EQUIVARIANT CHOW CLASSES OF MATRIX ORBIT CLOSURES

ANDREW BERGET AND ALEX FINK

ABSTRACT. Let G be the product $\mathrm{GL}_r(\mathbf{C}) \times (\mathbf{C}^\times)^n$. We show that the G-equivariant Chow class of a G orbit closure in the space of r-by-n matrices is determined by a matroid. To do this, we split the natural surjective map from the G equvariant Chow ring of the space of matrices to the torus equivariant Chow ring of the Grassmannian. The splitting takes the class of a Schubert variety to the corresponding factorial Schur polynomial, and also has the property that the class of a subvariety of the Grassmannian is mapped to the class of the closure of those matrices whose row span is in the variety.

1. INTRODUCTION

The first goal of this paper is to prove that the Chow class of a certain affine variety determined by a r-by-n matrix is a function of the matroid of that matrix. Specifically, given an r-by-n matrix v with complex entries, we let X_v° denote the set of those matrices that are projectively equivalent to v in the sense that they are of the form gvt^{-1}, where $g \in \mathrm{GL}_r(\mathbf{C})$, and $t \in \mathrm{GL}_n(\mathbf{C})$ is a diagonal matrix. Let G be the group consisting of pairs of matrices (g,t), which acts on the space $\mathbf{A}^{r\times n}$ of r-by-n matrices via the rule $(g,t)v = gvt^{-1}$. A matrix orbit closure X_v is the Zariski closure of X_v° in $\mathbf{A}^{r\times n}$; it is the G orbit closure of v. This variety determines a class in the G equivariant Chow ring of $\mathbf{A}^{r\times n}$. Theorem 4.3 states that this class depends only on the matroid of v.

This matroid invariance is a consequence of two results. The first result is the matroid invariance of the class of a torus orbit closure in the torus equivariant K-theory of the Grassmannian $G(r,n)$. This result was shown by Speyer [Spe09] and was used by Speyer and the second author to find a purely algebro-geometric interpretation of the Tutte polynomial [FS12]. The second result which our matroid invariance relies on deals with the relationship between the G equivariant Chow ring of $\mathbf{A}^{r\times n}$ and the torus equivariant Chow ring of $G(r,n)$, which we now explain.

The geometry of a particular subvariety Y of the Grassmannian $G(r,n)$ (or more generally, a partial flag variety) is of interest. To study it, one constructs a certain matrix analog of Y, defined to be the closure in $\mathbf{A}^{r\times n}$ of $\pi^{-1}(Y)$ where π is the projection from the space full rank r-by-n matrices to $G(r,n)$. Let X denote this matrix analog, which is a $\mathrm{GL}_r(\mathbf{C})$ invariant subvariety of $\mathbf{A}^{r\times n}$. Sometimes X can be effectively studied using the techniques of combinatorial commutative algebra, in the sense that its prime ideal is recognizable and a Gröbner basis can be produced from it. If the original variety Y had the action of a subgroup of $\mathrm{GL}_n(\mathbf{C})$ (acting on the Grassmannian in the usual way), then X has the action of the product of $\mathrm{GL}_r(\mathbf{C})$ and this group. Such analogs have been constructed for Schubert varieties,

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first by Fulton [Ful92] and later by Knutson, Miller [KM05] and their collaborators (e.g., [KMS06, KMY09]). Knutson considered the matrix analog of Richardson varieties in [Knu10]. Recently, Weyser and Yong have constructed such analogs for symmetric pair orbit closures in flag varieties [WY14].

The matrix analog of a torus orbit closure in $G(r,n)$ is precisely a variety of the form X_v, where v has rank r. In this case, the matrix analog appears to be, in some sense, more complicated than the original variety. Set-theoretic equations are known for X_v, but they are not known to generate its prime ideal. It is natural to ask if the apparent added complications are visible in various algebraic invariants of X_v. This is how our second main result of the paper arose. We will prove the following theorem.

Theorem. *The natural surjective map of $\mathbf{Z}[t_1,\dots,t_n]$-modules,*

$$A^*_G(\mathbf{A}^{r\times n}) \to A^*_T(G(r,n)),$$

has a splitting s that satisfies the following properties: For every closed, irreducible, T-invariant subvariety $Y \subset G(r,n)$,

- (i) $s([Y]_T) = [\overline{\pi^{-1}Y}]_G$,
- (ii) $s([Y]_T)$ *is a $\mathbf{Z}[t_1,\dots,t_n]$-linear combination of classes of matrix Schubert varieties $[X_\lambda]_G$.*

The structure of our paper is as follows. In Section 2.1 we provide the required background on equivariant Chow groups. In Section 3 we recall results of Fehér and Rimányi used to bound polynomial degrees in Chow classes, and use these results to prove Theorem 3.5, which is the theorem stated above. In Section 4 we use Theorem 3.5 to prove Theorem 4.3 on the matroid invariance of the class of X_v. Lastly, in Section 5 we use equivariant localization to compute the Chow class of a sufficiently generic torus orbit closure in $G(r,n)$, and use Theorem 4.3 to compute the Chow class of X_v when v has a uniform rank r matroid.

2. Equivariant Chow ring

2.1. **Background on Chow groups and rings.** A variety is a reduced and irreducible scheme over $\mathbf{C}$ and a subvariety is a closed subscheme which is a variety. Let X be a variety over $\mathbf{C}$. Assume that X has an action of a reductive linear algebraic group G. Our main references for equivariant Chow groups are [Bri97, EG98a].

Let V be a representation of G containing an open subvariety U which is the total space of a principal G-bundle. Such a representation always exists because G is reductive. The product $X \times U$ has a free G action, so the quotient space $X \times_G U := (X \times U)/G$ is a variety. Assume the codimension of U in V is larger than some integer k. The G-equivariant Chow group of X of degree k is defined as

$$A_k^G(X) := A_{k+\dim(V)-\dim(G)}(X \times_G U),$$

and this is independent of the choice of U. Here $A_k(-)$ is the usual Chow group of dimension k cycles on $-$, modulo rational equivalence. If $Y \subset X$ is a G-invariant subvariety of codimension k, then Y defines a class $[Y]_G := [Y \times_G U]$ in $A^k_G(X)$.

For all integers k, there is an exact sequence of groups

$$A_k^G(Y) \to A_k^G(X) \to A_k^G(X - Y) \to 0 \tag{1}$$

where the former map is pushforward and the latter is pullback. In general, any proper G-equivariant map $f : Y \to Z$ gives rise to a pushforward map $A_k^G(Y) \to$

$A_k^G(Z)$ and any flat G-equivariant map $X \to Y$ gives rise to a pullback map $A_k^G(Y) \to A_k^G(X)$.

Proposition 2.1. *Let $Y \subset X$ be a closed, irreducible, G-invariant subvariety of dimension d. Then,*

(i) *$[Y]_G$ freely generates* $\ker(A_d^G(X) \to A_d^G(X - Y))$,
(ii) *for all $j > d$, we have* $\ker(A_j^G(X) \to A_j^G(X - Y)) = 0$.

Proof. This follows because $A_j^G(Y) = 0$ for $j > d$ and because $A_d^G(Y)$ is freely generated by $[Y]_G$. □

Assume X is smooth. Write $A_G^k(X) = A^G_{\dim(X)-k}(X)$ and define

$$A_G^*(X) := \bigoplus_{k \geq 0} A_G^k(X).$$

Since X is smooth, this group can be endowed with the intersection product, for which the element $[X]_G \in A_G^0(X)$ is a multiplicative identity. The group $A_G^*(X)$ becomes a graded commutative ring called the G-equivariant Chow ring of X. This name reflects the fact that $A_G^*(X)$ is the Chow ring of $X \times_G U$.

When the open complement $X - Y \subset X$ is smooth, one obtains a surjective map of graded rings $A_G^*(X) \to A_G^*(X - Y)$.

Corollary 2.2. *Suppose that $Y \subset X$ is an irreducible G-invariant subvariety of codimension k with a smooth complement $X - Y$. Then the kernel of the pullback $A_G^*(X) \to A_G^*(X - Y)$ is a graded ideal satisfying:*

(i) *$[Y]_G$ freely generates* $\ker(A_G^k(X) \to A_G^k(X - Y))$, *and*
(ii) *for all $j < k$,* $\ker(A_G^j(X) \to A_G^j(X - Y)) = 0$.

Remark 2.3. The ideal $\ker(A_G^*(X) \to A_G^*(X - Y))$ is not necessarily principal.

2.2. **K-theory and Chow groups of affine spaces.** We will briefly need the torus equivariant K-theory of an affine space $\mathbf{A}$ and its relation to the equivariant Chow ring.

Let $K_0^G(X)$ denote the Grothendieck group of G-equivariant coherent sheaves on X. When X is smooth, this is generated by the classes of locally free sheaves, and this group becomes a ring with product being the tensor product of locally free sheaves.

When $X = \mathbf{A}$ is an affine space, then $K_0^G(\mathbf{A})$ is simply $K_0^G(\mathrm{pt})$ which is the representation ring of the group G. The class of a representation corresponds to a trivial bundle over $\mathbf{A}$ with G action determined by the representation. If G is a torus $(\mathbf{C}^\times)^m$ then $K_0^G(\mathbf{A})$ is a Laurent polynomial ring in m variables $\mathbf{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Similarly, the equivariant Chow ring of $\mathbf{A}$ is $\mathbf{Z}[t_1, \dots, t_m]$. Here, a trivial line bundle twisted by a character is mapped to its first equivariant Chern class.

If $Y \subset \mathbf{A}$ is a subvariety of $\mathbf{A}$ then we write $\mathcal{K}(Y)$ for the class of the structure sheaf of Y in $K_0^G(\mathbf{A})$. There is a recipe to obtain $[Y]_G$ from $\mathcal{K}(Y)$ [KMS06, Proposition 1.9].

Proposition 2.4 (Knutson–Miller–Shimozono)**.** *To obtain $[Y]_G$ from $\mathcal{K}(Y)$, first replace each t_i with $1 - t_i$ and expand the result as a formal power series in the t_i. Gather the monomials of lowest possible total degree, which will be the codimension of Y in $\mathbf{A}$. The result is $[Y]_G$.*

3. Splitting of the localization sequence

We now specialize the set-up of Section 2.1 to our main case of interest. Let $\mathbf{A}^{r\times n}$, $r \le n$, be the affine space of r-by-n matrices with entries in $\mathbf{C}$. This has an action by $G = \mathrm{GL}_r(\mathbf{C}) \times T$, where $T = (\mathbf{C}^\times)^n$ is the algebraic n-torus acting by $(g,t)\cdot m = gmt^{-1}$, viewing $t \in T$ as a diagonal matrix. For the remainder of our work G will denote this product of groups.

3.1. **Degree bound of Fehér and Rimányi.** The equivariant Chow ring of $\mathbf{A}^{r\times n}$ is the equivariant Chow ring of a point, since $\mathbf{A}^{r\times n}$ is a vector bundle over a point. We can succinctly describe this object [EG98a, Proposition 6]: It is the ring of Weyl group invariants of the polynomial ring over the character lattice of a maximal torus of G. Specifically,

$$A^*_G(\mathbf{A}^{r\times n}) = \mathbf{Z}[u_1,\dots,u_r,t_1,\dots,t_n]^{S_r},$$

where the symmetric group S_r acts by permuting the subscripts on the u variables. Here the u variables represent the characters of the diagonal torus in $\mathrm{GL}_r(\mathbf{C})$ and the t variables represent the characters of the torus T.

We consider the map $\mathbf{A}^{r\times n} \to \mathbf{A}^{(r-1)\times n}$ that forgets the last row of a matrix. Given $Y \subset \mathbf{A}^{r\times n}$ we let Y_0 denote the image of Y under the above map, which is itself closed and irreducible.

The following result relates the classes of Y and Y_0 and is a special case of a more general result due to Fehér and Rimányi [FR07, Theorem 2.1].

Theorem 3.1. *Let Y and Y_0 be as above, which we take to have codimension c and c_0, respectively. Write $[Y]_G \in A^c_G(\mathbf{A}^{(r+1)\times n})$ uniquely as*

$$[Y]_G = \sum_k p_k(u_1,\dots,u_{r-1},t_1,\dots,t_n)\cdot u_r^{c-k},$$

where p_k is a homogeneous polynomial of degree k. Write $G_0 = \mathrm{GL}_{r-1}(\mathbf{C})\times T \subset G$. Then for all $k \ge 0$, p_k is in the kernel of $A^*_{G_0}(\mathbf{A}^{(r-1)\times n}) \to A^*_{G_0}(\mathbf{A}^{(r-1)\times n} - Y_0)$. In particular, the degree of u_r in $[Y]_G$ is at most $c - c_0$.

Corollary 3.2. *Using the notation above, if Y is the closure of its full rank matrices then the degree of u_r in $[Y]_G$ is at most $n - r$.*

Proof. It suffices to verify that $c - c_0 \le n - r$, and this is equivalent to $\dim(Y) - \dim(Y_0) \ge r$. To verify this, we consider the non-empty open subset of Y_0 of rank $r-1$ matrices. The fiber of the natural map $Y \to Y_0$ over a matrix of rank $r-1$ is at least $(r-1)+1$ dimensional, since Y is $\mathrm{GL}_r(\mathbf{C})$ invariant. Since the dimension of this general fiber is precisely $\dim(Y) - \dim(Y_0)$, we are done. □

3.2. **Matrix Schubert varieties.** The **Schubert varieties** of the Grassmannian $G(r,n)$ are $B \subset \mathrm{GL}_n(\mathbf{C})$ orbit closures, where B is a Borel subgroup. Fixing such a B, the Schubert varieties are in bijection with partitions $\lambda = (\lambda_1 \ge \cdots \ge \lambda_r \ge 0)$ with $\lambda_1 \le n - r$. We denote the Schubert variety corresponding to λ by Ω_λ. Since the Schubert varieties Ω_λ form a stratification of $G(r,n)$, the classes $[\Omega_\lambda]_T$ form a $\mathbf{Z}[t_1,\dots,t_n]$-linear basis of $A^*_T(G(r,n))$.

For $r < n$, denote the set of $f$ull $r$ank r-by-n matrices by $\mathbf{A}^{\mathrm{fr}}$. Let $\pi : \mathbf{A}^{\mathrm{fr}} \to G(r,n)$ denote the projection map, which sends a matrix to the span of its rows. Define a **matrix Schubert variety** as $X_\lambda = \overline{\pi^{-1}(\Omega_\lambda)}$, where the closure takes place within $\mathbf{A}^{r\times n}$. The equivariant Chow classes of these varieties were computed by Knutson, Miller and Yong [KMY09]. It is important here that one computes the class of the matrix Schubert variety X_λ, and not a *representative* for the class of the Schubert variety Ω_λ. This is to say, one does computations in $A_G^*(\mathbf{A}^{r\times n})$ instead of a quotient of this ring.

Let λ be a partition, which we regard simultaneously as a decreasing sequence of non-negative integers, as above, and as a set $\lambda = \{c_{ij} : 1 \leq i \leq \lambda_j\}$. We say that $c_{ij} \in \lambda$ is above (or left) of $c_{k\ell} \in \lambda$ if $j < \ell$ (or $i < k$). A **tableau** is a function $\tau : \lambda \to \mathbf{N}^+$.

A tableau $\tau : \lambda \to \mathbf{N}^+$ is said to be **semistandard** provided that for all $c, d \in \lambda$

(i) if c lays to the left of d then $\tau(c) \leq \tau(d)$; and
(ii) if c lays above d then $\tau(c) < \tau(d)$.

We let $SST(\lambda, r)$ be the set of all semistandard tableaux $\tau : \lambda \to \{1, 2, \ldots, r\}$. The following result appears in [KMY09, Theorem 5.8], although its origins are much older.

Theorem 3.3 (Knutson–Miller–Yong)**.** *For any matrix Schubert variety $X_\lambda \subset \mathbf{A}^{r\times n}$,*

$$[X_\lambda]_G = \sum_{SST(\lambda,r)} \prod_{c_{ij}\in\lambda} (u_{\tau(c_{ij})} - t_{\tau(c_{ij})+j-i}).$$

The displayed polynomial is called a **factorial Schur polynomial**. It is important to note that since the partitions λ above have $\lambda_1 \leq n - r$, the degree of any u_i in $[X_\lambda]_G$ is at most $n - r$.

Corollary 3.4. *The following is a $\mathbf{Z}[t_1, \ldots, t_n]$-linear basis for $A_G^*(\mathbf{A}^{r\times n})$: The set of classes of matrix Schubert varieties together with the Schur polynomials $s_\lambda(u_1, \ldots, u_r)$ where $\lambda_1 \geq n - r + 1$.*

3.3. **Splitting of the localization sequence.** In this section we prove the following result.

Theorem 3.5. *The natural map of $\mathbf{Z}[t_1, \ldots, t_n]$-modules,*

$$A_G^*(\mathbf{A}^{r\times n}) \to A_G^*(\mathbf{A}^{\mathrm{fr}}) \approx A_T^*(G(r,n)),$$

has a splitting s that satisfies the following properties: For every T invariant subvariety $Y \subset G(r,n)$,

(i) $s([Y]_T) = [\overline{\pi^{-1}Y}]_G$,
(ii) *$s([Y]_T)$ is a $\mathbf{Z}[t_1, \ldots, t_n]$-linear combination of classes of matrix Schubert varieties $[X_\lambda]_G$. Equivalently, the Schur polynomial expansion of $s([Y]_T)$ is a linear combination of Schur polynomiala $s_\lambda(u)$ with $\lambda_1 \leq n - r$.*

Proof. A splitting s is uniquely determined by condition (ii). So it suffices to show that if $[Y]_T = \sum_\lambda q_\lambda[\Omega_\lambda]_T$ then $[\overline{\pi^{-1}Y}]_G = \sum_\lambda q_\lambda[X_\lambda]_G$.

Suppose that Y is a T-invariant subvariety of $G(r,n)$. It follows that $X = \overline{\pi^{-1}Y}$ satisfies the hypothesis of Corollary 3.2 and so the degree of u_r (and hence any u variable) in $[X]_G$ is at most $n - r$. Hence $[X]_G$ is a $\mathbf{Z}[t_1, \ldots, t_n]$-linear combination of classes of Schur polynomials s_λ with $\lambda_1 \leq n - r$ (*cf.* [FNR12, Theorem 7.4]). We conclude that $[X]_G$ is a linear combination of the the classes of the matrix Schubert varieties: $[X]_G = \sum_\lambda p_\lambda[X_\lambda]_G$.

We can uniquely write

$$[Y]_T = \sum q_\lambda[\Omega_\lambda]_T \in A_T^*(G(r,n)),$$

for some polynomials $q_\lambda \in \mathbf{Z}[t_1, \ldots, t_n]$. Since $\pi^* : A^*_T(G(r,n)) \to A^*_G(\mathbf{A}^{\text{fr}})$ is an isomorphism, this yields

$$[X^\circ]_G = \sum_\lambda q_\lambda [\pi^{-1}\Omega_\lambda]_G.$$

We claim that $[X]_G = \sum_\lambda q_\lambda [X_\lambda]_G$. To see this, note that

$$\sum_\lambda (q_\lambda - p_\lambda)[X_\lambda]_G \in \ker(A^*_G(\mathbf{A}^{r\times n}) \xrightarrow{i^*} A^*_G(\mathbf{A}^{\text{fr}})).$$

Applying i^* to this class gives $\sum_\lambda (q_\lambda - p_\lambda)[\pi^{-1}\Omega_\lambda] = 0$. However, the classes $[\pi^{-1}\Omega_\lambda]_G$ form a $\mathbf{Z}[t_1, \ldots, t_n]$-linear basis for $A^*_G(\mathbf{A}^{\text{fr}})$ so this means $q_\lambda = p_\lambda$. □

4. Matrix orbit closures and matroids

In this section we prove that the equivariant Chow class of a G-orbit closure in $\mathbf{A}^{r\times n}$ is determined by a matroid. We begin by stating the background we need from matroid theory.

4.1. **Matroid terminology.** Write $[n]$ for $\{1, 2, \ldots, n\}$ and $\binom{[n]}{r}$ for the set of size r subsets of $[n]$. Let $v \in \mathbf{A}^{r\times n}$ be any r-by-n matrix. The **matroid** of v, denoted $M(v)$, is the set of subsets $I \subset [n]$ where the column restricted matrix v_I has rank $|I|$. When the rank of v is k, so that k is the maximum cardinality of a set in $M(v)$, we say that $M(v)$ has rank k. In this case $M(v)$ is determined by its size k sets, which are called its bases.

Given a matroid M with ground set $[n]$ we define its **matroid base polytope** $P(M)$ as follows: Let $\{e_i\} \subset \mathbf{R}^n$ be the standard basis vectors and write $e_I = \sum_{i\in I} e_i$. We define $P(M)$ to be the convex hull in $\mathbf{R}^n$ of e_B where B ranges over the bases of M. The points e_B are actually the vertices of $P(M)$, and the convex hull of two vertices e_B and $e_{B'}$ forms an edge of $P(M)$ if and only B and B' differ by exactly one element [Edm70, GGMS87].

4.2. **Matroid invariance of Chow classes.** A **matrix orbit closure** is a G orbit closure of a point in $\mathbf{A}^{r\times n}$. We write X_v for the orbit closure of a matrix $v \in \mathbf{A}^{r\times n}$ and X°_v for the G orbit itself. When v is a rank r matrix we can project X°_v to the Grassmannian. The result is the T orbit closure of $\pi(v)$, which we denote by $Y_{\pi(v)}$.

The following result is proven by Speyer [Spe09, Proposition 12.5].

Theorem 4.1. *For any rank r matrix $v \in \mathbf{A}^{r\times n}$, the class of the structure sheaf of $Y_{\pi(v)}$ in the T-equivariant K-theory of $G(r,n)$ is determined by the matroid $M(v)$.*

As an immediate corollary we have:

Corollary 4.2. *The T-equivariant Chow class $[Y_{\pi(v)}]_T$ is determined by the matroid $M(v)$.*

Applying Theorem 3.5 gives:

Theorem 4.3. *Assume the matrix v has rank r. The class $[X_v]_G$ depends only on the matroid $M(v)$.*

The case when v has rank less than r can be obtained by taking a rank(v)-by-n matrix u with the same row span as v, considering X_u in a smaller matrix space and taking the G orbit closure of this variety in $\mathbf{A}^{r\times n}$ (*cf.* [FNR12, Theorem 7.5]).

5. Chow classes for uniform matroids

In this section our goal is to explicitly compute $[X_v]_G$ when v is a sufficiently general matrix in $\mathbf{A}^{r\times n}$. Here, sufficiently general means that the matroid $M(v)$ is *uniform*, i.e., no maximal minor of v vanishes. That such a formula exists is due to Theorem 4.3. To find this class, we will follow the idea of Theorem 3.5, and compute the Chow class of the toric variety $Y_{\pi(v)}$ in $A^*_T(G(r,n))$ and lift the result to $A^*_G(\mathbf{A}^{r\times n})$.

5.1. **Equivariant localization and the Grassmannian.** In order to state our main result we will need to gather some background material about equivariant localization and the Grassmannian.

The Grassmannian $G(r,n)$ has a finite set of T-fixed points: they are the r-dimensional coordinate subspaces of $\mathbf{C}^n$. We denote by x_B the fixed point in which the unique coordinate *not* fixed to zero is the one indexed by the set $B \in \binom{[n]}{r}$. The Plücker embedding embeds $G(r,n)$ equivariantly in $\mathbf{P}^{\binom{n}{r}-1} = \mathbf{P}\bigwedge^r \mathbf{C}^n$, with T action inherited from the natural one on $\mathbf{C}^n$. A fixed point x_B of $G(r,n)$ is sent to a coordinate point in $\mathbf{P}^{\binom{n}{r}-1}$, and the character by which T acts on the corresponding coordinate is $t^B = \prod_{i\in B} t_i$.

The inclusion ι of this discrete fixed set $G(r,n)^T$ into $G(r,n)$ induces a restriction map

$$\iota^* : A^*_T(G(r,n)) \to A^*_T(G(r,n)^T).$$

Its target $A^*_T(G(r,n)^T)$ is a direct sum of polynomial rings $A^*_T(\mathrm{pt}) = \mathbf{Z}[t_1 \dots, t_n]$, one for each fixed point. The restriction of the class of a T-equivariant subvariety to a fixed point x will equal the restriction of this class to an affine space $\mathbf{A}$ containing x on which T acts linearly, under the natural isomorphism $A^*_T(\mathbf{A}) = \mathbf{Z}[t_1 \dots, t_n] = A^*_T(\mathrm{pt})$. We will let $c|_x \in \mathbf{Z}[t_1, \dots, t_n]$ denote the restriction of the class c to x.

Since the Grassmannian is smooth and projective and T is a torus, results of Brion [Bri97, Theorems 3.2–3.4] imply that ι^* is injective and we can identify the image of ι^*. It consists of the tuples of polynomials $f = (f_B : B \in \binom{[n]}{r})$ such that

$$f_B - f_{B\cup j\setminus i} \in \langle t_j - t_i\rangle$$

for all $i \in B$ and $j \notin B$. Such results were also proved by Edidin and Graham [EG98b] and are closely related to the topological results of Goresky, Kottwitz and MacPherson [GKM98].

5.2. **Vector bundles on the Grassmannian.** We will let $\mathcal{S}$ denote the tautological rank r vector vector bundle over $G(r,n)$. It is a subbundle of the trivial bundle $\mathbf{C}^n$ and its fiber over $x \in G(r,n)$ is the r-dimensional subspace $x \subset \mathbf{C}^n$. The quotient bundle $\mathcal{Q}$ is $\mathbf{C}^n/\mathcal{S}$.

When we write a symmetric function of a vector bundle $\mathcal{E}$, we mean that symmetric function of its Chern roots, so that $e_k(\mathcal{E}) = c_k(\mathcal{E})$. For later reference, we give explicit expansions of $s_\nu(\mathcal{S}^\vee)$ and $s_\nu(\mathcal{Q})$, which are elements of $A^*_T(G(r,n))$, as polynomials in the variables u_i and t_j. The formulae are these:

$$s_\nu(S^\vee) = s_\nu(u),$$
$$s_\nu(\mathcal{Q}) = \omega\left(s_\nu(u,t)\right) = \sum_{\lambda,\mu} c^\nu_{\lambda,\mu} s_\lambda(t) s_{\mu'}(u).$$

Here ω is the usual operation on symmetric polynomials that transposes Schur polynomials, extended $\mathbf{Z}[t]$-linearly [Mac95, I.2.7]. On symmetric functions in infinitely many variables, ω is an involution; in our setting, it is an involution as long as no part of a partition exceeds r.

If $\mathcal{E}$ is a vector bundle on $G(r,n)$, a Schur polynomial s_λ of the Chern roots of $\mathcal{E}$ localizes at a fixed point to the sum of the characters by which T acts on the tangent space of $\mathbf{S}^\lambda(\mathcal{E}^\vee)$, where $\mathbf{S}^\lambda$ is a Schur functor. For the vector bundles $\mathcal{S}^\vee$ and $\mathcal{Q}$, the resulting localizations are

$$\begin{aligned} s_\lambda(\mathcal{S}^\vee)|_{x_B} &= s_\lambda(-t_i : i \in B), \\ s_\lambda(\mathcal{Q})|_{x_B} &= s_\lambda(t_j : j \notin B). \end{aligned}$$

5.3. **Statement of the formula.** We will use one piece of notation to succinctly state our theorem. The partition $(n-r-1)^{r-1}$ is the $(r-1)\times(n-r-1)$ rectangle, and if λ and μ are two partitions, then the Littlewood–Richardson coefficient $c_{\lambda\mu}^{(n-r-1)^{r-1}}$ equals 1 or 0, according to whether or not λ is the 180° rotated complement of μ within this rectangle. Given a partition λ fitting inside a $(r-1)\times(n-r-1)$ box, we let $\tilde\lambda$ denote the unique partition μ satisfying $c_{\lambda\mu}^{(n-r-1)^{r-1}} = 1$.

Theorem 5.1. *Given $v \in \mathbf{A}^{r\times n}$ whose matroid is uniform of rank r, the class of $Y_{\pi(v)}$ in $A_T^*(G(r,n))$ is*

$$[Y_{\pi(v)}]_T = \sum_{\lambda\subset(n-r-1)^{r-1}} s_\lambda(\mathcal{S}^\vee)s_{\tilde\lambda}(\mathcal{Q}). \tag{2}$$

The class of X_v in $A_G^*(\mathbf{A}^{r\times n})$ is

$$\begin{aligned}[X_v]_G &= \sum_{\substack{\lambda\subset(n-r-1)^{r-1}\\ \mu,\nu}} c_{\mu\nu}^{\tilde\lambda} s_\lambda(u)s_{\mu'}(t)s_\nu(u), \\ &= \omega(s_{(r-1)^{n-r-1}}(u,u,t)).\end{aligned}$$

5.4. **Proof of the formula.** The first step is to understand the class of $Y_{\pi(v)}$ localized at a T-fixed point.

Lemma 5.2. *The T-equivariant cohomology class of $Y_{\pi(v)}$ localized at x_B is*

$$[Y_{\pi(v)}]_T|_{x_B} = \prod_{i\in B, j\notin B}(t_j - t_i)\sum_{(i_1,\dots,i_n)}\frac{1}{(t_{i_2}-t_{i_1})(t_{i_3}-t_{i_2})\cdots(t_{i_n}-t_{i_{n-1}})}, \tag{3}$$

where the sums range over permutations $(i_1,\dots,i_n)\in S_n$ whose lex-first basis is B.

Note that the sum occurring in Lemma 5.2 is zero if B is not a basis of $M(v)$. When $M(v)$ is uniform, the sum ranges over those permutations that have the elements of B in their first r positions.

Proof. Following the approach of [FS12], we first identify $Y_{\pi(v)}$ as a toric variety. Viewing toric varieties as images of monomial maps [MS05, Chapters 7, 10], the normalization of $Y_{\pi(v)}$ is the toric variety of the polytope given as the convex hull of the characters corresponding to the T-fixed points it contains. By a result of White [Whi77, Theorem 2], the variety $Y_{\pi(v)}$ is already normal, and therefore *is* the toric variety just stated. The T-fixed points in $Y_{\pi(v)}$ are those x_B such that B is a basis of the matroid $M(v)$. The corresponding characters are $\{t^B :$

B is a basis of $M(v)\}$, whose convex hull is the matroid base polytope $P(M(v))$ of $M(v)$, defined in Section 4.

If the toric variety $Y_{\pi(v)}$ contains the fixed point x_B, then its restriction to the T-invariant translate of the big Schubert cell around x_B is the corresponding affine patch of $Y_{\pi(v)}$, in Fulton's construction: that is, it is the affine toric subvariety consisting of the orbits whose closures contain x_B. Explicitly, this affine scheme is $\operatorname{Spec} \mathbf{C}[C]$, where C is the semigroup of lattice points in the tangent cone to $P(M(v))$ at the vertex e_B. The T-equivariant K-theory class of $[Y_{\pi(v)}]_T|_{x_B}$ is then the product of $\operatorname{Hilb}(\mathbf{C}[C])$ with $\prod_{i\in B, j\notin B}(1-t_j/t_i)$. The Hilbert series $\operatorname{Hilb}(\mathbf{C}[C])$ is the finely-graded lattice point enumerator of C. We claim

$$\begin{aligned}\operatorname{Hilb}(\mathbf{C}[C]) &= \sum_{(i_1,\ldots,i_n)} \operatorname{Hilb}\operatorname{cone}(e_{i_2}-e_{i_1},\ldots,e_{i_n}-e_{i_{n-1}}) \\ &= \sum_{(i_1,\ldots,i_n)} \frac{1}{(1-t_{i_2}/t_{i_1})(1-t_{i_3}/t_{i_2})\cdots(1-t_{i_n}/t_{i_{n-1}})} \qquad (4)\end{aligned}$$

where the sums range over permutations $(i_1,\ldots,i_n) \in S_n$ whose lex-first basis is B. To see this, apply Brion's theorem to the triangulation of the dual of this cone into type A Weyl chambers. The cones in the triangulation are unimodular, and their lattice point generators are those given in the second line.

Altogether,

$$\mathcal{K}(Y_{\pi(v)})|_{x_B} = \prod_{i\in B, j\notin B}(1-t_j/t_i) \sum_{(i_1,\ldots,i_n)} \frac{1}{(1-t_{i_2}/t_{i_1})\cdots(1-t_{i_n}/t_{i_{n-1}})}$$

Using Proposition 2.4, this becomes the equation to be proved upon replacing each t_i with $1-t_i$, and then extracting the lowest degree term of the resulting power series. (Note that taking the lowest-degree term can be done one factor at a time.) □

Proof of Theorem 5.1. The second equation of the theorem follows from the first, by Theorem 3.5.

By equivariant localization, it is enough to show the claimed equality after restriction to each x_B, in $A_T^*(x_B) \cong \mathbf{Z}[t_1,\ldots,t_n]$. On one hand, the restriction of the right side of (2) at x_B is

$$\sum_\lambda s_\lambda(-t_i : i \in B)\, s_{\tilde\lambda}(t_j : j \notin B).$$

We massage the formula for $[Y_{\pi(v)}]_T|_{x_B}$ in Lemma 5.2, and show that it equals the above polynomial.

Let us temporarily write $f(i_1,\ldots,i_n)$ for $1/(t_{i_2}-t_{i_1})\cdots(t_{i_n}-t_{i_{n-1}})$. We have

$$\frac{f(i_1,\ldots,\widehat{i_s},\ldots,i_n)}{f(i_1,\ldots,i_n)} = \frac{t_{i_{s+1}}-t_{i_{s-1}}}{(t_{i_s}-t_{i_{s-1}})(t_{i_{s+1}}-t_{i_s})} = \frac{1}{t_{i_{s+1}}-t_{i_s}} - \frac{1}{t_{i_{s-1}}-t_{i_s}}$$

and similar identities when $s=1$ or $s=n$ in which the right hand term with an out-of-range index in it is deleted. Thus, if ℓ is a list of indices, we have a telescoping sum

$$\sum_{\substack{\ell' \,:\, \ell \text{ is } \ell' \text{ with } i \text{ dropped} \\ i \text{ precedes } j \text{ in } \ell'}} f(\ell') = \frac{f(\ell)}{t_j - t_i}.$$

Grouping the terms of the sum in (3) by i_r and repeatedly applying the above identity and its order-reversed counterpart, we get

$$(5)\qquad [Y_{\pi(v)}]_T|_{x_B} = \prod_{i\in B, j\notin B}(t_j - t_i)\ \cdot\ \sum_{i_r\in B}\left(\prod_{i\in B\setminus i_r}\frac{1}{t_{i-r}-t_i}\right)\left(\prod_{j\notin B}\frac{1}{t_j - t_{i_r}}\right).$$

We next invoke the following variant of the Cauchy identity:

$$\prod_{t\in T, v\in V}(t-v) = \sum_{\nu,\mu} c_{\nu\mu}^{(|V|)^{|T|}} s_\nu(t\in T)\, s_{\mu'}(-v\in V)$$

In our localized cohomology class, we combine the first and last products in (5) and apply the Cauchy identity with $(T,V) = (\{-t_i : i\in B\setminus i_r\}, \{-t_j : j\notin B\})$, giving

$$\sum_{i_r\in B}\left(\prod_{i\in B\setminus i_r}\frac{1}{t_{i_r}-t_i}\right)\left(\sum_{\nu,\mu} c_{\nu\mu}^{(n-r)^{r-1}}\frac{\det(-t_i^j)_{i\in B\setminus i_r}^{j=\nu_k+r-1-k}}{\det(-t_i^j)_{i\in B\setminus i_r}^{0\le j<r-1}}\cdot s_{\mu'}(t_j : j\notin B)\right)$$

where the s_ν is written as a ratio of determinants. Now we combine the remaining product in the above display into the Vandermonde determinant in the denominator. The sum over $i_r\in B$ can then be read as an expansion along the last row of the determinantal formula for $s_\lambda(-t_i : i\in B)$, where λ is obtained from ν by decrementing every part if ν has $r-1$ parts; if ν has fewer parts then the terms in this determinantal expansion cancel. For a given ν the only μ yielding a nonzero term is the one such that $c_{\nu,\mu}^{(n-r)^{r-1}}$ equals 1, i.e. so that ν and μ are complements in a $(r-1)\times(n-r)$ rectangle. In this event λ and μ are complements in a $(r-1)\times(n-r-1)$ rectangle, so our localized class is

$$\sum_{\lambda\subset(n-r-1)^{r-1}} s_\lambda(-t_i : i\in B)s_{\tilde\lambda}(t_j : j\notin B).$$

This agrees with the localization of (2) and the theorem follows. □

5.5. **Comparison to a formula of Klyachko.** There is another formula, due to Klyachko [Kly85], for the non-equivariant class of $Y_{\pi(v)}$ in $A^*(G(r,n))$ when v has a uniform rank r matroid.

Theorem 5.3 (Klyachko)**.** *Let v have a uniform, rank r matroid. Let $\lambda\subset(n-r)^r$ be a partition of $n-1$. Then the coefficient of $[\Omega_\lambda]$ in $[Y_{\pi(v)}]$ is*

$$\sum_{i=1}^r(-1)^i\binom{n}{i}s_\lambda(1^{r-i}).$$

Setting all the t variables equal to zero in Theorem 5.1 we obtain a different looking formula for the $\mathrm{GL}_r(\mathbf{C})$-equivariant Chow class of X_v:

$$[X_v]_{\mathrm{GL}_r(\mathbf{C})} = \sum_{\lambda\subset(n-r-1)^{(r-1)}} s_\lambda(u)s_{\tilde\lambda}(u).$$

As a consequence of this:

Corollary 5.4. *Let $v\in\mathbf{A}^{r\times n}$ have a uniform rank r matroid. The degree of the variety X_v is*

$$\sum_{\lambda\subset(n-r-1)^{r-1}} s_\lambda(1^r)s_{\tilde\lambda}(1^r).$$

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Department of Mathematics, Western Washington Universty, Bellingham, WA
E-mail address: andrew.berget@wwu.edu

Schoor of Mathematical Sciences, Queen Mary University of London, United Kingdom
E-mail address: a.fink@qmul.ac.uk