a pre-requisite for any analysis and highlights properties of certain earlier treatments which are symptomatic of their neglect of this.

Analytical work is combined with computer simulations of a larger econometric model. Higher order dynamics are the realised...
consequences of asset accumulation and the government budget constraint. The issue of bond-financed deficit instability is subject to extensive testing and the implications of divergent expectations mechanisms elicited.
The author would like to express his appreciation of the continual guidance and encouragement given by his supervisor, Professor David Currie. In addition, Professor Currie encouraged participation in the continuing study group meetings on optimal control between the Department of Economics at Queen Mary College and the Department of Applied Economics at the University of Cambridge. The author acknowledges the benefits gained from these.

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Needless to say, full appreciation is given for the ability to have access to the facilities at the CEPR together with having the opportunity for discussion with its participants who are too numerous to mention individually, but to whom a debt is gratefully acknowledged.

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Computer activity involved the use of recently developed software at Queen Mary College for the simulation of rational expectations models. Though this was very much in its experimental stages at the beginning of the research significant progress was made by, above all, the continual efforts of Mr. Ali Al-Nowaihi. To this person,
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Section 2.4 is based on research in collaboration with Professor David Currie, and chapter 4 is based on research in collaboration with Professor David Currie, Mr. Rod Whittaker and Mr. Simon Wren-Lewis. The author expresses his thanks to these persons for consenting to the inclusion of this material in the thesis.

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Needless to say, any remaining errors are the sole responsibility of the author.

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INTRODUCTION

The thesis is motivated by some interesting and important recent developments in macroeconomic theory and the theory of macroeconomic policy. For the most part, these developments have emerged as a realised consequence of the assimilation of the rational expectations hypothesis in macroeconomic policy evaluation. The thesis does not dispute the hypothesis, though some cautionary remarks will be elicited. Rational expectations is applied rigorously throughout on the grounds that it is a sophism to consider any alternative which eschews the implications of an intelligent economic system. As yet, rational expectations is the only hypothesis which embraces this in any formally definitive way. For a clear, critical and comprehensive appraisal of rational expectations, the reader is urged to consult the excellent studies by Begg (1982a), Minford and Peel (1983b) and Sheffrin (1983) (for further useful discussion and some doctrinal-historic perspectives see, for example, Kantor (1979); Buiter (1980b); Laidler (1981b); Meade (1981); Tobin (1981); Hahn (1982); Maddock and Carter (1982); Hoover (1984)).

For much of the thesis, small stylised models are constructed which permit analytical tractability. This is, of course, to incur criticism from those who regard such devices with scepticism: they have little or no relevance either in the understanding of the laws of motion of the system nor in practical policy making. These criticisms are vacuous here. Though the simple models which decorate the following pages are clearly unsuitable for practical policy making, it is stressed at the outset that it is not the intention that they should be considered as anything otherwise. The use of small analytical models serves efficiently the purpose of demon-
Stratifying succinctly issues which are entirely general, which may be obscured in much larger models and which are unlikely to be qualified in the latter. In this way, small models may yield insights into the operation of more complicated structures.

The thesis emphasizes the need and makes operational a particular methodology in macroeconomic policy evaluation. A policy is only optimal given the preferences of the controller and the system under control. It is unlikely to remain optimal in an alternative structure with different laws of motion and a different performance criteria. The testing of optimal policies across divergent systems is taken as a \textit{conditio sine qua non} of sensible policy evaluation.

A payoff matrix can be constructed which conveys important information about control rules that perform reasonably well irrespective of the particular dynamic system. Such policies may be termed model-robust rules and they derive their attractiveness from the eclectic methodology.

A significant and important ingredient in the thesis is the potentially crucial role of the information structure in determining the behaviour of the system. A desirable feature of the rational expectations hypothesis is that it draws attention to the information content of economic variables. The implications of this are far-reaching. Control rules now have scope to influence the system in two ways: the standard mechanism whereby control alters the dynamic response of the system; and an additional route by which control alters the information content of variables. Moreover, the actual implementation of controls is endowed with an additional complicated dimension since the controller might wish to avoid feedback on unobservable variables.
A summary of the thesis is outlined below. It is essentially comprised of two main parts. Part I contains chapters 1 - 4. Part II is primarily concerned with the informational assumptions in rational expectations models and comprises chapters 5 - 6.

Chapter 1 considers some recent developments in the application of optimal control theory to the derivation of optimal economic policies under rational expectations. Recent years have witnessed a proliferation of research in this area. The conclusions seem to be of fundamental importance and alter drastically the problem of policy evaluation. Yet some of these have been obscured by the pace at which the research has developed. The chapter attempts a comprehensive and detailed appraisal of the issues involved with the view to making these accessible to a wider audience than has hitherto been the case. The chapter forms a recent discussion paper by the author.

Chapter 2 is by way of a prelude to chapter 3. Both are concerned with making operational the aforementioned methodology regarding the search for model-robust control rules. The particular context is the issue of the optimal monetary instrument. In chapter 2, a brief critical review of the existing literature is given. It concludes that the resulting payoff matrix has negligible information value. An analytical framework which corrects for this is discussed informally. The main innovation in the chapter is a proposed resolution of the problem of indeterminacy under interest rate control. This is borrowed from some recent work in collaboration with the author's supervisor and can be found in a discussion paper.

Chapter 3 formalizes the taxonomy discussed in chapter 2. A systematic investigation into the instrument-control problem is performed. This yields a payoff matrix which conveys useful information about the possibility of model-robust policies. The chapter is a much extended version of a recent discussion paper by the author.
Chapter 4 adopts computational techniques for examining a larger econometric model. Asset accumulation and the government budget constraint are important components. The issue of bond-financed deficit instability is addressed and simulations of various exogenous shocks are performed. A comparison of divergent expectations mechanisms is the basic motivation. The chapter is based on a research paper forthcoming in the Economic Journal and written in collaboration with others whose identities are given in the acknowledgements.

Chapter 5 discusses the signal-extraction problem. A critical review of some recent literature is given. The novel element in the chapter is the formal clarification of the information contents of asset prices. This follows closely a recent discussion paper by the author, to be published in the Manchester School.

Chapter 6 contains a re-evaluation of the monetary instrument problem in the context of an information variable framework. The relative superiority of alternative control rules is shown to be non-invariant with respect to the information structure. This may qualify the favourability of certain policies derived from orthodox analysis. As before, the analysis can be found in a recent discussion paper.

Chapter 7 summarizes the preceding chapters. It suggests areas where we believe important research to lie in the future and records our final thoughts.

The thesis has a fairly high technical content, exploiting some relatively new techniques which might be unfamiliar to the reader. In view of this, the main text of the thesis is supplemented by a series of mathematical appendices. In each, the general technique is described and is followed by its application to the particular problems in the main text.
Appendix A summarizes the solution of the dynamic optimal control problem using the maximum principle and Hamiltonian dynamics.

Appendix B reviews the saddlepoint solution for rational expectations models.

Appendix C contains the undetermined coefficients technique for solving rational expectations models.

Appendix D gives a procedure for examining dynamic stability by computation of the test functions of the characteristic equation of the system.

Appendix E demonstrates the derivation of the asymptotic variance-covariance matrix of the state vector in a system of linear stochastic differential equations.

A bibliography is given at the end of the thesis.
1.1 Introduction

Recent years have witnessed an exciting and rapidly growing research programme on macroeconomic policy evaluation when private sector behaviour is forward-looking. The issues raised appear far-reaching and alter fundamentally the nature of the policy problem, focussing on such issues as credibility and reputation as important and endemic aspects of economic policy design. Above all, the major insight is the appreciation of the policy problem as a dynamic game between intelligent players - the government and the private sector - and the implications thereof. This chapter offers the first critical appraisal of this research with the view to putting matters into perspective and making the area accessible to a wider audience.

For the most part, the recent literature adopts an optimal control framework for the purpose of deriving optimal economic policies. There is no reason to believe, however, why the issues raised should not apply generally to the problem of implementing economic policies. If nothing else, the optimal control framework is a useful expositional device and is probably much more than this, being a valuable contribution to the economist's armoury. Briefly, optimal control theory is a branch of applied mathematics designed to compute an optimal strategy for directing a system to a desired state (useful introductions to optimal control theory include Aastroem (1970); Chow (1975); Intrilligator (1971); Meditch (1969); see also appendix A for further references). The deployment of this technique to economic problems was a consequence of the realisation of the apparent similarity between economic and physical systems. Both could be
summarized by an appropriate mathematical framework specifying the endogenous (state) and exogenous (forcing) variables, the control variables (instruments), the stochastic properties of the system, and the way in which the variables are related via a set of static and dynamic simultaneous equations. The derivation of an optimal control rule then proceeds by specifying a performance criteria and optimizing this subject to the structure of the system as completely described above. The attractiveness of this approach for economists was confirmed in the late 1970s when the growing interest was reflected in the content of the 1976-1977 volumes of the Annals of Economic and Social Measurement; in the symposium in the 1976 edition of the American Economic Review; in the establishment of PREM (Programme of Research into Econometric Methods) in the UK; and ultimately in the official committee report on policy optimization (see also Phillips (1954,1957); Johansen (1979); Pau (1979); Schupp (1979)).

About this time, however, the findings of a new breed of economists were becoming a popular focus of attention. Drawing from the seminal contribution by Muth (1961), a number of authors concentrated their efforts on investigating the implications of the rational expectations hypothesis for macroeconomic policy evaluation.

To begin with, the series of theoretical papers by Lucas (1972, 1975), Sargent (1973), Sargent and Wallace (1975, 1976) and Barro (1976) purported to demonstrate the superfluousness of macroeconomic policy in models embodying rational expectations – specifically the ineffectiveness of systematic stabilization policy in influencing the distribution of real variables – which, in terms of output at least therefore, meant the fallacy of applying optimal control techniques. Needless to say, a vast, and by now well-known, literature emerged with the view to prove and counter-prove the opposite and it was soon realised the importance of distinguishing between the rational
expectations assumption and the system in which this is embedded. By suitable re-specification of the model structure, the efficacy of control is entirely consistent with rational expectations and the policy ineffectiveness result merely a special case. A more detailed discussion of these issues is contained in chapter 2 (section 2.3)(1).

Though the prevailing orthodoxy survived the ineffectiveness proposition, rational expectations theorists were preparing to launch a much more fundamental and general attack which struck at the roots of conventional policy evaluation. Thus, Lucas (1976) delivered a devastating criticism of existing approaches which ignored the effects of different policy regimes on the estimated coefficients of an econometric model arising thorough private sector expectations. The critique is well-known and provides one of the most compelling reasons for adopting the rational expectations assumption in policy formulation(2).

Kydland and Prescott (1977) purported to strike a final blow to the established convention by highlighting an important dilemma in applying methods of the physical sciences to social phenomena - in particular, the application of standard optimal control techniques to the derivation of optimal economic policies when the economic system is characterised by forward-looking behaviour. Under such circumstances, the controller is playing a game against an intelligent private sector which behaviour is conditioned by its perceptions about the nature of the control to be applied. There is now a dilemma in implementing an optimal policy: an announcement by the controller stating his planned optimal strategy may lack credibility because he has an incentive to renge on the announcement subsequently. The optimal ex ante policy loses its optimality property because it is not believed and the controller is forced to pursue an inferior
policy. The ex post suboptimality of the ex ante optimal policy is known as the time inconsistency of this policy. It is this and its resolution which has inspired the recent surge of literature on macro-economic policy evaluation and optimal control theory which the current chapter is concerned to assess. Essentially, the policy problem in the absence of a credible optimal policy involves the search for control rules which, though sub-optimal with respect to the non-credible policy, are optimal within the subset of credible (time consistent) policies. Kydland and Prescott (1977) and Prescott (1977) regarded the time inconsistency property as the downfall of optimal control theory in economics. We believe this to be overly pessimistic: though the nature of the control problem becomes fundamentally more complicated, this does not preclude its usefulness.

A virtue of the recent literature is that, for the most part, the conclusions reached do not seem to be limited by model specificity and, to this extent, it offers potentially very important contributions to the theory of economic policy. An unfortunate outcome, however, is that some of these contributions have been obscured, partly by the pace at which the research has developed, partly by the lack of a unified terminology and partly by the high degree of technical sophistication. As yet, its precise importance, relevance and operational significance for practical policy making is unclear and there has been no attempt to put matters into perspective which is probably the reason why they remain principally the domain of a group of academics with little interest shown by others (a reflective discussion of some of the issues can be found in Currie (1985a, b)). We hope to make the issues accessible to a wider audience by offering what is hopefully an exhaustive account of the problems involved and by complementing mathematical rigour with economic intuition. In doing these, it has been our intention to contain the
formal solutions and proofs in a separate section and to concentrate the remaining sections on more informal discussion. This should allow those unfamiliar with the technical apparatus to grasp the main issues involved without having to understand the formal derivation of results, as well as providing a fairly rigorous mathematical framework for the more specialist researchers.

The chapter is structured as follows. Section 1.2 discusses the notions of time consistency and time inconsistency. It contains the standard control problem and then introduces a two-person dynamic game. Time inconsistency is seen to arise in standard game theory (which does not formally model rational expectations) where there is a dominant player, though we point out the implicit assumption of intelligent forward-looking behaviour and turn to a system which models rational expectations at the outset. Section 1.3 surveys the various proposed resolutions of time inconsistency along with addressing some other issues which have emerged in the literature. Equilibrium concepts are introduced and the notions of threat and reputation effects emphasized. This section concludes with an examination of some of the more substantive issues which are rarely addressed in the literature. For the more specialist and interested reader, section 1.4 provides a fairly comprehensive mathematical treatment of much of the issues discussed in the previous sections. Finally, section 1.5 contains some concluding remarks and suggestions about where we believe the important research lies.

1.2 Optimal Control and Dynamic Games: Time Consistency and Time Inconsistency

The discussion in section 1.1 indicated the fundamental difference between the control of physical systems and the control of social systems. Unlike the former, the latter has the intrinsic
feature that the system under control is intelligent and will attempt to anticipate the control which is to be applied. It is this inherent circularity that distinguishes the rational expectations approach to policy evaluation from the orthodox approach—a distinction between causal and non-causal systems. The correct way of addressing the policy problem then requires an appreciation of the strategic behaviour of economic groups. In the presence of forward-looking behaviour, it is not so much the derivation of optimal control rules which is of most interest to begin with, but rather their implementation. To be sure, the existence of an optimal policy is not at question. What is at stake is the operational significance of this policy, which may be zero because the policy lacks credibility. Then the derivation of alternative policies which are credible becomes important.

This section marks the beginning of an extensive discussion of some recent developments in the theory of optimal control under rational expectations. As a starting point, we find it useful to make the following remarks.

An important distinction in the optimal control literature is between open loop and closed loop control. In the former, the current and future values of control instruments are specified at the beginning of the planning horizon and are invariant with respect to any new information that becomes available during the course of this period. These are state independent (or non-contingent) rules. By contrast, the latter type of control makes the control instruments a function of new information previously unavailable at the beginning of the planning period. These are state dependent (or contingent) rules. In deterministic systems the distinction is innocuous: the optimal open loop policy is identical to the optimal closed loop policy in terms of yielding an equal value for the performance
measure. Closed loop control is generally regarded to be superior in stochastic systems because it permits a response to unforeseen contingencies(5).

Now, a possible source of confusion arises when the control problem takes on the features of a game, for the strategic behaviour of players can also be defined in terms of open loop and closed loop concepts (this will be clarified later). It is sufficient to note here the potential confusion which arises when the solution of an open loop game can be expressed in terms of a closed loop control rule. To avoid such confusion, we propose at the outset to retain open loop and closed loop concepts to characterize part of the strategic structure of the game and to define the eventual control rules which result therefrom in terms of either state independent (non-contingent) or state dependent (contingent) rules. Then each of these rules can be expressed formally as, respectively,

\[ w(t) = f^o[\Omega(t_0)] \quad (1.2.1) \]

\[ w(t) = f^c[\Omega(t)] \quad (1.2.2) \]

where \( w \) = vector of control variables

and \( \Omega \) is the information set with \( t_0 \) being the beginning of the planning horizon.

In what follows, deterministic systems occupy most of the discussion, serving as a convenient abstraction in most instances. These may, however, eschew potentially very important issues which arise in a stochastic environment and which may qualify certain of the results derived from the deterministic case. Accordingly, we also consider the stochastic control problem. In terms of the mathematical framework in section 1.4, the issues are formalised within a linear quadratic continuous time framework which is the
usual approach in the literature. The solution of the control problem then exploits the maximum (or minimum) principle developed by Pontryagin et al. (1962) and summarized in appendix A.

1.2(A) Time Consistency and the Principle of Optimality

The early view that optimal control constituted a valuable tool for economic policy evaluation was justified by the apparent similarity between physical and economic systems. Standard dynamic programming techniques were available from the engineering sciences which enabled computation of optimal economic policies. To be more precise, the standard optimal control problem is as follows.

Consider a general linear economic system with laws of motion that can be expressed in terms of the differential equation system

\[
\dot{y} = Ay + Bw
\]

where \( y = nx1 \) vector of predetermined state variables

\( w = mx1 \) vector of control variables.

All variables are measured as deviations from long-run equilibrium and we write \( y = y(t) \) and \( w = w(t) \) for notational convenience. Hence, \( \dot{y}(t) \) is the time derivative of \( y(t) \). \( A \) and \( B \) are time-invariant matrices with orders \( nxn \) and \( nxm \) respectively. Next, assume that the controller possesses a performance measure, \( J \), which is a positive semi-definite quadratic in the state and control:

\[
J = \int_{t_0}^{t_1} e^{-\rho t} (y^T Q y + w^T R w) dt \quad \rho > 0
\]

where \( \rho \) is a discount factor, \( Q \) is a symmetric positive semi-definite time-invariant matrix of order \( nxn \) and \( R \) is a symmetric positive
definite time-invariant matrix of order mxm. The planning horizon is \((t_0, t_1)\) with initial time, \(t_0\), given and terminal time being \(t_1\). Thus, the controller wishes to minimise deviations of the state and control variables from their long-run equilibrium levels\(^{(6)}\).

The optimal control problem is to choose a control sequence, 
\[ \{w(t_0), \ldots, w(t_1)\} = \{w(t)\}_{t_0}^{t_1} \]
which minimizes equation (1.2.4) subject to equation (1.2.3) and the \(n\) initial conditions, 
\[ y(t_0) = y_0. \]
Section 1.4(A) solves this problem formally where it is shown that the optimal control rule can be expressed either as a linear time-invariant feedback rule on the state vector, \(y(t)\), or as a linear time-varying rule on the initial state, \(y(t_0) = y_0\) (that is, a trajectory; equation (1.4.8)). It should also be noted that the control parameters derived using the maximum principle are merely the solutions to the steady state matrix Riccati equation obtained from more conventional methods (Proposition 1.4.1). The fundamental property of both types of control rule (contingent and non-contingent) is that they satisfy the principle of optimality (Proposition 1.4.2); the optimal policy computed at time \(t_i\), \(i > 0\), is merely the continuation of the original optimal plan computed at time \(t_0\), which is to say that the optimal policy is time consistent. Thus, if we denote by \(w(t|t_i)\) the optimal policy computed at time \(t_i\) then time consistency implies \(w(t|t_i) = w(t|t_0) (i > 0)\). This result is obtained in standard dynamic programming solutions which involve recursive backward substitution: \(J(t_1)\) is minimized with respect to \(w(t_1)\) taking as given all previous decisions and states; henceforth, \(J(t_j) (0 \leq j < 1)\) are minimized taking as given all previous decisions and states as before and, in addition, subject to the constraint that \(J(t_1) (j < i < 1)\) are minimized in an analogous way (a sketch of this is given in section 1.4(A), equations (1.4.13) - (1.4.14)).
What is important to note at this stage is that the time consistency property arises in the single controller problem inevitably and in standard dynamic programming solutions by construction. Actions at any time are based on the knowledge that subsequent actions are optimal and consistent with the initial action. Even though bygones are bygones, there is no incentive to change plans because the planned behaviour in subsequent periods is already optimal. Further discussion is contained below. A dilemma arises, however, when the control problem takes on the features of a dynamic game; for under such circumstances there may well exist an incentive to depart from the initial planned strategy. It is to this which we now turn.

1.2(B) Time Inconsistency and Game Theory

The possibility of there existing an incentive to renege on announced plans yields the time inconsistency problem. More formally, an optimal policy computed at time \( t_0 \) is time inconsistent if the optimal policy obtained by re-optimizing at time \( t_i \), \( i > 0 \), is not the continuation of the original plan formulated at \( t_0 \). Hence, time inconsistency implies \( w(t|t_1) \neq w(t|t_0) \) (\( i > 0 \)). This leads to problems of credibility as the controller's commitment to an announced policy is undermined.

The recent surge of interest in this and related issues has arisen from the realised consequences of incorporating rational expectations into economic models. As mentioned previously, however, explicit treatment of a rational expectations framework is by no means a necessary nor sufficient condition for time inconsistency to prevail. Time inconsistency is a possibility in any game-theoretic framework where the game is between intelligent players and rational expectations is merely one way of modelling the game, in which case the system is non-causal. There are other, more established methods,
too and the systems for these are entirely causal (see, for example, Simaan and Cruz (1973); Cruz (1975); Kydland (1975, 1977); Holly (1983); Miller and Salmon (1983, 1984b); Turner (1983); Cohen and Michel (1984)). In spite of this, however, rational expectations is not entirely innocuous for the time inconsistency issue. In fact, though time inconsistency can arise in systems which do not formally incorporate rational expectations into the structure of the model, sophisticated forward-looking behaviour is implicit. We clarify this below.

Consider, then, a two person game over the horizon \( t_0 \rightarrow t_1 \) and let us distinguish each player by the subscripts 1 and 2. Each player chooses a trajectory for his own controls (a strategy) \( (w_k(t))_{t_0}^{t_1} \) \((k = 1, 2)\) and the payoff to each player will generally be a function of his own controls and the controls of his rival:

\[
J_k = J_k((w_1(t))_{t_0}^{t_1}, (w_2(t))_{t_0}^{t_1}) \quad (k = 1, 2) \quad (1.2.5)
\]

The strategy of each player determines his control vector at any time. A general formulation of these strategies is

\[
w_k(t) = w_k(x(t)) \quad (k = 1, 2) \quad (1.2.6)
\]

which encompasses both open loop and closed loop strategies for the dynamic game. An open loop game is defined in terms of moves which are essentially constant strategies, \( w_k = \bar{w}_k(x(t)) \) \((k = 1, 2)\) say. Then player 1 makes forecasts of player 2's current and future actions, \( (\bar{w}_2(x(t)))_{t_0}^{t_1} \), and formulates his best response, \( (\bar{w}_1(x(t)))_{t_0}^{t_1} \), and vice versa. Clearly, such strategies are independent of the state at all times. By contrast, a closed loop game is defined in terms of rules for each player which, at any time, are functions of the state
at that time $w_k = w_k(y(t))$ ($k = 1, 2$). Analogously to before, player $t_1$ forecasts player 2's actions, $(w_2(y(t)))_{t_0}^{t_1}$, and formulates his best response, $(w_1(y(t)))_{t_0}^{t_1}$, and vice-versa.

The simplest way of incorporating game theory into optimal control problems is to invoke the Nash assumption. By this, each player takes the strategy of the other as given. This may be refined further according to whether the strategies are open loop or closed loop. An open loop Nash game involves each player taking as given the policy paths of his rival whilst in a closed loop Nash game each player treats as parametric his rival's control rule. The important difference between these types of strategies is that, in the latter, each player announces a rule describing how he will respond to any new information that may accrue during the course of the game. Thus, in computing their response, each player recognizes that this will influence the response of his rival because each player's actions influence the state and the state determines the control via the closed loop assumption.

In the types of games that interest us - games between the private sector and the government - the symmetry in the Nash assumption may be implausible. Specifically, the private sector can be considered as comprising many atomistic agents and, in this respect, the private sector may be regarded as passive vis à vis the government. The latter, however, assumes the role of a dominant player, recognizing the influence of its own actions on private sector behaviour. This type of strategic environment is encapsulated in the Stackelberg assumption (8). In a Stackelberg game therefore, not all participants take the strategies of others as given but rather some players recognize that the strategies of their rivals are non-variant with respect to their own behaviour; hence the distinction between leaders and followers and the asymmetry which is absent in the Nash game. In
our two person game, suppose that player 1 is the dominant player and player 2 is the follower. Then player 1 acts first, announcing his strategy in the knowledge that player 2 will condition his behaviour on this announcement. As before, the game can be refined further by assuming either open loop behaviour (player 2 takes as given the policy path of player 1 who takes into account the effect of this on player 2’s actions) or closed loop behaviour (player 2 treats as parametric the decision rule of player 1 who takes into account the effect of this on player 2’s decision rule).

Having introduced some relevant game-theoretic concepts, let us now examine how they relate to the optimal control problem and the issue of time inconsistency. Following Miller and Salmon (1983, 1984b), we consider a two person non zero-sum differential game over the horizon \([t_0, t_1]\) (a useful introduction to some of the issues can be found in Turner (1983) for static games). Strategic behaviour can be introduced into the system in equations (1.2.3) – (1.2.4) straightforwardly by distinguishing between the instruments under the control of each player:

\[
\dot{y} = Ay + \sum_{k=1}^{2} B_k w_k \tag{2.6}
\]

\[
J = \int_{t_0}^{t_1} e^{-\rho t} (y^T Q y + \sum_{h=1}^{2} w_h^T R_h w_h) dt \tag{k=1,2} \tag{2.7}
\]

where \(w_k\) is an \(n_k\) vector of control variables \((k=1,2)\)

and the orders of \(B_k\) and \(R_h\) are \(n_k \times n_k\) and \(m_h \times m_h\) respectively \((k, h = 1, 2)\)\(^9\). Section 1.4(B) solves the control problem for open loop Nash and Stackelberg games\(^{10}\).

In the Nash game, as for the single controller problem, the optimal control rule can be expressed in either state dependent form (with time-invariant coefficients) or as a control path \(^{11}\) (equation...
(1.4.21)). But the similarity between the Nash and single controller solutions does not stop here: more importantly, the optimal policy in the Nash game remains time consistent (Proposition 1.4.3); 
\[ w^N(t|t_1) = w^N(t|t_0^1)(i > 0). \]
In these respects, therefore, a game approach to optimal economic policy formulation does nothing to qualify the standard approach. The principle of optimality applies to both.

Matters are fundamentally different for the Stackelberg game, however. First, the control rule cannot be expressed as a linear time-invariant state dependent rule. If one is seeking to implement a rule in terms of the state vector, then this must be in the form of either a linear time varying contingent rule (equation (1.4.31) or a form of integral control (equation (1.4.32)). More significant is that the optimal policy in the Stackelberg game is no longer time consistent (Proposition 1.4.4), 
\[ w^S(t|t_1) \neq w^S(t|t_0^1)(i > 0), \]
and does not, therefore, satisfy the principle of optimality.

Thus, we have seen that a game against nature yields an optimal policy, \( w(t|t_0^1) \), which is dynamically consistent and though this continues to be true for a Nash game, \( w^N(t|t_0^1) \), it is not true for a Stackelberg game, \( w^S(t|t_0^1) \). Let us now be more specific about the source of time inconsistency. The reason for the differences in the properties of the above solutions is to be found in the strategic structure of each type of game. In a dominant player game, the leader announces his strategy first, \( w^S_1(t|t_0^1) \), the follower takes this as given and the leader appreciates his influence on the follower – in particular, the leader knows he is able to influence the follower currently by announcing what he plans to do in the future. He does just that and the announced plan, \( w^S_1(t|t_0^1) \), is optimal. Yet after the influence on the follower has been realised, bygones are bygones and the dominant player can profit by departing
from the initial plan. This means that there is necessarily a
distinction between the ex ante optimal policy \( \{ w_i(t|t_0) \}_{t_0}^{t_1} \), and the
ex post optimal policy, \( \{ w_i(t|t_1) \}_{t_0}^{t_1} \) (\( i > 0 \)). ex ante the optimal
policy is the announced plan; ex post the optimal policy is to depart
from this. The net result is a time inconsistent optimal policy invol-
vying the combination of announcement and reneging. We reflect further
on this below.

In contrast to the above, a Nash game precludes time inconsist-
ency because it destroys the asymmetry inherent in the Stackleberg
game; though players still take account of the future strategy of
their rival, neither exploits this, both of whom believe they have no
influence on their rival. As Simaan and Cruz (1973) and Cruz (1975)
point out, the Nash assumption safeguards each player against
attempts by others to seek further gains which arise due to the
ability to dupe others into believing a particular strategy. The
fact that there exists no profitable opportunities from departing
from an original plan makes the distinction between ex ante and
ex post optimal policies redundant and the problem effectively
reverts back to a game against nature.

It is seen, therefore, that time inconsistency is an established
property in standard game theory where one player is dominant. As
we have already mentioned, however, the apparent abstraction of
rational expectations is misleading. Fischer (1980) has recently
stressed, and the foregoing discussion (together with the proof in
section 1.4(B)) should have made clear that time inconsistency
hinges on intelligent forward-looking behaviour - specifically, the
ability of future expectations to influence current decisions. Thus,
rational expectations (or some forward-looking behaviour at a minimum)
plays the crucial role, making the original causal system become
non-causal.
It is important at this juncture to clarify further the notion of time inconsistency as this has been subject to different interpretations in the literature and may have led to some confusion. A non-trivial distinction to be made is between an optimal policy which is time inconsistent and the time inconsistent optimal policy. The former refers to the ex ante policy, \((w_1(t_0|t_0))_{t_0}^{t_1}\), which, in the Stackelberg game, is time inconsistent. The latter is the actually realised optimal policy which recognizes the time inconsistency of the ex ante plan. It comprises the announcement, \((w_1^S(t|t_0))_{t_0}^{t_1}\), coupled with the subsequent departure therefrom and is therefore the full optimal policy. The corollary of this is that the time inconsistent optimal policy is an optimal policy which is time consistent - it is the plan, \((w_1(t|t_0^S), \ldots, w_1(t|t_1^S), \ldots, w_1^S(t|t_1))_{t_0}^{t_1}\), from which there is no incentive to depart since it is the truly optimal strategy and therefore satisfies the principle of optimality.

What does not satisfy this principle is the ex ante optimal policy, \((w_1(t|t_0))_{t_0}^{t_1}\), which is not optimal ex post. This may suggest no conflict between optimality and consistency and, to the extent that the time inconsistent optimal policy can actually be implemented, this is true (in fact, it is the source of the perfect cheating solution discussed in section 1.3(A)).

There is, however, a fundamental problem in implementing both the ex ante policy and the full optimal policy. In short, the equilibrium is unlikely to be sustainable. Since the full optimal policy involves announcing the ex ante policy and subsequently reneging, it requires the announced policy to carry credibility. But intelligent agents surely foresee the incentive for the controller to renege on the announced policy which therefore lacks credibility. This is important because, in anticipation of the change in policy, private sector behaviour will be different from that upon which the full optimal
and ex ante policies are predicated. It is precisely this incentive to renege and agents' perception thereof which render both the full optimal and ex ante policies weak candidates as viable strategies. Moreover, lack of credibility may well manifest in greater uncertainty and an outcome inferior to that which would occur if the controller does not succumb to the temptation to cheat (that is, continues with ex ante policy).

Now, an immediate response to the above is identified by Holly (1983): since the time inconsistent (full) optimal policy is the truly optimal policy, why does the controller not simply announce this policy in the first place? The answer is that such an announcement deprives this policy of its optimality property. As we have already noted, the optimal policy in period t₀ is, in fact, \( w_{S(t|t_0)}^{t_0} \), so that the time inconsistent plan, \( (w_{S(t|t_0)}^{t_0}, \ldots, w_{S(t|t_1)}^{t_1})_{t_0} \), must involve at least as high a loss as \( w_{S(t|t_0)}^{t_0} \). The intuition is that the optimality property of the time inconsistent plan arises because agents expect the ex ante policy but are deceived; once the time inconsistent policy is made public, there is no deception.

Clearly, the problem of time inconsistency appears to be acute. The successful implementation of the full optimal policy requires continuous deception and the successful implementation of the ex ante optimal policy requires it to carry credibility. The rational expectations hypothesis would immediately emphasize the weakness of these and it is now that we turn to the control problem in an explicit rational expectations framework. As will become evident, there is a direct analogy between these models and the standard game-theoretic framework studied in this section.
1.2(C) Time-Inconsistency and Rational Expectations

The majority of work on time inconsistency has emphasized rational expectations. The seminal contribution by Kydland and Prescott (1977) has been extended and improved upon by a number of authors and some useful illustrations of time inconsistency in rational expectations models can be found in Calvo (1978), Buiter (1984a), Fischer (1980), Kydland and Prescott (1980) and Driffill (1982) (see also Buiter (1980a); Holly (1983); Holly and Zarrop (1981, 1983); Minford and Peel (1983); Sheffrin (1983)). The recent surge of interest in the issue is a reflection of the pioneering work of Buiter (1980a, 1981a, b, 1983, 1984a), Miller and Salmon (1983, 1984a,b,c), Currie and Levine (1983a, 1984a,b,c, 1985) and Levine and Currie (1983, 1984).

To incorporate rational expectations explicitly into the optimal control problem, we retain the system in equations (1.2.3) - (1.2.4) and partition the nx1 state vector such that \( \mathbf{y} = [\mathbf{z}^T \mathbf{x}^T]^T \). Here, \( \mathbf{z} = \mathbf{z}(t) \) is an \( n_1 \times 1 \) vector of predetermined (backward-looking) variables and \( \mathbf{x} = \mathbf{x}(t) \) is an \( n_2 \times 1 \) vector of non-predetermined (forward-looking, free or jump) variables \( (n_1 + n_2 = n) \). The control problem is as before: choose the plan \( \{\mathbf{u}(t)\}_{t=0}^{T} \) which minimizes equation (1.2.4) subject to equation (1.2.3) and section 1.4.(C) solves this constrained optimization for both an open loop and a closed loop Stackelberg game. One analogy with the standard open loop Stackelberg game in section 1.2(B) is that a control rule in terms of the predetermined state variables, \( \mathbf{z} \), must be either a time-varying feedback rule (equation (1.4.39)) or a form of integral control (equation (1.4.40)). Note, however, that \( \mathbf{z} \) are not the only state variables in the rational expectations model, the complete set of state variables including also the free variables, \( \mathbf{x} \), and though it is not possible to derive a time-invariant feedback rule on \( \mathbf{z} \), a time-invariant feedback rule on the entire state vector, \( [\mathbf{z}^T \mathbf{x}^T]^T \), is
possible (equation (1.4.41)). Levine (1985), however, shows that the policy rule cannot be announced in this form (we return to this later). In any event, rational expectations complicates the design of control rules and this insight owes much to the series of papers by Currie and Levine (1983, 1984a,b,c) and Levine and Curie (1983, 1984).

A second analogy with the standard Stackelberg game is that the optimal policy is time inconsistent (Proposition 1.4.5). This is seen immediately in the rational expectations model which encompasses the jump variables, $x_1$; these are forward-looking variables and hence reflect the announced policy of the controller. With respect to the standard dynamic programming solution, the time consistent policy derived from such an approach is sub-optimal because it ignores the non-causalities in the system (that is, it fails to take account of the relevance of future expectations for current decisions — Proposition 1.4.6). Thus, the optimality of the time consistent policy is a special case and, in general, the restrictions required for it will be absent (13).

To summarize this section we have seen that, though the full optimal policy is time consistent, its optimality property hinges on the capacity to secure agents' credibility of the announced \textit{ex ante} optimal plan which is time inconsistent. The incentive to renege and the requirement of successful continual deception render both the \textit{ex ante} and full optimal policies weak candidates as viable strategies. Clearly, then, attempting to control the system in the presence of forward-looking behaviour introduces additional complexities which are absent in the physical sciences. As mentioned earlier, Kydland and Prescott (1977) and Prescott (1977) interpreted this to mean the fundamental inapplicability of optimal control techniques to economic problems, though the next section will
indicate why this view may be overly pessimistic and how the problem of time inconsistency may have been overstated. Simultaneously, however, we shall also indicate reasons to believe why the problem of time inconsistency is likely to be much more real than apparent.

1.3 The Resolution, Importance and Existence of the Time Inconsistency Problem

Section 1.2 contained a discussion of the fundamental time inconsistency of optimal policies in models which envisage the controller playing a game against a sophisticated private sector. This property of optimal policies has inspired the recent surge of interest in the search for time consistent policies which, though likely to be sub-optimal, are credible. This section critically reviews the proposed alternative approaches for resolving time inconsistency and achieving time consistent equilibria. These take for granted that time inconsistency actually constitutes a problem though it is of some importance to clarify matters on this. In addition, we also address other, possible more substantive, issues which are rarely confronted in the existing literature.

1.3(A) Perfect Cheating and the Time Consistency of the Time Inconsistent Optimal Policy

Miller and Salmon (1983) and Currie and Levine (1985) show that a time consistent equilibrium exists if the controller is able to successfully and repeatedly delude the private sector (section 1.4(D) shows how this can be formalised for both open loop and closed loop games). This may appear rather peculiar since how is it possible for a policy which involves continuous reneging to be an optimal policy which is time consistent? In fact, the answer is quite simple when we recall our earlier observation that the time inconsistent (full)
optimal policy is an optimal policy which is time consistent. The combination of announcement and reneging in the perfect cheating solution is precisely the full optimal policy made operational and not only is this time consistent but the controller is also achieving his bliss point. Obviously, however, the perfect cheating solution is subject to the same criticism as the full optimal policy, requiring an implausibly gullible private sector, and this makes it unlikely that the equilibrium will be sustainable.

1.3(B) Precommitment and Rules versus Discretion

One obvious candidate as a way of enforcing the ex ante optimal policy, $\{w(t|t_0)\}_{t_0}^{t_1}$, is for the controller to bind his future actions such that this policy is actually pursued—for then this policy carries conviction. An interesting type of precommitment equilibria has been identified for a specific model by Lucas and Stokey (1983) who consider the problem of dynamic optimal taxation along the lines of Kydland and Prescott (1977, 1980) and Fischer (1980), but without capital. Time inconsistency arises in Lucas and Stokey (1983) because the government is able to influence the value of outstanding debt by manipulating the price level which makes the purchase of debt undesirable. In the absence of the ability to bind itself to the ex ante optimal tax policy, Lucas and Stokey (1983) consider the case in which all debt obligations are honoured and it is shown that this makes the ex ante tax policy enforceable (that is, credible and time consistent). An interesting implication of this is that for two sets of instruments under the direction of the controller, $w^1$ and $w^2$, which are related in some way, even if precommitment to the plan $\{w^1(t|t_0)\}_{t_0}^{t_1}$ is not feasible, a commitment to a particular plan $\{w^2(t|t_0)\}_{t_0}^{t_1}$ may ensure that $\{w^1(t|t_0)\}_{t_0}^{t_1}$ is enforced. Nonetheless, as Lucas and Stokey (1983) observe, there is nothing a priori to
indicate why some commitments are relatively easier than others and their particular analysis still rests on the crucial assumption that all debt obligations are honoured (see Fischer (1983) for further discussion). More generally, however, the essential problem with the precommitment solution seems to be the lack of enforceable contracts which renders any proposed precommitment suspect.

On a slightly different note, it is useful here to clarify the relationship between time inconsistency and the 'rules versus discretion' debate. The existence of time inconsistency has prompted the argument for precommitment to fixed rules which has been interpreted as an argument for non-contingent rules. Motivated by this, Buiter (1980a, 1981a,b) demonstrates, however, that non-causal models still imply the superiority of an 'innovation-contingent' policy over a non-contingent policy, with both policies being time inconsistent (see also Driffill (1981); Begg (1982b); Miller and Salmon (1983)). The logic of the argument is unquestionable but it is an error to infer support for non-contingent rules from time inconsistency in the above way in the first place. Obviously, some type of rule is desirable for economic policy but the desire to restrict future discretion (in order to enforce announcement) is a quite separate issue from the relative merits of contingent and non-contingent policies since both types of policies can suffer from time inconsistency (see, for example; Driffill (1981); Sheffrin (1983); Holly (1984)). To argue that time inconsistency lends support for rules rather than discretion is to associate rules with binding commitments and discretion with the absence thereof, and this is clearly distinct from the interpretation of rules as state independent policies and discretion as state dependent policies. As it stands, time inconsistency has nothing to say about the relative merits of these.
We hasten to point out, however, that there may be other circumstances under which time inconsistency does have some bearing on the issue. One possibility is in the case of uncertainty or, more specifically, when the controller and private sector have access to heterogeneous information. This is taken up in section 1.3(E) and we do not repeat that discussion here. A second possibility is that time inconsistency may be seen as an argument for ad hoc non-contingent rules which are not derived from an optimization process. When such policies are adopted there is no particular reason to suspect that the policy maker will renege since the optimization process which indicates the incentive to renege has been eschewed. Note, however, that this may work the other way: since the policy maker has chosen a policy arbitrarily, there is nothing to stop him from choosing a different policy in an equally arbitrary way. The outcome is unclear and the existing literature has nothing to contribute on the matter.

1.3(C) Nash Equilibria and Dynamic Programming

It will be recalled from section 1.2(B) that time consistency characterises standard dynamic games when strategic behaviour is regulated by the Nash assumption. Naturally, this has motivated investigations into whether time consistency is attainable in rational expectations models when the controller and private sector act as Nash players. Unfortunately, it is here that the terminology in the literature has a tendency to degenerate into mild chaos. Different solution procedures employed by different authors make it difficult enough to make comparisons and this is hindered even more by the lack of a unified terminology. Not only is it that many of the solutions are more-or-less identical, but that this has been noticeably overlooked.
Buiter (1983) and Cohen and Michel (1984) (see also Cohen and Michel (1985)) purport to discover time consistent solutions which are formalised in section 1.4(E). In both cases, the control rule can be formulated as a linear time-invariant feedback rule on the predetermined state variables, $z$, or as a control path (equations (1.4.54) and (1.4.75)). It turns out, however, that both of these solutions could have been anticipated from the conclusions of standards game theory. With respect to the Buiter (1983) solution, Miller and Salmon (1983) identified the equivalence between this and the open loop Nash solution in which the controller takes as given the level of the free state variables, $x^{(16)}$. Similarly, Currie and Levine (1985) have recently identified the Cohen and Michel (1984) solution — which is obtained using dynamic programming — with a type of closed loop Nash game in which the controller takes as given the reaction function of the private sector (Proposition 1.4.7). In fact, the closed loop Nash terminology used by Currie and Levine (1985) is slightly misleading since although the controller is behaving closed loop (treats as parametric the private sector's decision rule), the private sector is behaving open loop (takes as given the control vector of the controller). Miller and Salmon (1984b) make the same observation (as does d'Autume (1984) to which we return below) and to emphasize the point, let us denote the Cohen and Michel (1984) solution as quasi-closed loop Nash$^{(17)}$. In addition, Cohen and Michel (1984) and Miller and Salmon (1984b) denominate the dynamic programming solution interchangeably by open loop Stackelberg and recursive Stackelberg equilibria. Though these have some merit, both are somewhat of a misnomer for, as we have stated above (and as Miller and Salmon (1984b) appreciate), the controller is behaving closed loop. Similarly, the Stackelberg concept is only a half-truth though we suspect this to be rather more misleading. The leadership role is
only realised through the controller's perception of his influence on the state variables which feed through to the decision variables of the private sector via their reaction functions and this is quite distinct from the usual notion of a dominant player game. Moreover, both Cohen and Michel (1984) and Miller and Salmon (1984) explicitly state that the controller treats as parametric the private sector response function, an assumption which is closer to Nash than Stackelberg behaviour. Thus, the only asymmetry is the fact that the controller behaves closed loop and the private sector behaves open loop and we regard it as slightly misleading to designate the Stackelberg terminology purely on the basis of this asymmetry. Our view seems to have some support from Miller and Salmon (1984) themselves who observe that if the controller takes as given the level of the free state variables, $x$, (as opposed to the functional relationship between $x$ and $z$ in the reaction function), the solution reduces to the open loop Nash game. Intuitively this is appealing for it states that the open loop Nash solution is nested in the (quasi-) closed loop Nash solution in an obvious way (Proposition 1.4.8).

Let us now turn to some other matters. Typically, the question of time consistency propagates the search for appropriate equilibrium concepts, specifically Nash equilibria or, more precisely, perfect equilibria. The notion of perfection of equilibrium will be given a precise meaning below. For the moment we merely note that by an equilibrium we mean a situation such that given the behaviour of rivals, no single player can gain by departing from his own planned behaviour. Returning to the two person game of section 1.2(B), a Nash equilibrium is the couple $((w^*_1(t))_{t=0}^T, (w^*_2(t))_{t=0}^T)$ such that
Intuitively, a Nash equilibrium is a strategy for player 1, \((w^1(t))\) which is taken as given by player 2 and which is actually optimal for player 1, \((w^1_1(t))\), and vice versa. It is usual in the literature to refine the equilibrium concept further by introducing the notion of subgame perfection (see Kreps and Wilson (1982a); d'Autume (1984)): the strategy sequence \(((w^1(t))\), \((w^2(t))\)) is said to be a perfect Nash equilibrium if, at any juncture \(t_j\), \(0 < j < 1\), \(((w^1(t))\), \((w^2(t))\)) is a Nash equilibrium for the subgame in \(t_j\) given the history of the game in \(t_0, t_j\). Thus, consider a subgame induced by re-starting the entire game at some point. Then perfectness of equilibria guarantees an equilibrium of the subgame for any starting point, which is to say that the strategy sequence \(((w^1(t))\), \((w^2(t))\)) is subgame perfect if it is a Nash equilibrium for any subgame (Kreps and Wilson (1982a)).

Subgame perfection is related to the issue of time consistency in a straightforward way. As Cohen and Michel (1984) point out, their dynamic programming solution is, in fact, a perfect equilibrium. To see this intuitively, one merely has to realise the recursive structure of both of the solutions. A perfect equilibrium is time consistent because, for each subgame at any juncture of the entire game, the optimal strategy for the subgame is the strategy that was originally found to be optimal for the entire game. Now, d'Autume (1984) defines the dynamic programming time consistent solution as the outcome of the private sector behaving open loop and the controller behaving closed loop. The reader will immediately appreciate
the similarity between this and our quasi-closed loop Nash interpretation of Cohen and Michel (1984). In short, the time consistent equilibrium is merely a variant of the concept of perfect equilibria. We may also note a further (time consistent) equilibrium concept cited in the literature, namely a feedback Stackelberg solution (see Kydland (1975, 1977)). In order to avoid confusion with the closed loop Stackelberg solution (which is time inconsistent and therefore a weak candidate as an equilibrium concept) let us follow Cruz (1975) and substitute for feedback Stackelberg the title of Stackelberg equilibrium. Now, as would be expected, the time inconsistent open loop and closed loop Stackelberg solutions are superior to the time consistent Stackelberg equilibrium and Kydland (1977) reveals this to be due to the fact that the latter ignores the non-causalities in the system. Thus, again we appear to be back to the closed loop Nash game and, to support this, d'Autume (1984) observes the equivalence between Stackelberg equilibrium and perfect Nash equilibrium (21).

The foregoing discussion should, if nothing else, have highlighted (and hopefully in part resolved) the acute terminological problems in the literature. A closer inspection reveals that, for the most part, the conclusions reached by different authors amount to much the same. In any event, if all this amounts to demonstrating the time consistency of a backward recursive solution, we are a little sceptical about the importance of its contribution. The reason for our view is as follows.

The solutions discussed in this section may appear a little contrived. Invoking the Nash assumption negates time inconsistency but does so by changing the framework – it does not resolve time inconsistency which arises precisely because of the absence of Nash behaviour. The Nash assumption can be justified, however, by noting that if future discretion is unrestricted, agents will foresee the
incentive for the controller to renege which makes his announcements superfluous. He is then forced to take agents' actions as given and the outcome is a time consistent Nash equilibrium\(^{(22)}\). Though superficially appealing, this argument is very much incomplete and really avoids the main issue. In short, though the \textit{ex ante} optimal policy suffers from dynamic inconsistency, the controller will always do better by adhering to this policy than by pursuing the Nash policy. All of the solutions discussed so far have assumed that the leader ignores his previous actions because the follower does so. This avoids the case of \textit{repeated games} and the possibility of an equilibrium where it pays the controller to maintain previous policies in order to sustain a reputation. The most pressing problem confronting the controller is to convince agents of his commitment to the \textit{ex ante} optimal plan and what is required is a framework which indicates conditions under which the private sector begins to lose faith in the controller. The framework which immediately suggests itself is one which models \textit{reputation} endogenously.

1.3(D) Reputation and Credibility.

Some important recent work has shown that time inconsistency may be significantly alleviated if one endogenises this within a framework which entertains the notion of \textit{reputation}. It is precisely the threat of the loss of reputation (credibility) and reversion to an inferior (Nash) policy which may motivate the controller not to succumb to the temptation to renege on the announced \textit{ex ante} optimal plan. This plausible hypothesis enriches the game in an interesting way: players are now engaged in a \textit{repeated game} or \textit{supergame} (see, for example, Friedman (1971)) and the temptation to renege must be judged against the punishment for doing so, in which case it may pay the policy maker to invest in a reputation. As we shall see, in such games, the perfection of equilibrium becomes important.
An illuminating example of reputation building, and its implications for time inconsistency, is presented by Barro and Gordon (1983) and their analysis has been extended in important directions by Backus and Driffill (1984a,b, 1985) and Canzoneri (1985) (see also Driffill (1983, 1984); Blinder (1982); Loewy (1983); Anderson (1985)) to which we return later. For the moment, we convey the flavour of the original study.

Barro and Gordon (1983) assume a loss function for the controller of the form \( J_{1t} = q_1 \pi_t^2 + q_2 (\pi_t - \pi_t^e) (q_1 > 0, i = 1, 2) \) with \( \pi \) the actual rate of inflation and \( \pi^e \) the private sector's expected rate of inflation. Following d'Autume (1984) and Backus and Driffill (1984a,b, 1985), it is also useful to conceptualise the loss function for the private sector as \( J_{2t} = (\pi_t - \pi_t^e)^2 \). The choice (control) variable of the policy maker is \( \pi \) and the choice variable of the private sector is \( \pi^e \), the control problem being to minimize a discounted sum of \( J_{1t} \), \( J_{2t} \), with \( 0 < \lambda < 1 \) a discount factor. Time inconsistency arises in this model because by inducing a particular \( \pi_t^e \), the policy maker is able to exploit the gains from inflation. As we shall see, this means that an announced policy of \( \pi_t = 0 \) is simply not credible and there must be some positive inflation. An important limitation of Barro and Gordon (1983) (to which we return later) is that current effects on reputation do not influence future reputation. The advantage of this is that the optimization problem may be conducted for each period in isolation.

Barro and Gordon (1983) distinguish between three types of policies. The first is what they term a discretionary policy which obtains when agents perceive of the incentive for the controller to renge. It is, therefore, the outcome under the Nash assumption, being time consistent with \( \pi_t^e = \pi^N_t > 0 \) and \( J_{1t}^N > 0 \). The second policy is termed a policy rule and corresponds to the ex ante optimal
The third policy is called the cheating policy and is the time inconsistent (full) optimal policy actually implemented by virtue of the ability to perfectly cheat the private sector into believing the commitment to the ex ante policy. Hence, $\pi_t^e = 0$ but $\pi_t = \pi = \pi_t^N$ and $J_{1t} = - J_{1t}^N < 0$. These results accord with our previous intuition — in particular, $J_{1t}^N > J_{1t} > J_{1t}^*$. Obviously, however, since the cheating solution is non-feasible, neither is the ex ante policy and to see this formally, Barro and Gordon (1983) introduce the following enforcement rules: if private sector expectations are correct in one period ($\pi_{t-1} = \pi_{t-1}^e$), government credibility is enhanced and the private sector believes the commitment to the ex ante (zero inflation) policy next period ($\pi_t = \pi_t^*$); by contrast, forecast errors in one period ($\pi_{t-1} \neq \pi_{t-1}^e$) make agents distrust government announcements, reputation is lost and the Nash policy is enforced ($\pi_t = \pi_t^N$). Thus, the gains from reneging must be judged against the costs of losing reputation and these benefits and costs are quantifiable in a simple way. The temptation to cheat, $\eta_T$, say, is merely the difference in utility between adhering to the $\pi_t^*$ policy and cheating on this, $\eta_T = J_{1t}^* - J_{1t}$. The cost of cheating, $\eta_C$, say, is that the policy maker is forced to pursue the $\pi_t^N$ policy in the next period; since this is discounted at the rate $\lambda$, $\eta_C = \lambda(J_{1t}^N - J_{1t}^*).$

Now, the controller will adhere to the announced $\pi_t^*$ policy provided $\eta_C > \eta_T$ — otherwise he secures a net gain from cheating — and this provides the equilibrium condition. But if $0 < \lambda < 1$, $\eta_C < \eta_T$ and the equilibrium is unsustainable. The commonsense reason for this is that the costs of reneging come later and are therefore discounted; since agents know this, the $\pi_t^*$ policy is non-feasible and there must be positive inflation in the equilibrium. Note, however, that $\lambda = 1$ goes some way towards making $\pi_t^*$ credible. This is
intuitively appealing and is closely related to a recent finding by Currie and Levine (1985) in connection with stochastic systems (see section 1.3(E)). It occurs because the policy maker knows that by reneging he is forced to pursue the Nash policy in the future, but if the future is deemed important he will be more inclined to avoid reneging. The private sector knows this so that greater credibility is attached to the ex ante policy. The fact that $\pi_t^* = 0$ is generally unsustainable, however, means $\pi_t^* = \pi_t > 0$ in equilibrium but reputational considerations are still sufficient to avoid the $\pi_t^N$ policy.

Thus, the most important insight of Barro and Gordon (1983) is that a superior policy to the Nash strategy is enforceable when the policy maker is concerned with reputation. This implies that a resolution of time inconsistency exists which is already endogenous to the problem. As mentioned above, however, the most limiting aspect of Barro and Gordon (1983) is that reputation effects last for only one period which means that the only subgame perfect equilibrium is one which entails positive inflation. This is so because agents know the dominant strategy of the controller to involve inflating, but one might envisage the situation in which adherence to the zero inflation policy for some time builds a reputation for the controller for being tough.

In a series of papers, Backus and Driffill (1984a,b, 1985) incorporate asymmetric information into the Barro and Gordon (1983) framework by assuming that at least one player is unsure about the payoff matrix of his rival (see Kreps and Wilson (1982b)). In the case of one-sided uncertainty, this amounts to the private sector being uncertain of the relative magnitudes of the $q_i$ ($i = 1,2$) parameters in the controller's loss function so that it must learn about the type of controller it is playing against. Restricting the menu of choices to $\pi, \pi^e = (0,1)$, a controller that always inflates (always
plays $\pi = 1$) is termed a weak government (associated with relatively high (low) values of $q_2(q_1)$), and a controller that never inflates (always plays $\pi = 0$) is termed a hard government (associated with relatively low (high) values of $q_2(q_1)$). If the government's preferences are public knowledge, the two alternative equilibria are $\pi = \pi^e = 1$ or $\pi = \pi^e = 0$. Thus, an announced zero inflation policy may be credible if the government is believed to be hard; then a weak government may resist the temptation to inflate in order to build a reputation for being tough\(^{(25)}\).

The game involves the private sector assigning a probability, $P_t$, each period that the controller is hard (this measures the reputation for being tough) and both players choosing their best strategies given the strategies of others and the effect of current behaviour on future reputation. During the game, agents have perfect recall and the history of the game up to any stage conveys information about the type of policy maker which is processed and the probability $P_t$ is revised accordingly\(^{(26)}\). The equilibrium is subgame perfect (and therefore dynamically consistent) and some characteristics of the solution are as follows: for some periods after the start of the game the policy maker avoids inflating so that $\pi_t = \pi^e_t = 0$ with certainty and reputation is enhanced (in terms of output and inflation, there is full employment and zero inflation). After this, the controller still plays $\pi_t = 0$ but the private sector expects positive and zero inflation with equal probability (there is equal likelihood of full employment and unemployment with zero actual inflation). This occurs because a weak government is more inclined to inflate towards the end of the horizon. At some point, the controller inflates after which he loses his reputation and $\pi_t = \pi^e_t = 1$ (there is full employment and positive inflation). Thus, contrary to Barro and Gordon (1983), there is a period during which a zero inflation policy is
credible because it pays even a weak government to build a reputation for itself. The analysis has been generalised by Backus and Driffl (1984a, 1985) to incorporate two-sided uncertainty. The game is now a game of 'chicken' with each player testing the nerve of his rival. The theory predicts that a strong private sector (playing \( \pi^e = 1 \)) may induce tough action by a government concerned to reduce inflation in order to convince agents of its commitment and alleviate the costs of disinflation. On the other hand, a strong government may be forced to 'give in' and abandon its anti-inflationary stance if the output costs are too high. Hence, even a strong government may suffer output losses if its initial reputation is bad.

One might also mention here a recent approach by Anderson (1985) who envisages the problem of asymmetric information as a problem of adverse selection and the design of incentive compatibility schemes to enforce the true revelation of preferences. Analogous to the insurance literature, one can then imagine separating and pooling equilibria. In the case of a credibility problem (that is, when government announcements about its preferences cannot be relied upon) there is a pooling equilibrium with both hard and weak governments being offered the same private sector expectation. In the case of no credibility problem (that is, when the government can be relied upon to reveal its true preferences) there is a separating equilibrium with expectations being geared to the particular type of government. Though at first glance this appears to be an interesting approach to the credibility problem, we believe it to suffer from a fundamental drawback for, as we have already remarked for the Barro and Gordon (1983) model (which Anderson (1985) uses), if the controller is known to be weak the equilibrium is the inferior Nash equilibrium with \( \pi = \pi^e = 1 \). It is not at all obvious, therefore, that a truthful revelation of preferences is necessarily a Pareto improvement. And
one can go further than this. Barro (1976) has argued that, instead of exploiting any informational advantage, an equivalent solution would be for the controller to make his information set public; but the reputation framework seems to offer a plausible reason why an informational asymmetry will persist.

On a slightly more general note, it is important to note, here, the crucial role of perfection of equilibrium in the types of models considered in this section because the solution depends on the threat being credible; the policy maker must regard the loss of public confidence as a credible response of the private sector - and the threat is credible provided it is the optimal ex post response. There may be some problems here: first, such a punishment strategy may not, in fact, be credible because it may not be optimal if it involves a cost; second, as Friedman (1971) points out, the notion of a threat appears vacuous in non-cooperative games because the coalitions required to enforce the threat are ruled out by assumption. These issues can be dealt with fairly briefly. Kreps and Wilson (1982b) observe that the threat need not actually be optimal ex post; rather, all that is required is some probability that it is optimal ex post. In addition, coalitions to enforce the threat need not be required either; the policy maker merely has to appreciate a loss of public confidence if it reneges and, following Friedman (1971), one may replace the notion of threat by temptation. Then the policy maker may be tempted to renege but, in doing so, he forces his rivals into a safe position which, in the present context, amounts to avoiding the risk of believing the policy maker's announcements. The threat still exists, but is devoid of any connotations regarding collaboration. Finally, there is no cost of punishment in the above types of models since all that happens is a loss of confidence.
In conclusion, it is our view that the appreciation of reputational considerations constitutes a fundamentally important contribution to the time inconsistency literature and it is precisely within this framework that the issue should be addressed. There does seem however, some grounds for pessimism. First, the theory is devoid of any quantitative content. For example, Backus and Driffill (1984a, b, 1985) cite a number of applications of their analysis, especially as a descriptive device for recent UK experience. Thus, one argument is that the present government's commitment to an anti-inflationary policy was met with private sector scepticism which continued to expect higher inflation; a recession was the outcome. This has some merit but it does appear to suffer from a major drawback: put simply, it seems to have taken (or is taking) a surprisingly long time for the conviction of the government to be reflected in private sector behaviour. The players seem to have been playing the game for an implausible length of time and surely one of them would have tired by now. Second, perhaps the major drawback of the analysis is its operational significance. Even for the very simple models that have been outlined, the solution becomes extremely tedious and problems of generality are likely to limit the applicability of the theory to the actual policy making process for some time.

1.3(E) Stochastic Systems and Simple Rules

Thus far, we have largely avoided comment on the more realistic case in which the system is subject to stochastic perturbation. This may not be too serious since much of what has been said for the deterministic case applies equally to the stochastic case. There are, however, some points of interest which are only meaningful in the latter and some of these may qualify some of the conclusions reached for a deterministic environment. Thus, we begin by writing
the stochastic analogues to equations (1.2.3)–(1.2.4) as

\[ \begin{align*}
\dot{y} &= Aydt + Bwdt + du \\
E(J) &= \int_{t_0}^{t_1} e^{-\rho t} (y^T Qy + w^T Rw) dt
\end{align*} \] (1.3.2)

where \( du \) is an \( nx1 \) vector of stochastic disturbances, \( E(\cdot) \) is the expectations operator and \( dy \) is now \( dy = [dx^T (dx^e)^T]^T \) with \( dx^e = x^e(t+dt,t) - x(t) \). Note also that \( du = [(du^1)^T (du^2)^T]^T \) with \( du^1 \) an \( n_1 x 1 \) vector and \( du^2 \) an \( n_2 x 1 \) vector (28). Defining \( y^+ = y - y^* \) and \( w^+ = w - w^* \), Levine and Currie (1984) (see also Currie and Levine (1985)) show that

\[ \begin{align*}
\dot{y}^* &= Ay^* dt + Bw^* dt \\
\dot{y}^+ &= Ay^+ dt + Bw^+ dt + du \\
E(J) &= J^* + E(J^+)
\end{align*} \] (1.3.4–1.3.6)

with \( J^* \) and \( J^+ \) defined by

\[ J^* = \int_{t_0}^{t_1} e^{-\rho t} ((y^*)^T Qy^* + (w^*)^T Rw^*) dt \] (1.3.7)

\[ J^+ = \int_{t_0}^{t_1} e^{-\rho t} ((y^+)^T Qy^+ + (w^+)^T Rw^+) dt \] (1.3.8)

The control problem can then be performed in two parts: the first is to choose \( \{w^*(t)\}_{t_0}^{t_1} \) which minimizes \( J^* \) subject to equation (1.3.4) and is the deterministic control problem; the second is to choose \( \{w^+(t)\}_{t_0}^{t_1} \) which minimizes \( J^+ \) subject to equation (1.3.5) and is the stochastic control problem. The important property of the linear
quadratic framework is that it exhibits certainty equivalence for both the standard control problem and the rational expectations version (see Levine and Currie (1984)). Hence, \( w^* = w^+ = w \).

An important insight of Currie and Levine (1985), alluded to earlier, is the critical influence of the discount parameter, \( \rho \), on the time inconsistency property of the optimal ex ante policy in a Stackelberg game. Specifically, the sustainability of this policy becomes more likely as \( \rho \) falls (Proposition 1.4.10) and the intuition behind this is precisely because the stochastic system mimics a repeated game. Since reneging means a reversion to the inferior Nash policy and a weaker capacity to stabilize future shocks, the controller is more inclined to avoid the temptation to cheat as he gives higher weight to the future (and the private sector is aware of this). Yet the stochastic case should not merely be seen as a repetition of the reputation framework in section 1.3(D). On the contrary, it seems to offer rather more than this. Specifically, it appears to offer an attractive way to investigate reputational issues further which avoids the technical difficulties associated with the Kreps and Wilson (1982b) and Backus and Driffill (1984a,b, 1985) approach. In any event, the problem of time inconsistency appears to be overstated. There are good reasons to believe, however, why a stochastic environment may exacerbate the problem and these have received relatively little attention in the literature.

To begin with, recall from section 1.3(B) that, in a stochastic environment, the optimal policy permits a response to new information and is time inconsistent (see, for example, Buiter (1980a, 1981a,b); Begg (1982b); Miller and Salmon (1983)). Now suppose that the private sector is (at least temporarily) ignorant of contemporaneous stochastic perturbations. Then there is an immediate dilemma: when a shock occurs, the optimal response is to stabilize the disturbance
but the private sector, being unaware of this disturbance, may confuse the observed policy change with an attempt by the controller to pursue a cheating strategy. This will remain true even if the private sector knows the controller to be pursuing a contingent policy and, moreover, even if the controller publishes information about the extraneous event since the private sector cannot be sure of the truth of this information\(^{(29)}\). A specific example of this is given by Canzoneri (1984) who considers the case in which contemporaneous information about monetary shocks is available only to the controller. An ideal (time inconsistent) policy is to accommodate these shocks but the private sector's observation of the controller's adherence to the ideal policy is not verification of his honesty because he can exploit the ignorance of the private sector by announcing spurious information about the stochastic innovation. By doing this, he is able to make the cheating strategy observationally equivalent to the ideal policy.

What all this implies is the additional complexity on moving from a deterministic to a stochastic setting. In general, any observed policy change must be decomposed into that change arising from the optimal response to unforeseen contingencies and that change which reflects the dynamic inconsistency of the optimal policy. The outcome of this filtering problem is unclear: at one extreme, it may favour the policy maker since private sector uncertainty can be exploited; at the other extreme, however, greater confusion might induce agents to adopt a safe position and distrust any government announcements. In any event, consideration of purely deterministic systems can obscure some very real problems which have been rarely addressed in the literature and this has probably made the resolution of time inconsistency appear overly simple.
Let us now turn to a slightly different issue. As we have noted, the full optimal time inconsistent policy satisfies certainty equivalence. This is not true, however, of almost all policies and Levine and Currie (1984) have investigated this issue in some detail. They consider simple rules which are specified at the outset to be a linear time-variant feedback rule on the state vector, $\mathbf{y}$, so that $w = D_1 \mathbf{y}$, say. Then the control problem is to minimize equation (1.3.3) with respect to $D_1$. Equivalently, it can easily be shown that

$$\mathbf{d}z = Tz \mathbf{d}t + \mathbf{d}u^L$$ (see section 1.4(F)) in which case the $n_1 \times n_1$ asymptotic variance-covariance matrix of $\mathbf{z}$, $\mathbf{E}_z$, follows as

$$\mathbf{E}_z \mathbf{T}^T + \mathbf{T} \mathbf{E}_z + \mathbf{E} = 0$$ (1.3.9)

where $\mathbf{E}$ is the $n_1 \times n_1$ asymptotic variance-covariance matrix of $\mathbf{d}u^L$ (see Chow (1979)). The control problem is then to minimize $\mathbf{E}_z$ with respect to $D_1$ and section 1.4(F) shows how strategic behaviour can be incorporated into this problem (30). What we wish to note here is that the simple rule, $w = D_1 \mathbf{y}$, does not satisfy certainty equivalence which is immediately realised from equation (1.3.9) where the optimal value for $D_1$ is seen to generally depend on the stochastic properties of the system, $\mathbf{E}$. This might incline one to avoid deploying such rules as their performance is geared critically to the nature of stochastic perturbations and contemporaneous information about these disturbances is likely to be incomplete. There are, however, some strong arguments in favour of simple rules. First, it is quite possible to identify simple rules which perform reasonably well across a wide spectrum of shocks (this has occupied the attention of Levine and Currie (1983, 1984); Currie and Levine (1983a, 1984a,b,c); chapters 2 and 3 in this thesis). Second, the full optimal rule is likely to be very complex; simple rules are easier to understand which might be especially relevant when agents' behaviour rests...
partly on their understanding of policy. Third, the superiority of complex rules may well be due to their capacity to exploit the dynamic configuration of the system, but it is precisely this which the controller is least certain about. A structural shift may then reduce the attractiveness of complex rules and induce a search for more simple policies which are more robust. Fourth, in terms of the main concern of this paper, simple rules have a particularly attractive advantage in that they are likely to be easier to monitor which makes it easier to identify a policy maker who attempts to cheat. In any event, the stabilizing properties of complex and simple rules is just one aspect of the debate and the existing literature has little to say about the other, possibly more important, issues.

1.3(F) Final Remarks: Extensions, Informational Assumptions and the Existence of a Problem

The foregoing sections have elicited some recent contributions to the time inconsistency issue and its resolution. In the course of the discussion we have mentioned other points of interest which have emerged from the debate and, in this section, we comment on some further matters which have received at most relatively little attention. These issues are, however, of some substance and strike at the core of some of the underlying assumptions one often feels to be present but rarely revealed. To begin with, let us remark on some issues which have been touched upon in the literature.

In general, both Nash and Stackelberg solutions are inefficient because they are derived from non-cooperative behaviour (see, for example, Buiter (1980a); Miller and Salmon (1983)). Cooperation between players may also benefit the enforcement of a (otherwise time inconsistent) policy and there may, therefore, exist strong incentives to engage in cooperation. Note, however, that in the type
of games that we have considered, cooperation would imply the ability of agents to form coalitions - an assumption which is ruled out in the dominant player game. Indeed, collusive behaviour could change the nature of the problem dramatically as the game might degenerate into Stackelberg warfare. Now, for the most part, the issue of cooperation has been addressed in the context of seeking internationally coordinated policies between different nations (see, for example, Brandsma and Hughes-Hallett (1984); Hughes-Hallett (1984a, b); Miller and Salmon (1983, 1984a, c); Oudiz and Sachs (1984a, b); Sachs (1983); Eichengreen (1984); Turner (1984a, b); Laskar (1984); Currie and Levine (1985a, b); Levine and Currie (1985)) and this raises other matters. A general problem with cooperation is that the threat of punishment for breaking an agreement must carry credibility, but it is questionable that such is the case here. Currie (1985a), for example, has suggested that simple tit-for-tat strategies might suffice but this is to ignore the fact that countries differ in size and importance, and it is quite possible for some countries to be able to secure gains from cheating on an agreement and incur no penalty simply because either other countries cannot devise a tough enough punishment strategy or they may fear retaliation on a much larger scale. Moreover, even if policy makers were able to design a credible cooperative strategy, the conglomerate controller is still playing a game against the private sectors and the optimal policy will still be time inconsistent (see, for example, Miller and Salmon (1983)). In this respect, cooperation does nothing to alleviate the problem and might actually exacerbate it if collusion among policy makers makes it easier for them to cheat the private sector.

Turning to a slightly different matter, an alternative interpretation of the game is given by Oudiz and Sachs (1984b) who consider a supergame between successive governments involving the following
strategies: first, a government at time $t_0$ chooses the optimal ex ante policy, $(w(t|t_0))_{t_0}^{t_1}$ with pre-commitment provided all governments in the past have done so; second, if any government in the past selects a policy other than this, the government at time $t_0$ chooses a Nash policy, $(w^N(t|t_0))_{t_0}^{t_1}$. Thus, each government operates under the threat that its successors will revert to the Nash policy if it fails to adhere to the ex ante optimal policy. Provided this threat is credible, there are two alternative subgame perfect equilibria: an equilibrium in which all governments play the ex ante policy; and an equilibrium in which all governments play the Nash policy. Though the ex ante policy might be unsustainable if, at some time the temptation to depart from it outweighs the cost of doing so, the essential point remains, namely that in a model with reputation effects, a superior outcome to the Nash solution is attainable.

Let us now turn to some rather more substantive matters. It is commonly alleged in the literature, as if to demonstrate the force and generality of the proposition, that dynamic inconsistency arises even if the private sector and controller share identical preferences (see, for example, Rydland and Prescott (1977, 1980); Calvo (1978); Fischer (1980); Barro and Gordon (1983); Backus and Driffill (1984c)). Under such circumstances, however, it is difficult to envisage time inconsistency as posing any problem. The controller is surely suspected of reneging and in doing so he achieves a superior outcome, but the controller is entirely benevolent. He possesses a true social welfare function so that private agents are also better off when he reneges. Moreover, the private sector is not being cheated; it knows full well the intentions of the policy maker but it pays to plead ignorant and the temptation to renege is innocuous. What is required for time inconsistency to constitute a problem is that at least some persons are (or assign some positive probability to being)
made worse off following a departure from an announced plan. Needless to say, there seems to be sufficient justification that this is the case (for example, a divergence between the preferences of the private sector and controller; a divergence between private and social interest; lack of a true social welfare function; heterogeneous agents) and we believe, therefore, that there is a problem of time inconsistency, investigations into which being non-trivial pursuits.

A second point that we wish to make concerns the generalisation which takes explicit account of the possibility of differentially informed agents where, in particular, heterogeneity among individuals takes the form of different perceptions about the reputation of the controller. This has not received attention in the literature but is likely to be non-trivial and may undermine the threat effect for enforcing commitment to announced policies. Recall from section 1.3(D) that the threat need not actually be voiced nor collusion required for it to be credible; all that is required is for the policy maker to appreciate the loss of reputation incurred by reneging. Clearly, the threat is credible if all agents hold identical beliefs because each agent knows that his distrust of the policy maker is shared by others. But in the case of differential beliefs the controller may still have an incentive to renge in order to exploit those agents who remain convinced of his integrity. Moreover, those who perceive the incentive to renge are likely to profit at the expense of the ignorant. Note, however, that this may work the other way as the policy maker plays a game of double bluff: rather than exploiting the ignorant, he may exploit the wisdom of the other group by actually adhering to the announced policy— we now have the enforcement of the ex ante policy because of the absence of a credible threat. The precise outcome is unclear and it is likely to involve
players attempting to assess the beliefs and strengths of others in order to predict the intentions of the policy maker.

On a final point, a further important (but largely neglected) issue concerns the informational assumptions made in the literature. For the most part, the games that we have considered are full information differential games with every player knowing exactly the payoffs and strategies of each and every other player. A small amount of uncertainty would give substance to the problem of time inconsistency, enrich the issues involved and yield a closer approximation to reality, together with focusing on the implications of alternative information structures. Uncertainty about player's payoffs and consideration of stochastic systems are important contributions in this respect (see Backus and Driffill (1984a,b, 1985); Canzoneri (1985)). In any event, the essential questions that beg consideration are precisely those which are endemic to the credibility issue, namely the informational assumptions which condition agents' beliefs about the commitment to a particular policy regime.

1.4 A Mathematical Interpretation

By way of completeness, this section formalises much of the discussion in the previous sections. The optimal control problem is solved using the maximum (or minimum) principle developed by Pontryagin et al. (1962) and summarized in appendix A. The solution of systems which include non-predetermined variables follows from appendix B.

1.4(A) The Standard (Single Controller) Control Problem

\[ \begin{align*}
\min J = \frac{1}{2} \int_{t_0}^{t_1} e^{-ot} (y^T Q y + w^T R w) dt \\
\text{s.t. } \dot{y} = Ay + Bw
\end{align*} \]

problem definition \hspace{1cm} (1.4.1)
The Hamiltonian is given by:
\[ H = e^{-\rho t} (y^T Q y + w^T R w) + \mu^T (A y + B w) \]  
(1.4.2)

First order conditions are:
\[ \frac{\partial H}{\partial \mu} = \dot{y}, \quad \frac{\partial H}{\partial w} = 0 \Rightarrow w = -R^{-1} B^T \pi \]  
(1.4.3)

\[ \frac{\partial H}{\partial y} = \mu \Rightarrow \dot{\pi} = -Q y + A^T \pi \]

where \( \mu = \text{n x 1 vector of costate variables} \)
\( A = \rho I - A \)
and \( \mu = \mu(t) \) as before with \( I \) the \( n \times n \) identity matrix. It is useful to note here that the costate variables are the dynamic analogues of Lagrange multipliers. Hence, they represent the marginal contribution to the maximized value of the maximand with respect to a change in the constraint. We shall see below that this interpretation plays an important role in determining the boundary conditions when the control problem assumes game aspects. Substituting for \( w \) from equation (1.4.3) gives the adjoint system describing the evolution of \( y \) and \( \pi \) under optimal control:

\[
\begin{bmatrix}
\dot{y} \\
\dot{\pi}
\end{bmatrix} =
\begin{bmatrix}
A & E \\
-Q & A^T
\end{bmatrix}
\begin{bmatrix}
y \\
\pi
\end{bmatrix} = F
\begin{bmatrix}
y \\
\pi
\end{bmatrix}
\]  
(1.4.4)

where \( E = -R^{-1} B^T \).

The boundary conditions for equation (1.4.4) are \( y(t_0) = y_o \) (n initial conditions) and \( \pi(t_1) = 0 \) (n transversality conditions). The latter imposes the saddlepoint structure on equation (1.4.4) so that \( \pi \) contains non-predetermined costates associated with the predetermined \( y \) (see, for example, Miller and Salmon (1983)). Thus, the system in equation (1.4.4) is formally analogous to the standard state-space configuration of models embodying jump variables and the saddlepoint.
solution in appendix B applies. Denote by \( \Lambda \) the 2nx2n diagonal matrix of eigenvalues of \( F \) and by \( M \) the 2nx2n matrix of associated eigenvectors and note the canonical transform \( MF = \Lambda M \) or \( FM = \tilde{M} \Lambda \) with \( \tilde{M} = M^{-1} \). Then partitioning conformably with \( [Y \; \Pi] \),
\[
F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}
\]
and assuming a unique non-explosive solution, \( \Lambda_1 \) is the nxn diagonal matrix of stable eigenvalues and \( \Lambda_2 \) is the nxn diagonal matrix of unstable eigenvalues. The stable manifold is
\[
\Pi = -M_2^{-1}M_1 Y = M_2^{-1}M_1^* Y = -BY
\] (1.4.5)
so that substituting equation (1.4.5) into equation (1.4.4) and the expression for \( w \) in equation (1.4.3) gives the solution
\[
X(t) = e^{G(t-t_0)}X(t_0) = M_{11}^{-1}e^{\Lambda_1(t-t_0)}M^{-1}Y(t_0) = K(t)Y(t_0)
\] (1.4.6)
\[
\Pi(t) = -HK(t)Y(t_0)
\] (1.4.7)
\[
w(t) = R^{-1}B^TY(t) = R^{-1}B^THK(t)Y(t_0)
\] (1.4.8)
where \( G = F_{11} - F_{12}H \).

**PROPOSITION 1.4.1** : The nxn matrix \( H \) is the solution to the steady-state matrix Ricatti equation, \( H(\Lambda + EH) + Q - \Lambda^T R = 0 \).

**PROOF** : Let \( [Y^T(t) \; \Pi(t)]^T = \phi(t, t_0) \) be the unique solution of the system so that \( \phi(t, \cdot) \) is the 2nx2n state transition matrix. Then
and partitioning $\phi(t, t_1)$ conformably with $[y^T(t) \pi^T(t)]^T, y(t) = \phi_{11}(t, t_1)y(t_1)$ and $\pi(t) = \phi_{21}(t, t_1)y(t_1)$. Hence,

$$\pi(t) = \phi_{21}(t, t_1)\phi_{11}^{-1}(t, t_1)y(t_1) = Hy(t_1)$$

(compare with equation (1.4.5)). Substituting equation (1.4.10) into equation (1.4.4), $\dot{y} = (A + EH)y$, and from equation (1.4.3) and (1.4.10),

$$\dot{y} = By + B\dot{y} = (A^T H - Q)y.$$  

Then

$$(\dot{H} + H(A + EH) + Q - A^T H)y = 0$$

so that in steady state, $\dot{H} = 0$ and $H$ is given as in the Proposition Q.E.D.

**PROPOSITION 1.4.2:** The optimal policy is time consistent.

**PROOF:** Denote by $y(t_j|t_j)$ the value of $y$ at time $t_j$ resulting from an optimal policy implemented at time $t_j$. Then equation (A.1.6) gives

$$y(t|t_j) = M_{11} e^{\Lambda_1 (t-t_j)} M_{11} y(t_j|t_0) = M_{11} e^{\Lambda_1 (t-t_0)} M_{11} y(t_0)$$

$$= y(t|t_0)$$ Q.E.D.  

(1.4.12)

Standard dynamic programming techniques also yield optimal policies which are time consistent. An illustration of these techniques, to be returned to later in section 1.4(C), is the following two period optimization problem (see Kydland and Prescott (1977); Buiter (1980a); Holly and Zarrop (1983));
where the subscripts denote the time period. The terminal period optimization problem involves minimizing J with respect to \( w_2 \) treating \( y_1 \) and \( w_1 \) parametric which yields first order conditions,

\[
\frac{\partial J}{\partial w_2} = \frac{\partial J}{\partial y_2} \frac{\partial y_2}{\partial w_2} + \frac{\partial J}{\partial w_2} = 0 \Rightarrow w_2 = w_2^*(w_1). \tag{1.4.14}
\]

The first period optimization problem is to minimize J with respect to \( w_1 \) and subject to the constraint that \( w_2 = w_2^*(w_1) \). For our purposes, equation (1.4.14) will suffice.

### 1.4(B) The Two Person Nonzero-Sum Dynamic Game

**(a) Open Loop Nash**

\[
\begin{align*}
\text{min.} & \quad J_k = \int_{t_0}^{t_1} e^{-\rho t} \left( \sum_{h=1}^{T} T_{h} w_k + \sum_{h=1}^{T} T_{h} R_k w_k \right) dt \\
\text{s.t.} & \quad \dot{y}_k = Ay + \sum_{k=1}^{2} B_k w_k
\end{align*}
\]

Problem definition (players 1 and 2)

\[
H_k = \int_{t_0}^{t_1} e^{-\rho t} \left( \sum_{h=1}^{T} T_{h} w_k + \sum_{h=1}^{T} T_{h} R_k w_k \right) dt + \mu_k^T (Ay + \sum_{k=1}^{2} B_k w_k) \quad \text{Hamiltonian} \tag{1.4.15}
\]

\[
\begin{align*}
\frac{\partial H_k}{\partial \mu_k} = \dot{y}_k; & \quad \frac{\partial H_k}{\partial w_k} = 0 \Rightarrow w_k = -R_k^{-1}B_k \pi_k \\
-\frac{\partial H_k}{\partial y} = \dot{\pi}_k = & \quad \ddot{\pi}_k = -Q_k y + A^T \pi_k \quad \text{first order conditions} \tag{1.4.16}
\end{align*}
\]

with \( \pi_k = e^{\rho t} \mu_k \) (k=1,2). The expression for \( w_k \) in equation (1.4.17) is the reaction function for each player. It is each player's
optimal response given the controls of his rival. The adjoint system
describing the dynamic behaviour of the system under optimal open loop
Nash control is

\[
\begin{bmatrix}
Y^N \\
\pi_1 \\
\pi_2
\end{bmatrix} =
\begin{bmatrix}
A & E_1 & E_2 \\
-Q_1 & A^T & 0 \\
-Q_2 & 0 & A^T
\end{bmatrix}
\begin{bmatrix}
Y^N \\
\pi_1 \\
\pi_2
\end{bmatrix} = P^N
\begin{bmatrix}
Y^N \\
\pi_1 \\
\pi_2
\end{bmatrix}
\] (1.4.18)

where \( E_k = -B_k R_k^{-1} B_k^T \) (\( k = 1,2 \)).
The boundary conditions are the \( n \) initial conditions, \( Y(t_o) = Y_o \), and
the \( 2n \) transversality conditions, \( \pi_k(t_1) = 0 \) (\( k = 1,2 \)). Following
appendix B, therefore, the solution is

\[
Y^N(t) = e^{G^N(t-t_o)} Y(t_o) = \sum_{i=1}^{2n} A_i^N(t-t_o) N_{11}^{-1} Y(t_o) = R^N(t) Y(t_o)
\] (1.4.19)

\[
\pi^N(t) = -N K^N(t) Y(t_o)
\] (1.4.20)

\[
\pi^N_k(t) = R_k^{-1} (H^N_k)^T \pi(t) = R_k^{-1} B_k^T (H^N_k)^T \pi(t) Y(t_o)
\] (1.4.21)

where \( N = [\pi_1^T \pi_2^T]^T \) and \( H^N \) has been partitioned conformably with
\[
[\pi_1^T \pi_2^T]^T H^N = [((H^N)^1)^T ((H^N)^2)^T]^T.
\]

PROPOSITION 1.4.3 : The optimal policy is time consistent

PROOF : Repeat the proof of Proposition 1.4.2 using equation (1.4.19)
so that

\[
Y^N(t|t_j) = \sum_{i=1}^{2n} A_i^N(t-t_j) (M_{11}^N)^{-1} Y(t_j|t_o) = \sum_{i=1}^{2n} A_i^N(t-t_o) (M_{11}^N)^{-1} Y(t_o)
\]

\[
= Y^N(t|t_o) \ \text{Q.E.D.}
\] (1.4.22)
(b) *Open Loop Stackelberg*

Player 2 acts as a Nash Player so that equations (1.4.15) - (1.4.17) apply with k=2. In contrast, player 1 takes into account player 2's reaction function which is encapsulated analytically by player 1 optimizing conditional on the behaviour of player 2 described in equations (1.4.17) (that is, player 2's first order conditions). Then substituting for player 2's reaction function,

\[
\begin{align*}
\min J_1 &= \int_0^t e^{-\rho t} \left( YQ_1 Y + \pi_1 R_{11} \pi_1 + \pi_2 R_{12} \pi_2 \right) dt \\
\text{s.t.} \quad &\dot{Y} = AY + B_1 \pi_1 + E_2 \pi_2; \quad \dot{\pi}_2 = -Q_2 Y + A^T \pi_2 \\
\end{align*}
\]

Problem definition (player 1) (1.4.23)

\[
H = e^{-\rho t} \left( YQ_1 Y + \pi_1 R_{11} \pi_1 + \pi_2 R_{12} \pi_2 \right) + \mu_{11} (AY + B_1 \pi_1 + E_2 \pi_2) \\
+ \mu_{12} (-Q_2 Y + A^T \pi_2)
\]

Hamiltonian (1.4.24)

\[
\begin{align*}
\frac{\partial H}{\partial \mu_{11}} &= \dot{Y} \\
\frac{\partial H}{\partial \mu_{12}} &= \dot{\pi}_2 \\
\frac{\partial H}{\partial \pi_1} &= 0 \\
\mu_{11} &= -R_{11}^{-1} B_1^T \\
\end{align*}
\]

first order conditions (1.4.25)

\[
\begin{align*}
\frac{\partial \mu_{11}}{\partial Y} &= \dot{\pi}_1 \quad \frac{\partial \mu_{11}}{\partial \pi_1} = -Q_1 Y + A^T \pi_1 + Q_2 \pi_{12} \\
\frac{\partial \mu_{11}}{\partial \pi_2} &= \dot{\pi}_2 \quad \frac{\partial \mu_{11}}{\partial \pi_2} = -R_{12} \pi_2 - E_2^T \pi_1 + A \pi_{12} \\
\end{align*}
\]

where \( R_{12} = B_2^T R_{22}^{-1} R_{12} \) \( R_{12}^{-1} B_2 \) and \( \mu_{1k} = e^{\rho t} \mu_{1k} \) (k = 1, 2). It is now noted that the costate variable \( \mu_{12} \) is a costate on the non-predetermined costate variable of player 2, \( \pi_2 \). Just as the costates on predetermined variables are non-predetermined, likewise the costates on non-predetermined variables are predetermined. Thus, the adjoint system describing the dynamic behaviour of the system under optimal open loop Stackelberg control is
with 3n boundary conditions, \( y(t_o) = y_o \) and \( \pi_{11}(t_1) = \pi_2(t_1) = 0 \) as before so that there remains n further conditions associated with \( \pi_{12} \). Recall from above that \( \pi_{12} \) are costates on the non-predetermined costates of player 2, \( \pi_2 \), and are themselves, therefore, predetermined. In addition, since \( \pi_2 \) are jump variables, there is a freedom to choose any initial value for these, \( \pi_2(t_o) \), which will alter the value of the maximand, but this change must be zero at the optimum and since \( \pi_{12}(t_o) \) measures this change, \( \pi_{12}(t_o) = 0 \) provides the remaining n boundary conditions (see Driffill (1982)). The solution of the system now follows as

\[
\begin{bmatrix}
\dot{y}^s(t) \\
\pi_{12}(t)
\end{bmatrix}
= \begin{bmatrix}
A & 0 & E_1 & E_2 \\
0 & A & -E_2^T & -R_{12}
\end{bmatrix}
\begin{bmatrix}
\pi_{12} \\
\pi_{11} \\
\pi_2
\end{bmatrix}
= F^s
\begin{bmatrix}
\pi_{12} \\
\pi_{11} \\
\pi_2
\end{bmatrix}
(1.4.26)
\]

\[
y(t) = y(t_o)
\]

\[
\pi_{12}(t) = \frac{\partial}{\partial y} \left[ \begin{bmatrix}
\dot{y}^s \\
\pi_{12} \\
\pi_2
\end{bmatrix}
\right]
= \frac{\partial F^s}{\partial y}
\begin{bmatrix}
\dot{y}^s(t) \\
\pi_{12}(t)
\end{bmatrix}
(1.4.27)
\]

\[
\pi^s(t) = -\frac{\partial}{\partial y} K^s(t)
\begin{bmatrix}
y(t_o) \\
0
\end{bmatrix}
(1.4.28)
\]

where \( \pi^s(t) = [\pi_{11} \pi_2]^T \). To obtain expressions for \( w \) in terms of \( y \) proceed as follows.

Note first that \( \pi^s = -H^s[(y^s)^T \pi_{12}]^T \) defines the stable trajectory so that partitioning \( H^s \) as before, \( H^s = [(H^s)^1]^T (H^s)^2]^T \), we have

\[
\pi_{12} = -(H^s)^1[(y^s)^T \pi_{12}]^T \quad \text{and} \quad \pi_2 = -(H^s)^2[(y^s)^T \pi_{12}]^T.
\]

Substituting
into equations (1.4.17) and (1.4.25),

\[
\frac{w}{s} = \frac{-1}{\kappa} \frac{T}{\kappa} B_s(H) k_s \begin{bmatrix} y_s \\ \pi_{12} \end{bmatrix}
\]  

(1.4.29)

Then from equation (1.4.27),

\[
\frac{w}{s}(t) = \frac{-1}{\kappa} \frac{T}{\kappa} B_s(H) k_s(t) y(t_o)
\]  

(1.4.30)

with \(K^s(t) = [K^s_1(t) K^s_2(t)]\). An alternative way of deriving equation (1.4.30) is to note from equation (1.4.27) that \(y^s_s(t) = K^s_{11}(t)y(t_o)\) and \(\pi^s_{12}(t) = K^s_{21}(t)y(t_o)\) so that \(\pi^s_{12}(t) = K^s_{21}(t)(K^s_{11}(t))^{-1} y^s_s(t)\) and therefore

\[
\frac{w}{s}(t) = \frac{-1}{\kappa} \frac{T}{\kappa} B_s(H) k_s \begin{bmatrix} I \\ K^s_{21}(t)(K^s_{11}(t))^{-1} \end{bmatrix} y(t).
\]  

(1.4.31)

Finally, a further way of writing \(w\) is obtained by recalling that

\[
\frac{w}{s}(t) = G^s_{21} y^s_s(t) + G^s_{22} \pi^s_{12}(t)
\]

in which case

\[
\pi^s_{12}(t) = \int_{t_o}^{t} e^{G_{22}(t-r)} G_{21} y^s_s(\tau) \, d\tau \quad \text{(since \(\pi^s_{12}(t_o) = 0\))}
\]  

and

\[
\frac{w}{s}(t) = \frac{-1}{\kappa} \frac{T}{\kappa} B_s(H) k_s \begin{bmatrix} y_s(t) + \int_{t_o}^{t} e^{G_{22}(t-r)} G_{21} y^s_s(\tau) \, d\tau \\ \int_{t_o}^{t} e^{G_{22}(t-r)} G_{21} y^s_s(\tau) \, d\tau \end{bmatrix}
\]  

(1.4.32)

with \((H^s)^k = [H^s_{k1} H^s_{k2}] (k = 1, 2)\).

PROPOSITION 1.4.4 : The optimal policy is time inconsistent.

PROOF : Repeat the proof of Proposition 1.4.2 using equation (1.4.27)
and recall that it is always optimal to set $\pi_{12}(t) = 0$ at the date of optimization. Then

$$
\begin{align*}
\begin{bmatrix}
Y^s(t|t_j) \\
\pi_{12}(t|t_j)
\end{bmatrix} &= \left( M_{11} \Delta^*(t-t_j) (M_{11}^*)^{-1} \right) \begin{bmatrix}
Y^s(t_j) \\
0
\end{bmatrix} + \left( M_{12} \Delta^*(t-t_0) (M_{11}^*)^{-1} \right) \begin{bmatrix}
0
\end{bmatrix} \\
&= \begin{bmatrix}
Y^s(t|t_0) \\
\pi_{12}(t|t_0)
\end{bmatrix}
\end{align*}
$$

(1.4.33)

Formally, the source of time inconsistency is found in the follower's costates, $\pi_2$, which are forward-looking variables and reflect the announced policy of the dominant player who exploits this fact.

1.4(C) The Rational Expectations Dynamic Game

(a) Open Loop Stackelberg

The control problem is given in equation (1.4.1) - (1.4.3) yielding the adjoint system in equation (1.4.4). Note, however, that $y$ is now partitioned into $y = [z^T x^T]^T$ where $z$ is an $n_1 \times 1$ vector of predetermined variables and $x$ is an $n_2 \times 1$ vector of non-predetermined variables. In addition, $\pi$ is partitioned conformably with $[z^T x^T]^T$ so that $\pi = [\pi_{11}^T \pi_{12}^T]^T$ with $\pi_{11}$ an $n_1 \times 1$ vector of free costates (associated with $z$) and $\pi_{12}$ an $n_2 \times 1$ vector of predetermined costates (associated with $x$). The analogy with the standard Stackelberg game in section 1.4(B) is straightforward; $x$ contains forward looking variables and is the analogue to $\pi_2$ (where $\pi_2$ was the follower's vector of costates); the costates on $x$ are in $\pi_{12}$ which is the same as before (where $\pi_{12}$ was the leader's costates on $\pi_2$). Since both $y$ and $\pi$ contain both predetermined and non-determined variables, therefore, equation (1.4.4) is re-ordered to give the adjoint system for the rational expectations open loop Stackelberg game,
with 2n boundary conditions, \( z(t_0) = z_0, \pi_{11}(t_1) = x(t_1) = 0 \) and \( \pi_{12}(t_0) = 0 \). Then equation (1.4.34) has solution

\[
\begin{align*}
\begin{bmatrix}
\dot{z}^g(t) \\
\pi^g_{12}(t) \\
\pi^g_{11}(t) \\
\dot{x}^g(t)
\end{bmatrix}
&= e^{\Delta^g_1(t-t_0)}
\begin{bmatrix}
z(t_0) \\
0
\end{bmatrix}
= N_{11}^{-1}e^{\Delta^g_1(t-t_0)}(N_{11}^{-1})^{-1}
\begin{bmatrix}
z(t_0) \\
0
\end{bmatrix} \\
&= U^g(t)
\begin{bmatrix}
z(t_0) \\
0
\end{bmatrix}
\end{align*}
\] (1.4.35)

\[
\begin{align*}
\begin{bmatrix}
\pi^g_{11}(t) \\
x^g(t)
\end{bmatrix}
&= -P^gU^g(t)
\begin{bmatrix}
z(t_0) \\
0
\end{bmatrix}
\end{align*}
\] (1.4.36)

where the definitions of \( T, \Delta, N (N) \) and \( P \) should be clear from section 1.4(A). Analogous to the standard Stackelberg game in section 1.4(B), \( \varpi \) can be expressed in the following forms (see Levine and Currie (1984)). Using \( [\pi^T_{11}(z^g)^T] = -P^g[(z^g)^T \pi^T_{12}] \) (defining the saddlepath), \( \pi_{11} = -(P^g)^{-1}([z^g)^T \pi^T_{12}] = -((P^g_{11})^{-1} + (P^g_{12})^{-1}) \) with \( P^g = (P^g_{11})^{-1} + (P^g_{12})^{-1} \). So that using equation (1.4.3),

\[
\varpi^g = -R^{-1}B^T
\begin{bmatrix}
z^g \\
\pi^g_{12}
\end{bmatrix}
= -R^{-1}B^TV
\begin{bmatrix}
z^g \\
\pi^g_{12}
\end{bmatrix}
\] (1.4.37)

where \( B^T = [-B_{11}P^g_{11} - B_{12}P^g_{12}] \)

\[
V =
\begin{bmatrix}
P^g_{11} & P^g_{12} \\
P^g_{12} & I
\end{bmatrix}
\]
and B has been partitioned, B = [B₁ B₂]. Hence, from equation

(1.4.35)

\[ w^S(t) = - R^{-1} B^T U^S_1(t) z(t_o) \]  \hspace{1cm} (1.4.38)

with \( U^S_1(t) = [U^S_1(t) U^S_2(t)] \). Alternatively, note from equation

(1.4.35) that \( z^S(t) = U^S_{11}(t) z(t_o) \) and \( U^S_{12}(t) = U^S_{21}(t) z(t_o) \) so that

\[ U^S_{12}(t) = U^S_{21}(t)(U^S_{11}(t))^{-1} z(t) \]

and therefore

\[ w^S(t) = - R^{-1} B^T \begin{bmatrix} 1 \\ [U^S_{21}(t)(U^S_{11}(t))^{-1}] \end{bmatrix} z(t). \]  \hspace{1cm} (1.4.39)

In addition, since \( \Pi_{12}(t) = T^S_{21} z(t) + T^S_{22} \Pi_{12}(t) \) gives

\[ \Pi_{12}(t) = \int_{t_o}^{t} e^{T^S_{21}(t-r)T^S_{21}} z(t) dr \]  \hspace{1cm} (since \( \Pi_{12}(t_o) = 0 \)),

\[ w^S(t) = - R^{-1}(B^S^T U^S_1 + B^S^T \int_{t_o}^{t} e^{T^S_{21}(t-r)} T^S_{21} z(t) dr \]  \hspace{1cm} (1.4.40)

with \( B^S^T = [B^S_{12}] \). Finally, from the stable trajectory defined above, \( x = -(P^S)^2 [z \cdot T_{12}^T] T^S_{22} \pi_{12} = (P^S_{21} z + P^S_{22} \pi_{12}) \) so that \( \pi_{12} = -(P^S_{22})^{-1}(x^S + P^S_{21} z) \) and

\[ w^S(t) = - R^{-1} B^S^T \pi_{12}(t) = - R^{-1} B^S^T v^S_{12}(t) \]  \hspace{1cm} (1.4.41)

where \( B^S = [\overline{B} - \overline{B}_2 (P^S_{22})^{-1} P^S_{21} - \overline{B}_2 (P^S_{22})^{-1}] \),

\[ \overline{B} = \begin{bmatrix} -P^S_{11} + P^S_{12}(P^S_{22})^{-1} & P^S_{12} (P^S_{22})^{-1} \\ -(P^S_{22})^{-1} P^S_{21} & -(P^S_{22})^{-1} \end{bmatrix}. \]

PROPOSITION 1.4.5 : The optimal policy is time inconsistent.
PROOF: Repeat the proof of Proposition 1.4.2 using equation (1.4.34) and recall that it is always optimal to set $\pi_{12}(t) = 0$ at the date of optimization. Then

$$
\left[ \begin{array}{c} z^s(t|t_j) \\ \pi_{12}(t|t_j) \end{array} \right] = N_{11}^{-1} \Delta_1(t-t_j) N_{11}^{-1} \left[ \begin{array}{c} z(t_j) \\ 0 \end{array} \right] = N_{11}^{-1} \Delta_1(t-t_0) N_{11}^{-1} \left[ \begin{array}{c} z(t_0) \\ 0 \end{array} \right]
$$

(1.4.42)

PROPOSITION 1.4.6: The policy obtained from standard dynamic programming is generally sub-optimal.

PROOF: Recall the system in equation (1.4.13) and substitute for the $y_1$ relationship the non-causal relationship $y_1 = y_1(w_1, w_2)$. Then first order conditions for the terminal period optimization are

$$
\frac{\partial J}{\partial w_2} = \frac{\partial J}{\partial y_2} \frac{\partial y_2}{\partial w_2} + \frac{\partial J}{\partial w_2} + \frac{\partial y_1}{\partial w_2} (\frac{\partial J}{\partial y_1} + \frac{\partial J}{\partial y_2} \frac{\partial y_2}{\partial y_1}) = 0
$$

(1.4.43)

which reduces to equation (1.4.14) only for the special case in which $\frac{\partial y_1}{\partial w_2} = 0$ (or $\frac{\partial J}{\partial y_1} + \frac{\partial J}{\partial y_2} \frac{\partial y_2}{\partial y_1} = 0$) Q.E.D.

(b) Closed Loop Stackelberg

Currie and Levine (1985) conceptualise the reaction functions of the controller and private sector as follows. For the former, this is just the control rule in equation (1.4.3) which can be re-written as in equation (1.4.37). The reaction function for the private sector is the saddlepath relationship between $x$ and $[z^T \pi_{12}^T]^T$. In general, therefore, the reaction functions for the controller (player 1) and the private sector (player 2) are
\[ w = D_1 \begin{bmatrix} z \\ \pi_{12} \end{bmatrix}, \quad x = D_2 \begin{bmatrix} z \\ \pi_{12} \end{bmatrix} \] (1.4.44)

Note also from equation (1.4.35) that \( \dot{\pi}_{12} = C[Z^T \pi_{12}]^T \). Thus, partitioning \( B = [(B^1)^T(B^2)^T]^T \), \( D_k = [(D_{k1}^1)(D_{k1}^2)](k = 1,2) \) and \( C = [C_1C_2] \),

\[
\begin{align*}
\dot{\pi}_{12} &= \begin{bmatrix} A_{11}^1 B_1^1(D_1)_{11} + A_{12}^1(D_2)_{11} & B_1^1(D_1)_{12} + A_{12}^1(D_2)_{12} \\ C_1 & C_2 \end{bmatrix} \begin{bmatrix} z \\ \pi_{12} \end{bmatrix} = \begin{bmatrix} z \\ \pi_{12} \end{bmatrix}
\end{align*}
\] (1.4.45)

\[
J = v \int_{t=0}^{t_1} e^{-\rho t} [Z^T \pi_{12}] \begin{bmatrix} Z \\ \pi_{12} \end{bmatrix} dt
\] (1.4.46)

where \( Q = D_1^T R D_1 + Q \),

\[
Q = \begin{bmatrix} Q_{11}^1(1 + Q_{21}^1 + Q_{12}^2 + Q_{12}^2) & (D_2)^T Q_{22}^1(D_2) \\ (D_2)^T Q_{22}^2(D_2) & (D_2)^T Q_{22}^2(D_2) \end{bmatrix}
\]

The control problem is then to minimize equation (1.4.46) with respect to \( D_1 \) subject to equation (1.4.45). Note that the functional relationship between \( x \) and \( [Z^T \pi_{12}]^T \) in equation (1.4.44) summarised in \( D_2 \), is a function of \( D_1 \), indicating the dependence of the private sector's decision rule on the controller's policy rule.

1.4(D) Perfect Cheating

(a) Open Loop

Miller and Salmon (1983) show that the ability to perfectly cheat the private sector amounts to making the equation for \( \dot{x} \) redundant (since \( x \) reflects complete credibility of the announced policy and
is therefore effectively immune to the actual course of events) and recalling that continuous re-optimizing implies \( \pi_{12}(t) = 0 \) at all times so that \( \dot{\pi}_{12}(t) = 0 \). Then equation (1.4.34) becomes

\[
\begin{bmatrix}
\dot{z}^\text{PC} \\
\dot{\pi}^\text{PC}
\end{bmatrix} =
\begin{bmatrix}
A_{11} - A_{12}Q_2A_{22}^{-1}Q_21 & E_{11} + A_{12}Q_2A_{22}^{-1}A_T^{-1} \\
-Q_21A_{12}Q_2A_{22}^{-1}Q_21 & A_{11} - Q_21A_{12}Q_2A_{22}^{-1}
\end{bmatrix}
\begin{bmatrix}
\dot{z}^\text{PC} \\
\dot{\pi}^\text{PC}
\end{bmatrix} = L^\text{PC}
\begin{bmatrix}
\dot{z}^\text{PC} \\
\dot{\pi}^\text{PC}
\end{bmatrix}
\tag{1.4.47}
\]

with 2n boundary conditions, \( z(t_0) = z_0 \) and \( \pi_{11}(t_1) = 0 \), which has solution

\[
z^\text{PC}(t) = u^\text{PC}(t)z(t_0)
\tag{1.4.48}
\]

\[
\pi_{11}^\text{PC}(t) = -p^\text{PC}u^\text{PC}(t)z(t_0)
\tag{1.4.49}
\]

\[
w^\text{PC}(t) = R^{-1}B_1p^\text{PC}z(t) = R^{-1}B_1p^\text{PC}u^\text{PC}(t)z^\text{PC}(t_0)
\tag{1.4.50}
\]

which is obviously time consistent\(^{31}\).

(b) Closed Loop

Currie and Levine (1985) give the closed loop representation of perfect cheating as involving the controller announcing a policy in the form of equation (1.4.44) and deliberately re-optimizing at each time by setting \( \pi_{12}(t) = 0 \) which amounts to actually implementing

\[
w(t) = (D_1)_1z(t).
\]

1.4(E) Nash Equilibria

(a) Open Loop Nash and the Buiter (1983) Solution

Buiter (1983) exploits the fact that \( \pi_{12}(t) = 0 \) at all times for time consistency which amounts to simply deleting the row and the column corresponding to \( \pi_{12} \) in equation (1.4.34).
with \( z(t_0) = z_0 \) and \( \pi_{11}(t_1) = x(t_1) = 0 \). The solution is

\[
\dot{z}^N(t) = U^N(t)z(t_0)
\]

(1.4.52)

\[
\begin{bmatrix}
\pi_{11}(t) \\
\dot{z}(t)
\end{bmatrix} = -P^N(t)U(t)z(t_0)
\]

(1.4.53)

\[
\dot{w}^N(t) = -R^{-1}B_1'(t)R_1N(t)z(t) = -R^{-1}B_1'(t)R_1N(t)z(t_0)
\]

(1.4.54)

which is obviously time consistent(32).

PROPOSITION 1.4.6 : The Buiter (1983) solution is equivalent to the open loop Nash solution.

PROOF : Currie and Levine (1985) show that the open loop Nash game can be formalised as in equation (1.4.1) and the condition that \( x \) is given. Hence,

\[
H = \omega e^{-\rho t}(y^TQy + w^TRw) + \mu_{11}^T(A_{11}z + A_{12}x + B^lw)
\]

(1.4.55)

\[
\frac{\partial H}{\partial \mu_{11}} = \frac{\partial H}{\partial w} = 0 \Rightarrow w = -R^{-1}(B^l)^T\pi_{11}
\]

\[
-\frac{\partial H}{\partial z} - \dot{A}_{11} + \dot{\mu}_{11} = -Q_{11}z + \overline{A}_{11}^T\pi_{11} - Q_{12}^N
\]

(1.4.56)

and introducing the actual dynamics for \( x \) yields equation (1.4.51) with \( E_{21} = -B^2R^{-1}(B^l)^T \). Q.E.D.
Closed Loop Nash and the Cohen and Michel (1984)

(dynamic programming) solution

Cohen and Michel (1984) (generalised by Currie and Levine (1985)) show the continuous time dynamic programming solution to involve minimizing

\[ J_0 = \int e^{-\rho t} (Y^T Q + w^T R w) + \left( \frac{\partial J_1}{\partial z} \right)^2 \]

\[ J_1 = J_1(z(t)) = \int_0^t e^{-\rho (t-\tau)} (Y^T Q + w^T R w) d\tau \quad (1.4.57) \]

yielding a control rule of the form

\[ ^* N(t) = -R^{-1} (B^T)^T YZ(t) = D_t^* Z(t) \quad (1.4.58) \]

The matrix \( Y \) is a solution to a Ricatti equation derived as follows.

From equation (1.4.57)

\[ \frac{\partial J_1}{\partial t} = y^T Q y + w^T R w + \rho J_1 \]

\[ \frac{\partial (e^{-\rho t} J_1)}{\partial t} = e^{-\rho t} (Y^T Q + w^T R w). \quad (1.4.59) \]

Define \( J_1^O = J_1(z^O(t)) \) where \( z^O(t) = e^{-\rho t} z(t) \) so that

\[ \frac{\partial (e^{-\rho t} J_1)}{\partial t} = \frac{\partial J_1^O}{\partial t} = \left( \frac{\partial J_1}{\partial z} \right)^T z^O. \]

Then since \( z^O = -i\rho z^O + e^{-\rho t} z \)

and using \( \dot{z} = D_2^* z \) and \( w = D_1^* z \),

\[ \dot{z} = (A_{11} + A_{12} D_2^* + B D_1^*) z \]

\[ z^O = (A_{11} + A_{12} D_2^* + B D_1^* - i\rho I) z^O. \quad (1.4.60) \]

Setting \( \frac{\partial J_1}{\partial z} = -Yz^O \) then yields an expression for \( e^{\rho t} \frac{\partial (e^{-\rho t} J_1)}{\partial t} \).

An alternative expression for \( e^{\rho t} \frac{\partial (e^{-\rho t} J_1)}{\partial t} \) can be obtained directly from equation (1.4.59) so that equating these relationships gives
\[ Y^T(A_{11} + A_{12}D_2^{-2}Q_1) + (A_{11} + A_{12}D_2^{-2}Q_1)Y + Q_{11}Y - Y^TB^TR^{-1}(B^T)Y = 0 \]  

(1.4.61)

using the definition of \( D_1^* \) in equation (1.4.58) and where \( Q_{11}^* = Q_{11} + (D_2^*)^TQ_{21} + Q_{12}D_2^* + (D_2^*)^TQ_{22}D_2^* \).

PROPOSITION 1.4.7 : The Cohen and Michel (1984) dynamic programming solution is equivalent to a closed loop Nash solution.

PROOF : Recall the closed loop Stackelberg game in section 1.4(C).

For the Nash game, the optimization problem is to minimize equation (1.4.46) by choice of \( D_1 \) subject to equation (1.4.45) and the condition that the private sector response function is parametric (i.e., \( D_2 \) is given). This can also be expressed as minimizing equation (1.4.62) subject to equation (1.4.63) below which is in the standard form:

\[ J = \int_{t_0}^{t_1} e^{-\rho t} \left[ \begin{bmatrix} \pi_{12}^T & \pi_1^T \end{bmatrix} Q \begin{bmatrix} \pi_{12} \\ \pi_1 \end{bmatrix} + w^TRw \right] dt \]  

(1.4.62)

\[ \begin{bmatrix} \dot{\pi}_{12} \\ \dot{\pi}_1 \end{bmatrix} = \begin{bmatrix} A_{11} + A_{12}(D_2)^1 & A_{12}(D_2)^2 \\ C_1 & C_2 \end{bmatrix} \begin{bmatrix} \pi_2 \\ \pi_1 \end{bmatrix} + \begin{bmatrix} B^1 \\ 0 \end{bmatrix} w = \begin{bmatrix} \pi_2 \\ \pi_1 \end{bmatrix} + \begin{bmatrix} B^1 \\ 0 \end{bmatrix} w \]  

(1.4.63)

and the optimal control rule will be of the form

\[ \bar{w}N = -R^{-1}[(B^1)^T O]Y \begin{bmatrix} \pi_{12} \\ \pi_1 \end{bmatrix} \]  

(1.4.64)

with \( \bar{Y} \) the solution to the matrix Ricatti equation

\[ \bar{Y}(W + EY) + \bar{Q} - \bar{W}^TY = 0 \]  

(1.4.65)
where $\ddot{W} = \rho I - W$ and $E = -[(B_1)^T O]^{T} R^{-1}[(B_1)^T O]$. (This is easily verified by writing the Hamiltonian as

$$H = \sum e^{-pt} \left[ \begin{bmatrix} 2 \pi_{12}^T \pi_{12} \end{bmatrix} + \pi_{12}^T \pi_{12} \right] + H_{11} \left[ \begin{bmatrix} \pi \pi_{12} \end{bmatrix} + \begin{bmatrix} B_1^T \end{bmatrix} \pi_{12} \right] \right]$$

(1.4.66)

minimizing with respect to $H_{11}$, $W$ and $[\pi_{12}^T \pi_{12}]^T$ and noting, in particular, that $w = -R^{-1}[(B_1)^T O] \pi_{11}$. Then the adjoint of this system gives the saddlepath $\pi_{11}^* = Y[\pi_{12}^T \pi_{12}]^T$ which, when substituted into $w$ gives equation (1.4.64). Now, recall that the time consistent solution obtains if $\pi_{12}(t) = 0$ at all times. Then equation (1.4.64) becomes

$$\ddot{\pi}_{12}(t) = -R^{-1}[(B_1)^T \pi_{11}^* \ddot{\pi}_{12}(t)] = (D_1)^*_{11} \ddot{\pi}_{12}(t) \right)$$

(1.4.67)

with $Y_{11}$ the solution to

$$Y_{11}(W_{11} + E_{11} Y_{11} + \pi_{11} - \pi_{11}^T Y_{11}) = 0$$

(1.4.68)

and $E_{11} = -B_1 R^{-1}(B_1)^T$. Substituting for $W_{11} = A_{11} + A_{12}(D_2)_1$ from equation (1.4.63) into equation (1.4.68)

$$Y_{11}(A_{11} + A_{12}(D_2)_1 - \rho I) + (A_{11} + A_{12}(D_2)_1 - \rho I)^T Y_{11}$$

$$+ \pi_{11} - Y_{11} B_1 R^{-1}(B_1)^T Y_{11} = 0$$

(1.4.69)

so that with $(D_2)_1 = D_2^*$ (and therefore $\pi_{11} = \pi_{11}^*$) and with $Y$, $Y_{11}$ symmetric, equations (1.4.69) and (1.4.61) show that $Y = Y_{11}$, or that $Y$ in equation (1.4.58) satisfies the same matrix Ricatti equation as does $Y_{11}$ in equation (1.4.68). Q.E.D.
(c) Relationship Between the Butter (1983) and Cohen and Michel (1984) Solutions

An alternative way of deriving the Cohen and Michel (1984) solution is presented by these authors and Miller and Salmon (1984b). This particular method gives the following proposition.


PROOF: The Cohen and Michel (1984) solution proceeds with the controller treating as parametric the private sector's response function in equation (1.4.44) and ignoring \( \pi_{12} \) for time consistency. Hence, \( x = \{z \} \) say, with \( \xi \) an \( n_2 \times n_1 \) matrix which is treated as parametric by the controller. The analogues to equations (1.4.55) - (1.4.56) are therefore

\[
H = e^{-\alpha t} \left( TQz + w^T R w \right) + \mu_{11}^T (A_{11} + A_{12} \xi) z + B^T w \tag{1.4.70}
\]

\[
\frac{\partial H}{\partial \mu_{11}} = \dot{z} ; \quad \frac{\partial H}{\partial w} = \frac{\partial H}{\partial \xi} = 0 \Rightarrow w = -R^{-1}(B^T \pi_{11})
\]

\[
- \frac{\partial H}{\partial z} = \dot{\mu}_{11} \Rightarrow \dot{\mu}_{11} = -(Q_{11} + \xi^T Q_{21})z + (A_{11}^T - \xi^T A_{12})\pi_{11} - (Q_{12} + \xi^T Q_{22})x
\]

(1.4.71)

giving the adjoint system

\[
\begin{bmatrix}
\dot{z} \\
\dot{\mu}_{11} \\
\dot{\pi}_{11} \\
\dot{x}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & E_{11} & A_{12} \\
-(Q_{11} + \xi^T Q_{21}) & (A_{11}^T - \xi^T A_{12}) & -(Q_{12} + \xi^T Q_{22}) \\
A_{21} & E_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
z \\
\mu_{11} \\
\pi_{11} \\
x
\end{bmatrix}
= L
\begin{bmatrix}
z \\
\mu_{11} \\
\pi_{11} \\
x
\end{bmatrix}
\tag{1.4.72}
\]

with \( E_{21} = -B^T R^{-1}(B^T) \) and \( z(t_0) = z_0, \pi_{11}(t_1) = x(t_1) = 0 \). The
solution is

\[ \begin{array}{c}
\ddot{Z}(t) = U(t)Z(t_0) \\
\ddot{X}(t) = -P \ddot{U}(t)Z(t_0)
\end{array} \] (1.4.73)

\[ \begin{bmatrix}
\ddot{N}_{11}(t) \\
\ddot{N}_N(t)
\end{bmatrix} = -P \ddot{U}(t)Z(t_0) \] (1.4.74)

\[ \ddot{N}(t) = R^{-1}B_1(P)^T \ddot{N}(t) = R^{-1}B_1(P)^T U(t)Z(t_0) \] (1.4.75)

which is obviously time consistent. The Buiter (1983) solution obtains when the level of \( \dot{x} \) (as opposed to the functional relationship between \( \dot{x} \) and \( \dot{z} \)) is treated as given which occurs when \( \zeta = 0 \).

Q.E.D.

1.4(F) Stochastic Control

\[ \begin{align*}
\min E(J) & = \min \mathbb{E} \int_{t_0}^{T} e^{-\rho t} (y^T Q y + \dot{w}^T R \dot{w}) dt \\
\text{s.t.} & \quad \dot{y} = Ay + B \dot{w} + \rho \dot{w} + \dot{u}
\end{align*} \] (1.4.76)

As stated in section 1.3(E), the stochastic control problem can be redefined (see Levine and Curie (1984)) such that the optimisation is decomposed into deterministic and stochastic components. Levine and Curie (1984) show that the optimal control rule satisfies certainty equivalence.

PROPOSITION 1.4.9: The optimal policy is time inconsistent.

PROOF: An alternative proof of Proposition 1.4.5 which is useful for future purposes is as follows. Consider the deterministic case.

Then by the maximum principle, \( \frac{\partial J}{\partial y} = \pi^T \) so that \( J(t) = \pi^T y(t) \).

Recalling the saddlepath relationship in Section 1.4(C),
\( \pi_{11} = -(P_{11} z + P_{12} \pi_{12}) \) and  \( \pi = -(P_{21} z + P_{22} \pi_{12}) \). Note also that  
\( P_{11}^T = P_{11} \) and  \( P_{12}^T = -P_{21} \) (see Levine and Currie (1984)). Then the  
loss associated with the \textit{ex ante} optimal policy is

\[
J^*(t) = \text{tr}(T^T(t) P_{11}^T(t) \begin{bmatrix} -P_{11} & 0 \\ 0 & -P_{22} \end{bmatrix} \begin{bmatrix} \pi(t) \\
\pi_{12}(t) \end{bmatrix})
\]

\[
-\text{tr}(T^T(t) P_{11}^T(t) \pi(t)) = -\text{tr}(P_{11} z(t) + P_{22} \pi_{12}(t))
\]

(1.4.77)

where  \( Z(t) = z(t)^T(t) \),  \( \pi_{12}(t) = \pi_{12}(t)^T(t) \). But since  \( P_{22} \) is  
non-positive definite (see Levine and Currie (1984)) the policy  
becomes suboptimal at  \( t_j > t_0 \) because welfare loss can be reduced  
by an amount  \(-\text{tr}(P_{22} \pi_{12}(t))\) by re-optimising and setting  \( \pi_{12}(t) = 0 \).  
This holds for the stochastic case since equation (1.4.77) just  
becomes

\[
E(J^*(t)) = -\text{tr}(P_{11} Z(t) + \rho^{-1} \mathcal{L}) + P_{22} \pi_{12}(t))
\]

(1.4.78)

where  \( \mathcal{L} = \text{cov}(du^1); [(du^1)^T (du^2)^T]^T = du \). Q.E.D.

PROPOSITION 1.4.10 : In the stochastic system, if the rate of discount  
is sufficiently low, there is no incentive to renege and the optimal  
ex \textit{ante} policy is enforceable.

PROOF : First derive the welfare loss associated with the Nash  
policy. By analogy with equations (1.4.77) - (1.4.78),

\[
J^N(t) = \text{tr}(Y Z(t))
\]

(1.4.79)

\[
E(J^N(t)) = \text{tr}(Y Z(t) + \rho^{-1} \mathcal{L})
\]

(1.4.80)
where $Y$ is the solution to the Ricatti equation (1.4.61). In addition, from the proof of Proposition 1.4.9, the welfare loss associated with the time inconsistent (full optimal) policy is

$$\tilde{J}(t) = - \text{tr}(P_{11}Z(t)) \quad (1.4.81)$$

$$E(\tilde{J}(t)) = - \text{tr}(P_{11}(Z(t) + \rho^{-1} \Sigma)). \quad (1.4.82)$$

The temptation to reneg can now be measured by $\eta_T = E(\tilde{J}^*(t)) - E(\tilde{J}(t))$ whilst the cost of reneging is reversion to the Nash policy or $\eta_C = E(J^N(t)) - E(\tilde{J}(t))$. Equations (1.4.78) - (1.4.82) give

$$\eta_C = \text{tr}((Y + P_{11})(Z(t) + \rho^{-1} \Sigma) + P_{22} \Pi_{12}(t)) \quad (1.4.83)$$

$$- \text{tr}(P_{22} \Pi_{12}(t)). \quad (1.4.84)$$

Hence, as $\rho$ falls, $\eta_C$ increases until such point when there are no gains from reneging. Q.E.D.

Turning to the formulation of simple rules, assume no discounting so that

$$\min_w E(J) = E(Y^T Qy + w^T Rw)$$

subject to

$$dy = Aydt + Bw \ dt + du; \quad w = D_1 y \quad (1.4.83)$$

Substituting $w = D_1 y$,

$$dy = \tilde{A} y dt + du \quad (1.4.84)$$

where $\tilde{A} = A + BD_1$.

Assuming $\tilde{A}$ has the saddlepoint property, the stable trajectory is $x = D_2 z$ and the feedback rule becomes

$$w = D_1 z \quad (1.4.85)$$
where \( \bar{D}_1 = (D_1)_1 + (D_1)_2 D_2 \)

and \( D_1 = [(D_1)_1 (D_1)_2] \). Then \( x = D_2 z \) and equation \( (1.4.85) \) can be substituted into equation \( (1.4.83) \) so that

\[
\frac{dz}{dt} = \bar{T} z + d u^1
\]  

(1.4.86)

where \( \bar{T} = A_{11} + A_{12} D_2 + B^* D_1 \).

Following Chow (1979) the asymptotic variance of \( z \) can be computed as stated in Appendix E. Note that the solution will depend on the assumption made about strategic behaviour. Clearly, \( D_2 \) is a function of \( D_1 \). Hence, a closed loop Nash solution obtains if the controller treats as given the functional relationship, otherwise a closed loop Stackelberg solution is derived.

This completes our mathematical treatment of the issues discussed in sections 1.1 - 1.3.

1.5 **Summary and Concluding Remarks**

The chapter has been motivated by a recent rapidly growing research programme on macroeconomic policy design in the presence of forward-looking behaviour. The major insight of this research is the appreciation of the policy problem as a dynamic game between intelligent players - the government and the private sector - and the implications thereof. The scope of the paper reflects the rapid growth of interest in the area, the need to clarify some matters and the attempt to provide a comprehensive and unified discussion. It is hoped that we have been successful in achieving these and our final thoughts can be summarized fairly briefly as follows.

The fundamental dilemma of optimal policies when the controller assumes the role of a dominant player is that they are dynamically inconsistent (that is, do not satisfy the principle of optimality).
As such, there is always an incentive to renege on an announced policy which therefore lacks credibility and one must then seek policies which are optimal within the subset of credible (and hence time consistent policies). Time consistent equilibria can be obtained with precommitment and perfect cheating. We are sceptical about the former because the lack of enforceable contracts will always render any proposed precommitment suspect since the incentive to renege persists. The latter solution is simply inconceivable because it requires an implausibly gullible private sector. Researchers have then turned to examining Nash solutions which eschew the strategic asymmetry inherent in the Stackelberg game. Typically, these investigations have exploited the concept of perfect equilibria and the time consistency of backward-recursive optimization techniques analogous to standard dynamic programming approaches. Though time consistency is guaranteed in these frameworks, it is our view that there has been an over-preoccupation with them since they really avoid the main issue at stake which is the problem of enforcing the ex ante optimal policy. Moreover, it is precisely the loss of leadership and reversion to the inferior Nash solution which may motivate the controller to avoid the temptation to cheat. The reputation framework in which players are now engaged in a repeated game offers a resolution of time inconsistency which is already endogenous to the problem and we believe that this is the only plausible framework in which to address the issue. Unfortunately, the reputation framework appears to suffer (even for very simple models) from severe problems of tractability and is certainly devoid of any quantitative content. We urge research to be directed towards these matters if the reputation approach is to sustain and enhance the respect which we believe it deserves. An alternative approach for examining reputation issues which might prove particularly useful
here is to exploit the stochastic control problem which appears to mimic a repeated game.

Whilst on the subject of stochastic systems, it is work pointing out, however, that relatively little attention has been given to these. But the policy problem is likely to take on additional interesting dimensions under such circumstances as there is a potential confusion between an optimal response to stochastic perturbations and an attempt by the controller to cheat. The precise outcome is unclear and we believe that the stochastic case in general should be explored in more detail. It may turn out that consideration of purely deterministic systems has made the resolution of time inconsistency appear far more easy than is actually the case.

We also believe that greater insight could be gained by relaxing the informational assumptions that characterise most of the literature. An interest filtering problem would involve agents attempting to identify the intentions and current behaviour of others. The reputation of the policy maker may not be uniform across individuals and the credibility of threats could be undermined, though we have also suggested that this might actually help in enforcing the announced policy. In any event, preoccupation with full information games has probably meant a neglect of possibly more substantive issues associated with a greater degree of realism.

Our overall thoughts on the literature are guarded. There are sufficient grounds for regarding time inconsistency as posing an acute problem and we believe that the recent research has been crucial in yielding new insights into the problem of policy design. Note, however, that the literature has little or nothing to say about particular policy recommendations, rather it identifies general themes which are likely to apply globally. The ultimate anxiety is the question of its operational significance and whether the advances
made in control theory and macroeconomic policy evaluation will go unheeded by the afficianados of the policy making process. To enable the abstract ideas appear more relevant, empirical work is called for and the investigations into international cooperation and policy coordination seem especially useful here. Another potentially fruitful line of inquiry stressed by Chow (1976a, b) and Currie (1985) among others concerns the systematic investigation into policy rules which perform reasonably well not only in the face of different stochastic shocks but also across different models. Given the lack of consensus on the appropriate model of the economy, this would appear to be an especially useful area for research and chapters 2 and 3 in this thesis are directed towards this issue. To end on a more optimistic note, however, one may find some comfort in the pace at which the research is developing. If this continues we may find an answer sooner, rather than later, on whether, in fact, the research has merely been a game between economists. We suspect not.
Notes to Chapter One

This chapter can be found in Blackburn (1986a).

(1) One might argue that it is unfortunate that so much attention should have been devoted to the debate for it is an issue which rational expectations per se cannot resolve. At the same time, however, it provided the opportunity for economists to become familiar with the hypothesis and prompted awareness of other issues.

(2) As Buiter (1980b) observes, this does not necessarily rely on rational expectations but merely the fact that agents condition their expectations partly on their perceptions about the policy regime.

(3) Time inconsistency may also arise if tastes change, as identified by Strotz (1956) (see also Begg (1982b); Holly (1983)). This should be obvious and we are concerned with the more interesting case in which time inconsistency occurs in the absence of changing tastes.

(4) Causal dynamic models are models in which the current state of the system is determined solely by its present and past history. In non-causal models, however, the current state of the system is determined not only by its present and past history, but also by future expectations. The excellent volume by Chow (1975) focusses on causal models.

(5) Two further categories of control rules are work mentioning. First there is feedback control. This is more general than closed loop control (though the two terms are often used interchangeably) in the sense that it involves aspects of learning and probing. Second, there is sequential open loop control. In this, the controller responds to an observed departure from the optimal trajectory by recomputing an optimal open loop
plan. The fundamental difference between this type of response and the response exhibited by a closed loop rule is that the latter is entirely automatic. Clearly, sequential open loop control endows the controller with greater scope for discretion.

(6) Since it is not obvious the significance of the off-diagonal terms, it is commonly assumed that $Q$ and $R$ are diagonal. This is unimportant here. In addition, a term $y^T S w$ is also sometimes found in equation (1.2.4) where $S$ is an $n \times m$ time-invariant matrix. The inclusion of this adds nothing to the analysis and merely complicates the algebra.

(7) Generalization to the $n$-person game is straightforward, though tedious (see, for example, Miller and Salmon (1983)).

(8) Though collectively they may exert strong pressure on the government, individually each agent does not have such power and all agents know this. Moreover, it is unlikely that there is any mechanism through which any potential combined strength could be demonstrated. Indeed, if there was such a mechanism, the game could possibly degenerate into Stackelberg warfare. We return to this issue later.

(9) Obviously, $m_1 \neq m_2$ is possible.

(10) As shown in Section 1.4, there is a close relationship between the standard game-theoretic framework and the rational expectations framework. We consider both open loop and closed loop games in the latter and because of the close relationship, the closed loop game in the standard game is easily retrievable.

(11) Thus, resorting to the terminology often encountered in the literature, a closed loop decision rule can be a solution to an open loop game; hence the use of our own terminology.
Levine and Currie (1984) argue for the desirability of implementing control rules in terms of the state vector because they avoid feedback on the (possibly unobservable) costate variables (see Section 1.4 for the definition of costate variables). This argument has some merit, though it is not a complete solution to the problem: first, the controller could presumably merely divulge information about his costates; second, it is likely that contemporaneous information about some state variables is unavailable.

One condition is that the effect of changes in the initial state on the performance measure, both directly and indirectly through the influence of future expectations, is zero (see Proposition 1.4.6 and its proof). Buiter (1980a) observes that this is satisfied if the preference function is independent of control variables and Holly and Zarrop (1983) explain this by noting that, under such circumstances, the controller is able to costlessly achieve his targets and there is no gain from altering controls in subsequent periods (see also Turnovsky and Brock (1980) for a related observation). Begg (1982b) and Hillier and Malcomson (1984) observe that time inconsistency would disappear if there are sufficient numbers of instruments.

This is essentially because the government at time $t_1(i > 0)$ inherits an exogenous expenditure pattern and debt commitments and the budget constraint then dictates the tax policy which is the same as the tax policy computed at time $t_0$.

The 'innovation-contingent' policy embodies the state independent ex ante optimal policy plus a feedback on stochastic realizations. Buiter (1980a, 1981a,b) also shows that even a time consistent policy derived from standard dynamic
programming or *ad hoc* simple feedback rules may dominate the non-contingent (time inconsistent) policy because, though these rules eschew the non-causalities in the system, they permit a response to new information.

(16) Recall that these reflect the announced policy but, in the present case, the controller ignores his influence on his rival.

(17) It could also be called quasi-open loop Nash.

(18) In fact, we observed in section 2.2 that a closed loop Nash game has exactly the same feature only in this case it is symmetric, applying to both players.

(19) As d'Autume (1984, p.1) notes, "... one often finds in the literature hints that the quest for a time consistent policy is really the quest for a perfect equilibrium".

(20) The perfect equilibrium solution involves players making current decisions in the knowledge that future equilibrium decisions will be followed, a backward recursive structure which is analogous to standard dynamic programming.

(21) To confuse matters further, Backus and Driffill (1984c) propose a feedback Nash equilibrium as another equilibrium concept.

(22) Begg (1982b) has used this argument to justify the use of standard dynamic programming techniques.

(23) Thus, the policy maker dislikes inflation, but prefers more inflation to less. The benefits from inflation are really the benefits from surprise inflation, reflected in the term $q_2(\pi_t - \pi_t^e)$ and include expansions in output and employment. The costs of inflation, reflected in $q_1\pi_t^2$, are a little more difficult to identify but could comprise some distortionary effect. The private sector is assumed unable to influence aggregate variables and merely attempts to avoid making forecasting errors.
(24) \( \pi \) can be considered as a choice variable by envisaging some monetary instrument by which the controller can engineer a particular \( \pi \) (see Barro and Gordon (1983)). The treatment of \( \pi^e \) as a strategic variable may appear a little more contrived, though this interpretation should not be taken too literally. It is true that \( \pi^e \) cannot be chosen arbitrarily but must be rational and consistent with actual events as perceived by the private sector. The foregoing interpretation serves to emphasize the inter-relationship between players, the controller taking into account his influence on expectations which are conditioned by policy announcements.

(25) There need not be any pecuniary benefits of a strong reputation. The benefits would arise because if the controller is known to be weak, the solution is \( \pi = \pi^e = 1 \) with no gain in output. By inducing agents to believe he is tough, the controller may be able to sustain a \( \pi = 0 \) policy for some time and eventually exploit the low inflation expectations by engineering a higher actual inflation.

(26) Specifically, the probability of observing \( \pi_t = 0 \) in any period is \( \Pr(\pi_t = 0) = P_t + (1-P_t)g_t \) where \( g_t \) is the probability that a government will not inflate. Then given that \( \pi_t = 0 \) is actually observed, the probability next period that the government is hard is \( P_{t+1} = P_t / [P_t + (1-P_t)g_t] \).

(27) This is not strictly correct since reversion to the Nash solution might involve a cost.

(28) Note that \( du \) is cleansed of any autoregressive disturbances by suitable definition of the state vector, \( y \). Then \( du = \epsilon dt \), say where \( \epsilon \) has independent increments so that \( du \) is pure white noise.
(29) Holly and Zarrop (1961) identify a similar feature associated with sequential open loop control.

(30) As a digression, recall from section 1.2(C) that we pointed out that the full optimal rule is expressable in terms of a linear time-invariant feedback on just $z$. It is straightforward to show, however, that the simple rule, $w = Dy$, can be expressed as a time-invariant feedback on just $z$ (equation (1.4.85)). Levine (1984) observes this peculiarity to be due to the fact that the linear relationship between $w$ and $y$ in the full optimal rule does not constitute the optimal policy as announced by the policy maker because announcing the optimal policy in this form changes the dynamics of the system.

(31) For the standard Stackelberg game without rational expectations, the perfect cheating solution obtains by setting $\pi_{12}(t) = \pi_{12}(t) = 0$ and dropping the equation for $\pi_2$ in equation (1.4.26) (see Miller and Salmon (1983)).

(32) For the standard Stackelberg game without rational expectations, the Buiter (1983) solution obtains when the row and column corresponding to $\pi_{12}(t)$ in equation (1.4.26) is deleted.
CHAPTER 2

MONETARY INSTRUMENTS AND THE CONTROL PROBLEM:
AN INTRODUCTION AND APPRAISAL

2.1 Introduction

The issue of monetary control continues to receive widespread attention. In recent years most industrialised countries have witnessed the adoption of monetary targets as the anchor for macroeconomic policy. The arguments for and against such policies and the opinions expressed about the realised consequences thereof are numerous and varied. In this chapter we concentrate on one particular justification for their deployment in the context of a control framework.

To be specific, the issue is which potential monetary instrument variable the policy maker should choose for the purpose of stabilization policy. In particular, the set of alternatives includes the money stock, interest rates and the exchange rate. Control cannot be administered simultaneously over all three for the simple reason that at least one requires to be free in order to equilibrate the money market. It is well known, and will be proved below, that in a deterministic setting and provided the instruments do not enter into the policy maker's utility function, the problem of instrument choice is redundant. This is not true, however, in a stochastic environment; the choice of instrument is then non-trivial.

It will be recalled from chapter 1 that we emphasize the need to adopt a particular methodology for policy evaluation. This motivates the testing of alternative policies across divergent model structures with the view to seeking model-robust policy rules. This has been emphasized by Chow (1976a,b), Johansen (1979) and Currie (1985a,b). In chapter 3 we apply this methodology to the monetary instrument problem. It is useful, therefore, to be acquainted with existing
literature on the issue. This is the purpose of section 2.2. (The discussion is concerned with the standard and popular approach to the problem. This eschews potentially more interesting issues that will be commented upon in chapters 5 and 6.) A conclusion which emerges is that, despite the extent of research into the problem, the literature has little or nothing to say about model-robust policies. In chapter 3 no fewer than six different rational expectations models are employed in order to correct for this. A justification for our choice is given in section 2.3. Section 2.4 is a digression on an issue raised in the literature on the possibility of price level indeterminacy when control is administered over interest rates. Such an outcome would render this policy redundant as a viable strategy. We offer a resolution of this problem by demonstrating that there will generally exist a policy rule for which the solution of the system is uniquely determined even when the interest rate is arbitrarily fixed. The chapter concludes with section 2.5.

2.2 The Instrument Problem

Consider the stochastic IS-LM system

\begin{align*}
Y &= \beta_0 - \beta_1 R + \varepsilon_1 \\
M &= \gamma_0 + \gamma_1 Y - \gamma_2 R + \varepsilon_2
\end{align*}

\begin{align*}
\beta_i &> 0 \quad (i = 0,1) \\
\gamma_j &> 0 \quad (j = 0,1,2)
\end{align*}

where \( Y = \) real income \\
\( R = \) nominal (and real) interest rate \\
\( M = \) nominal (and real) stock of money \\
\( \varepsilon_k = \) stochastic disturbance \((k = 1,2)\)

and the upper case notation indicates that variables are not measured as deviations from long-run equilibrium. Equation (2.2.1) is the IS
curve; aggregate demand is negatively related to the rate of interest. Equation (2.2.2) is the LM curve; nominal money demand depends positively on real income and negatively on the rate of interest. The disturbance $\varepsilon_1$ is an expenditure shock; $\varepsilon_2$ represents random portfolio behaviour. We assume these to be Gaussian and independently distributed with asymptotic variance $\sigma_k^2$ $(k = 1, 2)$. The performance measure is

$$E(J) = \frac{1}{2}E(Y - Y^*)^2$$  \hspace{1cm} (2.2.3)$$

where an asterisk denotes the target value of a variable and the control problem is to minimize the value of equation (2.2.3) subject to the system in (2.2.1) - (2.2.2). The instrument is either $M$ or $R$. (Alternatively, one may question the direct controllability of the money stock and consider as a substitute the monetary base; the analysis that follows may then be applied — see, for example, Friedman (1975)).

The reduced form expressions for when the money stock or the interest rate is the instrument follow from equations (2.2.1) - (2.2.2) as

$$Y(M) = (\gamma_2 + \gamma_1 \beta_1)^{-1}(\gamma_2 \beta_o - \beta_1 (\gamma_o - M) + \gamma_2 \varepsilon_1 - \beta_1 \varepsilon_2)$$  \hspace{1cm} (2.2.4)$$

$$R(M) = (\gamma_2 + \gamma_1 \beta_1)^{-1}(\gamma_o - M + \gamma_1 \beta_o + \gamma_1 \varepsilon_1 + \varepsilon_2)$$  \hspace{1cm} (2.2.5)$$

$$Y(R) = \beta_o - \beta_1 R + \varepsilon_1$$  \hspace{1cm} (2.2.6)$$

$$M(R) = \gamma_o + \gamma_1 \beta_o - (\gamma_1 \beta_1 + \gamma_2)R + \gamma_1 \varepsilon_1 + \varepsilon_2$$  \hspace{1cm} (2.2.7)$$

where the notation emphasizes the particular policy regime. From
(2.2.4) and (2.2.6) the optimal values for $M$ and $R$, $M^*$ and $R^*$ respectively, are

$$M^* = \gamma_1^{-1}((\gamma_2 + \gamma_1^\beta_1)Y^* - \gamma_2\beta_0 + \beta_1\gamma_0) \quad (2.2.8)$$

$$R^* = -\gamma_1^{-1}(Y^* - \beta_0). \quad (2.2.9)$$

Equations (2.2.8) - (2.2.9) hold for both deterministic and stochastic cases. They are true for the latter by the certainty equivalence principle. Substituting (2.2.8) - (2.2.9) into (2.2.4) - (2.2.7) we find in particular

$$Y(M^*) = Y^* + (\gamma_2 + \gamma_1^\beta_1)^{-1}(\gamma_2\epsilon_1 - \beta_1\epsilon_2) \quad (2.2.10)$$

$$Y(R^*) = Y^* + \epsilon_1. \quad (2.2.11)$$

In the deterministic case ($\epsilon_k = 0 (k = 1, 2)$), therefore, the choice of instrument is innocuous: $Y(M^*) = Y(R^*) = Y^*$, and $R(M^*) = R^*$, $M(R^*) = M^*$ also. By contrast, the presence of stochastic shocks ($\epsilon_k \neq 0 (k = 1, 2)$) implies $Y(M^*) \neq Y(R^*) \neq Y^*$, and $R(M^*) \neq R^*$, $M(R^*) \neq M^*$ in addition. The choice of instrument is therefore non-trivial in the stochastic setting.

Poole (1970) provided the seminal contribution to the analysis of the monetary instrument problem (see also Holbrook and Shapiro (1970); Kareken (1970); LeRoy and Lindsey (1978)). His main conclusions can be obtained from equations (2.2.10) - (2.2.11) in conjunction with equation (2.2.3):

$$E(J(M^*)) = (\gamma_2 + \gamma_1^\beta_1)^{-2}(\gamma_2^2\sigma_1^2 + \beta_1^2\sigma_2^2) \quad (2.2.12)$$

$$E(J(R^*)) = \sigma_1^2 \quad (2.2.13)$$
where $E(\cdot)$ is the expectations operator. Thus, the relative merits of alternative instruments depend, in general, on the variance-covariance structure of stochastic disturbances and the structural parameters of the model. In particular, a relatively large value for $\sigma_1^2$ implies $E(J(M^*)) < E(J(R^*))$ whilst if $\sigma_2^2$ is more prominent, $E(J(M^*)) > E(J(R^*)) = 0$. In short, the optimal instrument is determined by the relative stability of the goods and money markets: controlling the money stock is optimal in the face of aggregate demand shocks; interest rate control is superior for the case of monetary volatility.

These results are explained as follows. Control over the money supply permits endogenous fluctuations in the interest rate. These variations are stabilizing for expenditure shocks, causing demand movements which partly offset the initial perturbation. By contrast, they allow monetary disturbances to spill over onto the goods market, which effect is neutralized by following an accommodative monetary policy. As shown by Poole (1970) such 'pure' policies can generally be improved upon by adopting a combination policy in which the money stock is contemporaneously related to the interest rate. The ability to respond to new information provides the reason for the superiority of this policy which encapsulates the pure policies as limiting cases\(^{(1)}\).

The above analysis has been extended in a number of important directions which overcome some of the weaknesses of the simple system in equations (2.2.1) - (2.2.2).

In a recent paper, Craine and Havenner (1981) investigate the case of supply shocks. An aggregate supply function relating output (positively) to prices is appended to equations (2.2.1) - (2.2.2) and the superiority of interest rate control is demonstrated. The reason is straightforward. Exogenous supply shocks cause incipient
movements in prices and interest rates. An accommodative monetary policy prevents interest rate fluctuations by exacerbating the initial price movement. This shifts aggregate demand and supply such that they equilibrate at the original level of output and the supply shock is completely neutralized. By contrast, and as a consequence, price volatility is greater than under money supply control. Moreover, the Poole (1970) criteria for demand and monetary disturbances carry over to the case of price stabilization. Thus, with regard to the latter two shocks, there is no conflict in optimal policies in terms of minimizing output and price fluctuations. This ceases to be true for supply disturbances. This was overlooked (or at a minimum not stressed) by Craine and Havenner (1981).

The analysis of the instrument problem in dynamic systems can be found in Poole (1970) and more rigorously in Sargent (1971) and Moore (1972). In the latter two, a modified multiplier-accelerator model is employed. The specification of policy describes lagged feedback rules some of which are in the derivative or proportional form, (see Phillips (1954)). The control problem is to minimize the value of (2.2.3) by appropriate choice of feedback parameters with the money stock and interest rate as alternative instruments. The dynamic structure may qualify the earlier results in two ways. First, the absence of lagged response in money demand appears important for the validity of these results. Second, pure policies which avoid feedback are likely to be especially unattractive when the system exhibits dynamic behaviour. This is another illustration of the view that contingent rules are generally superior to non-contingent rules by virtue of the dependency of control instruments on new information (and it always makes sense not to waste any information)(3). Once non-contingent policies are adopted, the usual
results of Poole (1970) apply. As Moore (1972) shows, the case for such rules is strengthened as the lags in the system decline. Under such circumstances, lagged feedback rules may be destabilizing because the operation of control is effectively out of phase with current perturbations.

The multiplier-accelerator model is adopted by Turnovsky (1975) in a further extension. The particular analysis by this author is tedious and the general point was demonstrated more succinctly by Kareken (1970). Both studies are motivated by the problem of control when the parameters of the system are stochastic. This encapsulates a further type of uncertainty facing the policy maker, originally explored by Brainard (1967). To incorporate this, Kareken (1970) modifies the basic structure in (2.2.1) - (2.2.2) such that $\beta_i$ ($i = 0, 1$) and $\gamma_j$ ($j = 0, 1, 2$) are random. The obvious implication is that the optimal policy is now determined not only by $\sigma_k^2$ ($k = 1, 2$) but also by $\sigma_{\beta_i}^2$ ($i = 0, 1$) and $\sigma_{\gamma_j}^2$ ($j = 0, 1, 2$), the variances of the stochastic coefficients. This is not surprising given our earlier remark concerning the relevance of parameter values for determining the optimal instrument$^4$.

As yet, the transmission mechanism of monetary policy operates exclusively through interest rate induced changes in aggregate demand. Moreover, fiscal policy has been entirely eschewed. Currie (1980c) conducts a preliminary investigation into the implications of wealth accumulation arising from endogenous shifts in the state of the government budget, which imbalances must be financed by the issue or retirement of asset stocks, either money or government debt. The IS and LM functions now have wealth as additional arguments and the economy fluctuates until stock-flow equilibrium obtains. There is a vast literature on the government budget constraint and chapter 4 addresses the issue explicitly. Suffice to remind the reader of
the well-known dynamic implications pertaining to the stability properties of the system. Thus, choosing the stock of bonds as the residual financing instrument is likely to generate cumulative explosive behaviour.

Currie (1980c) demonstrates that even in the face of expenditure disturbances, a money stock peg policy is likely to be qualified precisely because of the above possibility; money stock control requires residual budget financing by bonds so that for stability considerations, the optimal policy is likely to involve some monetary accommodation\(^{(5)}\).

An additional extension is the consideration of autocorrelated disturbances. This is not trivial for the reason that financing flows tend to be distributed over time and it is therefore of interest to consider systems which exhibit some persistence of fluctuations. Currie (1980c) shows that the degree of monetary accommodation is likely to increase with the degree of autocorrelation. Thus, as the degree of autocorrelation of expenditure disturbances falls, subsequent changes in asset holdings will be poorly correlated with the initial perturbation and will contribute additional noise to the system. The degree of monetary accommodation is determined by the private sector's optimal portfolio composition. As the degree of autocorrelation increases, however, the time phase aspects of budget financing tend to act as a stabilizing mechanism: persistent low levels of demand can be offset by residual financing through monetary expansion. In the limit, the interest rate is the control instrument and the automatic stabilizer role of money financing is fully exploited. There is, then, a reversal of the Poole (1970) criteria for expenditure shocks. (The case of money demand disturbances is not qualified.) The conclusion is important since the problem of bond financing instability is strikingly robust.
across a wide variety of divergent models. Nonetheless, the analysis by Currie (1980c) depends on the assumption that tax rates and government expenditure remain fixed. Though it is unlikely that fiscal parameters are sufficiently flexible to offset short-run fluctuations in the budget, the instability issue is really a long-run phenomenon. To the extent that fiscal parameters can eventually be altered, this may overcome the instability problem. In this respect, the analysis is rather specific and the results not without qualification.

The budget constraint plays a slightly different role in the study by Roper and Turnovsky (1980a). Assuming a continually balanced budget, the authors investigate the optimum monetary aggregate. To be clear, a general monetary aggregate, $Q$, is defined as a weighted sum of individual monetary assets. For the IS-LM system this has the form $Q = \lambda M + (1 - \lambda)PB$, $\lambda \neq 0$, where $B$ is the stock of government bonds and $P$ is their price. The control problem requires choosing the appropriate composition of $Q$ which involves optimizing with respect to $\lambda$. It is shown that $\lambda = \frac{1}{2}$ is optimal for monetary shocks and that this is inferior to $\lambda = 1$ in the face of demand shocks. The parallel with the standard analysis is that $\lambda = \frac{1}{2}$ is an interest rate peg and $\lambda = 1$ is a money supply peg. It is difficult to see how this analysis could be extended to the case of many assets including non-government monetary assets; for then both the absolute value and composition of $Q$ are no longer decision variables of the policy maker. Moreover, assuming a continually balanced budget abstracts from the financing problems discussed above and its relaxation makes the composition of $Q$ endogenous. For these reasons, this alternative approach to the instrument problem appears rather inferior to conventional analysis.
All of the foregoing research, with the exception of one, abstracts from expectations. The exception is Sargent (1971) but in this, expectations are merely extrapolations from the past. Craine and Havenner (1981) do address the issue but ignore it on the grounds that it is irrelevant for the instrument problem. Jansen (1984), however, shows the proposition to be true only when expectations are predetermined (that is, conditioned on a lagged information set). This is acknowledged subsequently by Craine and Havenner (1984).

In the classic paper by Sargent and Wallace (1975) the very existence of an instrument problem was questioned, even in a stochastic setting. The precise details of their model are described in section 2.3 and formalised in chapter 3 (section 3.2, equations (3.2.1a), (3.2.2a), (3.2.3a), (3.2.4) - (3.2.5)). Briefly, the IS-LM system is augmented by a supply function and flexible prices. Price expectations are crucial. When these expectations are assumed to be formed adaptively, the control problem is in the standard form. Substituting in rational expectations alters the problem drastically: first, interest rate control generates price level indeterminacy thus, second, the optimal policy inclines towards some money supply rule; but, third, the distribution of real variables are independent of any deterministic feedback rule; therefore, fourth, one might as well peg the money stock which eases computational difficulties for the private sector.

The third and fourth points have been touched upon in chapter 1 (section 1.1) and will receive further attention in section 2.3. The first is explained as follows. When control is administered over the interest rate, the money stock is the equilibrating variable in the money market. This is impounded in agents' expectations. To be sure, they know that whatever the price level is, the money supply
will adjust in order to ensure money market equilibrium; but then there are an infinity of price distributions which are entirely consistent with money market clearing. There is no anchor for price expectations and actual prices are therefore indeterminate.

Clearly, the conclusions of Sargent and Wallace (1975) imply a drastic re-evaluation of the instrument problem. They are not, however, without qualification. For the present, it is merely noted, first, that the policy ineffectiveness proposition is far from general and, second, that indeterminacy under an interest rate policy is not unequivocal. In fact, section 2.4 is concerned precisely with demonstrating the general determinacy of the system even under an interest rate peg when policy is specified appropriately. Thus, the instrument issue is still a non-trivial control problem.

In a Sargent and Wallace (1975) type framework, Woglom (1979) addresses explicitly the control problem under rational expectations. Though the early results are shown to be robust, the analysis offers little more to the literature: neither supply shocks nor price stabilization are dealt with. By contrast, Turnovsky (1980a) conducts a fairly rigorous investigation. The main innovation here is the assumption of differential information sets. Thus, whilst suppliers formulate expectations conditional on lagged information, investors have access to contemporaneous price information. The formal structure of the model is in all other respects identical to that of Sargent and Wallace (1975) and is set out in chapter 3 (section 3.2, equations (3.2.1c), (3.2.2c), (3.2.3c), (3.2.4) – (3.2.5)).

The usual results for expenditure and money demand shocks are valid. This is not true, however, for supply disturbances. Though a conflict of optimal policies (with respect to output and price stabilization criteria) still exists, the conflict is no longer
unambiguous. On the contrary, it is now possible that a money supply (interest rate) peg is optimal for output (price) stability. This is explained in terms of the information structure and is explicated in chapter 3 (section 3.3). The analysis is particularly interesting for this reason. Nonetheless, it is not entirely without criticism to which we return in part II of the thesis (chapter 5), which focus of attention is on the information structure.(7).

Three further papers dealing with rational expectations are those of Benavie (1983), Siegel (1983) and Fethke and Jackman (1984). The emphasis of Siegel (1983) is on the combination policy and Fethke and Jackman (1984) examine optimal wage indexation. For our purposes, nothing further is gained from either of these. Of more importance is the study by Benavie (1983) who shows a qualification of certain of the earlier results. The analytical framework is given in chapter 3 (section 3.2, equations (3.2.1b), (3.2.2b), (3.2.3b), (3.2.4) - (3.2.5)) and discussed informally in section 2.3. Some drawbacks of Benavie (1983) motivate us to investigate the issue in more detail.

It was remarked in section 2.1 that there is an alternative monetary instrument available, namely the exchange rate. Moreover, open economy models introduce additional noise by virtue of the impact of foreign or external disturbances. The control problem is therefore broadened in two respects.

Preliminary investigations using an open economy framework can be found in Turnovsky (1976), Parkin (1977), Boyer (1978) and Roper and Turnovsky (1980b). These consider the choice between fixed exchange rate (endogenous money supply) and floating exchange rate (exogenous money supply) regimes. Several criticisms, either individually or collectively, can be made of these studies. In Turnovsky (1976) and Parkin (1977), the relative merits of alternative policies are evaluated with reference to just the initial impact effect of exogenous
shocks. For the latter, this is unimportant since the model is entirely static. Emphasis on comparative statics by Turnovsky (1976), however, is more serious, given the dynamic configuration of the model. In addition, both Turnovsky (1976) and Boyer (1978) abstract from expectations which is probably of more importance in the former than in the latter whose framework is again static. Nonetheless, the literature on exchange rate dynamics clearly indicates the need to incorporate exchange rate expectations. A further limitation of Boyer (1978) is the restriction to just domestic disturbances. Though this means a direct comparison with Poole (1970) it has only limited appeal. Roper and Turnovsky (1980b) include foreign shocks in a model similar in spirit to that of Boyer (1978). Unfortunately, prices are still fixed. In spite of these drawbacks, however, these analyses yield some common results, which are worthwhile noting.

Boyer (1978) and Roper and Turnovsky (1980b) find that flexible exchange rates are optimal (in terms of output stabilization) in the face of domestic expenditure shocks. Turnovsky (1976), however, points out that this is critically dependent on the degree of capital mobility. Thus, if capital is highly mobile, the implications of demand shocks for the exchange rate focus on the capital account. Under an exogenous money stock policy, the exchange rate fluctuates and these fluctuations are stabilizing. If exchange rates are fixed, the ensuing change in reserves is destabilizing. These effects are reversed when the degree of capital mobility falls and the exchange rate response is then determined by the trade balance; for under such circumstances, the exchange rate response observed under perfect capital mobility is reversed. In the case of domestic money demand shocks, all three analyses reach the conclusion that a fixed exchange rate policy is superior because of the accommodating money supply
movements which prevent spillover onto the goods market. Turnovsky (1976) also examines a variety of foreign disturbances and indicates that the output response will be lower under flexible exchange rates for foreign trade and foreign price disturbances whilst the effect of shocks to capital flows is rather more ambiguous. Nonetheless, we are reluctant to put too much weight on the implications of these for the overall volatility of the system (10), (11).

Parkin (1978) and Weber (1981) employ an open economy Sargent and Wallace (1975) type framework. The latter can be considered a limiting case of the former. In fact, the emphasis in Parkin (1978) is on the precise mechanism of monetary control. Thus, the money supply ceases to be an instrument and becomes an intermediate target; control over the money stock is administered by some other variable such as the monetary base or interest rate. A number of control mechanisms are considered and the reader is referred to Parkin (1978) for a thorough discussion. Some general observations are as follows.

The results for expenditure and monetary shocks are as before. For supply volatility, and under perfect capital mobility, Weber (1981) shows the dependency of optimal policy on the income sensitivity of money demand and real exchange rate sensitivity of expenditure. Relatively high values for both of these favour money stock control. To see this, note that a supply disturbance induces opposite output and price responses. If money demand is sensitive to income fluctuations, interest rates tend to move in the same direction as the supply shock. This exerts a stabilizing effect. This is compounded by exchange rate movements. The more open is the economy, the greater the stabilization through the second channel. Exchange rate control has the obvious implications. Parkin (1978) notes similar complications for supply disturbances. In addition, this author shows that the domestic economy can be insulated entirely
from external shocks if exchange rates are pegged. This may appear somewhat surprising but is easily explained by the fact that, in this model, exchange rate control is coupled with either interest rate or monetary base control; the induced change in the stock of reserves can therefore have no consequences for output or prices.

Money supply and exchange rate control are the central elements in Artis (1981) and Artis and Currie (1981). The major contribution is the re-specification of supply behaviour which envisages the importance of cost-push elements in determining prices. Thus, import prices and exchange rate fluctuations feed through to domestic prices via a mark-up on domestic costs of production. Wages, in turn influenced by domestic prices, react back onto the system. The exchange rate-price-wage nexus provides the anchor for the results. Perfect capital mobility is assumed throughout.

Pegging the money stock is optimal for output stabilization in the face of demand shocks. In terms of price stability, however, the optimal policy depends critically upon the degree of openness of the economy. If the closed loop network of exchange rates, prices and wages is strong, optimal policy may incline towards an exchange rate peg in order to break the vicious circle. Needless to say, the implication of monetary shocks is unqualified. Aggregate supply disturbances exhibit the usual complications. When competitiveness effects an aggregate demand are important, a money supply peg is likely to exacerbate the initial price movement whilst the converse is true if the economy is relatively closed. Though shocks to capital flows deliver similar ambiguities, money stock control is unambiguously superior for foreign price disturbances.

A conclusion of Artis and Currie (1981) is the ambiguity of the comparative advantage of one policy over the other; more provocatively, exchange rate control appears just as good as money supply
control. In view of this, and the undesirable consequences of exchange rate volatility, the authors are led to advocate a policy regime which conditionalizes monetary policy on the behaviour of the exchange rate. The type of conditionality envisaged by Artis and Currie (1981) is encapsulated in their policy rule specifying a contemporaneous relationship between the money stock and the exchange rate. A much stronger form of conditionality takes the form of zones: money stock control is administered provided exchange rate fluctuations remain within certain pre-specified bands. Incipient exchange rate movements beyond these bands provoke abandonment of the monetary policy in order to alleviate exchange rate volatility. Though at first glance this might appear a viable strategy, it raises several crucial issues to which we are sympathetic. A complete and rigorous appraisal is beyond the scope of the thesis. It is useful, however, to note the salient points. This will serve to clarify the type of conditionality envisaged in Artis and Currie (1981) (and the majority of the monetary instrument literature) and the stronger type discussed above.

First, and most obvious, it is no good for the authorities to announce just the conditions under which the particular monetary regime would be abandoned. As a pre-requisite for this course of action, they must also disseminate information about the subsequent regime. In its absence, the conditional policy would serve only to increase the amount of confusion and encourage speculation about future policy. Second, it is not at all obvious the way in which the upper and lower limits of exchange rate variation would be determined. This is not just expressing caution over the use of arbitrary standards and the operational content of the policy. To be sure, there are more substantive problems. Presumably, the reason for adopting the conditional policy is to limit exchange rate
fluctuations. For this purpose, narrow bands are motivated; but then this may well require frequent shifts in policy regime to which the private sector must adapt quickly. Third, there is the likelihood that private speculation would be encouraged as agents observe the exchange rate approach its critical value. This, in turn, may generate large swings in competitiveness which is what the policy set out to prevent.

All the above is to say that the strong form of conditional policy is unlikely to be feasible. There is a very real difference between making control variables and control rules contingent on new information. We are sure that the above arguments could be formalized, though this is not attempted here. It would certainly represent an interesting line of enquiry.

For further discussion of money stock and exchange rate control the reader is referred to the fairly comprehensive, though non-analytical, study by Blundell-Wignall and Chouraqui (1983). Supplements to the analytical investigations using larger econometric models can be found in Currie and Karakitsos (1981, 1983) and Artis and Karakitsos (1982).

In concluding the section we make the following observations concerning the relative merits of alternative monetary instruments.

First, money supply control is generally superior to either interest rate or exchange rate control in the face of expenditure disturbances. This may be qualified in two main ways: sufficiently immobile international capital and a cost mark-up price determination reflecting foreign price and exchange rate pressures.

Second, for monetary volatility there is an unambiguous bias in favour of interest rate or exchange rate control. Indeed, accommodating these shocks prevents any spillover effects onto the goods market.
Third, supply and foreign disturbances appear to be rather less definitive in their implications for optimal policy.

Fourth, interest rate and exchange rate control are more closely related as the degree of capital mobility increases. Controlling the money stock implies equilibrating movements in the interest rate. The consequences of these are reinforced by the exchange rate.

Despite the apparent preoccupation with the issue, however, any conclusion regarding a model-robust policy (or lack thereof) is vacuous. To be sure, it is surprising that any consistency to be found in the variety of results is not more visible; for a major drawback of the existing literature is the similarity of the models employed. Thus, ignoring those analyses which abstract from rational expectations, the reader is invited to compare the remainder (Parkin (1977, 1978); Woglom (1979); Turnovsky (1980a); Weber (1981); Siegel (1983); Fethke and Jackman (1984)) (12). This is not trivial. It means that the payoff matrix associated with this literature has a negligible information content. In view of this, and the continued interest in the issue, chapter 3 performs a systematic and fairly comprehensive re-evaluation of the instrument problem. To do this, six models are employed with significantly divergent structures. As a prelude, section 2.3 that follows contains an informal discussion of the analytical framework and justifies the choice of models (13).

2.3 An Analytical Framework for a Non-Irrelevant Payoff Matrix

Early literature on rational expectations culminated in the well-known 'policy-ineffectiveness debate' (McCallum (1979b)). The proposition derived by Lucas (1972b, 1975), Sargent (1973), Sargent and Wallace (1975, 1976) and Barro (1976) was that systematic (and hence anticipatory) monetary policy is impotent in terms of influencing the distribution of real economic variables. Subsequent
research showed the proposition to be valid under only special assumptions. The debate has been amusingly characterized by Tobin (1980a) as 'the algebraic war about policy-ineffectiveness'. Rational expectations per se had no possibility of resolving the issue of neutrality, and the importance of distinguishing between the rational expectations assumption and the model in which it is embedded is now fully acknowledged (as emphasized, for example, in Laidler (1982)). We choose, as one basis for our robustness tests of the optimal instrument, the stylised models that emerged from the debate.

It is unnecessary to review the literature in any detail here. This has occupied the attention of many other authors and the issues involved should be familiar. Excellent surveys can be found in McCallum (1979b, 1980), Buitel (1980a,b, 1981b), Begg (1982a), Minford and Peel (1983b) and Sheffrin (1983) (see, also, Modigliani (1977); the symposium in the 1980 edition of the Journal of Money, Credit and Banking). Rather, we confine attention to certain aspects of the debate which are most relevant for our particular purposes. Thus, we are concerned with stabilization policy and weak neutrality, not strong neutrality (see, for example, Begg (1980); Hahn (1982) in addition to the aforementioned surveys). Moreover, and as usual, the models that we consider are linear (see, for example, Snower (1984)) and the implications of learning and convergence to rational expectations equilibria are abstracted from (see, for example, Taylor (1975); Shiller (1978); Friedman (1979)). The following preambles concentrates on four of the more well-known arguments against the orthogonality proposition. Anticipating somewhat, the orthogonality proposition rests on the following assumptions. First, there is market clearing and no institutional rigidities (all agents possess the same opportunity sets). Second, any disappointment in
plans is the result of information imperfections. In particular, supply decisions are motivated by perceived relative price changes. Third, all agents have identical information sets. Five of our chosen models are motivated by these arguments. In any event, the efficacy of stabilization policy is inevitable in our framework even for those models in the market-clearing tradition. This is by virtue of the description of policy which, in accordance with the monetary instrument literature, assumes a contemporaneous relationship between the money stock and the interest rate, whilst the private sector acts on the basis of lagged information.

Our first choice of model is that of Sargent and Wallace (1975) in which the demonstration of the neutrality proposition was simpler mundiss. The essential feature of this (and the models of Lucas (1972b, 1975), Sargent (1973), Sargent and Wallace (1976) and Barro (1976)) is the description of supply behaviour for a representative producer of a commodity in a market, \( h \). A more detailed description of this can be found in chapter 5 (section 5.2). Measuring variables as deviations from long run equilibrium, this has the formal structure:

\[
y^s_t(h) = \alpha(p_t(h) - p^{eh}_{t,t-1})
\]

(2.3.1)

where \( y(h) = \) natural logarithm of the real supply of the commodity in market \( h \).

\( p(h) = \) natural logarithm of the price of the commodity in market \( h \).

\( p = \) natural logarithm of the aggregate price level.

The term \( p^{eh}_{t,t-1} = \mathbb{E}(p_t|\mathcal{N}_{t-1}(h)) \), where \( \mathbb{E}(\cdot) \) is the mathematical conditional expectations operator and \( \mathcal{N}(h) \) is the
information set in market h. The motivation for equation (2.3.1) is, of course, that supply variations are the result of misperceived changes in relative prices\(^{(14)}\). Making the appropriate assumptions, equation (2.3.1) is aggregated over h to yield an aggregate supply function

\[
y'_t = \alpha(P_t - P_{t,t-1})
\]

where \(P_{t,t-1} = E(P_t | P_{t-1})\), the average expectation of the aggregate price level. Systematic feedback monetary policy is irrelevant for output here because the implications of this policy are entirely impounded in \(P_{t,t-1}^e\). By suitable re-specification of the model structure, however, effective stabilization policy can be shown to be entirely consistent with rational expectations.

Consider, first, the aggregate supply hypothesis in equation (2.3.2). A popular alternative derived by Lucas and Rapping (1969) (see, also, Sargent (1979); Benavie (1983); Minford and Peel (1983b)) is, in its most general form,

\[
y'_t = \alpha_1(P_t - \alpha_2 P_{t+1,t-1}) + \alpha_3(r_t - P_{t+1,t-1} + P^e_{t,t-1}) \]

\[
- \alpha_4(m_t - P_t)
\]

where \(r = \) nominal rate of interest
\(m = \) natural logarithm of the nominal stock of money.

Equation (2.3.3) is motivated in terms of the speculative inter-temporal aspects of labour supply in conjunction with a labour demand derived from a present value maximizing competitive firm. The signs of the parameters \(\alpha_j\) \((j = 2, 3, 4)\) are, in general, ambiguous because
of the income and substitution effects. Typically, however, it is assumed \( \alpha_j > 0 \) \((j = 2, 3, 4)\). It is also common to set \( \alpha_2 = 1 \) and \( \alpha_3 = \alpha_4 = 0 \). Systematic stabilization policy is effective in this framework because the supply function (2.3.3) breaks the block-recursiveness of the model.

The limiting case of \( \alpha_2 = 1, \alpha_3 = \alpha_4 = 0 \) has been criticized by a number of authors. McCallum (1978b) shows that neutrality re-obtains if future prices are appropriately discounted. Further modification by Minford and Peel (1981) shows the McCallum (1981b) result to be ephemeral. More importantly is the criticism by Fair (1978) who questions the combination of rational expectations with non-rational models; since there is no a priori reason for imposing \( \alpha_2 = 1, \alpha_3 = \alpha_4 = 0 \), to do so is ad hoc and departs from the attractive optimizing characteristics of equation (2.3.3). In a recent paper, Benavie (1983) demonstrates the importance of such restrictions in the monetary instrument problem. There are, however, some drawbacks of his analysis which motivate a more rigorous appraisal of the issue. Thus, this constitutes our second choice of model. To be sure, unlike Benavie (1983), we permit \( \alpha_4 \neq 0 \). This has some important implications which the former author necessarily overlooked. In addition, the precise transmission mechanism is outlined in greater detail. Finally, we continue to adopt the more popular approach by dating expectations at time \( t-1 \) (in contrast to Benavie (1983) in which the conditioning date is \( t' \)).

The existence of differential information sets is a second well-known way of obtaining policy effectiveness. This was pointed out in the original contributions by Sargent and Wallace (1975) and Barro (1976) in the context of an informational advantage in favour of the policy maker. There are, however, some features of this which render it a questionable base upon which to advance control rules.
Sargent and Wallace (1975) point out that in order for control to have its planned effects the controller must presumably know the precise form of the differential information structure. This is a general point pertaining to such information structures. A more substantive issue raised by Barro (1976) is that the controller could presumably achieve the same outcome by divulging any extra information he has to the private sector. The attractiveness of this would depend on the costs of making this information known and any difficulties for the private sector in processing, interpreting and making effective this information (relative to those incurred by the controller). Studies which avoid this particular information structure but retain the essential characteristics of differential information were originally conducted by Peel and Metcalfe (1979), Turnovsky (1980a) and Weiss (1980, 1982) (see, also, King (1982, 1983); Anderson (1983)). In these, the efficacy of stabilization policy emerges as a result of differentially informed private agents. In Peel and Metcalfe (1979) this takes the form of divergent expectations mechanisms (adaptive expectations in the labour market, rational expectations in financial markets) and this begs rather more questions than any that it answers. The most satisfactory analyses are those by Weiss (1980, 1982), King (1982, 1983) and Anderson (1983). We return to these in part II of the thesis (chapter 5).

We choose Turnovsky (1980a) as the third model partly because of its simplicity, and partly because the analysis based on which addresses the monetary instrument problem explicitly (see section 2.2). The innovation, it will be recalled, is the conditioning of investors' expectations at time \( t \). In terms of the IS curve,

\[
y^d_t = -\beta (x_t - p^e_{t+1,t} + p_t) \quad \beta > 0
\]  

(2.3.4)
where \( y^d \) = natural logarithm of real aggregate demand.

All other expectations in the model are conditioned on a lagged information set. Systematic policy is effective here because some expectations are no longer predetermined. The real interest rate is observed to fluctuate not only via variations in the nominal interest rate, but also through discrete jumps in actual and expected prices. Aggregate supply, however, is still described by equation (2.3.2) in which expectations are still predetermined. This permits the consequences of additional information about the real interest rate to induce unanticipated price movements. The particular specification (2.3.4) is subject to some criticisms. We remark on these in part II of the thesis (chapter 5).

The fourth choice of model is motivated by the negation of the policy ineffectiveness proposition when there exists non-market clearing or institutional rigidities. To some, the absence of these is the most objectionable feature of models which exhibit policy ineffectiveness (see, for example, Grossman (1982) for a critical appraisal). Phelps and Taylor (1977) showed that if firms set prices in advance of the period in which they apply, systematic monetary policy is able to influence the volatility of output. Subsequently, this occupied a fairly lively debate involving the construction of example and counter-example of the proposition (see, for example, McCallum (1977, 1978a, 1979a); Green and Honkapohja (1983); Nickerson (1984)).

In contrast to the above, rather less controversy has surrounded the popular analysis of Fischer (1977a) which interest was directed towards wage inertia. In this, wage stickiness is the outcome of long-term labour contracts which are both overlapping and non-contingent; at any time, at least part of the labour force is covered by pre-existing contracts which are independent of current news. The
stylized supply function for the two-period contract case has the well known form\(^{(17)}\)

\[
y^s_t = \frac{1}{2}(P_t - P^e_{t,t-1}) + \frac{1}{2}(P^e_t - P^e_{t,t-2})
\] (2.3.5)

which is the fourth choice of model. Clearly, non-neutrality obtains because some workers are locked in to pre-commitments which the policy maker is able to exploit. For one period contracts, of course, the supply function (2.3.5) reduces to just equation (2.3.2).

A criticism of the Fischer (1977a) framework relates to the persistence properties. An intrinsic feature is that exogenous shocks persist for only as long as the longest outstanding contract, after which time renegotiation nullifies any further persistence. This is at variance with the stylized facts of highly serially correlated movements in output and employment. A more pronounced disequilibrium framework motivates our fifth choice of model. The seminal contributions of Taylor (1979, 1980) combine the staggering of contracts with the assumption that one objective of workers is to maintain their relative wage positions. A contract wage for one particular cohort of workers then depends on previously negotiated contracts of other cohorts (which are still in operation), forecasts of future contracts of other cohorts (to be negotiated during the lifetime of the particular cohort’s contract), and expected excess demand pressure. Clearly, exogenous shocks persist even after the longest outstanding contract here because they are passed through a succession of contracts as workers attempt to maintain their relative wage position. This contrast, and the implications thereof, between the Fischer (1977a) and Taylor (1979, 1980) paradigms is stressed by Canzoneri and Underwood (1982) in their analysis of exchange rate dynamics.
The particular version of the disequilibrium contract framework that we adopt is that of Calvo (1981, 1982, 1983). Following Taylor (1979, 1980), the current contract price of a firm's output is a forward-looking moving average with exponentially declining weights of expected future aggregate prices and excess demand:

$$q(t) = \theta_1 \int_{t}^{\infty} (p(s) + \theta_2 y(s))e^{-\theta_1(s-t)} ds$$

(2.3.6)

where $q = $ natural logarithm of the contract price.

By contrast, actual aggregate prices are assumed to be a backward-looking exponentially declining moving average of past contract prices:

$$p(t) = \theta_1 \int_{t}^{\infty} q(s)e^{-\theta_1(t-s)} ds$$

(2.3.7)

Differentiating equations (2.3.6) - (2.3.7) with respect to time then yields the relevant structural equations,

$$dq = \theta_1(q - p - \theta_2 y) dt$$

(2.3.8)

$$dp = \theta_1(q - p) dt$$

(2.3.9)

(For some criticisms of the contract framework and further discussion, see, for example, Barro (1977); Fischer (1977).)

The final model that we consider is an open economy. Following Dornbusch (1976), uncovered interest parity determines capital flows. Unlike this author, however, we depart from fixed output and permit imperfect capital mobility (some analyses, related to ours, examining the implications of these can be found in Driskill and McCafferty (1980); Turnovsky and Bhandari (1982); Papell (1983)). The resulting framework is similar to those to be found in Bhandari (1981),
Bhandari, Driskill and Frenkel (1984), Driskill (1981), Frenkel and Rodriguez (1982) and Gazioglou (1984) in their analysis of exchange rate overshooting. It will be seen in chapter 3 (section 3.3) that the model exhibits some interesting features. In particular, it is not the degree of capital mobility per se which is important for dynamic stability but rather the relative importance of trade balance parameters. This is reflected in some possible perverse exchange rate responses to domestic price movements and is explained by the forward-looking behaviour of the private sector.

Chapter 3 (section 3.2) formalizes each of the above six models. The control problem is defined as minimizing some loss function subject to the constraints of these models. The choice variable is the parameter in the control rule which specifies a contemporaneous relationship between the nominal money stock and the nominal rate of interest. By virtue of this specification, it is necessarily the case that there is policy effectiveness, even in the Sargent and Wallace (1975) model. Moreover, encapsulated in this control rule are the polar cases of a money supply and an interest rate peg. One might also consider the alternative rule which embodies exchange rate control. We choose to focus on the money stock-interest rate rule partly because it is this which has received most attention in the literature and partly because of our own reservations about the viability of exchange rate control.

The purpose of chapter 3 is to test the optimality of different policies across the six models. By virtue of the significant differences between these, this yields a payoff matrix, the information value of which about model-robust policies is non-negligible. This is in contrast to the existing literature.
2.4 A Digression on Price Level Indeterminacy under Interest Rate Control: A Resolution of the Problem

It will be recalled from section 2.2 that Sargent and Wallace (1975) raise a serious objection to the pursuit of policies designed to control the interest rate. To be sure, an arbitrary choice of interest rate rule (such as a pegged interest rate) generates an indeterminate equilibrium price level (and an indeterminate equilibrium money stock) in models with forward-looking expectations. This is explained in terms of autonomous changes in expectations arising from private sector foresight of multiple money market equilibria associated with an accommodative monetary policy. Clearly, this would rule out an interest rate policy for the purpose of stabilization policy. Turnovsky (1980a) eschews the problem of indeterminacy by invoking stability and obtaining a finite price variance. In terms of price stabilization, therefore, the merits of an interest rate policy may still be evaluated. This seems rather peculiar for, as the author acknowledges, the equilibrium price level remains indeterminate and it is not clear how one is able to measure the variance around a non-unique equilibrium. Taylor (1977) observes that a finite (infinite) price variance and stability (instability) is true only in stochastic models for which the mean of the price level is well-defined. In any case, the fact that the price level is indeterminate under an interest rate policy would seem to rule out this policy as a viable strategy. Given this, it is not surprising that various attempts have been made to resolve this indeterminacy. These may be summarized briefly as follows (for further discussion, see McCallum (1984)).

Note first that the problem of indeterminacy would vanish if some variation in the interest rate were to be permitted. Thus, McCallum (1984) (see also Blackburn (1984a)) shows that a unique solution obtains under near, but not absolute, smoothing of the
interest rate (we show how this can be incorporated into our framework that follows). Though the outcome under such a policy would approximate a pure interest rate peg, the fundamental problem remains, namely the indeterminacy associated with absolute fixing of the interest rate. Attempts to resolve this problem appeal to some consistency requirement in the setting of interest rate and monetary targets. McCallum (1981) identifies a possible resolution of the indeterminacy problem by envisaging an interest rate policy with the view to having some specified effect on future money. Specifically, the target for the interest rate is set such as to make the expected value of the money stock equal to some desired level. Then expectations are tied to the monetary target and the equilibrium is well-defined. Dotsey and King (1983) similarly suppose that monetary expectations be consistent with a predeter- mined monetary target. In this case, however, the justification is that policy is revised sequentially and, in particular, an interest rate peg is defined for only one period. After this period has elapsed, the authorities re-establish the monetary target. Canzoneri, Henderson and Rogoff (1983) assume a given initial value for the money stock. In conjunction with the interest rate peg, this determines the initial mean price level. Then subsequent trend money is to be consistent with the initial condition and the interest rate peg\(^{(19)}\). As above, this provides an anchor for price expectations.

These attempted resolutions aim to deal at best with the problem of price level indeterminacy only for the case where the authorities are careful in their choice of interest rate peg, ensuring consistency between the choice of peg and the desired long-run money stock; but even in this, they are not successful. Even if initial conditions are chosen carefully, there is nothing to prevent a self-fulfilling
shift in private sector expectations concerning the expected long-run price level. So long as the authorities peg the interest rate, no matter at what level, such a shift will call forth an accommodating change in the money supply, underwriting the shift in expectations. Since this may occur at any instant in time, the price level is fundamentally indeterminate under an interest rate rule, irrespective of the choice of initial conditions.

In this section we show that there generally exists a policy rule which admits a unique well-defined solution even under an arbitrarily selected value for the interest rate. This policy comprises two components. The first is the usual contemporaneous relationship between the interest rate and the money stock. The second is the novel aspect of our analysis and involves a feedback on the forward price level expectations of the private sector. This serves to anchor expectations. In this second term there is also a pre-specified price level target. We show that, under such a rule, the long-run mean price level is equal to the authorities' price target and is therefore determinate. Moreover, along the unique convergent path to equilibrium, this policy rule is observationally equivalent to the simple contemporaneous feedback rule between the interest rate and money stock.

We illustrate our results using stripped-down versions of two well known rational expectations models. Section 2.4(A) contains a static model for which our results are most easily realised. Section 2.4(B) then generalizes the analysis for a dynamic model. This section also contains some intuition behind our results.

2.4(A) A Static Model

In this section we consider the following fixed output variant of the model of Sargent and Wallace (1975), as employed by
McCallum (1981, 1984)\(^{(20)}\):  

\[ y^s_t = \bar{y} + \epsilon_{1t} \]  

(2.4.1)  

\[ y^d_t = \beta_0 - \beta_1 (R_t - R^e_{t+1} + P_t) + \epsilon_{2t} \]  

\[ \beta_i > 0 \quad (i = 0, 1) \]  

(2.4.2)  

\[ M^d_t = \gamma_0 + P_t - \gamma_1 R_t + \gamma_2 Y_t + \epsilon_{3t} \]  

\[ \gamma_j > 0 \quad (i = 0, 1, 2) \]  

(2.4.3)  

\[ y^s_t = y^d_t = y^d_t \]  

(2.4.4)  

where  

\[ y^s \] = natural logarithm of aggregate supply  

\[ y^d \] = natural logarithm of aggregate demand  

\[ R \] = nominal rate of interest  

\[ P \] = natural logarithm of the aggregate price level  

\[ M^d \] = natural logarithm of the nominal demand for money  

\[ \epsilon_k \] = stochastic disturbances \((k = 1, 2, 3)\)  

and bars over variables indicate long-run mean values (hence the upper case notation). The term \( P^e_{t+1} \) is defined by \( P^e_{t+1} = E(P_{t+1} | \Omega_t) \) where \( E(\cdot) \) is the mathematical conditional expectations operator and \( \Omega \) is the information set. Stochastic disturbances, \( \epsilon_k \) \((k = 1, 2, 3)\) are assumed to be independently distributed white noise error terms.  

Equation (2.4.1) is an aggregate supply schedule with output equal to its mean value except for random error, \( \epsilon_{1t} \). Equation (2.4.2) is the aggregate demand schedule where aggregate expenditure depends negatively on the real rate of interest plus an exogenous shock, \( \epsilon_{2t} \). Equation (2.4.3) is a standard money demand function with the demand for nominal balances depending positively on prices
and income and negatively on the nominal interest rate. Random portfolio behaviour is captured by $\varepsilon_3$. Equation (2.4.4) gives the goods market equilibrium condition.

To close the model we introduce a policy rule, equation (2.4.5), and the equilibrium condition for the money market, equation (2.4.6):

$$R_t - \bar{R} = \phi_1(M_t - \tilde{M}) + \phi_2(P^e_{t+1}, t - \tilde{P})$$  \hspace{1cm} (2.4.5)

$$M^s_t = M^d_t = M_t$$  \hspace{1cm} (2.4.6)

where $M^g$ is natural logarithm of the nominal supply of money and tildes over variables denote target long-run values chosen by the authorities. Equation (2.4.5) is the main innovation of our analysis. It states that policy is comprised of two elements. The first is the standard contemporaneous relationship between the interest rate and the money stock with $\phi_1 = 0$ representing an arbitrary interest rate peg, $R_t = \bar{R}$. The second component of policy is a feedback on the deviation of forward price expectations from some price level target. The approximate solution alluded to earlier amounts to assuming $\phi_2 = 0$ in which case the mean price level is indeterminate if $\phi_1 = 0$. Uniqueness can, however, be obtained for $\phi_1$ arbitrarily close, but not equal, to zero. That is, near but not absolute smoothing of the interest rate is a well-specified policy.

In what follows we show that a non-zero value for $\phi_2$ is sufficient to eliminate indeterminacy even when $\phi_1 = 0$. When $\phi_1 \neq 0$, then $\phi_2$ can be zero as suggested by the literature. In both cases, the long-run mean price level is equal to the price level target, $\bar{P} = \tilde{P}$. It is then straightforward to show that equation (2.4.5) is observationally equivalent to the simple rule.
To solve the system (we exclude unstable solutions), we first eliminate $M_t$ from equation (2.4.5) using equations (2.4.3) and (2.4.6). Then $Y_t$ is eliminated using equations (2.4.1) and (2.4.4). This gives

$$R_t = (1 + \phi_1 \gamma_1)^{-1}((\tilde{R} + \phi_1(\gamma_0 + \gamma_2\bar{Y} - \bar{M}) - \phi_2 \bar{P})$$

$$+ \phi_1 \bar{P}_t + \phi_2 \bar{P}_{t+1} + \phi_1(\gamma_2 \epsilon_{1t} + \epsilon_{2t})).$$

(2.4.8)

An alternative expression for $R_t$ can be derived from equations (2.4.1) - (2.4.2) as

$$R_t = \bar{P}_{t+1} - \bar{P}_t + \beta_1^{-1}(\bar{P}_o - \bar{Y} + \epsilon_{2t} - \epsilon_{1t})$$

(2.4.9)

so that equating equations (2.4.8) - (2.4.9) yields

$$(1 + \phi_1 \gamma_1 - \phi_2)\bar{P}_{t+1} + \bar{P}_t = (1 + \phi_1 \gamma_1 + \phi_1)\bar{P}_t + \nu_t$$

$$+ (\tilde{R} + \phi_1(\gamma_0 + \gamma_2\bar{Y} - \bar{M}) - (1 + \phi_1 \gamma_1)\beta_1^{-1}(\bar{P}_o - \bar{Y}) - \phi_2 \bar{P})$$

(2.4.10)

where $\nu_t = (\phi_2 \gamma_2 + (1 + \phi_1 \gamma_1)\beta_1^{-1})\epsilon_{1t} - (1 + \phi_1 \gamma_1)\beta_1^{-1}\epsilon_{2t} + \phi_1 \epsilon_{3t}$.

Now note from equation (2.4.9) that long-run equilibrium implies

$$\bar{R} = \beta_1^{-1}(\bar{P}_o - \bar{Y}).$$

In addition, given $\bar{R}$ and $\bar{Y}$, equation (2.4.3) implies that the authorities' targets for $M$ and $P$ are related through the relationship $\tilde{M} = \gamma_0 + \bar{P} - \gamma_1 \bar{R} + \gamma_2 \bar{Y}$. Then equation (2.4.10) can be re-written as
\[(1 + \phi_1 \gamma_1 - \phi_2)P^e_{t+1, t} = (1 + \phi_1 \gamma_1 + \phi_1)P_t \nu_t - (\phi_1 + \phi_2)\tilde{P} \tag{2.4.11}\]

which, on taking means, gives

\[(\phi_1 + \phi_2)(\bar{P} - \tilde{P}) = 0 . \tag{2.4.12}\]

Equation (2.4.12) gives our main result. It shows that provided \(\phi_1 + \phi_2 \neq 0\), \(\bar{P} = \tilde{P}\); the mean price level is determinate and equal to the authorities' price level target. The limiting case of interest is \(\phi_1 = 0\). Under such circumstances the interest rate is pegged at \(\bar{R}\) (this continues to be true if \(\phi_2 \neq 0\) as shown below). Provided that \(\phi_2 \neq 0\), the mean of the price level is determinate, being equal to the authorities' long-run target irrespective of the value of \(\phi_1\) and hence including the case \(\phi_1 = 0\).

The solution of the model is easily obtained using the method of undetermined coefficients in appendix C applied to a trial solution (see, for example, Lucas (1972b, 1973, 1975); McCallum (1983); Minford and Peel (1983b)). Thus, from equation (2.4.11) we conjecture the solution

\[P_t = \bar{P} + \mu \nu_t \tag{2.4.13}\]

where \(\mu\) is a coefficient yet to be determined. Provided \(\phi_1 + \phi_2 \neq 0\), the solution to the system is obtained by substituting equation (2.4.13) (with \(P^e_{t+1, t} = \bar{P}\)) into equation (2.4.11) and equating coefficients to obtain \(\bar{P} = \tilde{P}\) and \(\mu = -(1 + \phi_1 \gamma_1 + \phi_1)^{-1}\). Hence,

\[P_t = \bar{P} - (1 + \phi_1 \gamma_1 + \phi_1)^{-1}\nu_t . \tag{2.4.14}\]
The interesting feature of equation (2.4.14) is that it does not depend on the particular choice of \( \phi_2 \). This is because \( P_{t+1}^e = \bar{P} = \bar{P} \) so that equation (2.4.5) and the behaviour of the system is observationally equivalent to the case \( \phi_2 = 0 \). The sole function of the second term in equation (2.4.5) is to anchor the long-run nominal equilibrium of the system, and it does not influence the short-run dynamic response of the system to exogenous disturbances. For the case \( \phi_1 = 0 \) and \( \phi_2 \neq 0 \), the rule is therefore observationally equivalent to a rule of pegging interest rates, whilst avoiding the difficulties of price level indeterminacy noted in the literature\(^{21}\).

Finally, we note that if disturbances are directly observable, equation (2.4.5) may be implemented as a feedback on these disturbances (and current prices) rather than on private sector expectations. To see this, using equation (2.4.11) we may re-write equation (2.4.5) as

\[
R_t - \bar{R} = \phi_1 (M_t - \bar{M}) + \phi_2 (1 + \phi_1 y_t - \phi_2)^{-1} [(1 + \phi_1 y_t + \phi_1)(P_t - \bar{P}) + v_t]
\]

(2.4.15)

which may be implemented if \( v_t \) is observable. It is straightforward to check that equation (2.4.15), like equation (2.4.5), serves to determine the mean long-run price level while being observationally equivalent to equation (2.4.7).

2.4(8) A Dynamic Model

The model of the previous section is particularly simple, being essentially static in character. It is therefore of interest to extend the analysis to a dynamic model, and here we do that in the context of the following discrete time version of the model in Dornbusch (1976):
\[ P_{t+1} - P_t = \lambda (S_t - P_t) + \varepsilon_{2t} \quad 0 < \lambda < 1 \]  
(2.4.16)

\[ M_t^d = \gamma_0 + P_t - \gamma_1 R_t + \varepsilon_{3t} \quad \gamma_i > 0 \quad (i = 0, 1) \]  
(2.4.17)

\[ S_{t+1,t}^e - S_t = R_t - R^f + \varepsilon_4 t \]  
(2.4.18)

where \( S \) = natural logarithm of the nominal exchange rate

\( R^f \) = exogenous foreign nominal rate of interest

\( \varepsilon_4 \) = stochastic disturbance

and all other variables are defined as before. Similarly, \( S_{t+1,t}^e = E(S_{t+1} | \Omega_t) \) and \( \varepsilon_4 \) is an independently distributed white noise error term.

Equation (2.4.16) is the inflation generating mechanism with prices responding sluggishly to real competitiveness and stochastic realizations, \( \varepsilon_4 \). Thus, the nominal exchange rate, \( S \), is the domestic currency price of foreign exchange so that a rise in \( S \) is a depreciation. Equation (2.4.17) is the money demand function. Equation (2.4.18) is the uncovered interest parity condition where the expected rate of depreciation is equal to the interest rate differential. Random deviations from interest parity are captured by \( \varepsilon_4 \).

For this model we assume a policy rule of the form

\[ R_t - \bar{R} = \phi_1 (M_t - \bar{M}) + \phi_2 ((P_t^e - \bar{P}) + \phi_3 (P_{t+1,t}^e - \bar{P})) \]  
(2.4.19)

and the model is closed by the money market equilibrium condition in equation (2.4.6).

The solution proceeds as follows. From equation (2.4.16),
\[ P_{t+1,t}^e = \lambda S_t + (1 - \lambda)P_t \quad (2.4.20) \]

\[ P_{t+2,t}^e = \lambda S_{t+1,t}^e + (1 - \lambda)(\lambda S_t + (1 - \lambda)P_t) \quad (2.4.21) \]

Substituting equations (2.4.20) - (2.4.21) into equation (2.4.19) and eliminating \( M_t \) using equations (2.4.6) and (2.4.17),

\[ R_t = (1 + \phi_1 y_t)^{-1}[(R + \phi_1(\gamma_o - M) - \phi_2(1 + \phi_3)\gamma_t) + \phi_2 \lambda S_{t+1,t}^e \]

\[ + (\phi_1 + \phi_2(1 - \lambda)(1 - \lambda + \phi_3))P_t + \phi_2 \lambda(1 - \lambda + \phi_3)S_t + \phi_1 \varepsilon_{2t}] \quad (2.4.22) \]

Now substitute equation (2.4.22) into equation (2.4.18) and note that \( R = R^f \). Then

\[ (1 + \phi_1 y_t)(S_{t+1,t}^e - S_t) = (\phi_1 + \phi_2(1 - \lambda)(1 - \lambda + \phi_3))P_t + \phi_2 \lambda S_{t+1,t}^e \]

\[ + \eta_t - (\phi_1(\gamma_1 R - \gamma_o + M) + \phi_2(1 + \phi_3)\gamma_t) \quad (2.4.23) \]

where \( \eta_t = \phi_1 \varepsilon_{2t} + (1 + \phi_1 y_t)\varepsilon_{3t} \).

In the long-run, \( S_{t+1,t}^e - S_t = 0 \) from equation (2.4.18) and \( S = \bar{P} \) from equation (2.4.16). In addition, given \( R, \bar{M} = \gamma_o + \bar{P} - \gamma_1 R \) from equation (2.4.17). Thus taking the long-run mean of equation (2.4.23) yields

\[ (\phi_1 + \phi_2(1 + \phi_3))(\bar{P} - \bar{P}) = 0 \quad (2.4.24) \]

which is our basic result for the dynamic model. Provided \( \phi_2 \neq 0 \) (and assuming \( \phi_3 \neq -1 \), \( \bar{P} = \bar{P} \) even when \( \phi_1 = 0 \) (that is, even when
the interest rate is arbitrarily fixed at $R$). $\phi_2 = 0$ is admissible for a well-defined solution only if $\phi_1 \neq 0$. These results accord with those obtained for the static model of section 2.4(A). It remains to show that the general policy in equation (2.4.19) is observationally equivalent to the simple rule in equation (2.4.7) and that the general rule may also be implemented as a feedback rule defined on disturbances and other current variables (provided the disturbances are observable). We prove these as follows.

Equation (2.4.23) can be written as

$$S_{t+1, t}^e = s_1 p_t + s_2 s_t + \ell_t \tag{2.4.25}$$

where

$$s_1 = (1 + \phi_1 y_1 - \phi_2 \lambda)^{-1}(\phi_1 + \phi_2 (1 - \lambda)(1 - \lambda + \phi_3))$$

$$s_2 = (1 + \phi_1 y_1 - \phi_2 \lambda)^{-1}(1 + \phi_1 y_1 + \phi_2 \lambda (1 - \lambda + \phi_3))$$

$$\ell_t = (1 + \phi_1 y_2 - \phi_2 \lambda)^{-1}(\eta_t - (\phi_1 + \phi_2 (1 + \phi_3))\beta).$$

Then the state-space representation of the system follows from equations (2.4.16) and (2.4.25) as

$$\begin{bmatrix} p_{t+1} \\ s_{t+1, t}^e \end{bmatrix} = \begin{bmatrix} 1 - \lambda & \lambda \\ s_1 & s_2 \end{bmatrix} \begin{bmatrix} p_t \\ s_t \end{bmatrix} + \begin{bmatrix} e_{2t} \\ \ell_t \end{bmatrix}. \tag{2.4.26}$$

The saddlepoint property requires that there be one eigenvalue which lie inside and one eigenvalue which lie outside the unit circle; $|\tau_1| < 1$ and $|\tau_2| > 1$, say. It is instructive to examine this in a little more detail. Thus, the characteristic equation of (2.4.26) is

$$\tau^2 - (1 - \lambda + s_2)\tau + (s_2 - \lambda s_2 - \lambda s_1) = 0 \tag{2.4.27}$$
and the precise conditions for saddlepoint stability can be derived by computing the test functions of equation (2.4.27) as demonstrated in appendix D. The condition is that there be one and only one sign change in the sequence

\[ \psi[(2-\lambda)(1+\phi_1 y_1) - \lambda(\phi_1 + \phi_2)], -\psi(\phi_1 + \phi_2(1 + \phi_3)] \] (2.4.28)

where \( \psi = [2(2-\lambda)(1+\phi_1 y_1) - \lambda(\phi_1 + \phi_2(1 - \phi_3))]^{-1} \).

Equivalently, the model has a unique convergent solution if \( \psi > 0 \) which we assume is satisfied. The indeterminacy problem can be seen directly from equation (2.4.28) by setting \( \phi_1 = \phi_2 = 0 \) in which case the relevant (third) test function has a zero value. This indicates the presence of a unit root as, for example, in the model analysed by Canzoneri, Henderson and Rogoff (1983), and in all the models that we employ in chapter 3\(^{(22)} \).

To be sure, imposing \( \phi_1 = \phi_2 = 0 \) gives \( \delta_1 = 0 \) and \( \delta_2 = 1 \) so that equation (2.4.27) reduces to

\[ f(\tau)(\bar{\tau}) = (\tau - 1)(\tau - (1 - \lambda)) \] (2.4.29)

where the notation emphasizes the interest rate peg policy. Thus, provided \( \phi_1 + \phi_2(1 + \phi_3) \neq 0 \), we have that

\[ p^{e}_{t+i,t} - \bar{p} = \tau^{i-j}_1 (p^{e}_{t+j,t} - \bar{p}), \quad i > j. \] (2.4.30)

Now choose \( \phi_3 = -\tau_1 \). In particular, equation (2.4.30) gives

\[ p^{e}_{t+2,t} - \bar{p} = -\phi_3 (p^{e}_{t+1,t} - \bar{p}). \] (2.4.31)

Then along the stable trajectory, the general policy rule in equation (2.4.19) is observationally equivalent to the simple rule in
equation (2.4.7). Moreover, $\tau_1$ does not depend on the choice of $\phi_2$. As before, $\phi_2$ acts solely to anchor the long-run equilibrium of the system, and the response of the system to short-run perturbations is identical under rules (2.4.7) and (2.4.19). The general policy rule works by anchoring expectations so as to be consistent with the saddlepath solution.

As for the model of section 2.4(A), if disturbances are directly observable, then rule (2.4.19) may be implemented as a rule defined not on private sector expectations, but on disturbances and other current variables. Thus, from equation (2.4.26),

$$p_{t+1}^e = (1 - \lambda)p_t + \lambda s_t$$  
(2.4.32)

$$p_{t+2}^e = ((1 - \lambda)^2 + \lambda s_1)p_t + \lambda(1 - \lambda + s_2)s_t + \lambda \ell_t$$  
(2.4.33)

so that equation (2.4.19) becomes

$$R_t - \bar{R} = \phi_1(m_t - \bar{M}) + \phi_2((1-\lambda)^2 + (1 - \lambda)\phi_3 + \lambda s_1)p_t$$

$$+ \phi_2\lambda(1 - \lambda + s_2 + \phi_3)s_t + \phi_2\lambda \ell_t - (\phi_2 + \phi_3)p.$$  
(2.4.34)

As before, the rule in equation (2.4.34) serves to anchor the long-run mean price level, while being observationally equivalent to equation (2.4.7) along the stable trajectory.

It is useful here to summarize this section and offer an interpretation of our results. We have provided a resolution of the problem of price level indeterminacy associated with an arbitrarily chosen peg for the interest rate. It has been accomplished by specifying a policy rule which involves feedback on forward price
expectations. This rule has some intuitive appeal. It states that if price expectations were to be higher than those implied by the unique saddlepath solution, interest rates would be raised to offset these expectations. Conversely, if price expectations were to be 'too low', interest rates would be reduced. In some respects, this is similar to the rules advanced by McCallum (1981) and Dotsey and King (1983). The difference is that in our analysis the threat of this rule serves to anchor expectations so that, in fact, no action is actually required. In their analyses, however, an arbitrarily chosen value for the interest rate peg continues to generate indeterminacy. In our analysis, an arbitrary interest rate peg is entirely consistent with a well-defined solution.

We showed how the results for a static model are robust with respect to a dynamic model. The latter possessed only first-order dynamics. The results are clearly generalisable, however, to higher order systems. The essential feature of the additional term in the policy rule is that it should be zero on the stable manifold consistent with the long-run target of the authorities, and non-zero off it. Such a term then serves to anchor expectations to this stable manifold, ensuring price level determinacy, while not influencing observed policy actions on this manifold.

An important property of our general policy rule is that the expectations feedback term acts solely to pin down the long-run state of the system and does not affect the response of the system to short-run shocks. In terms of the latter, this makes the general policy rule observationally equivalent to the simple contemporaneous feedback rule between interest rates and the money stock. Analyses employing the simple rule (as in the literature reviewed in section 2.2 and as in chapter 3) are, therefore, not necessarily misguided.
On a final note, we appreciate two fundamental conditions for our analysis to make sense. The first is that the announcement of the general policy rule must carry credibility. The second is that the authorities must have knowledge of either private sector expectations or the current values of certain variables and stochastic disturbances. An interesting line of inquiry would be to see the extent to which relaxation of these assumptions modify our results.

2.5 **Summary and Concluding Remarks**

The chapter has been motivated by the continuing interest in the monetary instruments issue and the control problem to which it gives rise. It is by way of a prelude to chapter 3 which tests the robustness of optimal instruments across a variety of divergent model structures. This has been made necessary owing to what we feel to be the rather inadequate treatment of the issue in the existing literature. The justification for this criticism is the outcome of the literature review that was given in section 2.2. We concluded that the similarity of the models employed in the literature made it difficult to say anything about the existence of model-robust policy rules.

In section 2.3 we constructed informally the analytical framework which is to be adopted in chapter 3. This discussed what are probably the six most well-known rational expectation models.

Section 2.4 took up the issue of indeterminacy associated with a policy which attempts to exogenise the current interest rate. We offered a resolution of this problem. We showed that there generally exists a feedback policy which admits a unique solution even when the interest rate is pegged at an arbitrary value. One component of this policy rule is the usual contemporaneous relationship between
the interest rate and the money stock. A second, and novel, element is a term which expresses a feedback on forward price expectations. Announcement of this general policy rule is sufficient to anchor these expectations. An identical policy is one in which the feedback on expectations is replaced by a feedback on the current values of certain variables and stochastic disturbances. The second component of the general policy rule serves solely to tie down the long-run state of the system and does not affect the response of the system to short-run perturbations. As shown formally, this makes the general rule observationally equivalent to the simple contemporaneous feedback rule between the interest rate and the money stock. This is attractive for the analysis in chapter 3; for we may solve the models under the assumption of the simple rule.
Notes to Chapter Two

(1) Clearly, as stated, the optimal instrument is also determined by the structural parameters. Thus, the loss associated with a money stock policy is a decreasing function of $\gamma_2$ since as $\gamma_2$ increases, the effect of money market disequilibrium on interest rates is reduced. By the same token, the loss associated with a money stock policy is an increasing function of $\gamma_2$; expenditure induced money demand fluctuations imply lower interest rate movements as $\gamma_2$ increases but it is precisely these variations which exert a stabilizing force on income. Obviously, the opposite can be said of the effect of $\gamma_1$ since the ratio of $\gamma_1$ to $\gamma_2$ determines the slope of the LM curve. Finally, the effect of $\beta_1$ is similarly straightforward analyse.

(2) A peculiar feature of the analytical model used by Craine and Havenner (1981) is the dependency of aggregate demand on the nominal interest rate despite the endogeneity of prices. Nonetheless, their results are shown not to be qualified in a larger econometric model.

(3) Recall that this was the reason for the superiority of the contemporaneous combination policy.

(4) In general, the optimal policy will depend on the covariances between different exogenous shocks, between different random coefficients and between exogenous shocks and random coefficients.

(5) Obviously, this depends on the assumption that tax rates and government expenditure remain fixed. We return to this issue below.
(6) As is usual, the impact of expenditure shocks can be reduced further by permitting some (countercyclical) monetary response. This requires $\lambda > 1$.

(7) It may also be noted that, in considering adaptive expectations, Turnovsky (1980) shows the instability of an interest rate peg policy. This occurs because the expectations mechanisms permits disturbances to feed through indefinitely. In addition, pegging the money stock may also generate explosive behaviour if the expectations adjustment coefficient is sufficiently large.

(8) In defence of this criticism, Turnovsky (1976) does compute the asymptotic variances for special cases. Nonetheless, for the most part, it is paradoxical that wealth accumulation and stock-flow interactions which are emphasized in the basic model are accorded no role in the comparative static exercises.

(9) As stated, the models of these particular authors are strikingly different. Turnovsky (1976) employs a fifteen equation system incorporating flexible output and prices together with stock-flow interactions. Boyer (1978) and Roper and Turnovsky (1980b) adopt a simple open economy IS-LM system.

(10) All of these results hold for price stability in Turnovsky (1976), where prices are determined by excess demand.

(11) As in Poole (1970), Boyer (1978) and Roper and Turnovsky (1980b) show the general superiority of a combination policy specifying a contemporaneous relationship between the money stock and the exchange rate. The policy is therefore analogous to a 'dirty float'.

(12) All of these employ a Sargent and Wallace (1975) framework or some minor variant thereof. An exception is Artis and Currie (1981).
(13) I have just discovered another paper on the instrument problem by Benavie and Froyen (1983) which shows that a developed financial sector may qualify some of the standard results, especially with respect to pure policies (see also, Benavie and Froyen (1984)).

(14) This supply function can also be motivated in terms of Phillips Curve arguments and wage contracts (see, for example, McCallum (1980); Fischer (1977a)).

(15) In terms of labour suppliers, this amounts to assuming that future goods and leisure are substitutes for current leisure, that leisure is not an inferior good, and that the substitution effect dominates.

(16) To be fair, the main concern of Peel and Metcalfe (1979) is dynamic stability, though our point is nonetheless true.

(17) The proportions pre-multiplying the parenthesized terms merely reflect the proportion of firms operating under contracts negotiated at different dates. This is innocuous.

(18) Interest rate control is defined here as policies which make the current interest rate exogenous. These include lagged feedback interest rate rules and obviously an interest rate peg.

(19) This is also discussed by Giavazzi and Wyplosz (1985) who observe that, since the equilibrium is undefined, the dynamics of the system are non-unique and depend upon the initial values of variables. This can obviously be overcome by appropriate choice of the initial conditions.

(20) The assumption that output is fixed is innocuous, serving only to simplify the algebra.
This is not to say that announcement of rule (2.4.5) is equivalent to announcement of rule (2.4.7). If the authorities announce the latter, then the usual indeterminacy obtains. The determinacy result obtained above arises by the authorities announcing rule (2.4.5) which turns out to be observationally equivalent to rule (2.4.7).

The continuous time analogue is the presence of a zero root, as observed by Giavazzi and Wyplosz (1985).
3.1 Introduction

The issue of the appropriate choice of monetary instrument for stabilization policy continues to attract a lively debate. The general conclusion offered by the literature is that this choice is determined, to a lesser or greater degree, by the particular variance-covariance structure of stochastic disturbance, the particular model, the values of the structural parameters of this model, and the particular objective function of the authorities. Unfortunately, the inexorable conclusion that emerged from chapter 2 (section 2.2) was that the existing literature has important drawbacks which render it inappropriate for diagnosing the relative merits of alternative monetary policies. In particular, if one is seeking 'model-robust' policy rules, the information value of the payoff matrix derived from the prevailing orthodoxy would be at best minimal. That this is so arises from the similarity of models examined in this literature. Since we argue this to be undesirable, it is of some necessity that it is rectified. It is the purpose of this chapter to conduct such corrective action.

This is not to suggest, of course, the unimportance of other issues. In particular, one might wish to investigate the possibility of control rules which perform well under a variety of different stochastic shocks (as opposed to different models). A full appreciation would then take into account the possibility that the actual shocks occurring are different from those expected. Such concern has occupied the interest of the series articles by Levine and Currie (1983, 1984) and Currie and Levine (1983a, 1984a,b,c). Though clearly of some importance, we feel it not to overshadow the problem
of model uncertainty: for suppose one discovered a policy which actually performed well in the face of a variety of disturbances. Then its usefulness for practical purposes is immediately undermined if the particular model conditioning the result is not reflective of the true system. A rigorous treatment of the policy problem would take account of both types of uncertainty simultaneously. This is beyond the scope of the thesis and we concentrate our attention on the possibly more interesting and relevant (but less addressed) issue of model uncertainty.

The analysis is motivated towards the construction of a payoff matrix with elements defining the optimal monetary instrument for particular types of stochastic disturbances and for each of the six rational expectations models discussed informally in chapter 2 (section 2.3). Owing to the significant differences between the model structures, the information content of the payoff matrix for indicating "model-robust" policy rules is far from non-zero. The analytical approach adopted in the chapter is in the spirit of the majority of the literature on the issue. In chapter 6 it will be seen that such an approach eschews potentially more important aspects of the problem.

Section 3.2 provides the formal schema in which the six rational expectations models are defined in terms of a broad taxonomic framework. Section 3.3 then solves each of the models and addresses the control problem. It should be noted that, from the discussion in chapter 2 (section 2.4), we discount the redundancy of the control problem arising as a consequence of price level indeterminacy under an interest rate peg.

During the course of the analysis certain additional issues arise as a by-product which are important in their own right. In particular, the open economy model exhibits some apparently perverse
features which are explained by the forward-looking nature of private sector expectations. These are particularly relevant for the stability properties of the model and indicate an error in thinking solely in terms of the absolute degree of capital mobility.

The profuse number of results derived in section 3.3 motivates the content of section 3.4 which constructs the payoff matrix and summarizes the main findings. Concluding remarks are contained in this section.

3.2 The Formal Taxonomy

A convenient and self-contained schema which captures the formal characteristics of each of the six models described informally in chapter 2 (section 2.3) is given below. The taxonomy comprises six partitions: equations describing aggregate demand; equations describing aggregate supply; expected inflation formation; money demand functions; the authorities' policy rule; and a balance of payments equation. All variables are defined collectively at the end. The use of both discrete and continuous time formulations reflects merely analytical convenience.

(a) Aggregate demand:

\[ y_d = -\beta (r - \pi_t) + \epsilon_t \]  
\[ y_d = -\beta (r - \pi) + \epsilon_1 \]  
\[ y_d = -\beta_1 (r - \pi) + \beta_2 (e - p) + \epsilon_1 \]  

\( \beta, \beta_i (i = 1, 2) > 0. \)
(b) Aggregate supply:

\[ y^s_t = \alpha(p_t - p^e_{t-1}) + \varepsilon_{2t} \]  
\[ y^s_t = \alpha_1(p_t - \alpha_2 p^e_{t+1, t-1}) + \alpha_3(x_t - \pi_t) \]  
\[ - \alpha_4(m_t - p_t) + \varepsilon_{2t} \]  
\[ y^s_t = \lambda_2(p_t - p^e_{t, t-1}) + \lambda_2(p_t - p^e_{t, t-2}) + \varepsilon_{2t} \]

(Fischer) \hspace{1cm} (3.2.2)

\[ dy = \lambda(y^d - y)dt \]  
\[ dp = \theta_1((q-p) + \varepsilon_2)dt \]  
\[ dq^e = \theta_1(q-p - \varepsilon_2 y)dt \]  
\[ dy = \lambda(y^d - y)dt \]  
\[ dp = \theta(y + \varepsilon_2)dt \]

(Open economy) \hspace{1cm} (3.2.2f)

\[ \alpha, \alpha_1, \theta, \theta_i (i = 1,2), \lambda > 0; \alpha_j (j = 2,3,4) \geq 0. \]

(c) Expected inflation:

\[ \pi_t = p^e_{t+1, t-1} - p^e_{t, t-1} \]  
\[ \pi_t = p^e_{t+1, t} - p_t \]

(Turnovsky) \hspace{1cm} (3.2.3)

\[ \pi = \dot{p}^e = \frac{dp^e}{dt} \]

(Open economy) \hspace{1cm} (3.2.3f)
(d) Money demand:

\[ m_t^d = \rho_t + \gamma_1 y_t - \gamma_2 r_t + \varepsilon_3 t \]

(Sargent-Wallace)  
(Lucas-Rapping)  
(Turnovsky)  
(Fischer)  
(3.2.4)  

\[ m^d = p + \gamma_1 y - \gamma_2 r + \varepsilon_3 \]

(Calvo)  
(Open economy)  

\[ \gamma_i (i = 1, 2) > 0. \]

(e) Money supply:

\[ m_t^s = \phi r_t \]

(Sargent-Wallace)  
(Lucas-Rapping)  
(Turnovsky)  
(Fischer)  
(3.2.5)  

\[ m = \phi r \]

(Calvo)  
(Open economy)  

\[ \phi \geq 0. \]

(f) Balance of payments

\[ \eta_1 (r - e^e) - \eta_2 y + \eta_3 (e - p) + \nu = 0 \]

(Open economy)  
(3.2.6)  

\[ \eta_k (k = 1, 2, 3) > 0. \]

where

- \( y^d \) = natural logarithm of real aggregate demand
- \( y^s \) = natural logarithm of real aggregate supply
- \( r \) = nominal rate of interest
- \( \pi \) = expected rate of inflation
- \( e \) = natural logarithm of the nominal exchange rate
- \( p \) = natural logarithm of the aggregate price level
\[ m^d \] = natural logarithm of the nominal demand for money
\[ m^s \] = natural logarithm of the nominal supply of money
\[ q \] = natural logarithm of the contract price
\[ \epsilon_k (k = 1,2,3), \nu \] = stochastic disturbances.

All variables are measured as deviations from their long-run equilibrium levels.

Aggregate demand is specified in equations (3.2.1) in part (a) as a negative function of the real rate of interest. The latter is given by equations (3.2.3) in part (c) as the difference between the nominal rate of interest and the expected rate of inflation. For the continuous time models we have \[ dp^e = p^e(t+d,t) - p(t) \]. In the open economy model, aggregate demand is also a (positive) function of the real exchange rate so that the nominal exchange rate, \( e \), is the domestic currency price of foreign exchange.

Equations (3.2.2) in part (b) describe aggregate supply. Both Sargent-Wallace and Turnovskv models employ the 'surprise' supply function. The Lucas-Rapping and Fischer models have similar supply structures though the important difference is obvious. For the Calvo model, the inflation and contract price determination equations discussed in chapter 2 (section 2.3) are supplemented with an expression which specifies a sluggish adjustment of actual output to aggregate demand pressure. In the open economy model, output is again demand determined and inflation responds directly to excess demand.

A standard money demand function is written in equations (3.2.4), part (d): the nominal demand for money depends positively on prices and output, and negatively on the nominal interest rate.

Equations (3.2.5) in part (e) describe the authorities policy behaviour for determining the nominal money supply. This is in terms of a contemporaneous relationship between the nominal money supply
and the nominal interest rate with no a priori sign attached to the policy feedback parameter $\phi$. In fact, the optimization process may well yield an optimal value $\phi = \phi^* < 0$. For most of what follows the polar cases $\phi = 0$ (money supply peg) and $\phi = \infty$ (interest rate peg) will be considered, though a derivation of $\phi^*$ is possible and is executed for the discrete time models. Recall that the discussion in chapter 2 (section 2.4) allows us to consider the simple rule defined in equation (3.2.5) for analytical convenience. Any unit or zero root arising as a consequence of an interest rate peg ($\phi = \infty$) may be ignored and treated as unstable.

Equation (3.2.6) in part (f) is the balance of payments equilibrium condition for the open economy model. The precise derivation of this follows Bhandari (1981), Bhandari, Driskill and Frenkel (1984), Driskill (1981), Frenkel and Rodriguez (1982), and Gazioglou (1984) (for the continuous time case). Thus, we postulate the capital and current accounts of the balance of payments as respectively

$$K = \eta_1 (r - \bar{e} + \nu_1) \tag{3.2.7}$$

$$T = -\eta_2 y + \eta_3 (e - p + \nu_3) \tag{3.2.8}$$

where $\bar{e}$, the expected rate of depreciation, is $\bar{e} = e^{\bar{e}}(t+dt,t) - e(t)$ and $\nu_h (h = 1,3)$ are stochastic disturbances. Summing equations (3.2.7) - (3.2.8) then yields (3.2.6) with $\nu = \eta_1 \nu_1 + \eta_3 \nu_3$. The flow view of capital movements implicit in equation (3.2.7) is probably innocuous given the results in Bhandari, Driskill and Frenkel (1984) and Gazioglou (1984).

Exogenous shocks in the taxonomy are denoted by $e_k (k = 1,2,3)$ and $\nu$. These have the following interpretations. In part (a), $e_1$
is an exogenous shock to aggregate expenditure; \( \varepsilon_2 \) in part (b) is a random supply shock; \( \varepsilon_3 \) in part (d) represents random portfolio behaviour; and \( \nu \) which impinges on the balance of payments in part (f) comprises shocks to capital and trade flows (see equations (3.2.7) - (3.2.8)).

We assume that \( \varepsilon_k (k = 1,2,3) \) and \( \nu \) are independently distributed. For the discrete-time models we also assume that each \( \varepsilon_k (k = 1,2,3) \) is purely white-noise with asymptotic variance \( \sigma_k^2 (k = 1,2,3) \). The continuous time analogue is that each \( \varepsilon_k (k = 1,2,3) \) has independent increments so that \( d\varepsilon_k (k = 1,2,3) \) is white noise. For the two external shocks, \( \nu_h (h = 1,3) \), we assume a first-order autoregressive process

\[
d\nu_h = -\rho(\nu_h dt - d\omega_h) \quad (h = 1,3)
\]

where \( \omega_h (h = 1,3) \) is independently distributed and has independent increments, and where the degree of positive autocorrelation is a decreasing function of \( \rho \). Thus, the composite disturbance \( \nu \) has the form

\[
d\nu = -\rho(\nu dt - d\omega)
\]

where \( d\omega = \eta_1 d\omega_1 + \eta_3 d\omega_3 \).

The autoregressive representation for \( \nu \) is used solely for analytical purposes.

In order that policies may be defined in terms of optimality, we assume a performance measure

\[
E(J) = t_2(\sigma_y^2 + (1 - a) \sigma_p^2) \quad 0 \leq a \leq 1
\]

where \( \sigma_y^2 \) and \( \sigma_p^2 \) are the asymptotic variances of output and prices respectively. Hence, the control problem involves minimizing the
value of (3.2.11) with respect to $\phi$ subject to the constraints imposed by each of the model structures. Clearly, $a = 0$ and $a = 1$ represent the extreme cases in which the authorities pursue only one objective.

This completes the formal taxonomy. We now turn to the control problem.

3.3 Robustness Tests of the Optimal Choice of Monetary Instrument

This section is divided into six parts, 3.3(A) - 3.3(F), each one being concerned to examine the control problem in each of the foregoing six models. Parts 3.3(A) - 3.3(D) analyse the discrete time models, the solution procedure for which is the method of undetermined coefficients, summarized in appendix C. Parts 3.3(E) - 3.3(F) contain the analyses for the continuous time models. The solution technique for these is the saddlepoint solution reviewed in appendix B.

3.3(A) The Control Problem in the Sargent-Wallace Model

The system is described by equations (3.2.1a), (3.2.2a), (3.2.3a), (3.2.4) - (3.2.5). We obtain a quasi-reduced form for prices as follows: set $m^d_t = m^s_t = m_t$ and use equations (3.2.4) - (3.2.5) to solve for $x_t$, substitute this expression into equation (3.2.1a) and eliminate $x_t$ using equation (3.2.3a); finally, set $y^d_t = y^s_t = y_t$ and apply equation (3.2.2a). Rearranging terms then yields

$$0 = (\beta + ac)p_t + (\beta - ac)p^e_{t-\tau,t-\tau - 1} - \beta p^e_{t+1,t-1} + u_t$$

(3.3.1)

where $b = (\phi + \gamma_2)^{-1}$

$c = 1 + \beta \gamma_1$

$u_t = -\epsilon_{1t} + \epsilon_{2t} + \beta \epsilon_{3t}$.
Positing a solution of the form

$$p_t = \sum_{i=0}^{\infty} \mu_i u_{t-1}$$  \hspace{3cm} (3.3.2)

and substituting into equation (3.3.1) gives the set of identities

$$\mu_0 = -\left(\beta \phi + \alpha c\right)^{-1}$$  \hspace{3cm} (3.3.3)

$$\mu_j = (1 + \beta)\mu_{j-1}, \quad j \geq 2.$$  \hspace{3cm} (3.3.4)

Clearly, $\phi = 0 \left(\beta = \gamma_2^{-1}\right)$ necessarily makes equation (3.3.4) an unstable process. Moreover, $\phi = 0 \left(\beta = 0\right)$ implies a unit root which, from the argument in chapter 2 (section 2.4), can be ignored. Hence, imposing stability, $\mu_j = 0 \left(j \geq 1\right)$ and the solution for prices is just

$$p_t = \mu_0 u_t.$$  \hspace{3cm} (3.3.5)

The solution for output is obtained by noting that

$$p_t - p_{t-1} = \mu_0 u_t$$  \hspace{3cm} so that from equation (3.2.2a),

$$y_t = \alpha u_t + \varepsilon_{2t}.$$  \hspace{3cm} (3.3.6)

From equations (3.3.5) - (3.3.6) and recalling the definitions of $\mu_0$ and $u_t$, the usual comparative statics emerge

$$\frac{\partial p_t}{\partial \varepsilon_{1t}} = -\mu_0 > 0, \quad \frac{\partial y_t}{\partial \varepsilon_{1t}} = -\alpha \mu_0 > 0$$

$$\frac{\partial p_t}{\partial \varepsilon_{2t}} = \mu_0 \phi < 0, \quad \frac{\partial y_t}{\partial \varepsilon_{2t}} = -\mu_0 \beta \phi > 0$$  \hspace{3cm} (3.3.7)

$$\frac{\partial p_t}{\partial \varepsilon_{3t}} = \mu_0 \beta \phi < 0, \quad \frac{\partial y_t}{\partial \varepsilon_{3t}} = \alpha \mu_0 \beta \phi < 0$$
which require no further comment. It is now apparent from equations (3.3.5) - (3.3.6) that if $\sigma_u^2$ is the asymptotic variance of $u$ and $\sigma_{uk}$ is the asymptotic covariance of $u$ and $e_k$ ($k = 1, 2, 3$), then

$$\sigma_p^2 = \mu_o^2 \sigma_u^2$$

(3.3.8)

$$\sigma_y^2 = \sigma_{e1}^2 \sigma_u^2 + \sigma_z^2 + 2\mu_o \sigma_{u2}$$

(3.3.9)

Table 3.3(A) summarizes the results of the control exercise. We set each $e_k = 0$ ($k = 1, 2, 3$) in turn and apply $\phi = 0$ (money supply peg, $\bar{m}$) and $\phi = \infty$ (interest rate peg, $\bar{r}$). Casual inspection of the direction of the inequalities reveals the Poole (1970) and Craine and Havenner (1981) results to be unchanged. The intuition is as usual: for expenditure shocks, fixing the money stock provides a stabilizing role for the endogenously determined interest rate; by contrast, this policy is inferior in the case of money demand disturbances because it permits these to spill over onto the good market; finally, though the foregoing optimal policies apply to both output and price stabilization criteria, there is a conflict in optimal policies in the face of supply shocks, with a money supply peg being superior (inferior) for price (output) stabilization (this conflict obtains, it will be recalled, because an accommodative monetary policy exacerbates the initial price movement which, since it is unanticipated, offsets the supply shock by virtue of the 'surprise' supply function).

It is instructive to show the relationship between the above results and those which are implied by a combination policy in which $\phi$ is determined optimally. To do this, it is convenient to rewrite the solutions (3.3.5) - (3.3.6) by multiplying through by $(\mathcal{A}^*)^{-1}/(\mathcal{A}^*)^{-1}$ such that
<table>
<thead>
<tr>
<th>Criteria</th>
<th>$\sigma_p^2$</th>
<th>$\sigma^2_y$</th>
</tr>
</thead>
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<tr>
<td></td>
<td>$\bar{m}$</td>
<td>$\bar{r}$</td>
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<tr>
<td><strong>Policy Rule</strong></td>
<td></td>
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<tr>
<td><strong>Disturbance</strong></td>
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</tr>
<tr>
<td>Expenditure</td>
<td>$\left[\frac{1}{\kappa_1}\right]^2 \sigma_1^2 &lt; \frac{\alpha}{\kappa_1}^2 \sigma_1^2$</td>
<td>$\left[\frac{1}{\alpha}\right]^2 \sigma_2^2 &lt; \left[\frac{\alpha}{\kappa_1}\right]^2 \sigma_2^2$</td>
</tr>
<tr>
<td>Supply</td>
<td>$\left[\frac{-(1+\beta_y^{-1}_2 \gamma_1)}{\kappa_1}\right]^2 \sigma_1^2 &lt; \left[\frac{-1}{\alpha}\right]^2 \sigma_2^2$</td>
<td>$\left[\frac{\beta_y^{-1}_2}{\kappa_1}\right]^2 \sigma_2^2 &gt; 0$</td>
</tr>
<tr>
<td>Money Demand</td>
<td>$\left[\frac{-\beta_y^{-1}_2}{\kappa_1}\right]^2 \sigma_3^2 &gt; 0$</td>
<td>$\left[\frac{-\alpha \beta_y^{-1}_2}{\kappa_1}\right]^2 \sigma_3^2 &gt; 0$</td>
</tr>
</tbody>
</table>

Table 3.3(A) : Optimal Monetary Instruments for the Sargent-Wallace model.

$$\kappa_1 = \beta_y^{-1}_2 + \alpha(1 + \beta_y^{-1}_2 \gamma_1)$$
\[ u_t = (\beta b)^{-1} u_t = -\beta^{-1}(\phi + \gamma_2) e_{1t} + (\beta^{-1}(\phi + \gamma_2) + \gamma_1) e_{2t} + e_{3t} \]  
(3.3.10)

\[ \mu_o = \frac{1}{(\beta b)^{-1}} \mu_o = -(1 + \alpha(\beta^{-1}(\phi + \gamma_2) + \gamma_1))^{-1} \]  
(3.3.11)

using the definitions of \( u_t \), \( b \), \( c \) and \( \mu_o \). The optimal value of \( \phi, \phi^*_A \), is now derived by minimizing the loss function in equation (3.2.11) with respect to \( \phi \). Hence, substituting for \( \sigma^2_y \) and \( \sigma^2_p \) from equations (3.3.8) - (3.3.9),

\[ E(\phi) = \frac{1}{2}(a(\alpha^2 \sigma_o^2 u^2 + \sigma^2_x + 2\alpha \sigma_o \sigma_j u_x) + (1-a)\mu_o \sigma^2_j) \]  
(3.3.12)

where \( \sigma^2_u \) and \( \sigma^2_{u_2} \) are appropriately defined from equation (3.3.10).

Thus

\[ \frac{\partial E(\phi)}{\partial \phi} = \frac{1}{2} \left[ \left( \frac{\partial \sigma_o^2 u^2}{\partial \phi} \right) \left( a\sigma^2 + (1-a) \right) + 2a\alpha \sigma \left( \frac{\sigma_o \sigma_j u_x}{\partial \phi} \right) \right] = 0 \]  
(3.3.13)

Some tedious computations yield the following expressions

\[ \frac{\partial [\mu^2_o \sigma^2_u]}{\partial \phi} = -2\mu_o^3 \beta^{-2}(\phi + \gamma_2)(1 + \alpha \gamma_1) \sigma^2_1 \]

\[ + \beta^{-1}(\beta^{-1}(\phi + \gamma_2) + \gamma_1) \sigma^2_2 - \alpha \beta^{-1} \sigma^2_3 \]  
(3.3.14)

\[ \frac{\partial [\mu_o \sigma_{u_2}]}{\partial \phi} = -\mu_o \beta^{-1} \sigma^2_2 \]  
(3.3.15)

Substituting equations (3.3.14) - (3.3.15) into equation (3.3.13), some further algebraic manipulation reveals \( \phi^*_A \) to be
\[
\phi_A^* = \frac{\alpha h_3 \sigma_3^2 - \beta^{-1} \gamma_2 h_1 \sigma_1^2 + h_2 \sigma_2^2}{H}
\]

where \( h_1 = h_3(1 + \alpha \gamma_1) \)

\( h_2 = \alpha \gamma_1 - (1-a)(\beta^{-1} \gamma_2 + \gamma_1) \)

\( h_3 = \alpha \gamma_1^2 + (1-a) \)

\( H = \beta^{-1}(h_1 \sigma_1^2 + (1-a)\sigma_2^2) \).

Consider, first, the case of expenditure shocks. Setting 
\( \sigma_2^2 = \sigma_3^2 = 0 \), we have \( \phi^*_A(\varepsilon_1) = - \gamma_2 \) and \( \sigma^2_y(\varepsilon_1) = \sigma^2_p(\varepsilon_1) = 0 \).

This is an immediate corollary of the superiority of a fixed money supply policy over a fixed interest rate policy which arises from the endogenous movements in the interest rate under the former. By following a counter-cyclical monetary response, the stabilizing interest rate fluctuations are reinforced.

For money demand disturbances, \( \sigma_2^2 \neq 0 \) and it is obvious that \( \phi^*_A(\varepsilon_3) \neq 0 \), which confirms our previous intuition.

Aggregate supply shocks imply the optimal policy 
\( \phi^*_A(\varepsilon_2) = (\beta^{-1}(1-a))^{-1}(a \gamma_1 - (1-a)(\beta^{-1} \gamma_2 + \gamma_1)) \) which demonstrates the conflict in optimal policies. Thus, when the sole objective is to minimize output fluctuations, \( a = 1 \) and an interest rate peg is optimal. By contrast, \( a = 0 \) implies a counter-cyclical monetary response for the reason discussed earlier. (Note, however, that in contrast to that policy, the optimal value for \( \phi \) here is constrained by the stability conditions. This is easily seen by substituting for \( \phi \) in equation (3.3.4) the value for \( \phi^*_A(\varepsilon_2) \) when \( a = 0 \), in which case a well-defined non-explosive solution is violated if \( 0 < (\beta \gamma_1)^{-1} < 2 \).)
general, when \( 0 < a < 1 \), the value for \( \phi^* \) will depend on the value for \( a \) together with the structural parameters of the model.

3.3.(B). The Control Problem in the Lucas-Rapping Model

The relevant equations are (3.2.1b), (3.2.2b), (3.2.3b), (3.2.4)-(3.2.5). In deriving the quasi-reduced form expression for prices, it is useful to note the supplementary expressions for \( y_t^s \) and \( y_t^d \). Thus, eliminate \( m_t^d = m_t^s = m_t \) from equations (3.2.4) and (3.2.2b) using equation (3.2.5); similarly, solve for \( r_t \) from equation (3.2.4) and substitute this expression into equations (3.2.1b) and (3.2.2b); finally, eliminate \( \pi_t \) using equation (3.2.3b). Then

\[
y_t^s = [\alpha_1 + \alpha_4 + (\alpha_3 - \alpha_4\phi)f(1 + \gamma_1(\alpha_1 + \alpha_4))]p_t
- [\alpha_1\alpha_2 + \alpha_3 + (\alpha_3 - \alpha_4\phi)\gamma_1(\alpha_1\alpha_2 + \alpha_3)]p_{t+1,t-1}^e
+ [\alpha_3 + (\alpha_3 - \alpha_4\phi)\gamma_1\alpha_3]p_{t,t-1}^e
+ (1 + (\alpha_3 - \alpha_4\phi)\gamma_1)p_{t+1,t-1}^e
+ (1 + (\alpha_3 - \alpha_4\phi)\gamma_1)\epsilon_{2t} + (\alpha_3 - \alpha_4\phi)\epsilon_{3t}
\]

\[
y_t^d = -\beta f(1 + \gamma_1(\alpha_1 + \alpha_4))p_t + \beta(1 + (\alpha_1\alpha_2 + \alpha_3)\gamma_1)p_{t+1,t-1}^e
- \beta(1 + \alpha_3\gamma_1)p_{t,t-1}^e + \epsilon_{1t} - \beta\gamma_1\epsilon_{2t} - \beta\epsilon_{3t}
\]

where \( f = (\gamma_2 - \alpha_3\gamma_1 + \phi(1 + \alpha_4\gamma_1))^{-1} \).

The quasi-reduced form expression for prices follows by equating equations (3.3.17) - (3.3.18):

\[
0 = ((\alpha_1 + \alpha_4)(1 + fg\gamma_1) + fg)p_t + (\alpha_3(1 + fg\gamma_1) + \beta)p_{t,t-1}^e
- ((\alpha_1\alpha_2 + \alpha_3)(1 + fg\gamma_1) + \beta)p_{t+1,t-1}^e + \epsilon_t
\]
where \( g = \beta + \alpha_3 - \alpha_4 \phi \)

\[
\nu_t = -\epsilon_{1t} + (1 + fg\gamma_1)\epsilon_{2t} + fge_{3t}.
\]

For a solution of the form

\[
P_t = \sum_{i=0}^{\infty} s_i \nu_{t-i}
\]

and substituting this into equation (3.3.19), the following identities obtain:

\[
\delta_0 = -((\alpha_1 + \alpha_4)(1 + fg\gamma_1) + fg)^{-1}
\]

\[
\delta_j = \frac{(\alpha_1 + \alpha_3 + \alpha_4)(1 + fg\gamma_1) + \beta + fg}{(\alpha_1 \alpha_2 + \alpha_3)(1 + fg\gamma_1) + \beta} \delta_{j-1}, \ j \geq 2.
\]

Thus, for a unique convergent solution, equation (3.3.22) must define an unstable process so that \( \delta_j = 0 \) \( (j \geq 1) \) by requirement. Unlike the Sargent-Wallace model in section 3.3(A), a money supply peg no longer guarantees a well-defined solution. This is most easily seen by setting \( \alpha_2 = 1 \) as usual. Then imposing \( \phi = 0 \), equation (3.3.22) can be written as

\[
\delta_j(\overline{m}) = 1 + \frac{\alpha_4(\gamma_2 + \beta\gamma_1) + \beta + \alpha_3}{\alpha_1(\gamma_2 + \beta\gamma_1) + \gamma_2(\beta + \alpha_3)} \delta_{j-1}(\overline{m}), \ j \geq 2.
\]

Clearly, the stability of equation (3.3.23) depends critically on all of the structural parameters of the system. The usual assumption that \( \alpha_1, \alpha_3, \alpha_4 > 0 \) is sufficient to yield a well-defined solution. Nonetheless, there is the possibility that this is violated if \( \alpha_3, \alpha_4 < 0 \). Naturally, we rule this out. Unlike the Sargent-Wallace
model again, an interest rate peg does not necessarily imply a unit root; invoking $\phi = \omega$ in equation (3.3.22) we have

$$\delta_j(r) = \frac{\alpha_1 + \alpha_3 + \beta(1 + \alpha_4 y_1)}{\alpha_1 \alpha_2 + \alpha_3 + \beta(1 + \alpha_4 y_1)} \delta_{j-1}(r), \quad j \geq 2 \quad (3.3.24)$$

Obviously, the usual assumption of $\alpha_2 = 1$ is necessary to generate a unit root; otherwise uniqueness and stability will depend on the structural parameters of the system. In any event, we assume the relevant condition to be satisfied. It may also be noted that for the orthodox version of the Lucas-Rapping model, $\alpha_2 = 1$, $\alpha_3 = \alpha_4 = 0$ so that $f = b$, $g = \beta$, and $\delta_o = \mu_o$, where $b$ and $\mu_o$ are defined in section 3.3(A). In addition, there is necessarily a unit root under an interest rate peg, and an unstable difference equation under a money stock peg. Anticipating somewhat, therefore, the orthodox Lucas-Rapping system will have identical results to those derived from the Sargent-Wallace model.

Thus, assuming a unique and non-explosive solution,

$$\delta_j = 0 \quad (j > 1)$$

and the price solution is

$$p_t = \delta_o v_t \quad (3.3.25)$$

where $\delta_o$ and $v_t$ are defined above. To obtain the solution for output recall that $\delta_j = 0 \quad (j > 1)$. Then from equation (3.3.20),

$$p_{t,t-1} = p_{t+1,t-1} = 0. \quad \text{Substituting these into equation (3.3.18),}$$

$$y_t = [(\alpha_1 + \alpha_4)(1 + f(\alpha_3 - \alpha_4 \phi) y_1) + f(\alpha_3 - \alpha_4 \phi)] \delta_o v_t + (1 + f(\alpha_3 - \alpha_4 \phi)y_1) \delta_2 t + f(\alpha_3 - \alpha_4 \phi) \delta_3 t \quad (3.3.26)$$

The impact effect of exogenous shocks are of rather more
interest here than in the Sargent-Wallace model, indicating some apparently perverse responses. From equations (3.3.25) - (3.3.26) and using the definition of \( v_t \), these are given by

\[
\begin{align*}
\frac{\partial p_t}{\partial \varepsilon_{1t}} &= -\delta_o, \\
\frac{\partial p_t}{\partial \varepsilon_{2t}} &= \delta_o (1 + f g y_1), \\
\frac{\partial p_t}{\partial \varepsilon_{3t}} &= \delta_o f g
\end{align*}
\]

\[
\frac{\partial y_t}{\partial \varepsilon_{1t}} = -[(\alpha_1 + \alpha_4)(1 + f(\alpha_3 - \alpha_4) y_1) + f(\alpha_3 - \alpha_4)] \delta_o
\]

(3.3.27)

\[
\frac{\partial y_t}{\partial \varepsilon_{2t}} = [(\alpha_1 + \alpha_4)(1 + f(\alpha_3 - \alpha_4) y_1) + f(\alpha_3 - \alpha_4)] \delta_o (1 + f g y_1)
\]

\[
+ (1 + f(\alpha_3 - \alpha_4) y_1)
\]

\[
\frac{\partial y_t}{\partial \varepsilon_{3t}} = [(\alpha_1 + \alpha_4)(1 + f(\alpha_3 - \alpha_4) y_1) + f(\alpha_3 - \alpha_4)] \delta_o f g
\]

\[
+ f(\alpha_3 - \alpha_4).
\]

The expressions in equation (3.3.27) appear rather forbidding but may be simplified greatly by substituting for \( f, g \) and \( \delta_o \). In terms of the latter,

\[
\delta_o = - \left[ \frac{\gamma_2 - \alpha_3 y_1 + \phi (1 + \alpha_4 y_1)}{(\alpha_1 + \alpha_4)(\gamma_2 + y_1 \beta) + (\beta + \alpha_3) + \alpha_1 \phi} \right]
\]

(3.3.28)

so that \( \frac{\partial p_t}{\partial \varepsilon_{1t}} \geq 0 \) according to whether \( \delta_o \leq 0 \), or a positive demand shock may actually reduce prices. Similar perverse results occur for money demand and supply disturbances. Rather than comment
further on these here, however, it is useful to return to them in conjunction with the control problem. For the moment, the following expressions are noted which are fairly straightforward, though rather tedious, to derive:

\[
\frac{\partial y_t}{\partial e_{1t}} = \frac{(\alpha_1 + \alpha_4)\gamma_2 + \alpha_1 \phi + \alpha_3}{(\alpha_1 + \alpha_4)(\gamma_2 + \gamma_1 \beta) + (\beta + \alpha_3) + \alpha_1 \phi} \tag{3.3.29}
\]

\[
\frac{\partial p_t}{\partial e_{1t}} = \left[ \frac{\gamma_2 + \gamma_1 \beta + \phi}{(\alpha_1 + \alpha_4)(\gamma_2 + \gamma_1 \beta) + (\beta + \alpha_3) + \alpha_1 \phi} \right] \tag{3.3.30}
\]

\[
\frac{\partial y_t}{\partial e_{2t}} = \frac{\beta}{(\alpha_1 + \alpha_4)(\gamma_2 + \gamma_1 \beta) + (\beta + \alpha_3) + \alpha_1 \phi} \tag{3.3.31}
\]

\[
\frac{\partial p_t}{\partial e_{2t}} = \left[ \frac{\beta + (\alpha_3 - \alpha_4 \phi)}{(\alpha_1 + \alpha_4)(\gamma_2 + \gamma_1 \beta) + (\beta + \alpha_3) + \alpha_1 \phi} \right] \tag{3.3.32}
\]

\[
\frac{\partial y_t}{\partial e_{3t}} = \left[ \frac{(\alpha_1 + \alpha_4)\beta}{(\alpha_1 + \alpha_4)(\gamma_2 + \gamma_1 \beta) + (\beta + \alpha_3) + \alpha_1 \phi} \right] \tag{3.3.33}
\]

It is of interest to note that all of the above responses reduce to those for the Sargent-Wallace model for the special case in which \(\alpha_3 = \alpha_4 = 0\) and the conventional Lucas-Rapping framework obtains. This illustrates the observation made earlier that for \(\alpha_3 = \alpha_4 = 0\), \(f = b\), \(g = \beta\) and \(\delta_0 = \mu_0\) where \(b\) and \(\mu_0\) are defined as before.

From equations (3.3.25) - (3.3.26) and recalling the definition of \(v_t\),
\[
\sigma_i^2 = \delta_i^2 \sigma_v^2 
\]

\[
\sigma_y^2 = \left[ (\alpha_1 + \alpha_4) (1 + f(\alpha_3 - \alpha_3^1) \gamma_1) + f(\alpha_3 - \alpha_4^1) \right]^2 \delta_i^2 \sigma_v^2 
\]

\[
+ (1 + f(\alpha_3 - \alpha_4^1) \gamma_1)^2 \sigma_2^2 + f(\alpha_3 - \alpha_4^1) \sigma_3^2 
\]

\[
+ 2[(\alpha_1 + \alpha_4)(1 + f(\alpha_3 - \alpha_4^1) \gamma_1) + f(\alpha_3 - \alpha_4^1)] \delta_i^2 f(\alpha_3 - \alpha_4^1) \sigma_v^3 
\]

\[
+ 2[(\alpha_1 + \alpha_4)(1 + f(\alpha_3 - \alpha_4^1) \gamma_1) + f(\alpha_3 - \alpha_4^1)] \delta_i^2 (1 
\]

\[
+ f(\alpha_3 - \alpha_4^1) \gamma_1) \sigma_v^2 
\]

where \( \sigma_v^2 \) and \( \sigma_v^k \) (\( k = 1, 2, 3 \)) have the usual interpretations.

The results of the control exercise are summarized in table 3.3(B). These are of some interest for they imply a reversal of optimal policies (relative to the Sargent-Wallace model) in some cases. The precise conditions for a particular policy to be optimal are rather involved, reflecting the high degree of simultaneity in the model. Nonetheless, some fairly intuitive remarks can be made with minimal algebraic complication. The following discussion is conducted with reference to the responses given in equations (3.3.29) - (3.3.33).

Thus, consider expenditure shocks. Then

\[
\sigma_P^2 (\bar{\omega}, \epsilon_1) \geq \sigma_P^2 (\bar{\epsilon}, \epsilon_1) \iff (\gamma_2 - \alpha_3 \gamma_1)^2 \sigma_1^2 - (1 + \alpha_4 \gamma_1)^2 \kappa_2^2 \geq 0 \quad (3.3.36)
\]

\[
\sigma_y^2 (\bar{\omega}, \epsilon_1) \geq \sigma_y^2 (\bar{\epsilon}, \epsilon_1) \iff ((\alpha_1 + \alpha_4) \gamma_2 + \alpha_3)^2 - \kappa_2^2 \geq 0 \quad (3.3.37)
\]
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<th>Criteria</th>
<th>$\sigma_p^2$</th>
<th>$\sigma_y^2$</th>
</tr>
</thead>
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<td>$\bar{x}$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Expenditure</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>$\frac{(1 + \alpha_4 \gamma_1)^2}{\alpha_1 \sigma_1}$</td>
<td>$\frac{(\alpha_1 + \alpha_4)(\gamma_2 + \gamma_1 \beta)}{\kappa_2 \sigma_1}$</td>
</tr>
<tr>
<td>Supply</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{-(\gamma_2 + \gamma_1 \beta)^2}{\kappa_2 \sigma_2}$</td>
<td>$\frac{-1^2}{\alpha_1 \sigma_2}$</td>
<td>$\frac{\beta^2}{\kappa_2 \sigma_2}$</td>
</tr>
<tr>
<td>Money Demand</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{-(\beta + \alpha_3)^2}{\kappa_2 \sigma_3}$</td>
<td>$\frac{\alpha_4^2}{\alpha_1 \sigma_3}$</td>
<td>$\frac{-(\alpha_1 + \alpha_4) \beta}{\kappa_2 \sigma_3}$</td>
</tr>
</tbody>
</table>

Table 3.3(B) : Optimal Monetary Instruments for the Lucas-Rapping model.

$$\kappa_2 = (\alpha_1 + \alpha_4)(\gamma_2 + \gamma_1 \beta) + (\beta + \alpha_3)$$
Assuming appropriate values for $a_3$, $a_4 > 0$, therefore, $\sigma_y^2(\bar{m}, \epsilon_1) < \sigma_y^2(\bar{r}, \epsilon_1)$ and $\sigma_y^2(\bar{m}, \epsilon_1) < \sigma_y^2(\bar{r}, \epsilon_1)$ and a money supply peg is optimal. The reason is as follows. If $a_3$, $a_4 > 0$ and the money supply is fixed, the initial rise in prices following a positive demand perturbation increases output not only through the usual unanticipated price effect, but also via the induced increase in real interest rates and induced fall in real balances. The rise in supply dampens the price movement which feeds back onto output. In fact, from (3.3.28) a sufficiently high value for $a_3 > 0$ implies $\sigma_y^2(\bar{r}, \epsilon_1) < 0$, whilst a sufficiently high value for $a_4 > 0$ requires $\phi > 0$ for this not to be offset. When interest rates are pegged, however, the stabilizing effects on and of prices are reduced: in particular, real interest rate and real money balance fluctuations are dampened. The opposite line of reasoning can be applied for the case in which $a_3$, $a_4 < 0$ imply the superiority of an interest rate peg.

Turning next to supply disturbances,

$$\sigma_y^2(\bar{m}, \epsilon_2) \geq \sigma_y^2(\bar{r}, \epsilon_2) \iff (\gamma_2 + \gamma_1 \phi)^2 \alpha_1 - \kappa_2^2 \geq 0 \quad (3.3.38)$$

whilst $\sigma_y^2(\bar{m}, \epsilon_2) > \sigma_y^2(\bar{r}, \epsilon_2) = 0$ unambiguously. Thus, sufficient values for $a_3$, $a_4 > 0$ imply $\sigma_y^2(\bar{m}, \epsilon_2) < \sigma_y^2(\bar{r}, \epsilon_2)$ and there is a conflict in optimal policies. This is consistent with the usual result. Nevertheless, there is still the possibility that $\sigma_y^2(\bar{m}, \epsilon_2) > \sigma_y^2(\bar{r}, \epsilon_2)$ if $a_3$, $a_4 < 0$ sufficiently. Under such circumstances, the initial tendency for prices to fall following a positive supply shock may be reversed as evidenced by equation (3.3.30). The reason is most easily seen by rewriting the denominator as $\alpha_1(\gamma_1 + \phi) + \alpha_3 + \beta(1 + (\alpha_1 + \alpha_4)\gamma_1)$ which indicates that
\( \partial p_t/\partial \varepsilon_{3t} > 0 \) is more likely as \( \alpha_3, \alpha_4 < 0 \) increase in absolute value and provided that \( \beta \) and \( \gamma_1 \) are also relatively large. In such a case, the tendency for interest rates to fall induces a potentially large increase in excess demand as \( \beta \) rises in value which tends to exert upward pressure on prices. Provided \( \gamma_1 \) is relatively large and \( \phi + 0 \) this may manifest itself in higher interest rates. With \( \alpha_3 < 0 \) sufficiently, this causes a contraction in output (see equation (3.3.31)) and further upward pressure on prices. In addition, with \( \alpha_4 < 0 \), the fall in real balances exacerbates the supply response. Adopting an accommodative monetary stance will offset the destabilizing effects of real interest rate and real balance variations. Hence, the conflict in optimal policies may be resolved.

Consider, finally, money demand disturbances for which we have

\[
\sigma_{Pm}^2(\varepsilon_3) > \sigma_{Pr}^2(\varepsilon_3) \iff (\beta + \alpha_3)^2 \sigma_1^2 - \alpha_4^2 \sigma_2^2 > 0 \tag{3.3.39}
\]

and \( \sigma_{ym}^2(\varepsilon_3) > \sigma_{yr}^2(\varepsilon_3) = 0 \). The interesting feature here is that the normal result, \( \sigma_{Pm}^2(\varepsilon_3) > \sigma_{Pr}^2(\varepsilon_3) = 0 \), holds only if \( \alpha_4 = 0 \). By contrast, \( \alpha_4 \neq 0 \) implies that money market disturbances are able to impinge directly on prices via the non-zero real balance effect. In particular, \( \alpha_4 > 0 \) sufficiently may even incline the optimal policy towards a money stock peg. This is because the incipient fall in prices following a random increase in money demand causes an increase in real balances. There is a further contraction in aggregate supply and the fall in prices is dampened. Alternatively, fixing the interest rate may cause real balances to rise temporarily to the extent that prices actually exhibit a net upward movement (in terms of equation (3.3.30), \( \partial p_t/\partial \varepsilon_{3t} > 0 \) is more likely as \( \alpha_4 > 0 \) increases and \( \phi + \infty \)).
Two further features of table 3.3.(B) are worth noting. First, when interest rates are fixed the variance of output in the face of each type of disturbance are identical to those for the Sargent-Wallace model. Second, when $a_3 = a_4 = 0$ (and $a_1 = a$), the equivalence between the two models is complete. In this respect, the generalized Lucas-Rapping framework offers an important contribution to the monetary instruments problem since it implies the possibility of a reversal in the usual results. Though Benavie (1983) was the first to suggest this conclusion, we feel that the current treatment is more satisfactory for a number of reasons. First, the precise economic mechanism underlying the results has been explicated in more detail. Second, and as a consequence, the current exposition is much more rigorous, yielding insights which were overlooked by Benavie (1983). Third, this author is very much mistaken when he abstracts from wealth effects on aggregate supply on the grounds that they are unimportant.

As in section 3.3(A), it is possible to derive an optimal value for $\phi$ which minimizes the loss function in equation (3.2.11). The simplest way of executing this here is to note that

$$E(J) = \frac{1}{2} \sum_{k=1}^{L} \left[ a \frac{\partial \sigma_k}{\partial \phi} \left( \frac{\partial y_t}{\partial \sigma_k} \right)^2 + (1-a) \frac{\partial \sigma_k}{\partial \phi} \right] (3.3.40)$$

in which case, the optimal value for $\phi$, $\phi^*_3$, is given implicitly by

$$\frac{\partial E(J)}{\partial \phi} = \frac{1}{2} \sum_{k=1}^{L} \left[ a \frac{\partial \sigma_k}{\partial \phi} \left( \frac{\partial y_t}{\partial \sigma_k} \right)^2 + (1-a) \frac{\partial \sigma_k}{\partial \phi} \right] = 0 \quad (3.3.41)$$
Thus, using equations (3.3.29) - (3.3.33) in (3.3.40),

\[ E(J) = \kappa_2^2 \left[ a_1^2 + \beta_1^2 + a_2^2 \right] \\
+ (1-a) \left[ a_1^2 + a_2^2 + a_3^2 \right] \]  

(3.3.42)

where

\[ \kappa_2 = (\alpha_1 + \alpha_4) \gamma_2 + \gamma_1 \beta + (\beta + \alpha_3) + \alpha_1 \phi \]

\[ a_1 = (\alpha_1 + \alpha_4) \gamma_2 + \gamma_3 + \alpha_1 \phi \]

\[ a_2 = (\alpha_1 + \alpha_4) \beta \]

\[ a_3 = \gamma_2 + \beta \gamma_1 + \phi. \]

It is useful to note that the expression for \( \kappa_2 \) can be rearranged such that

\[ \kappa_2 = a_1 + a_2 \gamma_1 + \beta = \alpha_1 \gamma_3 + \alpha_4 (\gamma_2 + \gamma_1 \beta) + (\beta + \alpha_3). \]

Equation (3.3.42) is now differentiated with respect to \( \phi \) after which some lengthy manipulation reveals \( \phi^*_B \) to be

\[ \phi^*_B = \frac{-[K_1 \gamma_2^2 + K_2 \gamma_1^2 + K_3 \gamma_3^2]}{K} \]  

(3.3.43)

where

\[ K_1 = a_0 \alpha \left( \gamma_2 + \alpha_3 \right) l_1 + (1-a)(\gamma_2 - \alpha_3 \gamma_1) l_2 \]

\[ K_2 = a_0 \beta^2 + (1-a)(\gamma_2 + \gamma_1 \beta) l_3 \]

\[ K_3 = a_0 (\gamma_2 + \gamma_1 \beta) - (1-a)(\beta + \alpha_3) l_4 \]

\[ l_1 = (\alpha_1 + \alpha_4) \gamma_1 + 1 \]

\[ l_2 = (1 + \alpha_4 \gamma_1)((\alpha_1 + \alpha_4) (\gamma_2 + \gamma_1 \beta) + \alpha_3 + \beta) + \alpha_1 (\alpha_3 \gamma_2 - \gamma_2) \]

\[ l_3 = \alpha_4 (\gamma_2 + \gamma_1 \beta) + \beta + \alpha_3 \]

\[ l_4 = \alpha_4 ((\alpha_1 + \alpha_4) (\gamma_2 + \gamma_1 \beta) + \alpha_4 (\alpha_3 + \beta)) + \alpha_1 (\alpha_3 + \beta) \]

\[ K = (a_0 \alpha \gamma_1^2 l_1 + (1-a)(1 + \alpha_4 \gamma_1) l_2 \gamma_2 + (1-a) l_3 \gamma_3 + (1-a) \alpha_1 \gamma_1 l_4) \]
The complexity of the expression (3.3.43) is again a symptom of the high degree of simultaneity in the model. For isolated expenditure shocks, however, (3.4.33) reduces to \( \Phi_B^*(e_1) = -(a_3\sigma_3^2 + (1-a)(1+\alpha_4\gamma_1)l_2)^{-1}K_1 \), in which case appropriate values for \( \alpha_3, \alpha_4 > 0 \) imply \( \Phi_B^*(e_1) < 0 \). This follows intuitively from the earlier results concerning the superiority of a money supply peg: a counter-cyclical monetary response enhances the stabilizing properties of endogenous interest-rate movements. Moreover, appropriate values for \( \alpha_3, \alpha_4 < 0 \) may imply \( \Phi_B^*(e_1) > 0 \).

For the case of supply shocks, \( \Phi_B^*(e_2) = -((1-a)l_3)^{-1}K_2 \) which, recalling the definition of \( K_2 \), indicates a source of conflict. When \( a = 1 \), it is obvious that an interest rate peg is optimal. When \( a = 0 \), a counter-cyclical response is required. The latter is interesting since it is independent of the parameters \( \alpha_3 \) and \( \alpha_4 \). For \( \alpha_3, \alpha_4 > 0 \), the counter-cyclical response is not surprising since it has already been shown that a fixed money stock policy is superior to a fixed interest rate policy. The reason why this appears still true for \( \alpha_3, \alpha_4 < 0 \) is not immediately obvious and is more than likely unsustainable as a policy which is conducive to stability.

Money demand volatility yields \( \Phi_B^*(e_3) = -((1-a)\sigma_4\gamma_4)^{-1}K_3 \) so that \( a = 1 \) implies the usual interest rate peg optimality whilst \( a = 0 \) is associated with either a pro-cyclical or counter-cyclical policy depending on whether \( \alpha_3, \alpha_4 \not< 0 \). The reason is that with \( \alpha_4 > 0 \), the fall in prices following an increase in money demand is dampened via the contraction in output resulting from the increase in real balances. This stabilizing effect can be enhanced by expanding the money stock. By contrast, \( \alpha_3 > 0 \) has the opposite effect to \( \alpha_4 > 0 \). Hence the appearance of \( \alpha_3 \) in the numerator of \( \Phi_B^*(e_3) \) in order that interest rate fluctuations are subject to greater stabilization as \( \alpha_3 > 0 \) increases.
The above optimal policies for price stabilization \((a = 0)\) coincide precisely with those derived by Benavie (1983)\(^1\). Those for output stabilization \((a = 1)\) are slightly different which is probably due to the fact that Benavie (1983) dates expectations at time \(t\).

In addition, by setting \(a_3 = a_4 = 0\) (the popular version of the Lucas-Rapping model), the optimal policies reduce to those for the Sargent-Wallace model. Though this was anticipated earlier, the reason has not been given. It is, in fact, simple; with \(a_3 = a_4 = 0\) nothing essential has been changed. Aggregate supply is determined exclusively by price movements and the replacement of \(p_{t,t-1}^e\) by \(p_{t+1,t-1}^e\) retains predetermined expectations.

3.3(C) The Control Problem in the Turnovsky Model

The model is given by equations (3.2.1c), (3.2.2c), (3.2.3c), (3.2.4) - (3.2.5). Proceeding as usual, obtain the expression

\[
0 = (\beta(1+b) + ac)p_t - \alpha c p_{t,t-1}^e - \beta p_{t+1,t}^e + u_t
\]

(3.3.44)

where \(b\), \(c\) and \(u_t\) are defined as before. For the solution

\[
p_t = \sum_{i=0}^{\infty} \xi_i u_{t-i}
\]

(3.3.45)

the relevant identities are

\[
0 = (\beta(1+b) + ac)\xi_0 - \beta \xi_1 + 1
\]

(3.3.46)

\[
\xi_j = (1 + b)\xi_{j-1}, \quad j > 2
\]

(3.3.47)

so that stability requires \(\xi_j = 0\) \((j > 1)\) and from (3.3.46)
\[ \xi_0 = -(\beta(1 + b) + \alpha c)^{-1}. \] (3.3.48)

Solutions for prices and output are therefore

\[ p_t = \xi_0 u_t \] (3.3.49)

\[ y_t = \alpha \xi_0 u_t + \epsilon_{2t} \] (3.3.50)

and it is easily verified that the partial derivatives, \( \partial p_t/\partial \epsilon_{kt} \) and \( \partial y_t/\partial \epsilon_{kt} \) \((k = 1, 2, 3)\), have the usual signs. The precise values of these terms can be obtained by substituting for \( \mu_\circ \) in equation (3.3.7) the expression for \( \xi_0 \) given in equation (3.3.48). It is also evident that

\[ \sigma^2_p = \xi_0^2 \sigma^2_u \] (3.3.51)

\[ \sigma^2_y = \alpha^2 \xi_0^2 \sigma^2_u + \sigma^2_2 + 2\alpha \xi_0 \sigma_{u2} \] (3.3.52)

from which the elements in table 3.3(C) emerge (\( k_l \) is defined as previously).

The results for expenditure and monetary disturbances are standard. It is of interest to note, however, that with the exception of \( \sigma^2_p(\tau, \epsilon_3) = \sigma^2_y(\tau, \epsilon_3) = 0 \), the variances of prices and output are always less than their Sargent-Wallace counterparts. This was overlooked in the original analysis by Turnovsky (1980) but is of some importance because it emphasizes the role of the information structure in determining economic behaviour. In short, the reason is that in the current model, some agents have access to contemporaneous information about aggregate prices. This knowledge conveys greater information about real interest rate movements which generates the
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<td>$\left[ \frac{1}{\beta + \alpha} \right]^{2-2}_{\sigma_1^2}$</td>
<td>$\left[ \frac{\alpha}{\beta + \kappa_1} \right]^{2-2}_{\sigma_1^2}$</td>
</tr>
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<td></td>
</tr>
<tr>
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<td>$\left[ \frac{-1}{\beta + \alpha} \right]^{2}_{\sigma_2^2}$</td>
<td>$\left[ \frac{\beta(1+\gamma_2 -1 \gamma_1)}{\beta + \kappa_1} \right]^{2}_{\sigma_2^2}$</td>
</tr>
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<td>Money Demand</td>
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<td></td>
</tr>
<tr>
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<td>$\left[ \frac{-\alpha \beta \gamma_2 -1 \gamma_1}{\beta + \kappa_1} \right]^{2}_{\sigma_3^2}$</td>
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Table 3.3(C): Optimal Monetary Instruments for the Turnovsky Model
above result. This theme is developed in more detail in part II of the thesis.

Nonetheless, the information structure plays a prominent role in the case of supply shocks for which the conflict of optimal policies is no longer unambiguous. In particular, it is a simple matter to show that

\[
\sigma_p^2(\bar{m}, \epsilon_2) \leq \sigma_p^2(\bar{r}, \epsilon_2), \quad \sigma_y^2(\bar{m}, \epsilon_2) \geq \sigma_y^2(\bar{r}, \epsilon_2) \iff \gamma_1 \leq \beta^{-1}. \quad (3.3.53)
\]

If \( \gamma_1 < \beta^{-1} \) the usual conflict obtains. By contrast \( \gamma_1 > \beta^{-1} \) implies a reversal of optimal policies. This is explained by the fact that the effect of having contemporaneous price information becomes more important as \( \gamma_1 \) and \( \beta \) increase in value. Thus, observing a fall in prices following a positive supply perturbation is equivalent to observing a greater reduction in real interest rates than observing just the fall in nominal interest rates. Prices are stabilized owing to the demand increase. It is easily verified, however, that \( \delta r_t / \delta \epsilon_{2t} \geq 0 \) according to whether \( \gamma_1 \beta \geq 1^{(2)} \). Thus, interest rates may actually increase following a positive supply shock and this will qualify the stabilizing powers of a money supply peg. This is obviously more likely as both \( \gamma_1 \) and \( \beta \) increase in value. The observed greater incipient fall in real interest rates raises aggregate demand and prices to the extent that the nominal interest rate increases. Pegging interest rates, however, prevents this and causes a greater stimulus to demand. The expenditure stimulus may be sufficient to stabilize prices at the cost of destabilizing output. The above results and the conditions in equation (3.3.53) are consistent with Turnovsky (1980)\(^{(3)}\). Unfortunately, this author offered no explanation for the ambiguity of the conflict.

The optimal value for \( \phi \) in a combination policy is derived as in section 3.3(A). Thus, for the solutions (3.3.49) - (3.3.50),
define \( \xi_0 = 1/(\beta \sigma_1)^{-1} \) and recall the definition of \( u_t \). Then

\[
\frac{\partial \xi_0}{\partial \sigma_2 u_t} = -2\xi_0^{-3}[\beta^{-2}(\phi + \gamma_2)(1 + \alpha \gamma_1)\sigma_1^{-2}
+ (\beta^{-1} - \gamma_1)(\beta^{-1}(-2(\phi + \gamma_2) + \gamma_1)\sigma_2^{-2} - (1 + \alpha \beta^{-1})\sigma_3^{-2}] \\
(3.3.54)
\]

\[
\frac{\partial \xi_0}{\partial u_t^2} = \xi_0^{-1}(\gamma_1 - \beta^{-1})\sigma_2^{-2} \\
(3.3.55)
\]

and

\[
\phi_C = \frac{M_3\sigma_3^{-2} - M_1\sigma_1^{-2} + M_2\sigma_2^{-2}}{M} \\
(3.3.56)
\]

where

\[
M_1 = \beta^{-2}\gamma_2(1 + \alpha \gamma_1)(\alpha \sigma_1^2 + (1-a))
\]
\[
M_2 = (\gamma_1 - \beta^{-1})((1-a)(\beta^{-1}\gamma_2 + \gamma_1) - a\alpha(\gamma_2 + 1))
\]
\[
M_3 = (1 + \alpha \beta^{-1})(\alpha \sigma_1^2 + (1-a))
\]
\[
M = \gamma_2^{-1}M_1\sigma_1^{-2} + (\beta^{-1} - \gamma_1)((1-a)\beta^{-1} - a\alpha)\sigma_2^{-2}.
\]

As would be expected, \( \phi_C^*(\varepsilon_1) \) and \( \phi_C^*(\varepsilon_3) \) are identical to \( \phi_A^*(\varepsilon_1) \) and \( \phi_A^*(\varepsilon_3) \) respectively. For supply shocks, by contrast, \( \phi_C^*(\varepsilon_2) = -((1-a)\beta^{-1} - a\alpha)^{-1}[(1-a)(\beta^{-1}\gamma_2 + \gamma_1) - a\alpha(\gamma_2 + 1)] \) so that a counter-cyclical monetary response is optimal regardless of whether \( a = 0 \) or \( a = 1 \). This is peculiar given that the particular optimal pure policies depend critically on \( \gamma_1 \) and \( \beta \). The explanation is that for \( a = 1 \) the counter-cyclical policy is not feasible under any circumstances whilst for \( a = 0 \) it is feasible only under certain circumstances. For the former case, substitution of \( \phi_C^*(\varepsilon_2) \) with \( a = 1 \) into equation (3.3.47) yields a zero root which violates the
unique solution. For the latter, repeating the same substitution with \( a = 0 \) reveals that \( \beta y_1 < 1 \) creates a similar problem (this is analogous to the condition derived in section 3.3(A)). Thus, provided \( \beta y_1 < 1 \), the counter-cyclical policy is indeed optimal for price stabilization. This supports the analysis above and the reason is as follows. Recall that \( \sigma t/\sigma_{2t} < 0 \) according to whether \( y_1 \beta \not\in 1 \). If \( y_1 \beta < 1 \) a money supply peg is superior to an interest rate peg. By permitting the money supply to increase in response to a fall in interest rates (following a positive supply shock), the additional further fall in interest rates stimulates aggregate demand further and stabilizes prices. This policy can become destabilizing as \( y_1 \) and \( \beta \) increase in value for then \( \sigma t/\sigma_{2t} > 0 \) and a counter-cyclical monetary response will exacerbate the initial price fall.

3.3(D) The Control Problem in the Fischer Model

Equations (3.2.1d) (3.2.2d), (3.2.3d) and (3.2.4) - (3.2.5) yield

\[
0 = (\beta b + c)p_t + (\beta - y_1 c)p_t^{e},t-1 - y_1 c p_t^{e},t-2 - \beta p_{t+1,t-1} + u_t
\]

(3.3.57)

where \( b, c \) and \( u_t \) are again defined as previously. Positing the solution

\[
p_t = \sum_{i=0}^{\infty} \psi u_{t-i}
\]

(3.3.58)

the identities are

\[
\psi = -(\beta b + c)^{-1}
\]

(3.3.59)

\[
o = (\beta (1+b) + y_1 c) \psi - \beta \psi_2
\]

(3.3.60)

\[
\psi_j = (1 + b) \psi_{j-1}, \quad j > 3
\]

(3.3.61)
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<td>$\frac{1}{\kappa_3}$</td>
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</tr>
<tr>
<td>Money Demand</td>
<td>$\frac{-1}{\kappa_3} \gamma_2$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Table 3.3(D): Optimal Monetary Instruments for the Fischer Model

$$\kappa_3 = \frac{1}{\gamma_2} + (1 + \frac{1}{\gamma_2} \gamma_1)$$
so that \( \psi_j = 0 \) (\( j > 2 \)) and \( \psi_1 = 0 \) from (3.3.60). The solution for prices is

\[
P_t = \psi_o u_t \tag{3.3.62}
\]

and noting that \( p_t - p_t^{e} = \psi_o u_t + \psi_1 u_{t-1} = \psi_o u_t \) in (3.2.2d), the solution for output is

\[
y_t = \psi_o u_t + \epsilon_{2t} \tag{3.3.63}
\]

Comparative statics are of the usual sign and obtained by substituting \( \psi_o \) from (3.3.59) in place of \( \mu_o \) in equation (3.3.7). Equations (3.3.62) - (3.3.63) yield

\[
\sigma_p^2 = \psi_o^2 \sigma_u^2 \tag{3.3.64}
\]

\[
\sigma_y^2 = \psi_o^2 \sigma_u^2 + \sigma_2^2 + 2\psi_o \sigma_u^2 \tag{3.3.65}
\]

and table 3.3(D) summarizes the results of the control exercises. Clearly, these are identical to those for the Sargent-Wallace model with \( \alpha = 1 \) and therefore require no further comment. It is equally unnecessary to compute the optimal value for \( \phi \).

3.3(E) The Control Problem in the Calvo Model

Using the solution procedure in appendix B the system described by equations (3.2.1e), (3.2.2e), (3.2.3e), (3.2.4)-(3.2.5) is reduced as follows: eliminate the non-dynamic relationships by first eliminating \( m^d = m^s = m \) and \( r \) in the usual way and substitute for \( y^d \) from equation (3.2.1e) and for \( \pi = \sigma_1(q-p) \) from equations (3.2.2e) and (3.2.3e). The resulting third-order differential equation system is written in state-space form as
\[
\begin{bmatrix}
\frac{dy}{dt} \\
\frac{dp}{dt} \\
\frac{dq}{dt}
\end{bmatrix} =
\begin{bmatrix}
-\lambda c & -\lambda \beta(b + \theta_1) & \lambda \beta \theta_1 \\
0 & -\theta_1 & \theta_1 \\
-\theta_1 \theta_2 & -\theta_1 & \theta_1
\end{bmatrix}
\begin{bmatrix}
y \\
p \\
q
\end{bmatrix} +
\begin{bmatrix}
du_1 \\
du_2 \\
du_3
\end{bmatrix}
\] (3.3.66)

where \( du_1 = [du_1, du_2]^T \)

\[
du_1 = \lambda(\varepsilon_1 - \alpha \varepsilon_3) dt = \lambda(\delta \varepsilon_1 - \alpha \beta \delta \varepsilon_3)
\]

\[
du_2 = \theta_1 \varepsilon_2 dt = \theta_1 \delta \varepsilon_2
\]

and \( b \) and \( c \) retain their previous definitions. The characteristic equation of (3.3.66) is

\[
f(\tau) = \tau^3 + \lambda c \tau^2 + \lambda \beta \theta_1^2 \tau - \lambda \beta \theta_1^2 \theta_2 = 0.
\] (3.3.67)

For this model \( y \) and \( p \) are predetermined and \( q \) is non-predetermined.

A unique non-explosive solution obtains, therefore, if \( \tau_1, \tau_2 < 0 \) and \( \tau_3 > 0 \), say. Examination of the test functions of the characteristic equation (see appendix D) shows that saddlepoint stability obtains provided there is one sign change in the sequence,

\[
1, \lambda c, \lambda \beta \theta_1^2 \theta_2 + \frac{\lambda \beta \theta_1^2 \theta_2}{\lambda c}, -\lambda \beta \theta_1^2 \theta_2
\] (3.3.68)

or \( \lambda \beta \theta_1^2 \theta_2 > 0 \). (An equality can be appended to the inequality because a zero value for the relevant test function merely indicates a zero root and this occurs under an interest rate peg (\( \phi = \infty \), \( b = 0 \) (see chapter 2, section 2.4).) There are two remaining eigenvalues with negative real part: \( f(\tau)(\tau) = \tau(\tau^2 + \lambda \tau + \lambda \beta \theta_1^2 \theta_2) = 0 \). Hence, saddlepoint stability obtains necessarily for both \( \phi = 0 \) and \( \phi = \infty \).
Equation (3.3.69) below gives the stable trajectory of the system:

\[ q = -m_{21}^y - m_{21}^p \]  \hspace{1cm} (3.3.69)

\[ m_{21}^y = \frac{-\theta_1 \theta_2}{\lambda c + \tau_3} \]  \hspace{1cm} (3.3.70)

\[ m_{21}^p = \frac{-\theta_1 - \lambda \beta (c + \theta_1) m_{21}^y}{\theta_1 + \tau_3} = \frac{\tau_3 - \theta_1 - \lambda \theta_1 m_{21}^y}{\theta_1} \]  \hspace{1cm} (3.3.71)

Thus,

\[ \frac{\partial q}{\partial y} = -m_{21}^y > 0 \] \hspace{1cm} \[ \frac{\partial q}{\partial p} = -m_{21}^p \geq 0 \]  \hspace{1cm} (3.3.72)

\( \frac{\partial q}{\partial y} > 0 \) necessarily because of the assumption that contract prices depend positively on excess demand. By contrast, \( \frac{\partial q}{\partial p} < 0 \) is possible despite a similar functional relationship. To see why this should be so substitute for \( m_{21}^y \) from equation (3.3.70) into equation (3.3.71) and suppose an interest rate peg \( (b = \tau_3 = 0; \ c = 1) \). Then \( m_{21}^p(\bar{r}) \geq 0 \) and \( \frac{\partial q}{\partial p}(\bar{r}) \leq 0 \) according to whether \( \theta_1 \theta_2 \geq 1 \). A positive price perturbation induces an expected rate of disinflation and future contractions in aggregate demand. The contraction increases with \( \beta \). The expectation of this implies negative pressure on the contract price which increases with \( \theta_2 \). Moreover, if contract prices actually do fall, expected disinflation continues via the inflation mechanism (3.2.3e) which is more likely as \( \theta_1 \) increases. In contrast, relatively low values for these parameters give the usual response, \( \frac{\partial q}{\partial p}(\bar{r}) > 0 \).

To examine the control problem, the system is reduced further to just two dynamic relationships in terms of \( y, p \) and the disturbances.
Substituting equation (3.3.69) into (3.3.66),

\[
\begin{bmatrix}
\frac{dy}{dt} \\
\frac{dp}{dt}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
y \\
p
\end{bmatrix} + du
\tag{3.3.73}
\]

where \( S_{11} = -\lambda(c + \beta_1 m_{21}^Y) \)

\( S_{12} = -\lambda \beta(b + \theta_1 + \theta_1 m_{21}^P) \)

\( S_{21} = -\theta_1 m_{21}^Y \)

\( S_{22} = -\theta_1 (1 + m_{21}^P) \)

\( \sigma_y^2 \) and \( \sigma_p^2 \) are now computed from equation (3.3.67) using the method described in appendix E. Hence, note that

\[
\Sigma_0 S^T + S \Sigma_0 + \Gamma = 0 \tag{3.3.74}
\]

\[

\begin{bmatrix}
\sigma_y^2 & \sigma_{yp} \\
\sigma_{yp} & \sigma_p^2
\end{bmatrix}
\]

\[
\Gamma =
\begin{bmatrix}
\sigma_1^2 & 0 \\
0 & \sigma_2^2
\end{bmatrix}
\tag{3.3.75}
\]

where, as usual, \( \sigma_{yp} = \sigma_{py} \) is the asymptotic covariance of \( y \) and \( p \)

and \( \sigma_h^2 \) is the asymptotic variance of \( du_h \) \((h = 1, 2)\). \( S \) is the \( 2 \times 2 \) dynamic matrix in equation (3.3.73). The matrix system in equation (3.3.74) defines a set of three independent simultaneous equations in \( \sigma_y^2, \sigma_p^2 \) and \( \sigma_{yp} \). These are given by

\[
\begin{bmatrix}
2S_{11} & 0 & 2S_{12} \\
0 & 2S_{22} & 2S_{21} \\
S_{21} & S_{12} & (S_{11} + S_{22})
\end{bmatrix}
\begin{bmatrix}
\sigma_y^2 \\
\sigma_p^2 \\
\sigma_{yp}
\end{bmatrix}
= 
\begin{bmatrix}
-\sigma_1^2 \\
-\sigma_2^2 \\
0
\end{bmatrix}
\tag{3.3.76}
\]
so that applying Cramer's Rule,

\[
\sigma_y^2 = \frac{(S_{12}S_{21} - S_{22}(S_{11} + S_{22}))\sigma_1^2 - S_{12}^2\sigma_2^2}{2\kappa_4} 
\]

(3.3.77)

\[
\sigma_p^2 = \frac{(S_{12}S_{21} - S_{11}(S_{11} + S_{22}))\sigma_2^2 - S_{21}^2\sigma_1^2}{2\kappa_4} 
\]

(3.3.78)

where \(\kappa_4 = (S_{11} + S_{22})(S_{11}S_{22} - S_{12}S_{21})\).

In addition, it is useful here to simplify the expression for \(\kappa_4\).

Two useful properties are \(S_{22} = -r_3 + \lambda\beta\theta_1^m\) and \(\theta_1^m = \lambda\beta\theta_2\theta_2^c(\lambda\theta + b)\) in which case \(|\kappa_4(r)| < |\kappa_4(\bar{m})|\).

Turning first to expenditure shocks,

\[
\sigma_y^2(e_1) = \frac{(S_{12}S_{21} - S_{22}(S_{11} + S_{22}))\lambda^2\sigma_1^2}{2\kappa_4} 
\]

(3.3.79)

\[
\sigma_p^2(e_1) = \left(-\frac{S_{21}^2}{\lambda}\right)\sigma_1^2 
\]

(3.3.80)

From the definition of \(m_{21}^y\) in equation (3.3.70), \(|m_{21}^y(r)| > |m_{21}^y(\bar{m})|\) so that \(S_{21}(r) > S_{21}(\bar{m})\). Thus, the numerator and denominator in equation (3.3.80) increases and decreases respectively as \(\phi\) increases. It follows that \(\sigma_p^2(r,e_1) > \sigma_p^2(\bar{m},e_1)\). Now consider \(\sigma_y^2(e_1)\) in equation (3.3.79). Then substituting for \(S_{1j} (i,j = 1,2)\) and rearranging gives
\[ \sigma_y^2(\varepsilon_1) = - \left[ \frac{\lambda \beta (b + \tau_3 - \lambda \theta m^y_{21}) \theta_1^2 \theta_2^2}{2\kappa_4 (\lambda c + \tau_3)} + \frac{(\tau_3 (\lambda c + \tau_3) + \lambda \theta \bar{m}^y)}{2\kappa_4} \right] \lambda^2 \sigma_1^{\theta-2} \] (3.3.81)

It is clear from the definition of \( \kappa_4 \) that the second term in \( \cdot \) is almost certain to increase with \( \phi \). The effect of \( \phi \) on the first term in \( \cdot \) is a little more difficult to discern since both \( b \) and \( \tau_3 \) decline with \( \phi \) whilst \( \bar{m}^y_{21} \) increases with \( \phi \). Nonetheless, the denominator unambiguously falls (absolutely) as \( \phi \) rises which suggests strongly that the whole term in \( \cdot \) increases. Then \( \sigma_y^2(\bar{m}, \varepsilon_1) > \sigma_y^2(m, \varepsilon_1) \). These results are supportive of the previous findings, though the intuition is slightly different here. A positive expenditure shock raises contract prices and, for a money supply peg, interest rates. The former implies subsequent price inflation. The rise in interest rates dampens output and, via the effect on contract prices, dampens aggregate price movements which otherwise tend to stimulate demand through real interest rates. An accommodative monetary policy, however, negates the stabilizing interest rate fluctuations and induces a further jump in the contract price because of the implied higher expected demand.

Supply shocks in the model are incorporated in the inflation mechanism (3.2.2e). From equations (3.3.77) - (3.3.78),

\[ \sigma_y^2(\varepsilon_2) = - \frac{S_{12}^2 \theta_1^2 \sigma_2^2}{2\kappa_4} \] (3.3.82)

\[ \sigma_p^2(\varepsilon_2) = \frac{(S_{12}^2 S_{21} - S_{11}(S_{11} + S_{22})) \theta_1^2 \sigma_2^2}{2\kappa_4} \] (3.3.83)
As before, we may re-write equations (3.3.82) - (3.3.83) as

\[
\sigma^2_y(\varepsilon_2) = -\frac{\lambda^2 \beta^2((b + \tau_3)(\lambda c + \tau_3) + \lambda \beta \theta_1 \theta_2)^2}{2\kappa_4(\lambda c + \tau_3)^2} \sigma^2_1 \sigma_2
\]  
\[\text{(3.3.84)}\]

\[
\sigma^2_p(\varepsilon_2) = -\left[\frac{\lambda \beta(b + \tau_3 - \lambda \beta \theta_1 \theta_2^y)}{2\kappa_4(\lambda c + \tau_3)} \theta_1^2 + \frac{\lambda(c(\lambda c + \tau_3) - \beta \theta_1 \theta_2)}{2\kappa_4}\right] \sigma^2_1 \sigma_2
\]  
\[\text{(3.3.85)}\]

in which case the most plausible outcome is \(\sigma^2_y(\overline{\varepsilon}, \varepsilon_2) > \sigma^2_y(\overline{\varepsilon}, \varepsilon_2)\) and \(\sigma^2_p(\overline{\varepsilon}, \varepsilon_2) > \sigma^2_p(\overline{\varepsilon}, \varepsilon_2)\). The reason is as follows. An initial positive perturbation to prices implies an expected rate of disinflation. The disinflation is sluggish because of the upward tendency of contract prices. Output effects will primarily follow the initial increase in the interest rate under a money supply peg. Expected lower demand tends to depress the contract price and stabilize prices. Output is stabilized by virtue of the lower interest rates. By contrast, accommodating monetary changes neutralizes these effects and the disturbance tends to be distributed over a longer period of time.

Implications of monetary volatility are captured in

\[
\sigma^2_y(\varepsilon_3) = \frac{(S_{12}S_{21} - S_{22}(S_{11} + S_{22}))\lambda^2 \beta^2 b^2 \sigma_3^2}{2\kappa_4} \]  
\[\text{(3.3.86)}\]

\[
\sigma^2_p(\varepsilon_3) = -\frac{S_{21}^2 \lambda^2 \beta^2 b^2 \sigma_3^2}{2\kappa_4} \]  
\[\text{(3.3.87)}\]

so that the usual result obtains, namely \(0 = \sigma^2_y(\overline{\varepsilon}, \varepsilon_3) < \sigma^2_y(\overline{\varepsilon}, \varepsilon_3)\) and \(0 = \sigma^2_p(\overline{\varepsilon}, \varepsilon_3) < \sigma^2_p(\overline{\varepsilon}, \varepsilon_3)\).

A derivation of an optimal value for \(\phi\), whilst possibly yielding greater insight, is obviously impossible for this system given its dynamic configuration.
3.3(F) The Control Problem in the Open Economy Model

The system described by equations (3.2.1f), (3.2.2f), (3.2.3f), (3.2.4) – (3.2.6) is reduced to a fourth-order dimension as follows: \( m^d = m^s = m, r \) and \( y^d \) are eliminated as before, as is \( \pi = \theta y \) from equations (3.2.2f); recall the process for \( \nu \) in equation (3.2.10) and include this in the state vector. Then

\[
\begin{bmatrix}
\frac{dy}{dt} \\
\frac{dp}{dt} \\
\frac{dv}{dt} \\
\frac{de}{dt}
\end{bmatrix} =
\begin{bmatrix}
-\lambda(c - \beta_1 \theta) & -\lambda(\beta_1 b + \beta_2) & 0 & \lambda \beta_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\rho & 0 \\
by_1 - n_1 & b - n_2 & 1 & n_2
\end{bmatrix}
\begin{bmatrix}
y \\
p \\
v \\
e
\end{bmatrix}
+ \begin{bmatrix}
du^1 \\
du^2
\end{bmatrix}
\]

(3.3.88)

where
\[
n_1 = \eta_1^{-1} \eta_2 \\
n_2 = \eta_1^{-1} \eta_3
\]

\[
du^1 = [du_1 \ du_2 \ du_3]^T \\
du^2 = du_4
\]

\[
du_1 = \lambda(\epsilon_1 - \beta_1 b \epsilon_3) dt = \lambda(\epsilon_1 - \beta_1 b \epsilon_3) \\
du_2 = \theta \epsilon_2 dt = \theta \epsilon_2 \\
du_3 = \rho \epsilon_4 dt = \rho \epsilon_4 \\
du_4 = b \epsilon_3 dt = b \epsilon_3 \\
v = \eta_1^{-1} \nu \\
\epsilon_4 = \eta_1^{-1} \nu
\]

and where \( b \) and \( c \) are defined as usual. The characteristic equation of (3.3.88) is
\[ q(\tau) = (\rho + \tau)(\tau^3 - (n_2 - \lambda(c - \beta_1 \theta))\tau) \]
\[ - \lambda[\beta_2(\gamma_1 - n_1) + n_2(c - \beta_1 \theta) - \theta(\beta_1 \beta + \beta_2)]\tau \]
\[ - \lambda\theta\beta(\beta_2 + \beta_1 n_2) = 0 \quad (3.3.89) \]

so that \(-\rho\), the autoregressive coefficient for the disturbance \(v\), is one of the stable eigenvalues, \(\tau_1\) say. Since \(y\) and \(p\) are predetermined and \(e\) is non-predetermined, saddlepoint stability then requires that \(\tau_2, \tau_3 < 0\) and \(\tau_4 > 0\), say. Examination of the \(\cdot\) term in equation (3.3.89) is sufficient to investigate this. Hence, computing the test functions of (3.3.89) (see appendix D), the saddlepoint property is satisfied if there is one and only one sign change in the sequence.

\[ -(n_2 - \lambda(c - \beta_1 \theta)) \]
\[ - \lambda[\beta_2(\gamma_1 - n_1) + n_2(c - \beta_1 \theta) - \theta(\beta_1 \beta + \beta_2)] - \frac{\lambda\theta\beta(\beta_2 + \beta_1 n_2)}{n_2 - \lambda(c - \beta_1 \theta)} \]
\[ - \lambda\theta\beta(\beta_2 + \beta_1 n_2) \quad (3.3.90) \]

It is of some interest to examine this condition in some detail here. In particular, an additional implication of the present analysis is that it is not the degree of capital mobility per se which matters for stability, but rather the relative magnitudes of current account parameters. Moreover, these are crucial in determining the impact effect of price level movements on the exchange rate which may exhibit perverse responses. As far as we know, these issues have hitherto gone unnoticed.

Consider, then, the second test function, \(-(n_2 - \lambda(c - \beta_1 \theta))\) and note that \(c - \beta_1 \theta > 0\) is a necessary condition for stability (this
is proved below). Then as the degree of capital mobility increases
\((n_1 \to \infty, n_2 \to 0)\), \(-n_2 - \lambda(c - \beta_1 \theta) > 0\), and the saddlepoint
condition reduces to just \(\lambda \Theta_b \beta_2 + \beta_1 n_2 > 0\) (where it is noted
again that an interest rate peg, \(\phi = \infty, b = 0\), implies a zero root).
Thus, a high degree of capital mobility is certainly conducive to a
well-defined solution. As \(n_1 \to 0 (n_2 > 0)\), however,
\(-n_2 - \lambda(c - \beta_1 \theta) < 0\) is more likely and it is necessary to examine
the behaviour of the third test function. It is clear that as \(n_2\)
increases, this test function also becomes negative, implying that
the saddlepoint condition reduces to that above, \(\lambda \Theta_b \beta_2 + \beta_1 n_2 > 0\).
Note, however, that \(n_1 \to 0\) also raises \(n_1\) which operates against
stability. This is the first indication that it is not the degree of
capital mobility per se which is important.

To elaborate further, suppose an interest rate peg. The zero
root is witnessed in the expression \(\cdot\) in equation (3.3.89):
\[g(\tau)(\bar{\tau}) = (\rho + \tau)\tau^2 - (n_2 - \lambda(1 - \beta_1 \theta))\tau - \lambda(n_2(1 - \beta_1 \theta) - \beta_2(n_1 + \theta)) = 0.\]
Then saddlepoint stability requires no change in the sequence
\[1, -(n_2 - \lambda(1 - \beta_1 \theta)), -\lambda(n_2(1 - \beta_1 \theta) - \beta_2(n_1 + \theta)).\] (3.3.91)

Perfect capital mobility implies \(n_1 = n_2 = 0\) so that provided
\(1 - \beta_1 \theta > 0\), the condition is satisfied. Since \(c > 1\), this verifies
the earlier proposition that \(c - \beta_1 \theta > 0\). For the case of imperfect
capital mobility it is seen that, whilst higher values for \(n_2\) tend
to violate stability, this is not true for increasing values of \(n_1\).
Thus, it is observed again that the absolute degree of capital
mobility is not the important factor; rather, it is the relative
magnitudes of the trade balance parameters, \(n_2\) and \(n_3\), which are
reflected in \(n_1\) and \(n_2\) respectively. Moreover, the way in which
the different parameters influence stability appears to change when
b ≠ 0 (flexible interest rates) high n₂ and low n₁ are conducive to stability; when b = 0 (fixed interest rates) the reverse is true. These results are of some interest. Some intuition about them is obtained below.

The stable trajectory is

\[ e = -m_{21}^Y v - m_{21}^P p - m_{21}^v v \] (3.3.92)

\[ \frac{m_{21}^Y}{\lambda \beta_2} = \frac{\tau_4 - n_2}{\lambda \beta_2} \] (3.3.93)

\[ \frac{m_{21}^p}{\tau_4} = \frac{b - n_2 - \lambda(\beta_1 b + \beta_2)m_{21}^Y}{\tau_4} = \frac{n_1 - b\gamma_1 + (\lambda(c - \beta_1 \theta) + \tau_4)m_{21}^Y}{\theta} \] (3.3.94)

\[ m_{21}^v = \frac{1}{\rho + \tau_4} \] (3.3.95)

so that

\[ \frac{\partial e}{\partial y} \geq 0, \quad \frac{\partial e}{\partial p} \geq 0, \quad \frac{\partial e}{\partial v} = -m_{21}^v < 0 \] (3.3.96)

\( \frac{\partial e}{\partial y} \geq 0 \) depending on the degree of capital mobility and the policy regime. Thus, perfect capital mobility (n₂ = 0) implies \( \frac{\partial e}{\partial y} < 0 \) because of the induced interest rate movements following output fluctuations. Obviously, \( \frac{\partial e}{\partial y(\bar{r})} = 0 \) under perfect capital mobility.

By contrast, n₂ ≠ 0 implies \( \frac{\partial e}{\partial y(\bar{r})} > 0 \) owing to the output-induced movements in the trade balance.

Similar arguments apply to the sign of \( \frac{\partial e}{\partial p} \). There are, however, some rather more interesting cases which relate back to the discussion of stability. Suppose an interest rate peg (b = 0, c = 1, \( \tau_4 = 0 \)) and recall the definition of \( m_{21}^Y \) in equation (3.3.93).
Then it is seen that $\frac{\partial e}{\partial p(\tilde{r})} \geq 0$ according to whether
\[ n_1 - \beta_2^{-1}(1 - \beta_2 \theta)n \leq 0. \]
The normal case is $\frac{\partial e}{\partial p(\tilde{r})} > 0$ because of fluctuations in real competitiveness. A necessary condition for this is $1 - \beta_1 \theta > 0$ which is recalled to be a necessary condition for stability. Alternatively, a relatively high (low) $n_1$ ($n_2$) implies $\frac{\partial e}{\partial p(\tilde{r})} < 0$ and, from the previous discussion, this is conducive to stability. An increase in prices, therefore, may actually cause an exchange rate appreciation even under imperfect capital mobility and fixed interest rates. The reason is as follows. An initial price increase induces an expected rate of disinflation which raises real interest rates and implies a subsequent output contraction. This effect increases with $\beta_1$. The expectation of lower income propagates an expected exchange rate appreciation which increases with $n_1$ ($n_2$). This is translated into a current appreciation. In terms of stability, this is beneficial because of the dampening force on prices.

Further insight is obtained by relaxing interest rate control. Then $m^D_{21}$ is given by (3.3.94) again. The noteworthy feature is the negative sign on $by_1$, implying a tendency for the exchange rate to depreciate following a price increase as interest rates become endogenous. The reason follows from above: expectation of lower demand induces the expectation of lower interest rates and a current exchange rate depreciation. In terms of stability, it is recalled that under flexible interest rates, relatively high $n_1$ may hinder stability; under such circumstances, the depreciation is exacerbated. Thus, for perfect capital mobility, $m^D_{21} = \theta^{-1}(-by_1 + (\lambda(c - \beta_1 \theta) + \tau_4) \tau_4 / \lambda \beta_2)$ which shows that the tendency for $\frac{\partial e}{\partial p} < 0$ is partly offset owing to the expectation effects via $-by_1$. 
The foregoing digression is useful for the insights it yields into the control problem. Thus, following appendix E, substitute equation (3.3.92) into (3.3.88) and obtain

$$\begin{bmatrix}
\frac{dy}{dt} \\
\frac{dp}{dt} \\
\frac{dv}{dt}
\end{bmatrix} = \begin{bmatrix}
1_{11} & 1_{12} & 1_{13} \\
0 & 0 & 0 \\
0 & 0 & -\rho
\end{bmatrix} \begin{bmatrix}
y \\
p \\
v
\end{bmatrix} dt + du^1$$  \hspace{1cm} (3.3.97)

where $1_{11} = -(\lambda(c - \beta_1 b) + \tau_4 - \eta_2)$

$1_{12} = -\lambda(\beta_1 b + \beta_2 (1 + m_{21}^D))$

$1_{13} = -\lambda \beta_2 m_{21}^D$.

Hence $\sigma_y^2$ and $\sigma_p^2$ are given implicitly by

$$\Gamma_o L^T + LL_o + \Gamma = 0$$ \hspace{1cm} (3.3.98)

$$\Gamma_o = \begin{bmatrix}
\sigma_y^2 & \sigma_{yp} & \sigma_{vy} \\
\sigma_{py} & \sigma_p^2 & \sigma_{pv} \\
\sigma_{vy} & \sigma_{vp} & \sigma_v^2
\end{bmatrix}, \quad L = \begin{bmatrix}
\sigma_1^2 & 0 & 0 \\
0 & \sigma_2^2 & 0 \\
0 & 0 & \sigma_3^2
\end{bmatrix}$$ \hspace{1cm} (3.3.99)

where $\sigma_{vy}^2 = \sigma_{vy}$ and $\sigma_{py}^2 = \sigma_{vp}$ are the appropriate covariances and $\sigma_h^2$ is the asymptotic variance of $du_n (n = 1, 2, 3)$. $L$ is the $3 \times 3$ dynamic matrix in equation (3.3.97). The system (3.3.98) describes six independent simultaneous equations which are solved such that

$$\sigma_y^2 = \frac{-\kappa_5 \sigma_1^2 + 1_{12} \kappa_5 \sigma_2^2 + 1_{13} \sigma_3^2}{2 \lambda_{11} \kappa_5}$$ \hspace{1cm} (3.3.100)
\[
\sigma_1^2 = \kappa_5 \sigma_1^2 + \left(1_{11} \sigma_1^2 - 1_{12} \right) \kappa_5 \sigma_1^2 + 1_{13} (1_{11} \sigma_1^2 - 1_1) \sigma_3^2
\]
\[
\frac{21_{11}^1 \kappa_5}{12} 
\]

where \( \kappa_5 = (1_{11} - \rho) \rho \sigma_1^2 + 1_{12} \).

For the case of aggregate demand shocks therefore,

\[
\sigma_2^2(y) = -\frac{\lambda}{2_{11}} \sigma_1^2
\]

\[
\sigma_1^2(y) = \frac{\lambda}{2_{11}} \sigma_1^2
\]

Clearly, \( |1_{11}(\bar{r})| < |1_{11}(\bar{m})| \) regardless of the degree of capital mobility (since \( \phi = 0 \) implies \( c = 1, \tau_4 = 0 \)). Hence, \( \sigma_2^2(-y, e_1) > \sigma_2^2(-y, e_1) \) regardless of the degree of capital mobility also. The reason is, of course, that endogenising interest rates endows this variable with a stabilising role which is compounded by exchange rate movements. The implications for \( \sigma_2^2(e_1) \) are rather more difficult to discern owing to the ambiguous effect of \( \phi \) on \( 1_{12} \). Nonetheless, one would expect \( \sigma_2^2(-y, e_1) > \sigma_2^2(-m, e_1) \) also since prices depend solely on income. This means either that \( |1_{12}(\bar{r})| < |1_{12}(\bar{m})| \), reinforcing \( \delta / \delta \phi \), or that if \( \delta / \delta \phi > 0 \), this is less than \( \delta / \delta \left| 1_{11} \right| / \delta \phi \) \( O(7) \). For perfect capital mobility \( (n_1 = n_2 = 0) \), \( m_{21}^P \) is given by

\[
(3.3.88) \text{as } m_{21}^P = \phi \left( -b y_1 - \lambda(\phi - \beta_1 \theta) + \tau_4 \right) / \beta_2 \] so that assuming the normal case \( m_{21}^P > 0, \delta e / \delta p < 0 \), \( m_{21}^P(\bar{m}) > m_{21}^P(\bar{r}) = 0 \) and \( |1_{12}(\bar{r})| < |1_{12}(\bar{m})| \). For imperfect capital mobility matters are slightly more complicated though the above argument applies.

Supply shocks yield the following:

\[
\sigma_2^2(y_2) = \frac{1_{12} \sigma_2^2}{2_{11}}
\]

\[
(3.3.104)
\]
Thus, whether $\sigma_{y}^{2}(\bar{r}, e_{2}) > \sigma_{y}^{2}(\bar{m}, e_{2})$ depends on the relative rates, $\partial|l_{12}|/\partial \phi$ and $\partial|l_{11}|/\partial \phi$. Consider, first, perfect capital mobility. Then recalling the definition of $m_{2}$, it is likely that $\partial|l_{12}|/\partial \phi < 0$ is greater than $\partial|l_{11}|/\partial \phi < 0$ and $\sigma_{y}^{2}(\bar{r}, e_{2}) < \sigma_{y}^{2}(\bar{m}, e_{2})$. The reason is that, for a money supply peg, the ensuing variation in interest rates, the exchange rate and expected inflation following a price perturbation reinforce each other causing output fluctuations. An accommodative monetary policy negates the first two of these effects. Such a policy, however, is unlikely to be optimal for price stabilization. Rewriting (3.3.105) as $\sigma_{p}^{2}(e_{2}) = (1^{2}l_{11}^{2} - l_{12})\sigma_{2}^{2}$ and using the above discussion, the term in parentheses tends to increase with $\phi$ and $\sigma_{p}^{2}(\bar{r}, e_{2}) > \sigma_{p}^{2}(\bar{m}, e_{2})$. The reason follows from that given for the case of output stabilization. In that, pegging the interest rate negates much of the spillover effects onto aggregate demand. In terms of price volatility, such effects are stabilizing, offsetting the initial price shock, by virtue of the inflation mechanism. Thus, there is the usual conflict of optimal policies.

Relaxing the assumption of perfect capital mobility complicates matters. It is certainly possible, however, that $\sigma_{y}^{2}(\bar{r}, e_{2}) > \sigma_{y}^{2}(\bar{m}, e_{2})$ in this case which implies a reversal in optimal policies. Intuitively, this can be seen by considering a positive price shock. Under an interest rate peg the exchange rate depreciates inducing an expansion in output. By contrast, fixing the money stock permits a stabilizing role for the upward movement in interest rates both directly, by the rise in real interest rates, and indirectly, by
dampening the exchange rate depreciation. Moreover, and in contrast to the perfect capital mobility case, the fixed money stock policy is also likely to be optimal for price stabilization; there is no conflict of optimal policies.

The implications of monetary volatility are unambiguous: thus,

\[ \sigma^2_y(e_3) = -\frac{\lambda \rho_1^2 \sigma_3^2}{21_{11}} \]  \hspace{1cm} (3.3.106)

\[ \sigma^2_p(e_3) = \frac{8\lambda \rho_1^2 \sigma_3^2}{21_{11112}} \]  \hspace{1cm} (3.3.107)

and pegging interest rates \( b = 0 \) implies \( \sigma^2_y(r, e_3) = \sigma^2_p(r, e_3) = 0. \)

The open economy is subject to a fourth type of stochastic shock, namely a foreign or external disturbance which incorporates random shifts in trade and capital flows. For this,

\[ \sigma^2_y(e_4) = \frac{12_13 \rho^{-1} \sigma_4^2}{21_{11} K_5} \]  \hspace{1cm} (3.3.108)

\[ \sigma^2_p(e_4) = \frac{12_13 (1^11 \rho^{-1} - 1) \rho^2 \sigma_4^2}{21_{11112} K_5} \]  \hspace{1cm} (3.3.109)

Clearly, from equation (3.3.95), \( \nu_{21}(\overline{r}) > \nu_{21}(\overline{m}) \) so that \( |l_{13}(\overline{r})| > |l_{13}(\overline{m})| \). Under perfect capital mobility, recalling the previous discussion about \( l_{11} \) and \( l_{12} \), it is also likely that \( |\kappa_5(\overline{r})| < |\kappa_5(\overline{m})| \). Hence, \( \sigma^2_y(r, e_4) > \sigma^2_y(m, e_4) \). In addition, rewriting (3.3.109) as \( \sigma^2_p(e_4) = \frac{12_13 \rho(1/l_{12} K_5 - \rho/l_{11112} K_5) \sigma_4^2}{21_{11112} K_5} \), shows that \( \sigma^2_p(r, e_4) > \sigma^2_p(m, e_4) \). The optimal policy is therefore a money supply peg. Considering a positive foreign shock which causes an exchange rate appreciation, there is a contractionary effect on demand and prices. Permitting a fall in interest rates offset this by reducing
real interest rates and dampening the appreciation. Less than perfectly mobile capital is unlikely to qualify these results.

In conclusion, it is noted that the foregoing has been predicated on the assumption that the normal cases hold. Perverse responses that were recorded earlier may, of course, alter some of the results. It is not our intention to pursue these further but merely note their implication.

3.4 Summary and Concluding Remarks

Table 3.4(A) provides a conspectus of the results derived in section 3.3. It constitutes the payoff matrix for the monetary instruments problem.

Casual inspection reveals a striking consistency in certain cases. In terms of model uncertainty, this means that it may not be all that important and ab uno disce omnes. At the same time, however, the payoff matrix illustrates vividly the possible and critical model dependency of the optimal policy.

Model robust policy rules are most apparent for expenditure and money demand shocks. With the exception of only one, the optimal policy is invariant with respect to divergent model structures. For the former disturbance this takes the form of a money stock peg; for the latter type of shock, an interest rate peg is optimal. These are explained generally by the endogenous movements in interest rates when the money supply is fixed and which are stabilizing in the face of demand shocks but destabilizing under monetary volatility. The exception is for the Lucas-Rapping model in which aggregate supply is determined partly by real interest rates and real money balances. These additional dimensions provide the reason for the ambiguity of optimal policy. A special case whereby the standard results re-obtain, however, is when these effects are absent. Though this is
<table>
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<tr>
<th>Model</th>
<th>Disturbance Criteria</th>
<th>Expenditure</th>
<th>Supply</th>
<th>Money Demand</th>
<th>External</th>
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<tr>
<td>Sargent-Wallace</td>
<td>$\sigma_p^2$</td>
<td>$\bar{m}$</td>
<td>$\bar{m}$</td>
<td>$\bar{r}$</td>
<td>$-$</td>
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<td>$\sigma_y^2$</td>
<td>$\bar{m}$</td>
<td>$\bar{r}$</td>
<td>$\bar{r}$</td>
<td>$-$</td>
</tr>
<tr>
<td>Lucas-Rapping</td>
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<td>$\bar{m}/\bar{r}$</td>
<td>$\bar{m}/\bar{r}$</td>
<td>$\bar{m}/\bar{r}$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y^2$</td>
<td>$\bar{m}/\bar{r}$</td>
<td>$\bar{r}$</td>
<td>$\bar{r}$</td>
<td>$-$</td>
</tr>
<tr>
<td>Turnovsky</td>
<td>$\sigma_p^2$</td>
<td>$\bar{m}$</td>
<td>$\bar{m}/\bar{r}$</td>
<td>$\bar{r}$</td>
<td>$-$</td>
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<td>$\bar{m}$</td>
<td>$\bar{m}/\bar{r}$</td>
<td>$\bar{r}$</td>
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<tr>
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<td>$\bar{r}$</td>
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<td>$\bar{r}$</td>
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</tr>
<tr>
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<td>$\bar{m}$</td>
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<tr>
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<td>$\bar{m}$</td>
<td>$\bar{m}/\bar{r}$</td>
<td>$\bar{r}$</td>
<td>$\bar{m}$</td>
</tr>
</tbody>
</table>

Table 3.4(A) : Payoff matrix for alternative optimal instruments
a limiting case, it is worth noting that it is the most popular and extensively used version of the Lucas-Rapping framework. If any importance is attributed to this fact, the robustness of optimal policies is complete.

The major problem of defining a model robust policy occurs for the case of supply shocks. The particular optimal policy under such circumstances depends in general on the structural parameters of the model, including those defining the policy maker's preferences. This is of some importance since it is probably precisely these types of disturbances which have become the predominant source of stochastic behaviour since the early 1970s. In view of this, any conclusion regarding the optimal choice of monetary policy must be tentative. Even in this single analysis of only two alternative policy regimes the need for testing robustness of policies is blatantly clear.

At the most general and important level, the major contribution of the chapter is its rigour and its comprehensive assessment of the monetary instruments problem. This reflects the earlier remarks about the drawback of existing literature. A proper evaluation of the issue must be conducted against the background of model uncertainty; the absence of this is the essential flaw in that literature. The singular characteristics of supply side disturbances were hinted at in early studies but a full appreciation of them can only be gained from the current chapter. These aspects, which are particular to supply shocks, have not hitherto been fully recognised. In addition, the deployment of the policy problem to richer dynamic models has been a useful extension, yielding insights otherwise overlooked. The role of forward-looking behaviour becomes far from trivial in these models.

In concluding the chapter, however, it is useful to note some features of the analysis that might be considered objectionable.
The first (and probably most obvious) objection that might be raised concerns the description of policy behaviour. Recalling equations (3.2.5), this assumes a contemporaneous relationship between the money stock and the interest rate. It is questionable the extent to which this mimics the true manner in which actual policy instruments are adjusted (see, for example, McCallum and Whittaker (1979) in connection with fiscal policy). Rather, delays in receiving, processing, and interpreting information may prompt consideration of a lagged feedback rule. Under such circumstances certain of our results may cease to be true especially for the discrete time models. This is because these models exhibit zero hysteresis. Thus, since all disturbances are assumed to be purely transitory, any lagged feedback rule will imply corrective action which is out of phase with the current perturbation (see, for example, Phillips (1957) for an early exposition). The propagation mechanism which converts random shocks into persistent deviations from equilibrium is precisely the policy rule so that the optimal policy is to avoid feedback\(^8\). This point is well-taken though it has been purposely ignored given that the purpose of the chapter was to concentrate solely on one particular class of policy rule.

Two further objections that may be noted relate to the shock structure. First, agents have been endowed with the knowledge that all stochastic disturbances are independently Gaussian distributed. Second, inference problems pertaining to the source of stochastic variation have been eschewed. With regard to the latter, chapter 6 shows some misleading implications of this abstraction.

In toto, the content of the chapter has suggested some clear implications for the evaluation of the monetary instruments problem. As far as we know, the chapter is the first to make operational the methodology that emphasizes the need to search for model robust
policies. We believe it to be an important contribution to the monetary instrument debate. Yet the particular preoccupation is entirely general. Our analysis should be followed up by similar policy evaluations for different problems. The methodology is attractive and constitutes a fruitful area for further research. It could be made even more attractive if, in addition to the search for model-robust policies, one introduced uncertainties about the source of stochastic fluctuation.
Notes to Chapter Three

This chapter is a much extended version of Blackburn (1984a).

See Blackburn (1985c).

(1) See Benavie (1983), pp. 157-160, with $a_3 = \sigma$, $a_4 = 0$, $\gamma_1 = \alpha_1$,
$\gamma_2 = \alpha_2$.

(2) Substitute the solutions (3.3.43) - (3.3.44) into the expression for $r_t$ obtained by equating $m^d_t = m^s_t = m_t$ and rearrange terms.

(3) See Turnovsky (1980a), p. 49, equations (24) - (25), with $\beta = |d_2|$, $\gamma_1 = \alpha_1$.

(4) In fact, $\tau_3 = 0$ is also admissible since this is a consequence of an interest rate peg which still yields a well-defined solution (see chapter 2, section 2.4). This is shown below.

(5) The reason for specifying the autoregressive process for $v$ is now clear: equation (3.3.92) has no white noise error terms. In this respect, the continuous time version of the solution loses information relative to the discrete time case (see appendix B). Since we wish to examine external disturbances it is therefore necessary to avoid defining these as purely white noise terms.

(6) It is noted that $l_{11} < 0$ is necessary for $\sigma_y^2(\epsilon_1) > 0$.

(7) $l_{12} < 0$ is necessary for $\sigma_p^2(\epsilon_1) > 0$.

(8) The reader is invited to verify this by substituting for the policy in equations (3.2.5), the rule $m_{t+1} = \phi r_t$. 
4.1 Introduction

A well-known drawback of the early treatments of monetary and fiscal policy is their failure to appreciate the interrelationships between these policies. The standard approach amounted to performing independent comparative static exercises with respect to changes in monetary and fiscal instruments in a simple IS-LM system which could be extended to incorporate price flexibility and international trade. The early papers by Ott and Ott (1965) and Christ (1967, 1968) (see also Meyer (1975)) highlighted the fallacy of this approach by pointing out the financing requirements associated with changes in the government's fiscal stance. Failure of the earlier analyses to incorporate these render their treatment of fiscal policy in general, and the crowding out issue in particular, inadequate. First, the effect of changes in government expenditure generally depends on the method of financing this expenditure. Second, the government budget constraint imposes restrictions on the number of degrees of freedom available to the authorities exogenously choosing values for monetary and fiscal instruments; with n instruments, there are only n-1 degrees of freedom because at least one instrument must be determined residually. A related additional (and probably the most important) implication of the government budget constraint continues to receive much attention and is the main focus of this chapter. This is as follows.

The government budget constraint simply requires that the total amount of government expenditure for all purposes be equal to the total amount of finance available to the government from all sources.
Government expenditure includes items such as direct government purchases of goods and services, debt interest payments and transfers. Sources of finance include taxes on income, taxes on debt interest, sales of government debt and monetary issue. The financing requirement endows the system with intrinsic dynamics arising from changes in asset stocks associated with a budget imbalance. These changes in asset stocks held by the private sector will induce second round effects which may offset or reinforce the initial impact of the budget imbalance. The steady state of the system is then characterized by stock-flow equilibrium(1). In the classic article by Blinder and Solow (1976) (see also Blinder and Solow (1974); Tobin and Buiter (1976)) the dynamic adjustment of a modified IS-LM system following a shift in the budget deficit away from equilibrium was shown to be critically dependent upon the financial policy adopted by the authorities. Most important is the probability of instability associated with a policy of issuing or retiring government debt to accommodate budget imbalances. This can be interpreted more provocatively by stating that if the system is stable, deficit financing by bonds is more expansionary than deficit financing by money. These results arise because of the appreciation of debt interest payments as an expense item in the government's accounts. Financing a deficit by issuing bonds forces up interest rates such that output expands by less than if the deficit is financed by money creation. This means that the induced tax receipts are lower under the former policy. Moreover, debt issue causes a larger volume of outstanding debt and greater interest payments. The outcome is an ever widening budget gap and the system explodes. Note that a similar instability occurs in the absence of interest payments in the government budget via the wealth effects operating in the goods (a stabilizing effect) and money markets (a destabilizing effect). In
any event, the critical stability condition requires a high wealth effect in the goods market relative to the wealth effect in the money market in order to generate a sufficient increase in output to close the budget gap.

The potential destabilizing properties of a bond financing policy has proved to be fairly robust to alternative model specifications. Flexible price models have been analysed by Christ (1978, 1979), Cohen and Leeuw (1980), Turnovsky (1980b) and Smith (1982), and extensions to the open economy have been performed by Scarth (1975, 1977) and Turnovsky (1976b). Further issues relating to the government budget constraint (and bond-financed deficit instability in particular) are discussed in Currie (1976a,b, 1977, 1980a,b,c, 1982, 1985b), and Scarth (1982). It is not our intention to review this literature here as this has occupied the attention of other authors (see, for example, Carlson and Spencer (1975); Buiter (1977); Currie (1978, 1981); the edited volume by Cook and Jackson (1979)). We should, however, make some comment on where we believe the existing literature is lacking since this is the motivation for our analysis that follows.

First, all of the literature cited above concentrates on small models which are analytically tractable. Though we welcome these as useful devices for important preliminary investigations, we believe that further research should be directed towards investigating somewhat larger systems with richer dynamic structures. This would obviously mean abandonment of analytical techniques and the use of numerical computer simulations. Clearly, the results obtained will then be dependent upon the particular values chosen for the parameters but a sensitivity analysis could test the robustness of the results across ranges of parameter values. The recent contributions by Nguyen and Turnovsky (1979, 1983), Turnovsky and Nguyen
(1980) and Camilleri, Nguyen and Campbell (1984) are steps forward in this direction and the results continue to imply the undesirability of bond financing. Second, there is still relatively little work on open economy models. Third, and likewise, relatively little attention has been given to the incorporation of rational expectations (exceptions to this are Turnovsky and Nguyen (1980), Camilleri, Nguyen and Campbell (1984) and Scarth (1980), though the latter continues to adopt an analytical approach). One immediate implication of introducing rational expectations is that the concentration on global stability in evaluating the stability of a system is inappropriate; rather, forward-looking behaviour stresses saddlepoint stability in determining a unique non-explosive solution.

This chapter is motivated by the above considerations. Specifically, we wish to analyse the stability of a fairly general open economy macroeconomic model which permits both perfect and imperfect capital mobility and which is able to incorporate either adaptive or rational expectations or both. In these ways, our contribution to the existing literature can be considered as two-fold: first, we examine the issue of bond-financed deficit instability in an open economy with a variety of sources of dynamic behaviour which mean the use of computer simulation techniques; second we inquire into the importance for this issue of different expectations mechanisms. At the moment, the literature has nothing to say about whether the probability of instability is likely to depend on the assumption one chooses to make about the way in which expectations are formed. In addition, we also examine the model's dynamic properties by subjecting it to exogenous perturbations and tracing through the trajectories for certain variables.
Section 4.2 sets out the model in its original non-linear form. This was developed by Whittaker and Wren-Lewis (1983) and is designed to encompass earlier, more specialised, models rather than to innovate. This chapter essentially extends the work in Whittaker and Wren-Lewis (1983) and Blackburn and Currie (1984), where the latter is itself an extension of the former. Whereas Nguyen and Turnovsky (1983) deal with a closed economy model with capital accumulation, and Camilleri, Nguyen and Campbell (1984) analyse an open economy with a fixed capital stock, Whittaker and Wren-Lewis (1983) allow for capital accumulation in an open economy framework. Expectations are formed extrapolatively, however, and the work reported in Blackburn and Currie (1984) focusses on the implications of rational expectations in the foreign exchange market. This chapter is the culmination of this research, combining both Whittaker and Wren-Lewis (1983) and Blackburn and Currie (1984), together with adding an additional dimension, namely rational expectations in the goods market. Thus, the model incorporates either global adaptive expectations, adaptive and rational expectations, or global rational expectations. A linearised version of the model is used for computer simulations. Section 4.3 presents a numerical analysis of stability using a range of plausible parameter values. Section 4.4 contains some simulations of permanent exogenous perturbations based on particular parameter sets. A summary and conclusion are contained in section 4.5.

4.2 The Model and the Simulation Framework

Whittaker and Wren-Lewis (1983) develop a fairly general macro-economic model for addressing the implications of alternative financial policies. It builds upon an open economy IS-LM system
permitting both perfect and imperfect capital mobility with a supply function for output and an expectations augmented Phillips curve generating inflation. Asset accumulation and the government budget constraint provide a rich source of dynamics and the analysis centres on a rule describing the method of financing budget imbalances. The model is closed by assumptions about expectations formation. The complete system comprises a set of thirteen static and dynamic non-linear simultaneous equations. These can be partitioned into two blocks: the first block contains equations (4.2.1) - (4.2.11), the second block consists of equations (4.2.12) - (4.2.13) and is the innovation of our analysis. We write the system as follows:

\[
\begin{align*}
y &= c + \dot{k} + \psi k + g + t \\
c &= \beta_1 (1 - \tau_1) y + \beta_2 ((1 - \tau_2) rb + h^f x^f - hr) - \beta_3 ((m + b - h) \pi + h^F \pi^F) - \beta_4 (x - \pi) + \beta_5 w + \beta_6 e \\
\dot{k} &= \lambda_1 (r - \pi) - k \\
t &= \sigma_1 w + \sigma_2 g - \sigma_3 y - \sigma_4 e \\
t - \dot{t} &= (r - \pi) h + (x^f - p^f) h - r s = 0 \\
a &= \eta_1 (r^f - r + \pi - \pi^f - e^f) + \eta_2 w - \eta_3 w^f + \eta_4 e \\
m &= \gamma_1 y + \gamma_2 w - \gamma_3 x \\
\dot{m} &= (p^* - p)(1 - \phi)m - (p^* - p) \phi b + \phi (m + b) \\
\psi &> 0 \\
\beta_i &> 0 (i = 1, \ldots, 6) \\
\lambda_j &> 0 (j = 1, 2) \\
\sigma_1 &> 0 (l = 1, \ldots, 4) \\
\eta_l &> 0 (l = 1, \ldots, 4) \\
\gamma_s &> 0 (s = 1, 2, 3) \\
0 &< \phi < 1
\end{align*}
\]
\[
\dot{m} + b = \dot{r} s + g - \tau_1 y + (1 - \tau_2)rb - p(m + b)
\]

\[0 < \tau_j < 1(j=1,2) \quad (4.2.9)\]

\[v = \mu k \quad \mu > 0 \quad (4.2.10)\]

\[w = m + b + a + k \quad (4.2.11)\]

\[
\dot{\pi} = \tilde{\theta}_1(p - \pi)
\]

\[\tilde{\theta}_j > 0 \quad (j=1,2) \quad (4.2.12A)\]

\[p = \tilde{\theta}_2(y - v) + \pi \quad (4.2.12R)\]

\[
\dot{q}^c = \tilde{\theta}_1(q - p) - \tilde{\theta}_2(y - v)
\]

\[\tilde{\theta}_j > 0 \quad (j=1,2) \quad (4.2.12R)\]

\[\pi = p \quad (4.2.13A)\]

\[
\dot{e}^e = (f - e)/e
\]

\[\delta > 0 \quad (4.2.13A)\]

\[\dot{f} = \delta(e - f) \quad (4.2.13R)\]

\[\dot{e}^e = \dot{e} \quad (4.2.13R)\]

where

- \(y\) = real output (aggregate demand)
- \(c\) = real consumption
- \(k\) = capital stock
- \(g\) = real government expenditure
- \(t\) = real trade balance
- \(r\) = domestic nominal rate of interest
- \(b\) = real value of interest bearing government debt
- \(a\) = \(h^f - h\) = net foreign assets
- \(h\) = domestic financial assets held by overseas sector
\( h_f \) = overseas financial assets held by domestic sector

\( \pi \) = expected domestic rate of inflation

\( m \) = real money stock

\( \pi_f \) = expected overseas rate of inflation

\( w \) = real domestic private sector wealth

\( e \) = real exchange rate

\( wd \) = world trade

\( p \) = domestic rate of inflation

\( r_f \) = overseas nominal rate of interest

\( p_f \) = overseas rate of inflation

\( rs \) = nominal value of overseas reserves (deflated by the output price deflator)

\( w_f \) = real overseas wealth

\( f \) = real exchange rate expected next period

\( v \) = real output supply

\( p^\ast \) = authorities 'inflation target'

\( q \) = rate of contract price inflation.

In addition, for any variable, \( x \), \( \dot{x} = \frac{dx}{dt} \) and the real exchange rate, \( e \), is measured such that a rise in \( e \) is a real appreciation. As mentioned in section 4.1, the model is intended to be fairly conventional and our main innovations are the introduction of international trade and capital transactions together with the incorporation of divergent expectations mechanisms. The model also reflects some features of the small analytical systems employed in the rest of the thesis.

Equation (4.2.1) is the national income identity which gives real aggregate demand as the sum of real consumption, real investment, real government spending and the real trade balance.

Equation (4.2.2) is the consumption function which relates real consumption to real disposable income, the inflation tax, real
interest rates, real wealth and the terms of trade\(^{(2)}\). Disposable income comprises post-tax earned income and net unearned income. The latter consists of post-tax interest income derived from government debt holdings, less the interest on the volume of government debt held by overseas residents, and the interest income obtained from holdings of overseas assets. The model is sufficiently general to allow for differences in the above tax rates, or \(\tau_1 \neq \tau_2\).

Similarly, the model permits divergent marginal propensities to consume out of post-tax labour income, \(\beta_1\), net interest income, \(\beta_2\), and the inflation tax, \(\beta_3\).

Equation (4.2.3) describes capital accumulation or investment. It captures the partial adjustment hypothesis with the desired capital stock depending inversely on the real rate of interest and depreciation incorporated into equation (4.2.1) by the term \(\psi k\).

Equation (4.2.4) gives the real trade balance which depends on world trade, real government expenditure, real domestic output and a measure of competitiveness given by the real exchange rate. The structure of the equation is such as to allow for different propensities to import out of domestic demand and domestic output. It is also noted that \(\sigma_4 > 0\) implies that the Marshall-Lerner conditions for a depreciation to improve the trade balance are satisfied.

The balance of payments identity is summarized in equation (4.2.5) and comprises the trade balance from equation (4.2.4), real net foreign asset accumulation, and changes in the authorities' foreign reserve holdings. The second of these is described more fully below. The important feature of this equation, following Scarth (1975), is the explicit account of interest payments on both domestic assets held overseas (denominated in domestic currency terms) and foreign assets held by domestic residents (denominated in 'world currency' terms), together with changes in the real value
of debt due to domestic and world inflation; hence the terms in
\((r - p)h\) and \((r^f - p^f)h^f\). Changes in the real value of the latter
group of assets due to fluctuations in the real exchange rate are
ignored for simplicity, as are taxes on these interest payments.

Equation (4.2.6) represents the net foreign asset demand func-
tion, the arguments of which are the real interest differential
(corrected for expected real exchange rate variations), real domestic
wealth, real foreign wealth and the real exchange rate. This
equation is motivated by considering the following functions
describing domestic demand for overseas assets and foreign demand
for domestic assets:

\[
h^f = \eta_{11}(r^f - r + \pi - \pi^f - e^e) + \eta_{14}w + \eta_{14}e \quad \eta_{11}, \eta_{14} > 0
\]

(4.2.14)

\[
h = -\eta_{21}(r^f - r + \pi - \pi^f - e^e) + \eta_{24}w^f - \eta_{24}e \quad \eta_{21}, \eta_{24} > 0
\]

(4.2.15)

Then the definition \(a = h^f - h\) gives equation (4.2.6) with
\(\eta_1 = \eta_{11} + \eta_{21}\) and \(\eta_4 = \eta_{14} + \eta_{24}\). Perfect capital mobility is not
assumed but may be approximated by attaching a high value to the
parameter \(\eta_1\). Thus, the model has some features in common with
portfolio balance models of open economies and the current account is
permitted a role in exchange rate determination (see, for example,
Dornbusch (1975); Dornbusch and Krugman (1976); Dornbusch and Fischer
(1980); Allen and Kenen (1980); Obstfeld and Rogoff (1983); the
series of papers in the 1976 edition of the Scandinavian Journal of
Economics).

Equation (4.2.7) is a standard money demand function with the
demand for real balances depending on real output, real wealth and
the nominal rate of interest.
The linchpin of the analysis that follows centres on equations (4.2.8) – (4.2.9). Equation (4.2.8) specifies a general financing rule for the government for which two extreme cases can be identified (the terminology follows Smith (1982)). The first is when $\phi = 0$ and represents a regime of bond financing in which the rate of growth of the nominal money supply is fixed at $p^*$. This policy rule is termed monetarist. The second regime occurs when $\phi = 1$ and the authorities pursue a policy of issuing or retiring money as the budget constraint dictates with the growth rate of interest bearing government debt now constrained to be $p^*$. For obvious reasons, this rule is termed bondist. It may be noted that in either regime, the equilibrium rate of inflation is $p^*$ so that $p^*$ may be regarded as the authorities' inflation target.

Equation (4.2.9) is the familiar budget constraint written in real terms and completes the description of government behaviour. Government bonds are assumed to be of the variable interest type rather than fixed interest perpetuities. We also assume the fiscal instrument to be government expenditure. We have defined government expenditure as expenditure on goods and services. Christ (1979) and Cohen and Leeuw (1980) have recently shown that the probability of instability associated with a monetarist financial policy may be reduced if government expenditure is defined to also include debt interest payments. Other definitions of government spending also exist, but we feel it would prejudge the issue of instability that this model is designed to explore by departing from the traditional definition of the fiscal instrument at this initial stage. There might be some interest in exploring other possibilities in future research.

Output is allowed to fluctuate in the long-run as well as the short-run via the production function in equation (4.2.10) which
relates output supply to the stock of capital. It may also be noted here that the model adopts the conventional view in the literature in the sense of regarding output as being purely demand determined (that is, actual output is always constrained to equal aggregate demand and not output supply). In these respects, therefore, the model loses some of its generality: firstly, by eschewing any explicit and important role for the labour market; and secondly, by permitting the principle of voluntary exchange to be operative on only the sellers' side of the market. Relaxation of these restrictions is another potentially interesting area for further research.

Equation (4.2.11) defines the identity for real domestic private sector wealth from which one may derive the demand for bonds as the residual asset demand function.

Equations (4.2.1) - (4.2.11) above represent the main structure of the model that will be used throughout our analysis. To this we append some further relationships describing the inflation mechanism and expectations formation. These constitute the main innovations of our analysis and the particular form of the relationships are determined by the assumptions about expectations formation. This second block of the model consists of equations (4.2.12) - (4.2.13). Each of these equations are partitioned according to the assumption about expectations. Thus, the suffix A denotes adaptive expectations and the suffix R denotes rational expectations. Equation (4.2.12A) describes the inflation process in terms of an expectations augmented Phillips curve with expectations being formed adaptively. Similarly, equation (4.2.13A) assumes adaptive expectations in the foreign exchange market. Replacing the adaptive expectations assumption by rational expectations is equivalent to invoking perfect foresight in this deterministic model so that $e^e = e$ and $\pi = p$. The former is suitable and equation (4.2.13R) applies; but the latter would make
supply and demand always equal were we to retain the augmented
Phillips curve. Not only is this implausible, but it would also tend
to render our stability analysis rather trivial. Hence, to incorpor-
ate rational expectations into the goods market, we use equation
(4.2.12R) which is a variant of the Calvo (1981, 1982, 1983) model
used in chapter 3 and described in chapter 2 (section 2.3). This
combines rational expectations with sluggish price adjustment due to
long-term contracts. The version adopted in equation (4.2.12R) is
slightly different from the usual formulation because we assume that
the sluggishness applies to the rate of change, rather than the
level, of prices. We adopt this specification in order to aid
comparison with the adaptive expectations case. Then actual infla-
tion adjusts sluggishly towards contract inflation which, in turn, is
given as a forward-looking discounted moving average of future
demand and inflation.

This completes the description of the model. Its core is given
by the main block comprising equations (4.2.1) - (4.2.11) and three
alternative versions of the system can be distinguished according to
which of the equations in (4.2.12) - (4.2.13) are chosen to comple-
ment the main block. The first version assumes global adaptive
expectations and is termed model A. The relevant equations are
(4.2.1) - (4.2.11), (4.2.12A) and (4.2.13A) and is the original
specification analysed in Whittaker and Wren-Lewis (1983). The
second variant assumes adaptive expectations in the goods market
and rational expectations in the foreign exchange market. This is
termed model A/R and is the subject of investigation in Blackburn
and Currie (1984). This type of divergent expectations has been
considered by Turnovsky (1980b) in a similar analysis and by Peel and
Metcalfe (1979) in a slightly different context. Though it is admit-
tedly rather arbitrary, it might serve as a first approximation to
the case in which asset markets adjust more quickly than goods markets and is in the spirit of the popular Dornbusch (1976) and Buiter and Miller (1981) models of exchange rate dynamics (see also Canzoneri and Underwood (1982); Obstfeld and Rogoff (1984); Wickens (1984)). The relevant equations for the A/R model are (4.2.1) - (4.2.11), (4.2.12A) and (4.2.13R). The third system is model R and assumes global rational expectations. The equations describing this model are (4.2.1) - (4.2.11), (4.2.12R) and (4.2.13R).

Our analysis that follows is based on a linear approximation of the model around its long-run equilibrium which amounts to linearizing equations (4.2.2), (4.2.5), (4.2.8) - (4.2.9) and (4.2.13A)). We also eliminate \( \dot{k} \) from equation (4.2.1) using equation (4.2.3) and substitute \( h^f = a + h \) in equations (4.2.2) and (4.2.5). Then denoting by \( \bar{x} \) the long-run equilibrium value of a variable \( x \), a Taylor's series expansion gives the linearized version of the system for computer simulations which we write in full as follows:

\[
y = c - \lambda_1 \lambda_2 r + \lambda_1 \lambda_2 \pi + (\psi - \lambda_1)k + g + t \tag{4.2.1}'
\]

\[
c = \beta_1 (1 - \tau_1) y + [\beta_2 (\bar{b}(1 - \tau_2) - \bar{n}) - \beta_4] r \\
+ (\beta_2 \bar{r}(1 - \tau_2) - \beta_3 \bar{\pi})b + (\beta_2 (\bar{r}^f - \bar{r}) + \beta_3 (\bar{\pi} - \bar{\pi}^f))h \\
+ (\beta_2 - \beta_3) \bar{t}^f a - \beta_3 \bar{m} - (\beta_3 (\bar{a} + \bar{b} - \bar{n}) - \beta_4) \pi \\
- \beta_3 (\bar{a} + \bar{n}) \pi^f + \beta_2 (\bar{a} + \bar{n}) r^f + \beta_5 w + \beta_6 e \tag{4.2.2}'
\]

\[
\dot{k} = - \lambda_1 \lambda_2 r + \lambda_1 \lambda_2 \pi - \lambda_1 k \tag{4.2.3}'
\]

\[
t = \sigma_1 w + \sigma_2 g - \sigma_3 y - \sigma_4 e \tag{4.2.4}'
\]
\[ t - \dot{a} - (\ddot{r} - \ddot{p} - \dddot{r} + \dddot{p})h - \ddot{h}r + \dot{h}p + (\dddot{a} + \dddot{h})r^2 \]
\[ = (\dddot{a} + \dddot{h})p^2 + (\dddot{r} - \dddot{p})a = 0 \quad (4.2.5) \]

\[ a = \eta_1 r^2 - \eta_1 r + \eta_1 r^2 - \eta_1 e^2 + \eta_2 w - \eta_3 w^2 + \eta_4 e \quad (4.2.6) \]

\[ m = \gamma_1 y + \gamma_2 w - \gamma_3 r \quad (4.2.7) \]

\[ (1 - \phi)\dot{m} - \phi \dot{b} = ((1 - \phi)\dot{m} - \phi \dot{b})p^* + (1 - \phi)(\dddot{p} - \dddot{p})m \]
\[ - \phi(\dddot{p} - \dddot{p})b + (\phi \dot{b} - (1 - \phi)\dot{m})p \quad (4.2.8) \]

\[ \dot{m} + \dddot{b} = q - \tau_1 y + (1 - \tau_2)\dddot{r}x + ((1 - \tau_2)\dddot{r} - \dddot{p})b - \dddot{p}m \]
\[ - (\dot{m} + \dddot{b})p \quad (4.2.9) \]

\[ v = \mu k \quad (4.2.10) \]

\[ w = m + \dot{b} + a + k \quad (4.2.11) \]

\[ \dot{\pi} = \theta_1 p - \theta_1 \pi \]
\[ p = \theta_2 y - \theta_2 v + \pi \quad (4.2.12A) \]

\[ \dot{p} = \theta_1 q - \theta_1 p \]
\[ \dot{q} = \theta_1 q - \theta_1 p - \theta_2 y + \theta_2 v \]
\[ \pi = p \quad (4.2.12R) \]

\[ e^e = f - e \quad (4.2.13A) \]
\[ \dot{f} = \delta e - \delta f \quad (4.2.13R) \]

\[ \dot{e}^e = e \quad (4.2.13R) \]
The solution of the system is performed via computer simulation using an earlier version of the PRISM rational expectations package described in Al-Nowaihi, Levine and Fontanelle (1985). We distinguish between those variables determined by static relationships and those variables determined by dynamic relationships. The system is then constructed on the computer in the form

\[
\begin{bmatrix}
\dot{u} \\
\dot{y}_d
\end{bmatrix} = \begin{bmatrix}
D1 & 0 \\
0 & D3
\end{bmatrix}
\begin{bmatrix}
u \\
y_d
\end{bmatrix} + D2 y_s 
\tag{4.2.16}
\]

\[
\begin{bmatrix}
u \\
y_d
\end{bmatrix} = \begin{bmatrix}
S1 & 0 \\
0 & S2
\end{bmatrix}
\begin{bmatrix}
u \\
y_d
\end{bmatrix} = \begin{bmatrix}
0
\end{bmatrix} 
\tag{4.2.17}
\]

where \(u\) = vector of exogenous variables
\(y_d\) = vector of endogenous variables determined by dynamic relationships
\(y_s\) = vector of endogenous variables determined by static relationships
and \(0\) is the null vector with \(D1, D2, D3, S1\) and \(S2\) matrices of structural parameters. In addition, and as usual, we partition the \(y_d\) vector such that

\[y_d = [Z^T, \bar{x}^T]^T\]

where \(Z\) is a vector of predetermined variables and \(\bar{x}\) is a vector of non-predetermined variables.

Equation (4.2.16) summarizes the dynamic relationships of the model and equation (4.2.17) summarizes the static relationships. These, together with the vectors of variables and the matrices of parameters, will generally depend on the choice of the three models described above. We may note, however, that equation (4.2.16) always includes equations (4.2.3)', (4.2.5)' and (4.2.8)' - (4.2.9)' which means that \(Z\) always includes the variables \(k, a, m\) and \(b\).

Similarly, equation (4.2.17) always contains equations (4.2.1)' - (4.2.2)', (4.2.4)', (4.2.7)' and (4.2.10)' - (4.2.11)' so that \(y_s\)
always contains the variables y, c, t, r, v and w. In addition the
vector of exogenous variables, \( \mathbf{u} \), is the same \( 8 \times 1 \) vector for all
three models and is given by \( \mathbf{u} = [p^T g h n^T r^T w d p^T w]^T \). We have
assumed that the exogenous variables follow identical and independent
autoregressive processes and have included them in the dynamic state
vector \( [u^T y_q^T]^T \). The autoregressive process is summarized by

\[
\dot{\mathbf{u}} = -P\mathbf{u}
\]  

(4.2.18)

where \( P \) is an \( 8 \times 8 \) diagonal matrix with elements

\[
\rho(i,j) = \begin{cases} 
0, & i \neq j \\
\rho > 0, & i = j 
\end{cases} \quad (i,j = 1, \ldots, 8) 
\]  

(4.2.19)

Thus, \( \rho > 0 \) implies that perturbations to exogenous variables are
temporary with the degree of autocorrelation falling as \( \rho \) increases.
When \( \rho = 0 \), exogenous shocks are permanent.

The complete vectors of variables for each model are presented in
table 4.2(A). The non-zero elements of the matrices, together with
the orders of these matrices, are given in tables 4.2(B) - 4.2(F).
The differences between the models in terms of these elements are
indicated by an asterisk. Our intention to make the models comparable
with each other means that many of these differences occur merely
because of the different orders of the matrices. Particular points
to note in deriving these matrices are as follows:

For model A, we eliminate \( e^e \) from equation (4.2.6)' using
equation (4.2.13A)' . We also introduce an extraneous variable \( x^o \),
which is a fictitious jump variable evolving according to \( \dot{x}^o = x^o \).
This variable plays absolutely no role in determining the behaviour
of the system and serves the sole purpose of making the model
compatible with the computer software. For model A/R, we drop
<table>
<thead>
<tr>
<th>Model</th>
<th>Vector</th>
<th>A</th>
<th>A/R</th>
<th>R</th>
</tr>
</thead>
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<td>$k$</td>
<td></td>
</tr>
<tr>
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</tr>
<tr>
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<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m$</td>
<td>$m$</td>
<td>$m$</td>
<td></td>
</tr>
<tr>
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<td>$b$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$f$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>$x_0$</td>
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<td>$[q]$</td>
<td>$[e]$</td>
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<td>$[y]$</td>
<td>$[y]$</td>
<td>$[y]$</td>
<td>$[y]$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
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<td>$c$</td>
<td>$c$</td>
</tr>
<tr>
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<td>$r$</td>
<td>$r$</td>
<td>$r$</td>
</tr>
<tr>
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<td>$p$</td>
<td>$p$</td>
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</tr>
<tr>
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<td>$v$</td>
<td>$v$</td>
<td>$v$</td>
</tr>
<tr>
<td></td>
<td>$w$</td>
<td>$w$</td>
<td>$w$</td>
<td>$w$</td>
</tr>
<tr>
<td></td>
<td>$t$</td>
<td>$t$</td>
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</tr>
</tbody>
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Table 4.2(A): Definitions of vectors under divergent model specifications.
<table>
<thead>
<tr>
<th>Model</th>
<th>A (15 x 15)</th>
<th>A/R (14 x 14)</th>
<th>R (15 x 15)</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(2,2)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(3,3)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(4,4)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(5,5)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(6,6)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(7,7)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(8,8)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(9,9)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(10,10)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(11,11)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(12,12)</td>
<td>1-φ</td>
<td>1-φ</td>
<td>1-φ</td>
</tr>
<tr>
<td>(12,13)</td>
<td>-φ</td>
<td>-φ</td>
<td>-φ</td>
</tr>
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</tr>
<tr>
<td>(14,14)</td>
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<tr>
<td>(15,15)*</td>
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Table 4.2(B): Non-zero elements in D1 matrix under divergent model specifications.
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<th>A/R (14 x 14)</th>
<th>R (15 x 15)</th>
</tr>
</thead>
<tbody>
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<td>$-\rho$</td>
<td>$-\rho$</td>
</tr>
<tr>
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<td>$-\rho$</td>
<td>$-\rho$</td>
</tr>
<tr>
<td>(3,3)</td>
<td>$-\rho$</td>
<td>$-\rho$</td>
<td>$-\rho$</td>
</tr>
<tr>
<td>(4,4)</td>
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<td>$-\rho$</td>
<td>$-\rho$</td>
</tr>
<tr>
<td>(5,5)</td>
<td>$-\rho$</td>
<td>$-\rho$</td>
<td>$-\rho$</td>
</tr>
<tr>
<td>(6,6)</td>
<td>$-\rho$</td>
<td>$-\rho$</td>
<td>$-\rho$</td>
</tr>
<tr>
<td>(7,7)</td>
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<td>$-\rho$</td>
<td>$-\rho$</td>
</tr>
<tr>
<td>(8,8)</td>
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<td>$-\rho$</td>
<td>$-\rho$</td>
</tr>
<tr>
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<td>$-\lambda_1$</td>
<td>$-\lambda_1$</td>
</tr>
<tr>
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<td>$\lambda_1\lambda_2$</td>
<td>$\lambda_1\lambda_2$</td>
<td>$\lambda_1\lambda_2$</td>
</tr>
<tr>
<td>(10,3)</td>
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<td>$\bar{p}+\bar{\tau}-\bar{p}^f$</td>
<td>$\bar{p}+\bar{\tau}-\bar{p}^f$</td>
</tr>
<tr>
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<td>$\bar{a}+\bar{\eta}$</td>
<td>$\bar{a}+\bar{\eta}$</td>
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<td>$\bar{\eta}$</td>
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<td>$-\theta_1$</td>
<td>$-\theta_1$</td>
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<td>$\theta_1$</td>
</tr>
<tr>
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<td>$(1-\phi)\bar{m}-\phi\bar{\eta}$</td>
<td>$(1-\phi)\bar{m}-\phi\bar{\eta}$</td>
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<tr>
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<td>$(1-\phi)(\bar{p}^*-\bar{p})$</td>
<td>$(1-\phi)(\bar{p}^*-\bar{p})$</td>
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<td>$-\phi(\bar{p}^*-\bar{p})$</td>
<td>$-\phi(\bar{p}^*-\bar{p})$</td>
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<td>$-\bar{m}-\bar{\eta}$</td>
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<td>$-\bar{p}$</td>
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<tr>
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<td>(1−τ_2)\bar{r}−\bar{p}</td>
<td>(1−τ_2)\bar{r}−\bar{p}</td>
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<td>----------------</td>
</tr>
<tr>
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<td>−\eta_1</td>
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</tr>
<tr>
<td>(14,5) *</td>
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</tr>
<tr>
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<td>−\eta_3</td>
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</tr>
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<td>−1</td>
<td>0</td>
</tr>
<tr>
<td>(14,11) *</td>
<td>0</td>
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<td>−\theta_1</td>
</tr>
<tr>
<td>(14,14) *</td>
<td>−s</td>
<td>\eta_4</td>
<td>\theta_1</td>
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<tr>
<td>(15,4) *</td>
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<td>−\eta_1</td>
</tr>
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<td>(15,5) *</td>
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<td>n/a</td>
<td>\eta_1</td>
</tr>
<tr>
<td>(15,8) *</td>
<td>0</td>
<td>n/a</td>
<td>−\eta_3</td>
</tr>
<tr>
<td>(15,10) *</td>
<td>0</td>
<td>n/a</td>
<td>−1</td>
</tr>
<tr>
<td>(15,11) *</td>
<td>0</td>
<td>n/a</td>
<td>\eta_1</td>
</tr>
<tr>
<td>(15,15) *</td>
<td>1</td>
<td>n/a</td>
<td>\eta_4</td>
</tr>
</tbody>
</table>

Table 4.2(C): Non-zero elements in D2 matrix under divergent model specifications.
Table 4.2(D): Non-zero elements in D3 matrix under divergent model specifications.
<table>
<thead>
<tr>
<th>Model Element</th>
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<th>A/R (7 x 7)</th>
<th>R (6 x 6)</th>
</tr>
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<tbody>
<tr>
<td>(1,1)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(1,2)</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(1,3)</td>
<td>$\lambda_1^2$</td>
<td>$\lambda_1^2$</td>
<td>$\lambda_1^2$</td>
</tr>
<tr>
<td>(1,6)*</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>(1,7)*</td>
<td>-1</td>
<td>-1</td>
<td>n/a</td>
</tr>
<tr>
<td>(2,1)</td>
<td>$-\rho_1(1-\tau_1)$</td>
<td>$-\rho_1(1-\tau_1)$</td>
<td>$-\rho_1(1-\tau_1)$</td>
</tr>
<tr>
<td>(2,2)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(2,3)</td>
<td>$\beta_4^2-\beta_2^2(b(1-\tau_2)-h)$</td>
<td>$\beta_4^2-\beta_2^2(b(1-\tau_2)-h)$</td>
<td>$\beta_4^2-\beta_2^2(b(1-\tau_2)-h)$</td>
</tr>
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<td>0</td>
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</tr>
<tr>
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<td>$-\rho_5$</td>
<td>$-\rho_5$</td>
<td>0</td>
</tr>
<tr>
<td>(2,8)*</td>
<td>$-\rho_6$</td>
<td>n/a</td>
<td>n/a</td>
</tr>
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<td>(3,1)</td>
<td>$-\gamma_1$</td>
<td>$-\gamma_1$</td>
<td>$-\gamma_1$</td>
</tr>
<tr>
<td>(3,3)</td>
<td>$\gamma_3$</td>
<td>$\gamma_3$</td>
<td>$\gamma_3$</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>$-\gamma_2$</td>
</tr>
<tr>
<td>(3,6)*</td>
<td>$-\gamma_2$</td>
<td>$-\gamma_2$</td>
<td>0</td>
</tr>
<tr>
<td>(4,1)*</td>
<td>$-\theta_2$</td>
<td>$-\theta_2$</td>
<td>0</td>
</tr>
<tr>
<td>(4,4)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(4,5)*</td>
<td>$\theta_2$</td>
<td>$\theta_2$</td>
<td>0</td>
</tr>
<tr>
<td>(5,5)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(6,1)*</td>
<td>0</td>
<td>0</td>
<td>$\sigma_3$</td>
</tr>
<tr>
<td>(6,6)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(7,1)*</td>
<td>$\sigma_3$</td>
<td>$\sigma_3$</td>
<td>n/a</td>
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<tr>
<td>(7,7)*</td>
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<td>1</td>
<td>n/a</td>
</tr>
<tr>
<td>(7,8)</td>
<td>$\sigma_4$</td>
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<td>n/a</td>
</tr>
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<td>-------</td>
<td>-----------</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
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<td>n/a</td>
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<td>(8,6)</td>
<td>$\eta_2$</td>
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<td>n/a</td>
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<td>(8,8)</td>
<td>$\eta_1 + \eta_4$</td>
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Table 4.2(E): Non-zero elements in $S_1$ matrix under divergent model specifications
<table>
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<th>Model</th>
<th>A (8 x 15)</th>
<th>A/R (7 x 14)</th>
<th>R (6 x 15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Element</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1,2)</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(1,9)</td>
<td>$\lambda_1 - \psi$</td>
<td>$\lambda_1 - \psi$</td>
<td>$\lambda_1 - \psi$</td>
</tr>
<tr>
<td>(1,11)</td>
<td>$-\lambda_1 \lambda_2$</td>
<td>$-\lambda_1 \lambda_2$</td>
<td>$-\lambda_1 \lambda_2$</td>
</tr>
<tr>
<td>(2,3)</td>
<td>$-\beta_2 (\bar{\tau} - \bar{x}) - \beta_3 (\bar{\pi} - \bar{\eta})$</td>
<td>$-\beta_2 (\bar{\tau} - \bar{x}) - \beta_3 (\bar{\pi} - \bar{\eta})$</td>
<td>$-\beta_2 (\bar{\tau} - \bar{x}) - \beta_3 (\bar{\pi} - \bar{\eta})$</td>
</tr>
<tr>
<td>(2,4)</td>
<td>$\beta_3 (\bar{\alpha} + h)$</td>
<td>$\beta_3 (\bar{\alpha} + h)$</td>
<td>$\beta_3 (\bar{\alpha} + h)$</td>
</tr>
<tr>
<td>(2,5)</td>
<td>$-\beta_2 (\bar{\alpha} + h)$</td>
<td>$-\beta_2 (\bar{\alpha} + h)$</td>
<td>$-\beta_2 (\bar{\alpha} + h)$</td>
</tr>
<tr>
<td>(2,10)</td>
<td>$-(\beta_2 - \beta_3) \bar{x}$</td>
<td>$-(\beta_2 - \beta_3) \bar{x}$</td>
<td>$-(\beta_2 - \beta_3) \bar{x}$</td>
</tr>
<tr>
<td>(2,11)</td>
<td>$\beta_3 (\bar{\alpha} + h) - \beta_4$</td>
<td>$\beta_3 (\bar{\alpha} + h) - \beta_4$</td>
<td>$\beta_3 (\bar{\alpha} + h) - \beta_4$</td>
</tr>
<tr>
<td>(2,12)</td>
<td>$\beta_3 \bar{\pi}$</td>
<td>$\beta_3 \bar{\pi}$</td>
<td>$\beta_3 \bar{\pi}$</td>
</tr>
<tr>
<td>(2,13)</td>
<td>$\beta_3 \bar{\pi} - \beta_2 \bar{x}(1 - \tau_2)$</td>
<td>$\beta_3 \bar{\pi} - \beta_2 \bar{x}(1 - \tau_2)$</td>
<td>$\beta_3 \bar{\pi} - \beta_2 \bar{x}(1 - \tau_2)$</td>
</tr>
<tr>
<td>(2,14) *</td>
<td>0</td>
<td>$-\beta_6$</td>
<td>0</td>
</tr>
<tr>
<td>(2,15) *</td>
<td>0</td>
<td>n/a</td>
<td>$-\beta_6$</td>
</tr>
<tr>
<td>(3,12)</td>
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<td>1</td>
</tr>
<tr>
<td>(4,9) *</td>
<td>0</td>
<td>0</td>
<td>$-\mu$</td>
</tr>
<tr>
<td>(4,11) *</td>
<td>$-\mu$</td>
<td>$-\mu$</td>
<td>$-\mu$</td>
</tr>
<tr>
<td>(5,9) *</td>
<td>0</td>
<td>0</td>
<td>$-\lambda$</td>
</tr>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>$-\lambda$</td>
</tr>
<tr>
<td>(6,2) *</td>
<td>0</td>
<td>0</td>
<td>$-\sigma_2$</td>
</tr>
<tr>
<td>(6,6) *</td>
<td>0</td>
<td>0</td>
<td>$-\sigma_1$</td>
</tr>
<tr>
<td>(6,9) *</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>(6,10) *</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>(6,12) *</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>(6.13)*</td>
<td>-1</td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>(6,15)*</td>
<td>0</td>
<td></td>
<td>n/a</td>
</tr>
<tr>
<td>(7,2)*</td>
<td>-σ₂</td>
<td></td>
<td>-σ₂</td>
</tr>
<tr>
<td>(7,6)*</td>
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<td>-σ₁</td>
</tr>
<tr>
<td>(7,14)*</td>
<td>0</td>
<td></td>
<td>σ₄</td>
</tr>
<tr>
<td>(8,4)*</td>
<td>-η₁</td>
<td></td>
<td>n/a</td>
</tr>
<tr>
<td>(8,5)*</td>
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<td>(8,8)*</td>
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<td>n/a</td>
</tr>
<tr>
<td>(8,14)*</td>
<td>-η₁</td>
<td></td>
<td>n/a</td>
</tr>
</tbody>
</table>

Table 4.2(F): Non-zero elements in matrix S₂ under divergent model specifications.
equation (4.2.13A)' and replace the equation for $\bar{e}^d$ with the rational expectations assumptions in equation (4.2.13R)', $\bar{e}^d = \bar{e}$. These modifications also apply to model R. In addition, for this model, we substitute equation (4.2.12R)' for equation (4.2.12A)' and impose the rational expectations assumption for prices so that $\pi = p$ throughout the model. All three systems are now in the required form for simulation.

4.3 Stability under Alternative Financial Policy Rules

In this section, we perform an analysis of stability of the model described in section 4.2 with the view to examining the possibility of instability associated with a monetarist financing rule and the implications for this of alternative expectations assumptions. Given the dynamic configuration of the model, a full algebraic analysis of stability is impossible. In Whittaker and Wren-Lewis (1983) and Whittaker, Wren-Lewis, Blackburn and Currie (1985), however, some manipulations permitted a preliminary algebraic investigation by concentrating on the necessary determinant condition for convergence. This indicated that a monetarist and bondist policy rule cannot both be stable at the same time so that one will necessarily be destabilizing. In fact, the necessary condition for a monetarist policy to be stable appears very stringent. We do not repeat the algebraic analysis in any detail here since our main concern is with eliciting the numerical results. It is useful, however, to comment briefly on certain properties of the analytical results as these provide some insight into the numerical investigation.

The main conclusion that emerges from the algebraic analysis is that stability is more likely under a monetarist policy if post-tax real interest rates are negative. This makes intuitive sense because
the critical destabilizing feature of this policy is that, with real interest rates positive, bond issue will raise the outstanding debt. A negative real interest rate will operate as a stabilizing force by reducing the real burden of interest payments. This is to say that if inflation rises faster than nominal interest rates (so that real interest rates fall), there is a greater likelihood that the stabilizing inflation tax on wealth will outweigh the destabilizing interest payments (see, for example, Nguyen and Turnovsky (1979); Turnovsky and Nguyen (1980)). Stability will be enhanced here as the degree of capital mobility rises (operating through the influence of wealth and the terms of trade on consumption) and as the interest sensitivity of consumption increases (operating via the real interest rate). Two final points worth mentioning about the algebraic analysis are as follows. First, the stability condition is independent of all the adjustment parameters in the model and is also independent of whether expectations are formed adaptively or rationally. The latter property is rather misleading since, as we demonstrate in the numerical simulations below, there are circumstances under which the stability of the model is non-invariant with respect to alternative expectations assumptions. Secondly, however, the numerical analysis strongly suggests that the interest payments on government debt will remain the dominant factor and that a monetarist policy will remain destabilizing.

We now turn to the central concern of this chapter, namely a numerical analysis of model stability using computer simulation techniques. This begins by defining a central parameter set for each model. Since the results will be dependent upon this set, it is necessary to test their sensitivity by varying the parameter values. Thus, we also define plausible ranges for some of the more uncertain or important parameters. We then attempt to mark out the
stable parameter space by setting one parameter at a time to its value at either end of this range, while keeping all other parameters at their central values. Finally, we use these results as a guide to further experimentation involving varying more than one parameter from its central value simultaneously.

The choice of parameter sets are termed low, central and high. These are chosen by Whittaker and Wren-Lewis (1983) with the central parameter set based on a version of the Treasury model. Equilibrium values of variables are based approximately on UK data for 1980. An important exception is the equilibrium value chosen for interest rates which, by being the same as the equilibrium value for inflation rates, implies a significantly negative post-tax real interest rate. The reason for choosing this value is that the algebraic analysis of stability discussed above implied that monetarist stability is unlikely to be compatible with positive real interest rates. Thus, there seemed little point in examining this area of the parameter space in much detail. All parameter values and equilibrium values are summarized in Tables 4.3(A) and 4.3(B) respectively.

Our stability analysis proceeds by computing the eigenvalues of the dynamic system. We may ignore the eight roots associated with the exogenous variables in u as these are not associated with the intrinsic dynamics of the system (hence the appearance of zero roots for permanent shocks, $\rho = 0$, may be ignored). The stability of the model then focuses on the remaining dynamic relationships for the endogenous variables. In rational expectations models, the convergence properties are defined in terms of saddlepoint stability (see appendix B for further discussion). This requires that there be as many stable eigenvalues (eigenvalues with negative real part) as there are predetermined variables, and as many unstable eigenvalues (eigenvalues with positive real part) as there are non-predetermined
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Equation</th>
<th>High</th>
<th>Central</th>
<th>Low</th>
</tr>
</thead>
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<td>$\psi$</td>
<td>(4.2.1)</td>
<td>-</td>
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<td>-</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>(4.2.2)</td>
<td>-</td>
<td>0.800</td>
<td>-</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>(4.2.2)</td>
<td>-</td>
<td>0.600</td>
<td>-</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>(4.2.2)</td>
<td>-</td>
<td>0.600</td>
<td>-</td>
</tr>
<tr>
<td>$\beta_4$</td>
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<td>0.000</td>
</tr>
<tr>
<td>$\beta_5$</td>
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<td>0.080</td>
<td>0.000</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>(4.2.2)</td>
<td>-</td>
<td>15.000</td>
<td>-</td>
</tr>
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<td>(4.2.3)</td>
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</tr>
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<td>15.000</td>
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</tr>
<tr>
<td>$\sigma_1$</td>
<td>(4.2.4)</td>
<td>-</td>
<td>30.000</td>
<td>-</td>
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<tr>
<td>$\sigma_2$</td>
<td>(4.2.4)</td>
<td>-</td>
<td>0.200</td>
<td>-</td>
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<tr>
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<td>(4.2.4)</td>
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<td>-</td>
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<tr>
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<td>28.000</td>
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</tr>
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<td>20.000</td>
<td>10.000</td>
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<td>$\eta_2$</td>
<td>(4.2.6)</td>
<td>-</td>
<td>0.000</td>
<td>-</td>
</tr>
<tr>
<td>$\eta_3$</td>
<td>(4.2.6)</td>
<td>-</td>
<td>0.000</td>
<td>-</td>
</tr>
<tr>
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<td>(4.2.6)</td>
<td>-</td>
<td>0.000</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma_1$</td>
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<td>0.039</td>
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</tr>
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<td>0/1</td>
<td>-</td>
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<td>-</td>
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<tr>
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<td>3.000</td>
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<td>0.005</td>
<td>0.002</td>
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<td>$\delta$</td>
<td>(4.2.13)</td>
<td>0.900</td>
<td>0.600</td>
<td>0.300</td>
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Table 4.3(A) : Parameter values for high, central and low parameter sets.
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<th>Variable</th>
<th>Equilibrium</th>
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<tbody>
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<td>$\bar{y}, \bar{v}$</td>
<td>100.0</td>
</tr>
<tr>
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</tr>
<tr>
<td>$\tau_1$</td>
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</tr>
<tr>
<td>$\tau_2$</td>
<td>0.6</td>
</tr>
<tr>
<td>$\bar{b}$</td>
<td>38.5</td>
</tr>
<tr>
<td>$\bar{m}$</td>
<td>6.5</td>
</tr>
<tr>
<td>$\bar{k}$</td>
<td>300.0</td>
</tr>
<tr>
<td>$\bar{a}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\bar{e}_{jwd}$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\bar{h}$</td>
<td>30.0</td>
</tr>
</tbody>
</table>

Table 4.3(B): Equilibrium values of variables.
variables. In model A, there are six dynamic relationships describing the evolution of predetermined variables plus an extraneous dynamic relationship associated with the fictitious jump variable, $x_0$. Hence, saddlepoint stability obtains if there are six stable roots and one unstable root (the latter will obviously be unity). The results reported for this model ignore the unstable root. Model A/R contains five dynamic equations for predetermined variables and one dynamic equation for a free variable which, in this case, is the exchange rate, $e$. Then saddlepoint stability requires five stable roots and one unstable root. Finally, model R also has five relationships describing the evolution of predetermined variables but there are now two jump variables; the contract price, $q$, and the exchange rate, $e$. In this case, a unique convergent solution requires five stable and two unstable eigenvalues.

Table 4.3(C) reports the results of the stability analysis for each parameter set under each financial policy, either monetarist ($\phi=0$) or bondist ($\phi=1$). These results are derived from the eigenvalues of the systems which are recorded in tables 4.3(D) - 4.3(J).

For the central parameter set, the models are all unstable under a monetarist policy. In fact, in each case, they contain one too many unstable roots. Under a bondist policy, however, the models are all stable. Section 4.4 examines the dynamic properties of the models under the central parameter set in more detail.

For the cases when a single parameter is varied from the central set, the notable result is the general instability of bond financing. This is true despite our deliberate inclusion of a negative post-tax real interest rate which is beneficial to the possibility of monetarist stability (we could not find any plausible parameter set that is stable under a policy when post-tax real interest rates are positive). There are, however, some interesting and important
exceptions to this general proposition. A very high value for the interest elasticity of consumption, $\beta_4$, ensures stability in all three cases. Systematic investigation revealed the critical value in models A and A/R to be $\beta_4 \approx 14$, whilst in model R this is $\beta_4 \approx 8$. This suggests that for $\beta_4 < 14$, models A and A/R suffer from instability because of the adaptive inflation expectations formation in the goods market. Note that $\beta_4 > 14$ is the only variant for which model A/R is stable under a monetarist rule. Models A/R and R are stable under bond financing if the degree of capital mobility, $n_1$, is increased sufficiently. It therefore appears that near-perfect capital mobility combined with rational expectations in the foreign exchange market increases the chances of monetarist stability. There is also a noticeable tendency for model R to offer greater probability of monetarist stability if the competitiveness effect on the trade balance, $\sigma_4$, is relative low. This supports the above result for variations in $\beta_4$: in particular, it strongly suggests that in the other models, it is the adaptive determination of inflation expectations which presents an additional obstacle to monetarist stability. Finally, the results of simultaneous variations of some parameters support the above findings for independent changes in the degree of capital mobility; a relatively high value for $n_1$ coupled with rational exchange rate expectations makes the system stable under bond financing.

Nonetheless, despite these differences, the broad conclusion remains that making the money stock exogenous is likely to generate problems of instability which are generally reduced if there is a move towards money financing. This conclusion is consistent with the consensus in the literature on the government budget constraint and suggests that, in most cases, it is not the result of assumptions about expectations, but rather concerns the interaction
<table>
<thead>
<tr>
<th>Policy</th>
<th>Monetarist ($\phi = 0$)</th>
<th>Bondist ($\phi = 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>A/R</td>
</tr>
<tr>
<td>Param. set</td>
<td>Central</td>
<td>Central</td>
</tr>
<tr>
<td></td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>$\sigma_4$ [High]</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>Low</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>$\lambda_1$ [High]</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>Low</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>$\lambda_2$ [High]</td>
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<td>u</td>
</tr>
<tr>
<td>Low</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>$\eta_1$ [High]</td>
<td>u</td>
<td>s</td>
</tr>
<tr>
<td>Low</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>$s$ [High]</td>
<td>u</td>
<td>n/a</td>
</tr>
<tr>
<td>Low</td>
<td>u</td>
<td>n/a</td>
</tr>
<tr>
<td>$\gamma_1$ [High]</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>Low</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>$\gamma_2$ [High]</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>Low</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>$\gamma_3$ [High]</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>Low</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>$\theta_1$ [High]</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>Low</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>Parameter</td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>-------------------</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>U</td>
<td>U</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>U</td>
<td>U</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>U</td>
<td>U</td>
</tr>
<tr>
<td>$\lambda_5$</td>
<td>U</td>
<td>S</td>
</tr>
<tr>
<td>$\eta_1$</td>
<td>U</td>
<td>S</td>
</tr>
<tr>
<td>$\eta_5$</td>
<td>U</td>
<td>S</td>
</tr>
</tbody>
</table>

Table 4.3(C): Stability under alternative parameter values and divergent model specifications.

$U = $ Unstable ; $S = $ Stable
<table>
<thead>
<tr>
<th>Monetarist (φ=0)</th>
<th>Bondist (φ=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.387 ± 0.902i</td>
<td>-3.842</td>
</tr>
<tr>
<td>-0.355 ± 0.184i</td>
<td>-0.291 ± 0.475i</td>
</tr>
<tr>
<td>0.114</td>
<td>-0.244 ± 0.148i</td>
</tr>
<tr>
<td>-0.165</td>
<td>0.114</td>
</tr>
</tbody>
</table>

Table 4.3(D)(i) : Eigenvalues for model A; central parameter set.

<table>
<thead>
<tr>
<th>Monetarist (φ=0)</th>
<th>Bondist (φ=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.103</td>
<td>-3.353</td>
</tr>
<tr>
<td>-0.466 ± 0.402i</td>
<td>1.104</td>
</tr>
<tr>
<td>-0.608</td>
<td>-0.662</td>
</tr>
<tr>
<td>0.125</td>
<td>-0.209 ± 0.165i</td>
</tr>
<tr>
<td>-0.165</td>
<td>0.125</td>
</tr>
</tbody>
</table>

Table 4.3(D)(ii) : Eigenvalues for model A/R; central parameter set.

<table>
<thead>
<tr>
<th>Monetarist (φ=0)</th>
<th>Bondist (φ=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.755 ± 0.462i</td>
<td>-3.317</td>
</tr>
<tr>
<td>-0.455</td>
<td>0.930</td>
</tr>
<tr>
<td>0.143</td>
<td>0.648</td>
</tr>
<tr>
<td>0.672</td>
<td>-0.592 ± 0.191i</td>
</tr>
<tr>
<td>-0.159</td>
<td>-0.334</td>
</tr>
<tr>
<td>0.962</td>
<td>-0.120</td>
</tr>
</tbody>
</table>

Table 4.3(D)(iii) : Eigenvalues for model R; central parameter set.

Table 4.3(D) : Eigenvalues for divergent model specifications; central parameter set.
<table>
<thead>
<tr>
<th>Policy</th>
<th>Monetarist ($\phi=0$)</th>
<th>Bondist ($\phi=1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>-0.765±1.4691</td>
<td>-0.636±0.4171</td>
</tr>
<tr>
<td></td>
<td>0.243</td>
<td>0.225±0.5071</td>
</tr>
<tr>
<td></td>
<td>-0.515</td>
<td>-0.281</td>
</tr>
<tr>
<td></td>
<td>-0.368</td>
<td>-0.051</td>
</tr>
<tr>
<td></td>
<td>-0.108</td>
<td></td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>-0.362±0.9081</td>
<td>-0.400±0.9041</td>
</tr>
<tr>
<td></td>
<td>-0.459±0.2341</td>
<td>-0.306±0.1221</td>
</tr>
<tr>
<td></td>
<td>-0.169</td>
<td>0.118</td>
</tr>
<tr>
<td></td>
<td>0.111</td>
<td>-0.156</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>-0.298±0.8771</td>
<td>-0.409±0.9091</td>
</tr>
<tr>
<td></td>
<td>-0.380±0.3511</td>
<td>-0.351±0.1001</td>
</tr>
<tr>
<td></td>
<td>-0.151</td>
<td>0.135</td>
</tr>
<tr>
<td></td>
<td>0.061</td>
<td>-0.175</td>
</tr>
<tr>
<td>$\eta_1$</td>
<td>0.121±0.6101</td>
<td>-0.904±0.8431</td>
</tr>
<tr>
<td></td>
<td>-0.290±0.1361</td>
<td>-0.369±0.2101</td>
</tr>
<tr>
<td></td>
<td>-0.167</td>
<td>0.136</td>
</tr>
<tr>
<td></td>
<td>-0.031</td>
<td>0.165</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>-0.375±1.0351</td>
<td>-0.418±0.7581</td>
</tr>
<tr>
<td></td>
<td>-0.378±0.2121</td>
<td>-0.312±0.0961</td>
</tr>
<tr>
<td></td>
<td>0.116</td>
<td>0.111</td>
</tr>
<tr>
<td></td>
<td>-0.165</td>
<td>-0.164</td>
</tr>
<tr>
<td></td>
<td>$\gamma_1$</td>
<td>$\gamma_2$</td>
</tr>
<tr>
<td>----</td>
<td>------------</td>
<td>------------</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>$-0.340 \pm 0.854i$</td>
<td>$-0.417 \pm 0.931i$</td>
</tr>
<tr>
<td></td>
<td>$0.114$</td>
<td>$0.114$</td>
</tr>
<tr>
<td></td>
<td>$-0.165$</td>
<td>$-0.165$</td>
</tr>
<tr>
<td></td>
<td>$-0.352 \pm 0.184i$</td>
<td>$-0.370 \pm 0.915i$</td>
</tr>
<tr>
<td></td>
<td>$-0.165$</td>
<td>$-0.165$</td>
</tr>
<tr>
<td></td>
<td>$0.117$</td>
<td>$0.112$</td>
</tr>
<tr>
<td></td>
<td>$-0.373 \pm 0.809i$</td>
<td>$-0.407 \pm 0.101i$</td>
</tr>
<tr>
<td></td>
<td>$0.061$</td>
<td>$0.155$</td>
</tr>
<tr>
<td></td>
<td>$0.022 \pm 2.157i$</td>
<td>$-0.141 \pm 0.096i$</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>$-0.158$</td>
<td>$-0.268$</td>
</tr>
<tr>
<td></td>
<td>$-0.123$</td>
<td>$0.092$</td>
</tr>
<tr>
<td></td>
<td>$-0.437 \pm 0.153i$</td>
<td>$-0.275 \pm 1.195i$</td>
</tr>
<tr>
<td></td>
<td>$-0.122$</td>
<td>$0.103$</td>
</tr>
<tr>
<td></td>
<td>$-0.158$</td>
<td>$-0.184$</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>$-0.426 \pm 0.160i$</td>
<td>$-0.256 \pm 0.171i$</td>
</tr>
<tr>
<td></td>
<td>$-0.122$</td>
<td>$0.103$</td>
</tr>
<tr>
<td></td>
<td>$-0.158$</td>
<td>$-0.184$</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>$-0.359 \pm 0.881i$</td>
<td>$-0.409 \pm 0.930i$</td>
</tr>
<tr>
<td></td>
<td>$-0.385 \pm 0.237i$</td>
<td>$-0.361 \pm 0.140i$</td>
</tr>
<tr>
<td></td>
<td>$-0.256$</td>
<td>$0.046 \pm 0.066i$</td>
</tr>
<tr>
<td></td>
<td>$0.134$</td>
<td>$0.279$</td>
</tr>
</tbody>
</table>

Table 4.3(E) : Eigenvalues for model A; high and low parameter sets.
<table>
<thead>
<tr>
<th>Policy Combination</th>
<th>Monetarist (ϕ=0)</th>
<th>Bondist (ϕ=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_2 ) High ( \beta_5 ) Low</td>
<td>(-0.330 \pm 0.903i)</td>
<td>(-4.151)</td>
</tr>
<tr>
<td>( \eta_1 ) High ( \beta_5 ) Low</td>
<td>(-0.357 \pm 0.309i)</td>
<td>(-0.377 \pm 0.498i)</td>
</tr>
<tr>
<td>( \lambda_2 ) High ( \eta_1 ) High ( \beta_5 ) Low</td>
<td>(-0.006 \pm 0.079i)</td>
<td>(-0.222 \pm 0.169i)</td>
</tr>
<tr>
<td>( \lambda_2 ) High ( \eta_1 ) High ( \beta_5 ) Low</td>
<td>(0.096 \pm 0.611i)</td>
<td>(-3.557)</td>
</tr>
<tr>
<td>( \lambda_2 ) High ( \eta_1 ) High ( \beta_5 ) Low</td>
<td>(-0.305 \pm 0.113i)</td>
<td>(0.230)</td>
</tr>
<tr>
<td>( \lambda_2 ) High ( \eta_1 ) High ( \beta_5 ) Low</td>
<td>(-0.006 \pm 0.079i)</td>
<td>(-0.016 \pm 0.027i)</td>
</tr>
<tr>
<td>( \lambda_2 ) High ( \eta_1 ) High ( \beta_5 ) Low</td>
<td>(0.096 \pm 0.611i)</td>
<td>(-0.222 \pm 0.169i)</td>
</tr>
</tbody>
</table>

Table 4.3(F): Eigenvalues for model A; simultaneous parameter variations.
<table>
<thead>
<tr>
<th>Policy</th>
<th>Monetarist ($\phi=0$)</th>
<th>Bondist ($\phi=1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>Set</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Param.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_4$</td>
<td>2.058</td>
<td>-1.078</td>
</tr>
<tr>
<td></td>
<td>-0.947±0.912i</td>
<td>0.082±0.6161</td>
</tr>
<tr>
<td></td>
<td>-0.353</td>
<td>0.642</td>
</tr>
<tr>
<td></td>
<td>0.252</td>
<td>-0.291</td>
</tr>
<tr>
<td></td>
<td>-0.108</td>
<td>-0.052</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>1.104</td>
<td>1.102</td>
</tr>
<tr>
<td></td>
<td>-0.827</td>
<td>-0.556±0.3601</td>
</tr>
<tr>
<td></td>
<td>-0.431±0.5231</td>
<td>-0.360</td>
</tr>
<tr>
<td></td>
<td>-0.169</td>
<td>0.129</td>
</tr>
<tr>
<td></td>
<td>0.120</td>
<td>-0.156</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>1.109</td>
<td>1.101</td>
</tr>
<tr>
<td></td>
<td>-0.864</td>
<td>-0.602±0.3901</td>
</tr>
<tr>
<td></td>
<td>-0.261±0.5831</td>
<td>-0.383</td>
</tr>
<tr>
<td></td>
<td>-0.152</td>
<td>0.149</td>
</tr>
<tr>
<td></td>
<td>0.063</td>
<td>-0.175</td>
</tr>
<tr>
<td>$\eta_1$</td>
<td>-0.223±0.6471</td>
<td>1.545</td>
</tr>
<tr>
<td></td>
<td>0.463</td>
<td>-1.174</td>
</tr>
<tr>
<td></td>
<td>-0.258</td>
<td>-0.413±0.2961</td>
</tr>
<tr>
<td></td>
<td>-0.171</td>
<td>0.143</td>
</tr>
<tr>
<td></td>
<td>-0.035</td>
<td>-0.165</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>1.097</td>
<td>1.106</td>
</tr>
<tr>
<td></td>
<td>-0.437±0.4171</td>
<td>-0.481±0.3901</td>
</tr>
<tr>
<td></td>
<td>-0.590</td>
<td>-0.621</td>
</tr>
<tr>
<td></td>
<td>0.125</td>
<td>0.124</td>
</tr>
<tr>
<td></td>
<td>-0.165</td>
<td>-0.165</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>1.101</td>
<td>1.105</td>
</tr>
<tr>
<td></td>
<td>-0.460±0.3811</td>
<td>-0.470±0.4221</td>
</tr>
<tr>
<td></td>
<td>-0.629</td>
<td>-0.591</td>
</tr>
<tr>
<td></td>
<td>0.127</td>
<td>0.122</td>
</tr>
<tr>
<td></td>
<td>-0.165</td>
<td>-0.165</td>
</tr>
</tbody>
</table>
Table 4.3(G) : Eigenvalues for model A/R; high and low parameter sets.

| $\gamma_3$ | 1.123 | 1.073 | -1.468 | -6.272 |
| -0.748 | -0.509±0.574i | 1.116 | 1.099 |
| -0.378±0.269i | -0.541 | -0.592 | -0.677 |
| -0.166 | 0.181 | -0.207±0.142i | -0.211±0.171i |
| 0.063 | -0.165 | -0.182 | -0.176 |
| -0.781±1.630i | 1.085 | -2.670 | -3.426 |
| 1.152 | -1.057 | -0.985±0.849i | 1.076 |
| -0.428 | -0.140±0.100i | 1.190 | -0.772 |
| -0.158 | -0.266 | -0.319 | -0.033±0.076i |
| 0.137 | 0.097 | -0.120 | -0.263 |
| -0.802±1.100i | 1.066 | -3.384 | -3.348 |
| 1.199 | -0.895 | 1.236 | 1.059 |
| -0.438 | -0.249±0.214i | -0.644±0.548i | -0.742 |
| -0.159 | -0.183 | -0.339 | -0.254 |
| 0.135 | 0.111 | -0.122 | -0.072±0.115i |
| $\theta_2$ | 1.099 | 1.108 | -3.342 | -3.365 |
| -0.763 | -0.575±0.422i | 1.100 | 1.110 |
| -0.386±0.458i | -0.455 | -0.752 | -0.527 |
| 0.147 | 0.049±0.068i | -0.151±0.254i | -0.341±0.175i |
| -0.258 | -0.280 | 0.037 |
### Table 4.3(H): Eigenvalues for model A/R; simultaneous parameter variations.

<table>
<thead>
<tr>
<th>Policy Combination</th>
<th>Monetarist ($\phi$=0)</th>
<th>Bondist ($\phi$=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_2$ High</td>
<td>1.114</td>
<td>-3.674</td>
</tr>
<tr>
<td>$\beta_5$ Low</td>
<td>-0.811</td>
<td>0.187±0.2111</td>
</tr>
<tr>
<td></td>
<td>-0.299±0.5531</td>
<td>-0.279</td>
</tr>
<tr>
<td></td>
<td>0.005±0.0801</td>
<td>-0.098</td>
</tr>
<tr>
<td></td>
<td>-0.257±0.6451</td>
<td>-0.068</td>
</tr>
<tr>
<td>$\eta_2$ High</td>
<td>0.447</td>
<td>0.156±0.2191</td>
</tr>
<tr>
<td>$\beta_5$ Low</td>
<td>-0.292</td>
<td>-0.097</td>
</tr>
<tr>
<td></td>
<td>-0.019±0.0281</td>
<td>-0.064</td>
</tr>
<tr>
<td></td>
<td>-0.191±0.7271</td>
<td>-3.674</td>
</tr>
<tr>
<td>$\lambda_2$ High</td>
<td>0.421</td>
<td>0.187±0.2111</td>
</tr>
<tr>
<td>$\eta_2$ High</td>
<td>-0.279</td>
<td>-0.279</td>
</tr>
<tr>
<td>$\beta_5$ Low</td>
<td>-0.023±0.0311</td>
<td>-0.098</td>
</tr>
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</table>

Table 4.3(H): Eigenvalues for model A/R; simultaneous parameter variations.
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<tr>
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<td>0.711</td>
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<tr>
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<td>0.112</td>
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<td>$\lambda_1$</td>
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<td>-0.793±0.4481</td>
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<td>-0.766</td>
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<tr>
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<td>-0.161</td>
<td>-0.288</td>
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<tr>
<td></td>
<td>0.967</td>
<td>0.959</td>
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<tr>
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<tr>
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<td>0.066</td>
<td>-0.165</td>
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<tr>
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<td>0.564±0.169</td>
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<tr>
<td></td>
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<td>-0.043±0.2461</td>
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<tr>
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<td>-0.035</td>
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<tr>
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<td>-0.168</td>
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<td>$\gamma_1$</td>
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<td>0.143</td>
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<td></td>
<td>0.669</td>
<td>0.674</td>
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<tr>
<td></td>
<td>-0.456</td>
<td>-0.159</td>
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<tr>
<td></td>
<td>0.955</td>
<td>0.960</td>
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<td>-0.748±0.477i</td>
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<td>0.145</td>
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<td>0.648</td>
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<tr>
<td>0.672</td>
<td>0.673</td>
<td>-0.587±0.193i</td>
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<td>-0.160</td>
<td>-0.120</td>
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<tr>
<td>0.960</td>
<td>0.964</td>
<td>-0.336</td>
</tr>
<tr>
<td>-0.744±0.337i</td>
<td>-0.769±0.589i</td>
<td>0.935</td>
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<tr>
<td>-0.428</td>
<td>0.695</td>
<td>-0.578±0.116i</td>
</tr>
<tr>
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<td>-0.160</td>
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<tr>
<td>0.068</td>
<td>0.977</td>
<td>-0.117</td>
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<tr>
<td>-1.144±1.096i</td>
<td>-0.955</td>
<td>-3.142</td>
</tr>
<tr>
<td>1.189±0.279i</td>
<td>1.026</td>
<td>1.120±0.364i</td>
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<td>-0.421</td>
<td>-0.327±0.168i</td>
<td>-1.025±0.508i</td>
</tr>
<tr>
<td>0.140</td>
<td>0.217</td>
<td>-0.311</td>
</tr>
<tr>
<td>-0.158</td>
<td>0.183</td>
<td>-0.116</td>
</tr>
<tr>
<td>-1.015±0.908</td>
<td>1.014</td>
<td>-3.230</td>
</tr>
<tr>
<td>1.062±0.267i</td>
<td>-0.800</td>
<td>1.004±0.314i</td>
</tr>
<tr>
<td>-0.424</td>
<td>-0.478±0.194</td>
<td>-0.863±0.451i</td>
</tr>
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<td>0.140</td>
<td>0.406</td>
<td>-0.313</td>
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<td>-0.158</td>
<td>0.148</td>
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<td>-0.160</td>
<td>-0.160</td>
<td>-0.128</td>
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<td>-0.690±0.447i</td>
<td>-0.798±0.491i</td>
<td>-3.306</td>
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<tr>
<td>0.928</td>
<td>0.052±0.076i</td>
<td>-0.388±0.144i</td>
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<tr>
<td>0.718</td>
<td>0.052±0.076i</td>
<td>-0.273</td>
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Table 4.3(I): Eigenvalues for model R; high and low parameter sets.
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<tr>
<th>Policy Combination</th>
<th>Monetarist ($\phi=0$)</th>
<th>Bondist ($\phi=1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_2$ High, $\beta_5$ Low</td>
<td>$-0.612 \pm 0.5861$</td>
<td>$-3.698$</td>
</tr>
<tr>
<td></td>
<td>$-0.730$</td>
<td>$1.032$</td>
</tr>
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<td></td>
<td>$1.014$</td>
<td>$0.378$</td>
</tr>
<tr>
<td></td>
<td>$0.666$</td>
<td>$-0.743$</td>
</tr>
<tr>
<td></td>
<td>$0.002 \pm 0.0811$</td>
<td>$-0.387 \pm 0.1401$</td>
</tr>
<tr>
<td></td>
<td>$0.075$</td>
<td></td>
</tr>
<tr>
<td>$\eta_2$ High, $\beta_5$ Low</td>
<td>$-0.516 \pm 0.6311$</td>
<td>$-3.255$</td>
</tr>
<tr>
<td></td>
<td>$0.544 \pm 0.1721$</td>
<td>$0.668$</td>
</tr>
<tr>
<td></td>
<td>$-0.292$</td>
<td>$-0.670 \pm 0.2031$</td>
</tr>
<tr>
<td></td>
<td>$-0.020 \pm 0.0281$</td>
<td>$-0.064$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-0.259 \pm 0.0091$</td>
</tr>
<tr>
<td>$\lambda_2$ High, $\eta_2$ High, $\beta_5$ Low</td>
<td>$-0.506 \pm 0.7361$</td>
<td>$-3.654$</td>
</tr>
<tr>
<td></td>
<td>$0.534 \pm 0.1241$</td>
<td>$0.587$</td>
</tr>
<tr>
<td></td>
<td>$-0.279$</td>
<td>$-0.060 \pm 0.2521$</td>
</tr>
<tr>
<td></td>
<td>$-0.023 \pm 0.0291$</td>
<td>$-0.067$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-0.237 \pm 0.016$</td>
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Table 4.3(J) : Eigenvalues for model R; simultaneous parameter variations.
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<th>Model</th>
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<th>A/R</th>
<th>R</th>
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<td>S</td>
<td>S</td>
<td>S</td>
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<tr>
<td>0.80</td>
<td>S</td>
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<td>0.70</td>
<td>S</td>
<td>S</td>
<td>S</td>
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<td>0.60</td>
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<td>S</td>
<td>S</td>
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<tr>
<td>0.50</td>
<td>S</td>
<td>S</td>
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<td>0.30</td>
<td>S</td>
<td>S</td>
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<td>0.20</td>
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<td>0.15</td>
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<td>U</td>
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Table 4.3(K): Stability under policy parameter variations and divergent model specifications.

U = Unstable ; S = Stable.
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<th>Model</th>
<th>A</th>
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<th>R</th>
</tr>
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<td>-0.303±0.4721</td>
<td>1.104</td>
<td>0.931</td>
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<tr>
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<td>-0.240±0.1421</td>
<td>-0.664</td>
<td>0.649</td>
</tr>
<tr>
<td></td>
<td>-0.178</td>
<td>-0.209±0.1571</td>
<td>-0.597±0.179</td>
</tr>
<tr>
<td>0.80</td>
<td>-3.092</td>
<td>-2.741</td>
<td>-2.713</td>
</tr>
<tr>
<td></td>
<td>-0.317±0.4701</td>
<td>1.104</td>
<td>0.933</td>
</tr>
<tr>
<td></td>
<td>-0.235±0.1341</td>
<td>-0.666</td>
<td>0.650</td>
</tr>
<tr>
<td></td>
<td>-0.180</td>
<td>-0.209±0.1471</td>
<td>-0.612±0.143</td>
</tr>
<tr>
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<td>-2.707</td>
<td>-2.433</td>
<td>-2.408</td>
</tr>
<tr>
<td></td>
<td>-0.336±0.4671</td>
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<td>0.933</td>
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<td>-0.231±0.1221</td>
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<td>0.650</td>
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<td>-0.612±0.143</td>
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<td>-0.647±0.0091</td>
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<td>-0.832 ± 0.4451</td>
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<td>-0.591 ± 0.3421</td>
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Table 4.3(L): Eigenvalues for divergent model specifications; policy parameter variations.
of asset stocks and flows in the system. Mixed financial rules may also be stable. We therefore examined the influence of the financing parameter, $\phi$, on stability in the region $0 \leq \phi \leq 1$. A selection of the results are reported in table 4.3(K), for which the eigenvalues are recorded in table 4.3(L), for the central parameter. They show that a small shift of $\phi$ away from zero makes the system stable: thus, stability obtains for $\phi > 0.15$ (our search was originally in steps of 0.05 so that the precise value of $\phi$ at which the system becomes stable cannot be identified, but it clearly lies in the region $0.1 < \phi < 0.15$). Interestingly, this is true for all models. Nevertheless, although the system is stable for $\phi > 0.15$, a low value of $\phi > 0.15$ may still be undesirable. This is motivated by inspection of the eigenvalues which reveals that values of $\phi$ in excess of, but close to 0.15, have an eigenvalue very close to zero. This will give rise to very slow convergence of the system and, when subject to shocks, the system will display a high degree of volatility. The next section, which examines the stochastic properties of the models, does not test the above prediction though we have no reason to believe that it would not be verified. It would be a simple matter for future research to investigate this.

4.4 Some Dynamic Properties of the Model

Section 4.3 indicated that model stability will be crucial in influencing the authorities' choice of financial policy rule. Stability is also, of course, a precondition for any meaningful discussion of the long-run properties of the model. In this section we convey the flavour of the stochastic properties of the system by subjecting it to two permanent disturbances: a unit permanent increase in the authorities' inflation target, $\pi^*$; and a unit permanent increase in government expenditure, $g$. As for the stability analysis in section 4.3, a preliminary algebraic treatment
of the long-run multipliers can be found in Whittaker and Wren-Lewis (1983) and Whittaker, Wren-Lewis, Blackburn and Currie (1985). This showed that the long-run multipliers are independent of both the adjustment parameters in the model and the expectations assumption. Needless to say, these assumptions will influence the dynamic path from one steady state to another and it is this adjustment that we wish to concentrate on here using computer simulations for a bondist policy and the central parameter set.

The financial policy (whether monetarist or bondist) determines the equilibrium rate of inflation which we have termed the authorities' inflation target, $p^*$. Figures 4.4(A) - 4.4(C) show the trajectories for output, inflation and the exchange rate following a unit increase in $p^*$. This is accomplished by a change in the nominal bond stock for given fiscal parameters so that both nominal and real interest rates rise. Higher real interest rates depresses demand. This initial fall is then reversed, partly because higher interest rates generate a larger budget deficit and increased interest income. Increased demand then moves inflation towards its long-run equilibrium level with a depreciating exchange rate. As inflation moves towards this level, real wealth is eroded which, coupled with the inflation tax, depresses demand towards its long run equilibrium which is below the original steady state value.

Figures 4.4(A) - 4.4(C) also illustrate the dynamic behaviour of output, inflation and the exchange rate for a money-financed permanent unit increase in government expenditure. Inflation increases and the real value of the bond stock falls so that both nominal and real interest rates are reduced. Lower domestic interest rates stimulates investment and consumption and induces a capital outflow. The budget moves into surplus because of higher tax receipts and lower debt interest payments and the increase in
Figure 4.4(A)(i) : Income trajectories for divergent model specifications; permanent unit increase in $p^*$. 

Figure 4.4(A)(ii) : Income trajectories for divergent model specifications; permanent unit increase in $g$. 

Figure 4.4(A) : Income trajectories for divergent model specifications under alternative permanent shocks.

--- model A; ------ model A/R; ---- model R
Figure 4.4(B)(i) : Inflation trajectories for divergent model specifications; permanent unit increase in $p^*$.  

Figure 4.4(B)(ii) : Inflation trajectories for divergent model specifications; permanent unit increase in $g$.  

Figure 4.4(B) : Inflation trajectories for divergent model specifications under alternative permanent shocks.  

--- model A; --- model A/R; --- model R
Figure 4.4(C)(i) : Exchange rate trajectories for divergent model specifications; permanent unit increase in $p^*$. 

Figure 4.4(C)(ii) : Exchange rate trajectories for divergent model specifications; permanent unit increase in $g$. 

Figure 4.4(C) : Exchange rate trajectories for divergent model specifications under alternative permanent shocks. 

--- model A; —— model A/R; —— model R
demand encourages an exchange rate depreciation which improves the trade balance such that the effect on the balance of payments arising from the capital outflow is neutralized. Over time, the effect of government expenditure is crowded out. The main reason for this is the decline in real wealth resulting from the budget surplus and higher inflation. Inflation continues to rise due to excess demand (and, in models A and A/R, due to the reinforcing effect of adaptive inflation expectations). Then the continual fall in real interest rates raises output supply above aggregate demand such that the inflationary pressure begins to ease. The fall in demand improves the trade balance and the exchange rate appreciates further. In the long-run, output is above its initial equilibrium level and so there is only partial crowding out.

Finally, it is worth noting some additional features of the above dynamics which are common to all models. In each case, the broad characteristics of the adjustment paths are strikingly similar. In addition, the initial exchange rate jumps in models A/R and R are relatively small (see, for example, Turnovsky and Nguyen (1980) for a similar observation). Nonetheless, it appears that the adjustment path in model R tends to be smoother and approaches equilibrium more rapidly. Thus, the introduction of rational expectations seems to smooth the process of adjustment and speeds it up (4).

4.5 Summary and Concluding Remarks

We have analysed the stability of a fairly general open economy model, incorporating both adaptive and rational expectations, under two stylised financial policy rules: a monetarist policy which fixes the rate of growth of nominal money; and a bondist policy which fixes the nominal rate of growth of government debt. The analysis suggests that as long as post-tax real interest rates are positive, the bondist rule is much more likely to be stable than the
monetarist rule. In this respect, we have confirmed the proposition familiar from the existing literature on the government budget constraint that a fixed monetary rule is likely to be destabilizing. If, however, post-tax real interest rates are negative, we found certain cases in which the expectations assumption and the open economy aspects of our model increased the possibility of monetarist stability. In particular, a model with rational expectations in the foreign exchange market and perfect capital mobility could be stable under this policy rule. In addition, rational expectations in the goods market also tends to aid monetarist stability as the interest sensitivity of consumption and the competitiveness effect on the trade balance fall. These suggest that it is the adaptive inflation expectations formation which presents an additional obstacle to bond financing stability. In spite of these interesting and important exceptions, a policy which involves pure bond financing of budget imbalances is generally unstable.

The long-run multipliers of the model are independent of the expectations assumptions but these assumptions will influence the dynamic response of the system to exogenous shocks. Our illustrative simulations for permanent changes in the authorities' inflation target and government expenditure indicated the broad similarity of dynamic adjustment under alternative expectations formation. Nonetheless, the approach to long-run equilibrium was far from monotonic and the paths were smoother if expectations were rational.

The analysis reported in this chapter could be extended in a number of directions. First, and most obvious, a wider spectrum of stochastic perturbations could be handled easily. These would take the form of different permanent shocks together with temporary disturbances. Second, as noted in section 4.2, some authors have obtained stability under a monetarist rule for analytical models
if government expenditure is appropriately defined (see, for example, Christ (1979); Cohen and Leeuw (1980)). We have no suggestions to make about the relative merits of alternative definitions of government spending, though there might be some interest in examining the implications of these different approaches using the framework set out in this chapter. Third, the model could be broadened in one important direction, namely the introduction of a more detailed description of supply behaviour. We suspect that our general conclusions will more-or-less remain intact, though the dynamic adjustment of the system may be altered. A particularly interesting approach would be to follow Artis and Currie (1981) who specify cost mark-up pressures including import prices and the exchange rate. This would increase the importance of the open economy aspects of the model.

Fourth, and in a slightly different vein, we intended to perform optimal control exercises on the model but the software was unfortunately unavailable. Similarly, we would also have welcomed the opportunity of examining some of the issues discussed in part II of the thesis on the information structure conditioning rational expectations in this system. As above, the software was unavailable. Given the fair degree of generality of the model and, in particular, the dynamic configuration resulting from asset accumulation and the government budget constraint, we believe that investigation into these matters would be of some interest. We hope to pursue this research shortly. Finally, as far as we know, the issues discussed in this chapter have not been analysed in the context of interdependent economies and we suggest this as another possibility for further research.
Notes to Chapter 4

This chapter is based on Whittaker, Wren-Lewis, Blackburn and Currie (1985) which is an extension of the research in Whittaker and Wren-Lewis (1983) and Blackburn and Currie (1984). The second of these papers provided the motivation for the other two.

(1) See, for example, Friedman (1978) for an analysis stressing portfolio behaviour.

(2) We assume that government bonds are net wealth. There is a well-known debate on the validity of this assumption and the reader is referred to Barro (1974), McCallum (1978c), Currie (1979) and Tobin (1980b). Our inclusion of bonds in net wealth is consistent with the majority of the budget constraint literature and is rather more plausible than their exclusion. The requirement that bonds be part of net wealth is not necessary for their appearance in the money demand function in equation (4.2.7) because of portfolio considerations (see Currie (1979)).

(3) Different forms of the simulation framework in equations (4.2.16) - (4.2.17) were used in earlier papers because of the different software. The particular configuration of the vectors and matrices can be found in Whittaker and Wren-Lewis (1983) and Blackburn and Currie (1984).

(4) Blundell-Wignall and Masson (1984) have observed a similar tendency in a slightly different context.
5.1 Introduction

By definition, the rational expectation hypothesis forces consideration of the amount and nature of information available to forecasting agents. This must be so; for in order to construct a rational expectation of some variable, \( x \), it is necessary to make explicit the information structure conditioning that expectation. Hence,

\[
x^e = E(x | \Omega)
\]

(5.1.1)

where \( E(\cdot) \) is the mathematical conditional expectations operator, requires some notion of what is actually meant by the information set, \( \Omega \). Unfortunately, the particular information structure is rarely explicated in sufficient detail. Neither also are the implications of the information structure for the results of an analysis suitably revealed. To be sure, the reader may have noticed in part I of the thesis the limited remarks about the factors conditioning expectations. At the same time, it is probably the case that many would have been unconcerned about this. After all, it is a familiar feature of much of the literature and one with which the reader is no doubt acquainted. One should not, however, infer from this the triviality of specifying the precise information structure in an economy. On the contrary, the lack of detail regarding this is a poor reflection of the literature and it is an unfortunate habit which economists have become accustomed to.

Though the formal definition of rational expectations has some merit, many would argue that it begs rather more questions than any
that it answers. Blind patriotism to the hypothesis is to eschew the very real problem of how agents actually learn the true process determining the laws of motion of the system (given that such exists). This is an issue which certain authors have taken as indication of the implausibility and inapplicability of rational expectations. Particular views on this can be found in Cyert and De Groot (1974), Taylor (1975), Shiller (1978), Friedman (1979a), Helm (1983), the symposium in the 1982 edition of the Journal of Economic Theory and the edited volume by Frydman and Phelps (1983).

Our own view on the matter is one of caution. We believe that problems of existence of and convergence to rational expectations equilibria with endogenous learning is an important area which has largely been neglected. Some of the strong conclusions reached so far may be qualified when the process of learning is integrated into the analytical framework. Nonetheless, it is possible that this research will open up a Pandora's box which makes practically anything possible. As a consequence, progress is likely to be exciting but slow and the conclusions must be judged with some caution. At the moment, the simplest way of modelling intelligent economic behaviour is to invoke the rational expectations assumption. Until such time emerges when we are more certain about some of the above issues, we must appreciate our original premise (maintained throughout the thesis) that rational expectations is still the only sensible assumption, at least in terms of policy evaluation.

There is, however, another interesting, and more directly realisable, line of inquiry. This is to examine the nature of the information structure in a rational expectations equilibrium itself. As noted above, analyses conducted on this assumption often neglect explicit consideration of the information structure. This is an important drawback for it obscures the potentially crucial role of
information in determining the dynamic response of the system.
The chapter is concerned with clarifying some of the issues involved.
The linchpin of this (and subsequent chapters) is the existence of
(only) partial ignorance. Let us be more specific.

In models embodying rational expectations, one may define the
information structure in terms of a number of characteristics.
Information sets may be homogeneous or heterogeneous (differentiated); they may be complete, incomplete or null. A popular
assumption is homogeneous and full information sets. By this is
meant that agents are endowed with complete information about all
variables in the system at all times and that this is true for all
agents. There is no lack of information and no individual or group
has an informational advantage. Though a useful assumption in some
cases, its obvious implausibility renders it redundant as a
reflection of actual situations. Similarly, the opposite extreme of
homogeneous and null information sets is an equal falsehood.
Intermediate between these polar cases is the assumption of
incomplete information sets which may be homogeneous or differenti-
tated. It would not be implausible to argue that these are probably
the closest approximations to the actual environment in which
agents find themselves. Thus, agents are likely to have access to
contemporaneous information about only a subset of the set of
economic variables. Note, however, that this does not necessarily
(and is unlikely to) imply zero information about currently unobserv-
able variables. To be sure, by observing contemporaneous movements
in some variables, informed inferences can be made about the beha-
viour of the unknown quantities. That this is so endows certain
variables with the role of conveyers of information — what we shall
term information variables. Agents can exploit the information
content of these variables. In the engineering literature, this type
of statistical inference problem is known as signal-extraction (that is, the extraction of auxiliary information from observed signals).

The general characterisation of the signal-extraction problem takes the form of the Kalman filter (see, for example, Meditch (1969); Aastroem (1970); Chow (1975, 1981)). Put simply, this provides a set of updating equations for sequentially revising the estimates of unobservable variables on the basis of innovations in news. By way of an introduction, the essence of the Kalman filter can be conveyed by considering the following general dynamic system:

\[ y_t = A_t y_{t-1} + u_t \]  \hspace{1cm} (5.1.2)

\[ s_t = C_t y_t + v_t \]  \hspace{1cm} (5.1.3)

where

\( y \) = vector of state variables

\( s \) = vector of observable variables

\( u, v \) = vectors of stochastic disturbances.

\( A \) and \( C \) are time-varying matrices and \( u \) and \( v \) are assumed to be independently and Gaussian distributed with variance-covariance matrices, \( \Sigma^u \) and \( \Sigma^v \) respectively. Equation (5.1.2) is the state equation describing the laws of motion of the system. Equation (5.1.3) is the measurement equation. The information set is

\[ \Omega_t = \{ s_{t-j}, \Gamma_{t-j}, \Sigma | j \geq 0 \} \]  \hspace{1cm} (5.1.4)

where \( \Gamma \) is a vector of time-varying structural parameters and \( \Sigma \) summarizes the variance-covariance properties of stochastic disturbances. The Kalman filter begins with the linear regression

\[ y^e_{t,t} - y^e_{t,t-1} = \Delta(s - s^e_{t,t-1}) \]  \hspace{1cm} (5.1.5)
where $\Delta$ is a time-varying matrix of regression coefficients and, for any variable, $X, X_{t,j} = E(X_t | X_{t-j}) (j = 0,1)$. Equation (5.1.5) shows that expectations about unobservable variables are updated according to unanticipated changes in information. Now, if we let $\Gamma_t, t-1 = E(\gamma_t - \gamma_{t-1})^2 (\gamma_t - \gamma_{t-1})^T$, then the Kalman filter may be summarized in the set of equations (5.1.6) - (5.1.10):

$$Y_{t,t}^e = (1 - \Delta_t C_t^T) Y_{t-1,t}^e + \Delta_t s_t \tag{5.1.6}$$

$$Y_{t-1,t} = \Delta_t Y_{t-1,1} \tag{5.1.7}$$

$$\Delta_t = \Gamma_t, t-1 C_t^T (C_t^T C_t + \Gamma^T)^{-1} \tag{5.1.8}$$

$$\Gamma_t, t-1 = \Delta_t \Gamma^T t-1, t-2 A_t + L_t^u \tag{5.1.9}$$

$$\Gamma_t, t = \Gamma_t, t-1 (I - C_t^T (C_t^T C_t + \Gamma^T)^{-1} C_t^T) \tag{5.1.10}$$

where $I$ is the identity matrix. Equations (5.1.6) - (5.1.10) are the set of updating equations and their derivation is fairly straightforward. Nonetheless, they are rather complicated for analytical purposes and the Kalman filter can be simplified greatly by assuming that full information eventually becomes available. For the types of problems that we will be addressing this is not an implausible assumption.

It is important to note that the filtering process is not merely an elaborate mathematical device with little substantive appeal. It is, in fact, endemic to rational expectations models which are characterised by partial information. Abstracting from it is to violate the fundamental tenet which states that sophisticated agents will not waste useful information. That it has been largely neglected in the literature is a non-trivial criticism.
Appreciation of the statistical inference problem has important implications for policy evaluation. Under such circumstances, control rules have scope to influence the system in two ways. The first is the standard mechanism, common to all control rules, whereby control alters the dynamic response of the system. The second occurs by virtue of the ability of control laws to alter the information content of variables. This can be considered as an extension of the Lucas (1976) critique on policy evaluation. Not only is the economic structure non-invariant with respect to changes in policy, which arises via the feedback through expectations, but neither also is the information structure from which these expectations are derived. In addition, the implementation of optimal policies takes on an additional complicated dimension since the controller may wish to formulate policy in terms of observable variables.

A further twist occurs when partial information is combined with the assumption of heterogeneous information. Consider the following scenario. A circularity inherent in the inference problem is that the observed signals are influenced by actions conditioned on inferences drawn from the signals. The importance of this is fully realized only when there exists differential information; for now, observed signals reflect the different perceptions of different agents. As Lucas (1975) notes, this makes the signals themselves an object of speculation. Agents must appreciate that information about unobservable variables is contaminated. They must form expectations about other’s expectations. The latter will be performing exactly the same computations. Then agents must form expectations about others’ expectations of their expectations and so on and so forth. This is an interesting problem and is likely to arise whenever there exists some idiosyncratic information. It motivates the idea of an informational game.
In summary, we hope that we have made clear the potentially critical importance of the information structure. The above issues will be returned to at greater length throughout this and subsequent chapters. The current chapter is concerned with the following.

Section 5.2 reviews critically some recent contributions to the field of signal-extraction. Section 5.3 is motivated by the need to clarify some issues which we feel have not been given adequate attention. Its main purpose is to provide a rigorous investigation into the information contents of the exchange rate and interest rates in a simple dynamic model. Section 5.4 concludes.

5.2 A Critique of Some Rational Expectations Models with Partial Information

The role of the information structure in determining the behaviour of the economy was demonstrated succinctly in the early pioneering work on rational expectations of Lucas (1972b, 1975), Sargent (1973), Sargent and Wallace (1975, 1976) and Barro (1976). For the most part, subsequent investigations have exploited the simplistic elegance of these studies with the view to reinforcing, generalizing and extending the insights thereof. The essence of this research may be summarized by considering the following type of statistical inference problem: if decisions are based on perceived relative price changes, agents must attempt to distinguish between local and aggregate shocks. The seminal papers by Lucas (1972b, 1975) envisaged the following well-known scenario.

A single commodity is transacted in various markets indexed by \( h = 1, \ldots, H \). These markets are spatially separated. At any time an individual agent finds himself in a particular market at random and can only trade in one market at each time. Thereafter, agents are again randomly distributed across markets. Information
circulates continuously and instantaneously within a market but is disseminated across locational boundaries sluggishly. Producers in a particular location observe their own local price but not the prices in other markets, information about which accrues with a one period lag. Supply decisions are based on perceived relative price changes: suppliers in each location must infer the extent to which an observed movement in their local price reflects a change in relative prices. This scheme should be familiar. In order to acquaint the reader with the mechanics of the inference problem and some of the issues to be discussed later, a useful illustrative device is a stripped-down version of this framework. This is contained in equations (5.2.1) - (5.2.3) (similar structures can be found in King (1981); Boschen and Grossman (1983)):

\[ y_s^t(h) = \alpha(p_t(h) - P_t^e) \quad (h = 1, \ldots, H) \quad \alpha > 0 \quad (5.2.1) \]

\[ y_d^t(h) = m_t - p_t(h) + \epsilon_t(h) \quad (h = 1, \ldots, H) \quad (5.2.2) \]

\[ m_t = \rho m_{t-1} + \nu_t \quad 0 < \rho < 1 \quad (5.2.3) \]

where \( y_s^t(h) \) = natural logarithm of the supply of the commodity in market \( h \) \( (h = 1, \ldots, H) \)

\( y_d^t(h) \) = natural logarithm of the demand for the commodity in market \( h \) \( (h = 1, \ldots, H) \)

\( p(h) \) = natural logarithm of the price of the commodity in market \( h \) \( (h = 1, \ldots, H) \)

\( p \) = natural logarithm of the aggregate price level

\( m \) = natural logarithm of the nominal stock of money

\( \epsilon(h), \nu \) = stochastic disturbances \( (h = 1, \ldots, H) \)

and all variables are measured as deviations from equilibrium.
Equations (5.2.1) - (5.2.3) should require little discussion. Equation (5.2.1) is the local supply function. The term 
\[ p_{t,t}^h \] (for \( h = 1, \ldots, H \)) is the expectation of the economy-wide price level, \( p_t \), formed by participants in location \( h \) (for \( h = 1, \ldots, H \)).

\[ p_{t,t}^h = E(p_t|\Omega_t^h) \] (for \( h = 1, \ldots, H \)) where \( E(\cdot) \) is the mathematical conditional expectations operator and \( \Omega_t^h \) is the information set in market \( h \) (for \( h = 1, \ldots, H \)). Equation (5.2.2) is a simple quantity theory of local demand. We shall discuss later the appearance of the aggregate money stock, \( m_t \), in this relationship rather than the local money stock, \( m_t^h \) (for \( h = 1, \ldots, H \)). Equation (5.2.3) specifies an exogenous non-cyclical autoregressive process for aggregate money.

Stochastic disturbances are denoted by \( \epsilon(h) \) (for \( h = 1, \ldots, H \)) and \( \nu \). The former is a local (market-specific) demand shock and the latter is a global (economy-wide) demand shock which is permanent if \( \rho = 1 \). Each disturbance is independently and Gaussian distributed with asymptotic variance of \( \epsilon(h) \) (for \( h = 1, \ldots, H \)) and \( \nu \) equal to \( \sigma^2 _\epsilon \) and \( \sigma^2 _\nu \) respectively.

Imposing the equilibrium condition \( y^s_t(h) - y^d_t(h) - y_t(h) \) (for \( h = 1, \ldots, H \)) and substituting for \( m_t \) from equation (5.2.3) yields

\[ p_t(h) = (1 + \alpha)^{-1}(\alpha p_{t,t}^h + \rho m_{t-1} + \nu_t + \epsilon_t(h)) \quad (h = 1, \ldots, H). \]

(5.2.4)

Now consider the information structure. At the most general and rigorous level, the current information set in each location is

\[ \Omega_t^F(h) = (p_t(h), y_t(h), m_t(h), p_{t-j}, y_{t-j}, m_{t-j}, p_{t-j}(h), y_{t-j}(h), m_{t-j}(h), \Gamma, \Sigma | j > 1) \quad (h = 1, \ldots, H) \]

(5.2.5)

where \( \Gamma \) is a vector of all structural parameters and \( \Sigma \) summarizes the variance-covariance structure of stochastic disturbances. Equation
(5.2.5) states that contemporaneous information is available about all local variables whilst information about aggregate variables accrues with a one period lag. Lucas (1972b, 1975) and Barro (1976) define the signal-extraction problem as involving the estimation of unobservable aggregate variables on the basis of local price movements. Thus, equation (5.2.4) constitutes the observation or measurement vector (in this case a scalar). It shows that observation of \( p_t(h) (h = 1, ..., H) \) conveys information about both local and global shocks. The reader might question the abstraction of local output, \( y_t(h) (h = 1, ..., H) \) as an information variable. The reason is that both \( p_t(h) \) and \( y_t(h) (h = 1, ..., H) \) move simultaneously and reflect the same information. Similarly, local money signals \( m_t(h) (h = 1, ..., H) \) seem also to be unexploited. This is rather more interesting. Note, first, that the inclusion of aggregate money in equation (5.2.2) is because of the assumption that \( m_t(h) (h = 1, ..., H) \) is always the same fraction of \( m_t \) (see, for example, Barro (1976)). Then if this is so, observation of \( m_t(h) (h = 1, ..., H) \) would surely amount to an accurate observation of \( v_t \). Lucas (1972b, 1975) avoids this by supposing that the proportionality between \( m_t(h) (h = 1, ..., H) \) and \( m_t \) is sufficiently uncertain as to make \( m_t(h) (h = 1, ..., H) \) an unreliable estimate of \( m_t \) movements\(^{(1)} \). This is a crucial assumption for, as Barro (1976) notes, without it the principal source of imperfect information disappears. We prove this formally below. For the moment, we follow convention and regard the available useful contemporaneous information to amount to an observation solely of local prices.

We shall term the information set in equation (5.2.5) the initial informational endowment. This is to be distinguished from the overall information available which comprises both this endowment and any auxiliary information extracted therefrom. The distinction
is not trivial. An important possibility which seems to have been overlooked in the literature is the perverse outcome where an increase in the informational endowment reduces the accuracy of some forecasts (that is, makes some auxiliary information more unreliable). Section 5.3 illustrates this.

We solve the system by applying the method of undetermined coefficients in appendix C to a trial solution (see, for example, Lucas (1972b, 1973, 1975); McCallum (1983); Minford and Peel (1983b)). Thus, equation (5.2.4) inspires consideration of the following solutions for local and economy-wide prices:

\[ p_t(h) = \mu_0 \varepsilon_t(h) + \mu_1 v_t + \mu_2 m_{t-1} \quad (h = 1, \ldots, H) \]  \hspace{1cm} (5.2.6)

\[ p_t = \mu_1 v_t + \mu_2 m_{t-1} \]  \hspace{1cm} (5.2.7)

where \( \mu_i \) \((i = 0, 1, 2)\) are coefficients to be determined and we have used the fact that \( \varepsilon_t(h) \) \((h = 1, \ldots, H)\) disappears on aggregating. Then equation (5.2.7) implies

\[ e_{h} \varepsilon_{h} = \mu_1 \varepsilon_{t,h} + \mu_2 m_{t-1} \]  \hspace{1cm} (5.2.8)

so that substituting equations (5.2.6) and (5.2.8) into equation (5.2.4) gives

\[ (1 + \alpha)(\mu_0 \varepsilon_t(h) + \mu_1 v_t + \mu_2 m_{t-1}) = \alpha(\mu_1 \varepsilon_{t,h} + \mu_2 m_{t-1}) + \rho m_{t-1} + \nu_t + \varepsilon_t(h) \quad (h = 1, \ldots, H). \]  \hspace{1cm} (5.2.9)

It is now necessary to compute the expectation \( \nu_{t,h}(h = 1, \ldots, H) \). This is found by solving the signal-extraction problem alluded to
earlier. Specifically, given the linear dependence of the local price signal on the stochastic shocks in equation (5.2.4), the optimal filtered estimates of $v_t$ and $e_t(h)$ ($h = 1, \ldots, H$) follow as

\[
\begin{bmatrix}
    v_{t,t}^{eh} \\
    e_{t,t}^{eh}(h)
\end{bmatrix} = \begin{bmatrix}
    \sigma_v^2 \\
    \sigma_e^2
\end{bmatrix} (\sigma_v^2 + \sigma_e^2)^{-1} (v_t + e_t(h)) \quad (h = 1, \ldots, H)
\]

or

\[
v_{t,t}^{eh} = \phi^v (v_t + e_t(h)) \quad (h = 1, \ldots, H) \quad (5.2.11)
\]

\[
e_{t,t}^{eh}(h) = \phi^e (v_t + e_t(h)) \quad (h = 1, \ldots, H) \quad (5.2.12)
\]

where $\phi^v = \sigma_v^2 (\sigma_v^2 + \sigma_e^2)^{-1}$ and $\phi^e = \sigma_e^2 (\sigma_v^2 + \sigma_e^2)^{-1}$.

The expressions in equations (5.2.11) - (5.2.12) have some intuitive appeal. Suppose that monetary noise dominates. Then $\sigma_v^2 + \sigma_e^2$ implies $\phi^e = e_{t,t}^{eh}(h) \rightarrow 0$, ($h = 1, \ldots, H$). Conversely, if money is relatively stable, $\sigma_v^2 \rightarrow 0$ and $\phi^v = v_{t,t}^{eh} \rightarrow 0$. Similar properties apply to variations in $\sigma_e^2$. Substituting equation (5.2.11) into equation (5.2.9) and equating coefficients now reveals the following identities:

\[
\mu_0 = \mu_1 = \frac{\sigma_v^2 + \sigma_e^2}{\sigma_v^2 + (1 + \alpha)\sigma_e^2} = \mu \quad (5.2.13)
\]

\[
\mu_2 = \rho \quad (5.2.14)
\]

where we have used $\phi^e = 1 - \phi^v$. Now denote by the superscript $F$ the
solution of the system under the filtering problem. Then substituting equations (5.2.13) - (5.2.14) into equations (5.2.6) - 5.2.7) gives the solutions for local and global prices under partial contemporaneous information. The solution for output follows by substituting \( p_{t(h)} = p_{t,t} = \mu(e_t(h) + \nu_t - \nu^{eh}_{t,t}(h = 1, \ldots, H) \) into equation (5.2.1). The resulting expressions are summarized in table 5.2(A).

Table 5.2(A) also contains the solution of the model under full contemporaneous information in which case there is no filtering problem; hence the superscript NF. Then \( e_{t,t}^{nh} = e_t(h) \) and \( \nu_{t,t}^{nh} = \nu_t(h = 1, \ldots, H) \) and the informational endowment is

\[
\Omega_{t}^{NF}(h) = (p_{t-j}(h), y_{t-j}(h), m_{t-j}(h), P_{t-j}, Y_{t-j}, M_{t-j}, (h = 1, \ldots, H). \tag{5.2.15}
\]

As shown, when agents are endowed with complete information, the familiar neutrality result obtains: both local and global prices incur the full impact of an aggregate demand shock whilst both local and global output remain invariant. By contrast, this is no longer true when information is restricted. In this case, agents are unable to decompose the local price signal into relative and aggregate perturbations. There is a change in perceived relative prices and output deviates from equilibrium. As a consequence, local and global prices change by less than their full information counterparts. In a similar vein, local output under responds to local shocks as agents misinfer an aggregate disturbance. Then local prices are stabilized by less than under full information. A further property of the solutions is the observation of Lucas (1973) that the slope coefficient in the supply schedule is a function of the stochastic properties of the exogenous variables. In particular, as monetary volatility increases, \( \sigma^2 / \omega \) and \( \phi^c + 0 \). This is to say
<table>
<thead>
<tr>
<th>Variables</th>
<th>Prices (P_t(h); P_t)</th>
<th>Output (Y_t(h); Y_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Partial (Ω^F_t(h))</td>
<td>P^F_t(h) = µ(ε_t(h) + ν_t) + ρm_{t-1}</td>
<td>y^F_t(h) = αφ(ε_t(h) + ν_t)</td>
</tr>
<tr>
<td></td>
<td>P^F_t = µν_t + ρm_{t-1}</td>
<td>y^F_t = αφν_t</td>
</tr>
<tr>
<td>Full (Ω^{NF}_t(h))</td>
<td>P^{NF}<em>t(h) = (1+α)^{-1}ε_t(h) + ν_t + ρm</em>{t-1}</td>
<td>y^{NF}_t(h) = α(1+α)^{-1}ε_t(h)</td>
</tr>
<tr>
<td></td>
<td>P^{NF}<em>t = ν_t + ρm</em>{t-1}</td>
<td>y^{NF}_t = 0</td>
</tr>
</tbody>
</table>

Table 5.2(A): Price and output solutions under divergent information structures.

\[
\phi = \frac{\sigma_\epsilon^2}{\sigma_\nu^2 + (1+\alpha)\sigma_\epsilon^2}
\]
that greater monetary volatility reduces the impact of unanticipated monetary shocks on output because agents interpret (correctly) the occurrence of these shocks\(^2\).

Further insight into the importance of the information structure can be gained by recalling our earlier remarks concerning the possibility that local money might convey information about aggregate money. Suppose, for example, that \( m_t(h) \) \((h = 1, \ldots, H)\) is related to \( m_t \) by \( m_t(h) = \Theta(h)m_t \) with \( 0 < \Theta(h) < 1 \) \((h = 1, \ldots, H)\) known to agents. Then we may write the observation vector as

\[
\begin{bmatrix}
  p_t(h) \\
  m_t(h)
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  0 & \Theta(h)
\end{bmatrix} \begin{bmatrix}
  p_t(h) \\
  m_t
\end{bmatrix} \quad (h = 1, \ldots, H) \quad (5.2.16)
\]

Denote by \( C \) the 2x2 right hand side matrix in equation (5.2.16) and partition this such that \( C = [C_1 \; C_2] \). Equation (5.2.16) may be re-written as

\[
\begin{bmatrix}
  p_t(h) \\
  m_t(h)
\end{bmatrix} = C_1(1 + \alpha)^{-1} \alpha p_{t-h} + Dm_{t-1} + Dv_t + C_1(1 + \alpha)^{-1} \epsilon_t(h) \quad (h = 1, \ldots, H) \quad (5.2.17)
\]

where \( D = C_1(1 + \alpha)^{-1}\Theta(h) + C_2 \).

As before, observation of \( p_t(h) \) and \( m_t(h) \) \((h = 1, \ldots, H)\) conveys information about both \( v_t \) and \( \epsilon_t(h) \) \((h = 1, \ldots, H)\). In this case, however, this information is absolute. To see this, note that equation (5.2.17) gives the optimal filtered estimates as
\[
\begin{bmatrix}
\nu_{t,t}^e h \\
e_{t,t(h)}^e h
\end{bmatrix}
= 
\begin{bmatrix}
\sigma_D^2 T \\
\sigma_C^2(1 + \alpha)^{-1}
\end{bmatrix}
\phi(D\nu_t + (1 + \alpha)^{-1} C_1 \nu_t(h))
\]

\[(h = 1, \ldots, H) \quad (5.2.18)\]

where \( \phi = (D\sigma_D^2 T + (1 + \alpha)^{-2} C_1 \sigma_C^2 T)^{-1} \).

It is now fairly straightforward to show that \( \nu_{t,t}^e h = \nu_t \) and \( \nu_{t,t(h)}^e = \nu_t(h) \) (\( h = 1, \ldots, H \)). The reason is that local money now conveys accurate information about aggregate money. This enables the extraction of perfect information about local shocks from local price signals. This is not a new observation (see, for example, Barro (1976)). We hope, however, to have illustrated explicitly the crucial influence of the information structure. In terms of the implications of the equilibrium models it is critical that agents have little faith in the information value of local money signals.

King (1981) and Boschen and Grossman (1983) steer an intermediate course between the above two scenarios. They assume the availability of some preliminary aggregate monetary estimate, \( \tilde{m}_t \), say, where \( \tilde{m}_t = m_t + \nu_{1t} \) and \( \nu_1 \) is a measurement error assumed to be white noise with asymptotic variance \( \sigma_{\nu_1}^2 \). The essential implications of partial information noted above are unchanged. Limiting cases of interest are when \( \sigma_{\nu_1}^2 \to 0 \) and \( \sigma_{\nu_1}^2 \to \infty \). In the former there is accurate contemporaneous aggregate monetary information. For our model this would imply full information \(^3\). In the latter, the monetary signal is sufficiently noisy as to induce agents to ignore it. The solution then collapses to the case in which only local price signals are exploited.

A further type of generalization is performed by Cukierman (1979). Here, it is assumed that agents are not restricted to just local price information but may sample prices in other markets. Moreover, the number of prices observed is a choice variable. The
rather obvious conclusion is that forecast errors are a decreasing function of the number of prices observed. In the limit, when agents have access to information about all prices, the solution reduces to the full information case though agents may choose not to gather full information if this involves a cost.

An important extension by Barro (1980) is the inclusion of on economy-wide capital market. In the model of equations (5.2.1) - (5.2.3) this might be accomplished by regarding equation (5.2.2) as a money demand function which depends on an economy-wide interest rate. Then an aggregate demand schedule may be posited to close the model. Barro (1980) follows his earlier paper (Barro (1976)) and assumes local commodity supply and demand to be determined by the anticipated real rate of interest plus a wealth variable which amounts to a forecast error of contemporaneous aggregate money movements. Supply (demand) is positively (negatively) related to the former and negatively (positively) related to the latter (4,5). The key aspect of this extension is that agents have access to an additional information variable, namely the economy-wide nominal interest rate which summarizes aggregate conditions. Information is still imperfect provided there is more than one global shock (6). Again, the familiar results obtain: neutrality emerges under conditions of perfect information; under imperfect information output over-responds to global shocks and under-responds to local shocks. The precise effect of stochastic perturbations depend, in general, on the structural parameters of the model and the reader is referred to Barro (1980) for further discussion. Our main comment on the analysis is favourable in the sense that it appreciates the possibility of a global price (that is, the interest rate) acting as an information variable. It is not implausible to argue that the most obvious candidates for information variables are indeed asset prices.
Information about these is likely to be available almost instantaneously and continuously, more so than is information about other aggregate variable such as aggregate prices and output. It is precisely this idea which has been encompassed in recent contributions to the literature. For the most part, these have borrowed the equilibrium frameworks of the earlier studies and applied them to an open economy setting. They may be summarized briefly as follows.

Saidi (1982) examines an equilibrium model of interdependent economies in the spirit of the monetary theory of the balance of payments (see, for example, Hahn (1959); Mussa (1974); Whitman (1975); Hacche and Townend (1981); Krueger (1983); the edited volume by Frenkel and Johnson (1976)). The markets previously indexed by \( h \) \((h = 1, \ldots, H)\) now represent identically structured economies operating under a fixed exchange rate regime. The formal structure is similar to Barro (1976, 1980). The set of information variables for each economy comprises economy-specific prices and the world interest rate (determined by international capital mobility). We should note here that the exchange rate is also contemporaneously observable. The fixed exchange rate assumption, however, deprives this variable of any useful information. This is in stark contrast to the case in which the exchange rate is permitted to float and illustrates vividly how control may influence the information structure. Certain subsequent research has been concerned to examine the implications of this. Saidi (1982) obtains results under full information which are consistent with the monetary theory. World monetary perturbations are neutral with respect to real variables whilst economy-specific shocks induce relative price changes and output fluctuations. When information is restricted the familiar modifications to these results obtain. In this respect the monetary theory of the balance of payments
is a special case which is realised only under conditions of full information. Otherwise, the response of the system is determined by agents' misperceptions and the reduced form coefficients are functions of the variances of exogenous variables.

Duck (1984) considers a two country model with both fixed and flexible exchange rates. Each country is characterized by the equilibrium physically separated markets paradigm. Demanders are located randomly in markets and international trade takes place when the citizens of one country find themselves in the markets of the other country. Under fixed exchange rates the balance of payments is given by the relative demands for each nation's commodity. This determines the exchange rate when permitted to float. Initial information is summarized in local prices and the global exchange rate and there are no interest bearing assets to act as additional information variables. In addition, as noted above, the exchange rate ceases to convey useful information when it is fixed. An implication of the analysis is that stabilizing monetary volatility is insufficient to negate output movements under fixed exchange rates but is sufficient when exchange rates are flexible. The reason is that, in the former regime, agents have access to only local price signals which are insufficient to infer the precise values of local shocks and foreign monetary disturbances. By contrast, observing exchange rate movements conveys full information about the latter. A similar but more rigorous analysis which allows capital mobility can be found in Lawrence (1984).

Let us now digress a little on some matters of importance. The role of asset prices as information variables is intuitively appealing. In this respect, there is an important peculiarity in the simple aggregative and widely employed model of Sargent and Wallace (1975) (see also Sargent (1973)). A stylized version of
this (but which retains the peculiarity) was given in chapter 3 (section 3.2, equations (3.2.1a), (3.2.2a), (3.2.3a), (3.2.4) - (3.2.5)). The information set conditioning expectations is lagged and comprises only lagged values of variables. Implicitly, however, the current nominal interest rate is assumed to be contemporaneously observable. Yet the value of this as a conveyer of information is not exploited. As we argued in section 5.1, this appears to be inconsistent with the rational expectations assumption. Moreover, this criticism is not trivial. Minford and Peel (1983a) have recently demonstrated some apparently perverse results when the information content of the interest rate is taken account of. In particular, a positive aggregate demand shock may reduce output if aggregate supply is sufficiently volatile. Then the observed rise in the interest rate is misinterpreted as a negative supply shock. Expected prices rise more than actual prices and output contracts. We also obtain this result in chapter 6 (section 6.3).

Even here, however, agents are neglecting some information for they still have local price observations. This might be important in the case of money demand shocks. An exogenous increased desire for liquidity raises interest rates and depresses prices. Demand and supply perturbations have the same qualitative effects: positive (negative) demand (supply) shocks tend to raise both prices and interest rates. This suggests that observation of both local prices and the global interest rate will convey full information about money demand shocks. We do not attempt to prove this intuition here but merely point it out as an interesting observation.

Nonetheless, it is possible to justify abstracting from local price signals. Suppose that local prices are subject to a wide variety of both local and global shocks. The interest rate
reflects only the latter. Then the contamination of local price signals may be sufficient to induce agents to ignore them. That is, local prices are so noisy that agents regard them to have negligible information value\(^{(7)}\). Agents merely take these prices as given and concentrate their efforts on extracting information from observable global variables. This is not implausible and may apply generally to the low information content of local variables. The argument gains force if one is primarily concerned with discerning the nature of aggregate shocks.

It is possible that the abstraction of local price signals is important in order to avoid some severe complications in the inference problem. More generally, we should be wary of private, or equivalently differential, information which may raise some issues of substance and which effect is to poison the simplicity of the analyses considered hitherto. Our argument can be illustrated most easily as follows. Let us augment the model in equations (5.2.1) - (5.2.3) with a local demand relationship. In addition, let equation (5.2.2) be aggregated and let us introduce the nominal rate of interest as an argument. Then local commodity demand and aggregate portfolio balance equations are written respectively as

\[
y^d_t(h) = -\beta(r_t - P^h_{t+1,t} + P^h_t) + \epsilon_t(h) + \eta_t \tag{5.2.19}
\]

\[
m^d_t = P_t + \gamma_1 y_t - \gamma_2 r_t \quad \gamma_j > 0 \quad (j = 1, 2) \tag{5.2.20}
\]

where \(r\) = nominal rate of interest rate

\(\epsilon(h), \eta\) = stochastic disturbances \((h = 1, \ldots, H)\).

Clearly, \(\epsilon(h)\) \((h = 1, \ldots, H)\) is a local demand shock and \(\eta\) is an aggregate demand shock. Now consider the information content of
the interest rate. Appropriate substitution and manipulation reduces equations (5.2.1), (5.2.3), (5.2.19) - (5.2.20) to

\[ p_t = a_1 p^e_{t,t} + a_2 p_{t+1,t} + a_3 m_{t-1} + a_4 v_t + a_5 n_t \]  \hspace{1cm} (5.2.21)

\[ r_t = \gamma_2^{-1}(b p_t - \gamma_1 p^e_{t,t} - \beta m_{t-1} - v_t) \]  \hspace{1cm} (5.2.22)

where \( a_1 = a_5 \alpha(1 + \gamma_2^{-1}\gamma_1 \beta) \)

\( a_2 = a_5 \beta \)

\( a_3 = a_4 \rho \)

\( a_4 = a_2 \gamma_2^{-1} \)

\( a_5 = (\beta(1 + \gamma_2^{-1}) + \alpha(1 + \gamma_2^{-1}\gamma_1 \beta))^{-1} \)

\( b = (1 + \gamma_1 \alpha) \)

and \( p^e_{t+s,t} \) \((s = 0,1)\) is the average price expectation across markets. Substituting equation (5.2.21) into equation (5.2.22) gives

\[ r_t = \gamma_2^{-1} \left[(ba_1 - \gamma_1 \alpha)p^e_{t,t} + ba_2 p^e_{t+1,t} + (ba_3 - \rho)m_{t-1} \right. \\
\left. + (ba_4 - 1)v_t + ba_5 n_t \right]. \]  \hspace{1cm} (5.2.23)

Equation (5.2.23) constitutes the interest rate signal of the observation vector \([p_t(h) r_t]^T\). The first three right hand side terms in the expression are assumed to be known and the problem is to infer the values of the unknown shocks, \( v_t \) and \( n_t \) (and \( f_t(h) \)). The crucial elements in the expressions are the terms \( p^e_{t+s,t} \) \((s = 0,1)\) which reflect average expectations across markets (or market expectations). It is these terms which require further comment.
If all agents possess identical information then \( p_{t+s,t}^{eh} \) \( P_{t+s,t}^e \) \( (S = 0,1; h = 1,\ldots,H) \) and market expectations are indeed known. This may be a critical assumption and is also ruled out in the equilibrium models. In particular, it is important to be aware that these models assume not only partial information but also differential information. This follows from the fact that, in addition to contemporaneous information about the global interest rate, each agent possesses a piece of private information, namely an observation of the local price. Then it is no longer true that \( p_{t+s,t}^{eh} = P_{t+s,t}^e \) \( (S = 0,1; h = 1,\ldots,H) \). From the perspective of any individual, therefore, the market price expectation is unknown.

Agents must form an estimate of this in order to infer the underlying shocks. Let us denote by \( p_{t+s,t}^{e(h)} \) agent h’s expectation of the market expectation \( (S = 0,1; h = 1,\ldots,H) \). The question is how each agent computes this expectation. This leads us to the following scenario outlined in section 5.1. Note, first, that an expectation of market price expectations is equivalent to an expectation of market expectations about stochastic disturbances. Then each agent must infer the actual values of disturbances and market expectations about these.

The approach to this problem adopted in the aforementioned literature is quite simple: since local prices are purely random (and since aggregate prices are merely an average of local prices), expectations thereof are unbiased estimates of the market price expectation. A more sophisticated approach is for each agent to recognize that others’ expectations differ from his own. The approach usually adopted then, to which we turn below, is to assume that market expectations are given; that is, each agent forecasts market expectations but assumes that he is unable to influence the economy-wide estimate. This is really a special case of the more
general approach to a differential information problem which we now describe. Each agent knows that his own inference-based action will influence his observations. Individually, these might only influence the local signal. Collectively, however, the inference-based actions of all agents will alter the global signal as well. This means that each agent appreciates that his observations reflect not only the underlying shocks to the system but also market expectations thereof (see equation (5.2.23)). The latter comprises expectations of other individuals who will similarly recognize this contamination of observations. Accordingly they will also form opinions about market expectations. This is not the end of the story. Agents now know that their observations reflect not only the expectations of others, but also other's expectations about their own expectation. They will then form expectations about this latter expectation and so on and so forth. This is an interesting problem and it motivates the notion of an informational game. Agents may take as given the expectations of others — a Nash informational game if you will; or they may recognize their influence on other's expectations — a Stackelberg assumption perhaps. This may apply equally to those models which abstract from a global information variable. A particular agent misinterpreting a relative change in his situation will alter output and this will change the local signal. An agent in a different location will be forming expectations about the price in others markets. Then he must recognize the above effect. The former individual will then appreciate this recognition. The problem is the same as before. In fact, Lucas (1975) appreciated precisely the point that we are making. He avoided addressing it by assuming that different agents' estimates are somehow 'pooled', yielding a common expectation applicable to all agents. The problem with this is that it is vacuous without eliciting how this pooling
takes place. After all, if agents are able themselves to pool their separate estimates, they are surely capable of divulging their own pieces of private information.

The upshot of our discussion is the potentially crucial distinction between the assumption of partial information and the dual assumption of partial and heterogeneous information. The full implications of this distinction are not always appreciated. Shifting from the former to the latter may transform substantially the nature of the signal-extraction problem. We shall have more to say about this later. The immediately proceeding discussion centres on those studies for which the added complication does not arise. Specifically, in these analyses, contemporaneous information takes the form of aggregate signals which are available to all agents. As before, we can summarize these analyses fairly briefly as follows.

Saidi (1980) considers a small open economy in which purchasing power parity holds. An intertemporal supply function in the spirit of Lucas and Rapping (1969) is introduced and the set of information variables comprises exchange rate and price observations. Note that the only stores of value in the model are domestic and foreign money so that the role of interest rates as information variables does not exist. The general conclusions are familiar and the analysis adds little to what has been said already. Merely note that imperfect information renders the insulation properties of flexible exchange rates no longer applicable and generates exchange rate overshooting or undershooting relative to the full information response. We are sceptical about the inclusion of aggregate prices in the set of information variables.

A more attractive analysis is performed by Kimbrough (1983a). It is motivated by a fundamental difference between fixed and flexible exchange rate regimes alluded to earlier. Unlike the former, the
latter makes the information content of the exchange rate nil. This might be an important factor when assessing the relative merits of the two systems. In fact, it is a similar notion which motivates our analysis in chapter 6 on the re-evaluation of the appropriate choice of monetary instrument (see chapters 2 and 3) when the interest rate performs the role of an information variable.

Kimbrough (1983a) assumes a simple aggregative model of a small open economy characterized by a money demand function (depending on the domestic price level and output) a supply function (determined by contemporaneous price forecast errors) and a demand relationship (a function of real competitiveness). As in Saidi (1980) interest bearing assets are excluded but contemporaneous information is available about the exchange rate. Two central conclusions are as follows. First, regardless of which type of shock occurs, if one type of disturbance predominates, flexible exchange rates are at least as good as fixed exchange rates in terms of minimizing the expected squared deviation of output around its full information level. Second, when domestic monetary shocks are accompanied by real shocks, flexible exchange rates are unambiguously superior. The reasons for these are as follows.

Recall from chapters 2 and 3 two implications of the monetary instruments literature. First, in the face of monetary disturbances, fixing the exchange rate will neutralize these shocks by permitting an accommodatory monetary response. Second, in the face of real disturbances, some flexibility in exchange rates is preferable. The latter continues to be true in Kimbrough (1983a). In addition, however, suppose that monetary shocks predominate. Then this is known to agents who correctly perceive no relative price change. Output remains invariant just as it does under fixed exchange rates. Moreover, if monetary shocks are accompanied by real shocks the
usual superior stabilizing properties of flexible exchange rates operates. Clearly, these add some qualification to the conclusions reached from a standard approach to the issue (which neglects consideration of the information content of the exchange rate).

Nonetheless, as Kimbrough (1983a) notes, it remains to be seen how robust the particular conclusions he reaches are when the model structure is enriched and a wider spectrum of disturbances considered.

The main drawback of Saidi (1980) and Kimbrough (1983a) is the absence of global interest rates (both domestic and foreign). Bhandari (1982) incorporates the interest rate's role as a conveyor of information together with permitting a preliminary estimate of the money stock as in King (1981) and Boschen and Grossman (1983). He also assumes a large closed foreign economy in order to trace the precise mechanisms whereby foreign shocks are transmitted to the domestic economy. A more fruitful line of inquiry might have been to allow for some symmetry in the interdependence between economies. Rather than commenting further on Bhandari (1982) and other similar studies, we wish to point out an important criticism of all of the analyses considered hitherto.

The foregoing research has a single important characteristic, namely the concentration on purely static models. Any dynamics are modelled by an arbitrary selection of the stochastic process determining the exogenous variables. For this reason, we are a little sceptical about the generalizability and importance of the results derived from these models. When different variables exhibit different degrees of sluggishness, exogenous shocks will impact at different speeds and the full implications of a shock will be distributed over time. It is precisely this which is likely to enrich the information content of variables. Consider, for
example, the abstraction of interest rates from the set of information variables. Kimbrough (1983a) observes that observation of foreign nominal interest rates will convey no information about domestic perturbations if the domestic economy is small (that is, foreign interest rates are treated as exogenous). This is true but neglects the usefulness of foreign interest rate observations as signals of foreign shocks. Now, assuming uncovered interest parity, Kimbrough (1983a) also argues that domestic interest rate observations will convey no extra information other than that which is already embodied in the exchange rate. This is true only in so far as stochastic perturbations impact on the interest rate and exchange rate simultaneously; but there are plausible circumstances under which this may not be so. All that is required is that there be some sluggishness in the domestic economy. Then foreign shocks may impact on the exchange rate contemporaneously but feed through to interest rates with a lag. This asymmetry in speeds of adjustment acts as a further source of information—an additional information variable if you like. Moreover, the assumption of uncovered interest parity makes domestic interest rates and the exchange rate move in the same direction. Relaxation of this will generally make the information contents of the exchange rate and interest rate differ. Some implications of these ideas are demonstrated in section 5.3. Currie and Levine (1983b) and Minford and Peel (1983a) have considered dynamic models in the spirit of the Dornbusch (1976) sticky price framework. As our analysis shows, their treatment of the information structure is rather misleading and obscures matters of some interest.

As we have noted previously, there is a natural progression from a purely partial information framework to one which combines this with the assumption of heterogeneous information. When agents
are both imperfectly and differentially informed, observed signals will reflect a linear combination (for a linear model) of the actual disturbances impinging on the system and the different expectations of these held by different agents. The existence and continuance of differential information is not, however, always justified. Hinford and Peel (1983a) express some concern over this. They argue that there is likely to develop a market which specializes in information if information is important for decisions. In any event, it is probably true that differential information will eventually be smoothed as the relatively uninformed agents acquire more information. Nonetheless, it is also possible that news is not disseminated to all agents instantaneously. An obvious example is the distinction between foreign exchange markets and goods or labour markets. Another is that agents are likely to be more aware of their own circumstances than are other agents and vice versa. Then an immediate consequence of an exogenous perturbation is to drive a wedge between the information available to agents operating in different markets. Investigations into a differential information structure are not, therefore, entirely trivial. At a minimum, they may yield some insight into the immediate response of the system to unforeseen events.

As mentioned earlier, the full implications of a differential information structure are not addressed in the literature. Forecasting others' forecasts is surely appreciated; but this is where the iterations stop. This is exemplified in King and Trehan (1983) who illustrate the type of solution technique employed in the literature. It is probably the case that the most interesting aspect of the problem is where agents perform further iterations. Then each individual forecasts others' forecasts about his own forecasts and so on and so forth. This strikes a direct analogy.
with some of the research into the process of learning the model\(^{(9)}\). Nonetheless, there are some interesting conclusions which emerge in the absence of this and which are likely to illustrate general points.

Much of the literature continues the theme of the early equilibrium models on neutrality and the efficacy of stabilization policy. With respect to the latter, chapter 2 (section 2.3) contained a summary of the debate. It was mentioned there that deterministic feedback policy is able to influence real variables if there exists differential information sets. An informational advantage in favour of the controller will obviously give potency to control rules. As Barro (1976) notes, however, the desirability of such action must be judged against the alternative course in which the controller divulges his extra information. In view of this, subsequent research on differential information assumes the policy maker to be no better informed than the most ignorant private agent. That is, differential information is assumed to exist within the private sector.

Turnovsky (1980) and Weiss (1980, 1982) are the popularly cited illustrations of the implications of the above scenario. The analytical framework of the former was described in chapter 3 (section 3.2, equations (3.2.1c), (3.2.2c), (3.2.3c), (3.2.4) - (3.2.5); see also the discussion in chapter 2, section 2.3). The efficacy of stabilization policy is a consequence of the informational advantage of demanders of goods who, unlike suppliers, have access to accurate contemporaneous price information. There are two major drawbacks of Turnovsky (1980): first, the signal-extraction problem is not addressed; second, it is debatable whether aggregate prices are contemporaneously observable. Such criticisms do not apply to Weiss (1980, 1982) who envisages the source of differential information to be the existence of local information.
Systematic stabilization policy is effective in this and other differential information models because of the ability to influence future expectations of differentially informed agents. A number of authors to be mentioned below have associated this with the idea that control operates by influencing the information content of globally observed variables. This follows from the fact that these variables reflect the expectations of all agents in the system (market expectations). The influence of control on these expectations is not uniform because of the idiosyncracies in information. Altering control will alter these (unknown) expectations and change the signal, thereby influencing the behaviour of agents conditioned on this signal. This has been termed the effectiveness of prospective feedback policy (the future response of policy to currently unknown events)\(^{(10)}\). Weiss (1980, 1982) shows that allocative efficiency can be improved by such policy via its ability to improve the information content of variables. In a Lucas (1972b, 1975) and Barro (1976) type framework, King (1982) demonstrates the same point. Differential information takes the form of differentially informed agents within a particular market. The global observable variable from the perspective of this market is the local price therein. More plausibly, perhaps, King (1983) adopts the Barro (1980) specification in which the informational asymmetry exists in the economy-wide capital market by virtue of market-specific goods price observations. It may be noted that, in this respect, the analysis by King (1983) is a much improved version of Barro (1980) who entirely eschews the problem of inferring market expectations.

Thus, in a differential information framework with signal-extraction, the operation of systematic feedback control is through market expectations. It is the prospect of reactions to current events which gives scope for effective stabilization policy even
in market clearing models because these current events are perceived
non-uniformly by different agents. It is possible, however, to turn
this around and suggest that such action could be detrimental.
Suppose that the first best solution is that which obtains under
full information. Imperfect and differential information will
obviously lead to a departure from this. Yet the full effect of
differential information may be exacerbated by prospective feed-
back actions which serve to confuse agents further about market
expectations. Of course, it may not exacerbate problems. In any
event, however, a superior outcome might be achieved if the con-
troller avoids feedback on currently unknown circumstances. This
idea is addressed by Anderson (1983) in the model of Barro (1980)
modified as follows. Suppliers and demanders of the commodity are
differentially informed about the conditions on the other side of
the market. Thus, all suppliers are assumed to know the aggregate
supply shock, \( v^s_t \), say, but not the aggregate demand shock, \( v^d_t \), say.
The converse holds for all demanders. At the same time, there is the
usual market-specific information about local prices, \( p_t(h) \)
\((h = 1, \ldots, H)\) which depend on local perturbations. Aggregate shocks
follow random walks;
\[
\begin{align*}
\nu^s_t &= \nu^s_{t-1} + \nu^s_t \\
\nu^d_t &= \nu^d_{t-1} + \nu^d_t
\end{align*}
\]
where \( \nu^s \) and \( \nu^d \) are white noise error terms. Then Anderson (1983) writes
the information sets for local suppliers and local demanders in the
form,
\[
\begin{align*}
\Gamma^s_t(h) &= \{ p_t(h), v^s_t, m_t, p_{t-j}(h), p_{t-j}, v^d_{t-j}, v^s_{t-j}, m_{t-j} \} \quad (h = 1, \ldots, H) \\
\Gamma^d_t(h) &= \{ p_t(h), v^d_t, m_t, p_{t-j}(h), p_{t-j}, v^d_{t-j}, v^s_{t-j}, m_{t-j} \} \quad (h = 1, \ldots, H).
\end{align*}
\]
The monetary feedback rule is

\[ m_{t+1} = m_t + \rho_1 v_{t-1}^s + \rho_2 v_t^s + \rho_3 v_{t-1}^d + \rho_4 v_t^d, \quad \rho_i \neq 0 \ (i = 1, 2, 3, 4). \]  

(5.2.26)

The primary source of differential information about future money is the fact that suppliers and demanders are differentially informed about aggregate supply and demand shocks. A pure prospective feedback policy occurs when \( \rho_1 = \rho_3 = 0 \) and \( \rho_2, \rho_4 \neq 0 \). A pure current feedback policy is when \( \rho_2 = \rho_4 = 0 \) and \( \rho_1, \rho_3 \neq 0 \). Within this framework, Anderson (1983) shows that, unlike Weiss (1980, 1982), a pure prospective feedback policy cannot attain the first best solution. By contrast, a pure current feedback policy is able to accomplish this. The reason relates to our previous intuition. Feedback on currently unknown variables imply different expectations of future money and prices between agents. The differential information structure is maintained. By contrast, reacting to commonly known variables implies uniform expectations across agents. In short, this policy removes the primary source of divergent expectations. In this respect, Anderson (1983) is strictly incorrect in his assertion that policy is not operating by improving the information content of signals. On the contrary, policy is operating in precisely this way. It definitely removes a source of confusion by inducing all agents to hold the same expectations.

Some of the above issues have been extended to an open economy framework by Kimbrough (1983b, 1984). Purchasing power parity holds in the aggregate but not in localised markets\(^{(12)}\). An economy-wide foreign exchange market provides the setting in which differential information manifests. Thus, in Kimbrough (1984) the set of information variables comprises local prices (foreign and domestic) and the global exchange rate. The motivation for the
analysis is as in Kimbrough (1983a), mentioned earlier, namely the fact that the exchange rate ceases to convey useful information when it is fixed. Monetary policy operates as above by influencing the expectations of differentially informed agents. When the exchange rate is fixed, however, monetary policy is entirely accommodating and the mechanism breaks down. What matters is exchange rate policy but this is impotent with respect to influencing real variables. Moreover, since exchange rate control reduces the information available to agents, a floating exchange rate regime is deemed to be superior (13).

Some comments are in order about these results. Note that under a flexible rate system, the money stock is predetermined. When exchange rates are fixed, the money stock must accommodate any pressure on the exchange rate. Then the information conveyed by fluctuating exchange rates will be reflected in the money stock when exchange rates are fixed. Moreover, it must be assumed that the controller is able to observe aggregate money in order to stabilize the exchange rate. Thus, it must also be assumed some reason for the controller not to divulge this information (see, for example, Barro (1976)). In addition, it would appear that exchange rate policy could gain effectiveness if private agents have observations of aggregate money. Then just as monetary policy influences the information content of the exchange rate under a floating system, exchange rate policy would influence the information content of money under the alternative regime. This is demonstrated in Kimbrough (1983b). It does not depend on there being accurate monetary information; some preliminary noisy estimate will suffice. Hence, the superiority of floating exchange rates hinges critically on the informational assumption one cares to make.
There is another point as well. Kimbrough (1983b, 1984) posits two conditions for policy to be effective in equilibrium models: first, it alters the information content of a globally observed signal; second, it does this by virtue of incomplete and differential information. That is, merely incomplete information is not enough. To some extent this might be true; but there is an error in the illustration used by Kimbrough (1983b, 1984). This involves considering the case in which local volatility is zero. Then the source of differential information disappears, policy becomes ineffective but there is still imperfect information regarding aggregate shocks. This example is fundamentally misleading; for if local volatility is zero and agents know this, local output will always remain invariant to policy. This follows because there can be no misperceptions about relative price changes. This is not entirely a trivial point. We make it in order to emphasize the difficulty in discerning the precise role of the information structure which is not always easy to isolate from the normal workings of a model. Even when it is possible, the interpretation may not always be unique.

To close this section, we wish to comment briefly on four remaining issues of some substance. The first relates to the implementation of policy in the presence of heterogeneous information. King (1982) observes that the formulation of optimal policy under such circumstances will increase the informational demands of the controller. In particular, Sargent and Wallace (1975) point out that for control to have its desired effects, the controller must be aware of the precise nature of the differential information structure. This is an important issue pertaining to practical policy making. In standard analytical work, however, it cannot be taken too seriously. In any rational expectations
model, with or without differential information, the policy maker is assumed to possess knowledge of the model's characteristics. Introducing differential information is just another characteristic; and if the policy maker is assumed able to work out the economic structure, he surely has the capacity to figure out the information structure. Nonetheless, we believe an explicit treatment of the problem in a properly specified framework might be of some interest. It is one way of relaxing the extreme information assumptions one often finds in the literature. In a related vein, there is also a serious matter of robustness. Even if one admits the formulation of policy on the basis of knowledge about the information structure, there is an interesting question concerning how robust particular policy rules are with respect to different informational assumptions. This is a focus of attention in chapter 3.

The second point is also concerned with differential information. Kimbrough (1985) has recently shown that the existence of an economy-wide futures market in the Barro (1980) model will re-establish policy ineffectiveness by virtue of endowing agents with complete information about market expectations. More generally, a sufficiently rich menu of economy-wide markets serves to de-contaminate observed signals of the expectations of differentially informed agents(14). It is plausible to argue that the number of such markets is, in fact, insufficient. Yet this is inadequate; for if the first best solution obtains under full information, there will presumably exist incentives for agents to establish these markets. This is another possibly interesting area for further research.

Third, most of the literature has concentrated on analysing partial and differential information in equilibrium models with the view to realizing their implications for neutrality and policy effectiveness. We believe that these types of analyses should now
be abandoned and the implications of signal-extraction examined in a wider spectrum of models and for further issues. A recent contribution in this respect is Kimbrough (1983c) who attempts to explain exchange rate overshooting and deviations from purchasing power parity in a model where agents in the foreign exchange market have superior information to agents in the goods market.

The final point we wish to make is that, though the information structure appears potentially crucial for determining the behaviour of a system, its precise quantitative importance remains to be seen. It may well be that a seemingly significant change in the information structure yields relatively little change in the behaviour of the system. Investigations into the quantitative importance of the informational assumptions would require numerical simulations and we ask for future research in this direction.

This concludes the present section. In addition, there is some further literature on signal-extraction with special reference to monetary control. This is related to the subject matter of chapter 6 and our comments are contained in that chapter (section 6.1).

5.3 Interest Parity, the Degree of Capital Mobility and the Information Contents of the Exchange Rate and the Interest Rate: Clarifications and Extensions*

This section explores in detail the role of the exchange rate and the interest rate as information variables under conditions of perfect and imperfect capital mobility. At a more general level, it illustrates the importance of a detailed investigation into the information structure as a pre-requisite for any analysis. Section 5.2 recorded some recent work in the field of partial information which indicated the potentially critical dependence of the behaviour of the economy on the information structure. A common assumption
of some of the literature is that the set of currently observable variables includes at a minimum all asset prices, in particular the exchange rate and interest rates. These are typically related through uncovered interest parity. As a consequence, it is often assumed that interest rates convey no extra information other than that already embodied in the exchange rate. The current section is motivated by this view. Formally, it states that the information contents of the exchange rate and interest rates are sufficiently similar as to make explicit treatment of observation of both unnecessary. We show that, despite the intimate relationship between these variables (via interest parity), observation of both conveys more information than does observation of just the exchange rate. At the same time, however, there is the interesting perverse possibility that enriching the information set in this way actually reduces the accuracy of forecasts about some shocks. In this case, the extra information is misleading. This motivates the distinction, mentioned in section 5.2, between the informational endowment and the overall information available. Our result implies that increases in the former do not necessarily yield more accurate auxiliary information about all stochastic disturbances. As far as we know, this has hitherto gone unnoticed. We also show the critical dependence of the amount and nature of information conveyed to agents on the degree of capital mobility and the nature of the shock structure assumed. Implications are drawn for the dynamic behaviour of prices and the exchange rate.

The commonly held belief about the similarity of the information contents of the exchange rate and interest rates is exemplified in Minford and Peel (1983a):
"... in this particular model no additional information on the shock structure is conveyed by the global interest rate. This is because the pieces of global information are structurally related via [interest parity]. Observation of [the exchange rate] implies observation of [the interest rate]." (15)

By contrast, Currie and Levine (1983b) make no reference whatsoever regarding their abstraction of the current interest rate from the set of information variables. In Kimbrough (1983a,b, 1984), the model is entirely static and exogenous shocks impact simultaneously on both the exchange rate and interest rates. In addition, as in Minford and Peel (1983a) there are no external (foreign) disturbances. Relaxation of these properties have important implications.

This section demonstrates that in the models employed by Currie and Levine (1983b) and Minford and Peel (1983a), the information contents of the exchange rate and the interest rate are identical only if the following conditions are satisfied: first, there is perfect capital mobility; second, either no external shocks exist or, if such shocks do exist, they take the form solely of foreign interest rate variations. In the Minford and Peel (1983a) model both of these conditions are satisfied. Hence, their proposition is true though only for these limiting cases. In Currie and Levine (1983b), however, imperfect capital mobility and foreign disturbances are explicit.

The model we use for demonstration is a fixed output discrete time variant of the small open economy described in chapter 3 (section 3.2, equations (3.2.1f), (3.2.2f), (3.2.3f), (3.2.4) – (3.2.6)). This is based on Dorbusch (1976) as modified by Bhandari (1981), Bhandari, Driskill and Frenkel (1984), Driskill (1981), Frenkel and Rodriguez (1982) and Gazioglu (1984) to incorporate imperfect capital mobility. The resulting structure is formally identical to Currie and Levine (1983b) and encompasses Minford and Peel (1983b) as a special case.
\[ p_{t+1} - p_t = \lambda(e_t - p_t) + \epsilon_{1t} \quad 0 < \lambda < 1 \quad (5.3.1) \]

\[ m_t^d = p_t - \gamma r_t + \epsilon_{2t} \quad \gamma > 0 \quad (5.3.2) \]

\[ \eta_1(x_t - e_{t+1,t}^e + e_t) + \eta_2(e_t - p_t) + \nu_t = 0 \quad \eta_i > 0 \quad (i = 1, 2) \quad (5.3.3) \]

where
- \( p \) = natural logarithm of the price level
- \( e \) = natural logarithm of the nominal exchange rate
- \( m^d \) = natural logarithm of the nominal demand for money
- \( r \) = nominal rate of interest
- \( \epsilon_k, \eta \) = stochastic disturbances (\( k = 1, 2 \)).

As usual, all variables are measured as deviations from long-run equilibrium and \( e_{t+1,t}^e = E(e_{t+1,t} | \Omega_t) \) where \( E(\cdot) \) is the mathematical conditional expectations operator and \( \Omega \) is the information set.

Equations (5.3.1) - (5.3.3) should be familiar. Equation (5.3.1) is the inflation generating mechanism subject to stochastic perturbation. The nominal exchange rate, \( e \), is therefore the domestic currency price of foreign exchange (a rise in \( e \) is a depreciation). Equation (5.3.2) is a stochastic money demand function. Equation (5.3.3) is the balance of payments equilibrium condition. Recalling the discussion in chapter 3 (section 3.2), this is derived from the sum of the capital and trade accounts, \( K_t \) and \( T_t \) respectively, where

\[ K_t = \eta_1(x_t - e_{t+1,t}^e + e_t + \nu_{1t}) \quad (5.3.4) \]

\[ T_t = \eta_2(e_t - p_t + \nu_{2t}) \quad (5.3.5) \]

where \( \nu_i \) = stochastic disturbances (\( i = 1, 2 \)).

Thus, \( \eta_1 = \infty \) approximates perfect capital mobility and \( \nu_1 \) and \( \nu_2 \)
are random shocks to capital and trade flows \(^{(16)}\). It is important to note that \( \nu_1 \) is not to be interpreted as a foreign interest rate disturbance. In what follows we shall be assuming all interest rates to be contemporaneously observable. Since foreign interest rates are exogenous in this model, \( \nu_1 \) would be known with certainty. The correct interpretation here is that \( \nu_1 \) captures non-systematic deviations from interest parity. In this way, under perfect capital mobility, the Minford and Peel (1983a) specification (which excludes foreign shocks) is considered a limiting case of the current framework. Finally, we assume that each disturbance is independently and Gaussian distributed.

The model is closed by assuming a fixed money stock so that money market equilibrium implies \( m_t^d = 0 \) in equation (5.3.2)\(^{(17)}\). Then equations (5.3.1) - (5.3.3) can be written in state-space form as

\[
\begin{bmatrix}
P_{t+1} \\
e_{t+1,t} \\
\end{bmatrix} = \begin{bmatrix}
1 - \lambda & \lambda \\
-1 & a_1 \\
\end{bmatrix} \begin{bmatrix}
P_t \\
e_t \\
\end{bmatrix} + \begin{bmatrix}
u^1_t \\
u^2_t \\
\end{bmatrix}
\]

(5.3.6)

where

\[
\begin{align*}
a_1 &= \gamma^{-1} - n, \\
a_2 &= 1 + n, \\
n &= \eta_1^{-1} \eta_2, \\
\nu^1_t &= \epsilon_{1t}, \\
\nu^2_t &= \gamma^{-1} \epsilon_{2t} + \epsilon_{3t}, \\
\epsilon_{3t} &= \eta_1^{-1} \nu_t
\end{align*}
\]

The characteristic equation of (5.3.6) is

\[
f(\tau) = \tau^2 - (2 + n - \lambda)\tau + (1 + n - \lambda (1 + \gamma^{-1})) = 0 \quad (5.3.7)
\]
from which

\[
\tau_1, \tau_2 = \frac{2(2 + n - \lambda)}{2(2 + n - \lambda) - \lambda \gamma^{-1}} \frac{-\lambda \gamma^{-1}}{2(2 + n - \lambda) - \lambda \gamma^{-1}}
\]  

(5.3.9)

Treating \(p\) as predetermined and \(e\) as non-predetermined, saddlepoint stability requires one stable root and one unstable root; \(|\tau_1| < 1\) and \(|\tau_2| > 1\), say. Following appendix D, this condition is satisfied if there is one (and only one) sign change in the sequence of test functions

\[
1, \frac{2(3 + n - \lambda)}{2(2 + n - \lambda) - \lambda \gamma^{-1}}, \frac{-\lambda \gamma^{-1}}{2(2 + n - \lambda) - \lambda \gamma^{-1}}
\]  

(5.3.9)

which implies the condition \(2(2 + n - \lambda) - \lambda \gamma^{-1} > 0\). Note that \(2 + n - \lambda > 1\). Thus, \(\tau_2 > 1\) unambiguously and regardless of the degree of capital mobility. By contrast, \(\tau_1 \geq 0\) depending in particular on the magnitude of \(\gamma\). Relatively low values of \(\gamma\) imply \(\tau_1 < 0\); but the test function condition is more susceptible to violation. Hence, noting the requirement for a sufficiently large value for \(\gamma\), we make the following assumption.

ASSUMPTION 5.3.1 : The system in equation (5.3.4) satisfies the saddlepoint property and \(0 < \tau_1 < 1\) and \(\tau_2 > 1\).

REMARK 5.3.1 : The assumption that \(0 < \tau_1 < 1\) rules out cyclical behaviour.

PROPOSITION 5.3.1 : The general quasi-solution of the system described by equations (5.3.1) - (5.3.3) can be written as
\[
e_t = -m_{21}p_t + b_1u_{t-1} + a_2^{-l} l e_{2t} + a_2^{-l} e_{3t} - b_1u_{t-1,t}
\]
\[+ b_2^{-l} e_{2t,t} + b_2 e_{3t,t} \]
\[\text{(5.3.10)} \]

\[
P_{t+1} = \tau_1 p_t + \lambda b_1 u_{t-1} - \lambda a_2^{-l} l e_{2t} - \lambda a_2^{-l} e_{3t} - \lambda b_1 u_{t-1,t}
\]
\[+ \lambda b_2^{-l} e_{2t} + \lambda b_2 e_{3t,t} + u_t \]
\[\text{(5.3.11)} \]

\[
e_{t+1} = \tau_1 p_t - \lambda b_1 u_{t-1} - a_2^{-l} a_1 u_t + \lambda a_2^{-l} l e_{2t} + \lambda a_2^{-l} e_{3t}
\]
\[- a_2^{-l} l e_{2t+1} - a_2^{-l} e_{3t+1} + \lambda b_1 u_{t-1,t} - \lambda b_2^{-l} e_{2t,t}
\]
\[- \lambda b_2 e_{3t,t} - b_1 u_{t,t+1} + b_2^{-l} e_{2t+1,t+1} + b_2 e_{3t+1,t+1} \]
\[\text{(5.3.12)} \]

where \[m_{21} = -(1 - \lambda - \tau_2)^{-1} a_1 = \lambda^{-1}(\tau_2 - a_2)\]
\[b_1 = m_{21} - a_2^{-l} a_1\]
\[b_2 = a_2^{-l} - \tau_2^{-l}\]
\[e = \lambda m_{21} + \tau_1\]

PROOF: See appendix B. Q.E.D.

The virtue of writing the solutions as in equations (5.3.10) - (5.3.12) is that we can immediately identify the differences (and implications thereof) in the information contents of the exchange rate and the interest rate. Intuitively, these differences will be realized in the optimal filtered estimates of the disturbances \(e_k\) \((k = 1, 2, 3)\). We consider four alternative information structures
(informational endowments) conditioning these expectations. These are given as follows

\[ \mathbf{\Omega}_{t}^{NF} = \{ e_{t-j}, r_{t-j}, \varphi_{t-j}, P_{t-j}, m_{t-j}, \Sigma, \Sigma \mid j > 0 \} \]  

\[ \mathbf{\Omega}_{t}^{P} = \{ e_{t}, e_{t-j}, r_{t-j}, \varphi_{t-j}, P_{t-j}, m_{t-j}, \Sigma, \Sigma \mid j > 1 \} \]  

\[ \mathbf{\Omega}_{t}^{P} = \{ e_{t}, r_{t-j}, \varphi_{t-j}, P_{t-j}, m_{t-j}, \Sigma, \Sigma \mid j > 1 \} \]  

\[ \mathbf{\Omega}_{t}^{P} = \{ r_{t}, r_{t-j}, \varphi_{t-j}, e_{t-j}, P_{t-j}, m_{t-j}, \Sigma, \Sigma \mid j > 1 \} \]  

where \( \varphi_{t} \) = foreign nominal rate of interest

and \( \Sigma \) and \( \Sigma \) represent a vector of all structural parameters and the variance-covariance characteristics of stochastic disturbances respectively. Equation (5.3.13) states that agents have full contemporaneous information about all variables. Equation (5.3.14a) restricts current information on just asset prices. Equations (5.3.14b) - (5.3.14c) restrict current information further such that only one type of asset price is contemporaneously observable. The superscripts NF and P refer respectively to the absence and presence of a filtering problem.

REMARK 5.3.2: The following features of equations (5.3.13) - (5.3.14c) are worth noting. First, the information sets are homogeneous. Thus, we avoid the complications introduced by heterogeneous information (see section 5.2). Second, the approach taken by
previous authors can be summarized as follows. The popular assumption is the informational endowment in equation (5.3.14a). Then a supposedly simplifying assumption is to ignore observations of $r_t$ and $r^f_t$. In what follows, we show this not to be merely a simplification. The abstraction leads to the information structure in equation (5.3.14b). This yields results which, even if qualitatively similar to, are at least quantitatively different from those derived from equation (5.3.14a).

REMARK 5.3.3 : The information structure in equation (5.3.14a) is consistent with the model specification. Actual prices appear in equation (5.3.1) because information about current prices is available in the next period. Alternatively one may consider a micro relationship in which market specific (known) prices respond sluggishly to market specific competitiveness. Aggregating yields equation (5.3.1). The latter interpretation can be applied to equations (5.3.3) and (5.3.4). Similarly, equation (5.3.2) reflects aggregated individual money demand functions. The relationship between the information structures in equations (5.3.14b) - (5.3.14c) and the model specification is more erroneous. We have no intention to justify these as each of them are meant to reflect supposedly simplifying assumptions. By considering these separately we can isolate the information contents of the exchange rate and the interest rate.

PROPOSITION 5.3.2 : For the three information structures defined in equations (5.3.14), the following expressions are true:

\[
\begin{align*}
\sigma t^{-1}_{u} &= \phi_{1}^{2} \left[ \sigma_{3}^{2} + \sigma_{2}^{2} \right] u_{t-1} - \sigma_{2}^{2} \sigma_{3}^{2} + \sigma_{3}^{2} \\
\sigma t^{e}_{3} &= \phi_{2}^{2} \left[ \sigma_{3}^{2} + \sigma_{2}^{2} \right] \sigma_{3}^{2} + \sigma_{3}^{2} \\
\sigma t^{g}_{3} &= \phi_{3}^{2} \left[ \sigma_{1}^{2} + \sigma_{2}^{2} \right] \sigma_{3}^{2} + \sigma_{3}^{2} \end{align*}
\]
\[ b^{1e}_{t-1,t} = b^{1e}_{t-1,t} + \varepsilon_3t + \gamma^{-1}\varepsilon_{2t} \]

\[ b^{e}_{2t,t} = b^{e}_{2t,t} + \varepsilon_3t + \gamma^{-1}\varepsilon_{2t} \]

\[ b^{e}_{3t,t} = b^{e}_{3t,t} + \varepsilon_3t + \gamma^{-1}\varepsilon_{2t} \] (5.3.15b)

\[ c^{1e}_{t-1,t} = c^{1e}_{t-1,t} + \varepsilon_2t \]

\[ c^{e}_{2t,t} = c^{e}_{2t,t} + \varepsilon_2t \] (5.3.15c)

\[ a^\phi = \left[ \begin{array}{cc} -2 & 2 \varepsilon_2^2 \\ \sigma_2^2 \left( \sigma_1^2 + \sigma_3^2 \right) + \sigma_1^2 \sigma_3^2 \end{array} \right]^{-1} \] (5.3.16a)

\[ b^\phi = \left[ \begin{array}{cc} \sigma_1^2 + \sigma_3^2 + \gamma^{-2}\sigma_2^2 \end{array} \right]^{-1} \] (5.3.16b)

\[ c^\phi = \left[ \sigma_1^2 + \sigma_2^2 \right]^{-1} \] (5.3.16c)

where \( \sigma_k^2 \) and \( \sigma_h^2 \) are the asymptotic variances of \( \varepsilon_k \) (\( k = 1,2,3 \)) and \( u^h \) \( h = 1,2 \) respectively.

**PROOF:** (See appendix B for further discussion.) Denote by \( A \) the 2 x 2 right hand side dynamic matrix in equation (5.3.6). Partition this such that \( A = [(A^1)^T \ (A^2)^T]^T \).

(a) For equations (5.3.15a) and (5.3.16a) denote by \( a^s_t \) the vector of currently observable variables. Then

\[ a^s_t = \begin{bmatrix} \varepsilon_t \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_t \\ e_t \end{bmatrix} + a^z_t \] (5.3.17)

where \( a^z_t = \begin{bmatrix} \gamma^{-1}\varepsilon_{2t} \end{bmatrix}^T \). Let \( C \) be the 2 x 2 right hand side matrix in equation (5.3.17). Partition this conformably with \( A \) such that \( C = [C_1 \ C_2] \). Following appendix B, equation (5.3.17) can be
re-written as

\[
\begin{align*}
\mathbf{a}_t &= \mathbf{F} \begin{bmatrix} \mathbf{a}_t \end{bmatrix}^T \mathbf{p}_{t-1} + C_2 a_2^{-1} e_{t+1} + F u_{t-1} - C_2 a_2^{-1} \mathbf{e}_t + v_t \\
&= \mathbf{F} \begin{bmatrix} \mathbf{a}_t \end{bmatrix}^T \mathbf{p}_{t-1} + C_2 a_2^{-1} e_{t+1} + F u_{t-1} - C_2 a_2^{-1} \mathbf{e}_t + G \mathbf{e}_{2t}
\end{align*}
\]

(5.2.18)

where \( F = C_1 - C_2 a_2^{-1} a_1 \), \( G = \begin{bmatrix} \gamma^{-1} & -a_2 & -1 \end{bmatrix} \) T and we have used the definitions of \( u_2^2 \) and \( v_t \). From equation (5.3.18), \( u_{t-1,t}^1 \), \( e_{a_3, t} \), and \( e_{a_2, t} \) are given by

\[
\begin{bmatrix}
\begin{array}{c}
u_{t-1,t}^1 \\
e_{a_3, t} \\
e_{a_2, t}
\end{array}
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
u_{t-1}^1 \\
\sigma_{2T}^2
\end{array}
\end{bmatrix} \Phi \begin{bmatrix}
\begin{array}{c}
\begin{bmatrix}
\begin{array}{c}
u_{t-1}^1 \\
\sigma_{2T}^2
\end{array}
\end{bmatrix}
\end{array}
\end{bmatrix} \begin{bmatrix}
\begin{array}{c}
u_{t-1}^1 \\
\sigma_{2T}^2
\end{array}
\end{bmatrix}
\]

(5.3.19)

where \( \Phi = \left( F_1^{2T} + a_2^{-2} C_2^{2} C_2^{2} + G \sigma_{2}^{2} G \right)^{-1} \). The 2 x 2 inverted matrix, \( \Phi \), is

\[
\Phi = \left( a_2^{-2} \right)^{-2} \left[ a_2^{-2} a_2^{-2} \right] \left[ a_2^{-2} a_2^{-2} \right]^{-1} \left[ a_2^{-2} a_2^{-2} \right]
\]

(5.3.20)

where \( a_2 \) is defined in equation (5.3.16a). Some tedious algebra then gives the expressions for \( \mathbf{u}_{t-1,t}^1 \), \( \mathbf{e}_{a_2, t} \), and \( \mathbf{e}_{a_3, t} \) in equations (5.3.15a).

(b) For equations (5.3.15b) and (5.3.16b), the vector of currently observable variables is the scalar
\[ \mathbf{b}_t^s = \mathbf{e}_t = [0 \ 1] \begin{bmatrix} \mathbf{p}_t \\ \mathbf{e}_t \end{bmatrix} \]  

(5.3.21)

which can be written equivalently as

\[ \begin{bmatrix} \mathbf{p}_t \\ \mathbf{e}_t \end{bmatrix} = -a_2^{-1} a_1^{-1} \begin{bmatrix} \mathbf{p}_{t-1} \\ \mathbf{e}_{t-1} \end{bmatrix} + a_2^{-1} \mathbf{e}_{t+1,t} - a_2^{-1} a_1^{-1} \mathbf{u}_{t-1} - a_2^{-1} \mathbf{u}_t \]

\[ \begin{bmatrix} \mathbf{p}_t \\ \mathbf{e}_t \end{bmatrix} = -a_2^{-1} a_1^{-1} \begin{bmatrix} \mathbf{p}_{t-1} \\ \mathbf{e}_{t-1} \end{bmatrix} + a_2^{-1} \mathbf{e}_{t+1,t} - a_2^{-1} a_1^{-1} \mathbf{u}_{t-1} - a_2^{-1} \mathbf{u}_t \]

(5.3.22)

using the definition of \( u_2^2 \). Hence,

\[
\begin{bmatrix}
1 e \\
b_{t-1,t} \ e^{e}
\end{bmatrix} = \begin{bmatrix}
\sigma_1^2 a_1 \\
\sigma_2 \\
b_{t-1,t} \ e^{e}
\end{bmatrix} \phi(a_1^{-1} u_{t-1} + e_{3t} + \gamma^{-1} \mathbf{e}_{2t})
\]

(5.3.23)

where \( \phi \) is defined in equation (5.3.16b), which gives the expressions for \( b_{t-1,t} \) and \( b_{t-1,t} \) in equations (5.3.15b).

(c) For equations (5.3.15c) and (5.3.16c), the measurement vector is

\[ \mathbf{c}_t^s = \mathbf{r}_t = [\gamma^{-1} \ 0] \begin{bmatrix} \mathbf{p}_t \\ \mathbf{e}_t \end{bmatrix} + c'_t \]

(5.3.24)

where \( c'_t = \mathbf{r}^{-1} \mathbf{e}_{2t} \). Equivalently,

\[ c_t^s = \gamma^{-1} \chi \begin{bmatrix} \mathbf{p}_{t-1} \\ \mathbf{e}_{t-1} \end{bmatrix} + \gamma^{-1} u_{t-1} + \gamma^{-1} \mathbf{e}_{2t} \]

(5.3.25)
using the definition of $c_{v_t}$, so that

\[
\begin{bmatrix}
1_e \\
1_{e(t-1,t)} \\
e \\
e_{2(t,t)}
\end{bmatrix} =
\begin{bmatrix}
\sigma_1^2 \\
\sigma_1 \\
-2 \\
\sigma_2
\end{bmatrix} \phi(u_{t-1}^1 + \epsilon_{2t})
\]

(5.3.26)

with $\phi$ defined in equation (5.3.16c). The expressions for $c_{v(t-1,t)}$ and $e_{2(t,t)}$ follow as in equations (5.3.15c). Q.E.D.

REMARK 5.3.4: It is obvious from equations (5.3.15) - (5.3.16) that the information contents of the exchange rate and the interest rate are, in general, different: $1_e, 1_{e(t-1,t)} \neq c_{v(t-1,t)}, e_{2(t,t)} \neq e_{2(t,t)}$. Particularly interesting results are stated in the following propositions (18).

PROPOSITION 5.3.3: Under conditions of perfect capital mobility, the information contents of the exchange rate and the interest rate are identical if there are no external shocks.

PROOF: Consider equations (5.3.15b) - (5.3.15c) and (5.3.16b) - (5.3.16c). Set $e_{3t} = \sigma_2 = 0$. Also, with $\eta_1 = \omega$, $n = 0$ and $a_1 = \gamma^{-1}$. It then follows that $b_{v(t-1,t)} = c_{v(t-1,t)}$ and $e_{2(t,t)} = e_{2(t,t)}$. Q.E.D.

PROPOSITION 5.3.4: Under conditions of perfect capital mobility, observation of both the current exchange rate and the current interest rate conveys complete information about external shocks.

PROOF: Consider $e_{3t,t}$ in equations (5.3.15a) and (5.3.16a). Set $\eta_1 = \omega$ so that $n = 0$. Then $e_{3t,t} = e_{3t}$. Q.E.D.

PROPOSITION 5.3.5: Under conditions of imperfect capital mobility, Proposition 5.3.3 ceases to be true.
PROOF : Repeat the proof of Proposition 5.3.3 with \( n_1 < n, \) \( n > 0 \) and 
\( a_1 = \gamma^{-1} - n. \) Then \( t e_{t-1,t} \neq t e_{t-1,t} \) and 
\( e_{2t,t} \neq e_{2t,t} \) obviously. Q.E.D.

PROPOSITION 5.3.6 : Under conditions of imperfect capital mobility, 
Proposition 5.3.4. ceases to be true.

PROOF : Repeat the proof of Proposition 5.3.4 with \( n_1 < n \) so that 
\( n > 0. \) Then \( e_{3t,t} \neq e_{3t} \) obviously. Q.E.D.

PROPOSITION 5.3.7 : Regardless of the degree of capital mobility, if 
there is only one internal shock and one external shock, observation 
of both the current exchange rate and the current interest rate 
conveys complete information about both disturbances.

PROOF : Consider the case in which \( e_{2t} = \sigma^2 = 0. \) Then from equation 
(5.3.15a) and (5.3.16a), \( a_{t-1,t} = u_{t-1} \) and \( e_{3t,t} = e_{3t}. \) Alternat-
ively, consider the case in which \( u_{t-1} = \sigma^2 = 0. \) Then from equations 
(5.3.15a) and (5.3.16a), \( a_{2t,t} = e_{2t} \) and \( e_{3t,t} = e_{3t}. \) Q.E.D.

The foregoing results have some economic intuition. The fact 
that there exists differences between the information contents of the 
exchange rate and the interest rate is obvious. The current exchange 
rate responds to all contemporaneous shocks. By contrast, the 
current interest rate responds only to contemporaneous domestic 
shocks. It also depends on last period's external shock via the 
sluggish price adjustment; but, by assumption, this is part of 
current information. That the current interest rate conveys no 
information about contemporaneous external disturbances is readily 
apparent from equation (5.3.25). This demonstrates the point made 
earlier regarding the implications of intrinsic dynamics.

Propositions 5.3.3 and 5.3.5 are a little less obvious but still 
fairly straightforward. When capital is perfectly mobile and there
are no external disturbances, fluctuations in the exchange rate are
dictated entirely by internal (domestic) perturbations operating
through the interest rate. Observation of the exchange rate is
therefore equivalent to observation of the interest rate. This is
obviously not the case if external shocks exist. When capital is
less than perfectly mobile, domestic price perturbations can cause
exchange rate movements independently of any interest rate variation
via their effect on competitiveness (and hence the trade balance).
In this way, the information contents of the exchange rate and the
interest rate differ.

Consider next Propositions 5.3.4 and 5.3.6. The former arises
since exchange rate fluctuations are again motivated exclusively
by interest rate movements plus, now, external shocks. Any variation
in the exchange rate over and above that implied by interest rate
changes are necessarily the result of foreign disturbances. By
contrast, as noted above, imperfect capital mobility permits domestic
price shocks to impinge directly on the exchange rate.

Finally, consider Proposition 5.3.7. For perfect capital
mobility, this is a corollary of Proposition 5.3.4 which implied that
agents have information about only a composite domestic shock.
Decomposition of this was not possible. Obviously, however, with
only one type of domestic shock, the problem of aggregation
disappears. For imperfect capital mobility, similar reasoning
applies. A single current internal disturbance causes a contemporaneous interest rate response whilst the effect of an external shock
on the interest rate is realized only after a lag.

The upshot of the foregoing results is that both the degree of
capital mobility and the nature of the shock structure have drastic
implications for the amount and nature of auxiliary information
extracted from observed signals. Previous work has overlooked these points. A corollary is the following.

COROLLARY 5.3.1: The accuracy of auxiliary information about external disturbances is an increasing function of the degree of capital mobility.

Corollary 5.3.1 follows from the fact that imperfect capital mobility contaminates the information content of observed signals by delivering an additional source of confusion via the direct effect of prices on the exchange rate through the trade balance.

Some implications of our results are illustrated in figures 5.3(A), 5.3(B) and 5.3(C). These show the dynamic behaviour of the exchange rate and prices under a variety of circumstances. Single domestic price, money demand and foreign disturbances are considered in isolation; \( u_{t-1}, e_{2t}, e_{3t} > 0 \). The trajectories are conditioned by the degree of capital mobility and the assumption about the initial informational endowment. For the latter, these assumptions are given by equations (5.3.13) - (5.3.14b). The trajectories are derived from the complete solutions of the system given as follows.

PROPOSITION 5.3.8: For each of the information structures defined in equations (5.3.13) - (5.3.14b), the complete solutions of the system can be written as

\[
\begin{align*}
\frac{NF}{e_t} &= -m_1 \frac{NF}{e_t} - \tau_2 \gamma e_{2t} - \tau_2 e_{3t} \\
\frac{NF}{P_{t+1}} &= \tau_1 \frac{NF}{e_t} - \lambda r_2 \gamma e_{2t} - \lambda r_2 e_{3t} + u_{t+1} \\
\frac{NF}{e_{t+1}} &= \tau_1 \frac{NF}{e_t} - m_2 u_{t+1} + fr_2 \gamma e_{2t} + fr_2 e_{3t} - \tau_2 \gamma e_{2t+1} - \tau_2 e_{3t+1} \\
&= \frac{NF}{e_{t+1}} - \frac{NF}{e_{t+1}}
\end{align*}
\]

(5.3.27)
\[ a_{t}^F = -m_{21}P_t + a\Theta^2(m_{21}a_1 - \tau_2^{-1})u_{t-1} + (-\gamma_2^{-1} - \\)
\[ a\Theta^3(m_{21} - \tau_2^{-1}a_1)e_{2t} + (-\tau_2^{-1} + a\Theta^3(m_{21} - \tau_2^{-1}a_1))e_{3t} + u_t^1 \]

\[ a_{t+1}^F = \tau_1 P_t + a\Theta^2\lambda(m_{21}a_1 - \tau_2^{-1})u_{t-1} + \\lambda(-\tau_2^{-1} - \\)
\[ a\Theta^3(m_{21} - \tau_2^{-1}a_1)e_{2t} + \\lambda(-\tau_2^{-1} + a\Theta^3(m_{21} - \tau_2^{-1}a_1))e_{3t} + u_t^1 \]

(5.3.28a)

\[ a_{t+1}^F = \tau_1(a_{t}^F) - a\Theta^2f(m_{21}a_1 - \tau_2^{-1})u_{t-1} - \\gamma_2^{-1} - \\)
\[ a\Theta^3(m_{21} - \tau_2^{-1}a_1)e_{2t} - f(-\tau_2^{-1} + a\Theta^3(m_{21} - \tau_2^{-1}a_1))e_{3t} + \\]
\[ + (-m_{21} + a\Theta^2m_{21}a_1 - \tau_2^{-1})u_t^1 + (-\tau_2^{-1} - \\)
\[ a\Theta^3(m_{21} - \tau_2^{-1}a_1)e_{2t+1} + (-\tau_2^{-1} + a\Theta^3(m_{21} - \tau_2^{-1}a_1))e_{3t+1} \]

(5.3.28b)

\[ b_{t}^F = -m_{21}P_t + b\Theta^2(m_{21}a_1 - \tau_2^{-1})u_{t-1} + (-\gamma_2^{-1} - \\)
\[ b\Theta^3(m_{21} - \tau_2^{-1}a_1)e_{2t} + (-\tau_2^{-1} - b\Theta^3(m_{21} - \tau_2^{-1}a_1))e_{3t} \]

\[ b_{t+1}^F = \tau_1 P_t + b\Theta^2\lambda(m_{21}a_1 - \tau_2^{-1})u_{t-1} + \\lambda(-\tau_2^{-1} - \\)
\[ b\Theta^3(m_{21} - \tau_2^{-1}a_1)e_{2t} + \\lambda(-\tau_2^{-1} - b\Theta^3(m_{21} - \tau_2^{-1}a_1))e_{3t} + u_t^1 \]

(5.3.28b)
where
\[ a e^2 = \phi a_1 \sigma_1^2 \sigma_3 \]
\[ a e^1 = \phi a_1 \sigma_1 \]
\[ a e^0 = \phi a_1 \]
\[ b e^2 = b a_1 (\gamma \sigma_1^2 + \sigma_3^2) \]
\[ b e^1 = b a_1 \gamma \sigma_1 \]
\[ b e^0 = b a_1 \sigma_1 \]

PROOF: For \( e^F_{t}, P_{t+1}^F \) and \( e^F_{t+1} \) in equations (5.3.27) set \( u^e_{t+1} = u^1_t \),
\( e^2 = e_2 \) and \( e^3 = e_3 \) in equations (5.3.10) - (5.3.12).

(a) For \( e^F_{t}, e^P_{t+1} \) and \( e^P_{t+1} \) in equations (5.3.28a) substitute into equations (5.3.10) - (5.3.12) the expressions for \( u^e_{t+1}, e^2 \) and \( e^3 \) given in equations (5.3.15a). Collect terms and obtain the coefficient expressions on each disturbance. In particular, in each expression,

\[ u^1_{t-1} = b_1 - a^1 b_1 + a^2 b_2 \]
\[ u^1_t = -a^1 a_1 - a^1 b_1 + a^2 b_2 \]
\[ e^2 = -a^2 \gamma - a^1 b_1 + a^2 b_2 \]
\[ e^3 = -a^2 + a^1 b_1 + a^2 b_2 \]  

where \( a^1 = a \phi a_1 (n^2 \sigma_2^2 + \sigma_3^2) \), \( a^2 = a \phi (\gamma \sigma_1^2 (n^2 \sigma_1^2 + \sigma_3^2) + n^2 \sigma_3^2) \),
\( a^0 = a \phi (\gamma \sigma_1 (n^2 + \sigma_3^2) + \sigma_1^2 \sigma_3^2) \); and \( a^2, a^1 \) and \( a^0 \) are defined previously. Now note that \( a^1 = 1 - a^2 a_1, a^2 = \gamma - a^1 a_1 \) and \( a^0 = 1 + a^1 a_1 \). Recalling the definitions of \( b_i \) \( (i = 1, 2) \),
appropriate substitution in equations (5.3.29) yields the coefficients on the disturbances shown in equations (5.3.28a).

(b) For $e_{F}^{F}$ and $e_{t+1}^{F}$ in equations (5.3.28b) substitute into equations (5.3.10) - (5.3.12) the expressions for $b_{1}, b_{2}$ and $b_{3}$ given in equations (5.3.15b). Collect terms and obtain the coefficient expressions on each disturbance. In particular, in each expression,

$$
\begin{align*}
\epsilon_{t-1} & : b_{1} - b_{1}^{1}b_{1} + b_{2}^{2}b_{2} \\
\epsilon_{t} & : -a_{1}^{1}a_{1} - b_{1}b_{1} + b_{2}^{2}b_{2} \\
\epsilon_{2} & : -a_{2}^{1}a_{1} - b_{1}b_{1} + b_{2}^{2}b_{2} \\
\epsilon_{3} & : -a_{2}^{1}a_{1} - b_{1}b_{1} + b_{2}^{2}b_{2}
\end{align*}
$$

(5.3.30)

where $b_{1}^{1} = b_{1}^{1}, b_{2}^{2} = b_{2}^{2}(\gamma - 2\sigma_{2}^{2} + \sigma_{3}^{2}), b_{3}^{3} = \phi(\gamma - 2\sigma_{2}^{2} + \sigma_{3}^{2})$, and $b_{2}^{2}, b_{1}^{1}$ and $b_{2}^{2}$ are defined previously. Now note that $b_{1}^{1} = 1 - b_{2}^{2}a_{1}^{1}, b_{2}^{2} = \gamma - 1 - b_{2}^{2}a_{1}^{1}$ and $b_{3}^{3} = 1 - b_{2}^{2}a_{1}^{1}$. Recalling the definitions of $b_{1}^{i}$ ($i = 1, 2$), appropriate substitution in equation (5.3.30) yields the coefficients on the disturbances shown in equations (5.3.28b). Q.E.D.

The diagrams constitute a pictorial representation of the following propositions.

**PROPOSITION 5.3.9**: In the presence of domestic price disturbances, observation of both the current exchange rate and the current interest rate conveys more information than does observation of just the current exchange rate.
PROOF: Denote by $e_t(u_{t-1}^L)$ the response of $e_t$ to $u_{t-1}^L > 0$. Then from equations (5.3.27) - (5.3.28b),

$$e^{NF}_t(u_{t-1}^L) = -m_{21}$$

$$e^{F}_t(u_{t-1}^L) = -m_{21} + a^2(m_{21}a_1 - \tau_2^{-1})$$

$$b^e_t(u_{t-1}^L) = -m_{21} + b^2(m_{21}a_1 - \tau_2^{-1})$$

$$b^{F}_t(u_{t-1}^L) = b^{F}_t(u_{t-1}^L) = (m_{21}a_1 - \tau_2^{-1})(b^2 - a^2)$$

$$= (m_{21}a_1 - \tau_2^{-1})\psi a_1\sigma^2$$

where $\psi = a^2 + n^{-1}a_2^2$. It is easily verified that $m_{21}^{-1} - \tau_2 > 0$. For $\eta_1 = \omega (n = 0)$, $m_{21}$, $a_1$, $a^2$, $b^2 > 0$. From equations (5.3.31) - (5.3.32b), $e^{NF}_t(u_{t-1}^L) < 0$ and $e^{F}_t(u_{t-1}^L)$,

$F^{1}\ e^{NF}_t(u_{t-1}^L) < e^{F}_t(u_{t-1}^L)$. In addition, from equation (5.3.33),

$F^{1}\ a^e_t(u_{t-1}^L) > e^{F}_t(u_{t-1}^L)$. Hence, $e^{NF}_t(u_{t-1}^L) < e^{F}_t(u_{t-1}^L) < b^{F}_t(u_{t-1}^L)$. For $\eta_1 = \omega (n > 0)$, $m_{21}$, $a_1$, $a^2$, $b^2 < 0$. From equations (5.3.31) - (5.3.32b), $e^{NF}_t(u_{t-1}^L) > 0$ and $e^{F}_t(u_{t-1}^L)$, $e^{F}_t(u_{t-1}^L) < e^{F}_t(u_{t-1}^L)$. In addition, equation (5.3.33) gives $b^{e}_t(u_{t-1}^L) < e^{F}_t(u_{t-1}^L)$. Hence,

$e^{NF}_t(u_{t-1}^L) > e^{F}_t(u_{t-1}^L) > b^{F}_t(u_{t-1}^L)$. Since the subsequent dynamics are determined by these initial responses, this establishes the Proposition. Q.E.D.

PROPOSITION 5.3.10: In the presence of foreign disturbances, observation of both the current exchange rate and the current interest rate conveys more information than does observation of just the current exchange rate if there is either perfect capital mobility or imperfect capital mobility accompanied with relatively low money demand volatility. Otherwise, observation of both the current exchange rate
and the current interest rate conveys less information than does observation of just the current exchange rate.

PROOF: Denote by $e_t(\varepsilon_{3t})$ the response of $e_t$ to $\varepsilon_{3t} > 0$. Then from equations (5.3.27) – (5.3.28b),

\[
\begin{align*}
e^{NF}_t(\varepsilon_{3t}) &= -\tau_2^{-1} \\
e^F_t(\varepsilon_{3t}) &= -\tau_2^{-1} + \sigma_1^{-1}(m_{21} - \tau_2^{-1}a_l) \\
e^F_t(\varepsilon_{3t}) &= -\tau_2^{-1} - \sigma_1^{-1}(m_{21} - \tau_2^{-1}a_l)
\end{align*}
\]

\[
\begin{align*}
&= -(m_{21} - \tau_2^{-1}a_l)(\sigma_1^{-1} + \sigma_1^{-1}) \\
&= -(m_{21} - \tau_2^{-1}a_l)\psi \sigma_1^2(a_1 \sigma_1^2 + \gamma^{-1-2})
\end{align*}
\]

It is easily verified that $m_{21} - \tau_2^{-1}a_l \geq 0 \iff a_l \geq 0$. For $\eta_2 = \infty$, $m_{21} < 0$, $a_l < 0$, $\sigma_1^{-1} > 0$ and $\sigma_1^{-1} = 0$. From equations (5.3.34) – (5.3.35b),

\[
\begin{align*}
e^{NF}_t(\varepsilon_{3t}) &= e^{F}_t(\varepsilon_{3t}) < 0 \quad \text{and} \quad e^{F}_t(\varepsilon_{3t}) < e^{NF}_t(\varepsilon_{3t}).
\end{align*}
\]

Obviously, $e^{NF}_t(\varepsilon_{3t}) = e^{F}_t(\varepsilon_{3t}) < e^{F}_t(\varepsilon_{3t})$. For $\eta_1 < \infty$, $m_{21}, a_l, \sigma_1^{-1} < 0$ and $\sigma_1^{-1} > 0$. From equations (5.3.34) – (5.3.35b),

\[
\begin{align*}
e^{NF}_t(\varepsilon_{3t}) &= e^{F}_t(\varepsilon_{3t}) < 0 \quad \text{and} \quad e^{F}_t(\varepsilon_{3t}) < e^{NF}_t(\varepsilon_{3t}). \quad \text{If} \quad |a_1 \sigma_1^2| > \gamma^{-1-2} \quad \text{equation (5.3.36) gives} \quad e^{F}_t(\varepsilon_{3t}) < e^{F}_t(\varepsilon_{3t}) \quad \text{so that} \quad e^{F}_t(\varepsilon_{3t}) = e^{F}_t(\varepsilon_{3t}) > e^{F}_t(\varepsilon_{3t}) \quad \text{and} \quad e^{NF}_t(\varepsilon_{3t}) > e^{NF}_t(\varepsilon_{3t}) > e^{NF}_t(\varepsilon_{3t}) > e^{NF}_t(\varepsilon_{3t}). \quad \text{Since the subsequent dynamics are determined by these initial responses, this establishes the Proposition. Q.E.D.}
\end{align*}
\]

PROPOSITION 5.3.11: In the presence of domestic money demand disturbances, observation of both the current exchange rate and the current interest rate conveys less information than does observation of just the current exchange rate if there is perfect capital mobility.
Otherwise, observation of both the current exchange rate and the current interest rate conveys information which is qualitatively different from the information conveyed by observation of just the current exchange rate.

PROOF: Denote by $e_t(e_{2t})$ the response of $e_t$ to $e_{2t} > 0$. Then from equations (5.3.27) - (5.3.28b),

$$e_t(e_{2t}) = -\tau_2 \gamma$$  \hspace{1cm} (5.3.37)

$$a_t(e_{2t}) = -\tau_2 \gamma - a^1(m_{21} - \tau_2 a_1)$$  \hspace{1cm} (5.3.38a)

$$b_t(e_{2t}) = -\tau_2 \gamma - b^1(m_{21} - \tau_2 a_1)$$  \hspace{1cm} (5.3.38b)

$$b_t(e_{2t}) - a_t(e_{2t}) = -(m_{21} - \tau_2 a_1)(b^1 - a^1)$$

$$= -(m_{21} - \tau_2 a_1)^2(a^2 - \sigma_1^2 - \sigma_3^2).$$ \hspace{1cm} (5.3.39)

Recall that $m_{21} - \tau_2 a_1 \geq 0 \iff a_1 \geq 0$. For $\eta < \infty (n = 0)$, $m_{21}, a_1, a^1, b^1 > 0$. From equations (5.3.37) - (5.3.38b), $e_t(e_{2t}) < 0$ and $a_t(e_{2t}), b_t(e_{2t}) < e_t(e_{2t})$. Equation (5.3.39) gives $b_t(e_{2t}) > a_t(e_{2t})$. Hence, $e_t(e_{2t}) > b_t(e_{2t}) > a_t(e_{2t})$.

For $\eta < \infty (n > 0)$, $m_{21}, a_1, b^1 < 0$ and $a^1 > 0$. From equations (5.3.37) - (5.3.38b), $e_t(e_{2t}) < 0$, $a_t(e_{2t}), b_t(e_{2t}) > e_t(e_{2t})$ and $b_t(e_{2t}) < e_t(e_{2t})$. Hence, $a_t(e_{2t}), b_t(e_{2t}) > e_t(e_{2t})$. Since the subsequent dynamics are determined by these initial responses, this establishes the Proposition. Q.E.D.

The economic intuition underlying the foregoing results is as follows. Figure 5.3(A) illustrates the case of a positive domestic price shock, $u_{t-1} > 0$. As shown in the proof of proposition 5.3.9, the exchange rate makes an instantaneous discrete appreciation under
perfect capital mobility and an instantaneous discrete depreciation under imperfect capital mobility. In both cases, the effect of partial information is to dampen the response. This is because agents misinterpret the signals given by the exchange rate and the interest rate as reflecting other shocks. For perfect capital mobility, this means that the expected future exchange rate is more depreciated relative to the full information case. Hence, the current exchange rate appreciation is lower than under full information. For imperfect capital mobility, the misinterpretation implies an expected future exchange rate which is relatively more appreciated. Then there is pressure for the current exchange rate to appreciate as well. Observation of both the current exchange rate and the current interest rate generates dynamic behaviour which is closer to the full information solution than that implied by observation of just the current exchange rate. The reason is that observable contemporaneous movements in the interest rate signal (correctly) the definite occurrence of domestic shocks.

Figure 5.3.(B) illustrates the case of a positive external disturbance, \( \varepsilon^e_3 > 0 \). The proof of Proposition 5.3.10 showed that the exchange rate appreciates. This is true regardless of the degree of capital mobility by construction. Moreover, the proof supports Propositions 5.3.4 and 5.3.6. For perfect capital mobility, contemporaneous information about both the exchange rate and the interest rate conveys perfect auxiliary information about the external disturbance. This is not true when the information content of the interest rate is ignored. Under such circumstances, agents infer a positive price shock which induces relatively more appreciated future exchange rates. This exacerbates the current appreciation. When capital is less than perfectly mobile, any restriction of the information set leads to a departure from the full information solution.
Figure 5.3(A)(i): Exchange rate and price level trajectories, $\eta_1 = \infty$

Figure 5.3(A)(ii): Exchange rate and price level trajectories, $\eta_1 < \infty$

Figure 5.3(A): Exchange rate and price level trajectories under divergent information structures, $\eta_1 < \infty$, $u_{t-1}^1 > 0$

---

$e^{NF}(u_{t-1})$; $p^{NF}(u_{t-1})$

$e^F(u_{t-1})$; $a^P(u_{t-1})$

$e^F(u_{t-1})$; $b^P(u_{t-1})$
Figure 5.3(B)(i) : Exchange rate and price level trajectories, $\eta_1 = \infty$

Figure 5.3(B)(ii) : Exchange rate and price level trajectories, $\eta_1 < \infty$

Figure 5.3(B) : Exchange rate and price level trajectories under divergent information structures, $\eta_1 < \infty$, $\epsilon_{3t} > 0$

$e^{NF}(\epsilon_{3t})$; $P^{NF}(\epsilon_{3t})$

$e^{P}(\epsilon_{3t})$; $P^{P}(\epsilon_{3t})$

$e^{P}(\epsilon_{3t})$; $P^{P}(\epsilon_{3t})$
Figure 5.3(C)(i) : Exchange rate and price level trajectories, $n_1 = \infty$

Figure 5.3(C)(ii) : Exchange rate and price level trajectories, $n_1 < \infty$

Figure 5.3(C) : Exchange rate and price level trajectories under divergent information structures, $n_1 < \infty$, $e_{2t} > 0$

- $e^{N_F}(e_{2t})$; $p^{N_F}(e_{2t})$
- $a_e^F(e_{2t})$; $a_p^F(e_{2t})$
- $b_e^F(e_{2t})$; $b_p^F(e_{2t})$
In particular, the effect of signal-extraction is to compound the initial appreciation. This occurs because agents infer a negative domestic price shock in which case expected future exchange rates are relatively more appreciated. Figure 5.3(B)(ii) shows the outcome where exchange rate and interest rate observations yield more accurate information than just exchange rate information by itself. Nonetheless, Proposition 5.3.10 and its proof identified the possibility that the opposite might occur. An interesting condition for this is that monetary volatility must be relatively large. The explanation is as follows. In response to a current external perturbation, agents observe no contemporaneous interest rate fluctuation. Then the inference of a negative price shock must be accompanied with the expectation of a positive money demand disturbance in order to be consistent with this observation. If money demand volatility is relatively large, then so must be the expectation of the negative price movement. From above, this exacerbates the initial exchange rate appreciation.

Turning, finally, to money demand disturbances, \( \epsilon_{2t} > 0 \), figure 5.3(C) is relevant. The proof of Proposition 5.3.11 showed that the exchange rate initially appreciates regardless of the degree of capital mobility. When capital is perfectly mobile, the effect of partial information is to exacerbate the response. This is because of the (mis)inference of a positive price movement (implying relatively more appreciated future exchange rates). The degree of exacerbation is greater when agents observe both asset prices than when they rely solely on the information content of the exchange rate. As before, then, greater initial information reduces the accuracy of forecasts. In this case, however, this peculiarity is unambiguously true. This can be explained as follows. If agents observe only the exchange rate, they assign probabilities to the
occurrence of each type of shock. The additional observation of the interest rate signals (correctly) the occurrence of domestic shocks; but this signal is actually misleading. Agents are now more inclined to perceive positive domestic price shocks. Then forward expectations serve to deliver the outcome. For the case of imperfect capital mobility we observe a further peculiarity: relative to the full information solution, observing both the exchange rate and the interest rate implies a less appreciated exchange rate whilst observing just the exchange rate implies a more appreciated exchange rate. This motivated the statement in Proposition 5.3.11, namely that the two initial information endowments convey qualitatively different auxiliary information. Thus, observing solely exchange rate variations, agents infer a negative price movement which exacerbates the current appreciation. This is, however, inconsistent with the rise in the interest rate when this variable is also observed. On the contrary, agents are more inclined to infer a positive price movement. Hence, the difference.

In terms of the behaviour of prices, the above forces the following observations.

COROLLARY 5.3.2 : Under conditions of perfect capital mobility, the volatility of prices is a decreasing function of the initial informational endowment in the face of domestic price and external disturbances. In the face of domestic money demand disturbances, the effect is ambiguous.

COROLLARY 5.3.3 : Under conditions of imperfect capital mobility, the volatility of prices is an increasing function of the initial informational endowment in the face of domestic price disturbances. In the face of domestic money demand and external disturbances, the effect is ambiguous.
In summary, we have shown some significant differences between the information contents of the exchange rate and the interest rate. These differences disappear only under special conditions. In addition, the amount and nature of auxilliary information is critically non-invariant with respect to both the degree of capital mobility and the particular shock structure. Most interestingly, perhaps, are the following: first, a greater initial informational endowment may reduce the accuracy of forecasts about some shocks; second, a greater initial informational endowment may change the qualitative nature of auxilliary information. This is not to say that the overall dynamic behaviour of the system moves further from the full information solution as the initial informational endowment increases. Taken together (that is, in the face of all shocks impinging simultaneously), it must be true that increasing the information set conveys more accurate information in the sense of yielding a solution which is closer to the full information case. This could only be proved numerically in our model. Nonetheless, our analysis indicates that increases in forecast accuracy may not apply to all forecasts but rather may apply to the forecasts of only some shocks. As far as we know, these points have hitherto been overlooked.

5.4 Summary and Concluding Remarks

The chapter has been by way of an introduction to the application of signal-extraction from the engineering sciences to economic problems. As for the theory of optimal control discussed in chapter 1, there is a fundamental difference between the engineering and economic problems; unlike the former, the latter involves the use of signal-extraction techniques by participants within the system itself. This has gained in popularity in recent years as
various researchers have become more aware of the potentially crucial role of the information structure in determining the behaviour of the system. Moreover, it is intrinsic to rational expectations models and is a criticism of those analyses which abstract from the problem. The assumption one chooses to make about the elements in the set of information variables is, of course, somewhat arbitrary. Yet a fairly unprovocative notion is that this set comprises at a minimum all asset prices. If private information is contemporaneously available, then there is an important additional dimension to the inference problem. More generally, idiosyncracies in the information structure contaminate observed signals because these signals now reflect the expectations of differentially informed agents. The existing literature has made a bold attempt to investigate the consequences of this. Nonetheless, we believe that the full implications of heterogenous information are yet to be realized. Further research would be helpful on this matter.

Though, in general, the existing literature has much to commend it, we hope that our discussion in section 5.2 has highlighted some important gaps. Of much concern is the types of models that have been employed. With the exception of only a few, these have contained no intrinsic dynamics. Yet asymmetric speeds of adjustment may serve to enrichen the information content of variables.

Section 5.3 contained a fairly rigorous investigation into the information contents of the exchange rate and the interest rate. It was motivated by the view that these are sufficiently similar as to require consideration of only one. We showed this view to be generally misconceived. We also illustrated the critical dependence of the information structure on the degree of capital mobility and the nature of the shock structure. In addition, two important results were that enrichening the initial informational endowment
might actually reduce the accuracy of forecasts about some shocks and may change the qualitative nature of auxiliary information. Both of these suggest that it is not just the amount of initial information that is necessarily important but also the nature of the endowment. Though an increase in the endowment can only ultimately improve the overall accuracy of auxiliary information, the forecast accuracy about some shocks might actually fall.

It should be stressed that the particular signal-extraction problem that we have addressed concerns the extraction of information about unobservable events by employing the actual system generating these events. This is to be distinguished from the more thorny problem of learning the system itself. Our views on this matter were elicited in section 5.1. We believe it to be a fundamentally important issue and offers an exciting challenge for further research to which we have unfortunately been unable to make any contribution in this thesis. The issues and problems raised in the context of our own preoccupation are likely to be helpful to and compounded by a framework which analyses the process of learning the system and the convergence to an equilibrium.

In summary, the chapter has demonstrated the potentially crucial interdependence between the system and the informational structure within that system. Though different assumptions can have drastic implications for the information structure, however, the precise importance of different information structures for determining the dynamic behaviour of the system requires numerical evaluation. We suggest that research into this should be taken up. The important implications for policy evaluation are as follows. First, the informational requirements of the controller are increased substantially. Not only must he be aware of the mechanics of the system, but he must also judge the precise informational characteristic
of the system. Second, control has an opportunity to influence the system by altering the information content of observed signals. These issues are addressed in the following chapters.
Notes to Chapter Five

(1) Alternatively, Barro (1976) suggests that even if agents observe variations in local money balances, these may merely reflect the transfer of money between locations.

(2) Note that this is different from the case in which the structural parameters of the model are functions of the stochastic properties of the disturbances. This effect arises via agents' utility maximization.

(3) In King (1981) and Boschen and Grossman (1983) there is still a signal-extraction problem because, in addition to local demand and aggregate monetary shocks, there is also a velocity disturbance which serves as a second economy-wide perturbation.

(4) This is essentially the Lucas and Rapping (1969) supply hypothesis discussed in chapter 2 (section 2.2).

(5) The real interest rate term is \( r_t = p_{t+1,t}^{eh} + p_t(h) \) where \( r \) is the economy-wide nominal interest rate. The (local) expectation of next period's aggregate price level \( p_{t+1,t}^{eh} \) reflects the assumption that agents are randomly located in markets in the next period. This requires agents to form estimates of the aggregate price level rather than just the price level in the market in which they are currently located.

(6) If not, then observation of local prices and the global interest rate will convey full information. Suppose that the local price and interest rate signals are \( p_t(h) : \delta_1 \nu_1 t + \delta_2 \nu_2 t + \delta_3 s_t(h) \) and \( r_t : \gamma_1 \nu_1 t + \gamma_2 \nu_2 t \) respectively. With either \( \gamma_1 = \delta_1 = 0 \) or \( \gamma_2 = \delta_2 = 0 \), observation of \( r_t \) conveys perfect information about the single aggregate shock and then \( p_t(h) \) yields perfect knowledge about \( \varepsilon_t(h) \). With \( \gamma_1', \delta_1 \neq 0 \) \((i = 1, 2)\) this is no longer true. In addition, only if
\( \gamma_1 = \gamma_2 = \gamma \) and \( \delta_1 = \delta_2 = \delta \) would there be no confusion between aggregate and local shocks. The only confusion is then between the type of aggregate shock. In general, \( \gamma_1', \delta_1 \neq 0, \gamma_1 \neq \gamma_2 \) and \( \delta_1 \neq \delta_2 \) so that the composite aggregate shock \( v_{1t} + v_{2t} \) and its decomposition are unknown.

(7) This is similar to the argument advanced earlier concerning the negligible information conveyed by local money about aggregate money.

(8) This is a weighted average of domestic and foreign prices corrected for exchange rate movements.

(9) This is to be distinguished from learning the nature of stochastic shocks impinging on the model. In this case, agents' inferences are conditioned by their knowledge of the model.

(10) This is distinguished from current feedback policy which is a response to known events.

(11) There is no consensus on this.

(12) In each market, \( h (h = 1, \ldots, H) \), agents choose a foreign trading partner. Purchasing power parity gives

\[
p_t(h) = e_t + p^f_t(h) (h = 1, \ldots, H),
\]

where \( e_t \) is the nominal exchange rate and \( p^f_t(h) \) is the price of the commodity in market \( h \)'s \( (h = 1, \ldots, H) \) trading partner. The latter is stochastic owing to market-specific shocks.

(13) In particular, the variance of output around its full information level is lower when exchange rates are flexible than when exchange rates are fixed (see also Kimbrough (1983a)). This assumes that the full information solution is first best.
(14) This is similar to the result of Grossman (1976). Formally, if \( n_1 \) is the number of linear combinations of aggregate shocks and average expectations and \( n_2 \) is the number of economy-wide markets in which differentially informed agents trade, the observation of the \( n_2 \) global signals will permit decomposition of the composite aggregate shocks and average expectations provided \( n_2 > n_1 \).

*This section draws on the analysis in Blackburn (1984b, 1985a)


(16) Recall that the flow view of capital movements in equation (5.3.4) is likely to be innocuous (see, for example, Bhandari, Driskill and Frenkel (1984); Gaziosglou (1984)).

(17) This assumption is innocuous. Any exogenous process for the money supply will suffice. Assuming a fixed money supply is merely the most convenient. In any specification, the important point is that exogenising the money stock leaves the interest rate free to respond to market conditions. This permits the interest rate to perform the role of an information variable. The implications of controlling the interest rate are studied in chapter 6.

(18) In what follows, \( n_1 = \omega \) implies \( e_{3t} = \nu_{1t} \) which is the exogenous shock to capital flows, interpreted as random deviations from interest parity.
CHAPTER SIX

MONETARY INSTRUMENTS AND INFORMATION VARIABLES: A RE-EVALUATION

6.1 Introduction

An important implication of recognizing the problem of signal-extraction is the ability of control rules to influence the system by altering the information content of observed signals. In chapter 5 (section 5.2) we saw how this might operate in the presence of heterogeneous information. Then contemporaneously observed variables will reflect the expectations of differentially informed agents which may be manipulated by choice of control parameters. Nonetheless, differential information is not necessary for control to work in this way. Provided agents exploit the information content of observed variables, control has the potential to alter behaviour by influencing this information. In this chapter, we investigate a particular application of this idea to the issues discussed in chapters 2 and 3. Those chapters examined the optimal choice of monetary instrument (money stock or interest rate) in a variety of different models and in the face of a range of alternative sources of stochastic fluctuation. In some cases we assumed full contemporaneous information whilst in others the information set was restricted. These made our analysis directly comparable to the existing literature. There is, however, an important criticism that can be made of the orthodox approach. Not only is the full information assumption unrealistic, but equally the precise modelling of imperfect information is inappropriate. To be specific, it entails an overly restricted information structure. That this is so is to merely be aware of the abstraction of the important role of some variables as conveyers of auxiliary information, in particular, the exchange rate and interest rates. Given this, there is a potentially serious gap; for if
control is administered over the interest rate such as to make this variable currently exogenous, it is deprived of any useful information.

This chapter is concerned to re-evaluate the monetary instrument problem in the light of these criticisms of the orthodox analysis. By way of a preamble to our analysis, it is useful to be acquainted with some further literature on signal-extraction which was not mentioned in chapter 5 because it is more relevant for the present concern.

A number of studies address the optimal filtering problem with reference to the conduct of monetary policy. Thus, Kareken, Muench and Wallace (1973), Friedman (1975, 1977) and LeRoy and Waud (1977) show how contemporaneous observations of some monetary aggregate yield useful information about the state of the economy which can be exploited by the controller to revise monetary policy (see also LeRoy and Lindsey (1978)). A similar investigation can be found in Mitchell (1982). On a less rigorous note, Friedman (1979b) identifies the types of information used in formulating monetary policy which might be useful to the private sector. The approach taken in this chapter differs from these studies and of more relevance are the papers by Dotsey and King (1983) and Canzoneri, Henderson and Rogoff (1983). In both, a Sargent and Wallace (1975) model is employed and the set of information variables comprises the global nominal interest rate(1). The authors show that a contemporaneous feedback rule involving the money stock and the interest rate is impotent with respect to influencing the distribution of output (this is identified by Canzoneri, Henderson and Rogoff (1983) in a particular case; a similar result is obtained by Siegel (1982) in a related context). The result is essentially a generalization of the policy ineffectiveness proposition to the case of currently dated
information sets. In short, no-one has an informational advantage
and the controller is feeding back onto a variable which is part of
the private sector's information set. A second result of Dotsey and
King (1983) is that an interest rate peg can never be superior to a
policy which permits some interest rate variation because the former
necessarily deprives agents of information. Though superficially
appealing, it is important to note that the validity of the conclu-
sion hinges on the assumption that the appropriate loss function
penalizes deviations around the full information solution. Though
this is likely to be appropriate in some circumstances, there is no
consensus about the general validity of this criteria. Canzoneri,
Henderson and Rogoff (1983) consider alternative information struct-
ures which differ according to which group (investors, suppliers or
the authorities) exploit the information content of the interest
rate. When suppliers do not use this information, the contemporaneous
feedback policy becomes effective; essentially, the authorities have
an informational advantage. The problem with this analysis is that
there is no reason why any group should not exploit the information
content of the interest rate. The assumptions that the authors make
are quite arbitrary. After all, the interest rate is globally
observed and it is rather meaningless to suppose that some agents
will not use the observation in formulating their predictions.

In the analysis that follows we consider two aggregative
models that we have previously employed in the thesis. We abstract
from private information (and hence differential information) by
regarding the set of information variables to comprise solely economy-
wide asset prices. In particular, in both models the interest rate
is treated as an information variable. The usefulness of this is
neutralized when it is pegged. This is the assumption made in Dotsey
and King (1983) and Canzoneri, Henderson and Rogoff (1983). These
authors fail to point out, however, that the auxiliary information embodied in the interest rate when this variable is free to respond to market conditions will be reflected in the accommodating money stock movements when it is fixed. In addition, it must also be assumed that the controller has contemporaneous information on both the interest rate and the money stock. In view of these, there is an implicit assumption that, for some reason or another, the controller does not divulge contemporaneous money supply accommodations. The virtue of the assumption is that it enables consideration of the plausible case in which exogenising the interest rate does indeed deprive agents of information. Nonetheless, we feel it necessary to elicit these points here as they are rarely revealed in the literature.

Some other preliminary remarks are as follows. We compute the solution of the systems under the standard assumption which abstracts from the filtering problem and for the case in which there is a role for some variables as information variables. The criteria we assume is the minimization of the expected squared gap between actual values and long-run equilibrium values of variables (2). It is also worth recalling our comments in chapter 2 (section 2.4) concerning the feasibility of an interest rate peg. In particular, this policy may cease to be viable if it generates indeterminacy. As our analysis in that chapter showed, however, this is unlikely. A pure interest rate peg is entirely consistent with a well-defined solution provided the controller announces a particular control rule which involves feedback on forward expectations. Moreover, and very conveniently, this general rule is observationally equivalent to the simple contemporaneous feedback rule between the money stock and the interest rate. Without loss of generality, therefore, we may restrict attention to just the latter. Determinacy under an interest
rate peg is necessary for the present analysis. The information content of the interest rate is nil only under this policy. Any amount of interest rate variation, however small, would render our analysis inoperative. Note, however, that we may be assuming an informational advantage in favour of the controller with this type of rule. Recall that the policy rule which admits a unique solution under an interest rate peg can be implemented in either the above form (a feedback on private sector expectations) or in the form of a feedback on current variables and current stochastic disturbances. Thus, for the latter, whilst the private sector will be assumed to have incomplete price information, the policy maker must be able to observe current prices and shocks\(^3\). Finally, we are concerned with examining the change in the relative superiority of one policy over another as one moves from the non-filtering to the filtering solutions. Thus, though a particular policy may remain optimal, its relative superiority is likely to change. Other considerations, such as the practical difficulty in actually controlling a particular policy variable, may then incline one to regard the other more or less favourably.

The remainder of the chapter is as follows. Section 6.2 considers an open economy. The set of information variables comprises the exchange rate and the interest rate. Exogenising the latter still enables extraction of some auxiliary information via the former. Some unambiguous conclusions emerge. Section 6.3 examines a closed economy model in which the set of information variables now comprises only the interest rate. Then control over this variable will deprive agents of any contemporaneous information. In this case, the results are more sensitive to the relative magnitudes of the structural parameters of the model. Concluding remarks are contained in section 6.4.
6.2 An Open Economy Model

The relevant system has been described in chapter 5 (section 5.3, equations (5.3.1)-(5.3.3)). To this we append the control rule given in chapter 3 (section 3.2, equation (3.2.5)) specifying a contemporaneous relationship between the money stock and the interest rate. Recapitulating, the formal structure is defined by equations (6.2.1) - (6.2.4) below:

\[ P_{t+1} - P_t = \lambda(e_t - P_t) + \epsilon_{lt} \quad 0 < \lambda < 1 \]  
\[ m^d_t = P_t - \gamma r_t + \epsilon_{lt} \quad \gamma > 0 \]  
\[ \eta_1(r_t - e_t^{e_{t+1,t}} + e_t) + \eta_2(e_t - P_t) + \nu_t = 0 \quad \eta_1 > 0(i=1,2) \]  
\[ m^s_t = \phi r_t \quad \phi \geq 0 \]

where
- \( p \) = natural logarithm of the price level
- \( e \) = natural logarithm of the nominal exchange rate
- \( m^d \) = natural logarithm of the nominal demand for money
- \( r \) = nominal rate of interest
- \( m^s \) = natural logarithm of the nominal supply of money
- \( \epsilon_k, \nu \) = stochastic disturbances (k=1,3).

As usual, all variables are measured as deviations from long-run equilibrium and \( e_{t+1,t}^e \) is defined by \( e_{t+1,t}^e = E(e_{t+1,t} | n_t) \) with \( E(\cdot) \) the mathematical conditional expectations operator and \( n \) the information set.

Equation (6.2.1) is the inflation mechanism where the nominal exchange rate, \( e \), is the domestic currency price of foreign exchange (a rise in \( e \) is a depreciation). Thus, prices respond sluggishly
to changes in real competitiveness. The disturbance $\epsilon_1$ is an expenditure (or price) shock. Equation (6.2.2) describes a standard money demand function depending on prices and interest rates, and subject to a stochastic perturbation, $\epsilon_3$. The balance of payments equilibrium condition is summarized in equation (6.2.3) where the disturbance $\nu$ comprises random shocks to capital and trade flows, $\nu_1$ and $\nu_2$ respectively: $\nu = \eta_1 \nu_1 + \eta_2 \nu_2$. As in chapter 5 (section 5.3), $\nu_1$ is interpreted as random deviations from interest parity.

It is also recalled that $\eta_1 = \infty$ approximates perfect capital mobility. As usual, all stochastic shocks are assumed to be independently distributed white noise processes. Finally, equation (6.2.4) specifies the control rule. The reader is reminded of the two polar cases $\phi=0$ and $\phi=\infty$. The former is a money stock peg and the latter an interest rate peg.

As stated in the introduction, we compute two solutions of the system. In the first, agents face no filtering problem. This is the standard approach to analysing this type of model and amounts to the assumption of full contemporaneous information about all variables. In the second, the information set is restricted and agents are forced to infer the behaviour of the unobservable variables. In both cases, the controller is assumed to be concerned with minimizing the expected squared deviation of the price level around its long-run equilibrium level. The loss function is therefore just

$$E(J) = \sigma_p^2$$

(6.2.5)

where $\sigma_p^2$ is the asymptotic variance of prices.

In order to derive the two solutions we follow appendix B and reduce the system to minimal dimension. Thus, eliminating $m_t^d = m_t^s = m_t$ and $r_t$ from equations (6.2.1) - (6.2.4) the
state-space representation is

\[
\begin{bmatrix}
P_{t+1} \\
e^e_{t+1, t}
\end{bmatrix} =
\begin{bmatrix}
1 - \lambda & \lambda \\
\alpha_1 & \alpha_2
\end{bmatrix}
\begin{bmatrix}
P_t \\
e_t
\end{bmatrix} +
\begin{bmatrix}
u^1_t \\
u^2_t
\end{bmatrix}
\tag{6.2.6}
\]

where \( \alpha_1 = b - n \)
\( \alpha_2 = l + n \)
\( n = \eta_1 - \eta_2 \)
\( b = (\phi + \gamma)^{-1} \)
\( u^1_t = e^1_t \)
\( u^2_t = be^3_t + e^4_t \)
\( e^4_t = \eta_1 e^1_t \).

The characteristic equation of (6.2.6) is

\[
f(\tau) = \tau^2 - (2n - \lambda)\tau + (1 + n - \lambda(1+b)) = 0
\tag{6.2.7}
\]

implying

\[
\tau_1, \tau_2 = \frac{(2n - \lambda) \pm \sqrt{(2n - \lambda)^2 - 4(1 + n - \lambda(1+b))}}{2}
\tag{6.2.8}
\]

Obviously \( p \) is predetermined and \( e \) is non-predetermined. Hence, \(|\tau_1| < 1 \) and \(|\tau_2| > 1 \), say, for saddlepoint stability. Equivalently, there must be one end only sign change in the sequence,

\[
\frac{1}{2(2n - \lambda) - \lambda b}, \frac{-\lambda b}{2(2n - \lambda) - \lambda b}
\tag{6.2.9}
\]

or \([2(2n - \lambda) - \lambda b]^{-1}\lambda b > 0 \).

The case of a money supply peg was examined in chapter 5 (section 5.3). For this we recall the following (set \( \phi = 0, \ b = \gamma^{-1} \).
Since $2+n-\lambda > 0$, $\tau_2(m) > 1$ unambiguously and regardless of the degree of capital mobility. In addition, relatively low values for $\gamma$ imply $\tau_1(m) < 0$, but violate the test function condition. Thus, assume a sufficiently high value for $\gamma$ in which case $0 < \tau_1(m) < 1$. For an interest rate peg ($\phi = 0$, $b = 0$), equation (6.2.7) reduces to $f(\tau)(t) = (1-\gamma)(1+n-\lambda-\tau) = 0$. Hence, there is a unit root (as indicated by the zero value for the relevant test function condition). Recalling the discussion in chapter 2 (section 2.4), we ignore this (treat it as unstable; $\tau_2(t) = 1$). Then $\tau_1(t) = 1+n-\lambda$. Provided $n < \lambda$, the system possesses a unique convergent solution. In this respect, the requirement of a well-defined solution under an interest rate peg imposes a lower limit on the degree of capital mobility. Under both policies, the fact that we assume $0 < \tau_1 < 1$ always precludes cyclical behaviour.

Appendix B gives the solutions for the exchange rate and prices as

\begin{equation}
\begin{aligned}
e_t &= -m_2p_t + d_1 u_{t-1} - a_2^{-1} b_3 e_t - a_2^{-1} e_t - d_1 u_{t-1, t} \\
&\quad + d_2 b_{3t, t} + d_2 e_{4t, t} \\
\end{aligned}
\tag{6.2.10}
\end{equation}

\begin{equation}
\begin{aligned}
p_{t+1} &= \tau_1 p_t + \lambda_1 u_{t-1} + u_t - \lambda a_2^{-1} b_3 e_t - \lambda a_2^{-1} e_t \\
&\quad - \lambda d_1 e_{t-1, t} + \lambda d_2 b_{3t, t} + \lambda d_2 e_{4t, t} \\
\end{aligned}
\tag{6.2.11}
\end{equation}

\begin{equation}
\begin{aligned}
e_{t+1} &= \tau_1 e_t - f d_1 u_{t-1} - a_2^{-1} a_1 u_t + f a_2^{-1} b_3 e_t + f a_2^{-1} e_t \\
&\quad - a_2^{-1} b_{3t+1} - a_2^{-1} e_{4t+1} + f d_1 u_{t-1, t} - f d_2 b_{3t, t} \\
&\quad - f d_2 e_{4t, t} - d_1 u_{t+1, t} + d_2 b_{3t+1, t+1} + d_2 e_{4t+1, t+1} \\
\end{aligned}
\tag{6.2.12}
\end{equation}
\[
m_{21} = \frac{-a_1}{1-\lambda - \tau_2} = \frac{-a_2 - a_1}{\lambda} \quad (6.2.13)
\]

where
\[
d_1 = m_{21} - a_2^{-1} a_1
\]
\[
d_2 = a_2^{-1} - \tau_2
\]
\[
f = \lambda m_{21} + \tau_1.
\]

6.2(A) Model Solution with No Filtering

The popular approach to examining this type of system is to assume full contemporaneous information about all variables (and hence all disturbances). The current information set is then simply

\[
\Omega_{t-NF} = \{e_{t-j}, \tau_{t-j}, r_{t-j}, p_{t-j}, m_{t-j}, \Gamma, \Sigma[i \geq 0]\} \quad (6.2.14)
\]

where \(r^f\) = foreign nominal rate of interest

with \(\Gamma\) a vector of all structural parameters and \(\Sigma\) a description of the variance-covariance structure of stochastic disturbances.

(The superscript NF refers to the absence of a filtering problem.)

The complete solution then obtains by setting \(u_{1e} = u_1, e_3^e = e_3\) and \(e_4^e = e_4\) in equations (6.2.10) - (6.2.12):

\[
e_{t-NF} = - m_{21} p_t - \tau_2^{-1} b e_{3t} - \tau_2^{-1} e_{4t} \quad (6.2.15)
\]

\[
p_{t+1-NF} = \tau_1 p_t - \lambda \tau_2^{-1} b e_{3t} - \lambda \tau_2^{-1} e_{4t} + u_{1t} \quad (6.2.16)
\]

\[
e_{t+1-NF} = \tau_1 e_{t-NF} - m_{21} u_{1t} + f \tau_2^{-1} b e_{3t} + f \tau_2^{-1} e_{4t} - \tau_2^{-1} b e_{3t+1} - \tau_2^{-1} e_{4t+1} \quad (6.2.17)
\]

As usual, equation (6.2.15) gives the jump variable, \(e\), as a linear combination of the predetermined variable, \(p\), and the
composite disturbance $u^2 = \varepsilon_3 + \varepsilon_4$. Thus

$$\frac{\partial e_t^{NF}}{\partial p_t} = -\frac{m_{21}}{2} \triangleq 0, \quad \frac{\partial e_t^{NF}}{\partial e_3_t} = -\tau_1^{-1}b \triangleq 0$$

$$\frac{\partial e_t^{NF}}{\partial e_4_t} = -\tau_2^{-1} \triangleq 0. \quad (6.2.18)$$

The responses in equation (6.2.18) are familiar from previous analyses (see chapter 3, section 3.3(F); chapter 5, section 5.3). Since $m_{21} \triangleq 0$ in equation (6.2.13), $\frac{\partial e_t^{NF}}{\partial p_t} \delta p_t \neq 0$ depending on the degree of capital mobility and the policy regime. For perfect capital mobility ($n = 0$) and a money supply peg ($b = \gamma^{-1}$), $m_{21}(\bar{m}) > 0$ and $\frac{\partial e_t^{NF}}{\partial p_t}(\bar{m}) < 0$ because of the induced interest rate variations following price movements. This is obviously neutralized if interest rates are fixed ($b = m_{21}(\bar{r}) = 0$). For the case of imperfect capital mobility it is possible that $m_{21}(\bar{m}) < 0$ so that $\frac{\partial e_t^{NF}}{\partial p_t}(\bar{m}) > 0$. This is due to price induced fluctuations in competitiveness operating on the trade balance. Clearly, $\frac{\partial e_t^{NF}}{\partial p_t}(\bar{r}) > 0$ for imperfect capital mobility. Perturbations to money demand have the usual effect which is completely negated when interest rates are pegged. Finally, foreign disturbances have the indicated effects by construction.

To complete the description of the full information (no filtering) case, we note the asymptotic variance of prices from equation (6.2.16) as

$$\sigma^2_{e^P} = \frac{-2 + \lambda^2(-\tau_2^{-1}b)^2 - 2\lambda^2(-\tau_2^{-1})^2 - 2}{1 - \tau_1^2} \sigma_1^2 + \lambda^2(-\tau_2^{-1})^2\sigma_4^2 \quad (6.2.19)$$

where $\sigma^2_k$ is the asymptotic variance of $e_k$ ($k = 1,3,4$). The reason
for writing $\sigma_p^2$ in the form of equation (6.2.19) will become clear in what follows.

6.2(B) Model Solution with Filtering

The more realistic informational assumption posits some partial ignorance. In particular, from the discussion in section 6.1, we assume

$$\Omega_t^F = \{e_t, r_t, r_t^f, e_{t-j}, r_{t-j}, r_{t-j}^f, p_{t-j}, m_{t-j}\}$$

Thus, contemporaneous information about only the exchange rate and interest rates is available with information about the money stock and prices accruing after a lag. Recall that this is entirely consistent with the specification of the model in which the actual money stock and actual prices appear in the behavioural equations (see chapter 5, section 5.3).

The important property of this type of information structure is its non-invariance with respect to alternative control regimes. To be specific, though the initial endowment of information remains the same, the amount and nature of auxiliary information are unlikely to remain unaltered as the controller's policy stance shifts. This is so because of the effect of control on the information contents of currently observable variables. In addition, chapter 5 (section 5.3) demonstrated the critical dependence of the nature of auxiliary information on the degree of capital mobility. These properties are returned to below.

(As a slight digression of some interest, we are forced to consider each policy independently. This is so because of the
following. Suppose that a general solution (with no restrictions on $\phi$) is computed. Suppose also another solution with $\phi = \infty$ imposed at the outset. Then imposing $\phi = \infty$ in the former yields expressions which are not the same as those in the latter. This is because the expressions for the optimal filtered estimates of the disturbances are actually identical in each case. This is peculiar since for $\phi = \infty$, the interest rate conveys no information. Clearly, the results obtained for the second procedure (imposing $\phi = \infty$ at the outset) are correct. The contradiction arises because of the discontinuity as one moves from the $\phi \to \infty$ case to the $\phi = \infty$ case; with $\phi$ close to, but not equal to infinity, $r_t$ still fluctuates and therefore provides useful information. The discontinuity at $\phi = \infty$, however, is when $r_t$ is fixed absolutely and is therefore redundant as an information variable. This feature applies also to the model of section 6.3. The interested reader is referred to appendix B for the formal demonstration.)

Consider, then, a money supply peg. The vector of currently observable variables is $s_t(m) = [r_t e_t]^T$. This is related to the state vector in equation (6.2.6) as

$$s_t(m) = \begin{bmatrix} r_t \\ e_t \end{bmatrix} = \begin{bmatrix} y^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_t \\ e_t \end{bmatrix} + \nu_t$$

(6.2.21)

where $\nu_t = (\gamma^{-1} e_{3t} 0)^T$

and we have used $b = \gamma^{-1}$ for $\phi = 0$. Denote by $A$ and $C$ the $2 \times 2$ right hand side matrices in equations (6.2.6) and (6.2.21) respectively. Partition these conformably such that $A = [(A_1^T A_2^T)^T]$ and $C = [C_1 C_2]$. Appendix B shows that equation (6.2.21) can be re-written as
\[ s_t(m) = FA^1 \begin{bmatrix} P_{t-1} \\ e_{t-1} \end{bmatrix} + C_2 a_2^{-1} e_{t+1,t} + F_{t-1}^1 \\
- C_2 a_2^{-1} u^2_t(m) + v_t \\
= FA^1 \begin{bmatrix} P_{t-1} \\ e_{t-1} \end{bmatrix} + C_2 a_2^{-1} e_{t+1,t} + F_{t-1}^1 \\
- C_2 a_2^{-1} e_{4t} + G e_{3t} \] (6.2.22)

where \[ P = C_1 - C_2 a_2^{-1} a_1(m) \]
\[ G = [\gamma^{-1} - a_2^{-1} \gamma^{-1}]^T \]

and the notation emphasises the dependence of terms on \( \phi = 0 \).

The reader will recognise this information structure as being identical to that in chapter 5 (section 5.3). Equation (6.2.22) shows that observation of both the current exchange rate and the current interest rate conveys information about all shocks impinging on the system. We recall two results obtained from chapter 5 (section 5.3): first, under perfect capital mobility, the external shock is known with certainty (Proposition 5.3.4); second, this is no longer true for imperfect capital mobility (Proposition 5.3.6).

Obviously, we need not repeat all of the steps involved in solving this system. The reader is, however, reminded of the salient features. We choose to do this partly because of the slightly different notation used here and partly for direct comparison with the interest rate peg case. (It will also avoid the necessity for continually referring back to chapter 5 (section 5.3)). The optimal estimates of the disturbances can be computed from equation (6.2.22) as
\[
\begin{bmatrix}
\varepsilon_{1,t-1,t(m)} \\
\varepsilon_{4t,t(m)} \\
\varepsilon_{3t,t(m)}
\end{bmatrix}
= \begin{bmatrix}
-k_2F^T \\
-k_2C_2 + k_2T \\
-k_2C_3 + k_2G
\end{bmatrix} \Pi(F_{t-1}^T - a_1^2c_2e_{4t}^T + G_{4t}^T) (6.2.23)
\]

where \( \Pi = \left( F_{t-1}^2T + a_2^2C_2 + a_2^2T + G_{3T}^2 \right)^{-1} \)

and recalling that \( u_{t-1}^1 = \varepsilon_{1t-1} \) so that \( \sigma_{u_1}^2 = \sigma_{\varepsilon_1}^2 \), where \( \sigma_{u_1}^2 \) is the

asymptotic variance of \( u^h \) (\( h = 1, 2 \)). The procedure summarised in

chapter 5 (section 5.3) then yields

\[
\begin{align*}
\varepsilon_{1,t-1,t(m)} &= \pi(m)\sigma_{1}^{-2}((\sigma_4^2 + \sigma_3^2)u_{t-1}^1 - \sigma_3\sigma_4^2e_{4t} + \sigma_3^2e_{3t}) (6.2.24) \\
\varepsilon_{3t,t(m)} &= \pi(m)\sigma_{3}^{-2}((\sigma_4^2 + \sigma_3^2)u_{t-1}^1 + \sigma_3\sigma_4^2e_{4t} + \sigma_3^2e_{3t}) (6.2.25) \\
\varepsilon_{4t,t(m)} &= \pi(m)\sigma_{4}^{-2}((\sigma_4^2 + \sigma_3^2)e_{4t} + \sigma_3\sigma_4^2e_{3t} - \sigma_3\sigma_4^2u_{t-1}^1) (6.2.26)
\end{align*}
\]

where \( \pi(m) = (\sigma_3^2(\sigma_4^2 + \sigma_3^2) + \sigma_3^2\sigma_4^2)^{-1} \).

Substituting equations (6.2.24) - (6.2.26) into equations

(6.2.10) - (6.2.12) gives

\[
\begin{align*}
e_t^F(m) &= -m_2^1(m)p_t + \Delta(m)u_{t-1}^1 + \Delta(m)e_{3t} + \Delta(m)e_{4t} (6.2.27) \\
p_{t+1}^F(m) &= \tau(m)p_t + \lambda\Delta(m)u_{t-1}^1 + \lambda\Delta(m)e_{3t} + \lambda\Delta(m)e_{4t} + u_t (6.2.28) \\
e_{t+1}^F(m) &= \tau(m)e_t^F(m) - f(m)\Delta(m)u_{t-1}^1 - f(m)\Delta(m)e_{3t} - f(m)\Delta(m)e_{4t} + \Delta(m)^u_{t}^1 + \Delta(m)^e_{3t} + \Delta(m)^e_{4t} + (6.2.29)
\end{align*}
\]

where the following definitions are noted:
\[\Delta(m) = d_1(m) - \phi^1(m)d_1(m) + \phi^2(m)d_2(m)\]

\[\Delta^*(m) = -a_2^{-1} \gamma^{-1} - \phi^1(m)d_1(m) + \phi^2(m)d_2(m)\]

\[\Delta(m) = -a_2^{-1} + \phi^1(m)d_1(m) + \phi^2(m)d_2(m)\]

\[\Delta^*(m) = -a_2^{-1} \alpha_1(m) - \phi^1(m)d_1(m) + \phi^2(m)d_2(m)\] (6.2.30)

\[\phi^1(m) = n(m)\sigma_1^{-2}(n^2\sigma_3^{-2} + \sigma_4^2) \quad \phi^2(m) = n(m)\sigma_1^{-2} \sigma_3^{-2}(n^2\sigma_1^{-2} + \sigma_4^2) + n\sigma_1^{-2}\sigma_4^{-2}\]

\[\phi^1(m) = n(m)\sigma_1^{-2} \sigma_3^{-2} \quad \phi^2(m) = n(m)\sigma_3^{-2}\gamma^{-1}(n^2\sigma_1^{-2} + \sigma_4^2) + n\sigma_1^{-2}\sigma_4^{-2}\] (6.2.31)

Now recall that \(\phi^1(m) = 1 - \phi^2(m)\alpha_1(m)\), \(\phi^2(m) = \gamma^{-1} - \phi^1(m)\alpha_1(m)\)
and \(\phi^2(m) = 1 + \phi^1(m)\alpha_1(m)\). Appropriate substitution into equations (6.2.27) - (6.2.29) finally reveals

\[e^F_t(m) = -m_2(m)p_t + \phi^2(m)h_1(m)u_t + [-\tau^{-1}_2(m)\gamma^{-1} - \phi^1(m)h_2(m)]e_{3t} + [-\tau^{-1}_2(m) + \phi^1(m)h_2(m)]e_{4t}\] (6.2.32)

\[p^F_{t+1}(m) = \tau_1(m)p_t + \lambda\phi^2(m)h_1(m)u_{t-1} + \lambda[-\tau^{-1}_2(m)\gamma^{-1} - \phi^1(m)h_2(m)]e_{3t} + \lambda[-\tau^{-1}_2(m) + \phi^1(m)h_2(m)]e_{4t} + u^1_t\] (6.2.33)

\[e^F_{t+1}(m) = \tau_1(m)e^F_t(m) - f(m)\phi^2(m)h_1(m)u_{t-1}
- f(m)[-\tau^{-1}_2(m)\gamma^{-1} - \phi^1(m)h_2(m)]e_{3t} - f(m)[-\tau^{-1}_2(m)]
+ \phi^1(m)h_2(m)]e_{4t} + [-m_2(m) + \phi^2(m)h_1(m)]u_t
+ [-\tau^{-1}_2(m)\gamma^{-1} - \phi^1(m)h_2(m)]e_{3t+1}
+ [-\tau^{-1}_2(m) + \phi^1(m)h_2(m)]e_{4t+1}\] (6.2.34)
where $h_1 = m_1 a_1 - 1 - 1$
\[ h_2 = m_2 - 1. \]

Equations (6.2.32) - (6.2.34) are the partial informational analogues to equations (6.2.15) - (6.2.17) for a money supply peg. Thus, the asymptotic variance of prices for the former is given from equation (6.2.33) as

\[
\sigma^2_{P_t} = \frac{[1 + \lambda \theta_t^2(m)h_1(m)]^2 + \lambda \theta_t^2(m)h_2(m)]^2}{1 - \tau_1^2(m)}
\]

When the controller fixes the interest rate, this variable is deprived of any useful information. Setting $\phi = \omega$ so that $b = \tau_t = 0$, the vector of observable variables is just the scalar $s_t(\bar{r}) = e_t$,

\[
s_t(\bar{r}) = e_t = [0 \ 1] \begin{bmatrix} p_t \\ e_t \end{bmatrix}.
\]

An equivalent expression is derived in appendix B as

\[
s_t(\bar{r}) = -a_2 a_1(\bar{r})A_1 \begin{bmatrix} p_t-1 \\ e_{t-1} \end{bmatrix} + a_2 e_t + a_2 a_1(\bar{r})u_{t-1} - a_2 e_{t-1}
\]

where the notation now emphasizes the dependence of terms on $\phi = \omega$.

Note that, unlike equation (6.2.22), equation (6.2.37) shows that there is no auxiliary information about money demand shocks.

The expectation of these is therefore zero. Note also a further interesting feature. Under conditions of perfect capital mobility, $a_1(\bar{r}) = 0$, $a_2 = 1$ and $s_t(\bar{r}) = e_t e_{t+1} e_t = e_t$; external shocks are
known with certainty but no information is conveyed about internal disturbances. These properties are explained as follows.

The absence of information about money demand shocks is the immediate consequence of an interest rate peg which completely nullifies these shocks. Similarly, for perfect capital mobility, domestic price shocks have no effects either. The exchange rate fluctuates only because of foreign disturbances. When capital is less mobile, domestic price shocks can impinge on the exchange rate directly via the trade balance. This served to deliver an additional source of confusion under a money stock peg. For an interest rate peg, however, it tends to alleviate the problem in the sense that information about prices is now forthcoming. Nonetheless, since price shocks are still not completely known, neither are foreign perturbations. In this respect, therefore, imperfect capital mobility complicates the inference problem by contaminating information about external shocks. Comparing the two policy regimes, this establishes the critical influence of control on the information structure.

On the basis of equation (6.2.37), we have

\[
\begin{bmatrix}
\le_{t-1, t}(\bar{r})
\\le_{4t, t}(\bar{r})
\end{bmatrix}
= \begin{bmatrix}
\le_{2} a_{1}(\bar{r})
\le_{2} a_{1}(\bar{r})
\end{bmatrix}
\begin{bmatrix}
\sigma_{1}(\bar{r})^{2} + \sigma_{4}^{2}
\sigma_{4}^{2}
\end{bmatrix}
\le_{1, t-1} + \varepsilon_{4t}
\]

or

\[
\le_{t-1, t}(\bar{r}) = \pi(\bar{r}) a_{1}(\bar{r}) u_{t-1} + \varepsilon_{4t}
\]

\[
\le_{4t, t}(\bar{r}) = \pi(\bar{r}) a_{1}(\bar{r}) u_{t-1} + \varepsilon_{4t}
\]

where \( \pi(\bar{r}) = (\sigma_{1}(\bar{r})^{2} + \sigma_{4}^{2})^{-1} \).
(Clearly, the above discussion on the consequences of perfect capital mobility is verified in equations (6.2.39) - (6.2.40) by setting \( a_1(\cdot) = 0 \); \( u_{t-1,t}(\cdot) = 0 \), \( \epsilon_{4t,t} = \epsilon_{4t} \).)

The solution now proceeds in the usual way. Substitute equations (6.2.39) - (6.2.40) into equations (6.2.10) - (6.2.12) for \( \phi = \infty (b = 0) \) and obtain

\[
e^F_t(\cdot) = -m_{21}(\cdot)p_t + \Delta(\cdot)u^1_{t-1} + \bar{\Delta}(\cdot)\epsilon_{4t} \tag{6.2.41}
\]

\[
p^F_{t+1}(\cdot) = r_1p_t + \lambda\Delta(\cdot)u^1_{t-1} + \lambda\bar{\Delta}(\cdot)\epsilon_{4t} + u^1_t \tag{6.2.42}
\]

\[
e^F_{t+1}(\cdot) = r_1^e_t(\cdot) + f(\cdot)\Delta(\cdot)u^1_{t-1} - f(\cdot)\bar{\Delta}(\cdot)\epsilon_{4t}
\]

\[
+ \Delta^*(\cdot)u^1_t + \bar{\Delta}(\cdot)\epsilon_{4t+1} \tag{6.2.43}
\]

with

\[
\Delta(\cdot) = d_1(\cdot) - \theta^1(\cdot)d_1(\cdot) + \theta^2(\cdot)d_2(\cdot)
\]

\[
\bar{\Delta}(\cdot) = -a_2^{-1}d_1(\cdot) + \bar{\theta}^2(\cdot)d_2(\cdot)
\]

\[
\Delta^*(\cdot) = -a_2^{-1}a_1(\cdot) - \theta^1(\cdot)d_1(\cdot) + \theta^2(\cdot)d_2(\cdot) \tag{6.2.44}
\]

\[
\theta^1(\cdot) = \pi(\cdot)a_1(\cdot)\sigma^2 \quad \theta^2(\cdot) = \pi(\cdot)a_1(\cdot)\sigma^4
\]

\[
\bar{\theta}^1(\cdot) = \pi(\cdot)a_1(\cdot)\sigma^2 \quad \bar{\theta}^2(\cdot) = \pi(\cdot)\sigma^4 \tag{6.2.45}
\]

Noting that \( \theta^1(\cdot) = 1 - \theta^2(\cdot)a_1^{-1}(\cdot) \) and \( \bar{\theta}^2(\cdot) = 1 - \bar{\theta}^1(\cdot)a_1(\cdot) \), simple manipulation in equations (6.2.41) - (6.2.43) then yields

\[
e^F_t(\cdot) = -m_{21}(\cdot)p_t + \theta^2(\cdot)h_1(\cdot)u^1_{t-1} + [-\tau_2^{-1}(\cdot) - \theta^1(\cdot)h_2(\cdot)]\epsilon_{4t} \tag{6.2.46}
\]
\[
\begin{align*}
\mathbf{F}_{t+1}(\overline{r}) &= \tau(\overline{r})p_t + \lambda \theta^2(\overline{r})h_1(\overline{r})u^{1}_{t-1} + \lambda[-\tau^{-1}(\overline{r}) - \Theta(\overline{r})h_2(\overline{r})]\varepsilon_{4t} \\
+ u^{1}_{t} & \quad (6.2.47) \\
\mathbf{F}_{t+1}(\overline{r}) &= \tau(\overline{r})e^{F}_t(\overline{r}) - f(\overline{r})\theta^2(\overline{r})h_1(\overline{r})u^{1}_{t-1} \\
- f(\overline{r})[-\tau^{-1}(\overline{r}) - \Theta(\overline{r})h_2(\overline{r})]\varepsilon_{4t} \\
+ [-\tau_{21}(\overline{r}) + \Theta^2(\overline{r})h_1(\overline{r})]u^{1}_{t-1} \\
+ [-\tau_{2}^{-1}(\overline{r}) - \Theta(\overline{r})h_2(\overline{r})h_2(\overline{r})]\varepsilon_{4t} & \quad (6.2.48)
\end{align*}
\]

with \( h_1 \) and \( h_2 \) defined as before and appropriately conditioned by \( \phi = \omega \).

Equations (6.2.46) - (6.2.48) are the partial information analogues to equations (6.2.15) - (6.2.17) for an interest rate peg. The most interesting feature about both of these is that for perfect capital mobility (\( a_1(\overline{r}) = m_{21}(\overline{r}) = h_2(\overline{r}) = \Theta(\overline{r}) = \Theta^2(\overline{r}) = \Theta^3(\overline{r}) = 0 \), \( a_2 = f(\overline{r}) = \tau_2^{-1}(\overline{r}) = 1 \)), they are identical even though the latter is derived on the assumption of full information. Thus, perfect capital mobility and an interest rate peg makes the solution of the system under partial information observational equivalent to that obtained under full information. The reason is as follows. When interest rates are pegged, there is no auxiliary information about internal disturbances; thus this policy means that such shocks have no implications for the exchange rate anyway, the behaviour of which is dictated solely by foreign shocks. Information about these is therefore absolute. In short, the absence of information about domestic shocks is innocuous.

To complete the description of the interest rate peg solution, it remains merely to note the asymptotic price variance obtained from equation (6.2.47):
Clearly, from equations (6.2.32) - (6.2.34) and (6.2.46) - (6.2.48), the effect of the partial information structure will, in general, be to modify the dynamic response of the system to exogenous shocks. The details of this are explicated below.

6.2(C) The Relative Merits of Monetary and Interest Rate Control

We now evaluate the relative merits of alternative control in the light of the inference problem discussed above. In particular, we will be concerned with examining the change in the relative superiority of alternative policies as the information structure shifts. Tables 6.2(A) and 6.2(B) illustrate. The precise determination of the inequalities is given below.

The first points to note are the standard results for the no filtering (full information) solution (see chapter 2, section 2.2; chapter 3, sections 3.3 - 3.4 for further discussion). Hence,

\[ \sigma_p^2(m, e_1) > \sigma_p^2(\bar{m}, e_1), \quad \sigma_p^2(m, e_3) > \sigma_p^2(\bar{m}, e_3) = 0 \quad \text{and} \quad \sigma_p^2(m, e_4) > \sigma_p^2(\bar{m}, e_4). \]

These can be obtained by applying the implicit function theorem to the polynomial in equation (6.2.7) to obtain \( \frac{\partial \tau_1}{\partial \phi} \) (i = 1, 2):

\[ \frac{\partial \tau_1}{\partial \phi} = \frac{\partial f(\tau_1)/\partial \phi}{\partial f(\tau_1)/\partial \tau_1} \quad (i = 1, 2). \]

(6.2.50)

It is easily verified that \( \frac{\partial \tau_1}{\partial \phi} > 0 \) and \( \frac{\partial \tau_2}{\partial \phi} < 0 \) or \( \tau_1(\bar{m}) < \tau_1(\bar{\tau}) \) and \( \tau_2(\bar{m}) > \tau_2(\bar{\tau}) \). The intuition is familiar. Under a money supply peg, a price shock causes interest rate fluctuations. For perfect capital mobility, the induced exchange rate variation
<table>
<thead>
<tr>
<th>Disturbance</th>
<th>No filtering ( \sigma_{p}^{2}N_{F}(m) )</th>
<th>Filtering ( \sigma_{p}^{2F}(m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expenditure</td>
<td>( \frac{1}{1 - \tau_{1}(m)} ) ( \sigma_{1}^{2} )</td>
<td>( \frac{[1 + \lambda \theta_{2}(m)h_{1}(\bar{m})]^{2}}{1 - \tau_{1}(m)} ) ( \sigma_{1}^{2} )</td>
</tr>
<tr>
<td>Money demand</td>
<td>( \frac{\lambda^{2}[-\tau_{2}(m)\gamma - 1]^{2}}{1 - \tau_{1}(m)} ) ( \sigma_{2}^{2} )</td>
<td>( \frac{\lambda^{2}[-\tau_{2}(m)\gamma - 1 - \theta_{2}(m)h_{2}(\bar{m})]^{2}}{1 - \tau_{1}(m)} ) ( \sigma_{3}^{2} )</td>
</tr>
<tr>
<td>External</td>
<td>( \frac{\lambda^{2}[-\tau_{2}(m)]^{2}}{1 - \tau_{1}(m)} ) ( \sigma_{4}^{2} )</td>
<td>( \frac{\lambda^{2}[-\tau_{2}(m) - \theta_{2}(m)h_{2}(\bar{m})]^{2}}{1 - \tau_{1}(m)} ) ( \sigma_{4}^{2} )</td>
</tr>
</tbody>
</table>

Table 6.2(A): Asymptotic price variances under divergent information structures, money supply peg.
<table>
<thead>
<tr>
<th>Disturbance</th>
<th>No filtering $\sigma_p^{2NP}(\bar{r})$</th>
<th>Filtering $\sigma_p^{2P}(\bar{r})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expenditure</td>
<td>$\frac{1}{1 - \tau_1^2(\bar{r})} \frac{s^2}{\sigma_1}$</td>
<td>$\frac{[1 + \lambda \theta^2(\bar{r})h_1(\bar{r})]^2}{1 - \tau_1^2(\bar{r})} \frac{s^2}{\sigma_1}$</td>
</tr>
<tr>
<td>Money demand</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>External</td>
<td>$\frac{\lambda^2[-\tau_2^{-1}(\bar{r})]^2}{1 - \tau_1^2(\bar{r})} \frac{s^2}{\sigma_4}$</td>
<td>$\frac{\lambda^2[-\tau_2^{-1}(\bar{r}) - \theta^1(\bar{r})h_2(\bar{r})]^2}{1 - \tau_1^2(\bar{r})} \frac{s^2}{\sigma_4}$</td>
</tr>
</tbody>
</table>

Table 6.2(B): Asymptotic price variances under divergent information structures, interest rate peg.
is stabilizing. For imperfect capital mobility, the effect of the price shock may induce destabilizing movements in the exchange rate. Permitting interest rates to vary will offset these. Fixing interest rates obviously neutralizes any monetary shocks. Foreign disturbances imply exchange rate fluctuations which feed into prices and interest rates. The optimal policy is therefore explained as for the case of domestic price shocks.

Of more interest are the implications of restricting the information set.

Consider, first, a money supply peg summarized in table 6.2(A) (see chapter 5, section 5.3, for further discussion). Then for expenditure shocks, the effect of the inference problem may be to increase or decrease the volatility of prices. It is easily verified that \( h_1(\cdot) > 0 \) unambiguously. Thus, for perfect capital mobility (\( \eta_1 = \omega, n = 0 \)), \( a_1(\cdot) \), \( \theta^2(\cdot) > 0 \) and \( \sigma_P^{2NF}(\cdot, \cdot) < \sigma_P^{2F}(\cdot, \cdot) \). By contrast, imperfect capital mobility (\( \eta_1 < \omega, n > 0 \)) implies \( a_1(\cdot), \theta^2(\cdot) < 0 \) so that \( \sigma_P^{2NF}(\cdot, \cdot) > \sigma_P^{2F}(\cdot, \cdot) \). These are explained as follows. From equations (6.2.15) and (6.2.32),

\[
\frac{\delta e^F_t}{\delta e^F_{it-1}} (\cdot) = \frac{\delta e^{NF}_t}{\delta e^{NF}_{it-1}} (\cdot) + \theta^2(\cdot)h_1(\cdot)
\]

where \( \frac{\delta e^{NF}_t}{\delta e^{NF}_{it-1}} (\cdot) = -m_{21}(\cdot) \neq 0 \). For perfect capital mobility, \( m_{21}(\cdot) > 0 \) so that the effect of the inference problem is to dampen the exchange rate appreciation. This occurs because agents misinfer the occurrence of other shocks. Expected future prices are lower than actual, and expected future exchange rates are less appreciated than actual. The current exchange rate therefore appreciates by less than under full information.

In terms of price volatility, the initial price perturbation is offset by less under partial information. For imperfect capital
mobility, \( m_{21}(\bar{m}) < 0 \) and the effect of imperfect information is to dampen the exchange rate depreciation. The reason is as above: current and future prices are underestimated inducing the expectation of a relatively more appreciated (or less depreciated) exchange rate. The incipient current depreciation is therefore partly offset. In this case, then, the less depreciated exchange rate means a greater stabilizing force on prices.

Turning to money demand shocks, note that \( a_1(\bar{m}) > 0 \) unambiguously but \( h_2(\bar{m}) \geq 0 \) according to whether \( a_1(\bar{m}) > 0 \). Hence, for perfect capital mobility (\( a_1(\bar{m}) > 0 \)), \( \sigma_{m, \varepsilon_3}(\bar{m}, \varepsilon_3) < \sigma_{m, \varepsilon_3}(\bar{m}, \varepsilon_3) \), whilst for imperfect capital mobility (\( a_1(\bar{m}) < 0 \)), \( \sigma_{m, \varepsilon_3}(\bar{m}, \varepsilon_3) > \sigma_{m, \varepsilon_3}(\bar{m}, \varepsilon_3) \). The intuition is obtained again from equations (6.2.15) and (6.2.32):

\[
\frac{\partial \varepsilon_t}{\partial \varepsilon_{3t}} (\bar{m}) = \frac{\partial \varepsilon_{3t}}{\partial \varepsilon_{3t}} (\bar{m}) - \frac{\partial \varepsilon_{3t}}{\partial \varepsilon_{3t}} (\bar{m}) h_2(\bar{m})
\]  

(6.2.52)

with \( \frac{\partial \varepsilon_{3t}}{\partial \varepsilon_{3t}} (\bar{m}) = -\tau_2(\bar{m})y^{-1} < 0 \). When capital is perfectly mobile, therefore, the exchange rate appreciation is exacerbated. Increased interest rates and an exchange rate appreciation signal (incorrectly) positive domestic price movements. Future expectations of relatively high prices and relatively more appreciated exchange rates then induce a further current appreciation over and above the full information response. It follows that prices are then destabilized by partial information. By contrast, imperfect capital mobility dampens the initial appreciation. The observed appreciation might be due to negative price shocks; but these are inconsistent with the observed rise in domestic interest rates. The latter imply the inference of positive price movements. Under imperfect capital mobility, this forces the expectation of relatively more depreciated exchange rates in the future, which is translated into a relatively lower current appreciation. This has the effect of stabilizing
subsequent price movements relative to the full information case.

Consider, finally, external shocks. In this case we have \( \delta_{1}(\bar{m}) = 0 \) for perfect capital mobility \((n = 0)\) and \( \delta_{1}(\bar{m}) > 0, h_2(\bar{m}) < 0 \) for imperfect capital mobility. Hence, in the former case \( \sigma_{p}^{2NF}(\bar{m}, \varepsilon_4) = \sigma_{p}^{2F}(\bar{m}, \varepsilon_4) \), whilst in the latter, \( \sigma_{p}^{2NF}(\bar{m}, \varepsilon_4) < \sigma_{p}^{2F}(\bar{m}, \varepsilon_4) \). Note again from equations (6.2.15) and (6.2.32) that

\[
\frac{\partial e^P}{\partial e^F}(\bar{m}) = \frac{\partial e_{4t}}{\partial e_{4t}}(\bar{m}) + \delta_{1}(\bar{m})h_2(\bar{m})
\]

(6.2.53)

with \( \frac{\partial e_{NF}}{\partial e_{4t}}(\bar{m}) = -\tau_2(\bar{m}) < 0 \). Perfect capital mobility implies an identical exchange rate appreciation under each information structure because agents have complete information about the external shock (see chapter 5, section 5.3, Proposition 5.3.4). This is no longer true when capital is imperfectly mobile, in which case the exchange rate appreciation is exacerbated. This is, of course, due to the misinterpretation of negative domestic price shocks, inviting the expectation of a relatively more appreciated future exchange rate. Obviously, then, price volatility is increased under partial information.

Table 6.2(B) contains the results for interest rate control. As we have seen, perfect capital mobility makes the informational assumption innocuous. This is realised in table 6.2(B) by noting that \( a_1(\bar{r}) = h_2(\bar{r}) = \delta^2(\bar{r}) = \delta^1(\bar{r}) = 0 \) for perfect capital mobility. Thus, \( \sigma_{p}^{2NF}(\bar{r}, \varepsilon_k) = \sigma_{p}^{2F}(\bar{r}, \varepsilon_k) \) \((k = 1, 3, 4)\). This is not generally the case for imperfect capital mobility. The exception where it does continue to be true is in the case of money demand shocks, for which \( \sigma_{p}^{2NF}(\bar{r}, \varepsilon_3) = \sigma_{p}^{2F}(\bar{r}, \varepsilon_3) = 0 \) as usual. For expenditure and external disturbances, however, the informational assumption makes a difference.
For the former (expenditure shocks), note that $h_1(r) > 0$ unambiguously. In addition, with imperfect capital mobility

$\alpha_1(r) < 0$, $\theta^2(r) < 0$. Hence $\sigma_p^{\text{NF}}(r, e_1) > \sigma_p^{\text{F}}(r, e_1)$. The reason is obtained from equations (6.2.15) and (6.2.46) whereby

$$
\frac{\partial e_t}{\partial e_{1t-1}} (r) = \frac{\partial e_t^{\text{NF}}}{\partial e_{1t-1}} (r) + \theta^2(r) h_1(r)
$$

(6.2.54)

with $\frac{\partial e_t^{\text{NF}}}{\partial e_{1t-1}} (r) = - m_2(r) > 0$. The effect of the inference problem is therefore to dampen the exchange rate depreciation. The reason should be familiar. The incipient depreciation is mistaken as symptotic of other shocks. Expected current and future prices are lower than actual and agents expect a relatively less depreciated future exchange rate. The current exchange rate reflects these expectations. With a less depreciated exchange rate, price volatility under partial information is obviously less than under full information.

Consider, finally, foreign disturbances. Then $h_2(r)$, $\theta^1(r) < 0$ for imperfect capital mobility; clearly, $\sigma_p^{\text{NF}}(r, e_4) > \sigma_p^{\text{F}}(r, e_4)$.

From equations (6.2.15) and (6.2.46)

$$
\frac{\partial e_t}{\partial e_{4t}} (r) = \frac{\partial e_t^{\text{NF}}}{\partial e_{4t}} (r) - \theta^1(r) h_2(r)
$$

(6.2.55)

with $\frac{\partial e_t^{\text{NF}}}{\partial e_{4t}} (r) = - \tau_2^1(r) < 0$. Hence partial information compounds the exchange rate appreciation; agents infer negative price shocks and the current exchange rate embodies the induced forward expectations. Prices then suffer greater volatility than in the full information case.

Increased clarity of the above results can, perhaps, be obtained from a diagrammatic representation. Figures 6.2(A), 6.2(B) and
Figure 6.2(A)(i) : Exchange rate and price level trajectories, money supply peg,

Figure 6.2(A)(ii) : Exchange rate and price level trajectories, interest rate peg

Figure 6.2(A) : Exchange rate and price level trajectories under divergent information structures and control, $e_{1-t_1} > 0$.

- $e^{NF}(e_1), p^{NF}(e_1), \eta_1 = \infty$
- $e^{NF}(e_1), p^{NF}(e_1), \eta_1 < \infty$
- $e^F(e_1), p^F(e_1), \eta_1 = \infty$
- $e^F(e_1), p^F(e_1), \eta_1 < \infty$
Figure 6.2(B)(i): Exchange rate and price level trajectories, money supply peg,

Figure 6.2(B)(ii): Exchange rate and price level trajectories, interest rate peg.

Figure 6.2(B): Exchange rate and price level trajectories under divergent information structures and control, $\varepsilon_3 t > 0$. 

- $e^{NF}(\varepsilon_3), p^{NF}(\varepsilon_3), \eta_1 = \infty$
- $e^{NF}(\varepsilon_3), p^{NF}(\varepsilon_3), \eta_1 < \infty$
- $e^{F}(\varepsilon_3), p^{F}(\varepsilon_3), \eta_1 = \infty$
- $e^{F}(\varepsilon_3), p^{F}(\varepsilon_3), \eta_1 < \infty$
Figure 6.2(C)(i) : Exchange rate and price level trajectories, money supply peg,

Figure 6.2(C)(ii) : Exchange rate and price level trajectories, interest rate peg

Figure 6.2(C) : Exchange rate and price level trajectories under divergent information structures and control, $e_{4t} > 0$.

- $e^{NF}(e_4); p^{NF}(e_4), \eta_1 = \infty$
- $e^{NF}(e_4); p^{NF}(e_4), \eta_1 < \infty$

- $e^{F}(e_4); p^{F}(e_4), \eta_1 = \infty$
- $e^{F}(e_4); p^{F}(e_4), \eta_1 < \infty$
6.2(C) illustrate the dynamic behaviour of prices for positive expenditure, money demand and external shocks, $\varepsilon_{t-1}, \varepsilon_t, \varepsilon_{4t} > 0$. Different assumptions about the information structure and degree of capital mobility are superimposed in alternative diagrams specifying the control regime. The dynamics support the algebraic analysis\(^{(5)}\). Together, they suggest the following conclusions.

Under conditions of perfect capital mobility and a money supply peg the effect of the inference problem is to unambiguously raise the volatility of prices. An exception is the case of external shocks for which there is no departure from the non-inference solution. By contrast, partial information has no effect whatsoever when interest rates are fixed. Under imperfect capital mobility, the inference problem has the same qualitative effects for both control regimes in the face of expenditure and external shocks. For the former, price volatility is reduced; for the latter, price volatility is increased. In the presence of money demand shocks, partial information has no effect when interest rates are pegged and reduces the price variance under a money stock peg. Collectively, these results suggest that, in terms of the relative merits of monetary and interest rate control, there may be a net gain to the latter. More provocatively it may pay the controller to deprive agents of information.

6.3 A Closed Economy Model

A notable feature of the open economy model in section 6.2 is that the set of information variables comprises two elements, the exchange rate and the interest rate. Thus, pegging the latter still permitted the dissemination of some auxiliary information via the former. In the closed economy model in this section we retain the assumption that the set of currently observable variables includes solely asset prices. For this model, however, the only asset price
is the interest rate. Interest rate control, therefore, now deprives
agents of any auxiliary information. It is of some interest to
examine the implications of this. In addition, we shall realize the
extent to which the results of section 6.2 may be qualified.

A useful and popular expositional framework is the Sargent and
Wallace (1975) model. This was described in chapter 3 (section 3.2,
equations (3.2.1a), (3.2.2a), (3.2.3a), (3.2.4)-(3.2.5)). We write
this as

\[
\begin{align*}
\bar{y}_t &= -\beta (r_t - p_{t+1,t} + p_{t,t}) + \epsilon_{1t} \quad \beta > 0 \quad (6.3.1) \\
\bar{y}_t &= \alpha (p_t - p_{t,t}) + \epsilon_{2t} \quad \alpha > 0 \quad (6.3.2) \\
\bar{m}_t &= p_t + \gamma_1 \bar{y}_t - \gamma_2 \bar{x}_t + \epsilon_{3t} \quad \gamma_j > 0 \quad (j=1,2) \quad (6.3.3) \\
\bar{m}_t &= \phi r_t \quad \phi \geq 0 \quad (6.3.4)
\end{align*}
\]

where \( \bar{y} \) = natural logarithm of real aggregate demand.
\( r \) = nominal rate of interest
\( p \) = natural logarithm of the price level
\( \bar{y} \) = natural logarithm of real aggregate supply
\( \bar{m} \) = natural logarithm of the nominal demand for money
\( \bar{m} \) = natural logarithm of the nominal supply of money
\( \epsilon_k \) = stochastic disturbances (k=1,2,3).

As usual all variables are measured as deviations from long run
equilibrium and \( p_{t+1,t} = E(p_{t+1} | \Omega_t) \) (i=0,1), where \( E(\cdot) \) is the
mathematical conditional expectations operator and \( \Omega \) is the
information set.

Equation (6.3.1) is the aggregate demand schedule relating
aggregate expenditure (negatively) to the real rate of interest ;
\( \epsilon_{1} \) is a random expenditure shock. Equation (6.3.2) is the supply
schedule whereby supply depends on perceived relative prices and stochastic perturbation, $\varepsilon_2$. Equation (6.3.3) is a money demand function depending on prices, output and interest rates, and subject to random fluctuation, $\varepsilon_3$. All shocks are independent white noise error terms with asymptotic variances $\sigma_k^2 (k = 1, 2, 3)$. Equation (6.3.4) is the monetary control rule with $\phi = 0$ and $\phi = \infty$ representing money supply and interest rate pegs respectively.

Within this framework, we distinguish three cases according to the assumption about the current information set, $\Omega_t$. These amount to the assumptions of full, partial and 'zero' contemporaneous information. The latter is the standard formulation of the model where the current information set includes only lagged values of variables (see chapter 3, section 3.2, equations (3.2.1a), (3.2.2a), (3.2.3a), (3.2.4) -(3.2.5)). Despite its popularity, this involves a peculiarity because implicit is the assumption that the current interest rate is currently observable; yet its value as an information variable is totally ignored. The solution is as if agents have no contemporaneous information. Both this and the full information specification abstract from the filtering problem. We return to these issues in more detail. For all informational assumptions, the objective function of the controller is

$$E(J) = \frac{1}{2} (a_0 \sigma_y^2 + (1 - a) \sigma_p^2) \quad 0 \leq a \leq 1 \quad (6.3.5)$$

where $\sigma_y^2$ and $\sigma_p^2$ are the asymptotic variances of output and prices respectively, and $a = 0$ and $a = 1$ are noted as the polar cases.

Note, first, that for all informational assumptions, equations (6.3.1) - (6.3.4) can be reduced to

$$p_{t+1,t}^e = kp_t + np_{t,t}^e + u_t \quad (6.3.6)$$
where \( k = b + aq \)

\[
\begin{align*}
    n &= 1 - aq \\
    q &= \beta^{-1} + \gamma_1 \\
    b &= (\phi + \gamma_2)^{-1} \\
    u_t &= -\beta^{-1}e_{1t} + qe_{2t} + be_{3t}.
\end{align*}
\]

Treating \( p \) as non-predicted, there is a unique convergent solution provided that there is one eigenvalue which lies outside the unit circle \(|\tau| > 1\), say. Taking expectations at time \( t \) through equation (6.3.6),

\[
\begin{align*}
    p_{t+1}^e &= (k + n) p_t^e + u_t^e \\
    \left(6.3.7\right)
\end{align*}
\]

Hence, \( \tau = k + n = 1 + b \\gamma^{-1} \). For a money supply peg \((\phi = 0)\), \( b = \gamma^{-1} \) and \( \tau(m) > 1 \). For an interest rate peg \((\phi = \infty)\), \( b = 0 \) and \( \tau(r) = 1 \). Ignoring the unit root, therefore, the system possesses a well-defined non-explosive solution irrespective of the policy regime.

Appendix B gives the solution for prices as

\[
\begin{align*}
    p_t &= k^{-1} \beta^{-1} e_{1t} - k^{-1} qe_{2t} - k^{-1} be_{3t} - s\beta^{-1} e_{1t} \\
    &\quad + sqe_{2t} + sb e_{3t} \\
    \left(6.3.8\right)
\end{align*}
\]

where \( s = k^{-1} - \tau^{-1} \)

and using the definition of \( u_t \). In addition, appendix B shows that \( p_t^e - p_t^e = -k^{-1} (u_t - u_t^e) \) so that substitution into equation (6.3.2) yields the solution for output as

\[
\begin{align*}
    y_t &= \alpha k^{-1} \beta^{-1} e_{1t} + k^{-1} be_{2t} - \alpha k^{-1} be_{3t} - \alpha k^{-1} \beta^{-1} e_{1t} \\
    &\quad + \alpha k^{-1} qe_{2t} + \alpha k^{-1} be_{3t} \\
    \left(6.3.9\right)
\end{align*}
\]
In what follows, the complete solutions for prices and output are obtained by deriving the expressions for the expectations terms, $e^e_{kt,t}$ ($k = 1,2,3$). Three solutions emerge according to the particular assumption we choose to make about the information set conditioning these expectations. The three alternative information structures are listed below in order of decreasing initial informational endowment.

6.3(A) Model Solution with No Filtering: Full Contemporaneous Information

The relevant information set is

$$\alpha^{NP}_t = \{\gamma_{t-j}, y_{t-j}, m_{t-j}, \Gamma, \Sigma | j > 0\} \quad (6.3.10)$$

where $\Gamma$ is a vector of all structural parameters and $\Sigma$ summarizes the variance-covariance structure of stochastic disturbances. Hence, information about all variables is instantaneously available. It follows that the solution of the system is to set $e^e_{kt,t} = e_{kt}$ in equations (6.3.8) - (6.3.9):

$$a^{NP}_t = \tau^{-1} \beta^{-1} e_{1t} - \tau^{-1} q e_{2t} - \tau^{-1} b e_{3t} \quad (6.3.11)$$

$$a^{NP}_t = e_{2t} \quad (6.3.12)$$

using the definitions of $s$ and $k$. The usual impact responses follow:
The consideration of the full information case may seem rather trivial. After all, output is entirely orthogonal to the system and (probably for precisely this reason) this informational assumption has received very little attention in the literature. The reason for including this case here is that it serves as a useful benchmark against which the other two cases can be judged. In particular, though both of the latter depart from the full information assumption, only the immediately proceeding structure is strictly correct. The remaining specification (which is the most popular version of the model) has some peculiarities. These render it inappropriate as a true reflection of the implications of a partial information framework.

From equations (6.3.11) - (6.3.12) we have

$$\left( \begin{array}{c} \frac{\partial a_{t}^{P\text{NF}}}{\partial \epsilon_{1t}} = -\tau^{-1} \beta^{-1} \gamma > 0 \ \\
\frac{\partial a_{t}^{P\text{NF}}}{\partial \epsilon_{2t}} = -\tau^{-1} \tau^{-1} q < 0 \ \\
\frac{\partial a_{t}^{P\text{NF}}}{\partial \epsilon_{3t}} = -\tau^{-1} \tau^{-1} b < 0 \end{array} \right) \quad \left( \begin{array}{c} \frac{\partial y_{t}^{\text{NF}}}{\partial \epsilon_{1t}} = 0 \\
\frac{\partial y_{t}^{\text{NF}}}{\partial \epsilon_{2t}} = 1 \\
\frac{\partial y_{t}^{\text{NF}}}{\partial \epsilon_{3t}} = 0 \end{array} \right) \quad (6.3.13)$$

$$a_{t}^{2\text{NF}} = (\tau^{-1} \beta^{-1}) \sigma_{1} + (\tau^{-1} q) \sigma_{2} + (\tau^{-1} b) \sigma_{3} \quad (6.3.14)$$

$$a_{t}^{2\text{NF}} = -\frac{\sigma_{2}}{\sigma_{2}} \quad (6.3.15)$$

6.3(B) Model Solution with Filtering

We retain the popular assumption that asset prices are the most obvious candidates for informational variables. Information about
prices, output and money is assumed to accrue with a lag. Hence,

\[ R^P_t = (r_t, r_{t-j}, P_{t-j}, \gamma_{t-j}, \beta_t, \cdots, \beta_{t-j}) \quad (6.3.16) \]

Note again that this information structure is entirely consistent with the model specification. Equations (6.3.1) - (6.3.3) reflect aggregated micro relationships. Individual demand for goods depends on the global (observable) interest rate and individually expected rate of inflation: aggregating yields equation (6.3.1) where \( P_{t+1,t}^{e} \) is the average expectation of the aggregate price level. Individual supply depends on local (observable) prices and individually expected aggregate prices: aggregating yields equation (6.3.2) where, as before, \( P_{t,t}^{e} \) is the average expectation of the aggregate price level. Individual money demand depends on observable prices, observable individual income and the observable interest rate: aggregating gives equation (6.3.3).

Having access to contemporaneous information about the interest rate endows agents with potential auxiliary information about the unobservable variables. To be specific, the vector of currently observable variables is the scalar \( s_t = r_t \) which is related to prices by

\[ s_t = r_t = \omega bp_t - \alpha_y b P_{t,t}^{e} + v_t \quad (6.3.17) \]

where \( \omega = 1 + \alpha_y \)

\[ v_t = \gamma_{1} b_{2t} + b_{3t} \]

Appendix B shows that equation (6.3.17) can be re-written as
\[ s_t = \omega \beta_{k+1} e_{t+1} - b(\omega \beta_{k+1} + \alpha \gamma_1) p_{t+1} + \omega \beta_{k+1} e_{1t} \]
\[ - \omega \beta_{k+1} \epsilon_{2t} - \omega \beta_{k+1} \epsilon_{3t} + \nu_t \]
\[ = \omega \beta_{k+1} e_{t+1} - b(\omega \beta_{k+1} + \alpha \gamma_1) p_{t+1} + \omega \beta_{k+1} e_{1t} \]
\[ + b(\gamma - \omega \beta_{k+1}) \epsilon_{2t} + b(1 - \omega \beta_{k+1}) \epsilon_{3t} \quad (6.3.18) \]

using the definition of \( \nu_t \). Equation (6.3.18) shows that observation of the current interest rate conveys information about all shocks impinging on the system except for when the interest rate is pegged. In the latter case, agents have no auxiliary information whatsoever. (It is useful to note again the necessity for considering money stock peg and interest rate peg policies independently. As in section 6.2, imposing \( \phi = \infty \) in the final expressions for \( \epsilon_{kt} \) (\( k = 1, 2, 3 \)) makes no difference. This is true despite the fact that an interest rate peg delivers zero auxiliary information. The reader is again referred to appendix B for the formal proof.)

Under monetary control, recalling the definitions of \( \omega, b, k, n \) and \( q \) and setting \( \phi = 0 \), equation (6.3.18) becomes

\[ s_t(\bar{\omega}) = \omega \gamma_2 \beta_{k+1}(\bar{m}) p_{t+1} - \gamma_2 \beta_{k+1}(\bar{m})(1 + \alpha(\gamma_1 - \beta^{-1})) p_{t+1} \]
\[ + \omega \gamma_2 \beta_{k+1}(\bar{m}) \beta^{-1} e_{1t} - \gamma_2 \beta_{k+1}(\bar{m}) \beta^{-1} \epsilon_{2t} \]
\[ + \alpha \gamma_2 \beta_{k+1}(\bar{m}) \beta^{-1} \epsilon_{3t} \quad (6.3.19) \]

where the notation emphasizes the dependence of terms on \( \phi = 0 \).

Thus, from equation (6.3.19), the optimal filtered estimates of the disturbances are given by
The solution of the system under a money supply peg now follows by substituting equations (6.3.21) - (6.3.23) into equations (6.3.8) - (6.3.9) for \( \phi = 0 \):

\[
\begin{align*}
\phi_p(m) &= -s(m)\psi(\beta^{-1}\omega_1^{-2} + q(m)\sigma_2^{-2} - \gamma_2^{-1}\alpha_3^{-2})(\omega e_{1t} - e_{2t} + \alpha e_{3t}) \\
\gamma_p(m) &= -s(m)\psi(\beta^{-1}\omega_1^{-2} + q(m)\sigma_2^{-2} - \gamma_2^{-1}\alpha_3^{-2})(\omega e_{1t} - e_{2t} + \alpha e_{3t})
\end{align*}
\] (6.3.24)

Equations (6.3.24) - (6.3.25) can be expressed in a more convenient form. Collecting terms in \( e_{kt} \) (k = 1,2,3),
\[ p_t^P(-m) = [k^{-1}(\bar{m})\beta^{-1} - s(\bar{m})\xi]e_{1t} + [-k^{-1}(\bar{m})q(\bar{m}) + s(\bar{m})\xi]e_{2t} + [-k^{-1}(\bar{m})\gamma_2^{-1} - s(\bar{m})\xi]e_{3t} \] (6.3.26)

\[ y_t^P(-m) = [\alpha k^{-1}(\bar{m})\beta^{-1} - \alpha k^{-1}(\bar{m})\xi]e_{1t} + [\gamma_2^{-1}k^{-1}(\bar{m}) + \alpha k^{-1}(\bar{m})\xi]e_{2t} + [-\alpha k^{-1}(\bar{m})\gamma_2^{-1} - \alpha k^{-1}(\bar{m})\xi]e_{3t} \] (6.3.27)

where \( \xi = \psi(6\beta^{-1}\sigma_1^2 + q(\bar{m})\sigma_2^2 - \gamma_2^{-1}\sigma_3^2) \)

\( \bar{\xi} = \psi(6\beta^{-1}\sigma_1^2 + q(\bar{m})\sigma_2^2 - \gamma_2^{-1}\sigma_3^2) \)

\( \bar{\xi} = \psi(6\beta^{-1}\sigma_1^2 + q(\bar{m})\sigma_2^2 - \gamma_2^{-1}\sigma_3^2) \)

Using the definitions of \( \omega \) and \( q(\bar{m}) \), substitute \( \psi(\omega\sigma_1^2 + \sigma_2^2) = 1 + \psi(\omega\sigma_1^2 - \alpha\sigma_3^2) \) into \( \xi \), \( \omega\sigma_1^2 = 1 - \psi(\omega\sigma_1^2 + \alpha\sigma_3^2) \) into \( \bar{\xi} \), and \( \omega\sigma_1^2 = 1 - \psi(\omega\sigma_1^2 + \alpha\sigma_3^2) \) into \( \bar{\xi} \). Some manipulation then reveals the following solutions for prices and output:

\[ p_t^P(-m) = [\tau^{-1}(\bar{m})\beta^{-1} + \tau^{-1}(\bar{m})\mu\psi(\gamma_1^{-2}\sigma_1^2 - \alpha\sigma_3^2)]e_{1t} + [-\tau^{-1}(\bar{m})q(\bar{m}) + \tau^{-1}(\bar{m})\mu\psi(\omega\gamma_1^{-2}\sigma_1^2 + \alpha\sigma_3^2)]e_{2t} + [-\tau^{-1}(\bar{m})\gamma_2^{-1} + \tau^{-1}(\bar{m})\mu\psi(\omega\gamma_1^{-2}\sigma_1^2 + \sigma_2^2)]e_{3t} \] (6.3.28)

\[ y_t^P(-m) = -\alpha\psi(\gamma_1^{-2}\sigma_1^2 - \alpha\sigma_3^2)e_{1t} + [1 - \alpha\psi(\omega\gamma_1^{-2}\sigma_1^2 + \alpha\sigma_3^2)]e_{2t} - \alpha\psi(\omega\sigma_1^2 + \sigma_2^2)e_{3t} \] (6.3.29)

where \( \mu = \alpha(\beta^{-1} + \gamma_2^{-1}\gamma_1) - 1 \).

Clearly, the impact effect of exogenous shocks are different from those under full information (see equations (6.3.11) - (6.3.13)). Rather than deliberate upon these here, we return to them in connection with the control exercise. For the latter, we note the asymptotic variances of prices and output from equations (6.3.28) - (6.3.29):
\[ \sigma^2_{p}(m) = \left[ \tau^{-1}(m)\beta^{-1} + \tau^{-1}(m)\mu\psi(\gamma_{1}\sigma_{2}^{-2} - \alpha\sigma_{3}^{-2}) \right]^{2-2} \sigma_{1}^{2} + \left[ \tau^{-1}(m)q(m) + \tau^{-1}(m)\mu\psi(\omega_{1}\sigma_{1}^{-2} - \alpha\sigma_{3}^{-2}) \right]^{2-2} \sigma_{2}^{2} + \left[ - \tau^{-1}(m)\gamma_{2}^{-1} + \tau^{-1}(m)\mu\psi(\omega_{1}^{2} + \alpha\sigma_{2}^{2}) \right]^{2-2} \sigma_{3}^{2} \] (6.3.30)

\[ \sigma^2_{y}(m) = \left[ - \alpha\psi(\gamma_{1}\sigma_{2}^{-2} - \alpha\sigma_{3}^{-2}) \right]^{2-2} \sigma_{1}^{2} + \left[ 1 - \alpha\psi(\omega_{1}\sigma_{1}^{-2} + \alpha\sigma_{3}^{-2}) \right]^{2-2} \sigma_{2}^{2} + \left[ - \alpha\psi(\omega_{1}^{2} + \alpha\sigma_{2}^{2}) \right]^{2-2} \sigma_{3}^{2} \] (6.3.31)

This completes the description of the filtering solution for a money supply peg. Under an interest rate peg, matters are substantially simplified. As we have seen, such a policy prevents the dissemination of any auxiliary information. Hence, \( \varepsilon_{kt, t} = 0 \) (\( k = 1, 2, 3 \)) and equations (6.3.8) – (6.3.9) reduce to

\[ p_{t}(\bar{r}) = k^{-1}(\bar{r})\beta^{-1}\varepsilon_{1t} - k^{-1}(\bar{r})q(\bar{r})\varepsilon_{2t} \] (6.3.32)

\[ y_{t}(\bar{r}) = \alpha k^{-1}(\bar{r})\beta^{-1}\varepsilon_{1t} \] (6.3.33)

imposing \( \phi = \alpha (b = 0) \). From equation (6.3.32) – (6.3.33), therefore,

\[ \sigma^2_{p}(\bar{r}) = (k^{-1}(\bar{r})\beta^{-1})^{2-2}\sigma_{1}^{2} + (-k^{-1}(\bar{r})q(\bar{r}))^{2-2}\sigma_{2}^{2} \] (6.3.34)

\[ \sigma^2_{y}(\bar{r}) = (\alpha k^{-1}(\bar{r})\beta^{-1})^{2-2}\sigma_{1}^{2} \] (6.3.35)

6.3(C) Model Solution with No Filtering: 'Zero' Contemporaneous Information

Implicit in analyses based on the orthodox version of the model is the assumption of equation (6.3.16). There is, however, an important difference between the two cases. This was mentioned in chapter 5 (section 5.2). The problem with the popular specification is that it involves an inconsistency. To be specific, it assumes an
overly restricted information set: whilst the current interest rate is implicitly assumed to be currently observable, the value of this as a conveyor of auxiliary information is ignored. To see this, recall that in the popular version expectations are actually conditioned on lagged information (see chapter 3, section 3.2, equations (3.2.1a), (3.2.2a), (3.2.3a), (3.2.4) - (3.2.5)). In terms of the current framework, this amounts to assuming a current information set of the form,

\[ \Omega_{kt}^{NF} = \{ r_{t-j}, p_{t-j}, y_{t-j}, m_{t-j}, \Gamma, \xi | j > 1 \} . \]  

(6.3.36)

Thus, the role of the interest rate as an information variable is entirely ignored.

The solution to this system follows by merely setting \( e_{kt,t} = 0 \) (k = 1, 2, 3) in equations (6.3.8) - (6.3.9)

\[ b_{kt}^{NF} = k^{-1} \beta^{-1} e_{1t} - k^{-1} q e_{2t} - k^{-1} b e_{3t} \]  

(6.3.37)

\[ b_{yt}^{NF} = \alpha k^{-1} \beta^{-1} e_{1t} + k^{-1} b e_{2t} - \alpha k^{-1} b e_{3t} . \]  

(6.3.38)

Hence,

\[ \sigma_{p}^{2NF} = (k^{-1} \beta^{-1} \sigma_{1}^{2} + (-k^{-1} q)^{2} \sigma_{2}^{2} + (-k^{-1} b)^{2} \sigma_{3}^{2} \]  

(6.3.39)

\[ \sigma_{y}^{2NF} = (\alpha k^{-1} \beta^{-1} \sigma_{1}^{2} + (k^{-1} b)^{2} \sigma_{2}^{2} + (-\alpha k^{-1} b)^{2} \sigma_{3}^{2} . \]  

(6.3.40)

The most notable feature of equations (6.3.37) - (6.3.40) is that they are identical to equations (6.3.32) - (6.3.35) for the case of an interest rate peg (\( \phi = \alpha, b = 0 \)). The reason is, of course, that by fixing the interest rate, agents are deprived of any auxiliary information. Equations (6.3.37) - (6.3.40) are predicated on the absence of auxiliary information regardless of the policy regime.
6.3(D) The Relative Merits of Monetary Interest Rate Control

As in section 6.2 we are primarily concerned with the effect of the informational assumption on the relative superiority of money stock and interest rate control. Tables 6.3(A) and 6.3(B) give the asymptotic variances of prices and output under each of the three informational structures and each of the two policy regimes for single isolated shocks. It is noted that \( \tau(\vec{r}) = 1, q(\vec{r}) = \beta^{-1} \) and \( k(\vec{r}) = \alpha \beta^{-1} \). In addition, it should also be noted that the inequalities in the table are not necessarily in sequence. This will become clear below and is emphasized by the parentheses.

We begin by comparing monetary and interest rate peg policies for the two solutions without filtering. Chapter 3 (section 3.3(A)) demonstrated the following results for the orthodox version of the model: pegging the money stock is optimal for both price and output stabilization in the face of expenditure shocks and is optimal for price stabilization in the face of supply shocks; by contrast, interest rate control is superior for both price and output stabilization under monetary shocks and is optimal for output stabilization under supply shocks. The intuition is recalled as follows. Aggregate demand disturbances cause variations in prices which induce output fluctuations. Fixing the money stock permits endogenous movements in the interest rate which dampen the demand shock and hence stabilize prices and output. This is obviously absent when the interest rate is fixed. Money demand shocks are completely neutralized by an accommodative monetary policy so that \( b_P \sigma_{p}^{2NF}(\vec{r}, \vec{e}_3) = b_P \sigma_{y}^{2NF}(\vec{r}, \vec{e}_3) = 0 \).

For aggregate supply disturbances, the initial price movement is exacerbated if interest rates are fixed; at the same time, these unanticipated price fluctuations are stabilizing in terms of output variation.
### Table 6.3(A)(i): Asymptotic price variances under divergent information structures, money supply peg.

<table>
<thead>
<tr>
<th>Disturbance</th>
<th>No Filtering</th>
<th>Filtering</th>
<th>No Filtering</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \frac{\beta^{-1}}{\gamma} \frac{2}{\sigma_1^2} \left( s \right) )</td>
<td>( \frac{\gamma^{-1} + \mu \psi(\gamma_1 \sigma_2^{-2} - \alpha_3^2)}{\tau(m)} \frac{2}{\sigma_1^2} \left( s \right) )</td>
<td>( \frac{\gamma^{-1}}{\gamma} \frac{2}{\sigma_1^2} \left( s \right) )</td>
</tr>
<tr>
<td>Expenditure</td>
<td>( \frac{q(m) - q^{-2}}{\sigma_2^2} \left( s \right) )</td>
<td>( \frac{q(m) - q^{-2}}{\sigma_2^2} \left( s \right) )</td>
<td>( \frac{q(m) - q^{-2}}{\sigma_2^2} \left( s \right) )</td>
</tr>
<tr>
<td>Supply</td>
<td>( \frac{\gamma^{-1}}{\gamma} \frac{2}{\sigma_3^2} \left( s \right) )</td>
<td>( \frac{\gamma^{-1} + \mu \psi(\gamma_1 \sigma_2^{-2} - \alpha_3^2)}{\tau(m)} \frac{2}{\sigma_3^2} \left( s \right) )</td>
<td>( \frac{\gamma^{-1}}{\gamma} \frac{2}{\sigma_3^2} \left( s \right) )</td>
</tr>
<tr>
<td>Money</td>
<td>( \frac{\gamma^{-1}}{\gamma} \frac{2}{\sigma_3^2} \left( s \right) )</td>
<td>( \frac{\gamma^{-1} + \mu \psi(\gamma_1 \sigma_2^{-2} - \alpha_3^2)}{\tau(m)} \frac{2}{\sigma_3^2} \left( s \right) )</td>
<td>( \frac{\gamma^{-1}}{\gamma} \frac{2}{\sigma_3^2} \left( s \right) )</td>
</tr>
</tbody>
</table>

### Table 6.3(A)(ii): Asymptotic output variances under divergent information structures, money supply peg.

<table>
<thead>
<tr>
<th>Disturbance</th>
<th>No Filtering</th>
<th>Filtering</th>
<th>No Filtering</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \sigma_Y^{-2} )</td>
<td>( \gamma^{-1} \frac{2}{\sigma_2^2} \left( s \right) )</td>
<td>( \gamma^{-1} \frac{2}{\sigma_3^2} \left( s \right) )</td>
</tr>
<tr>
<td>Expenditure</td>
<td>( \sigma_Y^{-2} )</td>
<td>( \gamma^{-1} \frac{2}{\sigma_2^2} \left( s \right) )</td>
<td>( \gamma^{-1} \frac{2}{\sigma_3^2} \left( s \right) )</td>
</tr>
<tr>
<td>Supply</td>
<td>( \sigma_Y^{-2} )</td>
<td>( \gamma^{-1} \frac{2}{\sigma_3^2} \left( s \right) )</td>
<td>( \gamma^{-1} \frac{2}{\sigma_3^2} \left( s \right) )</td>
</tr>
<tr>
<td>Money</td>
<td>( \sigma_Y^{-2} )</td>
<td>( \gamma^{-1} \frac{2}{\sigma_3^2} \left( s \right) )</td>
<td>( \gamma^{-1} \frac{2}{\sigma_3^2} \left( s \right) )</td>
</tr>
</tbody>
</table>

### Table 6.3(A): Asymptotic price and output variances under divergent information structures, money supply peg.
### Table 6.3(B)(i): Asymptotic price variances under divergent information structures, interest rate peg.

<table>
<thead>
<tr>
<th>Disturbance</th>
<th>No Filtering ( \sigma^2_{2NP} ) ( \sigma_y(r) )</th>
<th>Filtering ( \sigma^2_F ) ( \sigma_y(r) )</th>
<th>No Filtering ( \sigma^2_{2NP} ) ( \sigma_y(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expenditure</td>
<td>((\beta^{-1}) \sigma_1^2)</td>
<td>((\alpha^{-1}) \sigma_1^2)</td>
<td>((\alpha^{-1}) \sigma_1^2)</td>
</tr>
<tr>
<td>Supply</td>
<td>((-\beta^{-1}) \sigma_2^2)</td>
<td>((-\alpha^{-1}) \sigma_2^2)</td>
<td>((-\alpha^{-1}) \sigma_2^2)</td>
</tr>
<tr>
<td>Money</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Demand</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 6.3(B)(ii): Asymptotic output variances under divergent information structures, interest rate peg.

<table>
<thead>
<tr>
<th>Disturbance</th>
<th>No Filtering ( \sigma^2_{2NP} ) ( \sigma_y(r) )</th>
<th>Filtering ( \sigma^2_F ) ( \sigma_y(r) )</th>
<th>No Filtering ( \sigma^2_{2NP} ) ( \sigma_y(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expenditure</td>
<td>0</td>
<td>(-\sigma_1^2)</td>
<td>(-\sigma_1^2)</td>
</tr>
<tr>
<td>Supply</td>
<td>(-\sigma_2^2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Money</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Demand</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 6.3(B): Asymptotic price and output variances under divergent information structures, interest rate peg.
For the full information case certain of these results are qualified. Output is orthogonal to the system and the transmission mechanism is slightly different. Clearly, then, the control problem for output is redundant. In terms of price stabilization criteria the above conclusions continue to hold with the exception of the case of supply shocks. A positive demand shock raises prices. This is observed and is equivalent to an increase in the real interest rate. This is enhanced by fixing the money stock and permitting a rise in the nominal interest rate. The demand shock is therefore dampened. Money demand shocks continue to be nullified under an interest rate peg. The optimal policy for supply perturbations is ambiguous. The precise condition for a policy to be optimal is

\[ a_p^{2NF} (m, \varepsilon_2) \leq a_p^{2NF} (r, \varepsilon_2) \leftrightarrow \gamma_1 \leq \beta^{-1} \]  

(6.3.41)

which is identical to the condition derived in chapter 3 (section 3.3(C), equation (3.3.53)) for a slightly different model. The common feature of both that model and the current framework, however, is the existence of contemporaneous price information (at least for demanders of goods). The normal case is when

\[ a_p^{2NF} (m, \varepsilon_2) < a_p^{2NF} (r, \varepsilon_2) \]  

a positive supply shock induces negative price movements which consequences for the real interest rate are enhanced by permitting interest rates to fall. The expenditure stimulus then stabilizes prices. Nonetheless, it is possible for interest rates to increase following the supply shock. As in chapter 3 (section 3.3(C)), \( \partial r / \partial \varepsilon_2 \leq 0 \) according to whether \( \gamma_1 - \beta^{-1} \leq 0 \). Hence, the observed fall in real interest rates (which is greater than in the orthodox model in which there is zero contemporaneous price information) induces an expenditure stimulus which increases with \( \beta \). As \( \gamma_1 \) increases this may be sufficient for the nominal interest rate to exhibit a net upward movement. This
would qualify the stabilizing properties of a money supply peg. An
accommodative monetary policy, however, prevents this and induces a
further increase in aggregate demand, thereby reducing price
fluctuations. Hence, \( \sigma_{\text{p}}^{2 \text{NF}}(\bar{m}, \varepsilon_2) > \sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_2) \).

On a final note, we may compare the full information and orthodox
solutions. For expenditure shocks we have that

\[
\sigma_{\text{p}}^{2 \text{NF}}(\bar{m}, \varepsilon_1) \geq \sigma_{\text{p}}^{2 \text{NF}}(\bar{m}, \varepsilon_1) \iff \mu \geq 0
\]  
(6.3.42)

\[
\sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_1) \geq \sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_1) \iff \beta \leq \alpha
\]  
(6.3.43)

From the definition of \( \mu \), \( \sigma_{\text{p}}^{2 \text{NF}}(\bar{m}, \varepsilon_1) > \sigma_{\text{p}}^{2 \text{NF}}(\bar{m}, \varepsilon_1) \) is more likely as
\( \alpha \) increases and \( \beta \) falls which implies that \( \sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_1) > \sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_1) \).
For the former, a high value for \( \alpha \) in the orthodox model means a
large supply response to the unanticipated price movement. This
has the effect of stabilizing prices relative to the full information
case. As \( \beta \) increases, however, the stabilizing effect of the
observed variation in real interest rates in the full information
case may outweigh the stabilizing properties of supply fluctuations
in the orthodox model. Similar arguments apply to the relative
magnitudes of \( \sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_1) \) and \( \sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_1) \).

For supply shocks,

\[
\sigma_{\text{p}}^{2 \text{NF}}(\bar{m}, \varepsilon_2) \geq \sigma_{\text{p}}^{2 \text{NF}}(\bar{m}, \varepsilon_2) \iff \mu \geq 0
\]  
(6.3.44)

\[
\sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_2) \geq \sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_2) \iff \beta \leq \alpha
\]  
(6.3.45)

so that relatively high values for \( \alpha \) imply \( \sigma_{\text{p}}^{2 \text{NF}}(\bar{m}, \varepsilon_2) > \sigma_{\text{p}}^{2 \text{NF}}(\bar{m}, \varepsilon_2) \)
and \( \sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_2) > \sigma_{\text{p}}^{2 \text{NF}}(\bar{r}, \varepsilon_2) \), whilst relatively high values for \( \beta \)
imply the converse. The reason is similar to that given above for
expenditure shocks. For the former, the initial price perturbation
is partly offset in the orthodox model by virtue of the induced supply response. This increases with $\alpha$. In the full information framework, observable fluctuations in real interest rates are also stabilizing. This increases with $\beta$. Analogous arguments apply to $a_p^{2NF}(r, e_2)$ and $b_p^{2NF}(r, e_2)$.

Finally, money demand shocks yield similar results,

$$a_p^{2NF}(m, e_3) \leq b_p^{2NF}(m, e_3) \iff \mu \geq 0$$ (6.3.46)

whilst $a_p^{2NF}(r, e_3) = b_p^{2NF}(r, e_3) = 0$.

Thus far, the analysis suggests that for a particular policy regime the effect of the information structure is ambiguous in terms of the volatility of prices. More precisely, restricting the information set may increase or decrease the variance of prices depending on the relative magnitudes of structural parameters of the model. In addition, for the case of output volatility we have the unambiguous results: $a_y^{2NF}(m, e_1) = 0 < b_y^{2NF}(m, e_1)$, $a_y^{2NF}(m, e_2) = 0 < b_y^{2NF}(m, e_2)$, $a_y^{2NF}(m, e_3) = 0 < b_y^{2NF}(m, e_3)$, $a_y^{2NF}(r, e_1) = 0 < b_y^{2NF}(r, e_1)$, $a_y^{2NF}(r, e_2) > b_y^{2NF}(r, e_2) = 0$, $a_y^{2NF}(r, e_3) = b_y^{2NF}(r, e_3) = 0$.

Of rather more interest is the case in which agents attempt to exploit the information content of the interest rate. When interest rate control is administered, such an attempt is futile since the information value of this variable is zero. As we have seen, the solution reduces to the conventional version of the model and the foregoing discussion for the case of an interest rate peg applies. When the money stock is fixed, however, the interest rate is free to respond to current market conditions.

Thus, consider aggregate demand shocks. Then we may postulate

$$a_p^{2NF}(m, e_1) \leq a_p^{2NF}(m, e_1) \iff \mu(\gamma_1 \delta_0 - \alpha_2^2 + \alpha_3^2) \geq 0.$$


Obviously, \( \sigma_{2y}^N (m, e_1) = 0 < \sigma_{2y}^P (m, e_1) \) but note a peculiarity. From equation (6.3.29),

\[
\frac{\partial y^F_{it}(m)}{\partial \varepsilon_{it}} = - \alpha \psi(\gamma_1 \sigma_2^2 - \sigma_3^2)
\]

so that output may actually fall following a positive demand shock.

This was originally noted by Minford and Peel (1983a) and occurs for the following reason. If supply noise dominates, \( \sigma_2^2 \) is relatively large. The observed rise in interest rates signals (incorrectly) a negative supply disturbance. Expected prices are higher (than actual prices) and output falls. By contrast, if monetary volatility is relatively high, \( \sigma_3^2 \) is large and the normal positive output response is observed: agents misinterpret a positive money demand shock, expected prices are lower and output increases. Thus, consider the behaviour of prices. From equation (6.3.28)

\[
\frac{\partial p^F_{it}}{\partial \varepsilon_{it}} (m) = \frac{\delta(a^NF_{it}) (m)}{\delta \varepsilon_{it}} (m) + \frac{\mu \psi(\gamma_1 \sigma_2^2 - \sigma_3^2) }{\tau(m)}
\]

where \( \delta(a^NF_{it})/\delta \varepsilon_{it} (m) = \beta^{-1} \tau^{-1}(m) > 0 \) from equation (6.3.11).

Suppose \( \sigma_2^2 \) is relatively large. Suppose also \( \mu > 0 \), which occurs for high values of \( \alpha \) and low values of \( \beta \). Then the price increase is greater under partial information than under full information and \( \sigma_{2p}^N (m, e_1) < \sigma_{2p}^F (m, e_1) \). The reason is that, as we have seen, output contracts in the former case, imparting an upward bias to prices.

In addition, \( \sigma_{2p}^N (m, e_1) > \sigma_{2p}^F (m, e_1) \). This may be qualified as \( \alpha \) falls and \( \beta \) increases. Under such circumstances, the destabilizing effect of the output contraction in the partial information model is dampened.

Moreover, the expectation of higher prices compounds the rise in real interest rates and is greater than in the full information case. Both
of these tend to stabilize prices so that \( \sigma_{p}^{2NF}(m, e_{1}) > \sigma_{p}^{2NF}(m, e_{1}) \). From above, \( \sigma_{p}^{2NF}(m, e_{1}) < b_{p}^{2NF}(m, e_{1}) \). Hence, \( \sigma_{p}^{2NF}(m, e_{1}) < \sigma_{p}^{2NF}(m, e_{1}) < b_{p}^{2NF}(m, e_{1}) \).

Now consider the case in which \( \sigma_{3}^{2} \) is relatively large. Suppose also that \( \mu > 0 \). Then from equation (6.3.48), the price increase is greater under full information than under partial information. The reason is that output increases (see equation (6.3.47)) in this case. This stabilizes the initial price rise so that \( \sigma_{p}^{2NF}(m, e_{1}) > \sigma_{p}^{2NF}(m, e_{1}) \). In addition, \( \sigma_{p}^{2NF}(m, e_{1}) > b_{p}^{2NF}(m, e_{1}) \) so that \( \sigma_{p}^{2NF}(m, e_{1}) \leq b_{p}^{2NF}(m, e_{1}) \). As before, these results are likely to be reversed as \( \alpha \) falls and \( \beta \) rises. Under such circumstances, the stabilizing output response in the partial information case is dampened. Coupled with the lower real rate of interest relative to the full information case (because expected prices are lower) this implies \( \sigma_{p}^{2NF}(m, e_{1}) < \sigma_{p}^{2NF}(m, e_{1}) \). From above, \( \sigma_{p}^{2NF}(m, e_{1}) < b_{p}^{2NF}(m, e_{1}) \) and therefore \( \sigma_{p}^{2NF}(m, e_{1}) \leq b_{p}^{2NF}(m, e_{1}) \).

The above results for expenditure shocks demonstrate an important point raised in chapter 5 (section 5.3): access to more initial information does not necessarily imply more accurate auxiliary information. To be sure, we have seen that price volatility may be greater under full information than in the case where agents do not exploit the information content of the interest rate; yet when agents do exploit this, price volatility may actually be greater than the full information case. Similar conclusions arise elsewhere for demand shocks and for other disturbances.

Let us now examine the position for supply disturbances. Clearly, \( \sigma_{p}^{2NF}(m, e_{2}) \leq \sigma_{p}^{2NF}(m, e_{2}) \) according to whether \( \mu \leq 0 \). The price and output responses under partial information are given by equations (6.3.28) - (6.3.29) as
\[
\frac{\partial y_t}{\partial e_{2t}} = \frac{\partial (a_{y_t}^{NF})}{\partial e_{2t}}(m) - \alpha \psi(\omega_1 \sigma_1^2 + \sigma_3^2)
\]

(6.3.50)

with \(\partial (a_{p_t}^{NF})/\partial e_{2t}(m) = -q(m)\tau^{-1}(m) < 0\) and \(\partial (a_{y_t}^{NF})/\partial e_{2t}(m) = 1\) from equations (6.3.11) - (6.3.12). Clearly, then, the output response under partial information is less than under full information and

\[\sigma_{2F}(m, \epsilon_2) > \sigma_{2NF}(m, \epsilon_2).\]

The reason is, of course, that the induced unanticipated price fall following a positive supply shock dampens this shock. Now suppose that \(\mu > 0\) (\(\alpha\) high, \(\beta\) low). Then

\[\sigma_{2NF}(m, \epsilon_2) > \sigma_{2F}(m, \epsilon_2).\]

This occurs because of the dampened supply movement under partial information which stabilizes the downward pressure on prices. In addition, we have seen that \(\sigma_{2NF}(m, \epsilon_2) > b\sigma_{p}(m, \epsilon_2)\) so that \(\sigma_{2F}(m, \epsilon_2) \leq b\sigma_{2NF}(m, \epsilon_2)\). For \(\mu < 0\) (\(\alpha\) low, \(\beta\) high), \(\sigma_{2NF}(m, \epsilon_2) < \sigma_{2F}(m, \epsilon_2)\); as \(\alpha\) falls, the stabilizing effect of unanticipated price movements is reduced; as \(\beta\) rises, the observed greater fall in real interest rates under full information induces stabilizing demand movements. Since \(\sigma_{2NF}(m, \epsilon_2) < b\sigma_{p}(m, \epsilon_2)\) in this case, \(\sigma_{2F}(m, \epsilon_2) \geq b\sigma_{2NF}(m, \epsilon_2)\).

Consider, finally, money demand disturbances. Clearly, \(\sigma_{2NF}(m, \epsilon_3) < \sigma_{2F}(m, \epsilon_3)\) according to whether \(\mu \leq 0\). Price and output responses are given by equations (6.3.28) - (6.3.29) as

\[
\frac{\partial p_t}{\partial e_{3t}} = \frac{\partial (a_{p_t}^{NF})}{\partial e_{3t}}(m) + \frac{\mu \psi(\omega_1 \sigma_1^2 + \sigma_2^2)}{\tau(m)}
\]

(6.3.51)

\[
\frac{\partial y_t}{\partial e_{3t}} = -\alpha \psi(\omega_1 \sigma_1^2 + \sigma_2^2)
\]

(6.3.52)
where $\partial(\frac{\sigma_{y}^{NF}}{\partial \varepsilon_{2t}(m)} - \gamma_{2}^{-1} - 1(\bar{m}) < 0$ and $\partial(\frac{\sigma_{y}^{NF}}{\partial \varepsilon_{2t}(m)} - 0$ from equations (6.3.11) - 6.3.12). Thus, $\sigma_{y}^{NF}(\bar{m}, \varepsilon_{3}) = 0 < \sigma_{y}^{2F}(\bar{m}, \varepsilon_{3})$ obviously because of the unanticipated price movement. If $\mu > 0$ ($\alpha$ high, $\beta$ low) then $\sigma_{p}^{2NF}(\bar{m}, \varepsilon_{3}) > \sigma_{p}^{2F}(\bar{m}, \varepsilon_{3})$. The reason is familiar. The initial price fall is offset by the output contraction which is absent under full information. Since $\sigma_{p}^{2NF}(\bar{m}, \varepsilon_{3}) > b_{p}^{2NF}(\bar{m}, \varepsilon_{3})$, then $\sigma_{p}^{2F}(\bar{m}, \varepsilon_{3}) = b_{p}^{2NF}(\bar{m}, \varepsilon_{3})$. If $\mu < 0$ ($\alpha$ low, $\beta$ high), $\sigma_{p}^{2NF}(\bar{m}, \varepsilon_{3}) < \sigma_{p}^{2F}(\bar{m}, \varepsilon_{3})$ is more likely; the stabilizing effects of unanticipated price fluctuations are dampened and the observed lower real interest rates under full information induces stabilizing demand responds. In addition, $\sigma_{p}^{2NF}(\bar{m}, \varepsilon_{3}) < b_{p}^{2NF}(\bar{m}, \varepsilon_{3})$ from above so that $\sigma_{p}^{2F}(\bar{m}, \varepsilon_{3}) = b_{p}^{2NF}(\bar{m}, \varepsilon_{3})$.

In conclusion, the results for the closed economy are rather more ambiguous than the results obtained for the open economy model in section 6.2. Whether or not the existence of a filtering problem increases or decreases price and output volatility is critically dependent on the structural parameters of the model. Nonetheless, the salient point remains, namely that the information assumption is (at least potentially) far from innocuous and the relative superiority of monetary over interest rate control is non-invariant with respect to the information structure. As in the open economy model, there is still the possibility that it may pay to deprive agents of information.

6.4 Summary and Concluding Remarks

The chapter has been concerned to re-assess the optimal choice of monetary instrument under alternative assumptions about the information structure. Standard approaches to the issue (as in chapters 2 and 3) eschew the important role of some variables as
conveyers of information. This means that the effect of control on the information content of these variables is neglected.

We have considered two simple models. Section 6.2. contained an open economy model in which the set of information variables under partial contemporaneous information comprised the exchange rate and the interest rate. Some fairly unambiguous conclusions emerged which indicated that, for all the shocks taken together, the relative superiority of a money stock peg over an interest rate peg is reduced when the assumption of full contemporaneous information is relaxed and agents extract auxiliary information from currently observed asset prices. This is interesting because an interest rate peg makes the set of useful information variables contain only the exchange rate. In section 6.3 a closed economy model was analysed. Unlike the open economy model, exogenising the interest rate here deprived agents of any contemporaneous information. The results were more sensitive to the structural parameters than in the open economy framework. The general point remained, however, namely the potentially critical influence of the information structure on the relative superiority of one policy over another. An implication of both analyses is that it may pay to deprive agents of information by pegging the interest rate.

We believe that the content of the chapter should be extended to other issues. Then a clearer understanding of the relative merits of alternative policies would be forthcoming. Certain policies deemed to be favourable under one information structure may lose some of their attractiveness under an alternative assumption. Moreover, depriving agents of some initial information by controlling information variables should not be ruled out. As our own analysis indicates, this may prove to be a useful way of reducing the amplitude of stochastic fluctuations. A priori, nothing more can be said
on this matter. Further research is necessary in order to yield greater insights into these issues. In particular, it should be noted that, though our results certainly show that the relative merits of different policies are likely to change according to the informational assumption, they have little or nothing to say about the quantitative magnitude of this change. As mentioned in chapter 5, a general drawback of the literature on signal-extraction is that the precise quantitative importance of alternative informational assumptions is unknown. Attempting to measure the potential importance would seem to be particularly relevant for policy evaluation and we suggest this as an area for future research.
Notes to Chapter 6

(1) In Dotsey and King (1983) there is also local price information.

(2) This is, of course, to incur criticism from those who regard the appropriate welfare criteria to be the minimization of deviations from the full information solution. As mentioned above, however, there is no general agreement on this. In the model of section 6.3 (an equilibrium model), the full information criteria is most persuasive. Nevertheless, we seek to compare the analysis in this section with the orthodox approach and the latter takes deviations from long-run equilibrium as the minimand. Thus, for the sake of comparison, we adopt the latter criteria.

(3) We have not investigated whether or not the results of our policy rule of chapter 2 (section 2.4) are qualified with respect to different information structures.

(4) In fact, $\tau_2(\bar{m}) > \tau_2(\bar{r})$ is obvious, since $\tau_2(\bar{r}) = 1$, $\tau_2(\bar{m}) > 1$ is necessary for stability.

(5) In constructing the diagrams, it is noted that $\partial\tau_1/\partial n > 0$ ($i = 1, 2$) by applying the implicit function theorem to the polynomial in equation (6.2.7): $\partial\tau_1/\partial n = -[\partial f(\tau_1)/\partial \tau_1]^{-1}$ $[\partial f(\tau_1)/\partial n] (i = 1, 2)$. 
CHAPTER 7

REFLECTIONS AND CONCLUSIONS

The thesis has been concerned to examine and extend some recent developments in macroeconomic theory and the theory of macroeconomic policy. For the most part, these developments reflect the integration of the rational expectations hypothesis into economic analysis and the consequences thereof. We have not attempted a rigorous defence of this hypothesis. Though the formal definition of rational expectations is subject to some important criticisms (to which we are sympathetic and to which we shall return below), we have adopted the assumption throughout the thesis on the grounds that it is a fallacy to consider any existing alternative. Nowhere is this more persuasive than in the problem of economic policy evaluation. The Lucas (1976) critique is one example of this. Another is that economic policy should not be designed on the assumption that agents can be continually fooled. Yet another is that agents will use or attempt to forecast policy in formulating their expectations and plans. Surely, rational expectations is to impart an extreme form of intelligence to the system; but it is the only existing assumption which captures sophisticated expectations formation in a simple way and we believe this to be sufficient justification for adopting the hypothesis.

The thesis has been consciously written with two parts in mind. Part I contained chapters 1 - 4. A notable feature of these chapters was the minimal reference to the information structure conditioning expectations. In certain instances, however, we hinted at the potentially important role of information in determining the outcome of events. Part II consisted of chapters 5 - 6 and was intended to address explicitly the informational assumptions in rational
expectations models. It now remains for us to summarize the content of each of these chapters, to collect our thoughts on these and to suggest areas where we believe important research to lie in the future. The reader is also referred to the concluding sections of each chapter.

Chapter I was motivated by a recent rapidly growing research programme on general problems of macroeconomic policy design in the presence of forward-looking behaviour. This has been pioneered by Buiter (1980a, 1981a,b, 1983, 1984a), Miller and Salmon (1983, 1984a, b,c), Currie and Levine (1983a, 1984a,b,c, 1985) and Levine and Currie (1983, 1984). Our purpose was to provide a comprehensive appraisal of the research, to clarify matters and to make the area accessible to a wider audience. The chapter introduced the technique of dynamic optimal control and showed that the standard optimal control problem yields optimal policies which satisfy the principle of optimality or, equivalently, are time consistent. This strikes a direct analogy with the problems faced in the physical sciences. We then discussed the seminal contribution by Kydland and Prescott (1977) which illustrated vividly a reason why one might feel less enthusiastic about applying techniques from the physical sciences to social phenomena. The major insight was the appreciation of the economic policy problem as a game between intelligent players. Various game-theoretic concepts were introduced, and the distinction between the notions of Nash and Stackelberg behaviour elicited. Open loop and closed loop concepts could be appended to these to refine the strategic structure of the game further. We argued that a game between the private sector and the government is most plausibly described by the Stackelberg assumption whereby the government acts as the dominant player. The consequence of this was seen to be the distinction between the ex ante optimal policy and the ex post
optimal policy or, equivalently, to be the *time inconsistency* of the original optimal plan. This makes the original plan lose its attractiveness because its optimality property hinges on it carrying credibility; but this is unlikely to be accommodated because of private sector foresight in anticipating the incentive for the controller to renege on his announcement. We illustrated how the time inconsistency property had been established in standard game-theoretic analyses, without formal modelling of rational expectations, where players are engaged in a Stackelberg game (see, for example, Simaan and Cruz (1973); Cruz (1975); Kydland (1975, 1977)). Nonetheless, we also emphasized that these analyses implicitly rely on sophisticated forward-looking behaviour akin to rational expectations.

Given the time inconsistency of the *ex ante* optimal policy, we re-defined the policy problem as seeking the best policy within the subset of credible (and hence time consistent) policies. Precommitment makes the *ex ante* optimal policy fall within this set but we questioned its feasibility since there always remains an incentive to cheat. In addition, we commented on the erroneous association of the time inconsistency issue with the debate on rules versus discretion. A strategy of *perfect cheating*, which is time consistent, was identified with the actual implementation of the full optimal policy which requires continual deception. This was deemed to be too implausible to merit serious consideration. The asymmetry inherent in the Stackelberg game can be avoided by invoking the Nash assumption which yields a time consistent solution. The original contributions were by Buiter (1983), Cohen and Michel (1984) and Currie and Levine (1985) and we identified other approaches with these. We discussed various concepts of Nash equilibria, in particular *subgame perfection*, and noted that the search for time
consistency is often a search for perfect equilibria. In either case, the solution has a backward-recursive structure which effectively eschews the non-causalities in the system.

The Nash assumption alters the strategic structure of the game but we saw how it can be motivated by considering the consequences of reneging and the loss of reputation. Following this we suggested that the preoccupation with Nash equilibria and dynamic programming techniques overlooked the main issue at stake. Control rules resulting from these analyses are inferior to the ex ante optimal policy and the most pressing problem is precisely how to enforce the latter. We applauded the recent work by Barro and Gordon (1983) and Backus and Drifill (1984a,b, 1985) which address the policy problem in the context of repeated games or a supergame. Then the inferiority of the Nash solution provides the mechanism by which the ex ante optimal policy might be enforced. Uncertainty about rivals' preferences will enrich the game as players engage in a learning process. For us, this type of framework is the correct background against which the time inconsistency issue should be addressed. We pointed out, however, what seems to us to be a major drawback, namely the apparent difficulty in generalizing the analysis. Moreover, we also stressed that, though the reputation framework seems to be a useful descriptive device in some circumstances, it is devoid of any quantitative content. We urge research to be directed towards these matters if the reputation approach is to sustain and enhance the respect which we believe it deserves. An investigation into the stochastic control problem by Currie and Levine (1985) was seen to yield results akin to the reputation models and this might prove to be a convenient way of examining reputation issues, given its relative superiority over the reputation models in terms of its tractability.
A discussion on the control problem in stochastic systems also argued that much of the literature overlooks certain issues of substance which arise precisely because of the presence of stochastic behaviour. A feedback (or innovation-contingent) policy adds a further interesting dimension to the policy problem since there is a potential confusion between the optimal response to exogenous perturbations and the time inconsistency property of the optimal policy. Canzoneri (1984) is a paper in mind here. A priori, the net outcome is difficult to ascertain. We suggest that this area is relatively terra incognita and should be explored in more detail.

We also recorded some comments on the question of cooperation and questioned the validity of analysing this since it would appear to assume collusion between players which is always ruled out, otherwise the non-cooperative game might degenerate into Stackelberg warfare. In addition, though cooperation between countries might be plausible and has occupied much attention, we also pointed out that suitable penalty strategies for enforcing cooperation between countries may be unavailable given the different sizes of nations.

Some rather more substantive issues that we addressed are as follows. We noted that the literature takes as a fact that time inconsistency actually constitutes a problem. We argued that if the controller is truly able to accommodate private sector preferences, it is difficult to envisage any problem associated with time inconsistency. Rather, for there to be a problem, at least some players must perceive of the possibility that they will be made worse off should the controller renege on his announced strategy. We believe that there are sufficient reasons to believe this to be the case, most notably because of divergent preferences between the controller and the private sector or heterogeneity in the private sector, or a divergence between private and social interests. We then suggested
that a more realistic approach incorporating heterogenous players is likely to complicate the problem significantly. The threat effect of a lost reputation may be undermined and, indeed, may not merit serious consideration because it would rely on collusive behaviour. Further research into these matters would be helpful. Our final remarks focussed on the informational assumptions in the literature. We pointed out that, for the most part, the analyses deal with full information dynamic games. Though useful for illustrating general themes, we believe that greater uncertainty should now be incorporated. This might take the form of uncertainty about players' payoffs, players' strategies and the general stochastic properties of the system. There would then be an interesting filtering problem in identifying the intentions and current behaviour of rivals.

Our overall thoughts on the time inconsistency literature were guarded and remain so. We do not subscribe to the view expressed by Kydland and Prescott (1977) and Prescott (1977) that control theory (or optimal economic policy formulation in general for that matter) is redundant when the problem takes on the features of a dynamic game. It is certainly true that rational expectations modifies substantially the approach to economic policy design but it does not render it futile. There is a clear need for collaboration on empirical evaluations of the issues involved. A sustained research programme involving close cooperation between academics should go some way towards making the area less abstract and appear more directly relevant to the aficionados of the policy making process.

Chapter 2 was by way of a prelude to chapter 3. It took up some issues raised in chapter 1 relating to economic policy evaluation and the design of control rules. First, Chow (1976a,b), Johansen (1979) and Currie (1985a,b) have stressed the need to incorporate model
uncertainty into problems of economic policy evaluation by testing the performance of alternative control rules across different models. This would yield a payoff matrix conveying information about the possibility of model-robust policies. Second, there may be advantages in adopting simple control rules which must be offset by the fact that these rules do not satisfy the principle of certainty equivalence and will therefore depend on the stochastic properties of the system. Poole (1970) illustrated the latter when examining the optimal choice of monetary instrument. This particular issue continues to generate a lively debate and chapter 2 contained a brief review of the literature. Our conclusion was that the existing analyses shed little light on the possibility of model-robust policies because of the similarity of the models employed. To correct for this, we proposed an analytical framework comprising six different models. Following this, we turned to an issue raised by Sargent and Wallace (1975) concerning the possibility of non-uniqueness associated with a policy of controlling the interest rate and offered a resolution of this problem. This involved the announcement of a policy rule with feedback on private sector forward price expectations and we showed how this rule could be implemented in terms of a feedback on actual current prices and stochastic disturbances, provided that direct observation of these is possible. The important property of both rules was shown to be that they are observationally equivalent to the usual contemporaneous feedback rule between the interest rate and the money stock. Thus, the general rule serves solely to pin down the long-run equilibrium and does not affect the dynamic response of the system. We suspect that there exists a wide class of policy rules with similar characteristics to ours and it would be interesting to pursue investigation into this matter.
Chapter 3 formalized the analytical framework discussed in chapter 2 and performed a systematic investigation into the optimal choice of monetary instrument across divergent model structures. A payoff matrix was constructed and the particular implications for model-robust policies can be found in chapter 3. Reflecting on some more general issues raised by our analysis, we would emphasize the following. First, we continue to urge further research in a similar spirit to ours, examining different types of control rules and different models. Second, in the case of simple rules which do not satisfy the principle of optimality, this should be combined with a sensitivity analysis across a wide spectrum of stochastic disturbances. We also suggest that a particularly interesting and relevant line of enquiry would be to employ stripped-down versions of the existing large econometric models currently used in forecasting and policy simulations.

Chapter 4 addressed a popular and extensive literature on the implications of the government budget constraint for the coordination of fiscal and monetary policy. The original pioneers of this were Ott and Ott (1965) and Christ (1967, 1968). We recalled a major implication, pointed out by Blinder and Solow (1973), to be the potential instability of a financial policy which involves the financing of budget imbalances by issuing or retiring government debt; what we termed a monetarist financial policy rule. We listed a number of papers dealing with this issue and which indicated the robustness of monetarist instability with respect to different models. We also suggested some gaps in the literature, most notably the concentration on small analytical models and the relatively little research into open economy models and the consequences of alternative expectations assumptions. To investigate these matters, we used a fairly general model developed by Whittaker and
Wren-Lewis (1983) incorporating asset accumulation, the government budget constraint and international trade and capital flows, together with exchange rate and price expectations which might be determined either adaptively or rationally. This required abandoning analytical techniques in favour of computer simulations using a recently developed rational expectations software package at Queen Mary College. The issue of bond-financed deficit instability was subject to a fairly rigorous investigation with emphasis on a sensitivity analysis with respect to parameter variations. Our results indicated strongly the tendency for monetarist instability which was broadly independent of the expectations assumption and rather occurred via the usual stock-flow interactions in the system. Nonetheless, provided post-tax real interest rates are negative, our results also indicated how rational expectations might improve the chances of monetarist stability and how adaptive price expectations might present an additional obstacle to this. Finally, we reported some simulations of the model under exogenous shocks. The broad characteristics of the adjustment paths from one steady state to another were seen to be similar for different expectations assumptions, though these paths were far from monotonic. In addition, rational expectations appeared to smooth and speed up the adjustment of the system.

Our overall conclusions suggested some areas for future research on this model. In particular, the model is sufficiently manageable to permit modifications to its basic structure so that a number of extensions could be made. An interesting extension would be to incorporate a wage-price-exchange rate nexus in the spirit of Artis and Currie (1981) (see also Artis (1981)). Optimal control techniques could also be applied as could problems of signal-extraction. The latter would be especially interesting as current research on
this (to be discussed below) is restricted to small analytical models. We hope to pursue these matters at a later date when the computer software becomes available.

Chapter 5 marked the beginning of Part II of the thesis on some issues relating to the information structure in rational expectations models. Our focus of attention was on signal-extraction. This was defined as the extraction of auxiliary information about currently unobservable variables from contemporaneous observations of some other variables which we termed information variables. We argued that this problem is endemic to rational expectations models which assume that agents exploit all currently available information. The general approach to this problem takes the form of the Kalman filter which, we noted, is often simplified by making the assumption that information eventually becomes available after a lag. We pointed out that signal-extraction endows control rules with the potential to influence the system in two ways: the first is the standard mechanism whereby control alters the dynamic response of the system; the second occurs by virtue of the possibility for control rules to influence the information content of contemporaneous signals. In this respect, this is similar to the Lucas (1976) critique: not only is the economic structure non-invariant with respect to policy rules, but neither also is the information structure. A critical review of some rational expectations models with partial information which address the filtering problem was given. This began with a simple exposition of the early work by Lucas (1972b, 1975) and Barro (1976) in the context of equilibrium models and served to familiarise the reader with the signal-extraction technique. We saw that, for the most part, subsequent literature has continued to employ the equilibrium framework, or variants thereof, and we suggested that future research should be directed towards
examine rather different models. Throughout our discussion we continued to emphasize the potentially crucial role of the informational assumptions in determining the behaviour of the system and highlighted some drawbacks of certain analyses. We also saw how partial information is often associated with *differential information* and suggested how the problem of filtering auxiliary information might take on additional aspects of some interest and complexity. That this is so is because of the circularity inherent in the filtering exercise; actions based on inferences will influence the observations upon which these inferences are conditioned. With heterogeneous information, observed signals will reflect the expectations of differentially informed agents in addition to the underlying stochastic shocks. Then agents might be forced to make inferences about other agents' inferences who will be making similar inferences and so on and so forth. This tricky problem has the seeds of what we called an *informational game* and we believe that research into this might be quite interesting. We have already begun investigation ourselves though the preliminary nature of this prevents its inclusion in the thesis. Needless to say, we hope to continue our exploration in the future and we would welcome contributions from other sources.

Certain of our criticisms of the existing literature motivated an investigation into the information contents of the exchange rate and interest rates. In particular, we noted that most of the analyses employing open economy models assumed a static structure and uncovered interest parity. This meant that the information contents of the exchange rate and the interest rate were identical so that one needed to consider the extraction of information from observation of only one, usually the exchange rate. We wished to test the validity of this in a dynamic model which permitted both perfect and imperfect
capital mobility and which introduced a potentially unobservable foreign disturbance. We showed that, in general, the information contents of the exchange rate and the interest rate are different so that observing both variables conveys different information than does observing just one. We also showed that the amount and nature of auxiliary information is critically dependent upon the degree of capital mobility and the nature of the shock structure assumed. Part of the reason for these results focussed on the dynamic configuration of the model and suggested that dynamic adjustment might act as an additional source of information. Following this, we drew implications for the dynamic response of the exchange rate and prices under alternative information structures. In some cases we discovered the perverse outcome that a greater initial informational endowment may actually reduce the accuracy of forecasts about a particular disturbance being considered. In toto, however, more information must improve the expectational accuracy in which case observation of both the current exchange rate and the current interest rate conveys more information than does observation of just the current exchange rate.

Though the existing literature, and our own analysis, indicated the role of information in determining the behaviour of the system, there has been no quantitative evaluation of the importance of this. We stress that future research should be directed towards this aspect of the problem so that the precise significance (or lack thereof, perhaps) of the informational assumptions can be assessed.

Chapter 6 continued the theme of signal-extraction with particular emphasis on policy evaluation. We wished to examine the potential for control rules to influence the system by altering the information content of observed signals. An obvious analysis which immediately suggested itself was the issue of the optimal choice of monetary instrument which occupied chapters 2 and 3. A potentially
useful information variable here is the interest rate which ceases to perform this role when exogenised by the policy maker. Our purpose was to re-evaluate the monetary instruments problem when the potential information content of the interest rate is exploited. We showed that depriving agents of useful information by controlling the interest rate may actually improve performance by reducing output and price volatility. For an open economy model, the relative superiority of a money stock peg unambiguously declined when we incorporated the statistical inference problem. For a closed economy model, the results were more sensitive to the structural parameter values. We encourage further research along similar lines to ours and, as above, hope for some investigations into the quantitative importance of the issues.

To conclude our thoughts on information, we emphasize the following areas of interest.

First, an important extension to the existing literature would be to examine the signal-extraction problem in larger models than those considered hitherto. Numerical simulations would indicate the precise importance of the information structure in determining the dynamic behaviour of the system and there might be some interesting modifications to the adjustment paths when the solution of a system involves information extraction. We mentioned above that we hope to pursue this research using the model of chapter 4 which, though fairly general and encompassing a rich source of dynamics, is amenable to the extensions suggested here.

Second, we noted that the actual implementation of control rules is likely to take on additional dimensions when the system is characterized by partial information. There may, for example, be arguments which favour restricting control rules to involve feedback on only the information variables in the system; and with heterogenous
information, there is then the question of which particular information variables to feedback on. Whatever the virtues of alternative specifications and whether or not anything general can be stated (we suspect not) remains to be seen.

Thirdly, we pointed out that the type of signal-extraction problem which occupied most of our attention was predicated on the standard assumption that the true structure of the system is known with certainty. That is, imperfect information was analysed in a rational expectations equilibrium and knowledge of the system's structure conditioned the optimal filtered estimates of the currently unobservable variables. This had the attraction of revealing immediately the salient points that we wished to make. Nonetheless, the most fundamental issue remains the formal definition of rational expectations. We suggest that a most important area for future research is to address the problem of policy evaluation in a framework which models the process of learning and the convergence (or lack thereof) to a rational expectations equilibrium. The optimal policies derived (if possible) are likely to be different from the standard rules and the strong conclusions of conventional rational expectations models are likely to be qualified. Rational expectations can be seen here as an end-point on the horizon to which agents are aspiring; but the practical policy problem will involve consideration of the transition period in which agents are learning about the system. The beast which is likely to be unleashed on considering these issues might be difficult to tame but we encourage attempts to do so.

To summarize this chapter, it is useful to list our main proposals for further research as follows
(a) Time inconsistency:
   (i) Coordinated investigation into the quantitative importance of time inconsistency and related issues.
   (ii) Further analysis, including generalizations, of reputation models and investigations into the quantitative importance thereof.
   (iii) Relaxation of the informational assumptions.
   (iv) Further analysis of stochastic systems.

(b) Other issues in policy evaluation and design:
   (i) Further investigations seeking model-robust policies and the combination of these with a sensitivity analysis for simple rules with respect to different stochastic shocks.
   (ii) Further investigations into policy rules à la Blackburn and Currie (1985).
   (iii) Further analysis of policy influence on the information content of information variables and investigations into the quantitative importance thereof.
   (iv) Analysis of the implementation of control rules under partial and differential information.
   (v) Analysis of policy evaluation and design in the presence of learning.

(c) Other issues in information:
   (i) Investigations into the quantitative importance of informational assumptions.
   (ii) Application of signal-extraction to larger models.
   (iii) Analysis of heterogeneous information and the possibility of an informational game.
   (iv) Further investigations into problems of learning and convergence to rational expectations equilibria, especially for the purpose of policy evaluation.
The foregoing pages have contained discussions and formal analyses of a variety of issues. Some of these are probably more controversial than others but we hope to have captured the interest of the reader in all of them. We also hope that at least some of the above suggestions for future research will be pursued shortly.

On the above note, we conclude the thesis.
APPENDIX A

HAMILTONIAN DYNAMICS AND THE MAXIMUM (MINIMUM) PRINCIPLE FOR DYNAMIC OPTIMAL CONTROL

The appendix sketches the solution of the dynamic optimal control problem using the maximum (minimum) principle, as expounded rigorously in Pontryagin, Boltyanski, Gankrelidze and Mishchenko (1962) for continuous time processes. Further discussion of this, including the discrete time version, can be found in Katz (1962), Intrilligator (1971), Chow (1975), Lawden (1975) and Kamien and Schwarz (1981). The second of these is especially recommended. In addition to the foregoing, some useful introductions to optimal control theory include Meditch (1969), Aastroem (1970), and Chow (1981).

A.1 General Solution

The continuous time formulation of the optimal control problem begins by measuring time continuously over the interval $t_0 < t < t_1$. Then consider the dynamic system

\[ \dot{y}(t) = f(y(t), w(t)) \]  \hspace{1cm} (A.1.1)

or

\[ \dot{y}_j(t) = f_j(y_1(t), \ldots, y_n(t), w_1(t), \ldots, w_m(t)) \hspace{1cm} (j = 1, \ldots, n) \]  \hspace{1cm} (A.1.2)

where $y = n \times 1$ vector of state variables

$w = m \times 1$ vector of control variables

and $\dot{y}(t)$ is the time derivative of $y$, $\dot{y}(t) = dy(t)/dt$. The equations of motion given by the nth order differential equation system in
equations (A.1.1) - (A.1.2) are autonomous. Extensions to non-autonomous systems is a trivial exercise.

It is assumed that the mx1 control vector, \( w(t) \), includes only admissible controls which are piecewise continuous functions, encapsulating the possibility that controls may jump discontinuously across the control region. In addition, it is assumed that the functions \( f_j(\cdot) \) (j = 1, ..., n) are continuously differentiable.

Clearly, the mx1 state vector, \( y(t) \), is a point in the n-dimensional Euclidean space, \( \mathbb{E}^n \), so that the n-dimensional state trajectory, \( \{y(t)\}_{t_0}^{t_1} \), is a path of points in \( \mathbb{E}^n \) and is a continuous vector-valued function of time. Similarly, the mx1 control vector, \( w(t) \), is a point in the Euclidean m-space, \( \mathbb{E}^m \), so that the m-dimensional control trajectory, \( \{w(t)\}_{t_0}^{t_1} \), is a path of points in \( \mathbb{E}^m \) and is a piecewise continuous vector-valued function of time.

Now define a performance measure as the objective functional, \( J \), evaluated over the horizon \( t_0 \leq t \leq t_1 \):

\[
J = \int_{t_0}^{t_1} g(y(t), w(t)) \, dt \quad (A.1.3)
\]

\[
g(y(t), w(t)) = g(y_1(t), ..., y_n(t); w_1(t), ..., w_m(t)) \quad (A.1.4)
\]

where the function \( g(\cdot) \) is assumed to be continuously differentiable. This function captures the influence of the state and control trajectories within the relevant time interval on \( J \). It has been called the intermediate function. It is sometimes the case that a final function, \( g(y(t_1)) \), is appended to equation (A.1.3) to capture the influence on \( J \) of the terminal state. This extension is a trivial exercise. The interpretation of \( J \) is that it is a measure of system error plus control effort. The dependence of \( J \) on \( y(t) \) represents the loss associated with deviating from some target; the dependence of \( J \) on \( w(t) \) captures the intensity of control input.
The dynamic optimal control problem is now defined as

$$\max. \text{ (or min.) } J = \int_{t_0}^{t_1} g(y(t), w(t)) \, dt$$

(A.1.5)

subject to $\dot{y}(t) = f(y(t), w(t))$

$y(t_0) = y_0 \text{ given}$

where $y(t_0) = y_0$ given provides $n$ boundary conditions. The Lagrange functional, $L$, is therefore

$$L = J + \int_{t_0}^{t_1} \mu(t)[f(y(t), w(t)) - \dot{y}(t)] \, dt$$

$$= \int_{t_0}^{t_1} [g(y(t), w(t)) + \mu(t)[f(y(t), w(t)) - \dot{y}(t)]] \, dt$$

$$= \int_{t_0}^{t_1} [g(y(t), w(t)) + \mu(t)f(y(t), w(t)) + \mu(t)y(t)] \, dt$$

$$- [\mu(t_1)y(t_1) - \mu(t_0)y(t_0)]$$

(A.1.6)

where $\mu = 1 \times n$ vector of costate variables

and $-\mu(t)\dot{y}(t)$ has been integrated. The new variables that have been introduced are costate (multiplier, auxiliary, adjoint or dual) variables in the $1 \times n$ vector $\mu(t)$. These can be regarded as the dynamic analogues to Lagrange multipliers. In addition, note that since each $\mu_j (j = 1, \ldots, n)$ corresponds to a differential equation each $\mu_j (j = 1, \ldots, n)$ will vary continuously over time. Obviously, the $1 \times n$ costate vector, $\mu(t)$, is a point in $\mathbb{R}^n$ and the n-dimensional costate trajectory, $\{\mu(t)\}_{t_0}^{t_1}$ is a path of points in $\mathbb{R}^n$, being a continuous vector-valued function of time.

The maximum (or minimum) principle now proceeds by defining the Hamiltonian functional, $H$, where
\[ H = q(y(t), w(t)) + \mu(t)f(y(t), w(t)) \quad (A.1.7) \]

so that equation (A.1.6) becomes

\[ L = \int_{t_0}^{t_1} (H + \dot{\mu}(t)y(t)) \, dt - \left[ \mu(t_1)y(t_1) - \mu(t_0)y(t_0) \right] \quad (A.1.8) \]

Thus, consider an arbitrary variation in \((\mu(t))\) \(t_0^1\) to \((\mu(t) + \Delta\mu(t))\) \(t_0^1\). Then equation (A.1.6) gives

\[ \Delta L = \int_{t_0}^{t_1} \Delta\mu(t)[f(\cdot) - \dot{y}(t)] \, dt. \quad (A.1.9) \]

For a stationary point, \(\Delta L = 0\) so that \(f(\cdot) = \dot{y}(t)\). Moreover, from equation (A.1.7),

\[ \frac{\partial H}{\partial \mu(t)} = f(\cdot). \quad (A.1.10) \]

Now consider an arbitrary variation in \((w(t))\) \(t_0^1\) to \((w(t) + \Delta w(t))\) \(t_0^1\) and the associated change in \((y(t))\) \(t_0^1\) to \((y(t) + \Delta y(t))\) \(t_0^1\). Equation (A.1.8) gives

\[ \Delta L = \int_{t_0}^{t_1} \left[ \frac{\partial H}{\partial \Delta w(t)} \Delta w(t) + \left( \frac{\partial H}{\partial \Delta y(t)} + \dot{\mu}(t) \right) \Delta y(t) \right] \, dt - \mu(t_1)\Delta y(t_1). \quad (A.1.11) \]

For a stationary point, \(\Delta L = 0\) so that \(\partial H/\partial w(t) = 0\), \(-\partial H/\partial y(t) = \dot{\mu}(t)\)

and \(\mu(t_1) = 0\). Moreover, from equation (A.1.7),

\[ \frac{\partial H}{\partial w(t)} = \frac{\partial g(\cdot)}{\partial w(t)} + \mu(t) \frac{\partial f(\cdot)}{\partial w(t)} \quad (A.1.12) \]

\[ \frac{\partial H}{\partial y(t)} = \frac{\partial g(\cdot)}{\partial y(t)} + \mu(t) \frac{\partial f(\cdot)}{\partial y(t)} \quad (A.1.13) \]
The foregoing has sketched the maximum (minimum) principle. The necessary conditions for an optimum are summarized below:

\[
\frac{\partial H}{\partial u(t)} = f'(t) = \dot{y}(t) \tag{A.1.14}
\]

\[
\frac{\partial H}{\partial w(t)} = \frac{\partial g'(t)}{\partial w(t)} + \mu^T(t) \frac{\partial f'(t)}{\partial w(t)} = 0 \tag{A.1.15}
\]

\[
\frac{\partial H}{\partial y(t)} = \frac{\partial g'(t)}{\partial y(t)} + \mu^T(t) \frac{\partial f'(t)}{\partial y(t)} = -\ddot{y}(t) \tag{A.1.16}
\]

\[
\mu(t_1) = 0. \tag{A.1.17}
\]

Equations (A.1.14) - (A.1.16) are the canonical equations and describe 2n differential equations. There are n boundary conditions determined from initial time, \(y(t_0) = y_o\), and n transversality conditions determined from terminal time, \(\mu(t_1) = 0\), given in equation (A.1.17). When the system is non-causal, the boundary and transversality conditions need to be modified as stated in the main text.

A.2 Particular Solution I : System 1.2

The particular analogue to equation (A.1.1) is equation (1.2.1) with \(f(y(t), w(t)) = A_1y(t) + Bw(t)\). The particular analogue to equation (A.1.3) is equation (1.2.2) with \(g(y(t), w(t)) = (y^T(t)Qy(t) + w^T(t)Rw(t))\). The solution of the optimal control problem proceeds as described in detail in the main text and summarized for the general case in section A.1.
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APPENDIX B

SADDLEPOINT SOLUTION TECHNIQUE FOR
RATIONAL EXPECTATIONS MODELS

The appendix describes the solution procedure for rational expectations models as developed by Dixit (1980) for continuous time systems and by Blanchard and Kahn (1980) for discrete time systems. Further discussion, including generalizations and extensions, can be found in Blanchard (1982), Buiter (1982, 1984a, b) and Currie and Levine (1982). The signal extraction technique in the case of partial information follows Currie and Levine (1982) and Pearlman, Currie and Levine (1983).

B.1. General Solution I: (Full Information) Continuous Time

Time is measured continuously with initial time, to, given. Then consider the dynamic system

\[ \dot{y}(t) = Ay(t) + du(t) \] (B.1.1)

where \( y \) = nx1 vector of endogenous variables

\( du \) = nx1 vector of stochastic disturbances

and all variables are measured as deviations from long-run equilibrium. In addition, A is an nxn time-invariant matrix of coefficients. The nx1 state vector, \( y(t) \), and the nx1 disturbance vector, \( du(t) \), are partitioned conformably such that

\[
\begin{bmatrix}
    z(t) \\
    x(t)
\end{bmatrix}
\quad \begin{bmatrix}
    \dot{z}(t) \\
    \dot{x}(t)
\end{bmatrix}
\quad \begin{bmatrix}
    du(t)
\end{bmatrix}
\]

(B.1.2)

where \( z \) = n₁x1 vector of predetermined variables

\( x \) = n₂x1 vector of non-predetermined variables
\[ \begin{align*}
\mathbf{d}u^1, \mathbf{d}u^2 &= n_1 \times 1, n_2 \times 1 \text{ vectors of stochastic disturbances} \\
\text{and } n_1 + n_2 &= n.
\end{align*} \]

Note that the \( n \times 1 \) disturbance vector is given by \( \mathbf{d}u(t) = \mathbf{ed}t \) where \( \mathbf{ed}t \) is an \( n \times 1 \) vector of stochastic disturbances with independent increments so that \( \mathbf{d}u(t) \) is white noise. Any autoregressive disturbances are therefore incorporated by suitable enlargement of the state vector (or, more precisely, suitable enlargement of the vector of predetermined variables). Clearly, the \( n \)-dimensional column state vector, \( \mathbf{y}(t) \), is a point in the Euclidean \( n \)-space, \( \mathbb{E}^n \).

Let \( \Omega \) be the information set such that

\[ \Omega(t) = \{ \mathbf{y}(j), \mathcal{F}, \mathcal{E} | j \leq t \} \quad \text{(B.1.3)} \]

where \( \mathcal{F} \) is a vector of structural parameters and \( \mathcal{E} \) summarizes the variance-covariance structure of stochastic disturbances. Then \( \mathbf{d}x^e(t) \) is given by

\[ \begin{align*}
\mathbf{d}x^e(t) &= \mathbf{x}^e(t+dt,t) - \mathbf{x}(t) \\
\mathbf{x}^e(t+dt,t) &= \mathbb{E}(\mathbf{x}(t+dt) | \Omega(t))
\end{align*} \quad \text{(B.1.4)} \]

where \( \mathbb{E}(\cdot) \) is the mathematical conditional expectations operator. The information set in equation (B.1.3) assumes full contemporaneous information. The assumption of discrete lags in information availability is meaningless in the continuous time system where information becomes available continuously.

The solution technique now proceeds by defining

\[ \mathbf{g}(t) = \mathbf{Mx}(t), \mathbf{y}(t) = \mathbf{My}(t) \quad \text{(B.1.6)} \]
where \( q = nx1 \) vector of canonical variables

and \( M \) is an \( nxn \) matrix of left eigenvectors of \( A \) with \( \tilde{M} = M^{-1} \) (so that \( \tilde{M} \) is the matrix of right eigenvectors). As before, \( q(t) \) is partitioned such that

\[
q(t) = \begin{bmatrix} q^1(t) \\ q^2(t) \end{bmatrix}
\]

where \( q^1, q^2 = n_1x1, n_2x1 \) vectors of canonical variables.

Using a well-known canonical transform, therefore,

\[
M \Lambda = \Lambda M \quad ; \quad \tilde{M} \Lambda = \Lambda \tilde{M}
\]

where \( \Lambda \) is the \( nxn \) diagonal matrix of eigenvalues of \( A \). The \( nxn \) matrices \( A, M, \tilde{M} \) and \( \Lambda \) are all partitioned conformably with \( y(t) \) and \( q(t) \) so that

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}
\]

\[
\tilde{M} = M^{-1} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}
\]

where \( A_{11}, M_{11}, \tilde{M}_{11} \) and \( \Lambda_1 \) are \( n_1 \times n_1 \) matrices; \( A_{12}, M_{12} \) and \( \tilde{M}_{12} \) are \( n_1 \times n_2 \) matrices; \( A_{21}, M_{21} \) and \( \tilde{M}_{21} \) are \( n_2 \times n_1 \) matrices; and \( A_{22}, M_{22} \) and \( \tilde{M}_{22} \) are \( n_2 \times n_2 \) matrices. Then

\[
dq^e(t) = M \dot{Y}^e(t) = M \dot{Y}(t) dt = \Lambda M \dot{Y}(t) dt = \Lambda q(t) dt
\]
using the fact that \(du^e(t) = 0\). Focussing on the last \(n_2\) rows of equation (B.1.10),

\[
\frac{dq^e(t)}{dt} = \Lambda_2q^2(t)dt
\]  

(B.1.11)

so that replacing \(t\) by \(\tau + dr\) and taking expectations at time \(t\),

\[
\frac{dq^e(\tau + dr, t)}{dt} = \Lambda_2q^2(\tau + dr, t), \quad \tau > t.
\]  

(B.1.12)

Blanchard and Kahn (1980) show that for the system to have a unique convergent rational expectations solution, it must possess the saddlepoint property. This requires that there be \(n_1\) stable eigenvalues of \(\Lambda\) (\(n_1\) eigenvalues with negative real part) and \(n_2\) unstable eigenvalues of \(\Lambda\) (\(n_2\) eigenvalues with positive real part). It is assumed that this condition is satisfied in which case the only non-explosive solution to equation (B.1.12) is

\[
q^2(\tau + dr, t) = 0 \quad (\tau > t).
\]

Then for \(\tau = t\), and letting \(dt \to 0\), \(q^2(t) = 0\) and the definition of \(q(t)\) gives

\[
0 = \Lambda_2(M_{21}z(t) + M_{22}x(t))
\]  

(B.1.13)

or

\[
x(t) = -M_{22}^{-1}M_{21}z(t) = M_{21}^{-1}z(t).
\]  

(B.1.14)

Equation (B.1.14) defines the stable trajectory of the system. It gives the level of the non-predetermined (jump, free or forward-looking) variables as a linear combination of the predetermined (backward-looking) variables. The complete solution of the system now follows by substituting equation (B.1.14) into equation (B.1.1)
to obtain

\[ dz(t) = Gz(t) + du^1(t) \] (B.1.15)

where \( G = A_{11} - A_{12}M_{22}^{-1}M_{21} = M_{11}A_{11}^{-1} \).

Hence, equations (B.1.14) - (B.1.15) give

\[ z(t) = e^{G(t-t_0)}z(t_0) + \int_{t_0}^{t} e^{G(t-\tau)} du^1(\tau) \]
\[ = M_{11}e^{A_1(t-t_0)M_{11}^{-1}z(t_0)} + \int_{t_0}^{t} M_{11}e^{A_1(t-\tau)M_{11}^{-1}du^1(\tau)} \] (B.1.16)

\[ x(t) = -H_{22}M_{21}e^{G(t-t_0)}z(t_0) - M_{22}M_{21}^{-1}z(t_0) + \int_{t_0}^{t} M_{21}e^{A_2(t-\tau)M_{11}^{-1}du^1(\tau)} \]
\[ = M_{21}e^{A_1(t-t_0)M_{11}^{-1}z(t_0)} + \int_{t_0}^{t} M_{21}e^{A_1(t-\tau)M_{11}^{-1}du^1(\tau)} \] (B.1.17)

with \( z(t_0) = z_0 \) given.

B.2 General Solution II: (Full and Partial Information) Discrete Time

Time is measured discretely with initial time, \( t_0 \), given. Then consider the dynamic system

\[ y_{t+1} = Ay_t + By_t^e + u_t \] (B.2.1)

\[ s_t = Cy_t + Dy_t^e + v_t \] (B.2.2)

where \( y \) = \( n \times 1 \) vector of endogenous variables
\( s \) = \( p \times 1 \) vector of observation variables
\( u, v \) = \( n \times 1, p \times 1 \) vectors of stochastic disturbances
and all variables are measured as deviations from long-run equilibrium. In addition, A and B are nxn time-invariant matrices of coefficients and C and D are pxn time-invariant matrices of coefficients. The nx1 state vector, \( \mathbf{y}_t \), and the nx1 disturbance vector, \( \mathbf{u}_t \), are partitioned conformably such that

\[
\mathbf{y}_t = \begin{bmatrix} \mathbf{y}_t^e \\ \mathbf{x}_t \end{bmatrix}, \quad \mathbf{y}_{t+1} = \begin{bmatrix} \mathbf{y}_{t+1}^e \\ \mathbf{x}_{t+1} \end{bmatrix}, \quad \mathbf{u}_t = \begin{bmatrix} \mathbf{u}_t^1 \\ \mathbf{u}_t^2 \end{bmatrix}
\]

(B.2.3)

where

\[
\begin{align*}
\mathbf{y}^e &= \text{n}_1 \times 1 \text{ vector of predetermined variables} \\
\mathbf{x} &= \text{n}_2 \times 1 \text{ vector of non-predetermined variables} \\
\mathbf{u}_1^1, \mathbf{u}_2^2 &= \text{n}_1 \times 1, \text{n}_2 \times 1 \text{ vectors of stochastic disturbances}
\end{align*}
\]

and \( n_1 + n_2 = n \).

Equation (B.2.1) is entirely general for the class of models with a single information set. In particular, any dynamic system with a finite number of lags can be reduced to a first order difference equation by suitable redefinition of variables and appropriate enlargement of the state vector. Hence, autoregressive processes for stochastic disturbances are incorporated in the state vector (or, more precisely, in the vector of predetermined variables) so that \( \mathbf{u} \) is purely white noise. Clearly, the \( n \)-dimensional column vector, \( \mathbf{y} \), is a point in the Euclidean \( n \)-space, \( \mathbb{E}^n \).

Equation (B.2.2) defines the measurement equation with the currently observable variables given in the px1 observation vector, \( \mathbf{z} \), which is a point in the Euclidean p-space, \( \mathbb{E}^p \). The information set, \( \mathcal{N}_t \), is defined as

\[
\mathcal{N}_t = \{ \mathbf{z}_t, \mathbf{z}_{t-j}, \mathbf{y}_{t-j}, \mathbb{E}, \mathcal{E} \} (j \geq 1)
\]

(B.2.4)
where \( \Gamma \) is a vector of structural parameters and \( \Xi \) summarizes the variance-covariance structure of stochastic disturbances. Then\

\[
\begin{align*}
\mathbf{x}_{t+1,t}^e &= E(\mathbf{x}_{t+1} | \Xi_t) \\
\mathbf{x}_{t,t}^e &= E(\mathbf{x}_t | \Xi_t) \\
\mathbf{z}_{t,t}^e &= E(\mathbf{z}_t | \Xi_t)
\end{align*}
\]  

(B.2.5)  
(B.2.6)  
(B.2.7)

where \( E(\cdot) \) is the mathematical conditional expectations operator. Thus, contemporaneous information is available about only a subset of variables. Information about other variables accrues with a one period lag. Note, however, that equation (B.2.4) includes as a special case the possibility that some of the \( z \) and \( x \) are directly observable. The two polar cases are when all of the \( z \) and \( x \) are components of \( s \) (full information), and when none of the \( z \) and \( x \) are included in \( s \).

The solution proceeds by defining\

\[
\mathbf{q}_t = M\mathbf{x}_t, \quad \mathbf{y}_t = \mathbf{M}_t
\]  

(B.2.8)

where \( \mathbf{q} = n \times 1 \) vector of canonical variables

and \( \mathbf{M} \) is an \( n \times n \) matrix of left eigenvectors of \( \mathbf{A} + \mathbf{B} \) with \( \mathbf{M} = \mathbf{M}^{-1} \) (so that \( \mathbf{M} \) is the matrix of right eigenvectors). In addition, \( \mathbf{q}_t \) is partitioned as for \( \mathbf{y}_t \) such that\

\[
\mathbf{q}_t = \begin{bmatrix} q_1^t \\ q_2^t \end{bmatrix}
\]  

(B.2.9)

where \( q_1^t, q_2^t = n \times 1 \) vectors of canonical variables.
Then a well-known canonical transform states that

\[ M(A+B) = \Lambda M; \quad (A+B)M = M\Lambda \]  

(B.2.10)

where \( \Lambda \) is the \( nxn \) diagonal matrix of eigenvalues of \( A+B \). The \( nxn \) matrices \( A+B, M, M, \) and \( \Lambda \) are partitioned conformably with \( y_t \) and \( q_t \) so that

\[
\begin{bmatrix}
(A+B)_{11} & (A+B)_{12} \\
(A+B)_{21} & (A+B)_{22}
\end{bmatrix},
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\]

where \( (A+B)_{11}, M_{11}, M_{11}, \) and \( A_1 \) are \( n_1 \times n_1 \) matrices; \( (A+B)_{12}, M_{12} \) and \( M_{12} \) are \( n_1 \times n_2 \) matrices; \( (A+B)_{21}, M_{21}, M_{21}, \) and \( A_2 \) are \( n_2 \times n_1 \) matrices; and \( (A+B)_{22}, M_{22}, M_{22}, \) and \( \Lambda_2 \) are \( n_2 \times n_2 \) matrices. Then

\[
\begin{align*}
q_{e, t+1, t} &= M_y e_{t+1, t} \\
&= M(A+B) y_{e, t, t} + M u_{e, t, t} \\
&= \Lambda y_{e, t, t} + u_{e, t, t} \\
&= \Lambda q_{e, t, t} + u_{e, t, t}
\end{align*}
\]

(B.2.12)

with \( u_{e, t, t} = [0 \ (u^{2e}_{t, t})^T] \) since \( u^{1e}_{t, t} = 0 \). Focusing on the last \( n_2 \) rows of equation (B.2.12),
so that replacing $t$ by $t+j$ ($j \neq 1$) and taking expectations at time $t$,

$$\tilde{q}_{t+1+j,t} = A_2 \tilde{q}_{t+j,t}, \quad j \neq 1$$  \hfill (B.2.14)

using the fact that $u_{t+j,t}^2e = 0$ ($j \neq 1$).

Blanchard and Kahn (1980) show that for the system to have a unique non-explosive rational expectations solution, it must possess the saddlepoint property. This requires that there be $n_1$ stable eigenvalues of $A+B$ ($n_1$ eigenvalues which lie inside the unit circle) and $n_2$ unstable eigenvalues of $A+B$ ($n_2$ eigenvalues outside the unit circle). It is assumed that this condition is satisfied in which case the only non-explosive solution to equation (B.2.14) is $\tilde{q}_{t+j,t}^2e = 0$ ($j \neq 1$). Then for $j = 1$, the definition of $\tilde{q}_t$ gives

$$0 = A_2(M_{21}\tilde{z}_{e,t} + M_{22}\tilde{z}_{e,t}) + M_{22}u_{t}^2e$$  \hfill (B.2.15)

or

$$\tilde{x}_{t,t} = -M_{22}^{-1}M_{21}\tilde{z}_{e,t} - M_{22}^{-1}M_{22}u_{t}^2e.$$  \hfill (B.2.16)

In addition, equation (B.2.1) yields

$$z_{t} - z_{e,t} = u_{t-1} - u_{t-1,t}^1.$$  \hfill (B.2.17)

$$x_{t} - x_{e,t} = -A_{22}^{-1}A_{21}(z_{t} - z_{e,t}) - A_{22}^{-1}u_{t}^2 - u_{t}^2e.$$  \hfill (B.2.18)

so that substituting equations (B.2.16) - (B.2.17) into equation (B.2.18) gives
Equation (B.2.19) is now substituted into equation (B.2.1) to obtain

\[
X_t = \frac{Y_t}{M_{22,21}}z + (Y_t^{1e} - A^{1e}u_{t-1}^{1e} - (M^{1}M_{22,21} - A^{1e}M_{22,21})u_{t-1,1}^{1e}) - A^{1e}u_{t}^{1e} + (A^{1e}M_{22} - A^{1e}M_{22})u_{t}^{2e}.
\]  

(B.2.19)

Equation (B.2.20) is now substituted into equation (B.2.1) to obtain

\[
Z_{t+1} = Gz_t + A^{1e}u_{t-1}^{1e} - A^{1e}(M^{1}M_{22,21} - A^{1e}M_{22,21})u_{t-1,1}^{1e} - A^{1e}A_{12}u^{1e} + A^{1e}(A^{1e}M_{22} - A^{1e}M_{22})u_{t}^{2e} + u_{t}^{1e}.
\]  

(B.2.20)

where \( G = A^{1e}M_{22}^{-1}M_{22,21} = M_{11}^{1e}M_{22}^{-1} \).

B.2(A) Full Information

The full information solution proceeds by noting that \( \theta \) includes all of the \( z \) and \( x \). Then all stochastic disturbances are known with certainty. Imposing \( u^{1e}_t = u^{1e}_{t-1,1} \) and \( u^{2e}_t = u^{2e}_{t} \) in equations (B.2.19) - (B.2.20), therefore,

\[
X_t = -M^{-1}M_{22,21}z - M^{-1}A^{1e}M_{22,21}u^{2e}_t.
\]  

(B.2.21)

\[
Z_{t+1} = Gz_t - A^{1e}A_{12}^{-1}M_{22,21}u^{2e}_t + u^{1e}_t.
\]  

(B.2.22)

Equation (B.2.21) is the stable trajectory, giving the level of the non-predetermined (jump, free or forward-looking) variables as a linear combination of the predetermined (backward-looking) variables and stochastic disturbances.

B.2(B) Partial Information

The partial information solution arises when \( \theta \) does not include all of the \( z \) and \( x \). Then to solve the system requires the use of signal-extraction. The extraction of auxiliary information about stochastic disturbances proceeds as follows.
First, partition the pxn C matrix in equation (B.2.2) conformably with \( y_t \) such that \( C = [C_1 \ C_2] \) with \( C_1 \) and \( C_2 \) having orders pxn\(_1\) and pxn\(_2\) respectively. Then

\[
s_t = C_1 z_t + C_2 x_t + Dy^e_t + v_t. \quad (B.2.23)
\]

Similarly, partition the nxn B matrix in equation (B.2.1) such that \( B = [(B^1)^T \ (B^2)^T]^T \) with \( B^1 \) and \( B^2 \) having orders n\(_1\)xn and n\(_2\)xn respectively. The last n\(_2\) rows of equation (B.2.1) can be written as

\[
x_t = A_2^{-1} x_t^{e_{t+1}} + A_2^{-1} A_1 y_t - B^2 y^e_t + u^2_t. \quad (B.2.24)
\]

Finally, partition the nxn A matrix as above so that \( A = [(A^1)^T \ (A^2)^T]^T \) where \( A^1 \) and \( A^2 \) have orders n\(_1\)xn and n\(_2\)xn respectively. The first n\(_1\) rows of equation (B.2.1) are

\[
z_t = A^1 y_{t-1} + B^1 y^e_{t-1, t-1} + u^1_{t-1}. \quad (B.2.25)
\]

Substituting equations (B.2.24) - (B.2.25) into equation (B.2.23) gives

\[
s_t = C_1 A^1 y_{t-1} + B^1 y^e_{t-1, t-1} + u^1_{t-1} + Dy^e_t + v_t
\]

\[
- C_2^{-1} A_2^{-1} A_1 y_t - (C_1 - C_2 A_2^{-1} A_1) B^1 y^e_{t-1, t-1}
\]

\[
+ (D - C_2 A_2 A_1^{-1}) y^e_{t, t} + C_2^{-1} A_2 y^e_{t-1, t} + (C_1 - C_2 A_2 A_1^{-1}) u^1_{t-1}
\]

\[
- C_2^{-1} A_2 u^1_{t} + v_t. \quad (B.2.26)
\]

The only terms in equation (B.2.26) which are currently unknown are \( u^1_{t-1}, u^2_t \) and \( v_t \). It is assumed that these are independently...
distributed; if they are not, then the composite disturbances can be rearranged such that the resulting expression is a linear combination of independently distributed stochastic disturbances. Given the linear dependence of $g_t$ on $u^1_{t-1}$, $u^2_t$ and $v_t$, the optimal filtered estimates of these disturbances are given by

$$
\begin{bmatrix}
    u^1_{t-1, t} \\
    u^2_t \\
    v^e_t, t
\end{bmatrix} =
\begin{bmatrix}
    \Gamma_1 F^T \\
    -\Gamma_2 H^T \\
    \Gamma_v
\end{bmatrix}
\phi(Fu^1_{t-1} - Hu^2_t + v_t)$$

(B.2.27)

where $F = C_1 - C_2 A_2^{-1} A_{21}$

$H = C_2 A_2^{-1}

\phi = (FC_1 F^T + HC_2 H^T + \Gamma_v)^{-1}$

and $\Gamma_1$, $\Gamma_2$ and $\Gamma_v$ are the $n_1 \times n_1$, $n_2 \times n_2$ and $p \times p$ asymptotic variance-covariance matrices of $u^1$, $u^2$ and $v$ respectively. The orders of $F$, $H$ and $\phi$ are $p \times n_1$, $p \times n_2$ and $p \times p$ respectively.

The solution of the system under partial information now follows by substituting equation (B.2.27) into equations (B.2.19) -(B.2.20):

$$
X_t = -M_{22} M^{-1}_{21} Z_t + (M_{22}^{-1} M_{21} - A_{22}^{-1} A_{21}) u^1_{t-1},
\text{etc.}
$$

(B.2.28)

$$
Z_{t+1} = Gz_t + A_{12} (M_{22}^{-1} M_{21} - A_{22}^{-1} A_{21}) u^1_{t-1},
\text{etc.}
$$

(B.2.29)
or

\[ x_t = -M_{22}^{-1}M_{21}z_t + [M_{22}^{-1}M_{21} - A_{22}^{-1}A_{21} - (M_{22}^{-1}M_{21} - A_{22}^{-1}A_{21})C_1F^T \Phi F]
\]

\[- (A_{22}^{-1}M_{22}^{-1}A_{22})C_2H^T \Phi H]u_{t-1}^{1} \]

\[ + [-A_{22}^{-1} \pm A_{22}^{-1} - A_{22}^{-1}A_{21})C_1F^T \Phi H + (A_{22}^{-1}M_{22}^{-1}A_{22})C_2H^T \Phi H]u_{t}^{2} \]

\[ - [(M_{22}^{-1}M_{21} - A_{22}^{-1}A_{21})C_1F^T \Phi + (A_{22}^{-1}M_{22}^{-1}A_{22})C_2H^T \Phi]v_t \]

(B.2.30)

\[ z_{t+1} = Gz_t + A_{12}[M_{22}^{-1}M_{21} - A_{22}^{-1}A_{21} - (M_{22}^{-1}M_{21} - A_{22}^{-1}A_{21})C_1F^T \Phi F]
\]

\[- (A_{22}^{-1}M_{22}^{-1}A_{22})C_2H^T \Phi H]u_{t-1}^{1} + u_t^{1} \]

\[ + A_{12}[-A_{22}^{-1} \pm A_{22}^{-1} - A_{22}^{-1}A_{21})C_1F^T \Phi H + (A_{22}^{-1}M_{22}^{-1}A_{22})C_2H^T \Phi H]u_{t}^{2} \]

\[ - A_{12}[(M_{22}^{-1}M_{21} - A_{22}^{-1}A_{21})C_1F^T \Phi + (A_{22}^{-1}M_{22}^{-1}A_{22})C_2H^T \Phi]v_t. \]

(B.2.31)

Equations (B.2.28) and (B.2.30) define the stable trajectory of the system, giving the level of the non-predetermined (jump, free or forward-looking) variables as a linear combination of the predetermined (backward-looking) variables and stochastic disturbances.

B.3 Particular Solution I: System 3.3(E)

The particular analogue to equation (B.1.10) is equation (3.3.60) with \( A \) the 3x3 dynamic matrix, \( z = [y, p]^T \) and \( x = q \). Normalising so that \( M_{22} = 1 \), the definitions in equation (B.1.9) give the particular analogue to \( M^2 \) as \( M^2 = [M_{21}, 1] \) with \( M_{21} = [m_{21}^y \ m_{21}^D] \). Then the particular analogue to equation (B.1.14) is equation (3.3.63). The expressions for \( m_{21}^y \) and \( m_{22}^D \) in equations (3.3.64) - (3.3.65) follow from
The particular analogue to equation (B.1.15) is equation (3.3.67) which is derived by substituting equation (3.3.63) into equation (3.3.60). Thus,

\[
\begin{bmatrix}
\begin{pmatrix}
\frac{dy}{dt} \\
\frac{dp}{dt}
\end{pmatrix}
\end{bmatrix}
= \begin{bmatrix}
-\lambda c & -\lambda \beta_1 (b + \theta_1) \\
0 & -\theta_1
\end{bmatrix}
\begin{bmatrix}
Y \\
p
\end{bmatrix}
\]

which gives equation (3.3.67).

**B.4 Particular Solution II : System 3.3(F)**

The particular analogue to equation (B.1.1) is equation (3.3.82) with \( A \) the 4x4 dynamic matrix, \( z = [y \ p \ v]^T \) and \( x = e \). Normalising so that \( M_{22} = 1 \), the definitions in equation (B.1.9) give the particular analogue to \( M^2 \) as \( M^2 = [M_{21} \ 1] \) with \( M_{21} = [m^y_{21} \ m^p_{21} \ m^v_{21}] \). Then the particular analogue to equation (B.1.14) is equation (3.3.66). The expressions for \( m^y_{21}, m^p_{21} \) and \( m^v_{21} \) in equations (3.3.87) - (3.3.89) follow from

\[
\begin{bmatrix}
\begin{pmatrix}
\frac{dy}{dt} \\
\frac{dp}{dt} \\
\frac{dv}{dt}
\end{pmatrix}
\end{bmatrix}
= \begin{bmatrix}
-(\lambda (c - \beta_1 b + \beta_2) + \tau_4) & -\lambda (\beta_1 b + \beta_2) & 0 & \lambda \beta_2 \\
0 & -\tau_4 & 0 & 0 \\
0 & 0 & -(\rho + \tau_4) & 0 \\
by_1 - n_1 & b - n_2 & 1 & n_2 - \tau_4
\end{bmatrix}
\]
The particular analogue to equation (B.1.15) is equation (3.3.91) which is derived by substituting equation (3.3.86) into equation (3.3.82). Thus,

\[
\begin{bmatrix}
\frac{dy}{dt} \\
\frac{dp}{dt} \\
\frac{dv}{dt}
\end{bmatrix} =
\begin{bmatrix}
-\lambda (c-\beta_1 \theta) & -\lambda (\beta_1 b+\beta_2) & 0 \\
0 & 0 & 0 \\
0 & 0 & -\rho
\end{bmatrix}
\begin{bmatrix}
y \\
p \\
v
\end{bmatrix}
\]

which gives equation (3.3.91).

\[\text{B. 5 Particular Solution III : System 5.3}\]

The particular analogue to equation (B.2.1) is equation (5.3.6) with \(\mathbf{A}\) the 2x2 dynamic matrix, \(\mathbf{z} = p\) and \(\mathbf{x} = e\). Normalising so that \(m_{22} = 1\), the definitions in equation (B.2.11) give the particular analogue to \(M^2\) as \(M^2 = [m_{21} 1]\). Then the particular analogues to equations (B.2.19) - (B.2.20) are equations (5.3.10) - (5.3.11). To obtain equation (5.3.12), rewrite equations (5.3.10) - (5.3.11) as

\[
(1-\tau_1)\epsilon_{t+1} = -m_{21}(1-\tau_1)p_{t+1} + b_1(1-\tau_1)u^1_{t} -a_2^{-1}(1-\tau_1)\epsilon_{2t+1} \\
- a_2^{-1}(1-\tau_1)\epsilon_{3t+1} -b_1(1-\tau_1)u^1_{t,t+1} + b_2^{-1}(1-\tau_1)\epsilon^e_{2t+1,t+1} \\
+ b_2^{-1}(1-\tau_1)\epsilon^e_{3t+1,t+1}
\]

\[\text{(B.5.1)}\]

\[
(1-\tau_1)p_{t+1} = \lambda b_1 u^1_{t-1} -\lambda a_2^{-1} \epsilon^e_{2t} -\lambda a_2^{-1} \epsilon^e_{3t} -\lambda b_1 u^1_{t-1,t} \\
+ \lambda b_2^{-1} \epsilon^e_{2t,t} + \lambda b_2^{-1} \epsilon^e_{3t,t} + u^1_{t}
\]

\[\text{(B.5.2)}\]

where \(L\) is the lag operator. Then substituting equation (B.5.2) into
equation (B.5.1) gives equation (5.3.12). The expression for \( m_{21} \) follows as

\[
\begin{bmatrix}
1 & \lambda - \tau_2 \\
\lambda & a_2 - \tau_2 \\
1 & a_1
\end{bmatrix} = 0. \tag{B.5.3}
\]

The full information analogue to equation (B.2.4) is equation (5.3.13). The partial information analogues to equation (B.2.4) are equations (5.3.14a) - (5.3.14c). For the information set in equation (5.3.14a), the particular analogue to equation (B.2.2) is equation (5.3.17) with \( C \) the 2x2 right hand side matrix. Equation (5.3.19) is the particular analogue to equation (B.2.27) and is obtained by proceeding as in section B.2(B) to obtain equation (B.2.26) which has the particular analogue in equation (5.3.18). The algebra which gives equations (5.3.15a) follows by noting from equation (5.3.19),

\[
a^{2T}_{t-1, t} = \sigma^2_{1T}(F_{t-1}^{-1}a_2^{-1}C_2^{-1}t + G_{2T}) \tag{B.5.4}
\]

\[
e^{2T}_{a_2, t} = \sigma^2_{2T}(F_{t-1}^{-1}a_2^{-1}C_2^{-1}t + G_{2T}) \tag{B.5.5}
\]

\[
e^{2T}_{a_3, t} = \sigma^2_{3T}a^{-1}_2(\sigma_{a_2}^{-1}l + G_{2T}) \tag{B.5.6}
\]

Then

\[
\sigma^2_{1T}(F) = \left( \phi^{-2}a_2^{-2}\right)^{-1} \sigma^2_{1}(\gamma^{-1}a_2^{-2}\sigma_3^{-2}a_2^{-1}n^2_{\sigma_1}) \tag{B.5.7}
\]

\[
\sigma^2_{2T}(F) = \left( \phi^{-2}a_2^{-2}\right)^{-1} \sigma^2_{2}(\gamma^{-1}a_2^{-2}\sigma_3^{-2}a_2^{-1}n^2_{\sigma_1}) \tag{B.5.8}
\]

\[
\sigma^2_{3T} = \left( \phi^{-2}a_2^{-2}\right)^{-1} \sigma^2_{3}(\gamma^{-1}a_2^{-2}\sigma_3^{-2}a_2^{-1}n^2_{\sigma_1}) \tag{B.5.9}
\]
Substituting equations (B.5.7) - (B.5.9) into equations (B.5.4) - (B.5.6) gives equations (5.3.15a).

For the information set in equation (5.3.14b), the particular analogue to equation (B.2.2) is equation (5.3.21). Equation (5.3.23) is the particular analogue to equation (B.2.27) and is obtained by proceeding as in section B.2(B) to obtain equation (B.2.26) which has the particular analogue in equation (5.3.22).

For the information set in equation (5.3.14c), the particular analogue to equation (B.2.2) is equation (5.3.24). Equation (5.3.26) is the particular analogue to equation (B.2.27) and is obtained by proceeding as in section B.2(B) to obtain equation (B.2.26) which has the particular analogue in equation (5.3.25).

The particular full information solution which is the analogue to equations (B.2.21) - (B.2.22) are equations (5.3.279). The particular partial information solutions for the information sets in equations (5.3.14a) - (5.3.14b) which are the analogues to equations (B.2.28) - (B.2.31) are equations (5.3.28a) - (5.3.28b).

B.6 Particular Solution IV : System 6.2

The particular analogue to equation (B.2.1) is equation (6.2.6) with A the 2x2 dynamic matrix, \( z = p \) and \( x = q \). Normalising so that \( M_{22} = 1 \), the definitions in equation (B.2.11) give the particular analogue to \( M^2 \) as \( M^2 = [m_{21} \ 1] \). Then the particular analogues to equations (B.2.19) - (B.2.20) are equations (6.2.10) - (6.2.11). To obtain equation (6.2.12), rewrite equations (6.2.10) - (6.2.11) as

\[
(1-t_1L)e_{t+1} = -m_{21}(1-t_1L)p_{t+1} + d_1(1-t_1L)u_{t+1}^{le} -a_2^{-1}b(1-t_1L)e_{3t+1}^{e} -a_2^{-1}(1-t_1L)e_{4t+1}^{e} - d_1(1-t_1L)u_{t+1}^{le} + d_2b(1-t_1L)e_{3t+1,t+1}^{e} + d_2(1-t_1L)e_{4t+1,t+1}^{e}
\]  
(B.6.1)
(1-\tau_1 L)p_{t+1} = \lambda d_1 u_{t-1} + u_t - \lambda a_2^{-1} b e_{3t} - \lambda a_2^{-1} \epsilon_{4t} - \lambda d_1 u_{t-1, t} + \lambda d_2 b e_{3t, t} + \lambda d_2 \epsilon_{4t, t} \tag{B.6.2}

where \( L \) is the lag operator. Then substituting equation (B.6.2) into equation (B.6.1) gives equation (6.2.12). The expression for \( m_{21} \) follows from

\[
[m_{21} \text{1} \begin{bmatrix} 1-\lambda^{-\tau_2} & \lambda \\ a_1 & a_2^{-\tau_2} \end{bmatrix} = 0. \tag{B.6.3}
\]

The full information analogue to equation (B.2.4) is equation (6.2.14). The partial information analogue to equation (B.2.4) is equation (6.2.20). For a money supply peg, the particular analogue to equation (B.2.2) is equation (6.2.21) and the steps summarized in section B.5 apply. For an interest rate peg, the particular analogue to equation (B.2.2) is equation (6.2.36) with \( C \) the 1x2 right hand side vector. Equation (6.2.38) is the particular analogue to equation (B.2.27) and is obtained by proceeding as in section B.2(B) to obtain equation (B.2.26) which has the particular analogue in equation (6.2.37).

We noted in the main text that it is not possible to retrieve the interest rate peg case from a general expression which imposed no initial restriction on the value of \( \phi \) (and hence \( b \)). The formal proof of this is easy. In fact, all one needs to do is to replace \( \gamma^{-1} \) in the money supply peg case with \( b \). Then \( b \) appears only in the term \( b^2 a_2^{-2} \). Just as \( \gamma^{-2} a_2^{-2} \) cancelled out for the money supply peg case, so \( b^2 a_2^{-2} \) also cancels out here. The optimal estimates of the disturbances are still given by equations (6.2.24) - (6.2.25) and imposing \( \phi = \infty \) (\( b = 0 \)) makes no difference.
The particular full information solution which is the analogue to equations (B.2.21) - (B.2.22) are equations (6.2.15) - (6.2.16). The particular partial information solutions for the money supply peg and interest rate peg cases which are the analogues to equations (B.2.28) - (B.2.31) are equations (6.2.32) - (6.2.33) and (6.2.46) - (6.2.47).

B.7 Particular Solution V : System 6.3

The particular analogue to equation (B.2.1) is the scalar equation (6.3.6) with $A$ and $B$ the scalar parameters $k$ and $n$ respectively, and $X = p$. Since $z = u^1_t = 0$, the particular analogue to equation (B.2.19) is just equation (6.3.8).

The full information analogue to equation (B.2.4) is equation (6.3.10). The partial information analogues to equation (B.2.4) are equations (6.3.16) and (6.3.36). For the information set in equation (6.3.16) and a money supply peg, the particular analogue to equation (B.2.2) is equation (6.3.17) with $C$ and $D$ the scalar parameters $wb$ and $-\gamma_1 b$ respectively. Equation (6.3.20) is the particular analogue to equation (B.2.27) and is obtained by proceeding as in section B.2(B) to obtain equation (B.2.26) which has the particular analogue in equation (6.2.37).

In the main text we noted that it is not possible to retrieve the interest rate peg case from a general expression which imposed no initial restriction on the value of $\phi$ (and hence $b$). To prove this is to simply replace $\gamma^{-1}$ in the money supply peg case with $b$. Then just as $\gamma^{-2}$ cancelled out in the former, so does $b^2$ here. In fact, this can be seen directly from the general expression in equation (6.3.18). The optimal estimates of the disturbances are still given by equations (6.3.21) - (6.3.23) and imposing $\phi = \infty (b = 0)$ makes no difference.
The particular full information solution which is the analogue to equation (B.2.21) is equation (6.3.11). The particular partial information solution for the money supply peg and interest rate peg cases which are the analogues to equations (B.2.28) and (B.2.30) are equations (6.3.28) and (6.3.32).
APPENDIX C

UNDETERMINED COEFFICIENTS SOLUTION TECHNIQUE

FOR RATIONAL EXPECTATIONS MODELS

The appendix summarizes the solution procedure for discrete time rational expectations models as developed by Muth (1961). Further discussion, including extensions and generalizations, can be found in Taylor (1977, 1984), Aoki and Canzoneri (1979), Peel (1981), Minford and Peel (1983b) and Whiteman (1983).

C.1 General Solution

Time is measured discretely with initial time, $t_0$, given. Then consider the dynamic system

\[ \begin{align*}
0 &= \sum_{k=0}^{r_1} A_k y_{t-k} + \sum_{r=r_0}^{r_1} \sum_{s=s_0}^{s_1} A_{rs} y_{t+r,s} + Bu_t \\
& \quad \text{where } y = nxl \text{ vector of endogenous variables}
\end{align*} \tag{C.1.1} \]

\[ y = mxl \text{ vector of stochastic disturbances} \]

and all variables are measured as deviations from long-run equilibrium. In addition, $A_k \ (k = 0, 1)$ and $A_{rs} \ (r = r_0 \geq 0, \ldots, r_1; s = s_0 > 0, \ldots, s_1)$ are nxn time-invariant matrices of coefficients and $B$ is an nxm time-invariant matrix of coefficients. Finally, $0$ is the nxl null vector.

Equation (C.1.1) is entirely general. In particular, any dynamic system with a finite number of lags can be reduced to a first-order difference equation by suitable redefinition of variables and appropriate enlargement of the state vector. Hence, autoregressive processes for stochastic disturbances are incorporated into the state vector so that $y$ is purely white noise. Clearly, the n-dimensional column state vector, $y$, is a point in the Euclidean n-space.
Moreover, the absence of restrictions on expectations terms permits systems in which expectations differ according to both the date of viewpoint and the date of formation.

The information set is defined as

\[ n_{t+r-s} = \{ y_{t+r-j}, \Gamma, \Sigma | j > s; r = r_o \neq 0, \ldots, r_i; s = s_o > 0, \ldots, s_1 \} \]

(C.1.2)

where \( \Gamma \) is a vector of structural parameters and \( \Sigma \) summarizes the variance-covariance structure of stochastic disturbances. Then \( y_{t+r, t+r-s}^e \) is given by

\[ y_{t+r, t+r-s}^e = E(y_{t+r} | n_{t+r-s}), r = r_o \neq 0, \ldots, r_i; s = s_o > 0, \ldots, s_1 \]

(C.1.3)

where \( E(\cdot) \) is the mathematical conditional expectations operator.

The solution proceeds by positing a solution

\[ y_{t-k} = \sum_{i=0}^{\infty} \phi_{i} u_{t-k-i}, \quad k = 0, 1 \]

(C.1.4)

so that

\[ y_{t+r, t+r-s}^e = \sum_{i=s}^{\infty} \phi_{i} u_{t+r-i}, \quad r = r_o \neq 0, \ldots, r_i; s = s_o > 0, \ldots, s_1 \]

(C.1.5)

where \( \phi_i (i > 0) \) are nxm time-invariant matrices of coefficients to be determined. Substituting equations (C.1.4) - (C.1.5) into equation (C.1.1),

\[ q = \sum_{k=0}^{1} A_k (\sum_{i=0}^{\infty} \phi_i u_{t-k-i}) + \sum_{r=r_o \neq 0}^{r_1} \sum_{i=s}^{\infty} A_{rs} (\sum_{i=s_o > 0}^{\infty} \phi_i u_{t+r-i}) + B u_t. \]

(C.1.6)
Then the coefficients on each \( u_{t+j} \) \( (j \neq 0) \) are equated. The assumption that the system possesses a unique convergent rational expectations solution imposes restrictions on the \( \phi_i (i \geq 0) \) coefficients such that each \( \phi_i (i \geq 0) \) coefficient such that each \( \phi_i (i \geq 0) \) is uniquely determined. Then the solution for \( y_t \) follows from equation (C.1.4).

C.2 Particular Solution I : System 3.3(A)

The particular analogue to equation (C.1.1) is equation (3.3.1) such that \( k = 0 \), and \( \rho, \rho_1 = 0, 1, 2 \). In addition, \( A_0 = (\beta \beta + \alpha \alpha) \), \( A_{01} = (\beta \alpha) \), \( A_{12} = -\beta \) and \( B = 1 \). Setting \( \phi = \mu \) in equations (C.1.4) - (C.1.5), equation (3.3.2) obtains and the particular analogue to equation (C.1.6) is

\[
0 = (\beta \beta + \alpha \alpha) \sum_{i=0}^{\infty} \mu_i u_{t-1} + (\beta \alpha) \sum_{i=1}^{\infty} \mu_i u_{t-1} - \beta \sum_{i=2}^{\infty} \mu_i u_{t-1} + u_t
\]

(C.2.1)

from which the identities (3.3.3) - (3.3.4) follow.

C.3 Particular Solution II : System 3.3(B)

The particular analogue to equation (C.1.1) is equation (3.3.19) such that \( k = 0 \) and \( \rho, \rho_1 = 0, 1, 2 \). In addition, \( A_0 = ((\alpha_1 + \alpha_4)(1 + \beta \gamma_1) + \beta \gamma), A_{01} = (\alpha_3(1 + \beta \gamma_1) + \beta), A_{12} = ((\alpha_1 \alpha_2 + \alpha_3)(1 + \beta \gamma_1) + \beta) \) and \( B = 1 \). Setting \( \phi = \delta \) in equations (C.1.4) - (C.1.5), equation (3.3.20) obtains with the particular analogue to equation (C.1.6) being

\[
0 = ((\alpha_1 + \alpha_4)(1 + \beta \gamma_1) + \beta \gamma) \sum_{i=0}^{\infty} \delta_i v_{t-1} + (\alpha_3(1 + \beta \gamma_1) + \beta) \sum_{i=1}^{\infty} \delta_i v_{t-1}

- ((\alpha_1 \alpha_2 + \alpha_3)(1 + \beta \gamma_1) + \beta) \sum_{i=2}^{\infty} \delta_i v_{t-1} + v_t
\]

(C.3.1)

from which the identities (3.3.21) - (3.3.22) follow.
C.4 Particular Solution III : System 3.3(C)

Equation (3.3.38) is the particular analogue to equation (C.1.1) with \( k = 0 \) and \( r_2 = 0,1,11 \). Also, \( A_0 = (\beta(1+b) + ac) \), \( A_0 = -ac \), \( A_{11} = -\beta \) and \( B = 1 \). Setting \( \phi = \xi \) in equations (C.1.4) - (C.1.5), equation (3.3.39) is obtained and the particular analogue to equation (C.1.6) is

\[
0 = (\beta(1+b) + ac) \sum_{i=0}^{\infty} \xi u_{t-i} - \alpha \sum_{i=1}^{\infty} \xi u_{t-i} - \beta \sum_{i=1}^{\infty} \xi u_{t-i+1} + u_t \tag{C.4.1}
\]

and the identities (3.3.40) - (3.3.41) follow.

C.5 Particular Solution IV : System 3.3(D)

The particular analogue to equation (C.1.1) is equation (3.3.51) such that \( k = 0 \) and \( r_2 = 0,1,2,12 \). In addition, \( A_0 = (\alpha b + c) \), \( A_{01} = (\beta - bc) \), \( A_{02} = -bc \), \( A_{12} = -\beta \) and \( B = 1 \). With \( \phi = \psi \) in equations (C.1.4) - (C.1.5), equation (3.3.52) obtains and the particular analogue to equation (C.1.6) is

\[
0 = (\alpha b + c) \sum_{i=0}^{\infty} \psi u_{t-i} + (\beta - bc) \sum_{i=1}^{\infty} \psi u_{t-i} - \beta \sum_{i=2}^{\infty} \psi u_{t-i+1} + u_t \tag{C.5.1}
\]

from which the identities (3.3.53) - (3.3.55) follow.
APPENDIX D

STABILITY AND SADDLEPOINT STABILITY ANALYSIS
BY MEANS OF TEST FUNCTIONS

The appendix sets out the procedure developed by Routh (1905) and Frazer, Duncan and Collar (1938) for determining the conditions for the dynamic stability of a system by determining the number of unstable roots. For continuous time systems, this is equivalent to calculating the number of eigenvalues with positive real part; the discrete time analogue is the number of eigenvalues which lie outside the unit circle. The scheme proceeds by computing the test functions of the characteristic equation of the system.

D.1 General Solution I: Continuous Time

Consider the nth-order polynomial

$$f(\tau) = \sum_{i=0}^{n} a_i \tau^{n-i}$$

$$= a_0 \tau^n + a_1 \tau^{n-1} + \ldots + a_{n-1} \tau + a_n = 0 \quad a_0 > 0$$

(D.1.1)

where \(a_i (i=0, \ldots, n)\) are time-invariant coefficients. Then define the test determinants, \(D_i (i=0, \ldots, n)\), associated with this polynomial such that

\[
D_0 = a_0 ; \quad D_1 = a_1 ; \quad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} \\
D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} ; \ldots
\]
In addition, define also the test functions, $F_i$ $(i=0,\ldots,n)$, associated with the polynomial in equation (D.1.1) such that

$$F_0 = D_0; \quad F_1 = D_1; \quad F_2 = D_1^{-1}D_2$$
$$F_3 = D_2^{-1}D_3; \ldots; \quad F_i = D_{i-1}^{-1}D_i; \ldots; \quad F_n = D_{n-1}^{-1}D_n \quad (D.1.3)$$

Then Routh (1905) and Frazer, Duncan and Collar (1938) show that the number of eigenvalues with positive real part equals the number of sign changes in the sequence of test functions, $F_i$ $(i=0,\ldots,n)$.

D.2 General Solution II : Discrete Time

Consider the $n$th-order polynomial

$$f(\tau) = \sum_{i=0}^{n} a_i \tau^{n-i}$$
$$= a_0 \tau^n + a_1 \tau^{n-1} + \ldots + a_{n-1} \tau + a_n = 0 \quad (D.2.1)$$

where $a_i$ $(i=0,\ldots,n)$ are time-invariant coefficients. To examine
the stability of discrete time systems, equation (D.2.1) must be suitably transformed. Thus, let

\[
\tau = \frac{1 + \mu}{1 - \mu}; \quad \mu = \frac{\tau - 1}{\tau + 1}.
\]  

(D.2.2)

Then \( g(\mu) \) is the nth-order polynomial

\[
g(\mu) = f\left[\frac{1 + \mu}{1 - \mu}\right]
\]

\[
= (1-\mu)^{-n} \sum_{i=0}^{n} \alpha_i (1+\mu)^n - (1-\mu)^i
\]

\[
= (1-\mu)^{-n} \left[ \alpha_0 (1+\mu)^n + \alpha_1 (1+\mu)^{n-1}(1-\mu) + \ldots \right.
\]

\[
\ldots + \alpha_{n-1} (1+\mu)(1-\mu)^{n-1} + \alpha_n (1-\mu)^n \right] = 0.
\]  

(D.2.3)

Clearly, the mapping \( \tau \to \mu \) maps the unit circle to the negative real plane. Hence, the unstable discrete roots of \( f(\tau) \) correspond to the unstable continuous roots of \( g(\mu) \). It is assumed that \( \mu \neq 1 \), thereby excluding the possibility of a pole (discontinuity of \( g(\mu) \)) at \( \mu = 1 \).

The roots of \( g(\mu) \) are the roots of the expression in \([\cdot]\) in equation (D.2.3). Then

\[
g(\mu) = \sum_{i=0}^{n} \alpha_i (1+\mu)^{n-i}(1-\mu)^i
\]

\[
= \alpha_0 (1+\mu)^n + \alpha_1 (1+\mu)^{n-1}(1-\mu) + \ldots + \alpha_{n-1} (1+\mu)(1-\mu)^{n-1} + \alpha_n (1-\mu)^n
\]

\[
= \sum_{i=0}^{n} \delta_i \mu^{n-i}
\]

\[
= \delta_0 \mu^n + \delta_1 \mu^{n-1} + \ldots + \delta_{n-1} \mu + \delta_n = 0
\]  

(D.2.4)

Here, \( \delta_0 = \sum_{i=0}^{n} (-1)^i \alpha_i \) and \( \delta_n = \sum_{i=0}^{n} \alpha_i \), and equation (D.2.4) is
re-written as

\[
g(\mu) = \sum_{i=0}^{n} \beta_i \mu^{n-i}
\]

\[
= \beta_0 \mu^n + \beta_1 \mu^{n-1} + \ldots + \beta_{n-1} \mu + \beta_n = 0 \quad \beta_0 > 0
\]

(D.2.5)

Now define the test determinants, \( D_i \) \((i=0, \ldots, n)\), associated with the polynomial in equation (D.2.5) such that

\[
D_0 = \begin{vmatrix} \beta_0 \end{vmatrix} \quad D_1 = \begin{vmatrix} \beta_0 & \beta_1 \end{vmatrix} \quad D_2 = \begin{vmatrix} \beta_1 & \beta_0 \\ \beta_3 & \beta_2 \end{vmatrix}
\]

\[
D_3 = \begin{vmatrix} \beta_1 & \beta_0 & 0 \\ \beta_3 & \beta_2 & \beta_1 \\ \beta_5 & \beta_4 & \beta_3 \end{vmatrix} \quad \ldots \ldots
\]

\[
D_i = \begin{vmatrix} \beta_1 & \beta_0 & 0 & \ldots & 0 \\ \beta_3 & \beta_2 & \beta_1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{2i-1} & \beta_{2i-2} & \ldots & 0 & 0 \end{vmatrix}
\]

(D.2.6)

In addition, define also the test functions, \( F_i \) \((i=0, \ldots, n)\), associated with the polynomial in equation (D.2.5) such that
Routh (1905) and Frazer, Duncan and Collar (1938) now show that the number of eigenvalues which lie outside the unit circle equals the number of sign changes in the sequence of test functions, \( F_i \) (\( i = 0, \ldots, n \)).

**D.3 Particular Solution I : System 2.4(B)**

The particular analogue to equation (D.2.1) is equation (2.4.29) with \( \sigma_0 = 1; \sigma_1 = -(1-\lambda+\delta_2); \sigma_2 = (\delta_2-\lambda\delta_2-\lambda^2) \) so that \( i = 0,1,2 \) and \( n = 2 \). Then the particular analogue to equation (D.2.4) gives

\[
\delta_0 = (\sigma_0-\sigma_1+\sigma_2); \quad \delta_1 = 2(\sigma_0-\sigma_2); \quad \delta_2 = (\sigma_0+\sigma_1+\sigma_2).
\]

Dividing through by \( \delta_0 \) gives the particular analogue to equation (D.2.5) with \( \beta_0 = 1; \beta_1 = (\sigma_0-\sigma_1+\sigma_2)^{-1}2(\sigma_0-\sigma_2); \beta_2 = (\sigma_0-\sigma_1+\sigma_2)^{-1}(\sigma_0+\sigma_1+\sigma_2). \) The full expressions for \( \beta_1 \) and \( \beta_2 \) are respectively

\[
\beta_1 = \frac{(2-\lambda)(1+\phi_1\gamma_1) - \lambda(\phi_1+\phi_2)}{2(2-\lambda)(1+\phi_1\gamma_1) - \lambda(\phi_1+\phi_2)(1-\phi_3)} \quad \text{(D.3.1)}
\]

\[
\beta_2 = \frac{-\lambda(\phi_1+\phi_2)(1+\phi_3)}{2(2-\lambda)(1+\phi_1\gamma_1) - \lambda(\phi_1+\phi_2)(1-\phi_3)} \quad \text{(D.3.2)}
\]

and the particular test determinants in equation (D.2.6) are

\[
D_0 = 1; \quad D_1 = \beta_1; \quad D_2 = \beta_1\beta_2. \quad \text{(D.3.3)}
\]

The particular test functions in equation (D.2.7) are therefore

\[
F_0 = 1; \quad F_1 = \beta_1; \quad F_2 = \beta_2 \quad \text{(D.3.4)}
\]
which gives the saddlepath stability condition in equation (2.4.30) with $\beta_1$ and $\beta_2$ defined as in equations (D.3.1) – (D.3.2).

### D.4 Particular Solution II : System 3.3(E)

The particular analogue to equation (D.1.1) is equation (3.3.61) with $\alpha_0 = 1; \alpha_1 = \lambda c; \alpha_2 = \lambda \beta \theta_1^2 \theta_2; \alpha_3 = -\lambda \alpha \theta_1^2 \theta_2$ so that $i = 0,1,2,3$ and $n = 3$. Then the particular test determinants in equation (D.1.2) are

$$D_0 = 1; \quad D_1 = \alpha_1; \quad D_2 = \alpha_1 \alpha_2 - \alpha_0 \alpha_3; \quad D_3 = \alpha_3 (\alpha_1 \alpha_2 - \alpha_0 \alpha_3)$$

(D.4.1)

The particular test functions in equation (D.1.3) are therefore

$$F_0 = 1; \quad F_1 = \alpha_1; \quad F_2 = \alpha_2 - \alpha_1 \alpha_0 \alpha_3; \quad F_3 = \alpha_3$$

(D.4.2)

which gives the saddlepath stability condition in equation (3.3.62).

### D.5 Particular Solution III : System 3.3(F)

The particular analogue to equation (D.1.1) is equation (3.3.83) with $\alpha_0 = 1; \alpha_1 = -(n_2 - \lambda (c - \beta_1 \theta)); \alpha_2 = -\lambda [\beta_2 (by_1 - n_1) + n_2 (c - \beta_1 \theta) - \theta (\beta_1 b + \beta_2)]; \alpha_3 = -\lambda \theta b (\beta_2 + n_2 \beta_1)$ so that $i = 0,1,2,3$ and $n = 3$.

Then the particular test determinants in equation (D.1.2) are

$$D_0 = 1; \quad D_1 = \alpha_1; \quad D_2 = \alpha_1 \alpha_2 - \alpha_0 \alpha_3; \quad D_3 = \alpha_3 (\alpha_1 \alpha_2 - \alpha_0 \alpha_3)$$

(D.5.1)

The particular test functions in equation (D.1.3) are therefore

$$F_0 = 1; \quad F_1 = \alpha_1; \quad F_2 = \alpha_2 - \alpha_1 \alpha_0 \alpha_3; \quad F_3 = \alpha_3$$

(D.5.2)

which gives the saddlepath stability condition in equation (3.3.84).
D.6 Particular Solution IV : System 5.3

The particular analogue to equation (D.2.1) is equation (5.3.7) with \( \alpha_0 = 1; \alpha_1 = -(2+n-\lambda); \alpha_2 = (1+n-\lambda(1+\gamma^{-1})) \) so that \( i = 0,1,2 \) and \( n = 2. \) Then the particular analogue to equation (D.2.4) gives

\[
\delta_0 = (\alpha_0 - \alpha_1 + \alpha_2); \quad \delta_1 = 2(\alpha_0 - \alpha_2); \quad \delta_2 = (\alpha_0 + \alpha_1 + \alpha_2). \quad \text{(Dividing through by } \delta_0 \text{ gives the particular analogue to equation (D.2.5) with)}
\]

\[
\beta_0 = 1; \quad \beta_1 = (\alpha_0 - \alpha_1 + \alpha_2)^{-1} 2(\alpha_0 - \alpha_2); \quad \beta_2 = (\alpha_0 - \alpha_1 + \alpha_2)^{-1} (\alpha_0 + \alpha_1 + \alpha_2). \quad \text{(The full expressions for } \beta_1 \text{ and } \beta_2 \text{ are respectively)}
\]

\[
\beta_1 = \frac{2(\lambda(1+\gamma^{-1})-n)}{2(2+n-\lambda)-\lambda \gamma^{-1}}, \quad \text{(D.6.1)}
\]

\[
\beta_2 = \frac{-\lambda \gamma^{-1}}{2(2+n-\lambda)-\lambda \gamma^{-1}}, \quad \text{(D.6.2)}
\]

and the particular test determinants in equation (D.2.6) are

\[
D_0 = 1; \quad D_1 = \beta_1; \quad D_2 = \beta_1 \beta_2. \quad \text{(D.6.3)}
\]

The particular test functions in equation (D.2.7) are therefore

\[
F_0 = 1; \quad F_1 = \beta_1; \quad F_2 = \beta_2 \quad \text{(D.6.4)}
\]

which gives the saddlepath stability condition in equation (5.3.9) with \( \beta_1 \) and \( \beta_2 \) defined as in equations (D.6.1) - (D.6.2).

D.7 Particular Solution V : System 6.2

The particular analogue to equation (D.2.1) is equation (6.2.7) with \( \alpha_0 = 1; \alpha_1 = -(2+n-\lambda); \alpha_2 = (1+n-\lambda(1+b)) \) so that \( i = 0,1,2 \) and \( n = 2. \) Then the particular analogue to equation (D.2.4) gives

\[
\delta_0 = (\alpha_0 - \alpha_1 + \alpha_2); \quad \delta_1 = 2(\alpha_0 - \alpha_2); \quad \delta_2 = (\alpha_0 + \alpha_1 + \alpha_2). \quad \text{(Dividing through through by } \delta_0 \text{ gives the particular analogue to equation (D.2.5) with)}
\]

\[
\beta_0 = 1; \quad \beta_1 = (\alpha_0 - \alpha_1 + \alpha_2)^{-1} 2(\alpha_0 - \alpha_2); \quad \beta_2 = (\alpha_0 - \alpha_1 + \alpha_2)^{-1} (\alpha_0 + \alpha_1 + \alpha_2). \quad \text{(The full expressions for } \beta_1 \text{ and } \beta_2 \text{ are respectively)}
\]
by $\delta_0$ gives the particular analogue to equation (D.2.5) with

$$\beta_0 = 1; \quad \beta_1 = (\alpha_0 - \alpha_1 + \alpha_2)^{-1}2(\alpha_0 - \alpha_2); \quad \beta_2 = (\alpha_0 - \alpha_1 + \alpha_2)^{-1}(\alpha_0 + \alpha_1 + \alpha_2).$$

The full expressions for $\beta_1$ and $\beta_2$ are respectively

$$\beta_1 = \frac{2(\lambda(1+b)-n)}{2(2+n-\lambda)-\lambda b} \quad (D.7.1)$$

$$\beta_2 = \frac{-\lambda b}{2(2+n-\lambda)-\lambda b} \quad (D.7.2)$$

and the particular test determinants in equation (D.2.6) are

$$D_0 = 1; \quad D_1 = \beta_1; \quad D_2 = \beta_1\beta_2. \quad (D.7.3)$$

The particular test functions in equation (D.2.7) are therefore

$$F_0 = 1; \quad F_1 = \beta_1; \quad F_2 = \beta_2 \quad (D.7.4)$$

which gives the saddlepath stability condition in equation (6.2.9)

with $\beta_1$ and $\beta_2$ defined as in equations (D.7.1) - (D.7.2).
APPENDIX E

ASYMPTOTIC VARIANCE-COVARIANCE MATRIX FOR LINEAR SIMULTANEOUS STOCHASTIC DIFFERENTIAL EQUATION SYSTEMS

The appendix gives the procedure for deriving the asymptotic variance-covariance matrix of the state vector in a system of linear stochastic differential equations as developed by Chow (1979).

E.1 General Solution

Consider the stochastic differential equation system

\[ dy(t) = Ay(t)dt + du(t) \]  \hspace{1cm} (E.1.1)

where \( y \) = nx1 vector of predetermined variables

\( du \) = nx1 vector of stochastic disturbances

and A is an nxn time-invariant matrix of coefficients. Note that \( y(t) \) consists solely of predetermined variables. In rational expectations models where the initial state vector includes non-predicted variables, this must be reduced to a final state vector comprising only predetermined variables. Appendix B gives the procedure for this. In addition, note that the nx1 disturbance vector is given by \( du(t) = edt \) where \( edt \) is an nx1 vector of stochastic disturbances with independent increments so that \( du(t) \) is white noise. Any autoregressive disturbances are therefore incorporated by suitable expansion of the state vector. Clearly, the n-dimension column state vector, \( y(t) \), is a point in the n-dimensional Euclidean space, \( \mathbb{E}^n \).

Now define \( \Sigma \) as the nxn variance-covariance matrix of the nx1 state vector, \( y(t) \), and \( Edt \) as the nxn variance-covariance matrix of
the nxl disturbance vector, \( du(t) \). Also, recall the definition of \( dy(t) \) as

\[
\begin{align*}
dy(t) &= y(t+dt) - y(t). \\
\end{align*}
\]  
(E.1.2)

Then

\[
\begin{align*}
\ddot{\xi}(t) &= \mathbb{E}y(t)\mathbf{x}^T(t) \\
\ddot{\xi}(t+dt) &= \mathbb{E}y(t+dt)\mathbf{x}^T(t+dt) \\
\end{align*}
\]  
(E.1.3)

(E.1.4)

where \( \mathbb{E}(\cdot) \) is the expectations operator. It therefore follows that

\[
\begin{align*}
d\dot{\xi} &= \ddot{\xi}(t+dt) - \ddot{\xi}(t) \\
&= \mathbb{E}y(t+dt)\mathbf{x}^T(t+dt) - \mathbb{E}y(t)\mathbf{x}^T(t) \\
&= \mathbb{E}[y(t) + dy(t)][y(t) + dy(t)]^T - \mathbb{E}y(t)\mathbf{x}^T(t) \\
&= \mathbb{E}y(t)dy^T(t) + \mathbb{E}dy(t)y^T(t) + \mathbb{E}dy(t)dy^T(t) \\
&= \mathbb{E}[y(t)][Ay(t)dt + du(t)]^T + \mathbb{E}[Ay(t)dt + du(t)]y^T(t) \\
&\quad + \mathbb{E}[Ay(t)dt + du(t)][Ay(t)dt + du(t)]^T. \\
\end{align*}
\]  
(E.1.5)

Since \( y(t) \) and \( du(t) \) are independent, \( \mathbb{E}y(t)du^T(t) = \mathbb{E}du(t)y^T(t) = 0 \), so that equation (E.1.5) becomes

\[
\begin{align*}
d\dot{\xi} &= \mathbb{E}y(t)y^T(t)A^Tdt + A\mathbb{E}y(t)y^T(t)dt + \mathbb{E}du(t)du^T(t)dt \\
&= \mathbb{E}y(t)y^T(t)A^Tdt + \mathbb{E}y(t)y^T(t)dt + \mathbb{E}du(t)du^T(t)dt. \\
\end{align*}
\]  
(E.1.6)

Then dividing through by \( dt \),
\[
\frac{\ddot{\mathbf{r}}}{dt} = -A\dot{T} + A\dot{\mathbf{r}} + \mathbf{E}
\]

so that setting \(\frac{d\mathbf{r}}{dt} = 0\) gives the asymptotic variance-covariance matrix of \(\mathbf{y}(t)\), \(\Sigma_o\), as

\[
\Sigma_o A^T + A\Sigma_o + \mathbf{E} = 0.
\]

Note that since both \(\Sigma_o\) and \(\Sigma\) are \(nxn\) symmetric matrices, equation (E.1.8) provides \('n(n+1)\) independent simultaneous equations which can be solved for the \('n(n+1)\) independent elements of \(\Sigma_o\) in terms of the parameters of \(A\) and \(\Sigma\).

E.2 Particular Solution I : System 2.3(E)

The particular analogue to equation (E.1.1) is equation (3.3.67) with \(A\) the \(2x2\) matrix \(S\). Then the particular analogue to equation (E.1.8) is equation (3.3.68) such that

\[
\begin{bmatrix}
\sigma_y^2 & \sigma_{y'p} \\
\sigma_{yp} & \sigma_p^2
\end{bmatrix}
\begin{bmatrix}
S_{11} & S_{21} \\
S_{21} & S_{22}
\end{bmatrix}
+ \begin{bmatrix}
\sigma_y^2 & \sigma_{y'p} \\
\sigma_{yp} & \sigma_p^2
\end{bmatrix}
+ \begin{bmatrix}
\sigma_1^2 & 0 \\
0 & \sigma_2^2
\end{bmatrix}
= 0
\]

(E.2.1)

This gives the three independent simultaneous equations in \(\sigma_y^2\), \(\sigma_p^2\) and \(\sigma_{yp}'\)

\[
2S_{11}\sigma_y^2 + 2S_{12}\sigma_{yp} = -\sigma_1^2
\]

(E.2.2)

\[
2S_{22}\sigma_p^2 + 2S_{21}\sigma_{yp} = -\sigma_2^2
\]

(E.2.3)

\[
S_{21}\sigma_y^2 + S_{12}\sigma_p^2 + (S_{11} + S_{22})\sigma_{yp} = 0
\]

(E.2.4)

which can be written in matrix form as in equation (3.3.70).
E.3 Particular Solution II: System 3.3(F)

The particular analogue to equation (E.1.1) is equation (3.3.85) with A the 3x3 matrix, \( L \). Then the particular analogue to equation (E.1.8) is equation (3.3.86) such that

\[
\begin{bmatrix}
\sigma_y^2 & \sigma_{yp} & \sigma_{yv} \\
\sigma_{py}^2 & \sigma_p & \sigma_{pv} \\
\sigma_{vy}^2 & \sigma_{vp} & \sigma_v
\end{bmatrix}
\begin{bmatrix}
1_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_y^2 \\
\sigma_p \\
\sigma_v
\end{bmatrix}
\begin{bmatrix}
1_{11} & 1_{12} & 1_{13} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_{yp} \\
\sigma_p \\
\sigma_{pv}
\end{bmatrix}
\begin{bmatrix}
1_{13} & 0 & -\rho \\
0 & 0 & -\rho \\
0 & 0 & -\rho
\end{bmatrix}
\begin{bmatrix}
\sigma_{yv} \\
\sigma_{vp} \\
\sigma_v
\end{bmatrix}
\]

This gives six independent simultaneous equations in \( \sigma_y^2, \sigma_p^2, \sigma_v^2, \sigma_{yp}, \sigma_{yv} \), and \( \sigma_{pv} \):

\[
\begin{align*}
21_{11}\sigma_y^2 + 21_{12}\sigma_{yp} + 21_{13}\sigma_{yv} &= -\sigma_1^2 \\
\sigma_y^2 + 1_{11}\sigma_{yp} + 1_{12}\sigma_p^2 + 1_{13}\sigma_{vp} &= 0 \\
(1_{11} - \rho)\sigma_{yv} + 1_{12}\sigma_{vp} + 1_{13}\sigma_v^2 &= 0 \\
2\sigma_{yp} &= -\sigma_2^2 \\
-\rho\sigma_{pv} + \epsilon\sigma_{yv} &= 0 \\
-2\sigma_v^2 &= -\sigma_3^2.
\end{align*}
\]
From equations (E.3.5) – (E.3.7),

\[ \sigma_{yp} = -r \theta^{-1} \sigma_2 \]  
\[ \sigma_{pv} = \rho^{-1} \theta \sigma_{yv} \]  
\[ \sigma_v^2 = 4 \rho^{-1} \sigma_3 \]  

so that substituting equations (E.3.8) – (E.3.10) into equations (E.3.2) – (E.3.4),

\[ 21_{11} \sigma_y^2 - 1 \theta^{-1} \sigma_2 + 21_{13} \sigma_{yv} = -\sigma_1 \]  
\[ \theta \sigma_y^2 - 41_{11} \theta^{-1} \sigma_2 + 41_2 \sigma_p^2 + 1 \theta \sigma_{yv} = 0 \]  
\[ (1_{11} - \rho + 41_2 \theta^{-1}) \sigma_{yv} + 41_{13} \theta^{-1} \sigma_3^2 = 0 \]

Finally, multiplying through equation (E.3.13) by \( \rho \theta^{-1} \) and rearranging,

\[ 21_{11} \sigma_y^2 + 21_{13} \sigma_{yv} = -\sigma_1^2 + 41_2 \theta^{-1} \sigma_2^2 \]  
\[ \theta \sigma_y^2 + 41_2 \sigma_p^2 + 1 \theta \sigma_{yv} = 41_{11} \theta^{-1} \sigma_2^2 \]  
\[ ((1_{11} - \rho) \rho \theta^{-1} + 41_2) \sigma_{yv} = -41_{13} \theta^{-1} \sigma_3^2 \]  
or
Applying Cramer's Rule to equation (E.3.27) then gives the expressions for $\sigma_y^2$ and $\sigma_p^2$ in equations (3.3.88) – (3.3.89) respectively.
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