

# Backscattering-Induced Crossover in Deterministic Diffusion

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We study diffusion in a one-dimensional periodic array of scatterers modeled by a simple map. The chaotic scattering process for this map can be changed by a control parameter and exhibits the dynamics of a crisis in chaotic scattering. We show that the strong backscattering associated with the crisis mechanism induces a crossover which leads to different asymptotic laws for the parameter-dependent diffusion coefficient. These laws are obtained from exact diffusion coefficient results and are supported by simple random walk models. We argue that the main physical feature of the crossover should be present in many other dynamical systems with non-equilibrium transport.

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One of the basic mechanisms in the theory of chaotic dynamical systems are so-called crisis events, where the asymptotic dynamics of the system changes drastically with respect to the variation of a control parameter [1–3]. Recently, it was discovered that similar events occur in simple chaotic scattering systems when the scattering rules are varied. This phenomenon has been denoted as a crisis in chaotic scattering [4]. On the other hand, over the past few years a considerable literature has been developed in which the origin of transport in non-equilibrium statistical mechanics has been related to the characteristics of chaotic scattering processes [5]. One problem studied was deterministic diffusion in simple one-dimensional maps [6–8], where for some examples parameter-dependent diffusion coefficients have been computed by taking the complete equations of motion of the dynamical systems into account [9,10]. Similar one-dimensional maps have been proposed in Ref. [4] as simple models for a crisis in chaotic scattering. Thus, the aim of this Letter is to investigate whether a crisis in chaotic scattering has an impact on deterministic diffusion in simple one-dimensional maps.

In the following, we consider discrete one-dimensional piecewise linear chaotic maps with uniform slope,  $x_{n+1} = M_h(x_n)$ , where  $h$  is a control parameter, and  $x_n$  is the position of a point particle at the discrete time  $n$ .  $M_h(x)$  is continued periodically beyond the interval  $[0, 1]$  onto the real line by a lift of degree one,  $M_h(x + 1) = M_h(x) + 1$ . We assume that  $M_h(x)$  is anti-symmetric with respect to  $x = 0$ ,  $M_h(x) = -M_h(-x)$ , i.e., that there is no drift imposed on a point particle. As an example, we consider the sawtooth map sketched in Fig. 1. It was chosen as a periodic continuation of the map studied in Ref. [4] which exhibits a crisis in chaotic scattering. The control parameter is here the height  $h$  of the map which is related to the absolute value of the slope  $a$  by  $h = (a - 3)/4$ . The diffusive properties of similar maps have been studied in Refs. [6,8]. For this sawtooth map the parameter-dependent diffusion coefficient has been computed by solving the Frobenius-Perron equation of the dynamical system [2],

$$\rho_{n+1}(x) = \int dy \rho_n(y) \delta(x - M_h(y)) \quad , \quad (1)$$

where  $\rho_n(x)$  is the probability density for points on the real line, and  $M_h(y)$  is the map under consideration. There exists a dense set of parameter values  $h$  for which one can construct Markov partitions for the map, and for each of these parameter values Eq. (1) can be written as a matrix equation [9,10],

$$\rho_{n+1} = (1/|a|)T \rho_n \quad . \quad (2)$$

$\rho_n$  represents a column vector of the probability densities in each part of the Markov partition at time  $n$ , and  $T$  is a topological transition matrix which can be obtained from the Markov partition. However, instead of solving the eigenvalue problem of  $T$  [9], here solutions for the probability density vector  $\rho_n$  have been obtained by iterating Eq. (2),

$$\rho_{n+1} = (1/|a|^n)T^n \rho_0 \quad . \quad (3)$$

Starting with any probability density vector  $\rho_0$  this iteration method enables us to compute the exact time-dependent probability density  $\rho_n$  at any time step  $n$  and all other dynamical quantities based on probability density averages for maps of the type of  $M_h(x)$  [10–12]. In particular, it provides an efficient way to calculate diffusion coefficients by employing an Einstein formula,

$$D(h) = \lim_{n \rightarrow \infty} \frac{1}{2n} \int dx \rho_n(x) x^2 \quad , \quad (4)$$

where the integral is the second moment of the time-dependent probability density [13].

Fig. 1 shows a log-log plot of the diffusion coefficient as a function of  $h$  up to  $h = 3.5$ . Included are four curves which describe the coarse-grained behavior of the exact results. For integer values of the height the diffusion coefficient can be computed analytically by applying the eigenvalue method of Ref. [9], and we get

$$D(h) = \frac{2h^3 + 3h^2 + h}{12h + 9} \rightarrow \frac{h^2}{6} \quad (h \rightarrow \infty), \quad h \in N. \quad (5)$$

The two *dashed* curves give approximate limits for the oscillations of the exact diffusion coefficient in the range  $h > h_c$ . They are obtained by fitting the diffusion coefficient with the functional form of Eq. (5) at  $h = (2k+1)/2$  and  $h = (4k+3)/4$ ,  $k \in N_0$ , for the upper and lower curve, respectively. The two *dotted* curves show two simple random walk approximations. For large heights the distance a point particle travels at one time step by moving from one unit interval to another is taken into account exactly [2], and we get

$$D_{rw1}(h) = \int_0^{1/2} dx (M_h(x) - x)^2 \rightarrow \frac{h^2}{6} \quad (h \rightarrow \infty), \quad (6)$$

which gives the dotted line plotted for  $h > h_c$ . For small heights the absolute value of the distance is approximated to either zero or one, depending on whether the particle remains on a unit interval or moves from one unit interval to the next [7]. This leads to

$$D_{rw2}(h) = \frac{2h}{4h+3} \rightarrow \frac{2}{3}h \quad (h \rightarrow 0). \quad (7)$$

These approximations indicate three different regions of coarse-grained behavior for the exact diffusion coefficient: The first one is a simple initial region, where the diffusion coefficient behaves linearly for small heights. For  $h \geq h_c$  it decreases slightly on increasing the height. Finally, for  $h \geq 0.5$  it starts to grow quadratically in the height, but with strong oscillations on a fine scale. The transition between the two different types of asymptotic coarse-grained behavior, which occurs in the intermediate region of  $h_c \leq h \leq 0.5$ , can be understood by referring

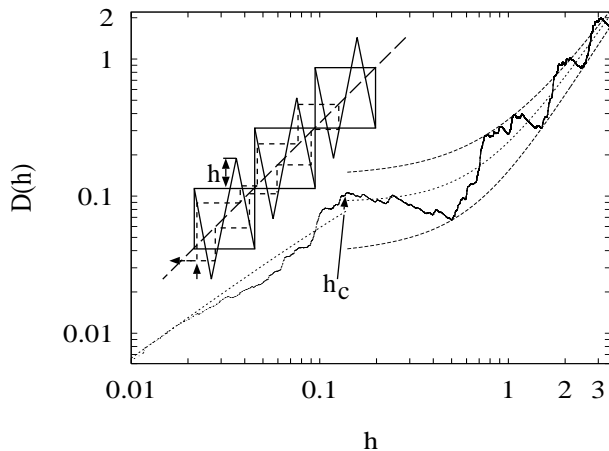


FIG. 1. Double logarithmic plot of the diffusion coefficient  $D(h)$  with respect to the height  $h$  for the sawtooth map shown in the figure. The graph is based on 38,889 single data points. Two random walk solutions (dotted lines) and two curves which approximately give the boundaries of the oscillations of  $D(h)$  for values of  $h$  above the crisis point  $h_c$  (dashed lines) are included.

to the action of certain microscopic scattering mechanisms of the map, which are closely connected to the dynamics of a crisis in chaotic scattering.

These scattering mechanisms are introduced in Fig. 2, where certain regions of the map have been distinguished by shaded squares and triangles: The triangles refer to parts where points of one unit interval get mapped from that particular interval into another unit interval. Additionally, if points enter a square they preferably move into the triangular escape region above or below the respective square after some iterations. These squares are identical to the squares of an analogous scattering model, where they provide the fundamental mechanism for a crisis in chaotic scattering [4]. The abbreviations *f* (“forward”) and *b* (“backward”) in these scattering regions refer to the dynamics of the critical point of the map, which is indicated by a small circle. Its first iteration is shown by the dashed line with the arrows. At its second iteration, and by increasing the height  $h$  continuously up from zero, the orbit of the critical point, denoted as the *critical orbit* in the following, travels along the graph of the map in the next right box from the upper left to the lower right, as indicated by bold black arrows. This way, the critical orbit explores all the different scattering regions of the map as  $h$  is increased from zero. If it hits a region labeled by a *b* it is in a position to get *backscattered* into the box to the left. *Vice versa*, if the orbit enters an *f* region it is in a preferable position to move further *forward* to the next box to the right. The critical point indicated in Fig. 2 is part of a forward scattering region. Note that there is a dense set of points around the critical point

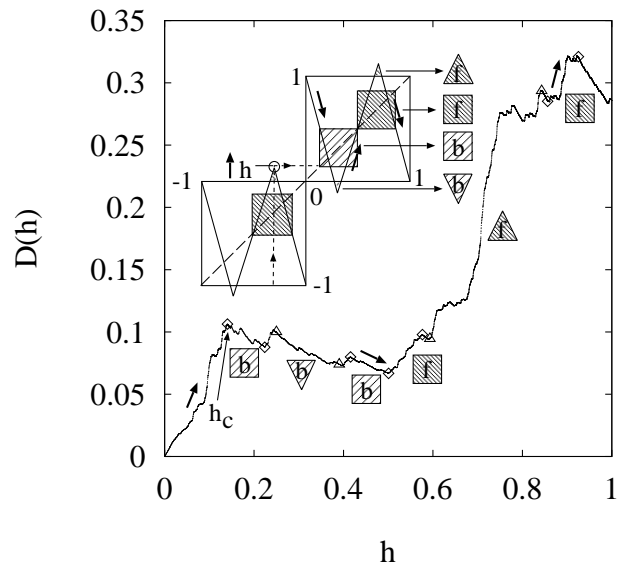


FIG. 2. Chaotic scattering in the sawtooth map and its connection to the behavior of the diffusion coefficient  $D(h)$ . Certain microscopic scattering mechanisms of the map are identified by shaded squares and triangles. The same symbols are shown along the  $D(h)$  curve, where they indicate the impact of the respective scattering regions on the diffusion coefficient. The graph consists of 10,268 single data points.

which exhibits the same dynamics, at least for the first few iterations. An event dynamically analogous to the one of a crisis in chaotic scattering now occurs at the *crisis point*, defined by the parameter value of the height  $h_c$  for which the critical orbit of the forward scattering region hits the boundary of a backward scattering region in the next right box after one iteration for the first time. This case is illustrated in Fig. 2 and, for the map shown, is determined by  $h_c = (\sqrt{17} - 3)/8 \simeq 0.1404$ . We remark that this process is *topologically* not identical to a crisis in chaotic scattering in the sense that a crisis is generated by the merging of two formerly isolated invariant sets in the phase space. Nevertheless, we argue that *dynamically* this process provides the same characteristics as a crisis in chaotic scattering, especially the onset of strong backscattering [4].

The squares and the triangles along the diffusion coefficient curve now refer to parameter regions where the critical orbit gets mapped into the respective scattering regions. The different symbols on the curve denote boundary points where the critical orbit enters, or leaves, these regions. The diffusion coefficient clearly decreases globally if the critical orbit enters a backscattering region, and it increases globally if it exhibits forward scattering. Hence, the different microscopic scattering mechanisms defined above are connected to regions in the macroscopic diffusion coefficient which exhibit different parameter-dependent behavior. Most importantly, the crisis point  $h_c$  corresponds to the first strong local maximum of the curve, as shown in Fig. 2. Magnifications reveal that the fine structure of the curve changes significantly just below and above the crisis value, and that the curve is fractal [9]. This can be understood in detail by refining the procedure explained above and is reported elsewhere [10,11]. Moreover, in Fig. 1 the crisis point  $h_c$  separates the regions which are described by the two different random walk models and which show significantly different coarse-grained behavior.

We conclude that the onset of strong backscattering affects the diffusion coefficient of the sawtooth map not only on a fine scale, but also on a coarse-grained scale. This phenomenon may be understood as a *backscattering-induced crossover in deterministic diffusion*. However, such a dynamical event must eventually take place in any map of the type of  $M_h(x)$  at a certain parameter value, independently of its special functional form and independently of the special topological characteristics of a crisis in chaotic scattering. This can be checked by identifying the forward and backward scattering regions of a map and applying the definition of the crisis point given above. As an example, the piecewise linear, discontinuous, non-sawtooth map studied in Ref. [9] has been analyzed. It turns out that the respective crisis point of the map again corresponds to the first strong local maximum of the diffusion coefficient curve, and that again the crisis point is related to a change between two different laws for the parameter-dependent diffusion coefficient on a coarse-grained scale: As before, the diffusion coefficient

grows linearly for small values of the height, although with a slope different from that obtained for the sawtooth map. It lacks a broad crossover region right above the crisis point, but as in case of the sawtooth map it increases quadratically with a factor of 1/6 in the limit of large heights. Analogous results for asymptotic diffusion coefficients have been reported in Ref. [6] for a variety of other maps of the type of  $M_h(x)$ , although in this previous work diffusion coefficients could be computed exactly only for special values of the height. This leads us to conjecture that a backscattering-induced crossover, as described above, is typical for diffusive maps of the type of  $M_h(x)$ . Moreover, we suspect that for these maps the parameter-dependent coarse-grained diffusion coefficient always decreases linearly in the limit of small heights, and that it increases quadratically in the limit of large heights with a universal factor of 1/6. Similar results concerning the asymptotic coarse-grained behavior of diffusion coefficients may be found in certain classes of two-dimensional maps, as, e.g., in standard and sawtooth maps, where quadratic laws in the limit of large control parameters have already been obtained [2,14].

We suppose that the main physical feature of the crossover, i.e., the connection between the onset of strong microscopic backscattering and a change in the behavior of macroscopic parameter-dependent transport coefficients, is quite common not only in discrete one- and two-dimensional models, but also in higher-dimensional Hamiltonian systems: For example, in Ref. [15] diffusion of an electron in a two-dimensional crystal, modeled by Coulombic periodic potentials, has been studied. For this class of dynamical systems an energy threshold has been proved to exist above which the diffusion coefficient increases with a power law in the particle energy, whereas below this threshold no diffusion coefficient exists. In Ref. [4] it has been found that related models exhibit a crisis in chaotic scattering. Hence, we conjecture that the existence of this energy threshold is linked to the dynamics of a crisis in chaotic scattering as in case of the backscattering-induced crossover discussed above. Furthermore, for the models with a crisis in chaotic scattering the significance of orbiting collisions has been pointed out, which indicate the onset of strong backscattering [4]. However, orbiting collisions have already been discussed extensively for fluid systems at low densities, where particles interact via Lennard-Jones potentials, and a qualitative connection between the onset of these collisions and a small change in the temperature-dependent behavior of transport coefficients has been established [16]. Thus, physically a direct line may be drawn from the dynamical origin of crises in dynamical systems over the occurrence of certain microscopic chaotic scattering processes to a specific coarse-grained behavior of transport coefficients, which may be related to macroscopic dynamical crossover phenomena, or in certain cases possibly even to dynamical phase transitions.

We conclude with a few remarks: (a) The sawtooth map under consideration here was chosen as a diffusive

version of a one-dimensional dynamical system which exhibits a crisis in chaotic scattering. A crossover in the parameter-dependent diffusion coefficient of this map has been found which is generated by the dynamical mechanism of this crisis event. It affects the diffusion coefficient not only on a fine scale, but also on a coarse-grained scale and leads to two different algebraic laws for the asymptotic diffusion coefficient. (b) These different algebraic laws, i.e., a linear decrease of the diffusion coefficient for small heights and a quadratic increase with a factor of 1/6 in the limit of large height, are suspected to be universal for maps of the type of  $M_h(x)$ . (c) The crossover is understood physically by relating the onset of strong microscopic backscattering to a change in the behavior of the macroscopic parameter-dependent transport coefficient. This main physical feature is conjectured to be quite common in dynamical systems which exhibit non-equilibrium transport.

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- [13] The iteration method leads to values for  $D(h)$  which are precise up to an order of  $10^{-7}$  after maximally 15 iteration steps. Therefore, no error bars appear for the results presented in Figs. 1 and 2. Another approach to compute diffusion coefficients for maps of the type of  $M_h(x)$  is based on a Green-Kubo formula and relates the parameter-dependent diffusion coefficient to a class of fractal functions which is defined by certain functional equations [10,11]. An even more efficient method has been developed by J. Groeneveld (to be published).
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