Connected Rigidity Matroids and Unique Realizations of Graphs

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Abstract

A d-dimensional *framework* is a straight line realization of a graph G in $\mathbb{R}^d$. We shall only consider *generic* frameworks, in which the co-ordinates of all the vertices of G are algebraically independent. Two frameworks for G are *equivalent* if corresponding edges in the two frameworks have the same length. A framework is a *unique realization* of G in $\mathbb{R}^d$ if every equivalent framework can be obtained from it by an isometry of $\mathbb{R}^d$. Bruce Hendrickson proved that if G has a unique realization in $\mathbb{R}^d$ then G is $(d+1)$-connected and redundantly rigid. He conjectured that every realization of a $(d+1)$-connected and redundantly rigid graph in $\mathbb{R}^d$ is unique. This conjecture is true for $d = 1$ but was disproved by Robert Connelly for $d \geq 3$. We resolve the remaining open case by showing that Hendrickson's conjecture is true for $d = 2$. As a corollary we deduce that every realization of a 6-connected graph as a 2-dimensional generic framework is a unique realization. Our proof is based on a new inductive characterization of 3-connected graphs whose rigidity matroid is connected.

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1 Introduction

We shall consider finite graphs without loops, multiple edges or isolated vertices. A d-dimensional *framework* is a pair (G,p), where $G=(V,E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line realization of G in $\mathbb{R}^d$. Two frameworks (G,p) and (G,q) are *equivalent* if $||p(u)-p(v)||=||q(u)-q(v)||$ holds for all pairs u,v with $uv\in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G,p), (G,q) are *congruent* if $||p(u)-p(v)||=||q(u)-q(v)||$ holds for all pairs u,v with $u,v\in V$. This is the same as saying that (G,q) can be obtained from (G,p) by an isometry of $\mathbb{R}^d$. We shall say that (G,p) is a *unique realization* of G in $\mathbb{R}^d$ if every framework which is equivalent to (G,p) is congruent to (G,p), see Figure 1.1.

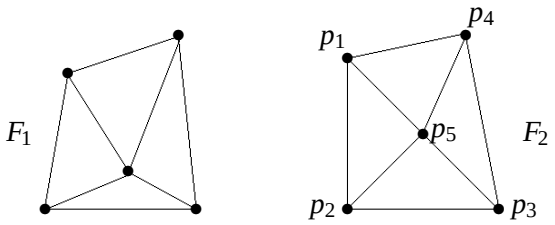

Figure 1.1. Two realizations of the same graph G in $\mathbb{R}^2$: F_1 is a unique realization, F_2 is not since we can obtain a realization of G which is equivalent but not congruent to F_2 by reflecting p_2 in the line through p_1,p_5,p_3.

The *unique realization problem* is to decide whether a given realization is unique. Saxe [19] proved that this problem is NP-hard. We obtain a problem of different type, however, if we exclude 'degenerate' cases. A framework (G,p) is said to be *generic* if the coordinates of all the points are algebraically independent over the rationals. Note that the framework F_2 of Figure 1.1 is not generic since the three points p_1,p_5,p_3 all lie on the same line. In what follows we shall consider the unique realization problem for generic frameworks.

A simple necessary condition for unique realization of generic frameworks is rigidity. The framework (G,p) is *rigid* if there exists an $\varepsilon>0$ such that if (G,q) is equivalent to (G,p) and $||p(u)-q(u)||<\varepsilon$ for all $v\in V$ then (G,q) is congruent to (G,p). Intuitively, this means that if we think of a d-dimensional framework (G,p) as a collection of bars and joints where points correspond to joints and each edge to a rigid bar joining its end-points, then the framework is rigid if it has no

non-trivial continuous deformations (see also [9],[23, Section 3.2]). It is known [23] that rigidity is a generic property, that is, the rigidity of (G,p) depends only on the graph G, if (G,p) is generic. We say that the graph G is rigid in $\mathbb{R}^d$ if every generic realization of G in $\mathbb{R}^d$ is rigid. (A combinatorial definition for the rigidity of G in $\mathbb{R}^2$ will be given in Section 2 of this paper. We refer the reader to [23, 24] for a detailed survey of the rigidity of d-dimensional frameworks.)

The necessary condition of rigidity was strengthened by Hendrickson [13] as follows. A graph G is *redundantly rigid* in $\mathbb{R}^d$ if deleting any edge of G results in a graph which is rigid in $\mathbb{R}^d$. By using methods from differential topology, Hendrickson proved that the redundant rigidity of G is a stronger necessary condition for the unique realizability of a generic framework (G,p).

Hendrickson [13] also pointed out that the $(d+1)$-connectivity of G is another necessary condition for a d-dimensional generic framework (G,p) to be a unique realization of G: if G has at least $d+2$ vertices and has a vertex separator S of size d, then we can obtain a framework which is equivalent but not congruent to (G,p) by reflecting one component of $G-S$ along the hyperplane spanned by $p(S)$. Similarly, if (G,p) is a unique realization of G and G has at most $d+1$ vertices then G is a complete graph. Summarising we have

Theorem 1.1 *[13] If a generic framework (G,p) is a unique realization of G in $\mathbb{R}^d$ then either G is a complete graph with at most $d+1$ vertices, or the following conditions hold:*
(a) G is $(d+1)$–connected, and
(b) G is redundantly rigid.

Hendrickson [11, 12, 13] conjectured that conditions (a) and (b) are sufficient to guarantee that any generic framework (G,p) is a unique realization of G. This conjecture is easy to prove for $d=1$ since G is rigid in $\mathbb{R}$ if and only if G is connected; G is redundantly rigid in $\mathbb{R}$ if and only if G is 2-edge-connected; and (G,p) is a unique generic realization of G in $\mathbb{R}$ if and only if G is 2-connected. On the other hand, Connelly [4] has shown that Hendrickson's conjecture is false for $d \geq 3$. We shall settle the remaining case by showing that the conjecture is true for $d=2$. As a corollary we deduce that unique realizability is also a generic property, that is to say the unique realizability of a 2-dimensional generic framework (G,p) depends only on the graph G. Note that it is not known whether unique realizability is a generic property in $\mathbb{R}^d$ for $d \geq 3$. Following Connelly [4], we say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every generic realization of G in $\mathbb{R}^d$ is a unique realization. Our solution of the conjecture implies that G is globally rigid in $\mathbb{R}^2$ if and only if G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid. Globally rigid graphs have several diverse applications, e.g. in distance

geometry [7], molecular conformation [12, 14], and localization problems in sensor networks [8].

Our proof of the conjecture is based on an inductive construction for all 3-connected redundantly rigid graphs. We shall show that every graph in this family can be built up from K_4 (which is globally rigid) by an appropriate sequence of operations, where each of the two operations we use preserves global rigidity.

One operation is *edge addition*: we add a new edge connecting some pair of non-adjacent vertices. The other is *1-extension*: we subdivide an edge uv by a new vertex z, and add a new edge zw for some $w \neq u, v$. Clearly, the first operation preserves global rigidity. So does the second. This fact follows from a deep result of Connelly, first proved in the 1980's (see [12]), and recently published in [5]. Connelly developed a sufficient condition for a generic framework in $\mathbb{R}^d$ to be a unique realization in terms of the rank of its 'stress matrix' (see also [3]). Based on this condition, he proved that if G is obtained from K_4 by a sequence of edge additions and 1-extensions then G is globally rigid in $\mathbb{R}^2$.

In what follows we shall assume that $d = 2$. In this case both conditions in Hendrickson's conjecture can be characterized (and efficiently tested) by purely combinatorial methods. This is straightforward for 3-connectivity. In the case of redundant rigidity, the combinatorial characterization and algorithm are based on the following result of Laman [16]. For a graph (G,E) and a subset $X \subseteq V$ let $i_G(X)$ (or simply $i(X)$ when it is obvious to which graph we are referring) denote the number of edges in the subgraph induced by X in G. The graph G is said to be *minimally rigid* if G is rigid, and $G-e$ is not rigid for all $e \in E$.

Theorem 1.2 *[16] A graph $G = (V,E)$ is minimally rigid in $\mathbb{R}^2$ if and only if $|E| = 2|V| - 3$ and*

$$i(X) \leq 2|X| - 3 \text{ for all } X \subset V \text{ with } |X| \geq 2. \tag{1}$$

Note that a graph is rigid if and only if it has a minimally rigid spanning subgraph.

It can be seen from Theorem 1.2 that a redundantly rigid graph $G = (V,E)$ will have at least four vertices and at least $2|V| - 2$ edges. We call graphs which are redundantly rigid and have this minimum number of edges *M-circuits*, see Figure 1.2. Motivated by Hendrickson's conjecture, Connelly conjectured (see e.g. [10, p.99], [23, p.188]) in the 1980's that all 3-connected M-circuits can be obtained from K_4 by 1-extensions. It is easy to see that the 1-extension operation preserves 3-connectivity and that it creates an M-circuit from an M-circuit. The other direction is more difficult. It is equivalent to saying that every 3-connected M-circuit on at least five vertices has a vertex of degree three which can be "suppressed" by the inverse operation to 1-extension, so that the resulting graph is a smaller 3-connected M-circuit.

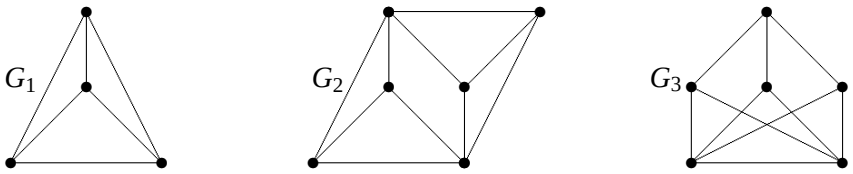

Figure 1.2. Three examples of M-circuits

The inverse operation to 1-extension is called *splitting*: it chooses a vertex v of degree three in a graph G, deletes v (and the edges incident to v) and adds a new edge connecting two non-adjacent neighbours of v. If G is a 3-connected M-circuit with at least five vertices and at least one of the splittings of v results in a 3-connected M-circuit, then we say that the vertex v is *feasible*. It can be seen that each M-circuit G has at least four vertices of degree three. It is not true, however, that each vertex of degree three in G is feasible. The existence of such a vertex was verified by Berg and the second author [1] in their recent solution to Connelly's conjecture.

In this paper we shall show that every 3-connected redundantly rigid graph can be obtained from K_4 by edge additions and 1-extensions by extending the methods in [1]. We show that every 3-connected redundantly rigid graph G on at least five vertices either contains an edge e such that $G-e$ is 3-connected and redundantly rigid, or a vertex v of degree three such that some splitting of v in G results in a graph which is 3-connected and redundantly rigid.

The structure of the paper is as follows. In Section 2 we review elementary results on rigidity: we define the rigidity matroid of a graph and use it to give combinatorial definitions for when a graph is rigid, redundantly rigid or an M-circuit. In Section 3 we characterize M-connected graphs (graphs with a connected rigidity matroid). Section 4 describes and extends lemmas from [1] on splitting in M-circuits. In Section 5, we use the concept of an ear decomposition of a matroid to extend the splitting theorem of [1] from M-circuits to M-connected graphs. We use this in Section 6 to obtain our above mentioned recursive construction for 3-connected redundantly rigid graphs. This verifies Hendrickson's conjecture. This, and other corollaries on global rigidity are included in Section 7.

2 Rigid graphs and the rigidity matroid

In this section we prove a number of preliminary lemmas and basic results, most of which are known. Our goal is to make the paper self-contained and to give a unified picture of these frequently used statements. Our proofs are based on Laman's theorem and use only graph theoretical arguments. Some of these results can be found in [10, 17, 21, 23, 24].

Let $G = (V,E)$ be a graph. Let F be a non-empty subset of E, U be the set of vertices incident with F, and $H = (U,F)$ be the subgraph of G induced by F. We say that F is *independent* if

$$i_H(X) \leq 2|X| - 3 \text{ for all } X \subseteq V(H) \text{ with } |X| \geq 2. \tag{2}$$

The empty set is also defined to be independent. The *rigidity matroid* $\mathcal{M}(G) = (E, I)$ is defined on the edge set of G by

$$I = \{F \subseteq E : F \text{ is independent in } G\}.$$

To see that $\mathcal{M}(G)$ is indeed a matroid, we shall verify that the following three matroid axioms are satisfied. (For basic matroid definitions not given here the reader may consult the book [18].)

(M1) $\emptyset \in I$,

(M2) if $D \subset F \in I$ then $D \in I$,

(M3) for every $E' \subseteq E$ the maximal independent subsets of E' have the same cardinality.

Let $G = (V,E)$ be a graph. For $X,Y,Z \subset V$, let $G[X]$ be the induced subgraph of G on vertex set X and $E_G(X)$ be the set of edges of $G[X]$. We simply use $E(X)$ if the graph is clear from the context. Let $d(X,Y) = |E(X \cup Y) - (E(X) \cup E(Y))|$, and $d(X,Y,Z) = |E(X \cup Y \cup Z) - (E(X) \cup E(Y) \cup E(Z))|$. We define the *degree* of X by $d(X) = d(X,V-X)$. Thus $d(X,Y)$ is the number of edges between $X-Y$ and $Y-X$ and $d(X)$ is the number of edges with precisely one endvertex in X. The degree of a vertex v is simply denoted by $d(v)$. We shall need the following equalities, which are easy to check by counting the contribution of an edge to each of their two sides.

Lemma 2.1 *Let G be a graph and $X,Y \subseteq V(G)$. Then*

$$i(X) + i(Y) + d(X,Y) = i(X \cup Y) + i(X \cap Y). \tag{3}$$

Lemma 2.2 *Let G be a graph and $X,Y,Z \subseteq V(G)$. Then*

$$\begin{aligned} i(X)+i(Y)+i(Z)+d(X,Y,Z) &= i(X\cup Y\cup Z)+i(X\cap Y)+i(X\cap Z)+ \\ & \quad i(Y\cap Z)-i(X\cap Y\cap Z). \end{aligned}$$

We say that the graph $H=(V,F)$ is *M-independent* if F is independent in $\mathcal{M}(H)$. We call a set $X \subseteq V$ *critical* if $i(X) = 2|X|-3$ holds.

Lemma 2.3 *Let $H=(V,F)$ be M-independent and let $X,Y \subset V$ be critical sets in H with $|X\cap Y| \geq 2$. Then $X\cap Y$ and $X\cup Y$ are also critical, and $d(X,Y)=0$.*

Proof: Since H is M-independent, (2) holds. By (3) we have
$2|X|-3+2|Y|-3 = i(X)+i(Y) = i(X\cap Y)+i(X\cup Y)-d(X,Y) \leq$
$2|X\cap Y|-3+2|X\cup Y|-3-d(X,Y) = 2|X|-3+2|Y|-3-d(X,Y)$. Thus $d(X,Y) =$
0 and equality holds everywhere. Therefore $X\cap Y$ and $X\cup Y$ are also critical. •

Lemma 2.4 *Let $G=(V,E')$ be a graph with $|E'| \geq 1$ and let $F \subseteq E'$ be a maximal independent subset of E'. Then*

$$|F| = \min\{\sum_{i=1}^{t}(2|X_i|-3)\} \tag{4}$$

where the minimum is taken over all collections of subsets $\{X_1,X_2,\ldots,X_t\}$ of V such that $\{E_G(X_1),E_G(X_2),\ldots,E_G(X_t)\}$ partitions E'.

Proof: Since F is independent, we have $|F\cap E_G(X_i)| \leq 2|X_i|-3$ for all $1 \leq i \leq t$. Thus $|F| \leq \sum_{i=1}^{t}(2|X_i|-3)$ for any collection of subsets $\{X_1,X_2,...,X_t\}$ satisfying the hypothesis of the lemma.

To see that equality can be attained, let H be the subgraph of G induced by F. Consider the maximal critical sets $X_1,X_2,...,X_t$ in H. By Lemma 2.3 we have $|X_i\cap X_j| \leq 1$ for all $1 \leq i < j \leq t$. Since every single edge of F induces a critical set, it follows that $\{E_H(X_1),E_H(X_2),...,E_H(X_t)\}$ is a partition of F. Thus

$$|F| = \sum_{1}^{t}|E_H(X_i)| = \sum_{1}^{t}(2|X_i|-3).$$

To complete the proof we show that $\{E_G(X_1),E_G(X_2),...,E_G(X_t)\}$ is a partition of E'. Choose $uv \in E'-F$. Since F is a maximal independent subset of E', $F+uv$ is dependent. Thus there exists a set $X \subseteq V$ such that $u,v \in X$ and $i_H(X) = 2|X|-3$. Hence X is a critical set in H. This implies that $X \subseteq X_i$ and hence $uv \in E_G(X_i)$ for

some $1 \leq i \leq t$. •

It follows from the definition of independence that $\mathcal{M}(G)$ satisfies axioms (M1) and (M2). Lemma 2.4 implies that $\mathcal{M}(G)$ also satisfies (M3). It also determines the rank function of $\mathcal{M}(G)$, which we shall denote by r_G or simply by r.

Corollary 2.5 *Let $G = (V,E)$ be a graph. Then $\mathcal{M}(G)$ is a matroid, in which the rank of a non-empty set $E' \subseteq E$ of edges is given by*

$$r(E') = \min\{\sum_{i=1}^{t}(2|X_i| - 3)\}$$

where the minimum is taken over all collections of subsets $\{X_1, X_2, \ldots, X_t\}$ of V such that $\{E_G(X_1), E_G(X_2), \ldots, E_G(X_t)\}$ partitions E'.

We say that a graph $G = (V,E)$ is *rigid* if $r(E) = 2|V| - 3$ in $\mathcal{M}(G)$. The graph G is *minimally rigid* if it is rigid and $|E| = 2|V| - 3$. Thus, if G is rigid and $H = (V,E')$ is a spanning subgraph of G, then H is minimally rigid if and only if E' is a base in $\mathcal{M}(G)$. Theorem 1.2 ensures that these definitions agree with the geometric definitions for rigidity given in Section 1.

A *k-separation* of a graph $H = (V,E)$ is a pair (H_1, H_2) of edge-disjoint subgraphs of G each with at least $k+1$ vertices such that $H = H_1 \cup H_2$ and $|V(H_1) \cap V(H_2)| = k$. The graph H is said to be *k-connected* if it has at least $k+1$ vertices and has no j-separation for all $0 \leq j \leq k-1$. If (H_1, H_2) is a k-separation of H, then we say that $V(H_1) \cap V(H_2)$ is a *k-separator* of H.

2.1 Minimally rigid graphs

We first investigate the connectivity properties of minimally rigid graphs.

Lemma 2.6 *Let $G = (V,E)$ be minimally rigid with $|V| \geq 3$. Then*
(a) G is 2-connected.
(b) For every $\emptyset \neq X \subset V$ we have $d(X) \geq 2$ and if $d(X) = 2$ holds then either $|X| = 1$ or $|V - X| = 1$.

Proof: Suppose that for some $v \in V$ the graph $G - v$ is disconnected and let $A \cup B$ be a partition of $V - v$ with $d(A,B) = 0$. Then (2) gives $|E| = 2|V| - 3 = i(A + v) + i(B + v) \leq 2(|A| + 1) - 3 + 2(|B| + 1) - 3 = 2(|A| + |B| + 1) - 4 = 2|V| - 4$, a contradiction. This proves (a).

Using (a), we have $d(X) \geq 2$ for every $\emptyset \neq X \subset V$. Suppose $|X|, |V - X| \geq 2$. By (2) we obtain $|E| = i(X) + i(V - X) + d(X) \leq 2|X| - 3 + 2|V - X| - 3 + d(X) =$

$2|V|-6+d(X)=|E|-3+d(X)$. This implies $d(X)\geq 3$ and proves (b). •

Let v be a vertex in a graph G with $d(v)=3$ and $N(v)=\{u,w,z\}$. Recall that the operation *splitting* means deleting v (and the edges incident to v) and adding a new edge, say uw, connecting two non-adjacent vertices of $N(v)$. The resulting graph is denoted by $G_v^{u,w}$ and we say that the splitting is made *on the pair* uv,wv. Note that v can be split in at most three different ways, see Figure 2.1.

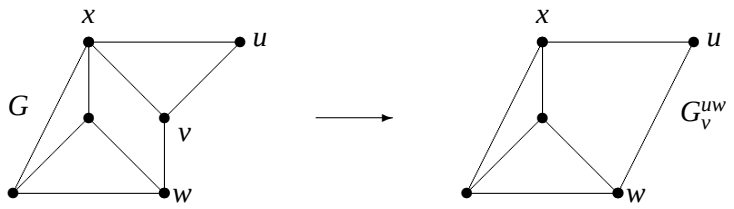

Figure 2.1. There are only two possible splittings of v in G. Splitting v on uv,wv results in the graph $G_v^{u,w}$.

Let $G=(V,E)$ be minimally rigid and let v be a vertex with $d(v)=3$. Splitting v on the pair uv,wv is said to be *suitable* if $G_v^{u,w}$ is minimally rigid. Note that in Figure 2.1, splitting v on uv,wv is suitable in G, but splitting v on xv,wv is not. We call a vertex v *suitable* if there is a suitable splitting at v. We shall show that every vertex of degree three in a minimally rigid graph is suitable.

Lemma 2.7 *Let $G=(V,E)$ be minimally rigid and let $X,Y,Z\subset V$ be critical sets in G with $|X\cap Y|=|X\cap Z|=|Y\cap Z|=1$ and $X\cap Y\cap Z=\emptyset$. Then $X\cup Y\cup Z$ is critical, and $d(X,Y,Z)=0$.*

Proof: Since G is minimally rigid and our sets are critical, Lemma 2.2 gives $2|X|-3+2|Y|-3+2|Z|-3+d(X,Y,Z)=i(X)+i(Y)+i(Z)+d(X,Y,Z)\leq i(X\cup Y\cup Z)\leq 2(|X\cup Y\cup Z|)-3=2(|X|+|Y|+|Z|-3)-3=2|X|-3+2|Y|-3+2|Z|-3$. Hence $d(X,Y,Z)=0$ and equality holds everywhere. Thus $X\cup Y\cup Z$ is critical. •

For $X\subseteq V$ let $N(X)$ denote the set of *neighbours* of X (that is, $N(X):=\{v\in V-X : uv\in E \text{ for some } u\in X\}$).

Lemma 2.8 *Let v be a vertex in a minimally rigid graph $G = (V,E)$.*
(a) If $d(v) = 2$ then $G - v$ is minimally rigid.
(b) If $d(v) = 3$ then v is suitable.

Proof: Part (a) follows easily from (2) and from the definition of minimally rigid graphs.

To prove (b) let $N(v) = \{u,w,z\}$. It is easy to see that splitting v on the pair uv, wv is not suitable if and only if there exists a critical set $X \subset V$ with $u,w \in X$ and $v,z \notin X$. Also observe that no critical set $Z \subseteq V - v$ can satisfy $d(v,Z) \geq 3$, since otherwise $E(G[Z \cup \{v\}])$ is not independent in G, contradicting the fact that G is minimally rigid. Thus if v is not suitable then there exist maximal critical sets $X_{uw}, X_{uz}, X_{wz} \subset V - v$ each containing precisely two neighbours ($\{u,w\}, \{u,z\}, \{w,z\}$, resp.) of v. By Lemma 2.3 and the maximality of these sets we must have $|X_{uw} \cap X_{uz}| = |X_{uw} \cap X_{wz}| = |X_{uz} \cap X_{wz}| = 1$. Thus, by Lemma 2.7 the set $Y := X_{uw} \cup X_{uz} \cup X_{wz}$ is also critical. Since $N(v) \subseteq Y$, we have $d(v,Y) \geq 3$. This is impossible by our previous observation. Therefore v is suitable. •

The minimally rigid graph $K_4 - e$ shows that among the three possible splittings at a vertex of degree three there may be only one which is suitable.

We now define the reverse operations of vertex deletion and vertex splitting used in Lemma 2.8. The operation 0-*extension* adds a new vertex v and two edges vu, vw with $u \neq w$. The operation 1-*extension* subdivides an edge uw by a new vertex v and adds a new edge vz for some $z \neq u,w$. (Thus, in Figure 2.1, G is a 1-extension of G_v^{uw}.) An *extension* is either a 0-extension or a 1-extension. The next lemma follows easily from (2).

Lemma 2.9 *Let G be minimally rigid and let G' be obtained from G by an extension. Then G' is minimally rigid.*

Theorem 2.10 *Let $G = (V,E)$ be minimally rigid and let $G' = (V',E')$ be a minimally rigid subgraph of G. Then G can be obtained from G' by a sequence of extensions.*

Proof: We shall prove that G' can be obtained from G by a sequence of vertex splittings and deletions of vertices (of degree two). The theorem will then follow since these are the inverse operations of extensions.

The proof is by induction on $|V - V'|$. Since G' is rigid and G is minimally rigid, G' must be an induced subgraph of G. Thus the theorem holds trivially when $|V - V'| = 0$. Now suppose that $Y = V - V' \neq \emptyset$. Since G' and G are minimally rigid, it is easy to see that $|E - E'| = 2|Y|$ holds. Therefore, if $|Y| = 1$, then we

must have $d(v) = 2$ for the unique vertex $v \in Y$. Hence G' can be obtained from G by deleting a vertex of degree two. Thus we may assume that $|Y| \geq 2$.

Claim 2.11 *If $|Y| \geq 2$ then $\sum_{v \in Y} d(v) \leq 4|Y| - 3$.*

Proof: Since $|V'| \geq 2$ and $|V - V'| \geq 2$, we can apply Lemma 2.6(b) to deduce that $d(Y) \geq 3$. Since $i(Y) + d(Y) = |E - E'| = 2|Y|$, we obtain

$$\sum_{v \in Y} d(v) = 2i(Y) + d(Y) = 4|Y| - d(Y) \leq 4|Y| - 3.$$

•

It follows from Claim 2.11 (and from the fact that the minimum degree in G is at least two) that there is a vertex $v \in Y$ with $2 \leq d(v) \leq 3$. Now Lemma 2.8 implies that either $H = G - v$ or $H = G_v^{u,w}$ is minimally rigid and is such that G' is a subgraph of H and $|V(H) - V(G')| < |V(G) - V(G')|$. The theorem now follows by induction. •

By choosing G' to be an arbitrary edge of G we obtain the following constructive characterization of minimally rigid graphs (called the Henneberg or Henneberg-Laman construction, c.f. [15, 16, 21]).

Corollary 2.12 *$G = (V, E)$ is minimally rigid if and only if G can be obtained from K_2 by a sequence of extensions.*

Theorem 2.13 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two minimally rigid graphs with $|V_1 \cap V_2| \geq 2$. Then $G_1 \cup G_2$ is rigid. Moreover, if $G_1 \cap G_2$ is minimally rigid then $G_1 \cup G_2$ is minimally rigid as well.*

Proof: Let F' be a maximal independent set in $\mathcal{M}(G_1 \cap G_2)$. Let K be the complete graph with vertex set $V(G_1 \cap G_2)$ and F be a base of $\mathcal{M}(K)$ containing F'. Let H be a minimally rigid spanning subgraph of $G_2 + (F - F')$ which contains F. Such an H exists, since G_2, and hence $G_2 + (F - F')$, is rigid. (To see that F and H exist we use the fact that any independent set in a matroid can be extended to a base.) Now Theorem 2.10 implies that H can be obtained by a sequence of extensions from $(V_1 \cap V_2, F)$. The same sequence of extensions, applied to G_1, yields a minimally rigid spanning subgraph of $G_1 \cup G_2$ by Lemma 2.9. This proves that $G_1 \cup G_2$ is rigid.

The second assertion follows from the fact that if $G_1 \cap G_2$ is minimally rigid then $F = F'$ and $H = G_2$. •

Corollary 2.14 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two rigid graphs with $|V_1 \cap V_2| \geq 2$. Then $G_1 \cup G_2$ is rigid.*

Let $G = (V, E)$ be a graph. Since every edge of G induces a rigid subgraph of G, Corollary 2.14 implies that the maximal rigid subgraphs $R_1, R_2, ..., R_t$ (called the *rigid components of G*) of G are pairwise edge-disjoint and $E(R_1), E(R_2), ..., E(R_t)$ is a partition of E. Thus a graph is rigid if and only if it has precisely one rigid component.

2.2 M-circuits and redundantly rigid graphs

Given a graph $G = (V, E)$, a subgraph $H = (W, C)$ is said to be an *M-circuit* in G if C is a circuit (i.e. a minimal dependent set) in $\mathcal{M}(G)$. In particular, G is an *M-circuit* if E is a circuit in $\mathcal{M}(G)$. For example, K_4, $K_{3,3}$ plus an edge, and $K_{3,4}$ are all M-circuits. Using (2) we may deduce:

Lemma 2.15 *Let $G = (V, E)$ be a graph. The following statements are equivalent.*
(a) G is an M-circuit.
(b) $|E| = 2|V| - 2$ and $G - e$ is minimally rigid for all $e \in E$.
(c) $|E| = 2|V| - 2$ and $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $2 \leq |X| \leq |V| - 1$.

We shall need the following elementary properties of M-circuits which can be derived in a similar way to Lemma 2.6.

Lemma 2.16 *[1, Lemma 2.4] Let $H = (V, E)$ be an M-circuit.*
(a) For every $\emptyset \neq X \subset V$ we have $d(X) \geq 3$ and if $d(X) = 3$ holds then either $|X| = 1$ or $|V - X| = 1$.
(b) If $X \subset V$ is critical with $|X| \geq 3$ then $H[X]$ is 2-connected.

Let $H = (V, E)$ be a 2-connected graph and suppose that (H_1, H_2) is a 2-separation of G with $V(H_1) \cap V(H_2) = \{a, b\}$. For $1 \leq i \leq 2$, let $H_i' = H_i + ab$ if $ab \notin E(H_i)$ and otherwise put $H_i' = H_i$. We say that H_1, H_2 are the *cleavage graphs* obtained by *cleaving G along* $\{a, b\}$. Given two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$ and two designated edges $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$, the 2-*sum* of H_1 and H_2 (along the edge pair u_1v_1, u_2v_2), denoted by $H_1 \oplus_2 H_2$, is the graph obtained from $H_1 - u_1v_1$ and $H_2 - u_2v_2$ by identifying u_1 with u_2 and v_1 with v_2. These definitions are illustrated by the graphs G_1, G_2 of Figure 1.2. If we cleave G_2 along its unique 2-seperator we obtain two copies of G_1, say H_1 and H_2, and $G_2 = H_1 \oplus_2 H_2$.

We shall use the following results on 2-sums and cleaving.

Lemma 2.17 *[1, Lemma 4.1] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be M-circuits and let $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$. Then the 2-sum $G_1 \oplus_2 G_2$ along the edge pair u_1v_1, u_2v_2 is an M-circuit.*

Lemma 2.18 *[1, Lemmas 2.4(c), 4.2] Let $G = (V, E)$ be an M-circuit and $\{a, b\}$ be a 2-separator of G. Then $ab \notin E$. Furthermore, if G' and G'' are the graphs obtained from G by cleaving G along $\{a, b\}$ then G' and G'' are both M-circuits.*

Recall that a graph G is *redundantly rigid* if G has at least two edges and $G - e$ is rigid for all $e \in E$. M-circuits are examples of (minimally) redundantly rigid graphs. Note also that a graph G is redundantly rigid if and only if G is rigid and each edge of G belongs to a circuit in $\mathcal{M}(G)$ i.e. an M-circuit of G.

It follows from Corollary 2.14 that any two maximal redundantly rigid subgraphs of a graph $G = (V, E)$ can have at most one vertex in common, and hence are edge-disjoint. Defining a *redundantly rigid component* of G to be either a maximal redundantly rigid subgraph of G, or a subgraph induced by an edge which belongs to no M-circuit of G, we deduce that the redundantly rigid components of G partition E. Since each redundantly rigid component is rigid, this partition is a refinement of the partition of E given by the rigid components of G.

We shall need two elementary lemmas on redundant rigidity.

Lemma 2.19 *If G is redundantly rigid and G' is obtained from G by an edge addition or a 1-extension, then G' is redundantly rigid.*

Proof: This follows from the definition of redundant rigidity and the facts that edge additions, 0-extensions and 1-extensions preserve rigidity. •

Lemma 2.20 *If G is redundantly rigid and $\{u, v\}$ is a 2-separator in G then $d(u), d(v) \geq 4$.*

Proof: Suppose $d(u) \leq 3$. Then we can choose an edge e incident to u such that $G - e$ is not 2-connected. By Lemma 2.6(a), $G - e$ is not rigid. This contradicts the redundant rigidity of G. •

3 Graphs with a connected rigidity matroid

Given a matroid $\mathcal{M} = (E, I)$, we define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that

this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*. If $\mathcal{M}$ has components $E_1, E_2, \ldots, E_t$ and $\mathcal{M}_i$ is the matroid restriction of $\mathcal{M}$ onto E_i then $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \ldots \oplus \mathcal{M}_t$, where $\oplus$ denotes the direct sum of matroids, see [18].

We say that a graph $G = (V, E)$ is *M-connected* if $\mathcal{M}(G)$ is connected. For example, $K_{3,m}$ is M-connected for all $m \geq 4$. The *M-components* of G are the subgraphs of G induced by the components of $\mathcal{M}(G)$.

Lemma 3.1 *Suppose that G is M-connected. Then G is redundantly rigid.*

Proof: G is rigid, since otherwise G has at least two rigid components and hence at least two M-components. Since $\mathcal{M}(G)$ is connected, every edge e is contained in a circuit of $\mathcal{M}(G)$. Thus G is redundantly rigid. •

Since the M-components of G are redundantly rigid by Lemma 3.1, the partition of $E(G)$ given by the M-components is a refinement of the partition given by the redundantly rigid components and hence a further refinement of the partition given by the rigid components, see Figure 3.1.

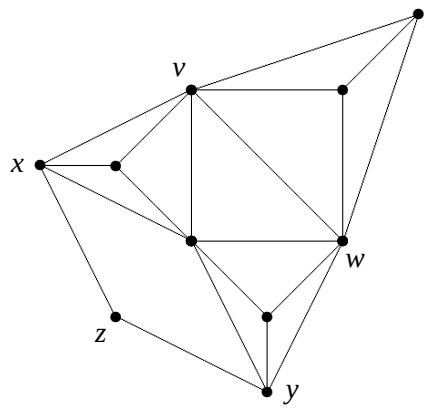

Figure 3.1. This graph is rigid so has exactly one rigid component. There are three redundantly rigid components, consisting of the union of the three copies of K_4, and the remaining two copies of K_2. There are five M-connected components: each of the three copies of K_4, and the remaining two copies of K_2.

Furthermore, $\mathcal{M}(G)$ can be expressed as the direct sum of the rigidity matroids of the rigid components of G, the redundantly rigid components of G, or the M-components of G.

The main result of this section (Theorem 3.7 below) characterizes M-connected graphs. We say that a graph G is *nearly* 3*-connected* if G can be made 3-connected by adding at most one new edge.

Theorem 3.2 *Suppose that G is nearly 3-connected and redundantly rigid. Then G is M-connected.*

Proof: For a contradiction suppose that G is not M-connected and let H_1, H_2,...,H_q be the M-components of G. Let $X_i = V(H_i) - \cup_{j \neq i} V(H_j)$ denote the set of vertices belonging to no other M-component than H_i, and let $Y_i = V(H_i) - X_i$ for $1 \leq i \leq q$. Let $n_i = |V(H_i)|$, $x_i = |X_i|$, $y_i = |Y_i|$. Clearly, $n_i = x_i + y_i$ and $|V| = \sum_{i=1}^{q} x_i + |\cup_{i=1}^{q} Y_i|$. Moreover, we have $\sum_{i=1}^{q} y_i \geq 2|\cup_{i=1}^{q} Y_i|$. Since every edge of G is in some M-circuit, and every M-circuit has at least four vertices, we have that $n_i \geq 4$ for $1 \leq i \leq q$. Furthermore, since G is nearly 3-connected, $y_i \geq 2$ for all $1 \leq i \leq q$, and $y_i \geq 3$ for all but at most two M-components.

Let us choose a base B_i in each rigidity matroid $\mathcal{M}(H_i)$. Using the above inequalities we have

$$\begin{aligned} |\cup_{i=1}^{q} B_i| &= \sum_{i=1}^{q} |B_i| = \sum_{i=1}^{q} (2n_i - 3) = 2\sum_{i=1}^{q} n_i - 3q \geq \\ & (2\sum_{i=1}^{q} x_i + \sum_{i=1}^{q} y_i) + \sum_{i=1}^{q} y_i - 3q \geq 2|V| + 3q - 2 - 3q = 2|V| - 2. \end{aligned}$$

Since $\mathcal{M}(G)$ has rank $2|V|-3$, this implies that $\cup_{i=1}^{q} B_i$ contains a circuit, contradicting the fact that the B_i's are bases for the $\mathcal{M}(H_i)$'s and $\mathcal{M}(G) = \oplus_{i=1}^{q} \mathcal{M}(H_i)$. •

A graph G is *birigid* if $G - v$ is rigid for all $v \in V(G)$. It was shown by Servatius [20, Theorem 2.2] (using a similar argument to our proof of Theorem 3.2) that every birigid graph is M-connected. Theorem 3.2 extends this result, since birigid graphs are clearly 3-connected and redundantly rigid. The wheels (on at least 5 vertices) are 3-connected redundantly rigid graphs which are not birigid. This shows that the extension is proper.

We need the following results to complete our characterization of M-connected graphs. The first two lemmas follow from Lemmas 2.17 and 2.18, respectively.

Lemma 3.3 *Suppose G_1 and G_2 are M-connected. Then $G_1 \oplus_2 G_2$ is M-connected.*

Lemma 3.4 *Suppose G_1 and G_2 are obtained from G by cleaving G along a 2-separator. If G is M-connected then G_1 and G_2 are also M-connected.*

Let $G=(V,E)$ be a 2-connected graph, $c \geq 3$ be an integer, and let $(X_1, X_2, ..., X_c)$ be cyclically ordered subsets of V satisfying (by taking $X_{c+1} = X_1$):
(i) $|X_i \cap X_j| = 1$, for $|i-j| = 1$, and $X_i \cap X_j = \emptyset$ for $|i-j| \geq 2$, and
(ii) $\{E(X_1), E(X_2), ..., E(X_c)\}$ is a partition of E.
Then we say that $(X_1, X_2, ..., X_c)$ is a *polygon (of size c)* in G. (The graph in Figure 3.1 is a polygon of size 3, where the sets X_1, X_2, X_3 are given by the vertex sets of its 2-rigid components.) It is easy to see that if u and v are distinct vertices with $\{u\} = X_{i-1} \cap X_i$ and $\{v\} = X_j \cap X_{j+1}$, for some $1 \leq i, j \leq c$, then either $\{u,v\}$ is a 2-separator in G or $i = j$ and $X_i = \{u,v\}$.

Lemma 3.5 *Suppose that $G=(V,E)$ has a polygon of size c. Then*
(a) G is not M-connected.
(b) If $c \geq 4$ then G is not rigid.

Proof: Let $X_1, X_2, ..., X_c$ be a polygon and let $E_i = E(X_i)$ for $1 \leq i \leq c$. Note that $E_1, E_2, ..., E_c$ is a partition of E. Using the polygon structure we obtain

$$r(E) \leq \sum_{i=1}^{c} r(E_i) \leq \sum_{i=1}^{c} (2|X_i| - 3) = 2|V| + 2c - 3c = 2|V| - c. \qquad (5)$$

Thus for $c \geq 4$ we have $r(E) \leq 2|V| - 4$, and hence G is not rigid. This proves (b). To prove (a) suppose that G is M-connected. Then G is rigid and $r(E) = 2|V| - 3$. By (b) this yields $c = 3$. Moreover, equality must hold everywhere in (5). Thus $r(E) = \sum_{i=1}^{c} r(E_i)$. It follows that no two edges in different sets E_i belong to an M-circuit, see [18, Proposition 4.2.1]. This contradicts the fact that $\mathcal{M}(G)$ is a connected matroid. •

We say that a 2-separator $\{x_1, x_2\}$ *crosses* another 2-separator $\{y_1, y_2\}$ in a 2-connected graph G, if x_1 and x_2 are in different components of $G - \{y_1, y_2\}$. It is easy to see that if $\{x_1, x_2\}$ crosses $\{y_1, y_2\}$ then $\{y_1, y_2\}$ crosses $\{x_1, x_2\}$. Thus, we can say that these 2-separators are *crossing*. It is also easy to see that crossing 2-separators induce a polygon of size four in G. Thus Lemma 3.5(a) has the following corollary:

Lemma 3.6 *Suppose that G is rigid (and hence 2-connected). Then there are no crossing 2-separators in G.*

Let $G = (V,E)$ be a 2-connected graph with no crossing 2-separators. The *cleavage units* of G are the graphs obtained by recursively cleaving G along each of its 2-separators. Since G has no crossing 2-separators this sequence of operations is uniquely defined and results in a unique set of graphs each of which have no 2-separators. Thus each cleavage unit of G is either 3-connected or else a complete graph on three vertices. (The graph G in Figure 3.1 has three cleavage units, obtained by cleaving G along the 2-separators $\{v,w\}$ and $\{x,y\}$.) The stronger hypothesis that G has no polygons will imply that each cleavage unit of G is a 3-connected graph. In this case, an equivalent definition for the cleavage units is to first construct the *augmented graph* $\hat{G}$ from G by adding all edges uv for which $\{u,v\}$ is a 2-separator of G and $uv \notin E$, and then take the cleavage units to be the maximal 3-connected subgraphs of $\hat{G}$. (These definitions are a special case of a general decomposition theory for 2-connected graphs due to Tutte [22].)

Theorem 3.7 *A graph G is M-connected if and only if it is 2-connected, has no polygon, and each of its cleavage units is redundantly rigid.*

Proof: If G is M-connected, then G is rigid and hence 2-connected by Lemma 2.6(a), G has no polygons by Lemma 3.5(a), each cleavage unit of G is M-connected by Lemma 3.4, and hence each cleavage unit is redundantly rigid by Lemma 3.1. On the other hand, if G is 2-connected, has no polygons and each cleavage unit is redundantly rigid, then each cleavage unit is M-connected by Theorem 3.2, and G is M-connected by Lemma 3.3. •

The weaker hypothesis that G is 2-connected, has no polygons, and is redundantly rigid is not sufficient to imply that G is M-connected. This can be seen by considering the graph G obtained from the triangular prism H by replacing each edge v_iv_j of H by a complete graph with vertex set $\{v_i, v_j, v_i', v_j'\}$, where $v_i', v_j' \notin V(H)$. The graph G is redundantly rigid since it is rigid and every edge belongs to an M-circuit (a complete graph on four vertices). To see that G is not M-connected we first note that H is minimally rigid and hence it is not redundantly rigid. We may now deduce that G is not M-connected since H is a cleavage unit of G, and every cleavage unit of an M-connected graph is M-connected by Lemma 3.4.

We close this section by obtaining two further results on M-connectivity which we will need later.

Lemma 3.8 *Let $G = (V,E)$ be a 2-connected graph and $\{u,v\}$ be a 2-separator of G such that $uv \in E$. Then G is M-connected if and only if $G - uv$ is M-connected.*

Proof: First suppose that $G-uv$ is M-connected. Then $G-uv$ is rigid by Lemma 3.1, and hence there exists an M-circuit H in G with $uv \in E(H)$. The M-connectivity of G now follows from the transitivity of the relation on E which defines the M-connected components. To see the other direction suppose that G is M-connected and let (G_1,G_2) be a 2-separation of G with $V(G_1)\cap V(G_2)=\{u,v\}$ and let $e,f \in E(G-uv)$. We shall prove that there is an M-circuit H in $G-uv$ which contains e and f. Since G is M-connected, there is an M-circuit H' with $e,f \in E(H')$. If $uv \notin E(H')$ then we are done by choosing $H=H'$. Note that if $E(H')$ intersects both sides of the 2-separation (in particular, if e and f belong to different G_i's) then $\{u,v\}$ is also a 2-separator of H' and hence $uv \notin E(H')$ by Lemma 2.18. Thus we may suppose, without loss of generality, that $e,f \in E(G_1)$, $uv \in E(H')$, and $E(H')\cap E(G_2-uv)=\emptyset$. Let $g \in E(G_2)-uv$. Since G is M-connected, there is an M-circuit H'' in G with $e,g \in E(H'')$. Let H_1 and H_2 be the subgraphs of G obtained by cleaving H'' along $\{u,v\}$, where $e \in E(H_1)$ and $g \in E(H_2)$. Then H_2 is an M-circuit by Lemma 2.18. Now $H=H'\oplus_2 H_2$ is the desired M-circuit in G with $e,f \in E(H)$ by Lemma 2.17. •

Lemma 3.9 *If G is M-connected and G' is obtained from G by an edge addition or a 1-extension, then G' is M-connected.*

Proof: First suppose that G' is obtained from G by adding an edge e. Since G is M-connected, it is rigid by Lemma 3.1. Thus there is an M-circuit H in G' with $e \in E(H)$. Now the M-connectivity of G' follows from transitivity.

Next consider the case when G' is obtained from G by a 1-extension which subdivides an edge uw of G by a new vertex v and adds a new edge vz for some $z \notin \{u,w\}$. Let $f \in E(G)$ be an edge which is incident with z. Since $f \neq uw$, we also have $f \in E(G')$. We shall prove that for all edges $g \in E(G')-f$ there exists an M-circuit H in G' with $f,g \in E(H)$. This will imply that G' is M-connected by transitivity.

If $g \in E(G)$ then there is an M-circuit H' in G with $f,g \in E(H')$. If $uw \notin E(H')$ then we are done by choosing $H=H'$. Otherwise we let H be the 1-extension of H' (on the edge uw and vertex z), which is a subgraph of G', and is also an M-circuit by Lemma 2.19. Finally, if $g \notin E(G)$, that is, if $g \in \{vu,vw,vz\}$, then we take an M-circuit H'' of G with $uw, f \in E(H'')$ and let H be the 1-extension of H'' (on the edge uw and vertex z). As above, H is an M-circuit of G' with $f,g \in E(H)$. •

4 Admissible splittings in M-circuits

Let $G = (V,E)$ be a graph and let $V_3 = \{v \in V : d(v) = 3\}$. We will refer to vertices in V_3 as *nodes* of G and to the subgraph $G[V_3]$ as the *node-subgraph* of G. A node of G with degree at most one (exactly two) in the node-subgraph of G is called a *leaf node* (*series node*, respectively). A *wheel* $W_n = (V,E)$ is a graph on $n \geq 4$ vertices which has a vertex z which is adjacent to all the other vertices and for which $W_n[V-z]$ is a cycle. Thus the node-subgraph of a wheel W_n with $n \geq 5$ is a cycle. It was shown in [1, Lemma 2.1] that if G is an M-circuit then either G is a wheel or $G[V_3]$ is a forest. The proof can be extended to M-connected graphs to give:

Lemma 4.1 *Let G be M-connected. If G is not a wheel, then the nodes of G induce a forest in G.*

We also need two results on M-circuits from [1]. The proof of the first lemma is similar to that of Lemma 2.3.

Lemma 4.2 *[1, Lemma 2.3] Let $H = (V,E)$ be an M-circuit and let $X,Y \subset V$ be critical sets with $|X \cap Y| \geq 2$ and $|X \cup Y| \leq |V| - 1$. Then $X \cap Y$ and $X \cup Y$ are both critical, and $d(X,Y) = 0$.*

Lemma 4.3 *[1, Lemma 2.5] Let $H = (V,E)$ be an M-circuit and let $X \subset V$ be a critical set. Then $V - X$ contains at least one node of H. Furthermore, if $|V - X| \geq 2$, then $V - X$ contains at least two nodes of H.*

We shall say that splitting a node v in an M-connected graph is *admissible* if it preserves M-connectivity, that v is an *admissible node* if it has an admissible splitting, and otherwise that v is *non-admissible*. Note that an admissible splitting in an M-circuit results in an M-connected graph with $|E| = 2|V| - 2$, and hence results in another M-circuit. The following result follows easily from Lemma 2.15.

Lemma 4.4 *[1, Lemma 3.1] Let $H = (V,E)$ be an M-circuit and v be a node in G with $N(v) = \{u,w,z\}$. Then splitting v on the pair uv, wv is not admissible if and only if there is a critical set $X \subset V$ with $u,w \in X$ and $v,z \notin X$.*

If v is a node in a graph G with $N(v) = \{u,w,z\}$ and X is a critical set with $u,w \in X$ and $v,z \notin X$ then we call X a *v-critical set on $\{u,w\}$*, or simply a *v-critical set*. If X is a v-critical set on $\{u,w\}$ for some node v with $N(v) = \{u,w,z\}$, and $d(z) \geq 4$, then X is said to be *node-critical*.

Our next lemma extends [1, Lemma 3.2].

Lemma 4.5 *Let $H = (V,E)$ be an M-circuit, $|V| \geq 5$, and v be a non-admissible leaf node in H with $N(v) = \{x,y,z\}$. Suppose that no two neighbours of v are a 2-separator in H.*
(a) If z is a node of H then for any pair X,Y of v-critical sets on $\{y,z\}$, and $\{x,z\}$, respectively, we have $|X \cap Y| \geq 2$ and $X \cup Y = V(H) - v$.
(b) If v is not adjacent to a node then there exist two v-critical sets X_1, X_2 with $|X_1 \cap X_2| \geq 2$, $X_1 \cup X_2 = V(H) - v$.

Proof: (a) If the edges xz and yz are both present in $E(H)$ then, since z is a node of H and $|V| \geq 5$, $\{x,y\}$ is a 2-separator, contradicting an hypothesis of the lemma. Thus we may assume, without loss of generality, that $yz \notin E$. Then for the v-critical set X on y,z we must have $|X| \geq 3$. By Lemma 2.16(b) $H[X]$ is 2-connected, and hence z has two neighbours in X. If z has no neighbours in Y then $xz \notin E(Y)$, $|Y| \geq 3$, and z is an isolated vertex in $H[Y]$. This would contradict Lemma 2.16(b). Hence z has a neighbour in Y. Since z is a node and has two neighbours in X, this implies that $|X \cap Y| \geq 2$. By Lemma 4.2 this gives that $X \cup Y$ is also critical. Since $d(v, X \cup Y) \geq 3$, Lemma 2.15 implies that $X \cup Y = V(H) - v$. Thus (a) holds.

(b) Since v is non-admissible, Lemma 4.4 implies that there exist three v-critical sets X,Y,Z on $\{y,z\}$, $\{x,z\}$ and $\{x,y\}$, respectively. Suppose that no two of these sets intersect each other in at least two vertices. Then we also have $X \cap Y \cap Z = \emptyset$. Lemma 2.2 implies that $X \cup Y \cup Z$ is critical and $d(X,Y,Z) = 0$. Since $d(v, X \cup Y \cup Z) = 3$, we deduce that $X \cup Y \cup Z = V - v$ (otherwise $(X \cup Y \cup Z) + v$ violates Lemma 2.15). Since $|V| \geq 5$, at least one of the three critical sets X,Y,Z (say, X) satisfies $|X| \geq 3$. But we have $d(X,Y,Z) = 0$, and hence $\{y,z\}$ is a 2-separator in H, contradicting an hypothesis of the lemma. This contradiction shows that we can choose two sets $X_1, X_2 \in \{X,Y,Z\}$ with $|X_1 \cap X_2| \geq 2$. Then $X_1 \cup X_2$ is critical by Lemma 4.2 and so $X_1 \cup X_2 = V - v$ follows, using Lemma 2.15 and $d(v, X \cup Y) = 3$. Thus (b) holds. •

The next lemma extends [1, Lemma 3.3].

Lemma 4.6 *Let $H = (V,E)$ be an M-circuit which is not a wheel, and let v be a node. Let $N(v) = \{x,y,z\}$ and let X be a v-critical set on x,y with $d(z) \geq 4$ and $|X| \geq 3$. Suppose that either*
(a) there is a non-admissible series node $u \in V - X - v$ with exactly one neighbour w in X, and w is a node, or
(b) there is a non-admissible leaf node $t \in V - X - v$.
Then either there is a 2-separation (H_1, H_2) of H with $X \subseteq V(H_1)$ or there is a node-critical set X^* with X properly contained in X^*.

Proof: Suppose first that (a) occurs and let $N(u) = \{w, p, q\}$. By our assumption $N(u) \cap X = \{w\}$ and $d(w) = 3$. Since u is a series node, we can assume that $d(p) = 3$ and $d(q) \geq 4$. Since u is non-admissible, there exists a u-critical set Y on $\{w, p\}$ by Lemma 4.4. Now H is not a wheel, and hence the node subgraph of H contains no cycles by Lemma 4.1. Thus $pw \notin E$ and hence $|Y| \geq 3$. This implies, by Lemma 2.6(a), that $G[Y]$ is 2-connected, and hence Y contains two neighbours of w. Since $|X| \geq 3$, Lemma 2.6(a) implies that $G[X]$ is 2-connected, and hence at least one of the neighbours of w in Y must be in X. Thus $|X \cap Y| \geq 2$. Let $X^* = X \cup Y$. We have $X^* \subseteq V - u - q$, and Lemma 4.2 implies that X^* is a u-critical set on $\{w, p\}$. Since $d(q) \geq 4$ and $p \notin X$, the set X^* is a node-critical set which properly contains X.

We next suppose that (b) occurs. We must have $|N(t) \cap X| \leq 2$, since $|N(t) \cap X| = 3$ would imply that $X + t$ violates Lemma 2.15(c). If $|N(t) \cap X| = 2$ then $X + t$ is also critical and by choosing $X^* = X + t$ the lemma follows. Thus we may assume that $|N(t) \cap X| \leq 1$.

Since t is a non-admissible leaf node, Lemma 4.5 implies that either there is a 2-separator consisting of two neighbours of t or there exist two t-critical sets Y_1 and Y_2 with $Y_1 \cup Y_2 = V - t$, $|Y_1 \cap Y_2| \geq 2$, and so that if t has a neighbour r which is a node then $r \in Y_1 \cap Y_2$. In the former case we are done (since $G[X]$ is 2-connected by Lemma 2.6(a) and hence X is contained in one side of the corresponding 2-separation). Suppose that the latter case holds. Note that Y_1 and Y_2 are node-critical since t is a leaf node and $|Y_1|, |Y_2| \geq 3$. Since $Y_1 \cup Y_2 = V - t$, $t \notin X$, and $|X| \geq 3$, we have $|X \cap Y_1| \geq 2$ or $|X \cap Y_2| \geq 2$. Let us assume, without loss of generality, that $|X \cap Y_1| \geq 2$ holds. By Lemma 4.2, $X \cup Y_1$ is a critical set. If $N(t) \cap X \subseteq Y_1$, then the lemma follows by choosing $X^* = X \cup Y_1$. (The set X^* is t-critical and the unique neighbour of t in $V - X^*$ has degree four in H.)

Thus we may assume that $N(t) \cap X = \{s\}$ and $s \notin Y_1$ holds. This implies that $d(s) \geq 4$, since if $d(s) = 3$ then we have $s \in Y_1 \cap Y_2$ as noted above. Since $Y_1 \cup Y_2 = V - t$, we have $s \in Y_2$. Hence if $|X \cap Y_2| \geq 2$ then we are done, as above, by choosing the t-critical set $X^* = X \cup Y_2$. Thus, we may suppose that $|X \cap Y_2| = 1$. Since $d(t, X \cup Y_1) = 3$, and $X \cup Y_1$ is critical, Lemma 2.15 implies $X \cup Y_1 = V - t$. Since $Y_1 \cup Y_2 = V - t$, we have $(X - s) \subseteq Y_1$. Thus $V - Y_1 = \{s, t\}$. This contradicts Lemma 4.3, since $d(s) \geq 4$, and completes the proof of the lemma. •

5 Ear decompositions and admissible splittings in M-connected graphs

Let $\mathcal{M} = (E, I)$ be a matroid and let $C_1, C_2, ..., C_t$ be a non-empty sequence of circuits of $\mathcal{M}$. Let $D_j = C_1 \cup C_2 \cup ... \cup C_j$ for $1 \leq j \leq t$. We say that $C_1, C_2, ..., C_t$ is a *partial ear decomposition* of $\mathcal{M}$ if for all $2 \leq i \leq t$ the following properties hold:

(E1) $C_i \cap D_{i-1} \neq \emptyset$,

(E2) $C_i - D_{i-1} \neq \emptyset$,

(E3) no circuit C_i' satisfying (E1) and (E2) has $C_i' - D_{i-1}$ properly contained in $C_i - D_{i-1}$.

The set $C_i - D_{i-1}$ is called the *lobe* of circuit C_i, and is denoted by $\tilde{C}_i$. An *ear decomposition* of $\mathcal{M}$ is a partial ear decomposition with $D_t = E$. As an example, we construct an ear-decomposition C_1, C_2, C_3 of the rigidity matroid of the graph obtained from $K_{3,5}$ by adding an edge, see Figure 5.1.

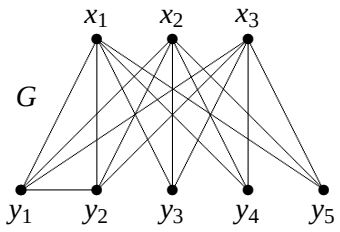

Figure 5.1. If $C_1 = E(G - y_1)$, $C_2 = E(G - y_2)$ and $C_3 = E(G - \{y_4, y_5\})$, then C_1, C_2, C_3 is an ear decomposition of the rigidity matroid of G. We have $\tilde{C}_2 = \{x_1y_1, x_2y_1, x_3y_1\}$ and $\tilde{C}_3 = \{y_1y_2\}$.

We need the following facts about ear decompositions. The proof of (a) and (b) in the next lemma can be found in [6]. The proof of (c) is easy and is omitted.

Lemma 5.1 *Let $\mathcal{M}$ be a matroid with rank function r. Then*
(a) $\mathcal{M}$ is connected if and only if $\mathcal{M}$ has an ear decomposition.
(b) If $\mathcal{M}$ is connected then any partial ear decomposition of $\mathcal{M}$ can be extended to an ear decomposition of $\mathcal{M}$.
(c) If $C_1, C_2, ..., C_t$ is an ear decomposition of $\mathcal{M}$ then

$$r(D_i) - r(D_{i-1}) = |\tilde{C}_i| - 1 \quad \textit{for} \quad 2 \leq i \leq t. \tag{6}$$

Lemma 5.2 *Let $G=(V,E)$ be an M-connected graph and $H_1,H_2,\ldots,H_t$ be the M-circuits of G induced by an ear decomposition $C_1,C_2,...,C_t$ of $\mathcal{M}(G)$ with $t\geq 2$. Let $Y=V(H_t)-\cup_{i=1}^{t-1}V(H_i)$, and let $X=V(H_t)-Y$. Then:*
(a) Either $Y=\emptyset$ and $|\tilde{C}_t|=1$, or $Y\neq\emptyset$ and every edge $e\in\tilde{C}_t$ is incident to Y.
(b) $|\tilde{C}_t|=2|Y|+1$.
(c) If $Y\neq\emptyset$ then X is critical in H_t.
(d) $G[Y]$ is connected.
(e) If G is 3-connected then $|X|\geq 3$.

Proof: Since M-connected graphs are rigid, it follows that G, $\cup_{i=1}^{t-1}H_i$, and H_t are all rigid. Thus (E3) implies that (a) holds. Furthermore, $r(E)=2|V|-3$ and $r(\cup_{i=1}^{t-1}C_i)=2|V-Y|-3$. By Lemma 5.1(c) this implies that $2|Y|=|\tilde{C}_t|-1$. This gives (b).

Since H_t is an M-circuit, we have $|E(H_t)|=2|V(H_t)|-2$. Hence, since $|X|\geq 2$, (b) implies that X is critical in H_t and hence (c) holds.

To prove (d) suppose that Y can be partitioned into two non-empty sets Y_1,Y_2 with $d(Y_1,Y_2)=0$. Since X is critical and H_t is an M-circuit, we must have $i(Y_j)+d(Y_j,X)\leq 2|Y_j|$ for $j=1,2$. This gives $|\tilde{C}_t|=\sum_{j=1}^{2}i(Y_j)+d(Y_j,X)\leq 2(|Y_1|+|Y_2|)\leq 2|Y|$, contradicting (b). Property (e) follows from the fact that either $Y\neq\emptyset$ and X is a separator in G (using (c)), or $Y=\emptyset$ and $|X|=|V(H_t)|\geq 4$ (since H_t is an M-circuit). •

Let $H=(V,E)$ be an M-circuit, v be a node of H, $N(v)=\{x,y,z\}$, and suppose that $xy\notin E$. Since $H-vz$ is rigid, $H-v$ is rigid by Lemma 2.8(a). Thus $H_v^{x,y}=H-v+xy$ is rigid. Since $|V(H_v^{x,y})|=2|E(H_v^{x,y})|-2$, $H_v^{x,y}$ contains a unique M-circuit J. We have $J=H_v^{x,y}$ if and only if the splitting of v on vx,vy is admissible. If not, $V(J)$ is the minimal v-critical set on $\{x,y\}$ in H.

Lemma 5.3 *Let $G=(V,E)$ be an M-connected graph and $H_1,H_2,\ldots,H_t$ be the M-circuits of G induced by an ear decomposition $C_1,C_2,...,C_t$ of $\mathcal{M}(G)$ with $t\geq 2$. Let $Y=V(H_t)-\cup_{i=1}^{t-1}V(H_i)$ and $X=V(H_t)-Y$. Let v be a node of G in Y, and let $x,y\in N(v)$ with $x\notin X$ and $xy\notin E$. Let J be the unique M-circuit in $(H_t)_v^{x,y}$ and $C=E(J)$. If $C\cap E_{H_t}(X)\neq\emptyset$ and $E((H_t)_v^{x,y})-E_{H_t}(X)\subset C$ then splitting v on vx,vy is admissible in G.*

Proof: Let $N(v)=\{x,y,z\}$. It suffices to show that $C_1,C_2,...,C_{t-1},C$ is an ear-decomposition of $\mathcal{M}(G_v^{x,y})$ since this will imply that $G_v^{x,y}$ is M-connected. Let $D_{t-1}=\cup_{i=1}^{t-1}C_i$. Then $E_{H_t}(X)\subseteq D_{t-1}$ by Lemma 5.2(a). Since $E((H_t)_v^{x,y})-E_{H_t}(X)\subset C$, $D_{t-1}\cup C=E(G_v^{x,y})$. Properties (E1), (E2) and (E3) are clearly satisfied for $2\leq i\leq t-1$. Property (E1) follows for '$i=t$' from the hypothesis that $C\cap E_{H_t}(X)\neq\emptyset$

and the fact that $E_{H_t}(X) \subseteq D_{t-1}$. Property (E2) holds for '$i = t$' since $xy \in C - D_{t-1}$. To see that (E3) holds for '$i = t$' we proceed by contradiction. Suppose that there is an M-circuit J' where $C' = E(J')$ satisfies $C' \cap D_{t-1} \neq \emptyset \neq C' - D_{t-1}$ and $C' - D_{t-1} \subset C - D_{t-1}$. Since $C_1, C_2, \ldots, C_t$ satisfies (E3), we must have $xy \in C'$. Let J'' be obtained from J' by a 1-extension, which deletes the edge xy, adds a new vertex v, and the edges vx, vy, vz. Then J'' is an M-circuit and $C'' = E(J'')$ violates (E3) with respect to the ear decomposition $C_1, C_2, \ldots, C_t$ of $\mathcal{M}(G)$, a contradiction. •

Note that if splitting v along vx, vy is admissible in H_t, then the hypotheses of Lemma 5.3 are trivially satisfied since we have $J = (H_t)_v^{x,y}$. Thus an admissible splitting of v in H_t is admissible in G. However, it is possible a non-admissible splitting of v in H_t still satisfies the hypotheses of Lemma 5.3 and hence is admissible in G.

Theorem 5.4 *Let $G = (V, E)$ be a 3-connected M-connected graph which is not an M-circuit. Let $H_1, H_2, \ldots, H_t$ be the M-circuits of G induced by an ear decomposition $C_1, C_2, ..., C_t$ of $\mathcal{M}(G)$. Suppose that $G - e$ is not M-connected for all $e \in \tilde{C}_t$ and for all but at most two edges of C_t. Then $V(H_t) - \cup_{i=1}^{t-1} V(H_i)$ contains an admissible node of G.*

Proof: Suppose the theorem is false and let G be a counterexample. Since G is not an M-circuit, we have $t \geq 2$. Let $Y = V(H_t) - \cup_{i=1}^{t-1} V(H_i)$, $X = V(H_t) - Y$. Since $G - e$ is not M-connected for all $e \in \tilde{C}_t$, we have $Y \neq \emptyset$ by Lemma 5.2(a). Let $L = \cup_{i=1}^{t-1} V(H_i)$. Since G is 3-connected, we have $|X| \geq 3$ by Lemma 5.2(e). Note that every edge $e \in \tilde{C}_t$ is incident to Y by Lemma 5.2(a).

By Lemmas 4.3 and 5.2(c), Y contains a node. Since G is not an M-circuit, G is not a wheel. Lemma 4.1 implies that we can choose a node v of G in Y such that v is a leaf in $G[Y \cap V_3] = H_t[Y \cap V_3]$, where V_3 is the set of nodes of G. Let $N(v) = \{x, y, z\}$.

Claim 5.5 *v does not have three neighbours in X.*

Proof: For a contradiction suppose $N(v) \subset X$. Then, by Lemma 5.2(d), we must have $Y = \{v\}$. By the hypothesis of the theorem there exists a pair of neighbours of v, say $x, y \in N(v)$, such that either $xy \notin E$ or $xy \in E$ and $G - xy$ is not M-connected. In the former case splitting v on the pair vx, vy gives $L + xy$, which is M-connected by Lemma 3.9. Thus v is an admissible node of G. In the latter case $G - xy$ is a 1-extension of L. Thus $G - xy$ is M-connected by Lemma 3.9, a contradiction. •

Claim 5.6 *v does not have two neighbours in X.*

Proof: Let $N(v) \cap X = \{x,y\}$. If splitting v along xz or yz is admissible in H_t then by Lemma 5.3 it is an admissible split in G. Hence, by Lemma 4.4, we may assume that there exist two minimal critical sets X_1, X_2 in H_t with $x,z \in X_1$ and $y,z \in X_2$. Note that the minimality of X_1 implies that the unique M-circuit J in $(H_t)_v^{x,z}$ satisfies $V(J) = X_1$. Let $C = E(J)$.

Suppose $|X \cap X_1| \geq 2$. Then $X \cup X_1$ and $X \cap X_1$ are critical and $d(X, X_1) = 0$ by Lemma 4.2. Since $d(v, X \cup X_1) = 3$, Lemma 2.15 now implies that $X \cup X_1 = H_t - v$. Hence $(E((H_t)_v^{x,z}) - E(X)) \subseteq C$. Since $X \cap X_1$ is critical, $H_t[X \cap X_1]$ is connected (it is either K_2 or is 2-connected by Lemma 2.6(a)) and hence $E(X) \cap C \neq \emptyset$. Thus v is admissible in G by Lemma 5.3. Hence $X \cap X_1 = \{x\}$ and, by symmetry, $X \cap X_2 = \{y\}$.

If $|X_1 \cap X_2| \geq 2$ then $X_1 \cup X_2 = V(H_t) - v$ and $\{x,y\}$ is a 2-separator in G. This contradicts the 3-connectivity of G and hence $|X_1 \cap X_2| = 1$. Now Lemma 2.2 implies that $d(X, X_1, X_2) = 0$. This again implies that $\{x,y\}$ is a 2-separator in G, and gives a contradiction. •

Claim 5.7 *There is a v-critical set $X' \subset V(H_t)$ such that X' is node-critical in H_t and $X \subseteq X'$.*

Proof: It follows from Claims 5.5, 5.6 that v has at most one neighbour in X.

Case 1 *v has exactly one neighbour, say x, in X.*
Since v is a leaf in $H_t[Y \cap V_3]$, we may assume without loss of generality that $d_{H_t}(y) \geq 4$. If splitting v along xz or yz is admissible in H_t then by Lemma 5.3 it is an admissible split in G. Hence, by Lemma 4.4, we may assume that there exist two minimal critical sets X_1, X_2 in H_t with $x,z \in X_1$ and $y,z \in X_2$. If $|X \cap X_1| \geq 2$ then Lemma 4.2 implies that $X \cup X_1$ is the desired v-critical, node critical set containing X in H_t. Hence

$$X \cap X_1 = \{x\}. \tag{7}$$

Suppose $|X \cap X_2| \geq 2$. Then Lemma 4.2 implies that $X \cup X_2$ and $X \cap X_2$ are critical and $d(X, X_2) = 0$. Since $N(v) \subseteq X \cup X_2$, Lemma 2.15 gives $X \cup X_2 = V(H_t) - v$. Hence the unique circuit $J = (X_2, C)$ in $(H_t)_v^{y,z}$ satisfies $(E((H_t)_v^{y,z}) - E(X)) \subseteq C$ and $E(X) \cap C \neq \emptyset$ (because $X \cap X_2$ is also critical, so $H_t[X \cap X_2]$ is connected). Thus v is admissible in G by Lemma 5.3. Hence

$$|X \cap X_2| \leq 1. \tag{8}$$

If $|X_1 \cap X_2| \geq 2$ then we may deduce as above that $X_1 \cup X_2 = V(H_t) - v$ must hold. Since $|X| \geq 3$, this contradicts either (7) or (8). Thus $X_1 \cap X_2 = \{z\}$. Since

$H_t[X_1], H_t[X_2]$ are minimally rigid, Lemma 2.6(a) implies that either $d_{H_t}(z) \geq 4$; or $d_{H_t}(z) = 3$, $|X_1| = 2 = |X_2|$ and $xz, yz \in E$. The second alternative would imply that $\{x,y\}$ is a 2-separator in G, and contradict the fact that G is 3-connected. Thus $d_{H_t}(z) \geq 4$. We now choose a critical set X_3 in H_t with $x,y \in X_3$ (if it did not exist then splitting v along xy would be admissible in G). By symmetry we have $|X_3 \cap X_2| = 1$. If $|X_3 \cap X| \geq 2$ then $X \cup X_3$ is the desired v-critical, node-critical set. Hence $|X_3 \cap X| = 1$ and Lemma 2.2 gives that $X_1 \cup X_2 \cup X_3$ is critical. Hence $X_1 \cup X_2 \cup X_3 = V(H_t) - v$. We may now deduce that $|X| \leq 2$, since $X \subseteq X_1 \cup X_2 \cup X_3$ and $X \cap (X_1 \cup X_3) = \{x\}$ and $|X \cap X_2| \leq 1$. This contradicts the fact that $|X| \geq 3$.

Case 2 $N(v) \cap X = \emptyset$.
We have $x,y,z \in Y$. Since v is a leaf in $H_t[Y \cap V_3]$ we may assume, without loss of generality, that $d_{H_t}(x) \geq 4$ and $d_{H_t}(y) \geq 4$. Lemma 5.3 implies that v is not splittable along yz or zx. Thus there exist minimal critical sets X_1 and X_2 in H_t on $\{y,z\}$ and $\{z,x\}$ respectively. If two neighbours of v form a 2-separator in H_t, then the fact that $H_t[X]$ is connected by Lemma 5.2(c) implies that this will also be a 2-separator in G. This contradicts the 3-connectivity of G. Lemma 4.5 now implies that $|X_1 \cap X_2| \geq 2$ and $X_1 \cup X_2 = V(H_t) - v$ (possibly after renaming x,y,z in the case when $d_{H_t}(z) \geq 4$). Since $|X| \geq 3$, we may assume by symmetry that $|X_1 \cap X| \geq 2$. Now Lemma 4.2 implies that $X \cup X_1$ is the required v-critical, node critical set containing X. •

Choose a maximal v-critical and node-critical set $X^* \subset V(H_t)$ with $X \subseteq X^*$. By applying Lemma 4.3 to the critical set $X^* \cup \{v\}$, we deduce that $H_t - X^* - v$ contains a node. Lemma 4.1 now implies that we may choose a leaf w in $H_t[V_3 - X^* - v]$. Then w has at most one neighbour in X^* (otherwise $X^* + w$ would either contradict Lemma 2.15 or be a larger v-critical, node critical set than X^*.) Thus w is either a leaf in $H_t[V_3]$ or is a series node with a unique neighbour r in X^*, such that r is a node. Using Lemma 4.6, the 3-connectivity of G and the maximality of X^*, we can deduce that w is admissible in H_t (and hence in G). This proves the theorem. •

We shall also need

Theorem 5.8 *[1, Theorem 3.8] Let G be a* 3*-connected M-circuit with at least five vertices. Then either G has three non-adjacent admissible nodes or G has four admissible nodes.*

Theorems 5.4 and 5.8, and Lemmas 3.3 and 3.4 imply the following extension of [1, Theorem 4.4].

Corollary 5.9 *$G=(V,E)$ is M-connected if and only if G is a connected graph obtained from disjoint copies of K_4's by recursively applying edge additions and* 1*-extensions within a connected component, and taking* 2*-sums of different connected components.*

6 Bricks

A graph G is a *brick* if it is 3-connected and M-connected. A brick $G=(V,E)$ is said to be *minimal* if $G-e$ is not a brick for all $e\in E$. An edge f of G is *admissible* if $G-f$ is M-connected. A node v of G is *feasible* if G_v is a brick for some splitting G_v of G at v. A *fragment* in a 2-connected graph H is a set $X\subseteq V(H)$ such that $|N_H(X)|=2$ and $1\leq|X|\leq|V(H)|-3$. Let S be a 2-separator in H, $x,y\in V(H)$ and $e\in E(H)$. We say that S *separates x and y* if x and y belong to different components of $H-S$. We say that S *separates x and e* if either x and e belong to different components of $H-S$, or e is an edge from S to a component of $H-S$ which does not contain x.

Theorem 6.1 *Let $G=(V,E)$ be a minimal brick. If $G\neq K_4$ then G has a feasible node.*

Proof: We proceed by contradiction. Suppose the theorem is false and let G be a counterexample with as few vertices as possible. If $G-e$ is not M-connected for all $e\in E$ (in particular, if G is an M-circuit) then G has an admissible splitting $G_w^{x,y}$ by Theorems 5.4 and 5.8. Since G is a counterexample to the theorem, $G'=G_w^{x,y}$ is not 3-connected. On the other hand, if G is not minimally M-connected, then G has an admissible edge f. Since G is a minimal brick, $G'=G-f$ is not 3-connected. We now consider all possible choices for an admissible splitting and an admissible edge, and choose one such that some fragment X of the resulting M-connected graph G' is minimal with respect to inclusion.

We shall prove that X contains a feasible node of G. Since G' is M-connected, G' has minimum degree at least three and hence $|X|\geq 2$. By Lemma 3.5, G' has no polygons. Let $S:=N_{G'}(X)=\{u,v\}$. Let H,L be the cleavage graphs obtained by cleaving G' at $\{u,v\}$, where $X=V(H)-\{u,v\}$. Note that the minimality of X and the fact that G' has no polygons imply that H is a cleavage unit of G', and the 3-connectivity of G implies that $L-\{u,v\}$ is connected.

If $G'=G_w^{x,y}$ and $N(w)=\{x,y,z\}$, then let $V^*(H)=X-\{x,y,z\}$ and $E^*(H)=(E(H)\cap E(G))$. (The 3-connectivity of G implies that either $x,y\in X\cup S$ and $z\in V(L)-S$, or $x,y\in V(L)$ and $z\in X$.) On the other hand, if $G'=G-f$ and $f=yz$, then let $V^*(H)=X-\{y,z\}$ and $E^*(H)=E(H)-uv$. (The 3-connectivity of G implies that $\{y,z\}\cap X\neq\emptyset$ and $\{y,z\}\cap(V(L)-S)\neq\emptyset$.) Note that $E(H)-E^*(H)=$

$\{uv, xy\}$ if $G' = G_w^{x,y}$ and $x, y \in V(H)$. Otherwise $E(H) - E^*(H) = \{uv\}$. Let $\theta = xy$ if $G' = G_w^{x,y}$ and $xy \in E(H)$, let $\theta = z$ if $G' = G_w^{x,y}$ and $xy \notin E(H)$, and let θ be the unique vertex of X which is incident to f in G if $G' = G - f$.

Claim 6.2 *H is 3-connected.*

Proof: This follows since G' has no polygons and hence all its cleavage units are 3-connected. •

Claim 6.3 *$uv \notin E(G)$.*

Proof: Suppose $uv \in E(G)$. Since G' is M-connected, and $\{u, v\}$ is a 2-separator, Lemma 3.8 implies that $G' - uv$ is M-connected. Since $G - uv$ is obtained from $G' - uv$ by either an edge addition or a 1-extension, $G - uv$ is M-connected by Lemma 3.9. Futhermore, $G' - uv$ contains three internally disjoint uv-paths (two in $H - uv$ by Claim 6.2 and one in $L - uv$). Thus $G - uv$ has three internally disjoint uv-paths and the 3-connectivity of G implies that $G - uv$ is 3-connected. This contradicts the fact that G is a minimal brick. •

Claim 6.4 *H and L are M-connected.*

Proof: This follows from Lemma 3.4 since G' is M-connected and H and L are obtained by cleaving G' along the 2-separator $\{u, v\}$. •

Claim 6.5 *Suppose that $G - e$ is M-connected for some $e \in E^*(H)$. Then $H - \{u, v, e\}$ is connected.*

Proof: Suppose $H - \{u, v, e\}$ has two components H_1, H_2. Choose $i \in \{1, 2\}$ such that $\theta \notin V(H_i) \cup E(H_i)$. Then $V(H_i)$ is a fragment of $G - e$ which is properly contained in X. This contradicts the choice of G' and X. •

Claim 6.6 *$G - e$ is not M-connected for all $e \in E^*(H)$.*

Proof: Suppose that $G - e$ is M-connected for some edge $e = ab \in E^*(H)$. Since G is a minimal brick, $G - e$ is not 3-connected. Let T be a 2-separator in $G - e$. Since G is 3-connected, T separates a and b. If $G' = G_w^{x,y}$ then Lemma 2.20 implies that $w \notin T$.

Since G' is M-connected, it is redundantly rigid. Hence the graph $G'' = G' - e$ is rigid. Thus G'' is 2-connected by Lemma 2.6(a). Clearly, T and S are 2-separators in G''. By Lemma 3.6, T and S do not cross in G''. By Claim 6.5, $H - \{u,v,e\}$ is connected. Since $a,b \in X \cup S$ and T separates a and b in $G-e$, we have $T \cap X \neq \emptyset$ and $G''[X]$ is a component of $G'' - S$. Since T and S do not cross, we have $T \cap (V - X - S) = \emptyset$. Since $L - S$ is connected, some component J' of $G'' - T = G' - e - T$ contains $V - X - S$. Let J be the component of $G - e - T$ which contains $V - X - S$. Then $V - X \subset V(J) \cup N_{G-e}(J)$. Moreover, if $G' = G_w$, then the neighbour(s) of w in X are contained in $V(J) \cup N_{G-e}(J)$, and, if $G' = G - f$ then the endvertex of f in X is contained in $V(J) \cup N_{G-e}(J)$. This implies in both cases that the vertex set of the component of $G - e - T$ distinct from J is a proper subset of X. This contradicts the minimality of X. •

Claim 6.7 *$H - e$ is not M-connected for all $e \in E^*(H)$.*

Proof: Suppose $H - e$ is M-connected. Then $G' - e = (H - e) \oplus_2 L$ and $G' - e$ is M-connected by Claim 6.4 and Lemma 3.3. Since, by Lemma 3.9, the property of being M-connected is preserved by edge addition and 1-extension, it follows that $G - e$ is M-connected. This contradicts Claim 6.6. •

Note that if $p \in V^*(H)$ is a node of G then $p \notin \{u,v\}$ since u and v have degree at least four in G' by Lemma 2.20, and hence also in G.

Claim 6.8 *Suppose $p \in V^*(H)$ is a node of G, $N_G(p) = \{q,s,t\}$, and $G_p^{s,t}$ is M-connected. Then $p \notin \{u,v\}$ and $H_p^{s,t} - \{u,v\}$ is connected.*

Proof: Suppose $H_p^{s,t} - \{u,v\}$ is disconnected. Then $H - \{u,v\}$ has a 1-separation (H_1,H_2) where $V(H_1) \cap V(H_2) = \{p\}$, $s,t \in V(H_1)$ and $q \in V(H_2)$. Choose $i \in \{1,2\}$ such that $\theta \notin V(H_i) \cup E(H_i)$. Then $V(H_i) - p$ is a fragment of $G_p^{s,t}$ which is properly contained in X. This contradicts the choice of G' and X. •

Claim 6.9 *G_p is not M-connected for all nodes p of G in $V^*(H)$.*

Proof: Suppose that $G_p = G_p^{s,t}$ is M-connected for some node p of G in $V^*(H)$, with $N_G(p) = \{q,s,t\}$. Since G is a counterexample to the theorem, $G_p^{s,t}$ is not 3-connected. Let T be a 2-separator in $G_p^{s,t}$. Since G is 3-connected, T separates st and q. If $G' = G_w^{x,y}$ then Lemma 2.20 implies that $w \notin T$.

Since G' is M-connected, it is redundantly rigid. Hence $G' - pq$ is rigid. Since $G' - p$ is obtained from $G' - pq$ by deleting a vertex of degree two, it is rigid by

Lemma 2.8(b). Since $G'' = (G')_p^{s,t}$ is obtained from $G' - p$ by an edge addition, it is also rigid. Thus G'' is 2-connected by Lemma 2.6(a). Clearly, T and S are 2-separators in G''. By Lemma 3.6, T and S do not cross in G''. By Claim 6.8, $H_p^{s,t} - \{u,v\}$ is connected. Since $q,s,t \in X \cup S$ and T separates st and q in G'', we have $T \cap X \neq \emptyset$ and $G''[X-p]$ is a component of $G'' - S$. Since T and S do not cross, we must have $T \cap (V - X - S) = \emptyset$. Hence some component J' of $G'' - T = G'_p - T$ contains $V - X - S$. Let J be the component of $G_p^{s,t} - T$ which contains $V - X - S$. Thus $V - X \subset V(J) \cup N_{G_p}(J)$. Moreover, if $G' = G_w$, then the neighbour(s) of w in X are also contained in $V(J) \cup N_{G_p}(J)$, and, if $G' = G - f$ then the endvertex of f in X is contained in $V(J) \cup N_{G_p}(J)$. This implies in both cases that the vertex set of the component of $G_p - T$ which is distinct from J is a proper subset of X. This contradicts the minimality of X. •

Claim 6.10 *H_p is not M-connected for all nodes p of G in $V^*(H)$.*

Proof: Suppose H_p is M-connected. Then $G'_p = H_p \oplus_2 L$ and G'_p is M-connected by Claim 6.4 and Lemma 3.3. Since the property of being M-connected is preserved by edge addition and 1-extension, it follows that G_p is M-connected. This contradicts Claim 6.9. •

Claim 6.11 *H is an M-circuit.*

Proof: Suppose H is not an M-circuit. Since H is M-connected by Claim 6.4, there exists an M-circuit H_1 in H which contains uv and θ. By Lemma 5.1(b) we may extend $C_1 = E(H_1)$ to an ear-decomposition of $C_1, C_2, ..., C_t$ of $\mathcal{M}(H)$. By Claim 6.7 $H - e$ is not M-connected for all but at most two edges of H since $E(H) - E^*(H) \subseteq \{uv, xy\}$. Then it follows from Claim 6.2 and Theorem 5.4 that $H_t - \cup_{i=1}^{t-1} H_i$ contains an admissible node p of G in $V^*(H)$. This contradicts Claim 6.10. •

Claim 6.12 *H is isomorphic to K_4.*

Proof: Suppose H is not isomorphic to K_4. By Claim 6.10, no node of H in $V^*(H)$ is admissible in H. Since $uv \in E(H)$, Claim 6.2 and Theorem 5.8 imply that $G' = G_w^{x,y}$, $x,y \in V(H)$, and u,v,x,y are the only admissible nodes in H. We shall show that x is a feasible node in G.

Since x is an admissible node of H, $H_x^{s,t}$ is M-connected for some $s,t \in N_H(x)$. Let $N_H(x) = \{q,s,t\}$. Since xy is an edge of H and y is a node of H, we must have

$y \in \{s,t\}$. Without loss of generality, $y = t$. Since $(G')_x^{s,y} = H_x^{s,y} \oplus_2 L$, Claim 6.4 and Lemma 3.3 imply that $(G')_x^{s,y}$ is M-connected. Since $G_x^{s,w}$ is a 1-extension of $(G')_x^{s,y}$ and since the property of being M-connected is preserved by 1-extension (by Lemma 3.9), it follows that $G_x^{s,w}$ is M-connected.

Suppose $H_x^{s,y} - \{u,v\}$ is disconnected. Then $H - \{u,v\}$ has a 1-separation (H_1,H_2) where $V(H_1) \cap V(H_2) = \{x\}$, $s,y \in V(H_1)$ and $q \in V(H_2)$. Then $V(H_2) - x$ is a fragment of $G_x^{s,w}$ which is properly contained in X. This contradicts the choice of G' and X. Thus $H_x^{s,y} - \{u,v\}$ is connected.

Since G is a counterexample to the theorem, $G_x^{s,w}$ is not 3-connected. Let T be a 2-separator in $G_x^{s,w}$. Since G is 3-connected, T separates sw and q. Since G' is M-connected, it is redundantly rigid. Hence $G' - xq$ is rigid. Since $G' - x$ is obtained from $G' - xq$ by deleting a vertex of degree two, it is rigid by Lemma 2.8(b). Since $G'' = (G')_x^{sy}$ is obtained from $G' - x$ by an edge addition, it is also rigid. Thus G'' is 2-connected by Lemma 2.6(a). Clearly, T and S are 2-separators in G'' and T separates sy and q in G''. By Lemma 3.6, T and S do not cross. Since $H_x^{s,y} - \{u,v\}$ is connected, $q,s,y \in X \cup S$, and T separates sy and q in G'', we have $T \cap X \neq \emptyset$ and $G''[X - x]$ is a component of $G'' - S$. Since T and S do not cross, we must have $T \cap (V - X - S) = \emptyset$. Hence some component J' of $G'' - T = (G')_x^{s,y} - T$ contains $V - X - S$. Let J be the component of $G_x^{s,y} - T$ which contains $V - X - S$. Then $V - X \subset V(J) \cup N_{G_x^{s,y}}(J)$. Moreover, w and y are also contained in $V(J) \cup N_{G_x^{s,y}}(J)$. This implies that the vertex set of the component of $G_x^{s,y} - T$ which is distinct from J is a proper subset of X. This contradicts the minimality of X. •

Claim 6.13 *$G' = G_w^{x,y}$, $x,y \in V(H)$, and hence $\theta = xy \in E(H)$.*

Proof: Suppose that the claim is false. Then θ is a vertex in X, and $V(H) = \{u,v,\theta,t\}$. Then t is a node of G. We shall show that $G_t^{u,v}$ is a brick. Note that $uv \notin E(G)$ by Claim 6.3. Note further that $G_t^{u,v}$ can be obtained from L by a sequence of either one 1-extension and one edge-addition (if $G' = G - f$), or two 1-extensions and one edge-addition (if $G' = G_w^{x,y}$). Since L is M-connected by Claim 6.4, it follows from Lemma 3.9 that $G_t^{u,v}$ is M-connected. Since θ is adjacent to u and v, there is no 2-separation separating θ from uv in $G_t^{u,v}$. Thus $G_t^{u,v}$ is 3-connected and hence is a brick. •

Claim 6.14 *$X \neq \{x,y\}$.*

Proof: Suppose that $X = \{x,y\}$. Then x,y are nodes of G. We shall show that $G_x^{w,v}$ is a brick. Note that $wv \notin E(G)$ since the neighbour of w distinct from x,y belongs

to $V-X-S$. Note further that $G_x^{w,v}$ can be obtained from L by a sequence of two 1-extensions. Since L is M-connected by Claim 6.4, it follows from Lemma 3.9 that $G_x^{w,v}$ is M-connected. Suppose that $G_x^{w,v}$ is not 3-connected. Then there is a 2-separator T in $G_x^{w,v}$, separating u and wv. Since u,w, and v are all neighbours of y in $G_x^{w,v}$, we must have $y \in T$. Since $G_x^{w,v}$ is M-connected and y is a node in $G_x^{w,v}$, this contradicts Lemma 2.20. Thus $G_x^{w,v}$ is 3-connected and hence is a brick. •

We can now complete the proof of the theorem. Using Claims 6.13 and 6.14, and relabelling if necessary, we may suppose that $X = \{x,t\}$ and $S = \{u,y\}$. Thus x is a node of G. We shall show that $G_x^{w,t}$ is a brick. Note that $wt \notin E(G)$ since the neighbour of w distinct from x,y belongs to $V-X-S$. Note further that $G_x^{w,t}$ can be obtained from L by a sequence of two 1-extensions. Since L is M-connected by Claim 6.4, it follows from Lemma 3.9 that $G_x^{w,t}$ is M-connected. Suppose that $G_x^{w,t}$ is not 3-connected. Then there is a 2-separator T in $G_x^{w,t}$, separating u and wt. Since ut is an edge of $G_x^{w,t}$, we must have $t \in T$. Since $G_x^{w,t}$ is M-connected and t is a node in $G_x^{w,t}$, this contradicts Lemma 2.20. Thus $G_x^{w,t}$ is 3-connected and hence is a brick. •

We have the following corollaries:

Theorem 6.15 *$G = (V,E)$ is a brick if and only if G can be obtained from K_4 by 1-extensions and edge additions.*

Proof: Since K_4 is M-connected, sufficiency follows from Lemma 3.9, and the fact that edge addition and 1-extension preserve 3-connectivity. Necessity follows easily by induction on $|E|$, using Theorem 6.1. •

We illustrate Theorem 6.15 by constructing the minimal brick $K_{3,5}$ from K_4, see Figure 6.1.

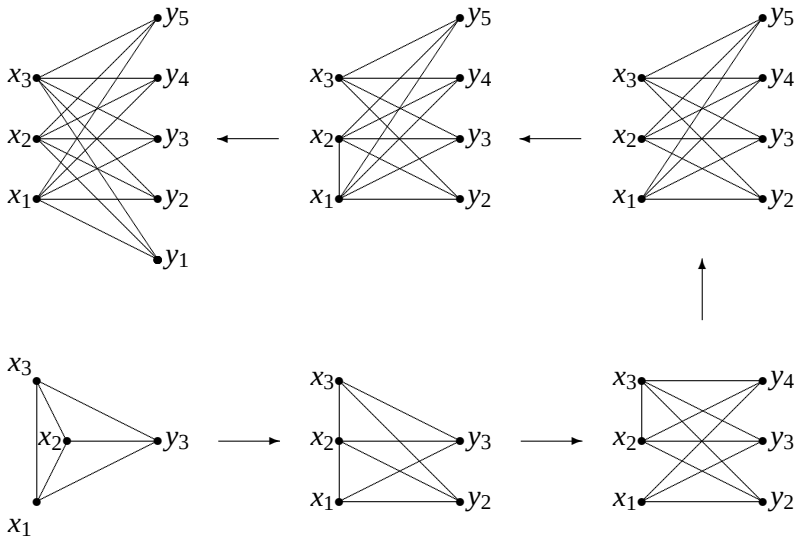

Figure 6.1. A construction of $K_{3,5}$ from K_4 using 1-extensions and edge additions.

Since $K_{3,5}$ is minimal, the first and last operations used in the construction must be 1-extensions. Since $K_{3,5}$ is not an M-circuit, at least one operation in the construction must be an edge addition. This shows that one may need to alternate between the two operations of Theorem 6.15 while building up a brick from K_4.

7 Globally rigid graphs in $\mathbb{R}^2$

Theorem 3.2 implies that a graph is a brick if and only if it is redundantly rigid and 3-connected. Thus Theorem 6.15 gives an inductive construction for redundantly rigid 3-connected graphs. It follows from the result of Connelly [5, Theorem 1.5] that any graph which can be obtained from K_4 by edge additions and 1-extensions is globally rigid in $\mathbb{R}^2$. By using Theorems 6.15 and 1.1 we can now characterise globally rigid graphs, and hence verifiy Hendrickson's conjecture, in dimension two.

Theorem 7.1 *Let G be a graph. Then G is globally rigid in $\mathbb{R}^2$ if and only if either G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid.*

Note that the special case of Theorem 7.1 when $|E(G)| = 2|V(G)| - 2$ was proved earlier in [1, Theorem 6.1].

It follows from Theorem 7.1 that global rigidity of frameworks is a generic property in $\mathbb{R}^2$. Lovász and Yemini [17] proved that 6-connected graphs are redundantly rigid (and that this bound is best possible). With this result and Theorem 7.1 we can show that sufficiently highly connected graphs are globally rigid. In fact, the same degree of connectivity suffices.

Theorem 7.2 *Let G be* 6*-connected. Then G is globally rigid in $\mathbb{R}^2$.*

This solves [10, Open question 4.47]. As we noted earlier, there exist efficient algorithms for testing 3-connectivity and redundant rigidity, and hence global rigidity in $\mathbb{R}^2$. See [2] for more details on the algorithmic aspects.

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