Killing spinors as a characterisation of rotating black hole spacetimes

Michael J. Cole [*,1,2] and Juan A. Valiente Kroon [†,1]

[1] *School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom.*

April 19, 2016

Abstract

We investigate the implications of the existence of Killing spinors in a spacetime. In particular, we show that in vacuum and electrovacuum a Killing spinor, along with some assumptions on the associated Killing vector in an asymptotic region, guarantees that the spacetime is locally isometric to the Kerr or Kerr-Newman solutions. We show that the characterisation of these spacetimes in terms of Killing spinors is an alternative expression of characterisation results of Mars (Kerr) and Wong (Kerr-Newman) involving restrictions on the Weyl curvature and matter content.

Keywords: Spinorial methods, black holes, Killing spinors, Kerr-Newman solution, invariant characterisations

PACS: 04.20.Jb, 04.20.Gz, 04.40.Nr, 04.20.Ha

1 Introduction

The *Kerr spacetime*, describing a rotating black hole in vacuum, is one of the most interesting exact solutions to the Einstein field equations. As well as having physical relevance, the existence of various incarnations of uniqueness theorems (see e.g. [8] and references within for a survey of this vast topic) has cemented its place as one of the most important vacuum solutions mathematically. There also exist generalisations to spacetimes containing restricted forms of matter —for example, the Kerr-Newman solution to the Einstein-Maxwell equations. Although less physically interesting than the vacuum case, these solutions still retain many interesting features of the Kerr solution, including uniqueness under further assumptions on the matter content. Thus, these generalisations still retain a mathematical importance.

The remarks in the previous paragraph justify the attention given to finding *characterisations of the Kerr spacetime* and its relations —see e.g. [15, 10]. Such characterisations can be used to study various open questions about these black hole spacetimes. For example, they can be used to reformulate uniqueness theorems and clarify relations between them; study the stability of the solutions, by indicating the behaviour of perturbations; and illustrate the special characteristics of these particular solutions, in particular through the use of symmetries —see e.g. [1] for a recent discussion on these and related ideas. The last of these is elegantly achieved through the use of *Killing spinors*. Closely related to *Killing-Yano tensors*, these spinorial objects represent "hidden symmetries" of the spacetime, which cannot be represented using Killing vectors. It has been shown previously (see [3, 4, 5]) that a vacuum spacetime admitting a Killing spinor, along with conditions on the Weyl curvature and an asymptotic condition, must be isometric to the Kerr spacetime. This result crucially depends on a result of Mars (see [16]) which uses the structure of

---

the Weyl tensor, and its relation to the Killing vectors of the spacetime, to characterise the Kerr solution in a way that exploits to the maximum possible extent the asymptotic flatness of the spacetime —more precisely, it is required that the selfdual Killing form of the stationary Killing vector is an eigenform of the selfdual Weyl tensor.

The characterisation of the Kerr spacetime by Mars given in [16] relies on a previous characterisation of this solution to the vacuum Einstein field equations in terms of the vanishing of the so-called *Mars-Simon tensor* —see [15]. Interestingly, the latter characterisation has been generalised to the electrovacuum case by Wong [22] assuming some restrictions on the matter content. This characterisation is not optimal —in the sense that it assumes the existence of certain relations among the relevant geometric objects; by contrast, in [15], the existence of the vacuum counterpart of these relations is a consequence of the characterisation. Nevertheless, as a consequence of the analysis in [22], one may expect that the Kerr-Newman solution can be characterised by the use of Killing spinors in a similar way to the vacuum case. The characterisations in both [15] and [22] come in both a *local version* (in which certain constraints arising in the characterisation are fixed by evaluating them at finite points of the manifold) and a *global version* (in which asymptotic flatness is used to fix the value of the constants). Remarkably, the generalisation of the characterisation in [16] to the electrovacuum case has, so far, not been obtained.

The purpose of this article is to revisit the characterisation of the Kerr spacetime using Killing spinors and then generalise to the electrovacuum case using Wong's result in [22]. Our analysis suggests that Wong's result can be strengthened to obtain a characterisation of the Kerr-Newman spacetime more in the spirit of Mars's original result in [15] and, in turn, use this result to obtain a generalisation of the analysis of [16] in which the Kerr-Newman spacetime is characterised in an optimal way by a combination of local and global assumptions.

Outline of the article. This paper is organised as follows. In section 2 we give an introduction to Killing spinors, their relation to Killing vectors and investigate the implications on the curvature of the spacetime. We will spend some time defining 1-forms and potentials which are useful in the characterisations later on. In section 3, we define the required asymptotic conditions needed for the characterisation theorems. Then, in section 4, we show that the conditions of the characterisation result of Mars [16] are satisfied when the spacetime admits an appropriate Killing spinor. Finally, section 5 shows the same for Wong's characterisation of the Kerr-Newman spacetime —i.e. the existence of an appropriate Killing spinor on a electrovacuum spacetime guarantees that the solution is Kerr-Newman up to an isometry.

Conventions

In what follows, $(\mathcal{M}, \boldsymbol{g})$ will denote an electrovacuum spacetime satisfying the Einstein equations with vanishing cosmological constant. The signature of the metric in this article will be $(+, -, -, -)$, to be consistent with most of the existing literature using spinors. We use the spinorial conventions of [18]. The lowercase Latin letters a, b, $c, \ldots$ are used as abstract spacetime tensor indices while the uppercase letters A, B, $C, \ldots$ will serve as abstract spinor indices. The Greek letters μ, ν, $\lambda, \ldots$ will be used as spacetime coordinate indices while α, β, $\gamma, \ldots$ will serve as spatial coordinate indices.

Our conventions for the curvature are that

$$\nabla_c \nabla_d u^b - \nabla_d \nabla_c u^b = R_{dca}{}^b u^a.$$

The curvature spinors Ψ_{ABCD} and $\Phi_{ABA'B'}$ are defined by the relations

$$\Box_{AB}\xi_C = \Psi_{ABCD}\xi^D - 2\Lambda\xi_{(A}\epsilon_{B)C}, \qquad \Box_{A'B'}\xi_C = \xi^D \Phi_{CDA'B'},$$

where $\Box_{AB} \equiv \nabla_{A'(A}\nabla_{B)}{}^{A'}$. Given an antisymmetric rank 2 tensor F_{ab}, its Hodge dual is defined by

$$F^\star_{ab} \equiv \frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}.$$

The self-dual version of F_{ab} is then defined by

$$\mathcal{F}_{ab} \equiv F_{ab} + \mathrm{i}F^\star{}_{ab}.$$

2

2 Killing spinors

The purpose of this section is to provide a summary of the basic theory of Killing spinors in electrovacuum spacetimes —see [11, 12, 13]. Throughout we assume that $(\mathcal{M}, \boldsymbol{g})$ denotes an electrovacuum spacetime. In spinorial notation the Einstein-Maxwell equations read

$$\Phi_{ABA'B'} = 2\phi_{AB}\bar{\phi}_{A'B'}, \qquad \Lambda = 0$$

where $\phi_{AB} = \phi_{(AB)}$ is the Maxwell spinor satisfying

$$\nabla^A{}_{A'}\phi_{AB} = 0. \tag{1}$$

The Bianchi identity in electrovacuum spacetimes takes the form

$$\nabla^A{}_{A'}\Psi_{ABCD} = 2\bar{\phi}_{A'B'}\nabla_B{}^{B'}\phi_{CD}. \tag{2}$$

2.1 Basic equations

A *Killing spinor* is a valence-2 symmetric spinor κ_{AB} satisfying the equation

$$\nabla_{A'(A}\kappa_{BC)} = 0. \tag{3}$$

By taking a further contracted derivative of this equation, it can be shown that a solution to equation (3) must also satisfy the integrability condition

$$\kappa_{(A}{}^F\Psi_{BCD)F} = 0 \tag{4}$$

where Ψ_{ABCD} is the Weyl spinor, a completely symmetric spinor which is the spinorial equivalent of the Weyl tensor. This condition restricts the form of the Weyl spinor as it requires that

$$\Psi_{ABCD} \propto \kappa_{(AB}\kappa_{CD)}.$$

This proportionality condition forces the spacetime to be of Petrov type D, N or O (i.e. conformally flat). In particular, if a non-vanishing Killing spinor has a repeated principal spinor α_A so that $\kappa_{AB} = \alpha_{(A}\alpha_{B)}$, then the Weyl spinor has four repeated null directions, and so it is of Petrov type N. If the Killing spinor is algebraically general, i.e. there exist α_A and β_B such that $\kappa_{AB} = \alpha_{(A}\beta_{B)}$, then the Weyl spinor has two pairs of repeated null directions, and so it is of Petrov type D.

Algebraically general Killing spinors

In the case that the Killing spinor κ_{AB} is algebraically general, we can use the principal spinors α_A and β_B to form a normalised spin dyad which we will denote by $\{o^A, \iota^B\}$ and such that $o_A\iota^A = 1$. The Killing spinor κ_{AB} is then expanded in terms of the basis as

$$\kappa_{AB} = \varkappa o_{(A}\iota_{B)} \tag{5}$$

for some factor of proportionality $\varkappa$. Due to equation (4), the Weyl spinor can be expanded in a similar way as

$$\Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)} \tag{6}$$

for some factor of proportionality ψ.

The substitution of expression (5) in the Killing spinor equation (3) implies restrictions on the Newman-Penrose (NP) spin connection coefficients. Namely, one has that

$$\kappa = \lambda = \nu = \sigma = 0,$$

consistent with the fact that the spacetime is, at least, of Petrov type D.

2.2 The Killing vector associated to a Killing spinor

A Killing spinor κ_{AB} can be used to define the spinorial counterpart $\xi_{AA'}$ of a (possibly complex) vector via the relation

$$\xi_{AA'} \equiv \nabla^C{}_{A'}\kappa_{AC}. \tag{7}$$

It can be shown, using the Killing spinor equation (3) and commuting covariant derivatives, that $\xi_{AA'}$ satisfies the equation

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = -6\kappa_{(A}{}^C\Phi_{B)CA'B'}.$$

Therefore, if

$$\kappa_{(A}{}^C\Phi_{B)CA'B'} = 0 \tag{8}$$

then $\xi_{AA'}$ is the spinorial counterpart of a (possibly complex) Killing vector in the spacetime. In what follows, we call condition (8) the *matter alignment condition.* In the particular case of an electrovacuum spacetime the matter alignment condition takes the form

$$\kappa_{(A}{}^C\phi_{B)C} = 0 \tag{9}$$

implying that the spinors κ_{AB} and ϕ_{AB} are proportional to each other. Thus, in terms of the basis dyad $\{o, \iota\}$ used to express equation (5) one can write

$$\phi_{AB} = \varphi o_{(A}\iota_{B)} \tag{10}$$

with φ a proportionality constant.

As discussed in [19], the notion of a Lie derivative is, in general, not well defined for spinors. However, in the case of a Hermitian spinor $\xi^{AA'}$ associated to a real Killing vector, there exists a consistent expression which can be used to obtain the spinorial counterpart of the condition $\mathcal{L}_\xi F_{ab} = 0$, stating that the Killing vector ξ^a is also a symmetry of the Faraday tensor —namely:

$$\mathcal{L}_\xi\phi_{AB} \equiv \xi^{CC'}\nabla_{CC'}\phi_{AB} + \phi_{C(A}\nabla_{B)C'}\xi^{CC'}. \tag{11}$$

The Maxwell spinor will be said to *inherit the symmetry generated by the Killing vector ξ^a* if $\mathcal{L}_\xi\phi_{AB} = 0$ —recall that the Maxwell spinor ϕ_{AB} is related to the Faraday tensor via the relation

$$\mathcal{F}_{AA'BB'} = 2\phi_{AB}\epsilon_{A'B'}$$

where $\mathcal{F}_{AA'BB'}$ denotes the spinorial counterpart of the self-dual Faraday tensor $\mathcal{F}_{ab} = F_{ab} + \mathrm{i}F^\star_{ab}$.

Remark. In Section 2.5.3 it will be shown that in an electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ endowed with a Killing spinor κ_{AB} such that $\xi_{AA'}$ is Hermitian and ϕ_{AB} and κ_{AB} satisfy the alignment condition (9) then ϕ_{AB} inherits the symmetry of the spacetime.

2.3 Relation to Killing-Yano tensors

If a spacetime $(\mathcal{M}, \boldsymbol{g})$ admits a Killing spinor κ_{AB}, and the vector $\xi^{AA'}$ defined by (7) satisfies $\xi^{AA'} = \bar{\xi}^{AA'}$ (i.e. is a real vector), then one can construct a real, valence-2 antisymmetric tensor Y_{ab} as the tensorial counterpart of the spinorial relation

$$Y_{AA'BB'} \equiv \mathrm{i}\left(\kappa_{AB}\epsilon_{A'B'} - \bar{\kappa}_{A'B'}\epsilon_{AB}\right)$$

which, as a consequence of (3), satisfies the Killing-Yano equation

$$\nabla_{(a}Y_{b)c} = 0.$$

Such a tensor is called a *Killing-Yano tensor.* Conversely, if a spacetime admits a Killing-Yano tensor Y_{ab}, one can construct a valence-2 symmetric spinor κ_{AB} from the relation

$$\kappa_{AB} \equiv -\frac{\mathrm{i}}{4}\epsilon^{A'B'}\left(Y_{AA'BB'} + \mathrm{i}Y^*_{AA'BB'}\right)$$

which satisfies the Killing spinor equation (3) —see e.g. [19], Section 6.7 page 107; also [17].

Remark. The existence of a Killing-Yano tensor for the Kerr-Newman spacetime is a key ingredient to show the integrability of the Hamilton-Jacobi equations for geodesic motion, and the separability of the Maxwell equations and the Dirac equation on the Kerr-Newman spacetime —see e.g. [14] or [21] for further details.

2.4 The Killing form

In the reminder of this section assume that the matter alignment condition (8) is satisfied, so that $\xi_{AA'}$ is the spinorial counterpart of a Killing vector. Moreover, *assume that $\xi_{AA'}$ is a Hermitian spinor so that, in fact, it is the spinorial counterpart of a real vector.* Then, define the spinorial counterpart of the *Killing form* of ξ^a, namely

$$H_{ab} \equiv \nabla_{[a}\xi_{b]} = \nabla_a \xi_b$$

by

$$H_{AA'BB'} \equiv \nabla_{AA'}\xi_{BB'}.$$

As a consequence of the antisymmetry in the pairs $_{AA'}$ and $_{BB'}$ it can be decomposed into irreducible parts as

$$H_{AA'BB'} = \eta_{AB}\epsilon_{A'B'} + \bar{\eta}_{A'B'}\epsilon_{AB} \tag{12}$$

where η_{AB} is a symmetric spinor —the *Killing form spinor*. In the sequel, we will require the self-dual part of $H_{AA'BB'}$, denoted by $\mathcal{H}_{AA'BB'}$, and defined by

$$\mathcal{H}_{AA'BB'} \equiv H_{AA'BB'} + \mathrm{i}H^\star_{AA'BB'}.$$

A direct calculation then yields

$$\mathcal{H}_{AA'BB'} = 2\eta_{AB}\epsilon_{A'B'}.$$

Using equation (12), the spinor η_{AB} can be expressed in terms of the Killing vector as

$$\eta_{AB} = \frac{1}{2}\nabla_{AA'}\xi_B{}^{A'}. \tag{13}$$

Then, by using (7), this can be expanded in terms of the Killing spinor so that

$$\eta_{AB} = -\frac{3}{4}\Psi_{ABCD}\kappa^{CD}. \tag{14}$$

Expansions for the algebraically general case

Assuming, now that κ_{AB} is algebraically general, using the basis expansions of both κ_{AB} and Ψ_{ABCD}, we can find the basis expansion of η_{AB}:

$$\eta_{AB} = \frac{1}{4}\varkappa\psi o_{(A}\iota_{B)} = \eta o_{(A}\iota_{B)} \tag{15}$$

where

$$\eta \equiv \frac{1}{4}\varkappa\psi. \tag{16}$$

2.5 The Ernst forms and potentials

Throughout this section let ξ^a denote a real Killing vector on the electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g})$. A well-known consequence of the Killing equation

$$\nabla_a\xi_b + \nabla_b\xi_a = 0$$

and the definition of the Riemann tensor in terms of commutators of covariant derivatives is that

$$\nabla_a\nabla_b\xi_c = R_{cba}{}^d\xi_d. \tag{17}$$

The *Ernst form* of the Killing vector ξ^a is defined as

$$\chi_a = 2\xi^b\mathcal{H}_{ba} \tag{18}$$

Several properties of the Ernst form follow from the identity (17) recast as

$$\nabla_a\mathcal{H}_{bc} = \mathcal{R}_{cba}{}^d\xi_d \tag{19}$$

where $\mathcal{R}_{abcd}$ denotes the *self-dual* Riemann tensor. From expression (19) it follows, using the identity

$$^*R_{[abc]d} = \frac{1}{3}\epsilon_{abce}R^e{}_d,$$

that

$$\nabla_{[a}\mathcal{H}_{bc]} = \frac{1}{3}\epsilon_{cbae}R^e{}_d\xi^d, \qquad \nabla^a\mathcal{H}_{ab} = -R_{ba}\xi^a.$$

A further computation using the above identities and the definition of the Ernst form, equation (18), yields

$$\nabla_a\chi_b - \nabla_b\chi_a = -2\epsilon_{cbae}\xi^c R^e{}_d\xi^d. \tag{20}$$

2.5.1 The vacuum case

In *vacuum* $\mathcal{R}_{abcd} = \mathcal{C}_{abcd}$, where $\mathcal{C}_{abcd}$ denotes the self-dual Weyl tensor, and so from the symmetries of the Weyl tensor one concludes that

$$\nabla_a\chi_b - \nabla_b\chi_a = 0.$$

Consequently, in vacuum the Ernst form is closed and thus, locally exact so that there exists a scalar, the *Ernst potential* χ, such that

$$\chi_a = \nabla_a\chi.$$

Let now $\xi_{AA'}$ denote the (Hermitian) spinorial counterpart of the real Killing vector ξ^a. If $\xi_{AA'}$ arises from a Killing spinor through relation (7), it follows from the spinor decomposition of $\mathcal{H}_{AA'BB'}$ that the spinorial counterpart $\chi_{AA'}$ of the Ernst form χ_a is given by

$$\chi_{AA'} = 4\eta_{AB}\xi^B{}_{A'}$$
$$= 3\kappa^{CF}\Psi_{ABCF}\nabla_{DA'}\kappa^{DB}.$$

2.5.2 The electrovacuum case

In the electrovacuum case the Ernst form is no longer exact —cf. equation (20). However, if the Faraday tensor inherits the symmetry of the spacetime —i.e. $\mathcal{L}_\xi F_{ab} = 0$— then it is possible to construct a further 1-form, the so-called *electromagnetic Ernst form*, which can be shown to be closed. In analogy to the definition in (18) one sets

$$\varsigma_a \equiv 2\xi^b\mathcal{F}_{ba}. \tag{21}$$

A computation then shows that

$$\nabla_a\varsigma_b - \nabla_b\varsigma_a = 2\mathcal{L}_\xi\mathcal{F}_{ab}.$$

Thus, as claimed, if $\mathcal{L}_\xi F_{ab} = 0$ then ς_a is closed and, accordingly locally exact so that there exists a scalar, the *electromagnetic Ernst potential* ς such that

$$\varsigma_a = \nabla_a\varsigma.$$

The spinorial version of equation (21) can be readily be found to be

$$\sigma_{AA'} = 4\phi_{AB}\nabla^Q{}_{A'}\kappa^B{}_Q.$$

2.5.3 Expansions in the algebraically general case

Consider now the case of an algebraically general spinor κ_{AB} such that $\xi_{AA'}$ as given by equation (7) is Hermitian. In order to find the full basis expansions of $\chi_{AA'}$ and $\varsigma_{AA'}$ we need to calculate

the derivative of the proportionality factor $\varkappa$. First, note the expressions for the derivatives of the spin basis vectors in terms of the spin coefficients of the Newman-Penrose formalism:

$$\nabla_{AA'}o_B = -\alpha o_A o_B \bar{\iota}_{A'} - \beta \iota_A o_B \bar{o}_{A'} + \gamma o_A o_B \bar{o}_{A'} + \epsilon \iota_A o_B \bar{\iota}_{A'}$$
$$- \kappa \iota_A \iota_B \bar{\iota}_{A'} + \rho o_A \iota_B \bar{\iota}_{A'} + \sigma \iota_A \iota_B \bar{o}_{A'} - \tau o_A \iota_B \bar{o}_{A'}, \tag{22a}$$
$$\nabla_{AA'}\iota_B = \alpha o_A \iota_B \bar{\iota}_{A'} + \beta \iota_A \iota_B \bar{o}_{A'} - \gamma o_A \iota_B \bar{o}_{A'} - \epsilon \iota_A \iota_B \bar{\iota}_{A'}$$
$$- \lambda o_A o_B \bar{\iota}_{A'} - \mu \iota_A o_B \bar{o}_{A'} + \nu o_A o_B \bar{o}_{A'} + \pi \iota_A o_B \bar{\iota}_{A'}. \tag{22b}$$

Substituting the basis expansion for the Killing spinor into the Killing spinor equation, using expressions (22a)-(22b) and the relation $\epsilon_{AB} = o_A \iota_B - \iota_A o_B$, we find that

$$\nabla_{AA'}\varkappa = \varkappa\left(\mu o_A \bar{o}_{A'} - \pi o_A \bar{\iota}_{A'} + \tau \iota_A \bar{o}_{A'} - \rho \iota_A \bar{\iota}_{A'}\right). \tag{23}$$

The expressions obtained in the previous paragraphs allow one to obtain an expression of the Killing spinor in terms of the spin basis. A calculation starting from the definition (7) readily yields the expression

$$\xi_{AA'} = -\frac{3}{2}\varkappa\left(\mu o_A \bar{o}_{A'} - \pi o_A \bar{\iota}_{A'} - \tau \iota_A \bar{o}_{A'} + \rho \iota_A \bar{\iota}_{A'}\right).$$

If $\xi_{AA'}$ is a Hermitian spinor, i.e. $\xi_{AA'} = \bar{\xi}_{AA'}$, then the previous expression implies

$$\bar{\mu}\bar{\varkappa} = \mu\varkappa, \qquad \bar{\tau}\bar{\varkappa} = \varkappa\pi, \qquad \bar{\rho}\bar{\varkappa} = \varkappa\rho. \tag{24}$$

The vacuum case. Using the previous expression along with the basis expansions for κ_{AB} and Ψ_{ABCD}, *in vacuum*, the Ernst form can be expanded as

$$\chi_{AA'} = \frac{3}{4}\varkappa^2\psi\left(\mu o_A \bar{o}_{A'} - \pi o_A \bar{\iota}_{A'} + \tau \iota_A \bar{o}_{A'} - \rho \iota_A \bar{\iota}_{A'}\right). \tag{25}$$

Intuitively, one would expect it should be possible to express the Ernst form χ in terms of the scalars $\varkappa$ and ψ. As it will be seen in Section 4, the characterisation of the Kerr spacetime given by Theorem 1 suggests that a combination of the form $\mathfrak{c} + \frac{3}{4}\varkappa^2\psi$ with $\mathfrak{c}$ a constant is a suitable candidate. In order to compute the derivative of this expression one needs an expression for $\nabla_{AA'}\psi$. The latter can be obtained from the vacuum Bianchi identity

$$\nabla^A{}_{A'}\Psi_{ABCD} = 0.$$

Substituting the basis expansion for the Weyl spinor into the above relation, using equations (22a) and (22b), collecting terms and finally making use of $\epsilon_{AB} = o_A \iota_B - \iota_A o_B$ one obtains

$$\nabla_{AA'}\psi = -3\psi\left(\mu o_A \bar{o}_{A'} - \pi o_A \bar{\iota}_{A'} + \tau \iota_A \bar{o}_{A'} - \rho \iota_A \bar{\iota}_{A'}\right). \tag{26}$$

Combining the latter with expression (23) for $\nabla_{AA'}\varkappa$ one finds that

$$\nabla_{AA'}\left(\mathfrak{c} - \frac{3}{4}\varkappa^2\psi\right) = \chi_{AA'}$$

so that one can indeed set

$$\chi = \mathfrak{c} - \frac{3}{4}\varkappa^2\psi \qquad \text{for some} \quad \mathfrak{c} \in \mathbb{C}.$$

This expression can be simplified using the following observation: combining expressions for $\nabla_{AA'}\psi$ and $\nabla_{AA'}\varkappa$ given by equations (26) and (23), respectively, one finds that

$$\nabla_{AA'}\left(\varkappa^3\psi\right) = 0;$$

accordingly, the combination $\varkappa^3\psi$ is a constant. Therefore, one has that

$$\varkappa^3\psi = \mathfrak{M} \tag{27}$$

with $\mathfrak{M}$ a (possibly complex) constant and one has

$$\chi = \mathfrak{c} - \frac{3\mathfrak{M}}{4\varkappa}. \tag{28}$$

The electrovacuum case. From the electrovacuum Bianchi identity in the form (2) a calculation yields

$$\nabla_{AA'}\psi = -3\left(\psi + 2\varphi\bar{\varphi}\right)\mu o_A\bar{o}_{A'} + 3\left(\psi - 2\varphi\bar{\varphi}\right)\pi o_A\bar{\iota}_{A'}$$
$$-3\left(\psi - 2\varphi\bar{\varphi}\right)\tau\iota_A\bar{o}_{A'} + 3\left(\psi + 2\varphi\bar{\varphi}\right)\rho\iota_A\bar{\iota}_{A'}.$$

Similarly, from the Maxwell equations (1) and the derivatives of the basis vectors given by equations (22a) and (22b) one finds that

$$\nabla_{AA'}\varphi = -2\varphi\left(\mu o_A\bar{o}_{A'} - \pi o_A\bar{\iota}_{A'} + \tau\iota_A\bar{o}_{A'} - \rho\iota_A\bar{\iota}_{A'}\right). \tag{29}$$

Thus, a further calculation using the previous expressions yields the following explicit expression for the electromagnetic Ernst potential:

$$\varsigma_{AA'} = 3\varkappa\varphi\left(\mu o_A\bar{o}_{A'} - \pi o_A\bar{\iota}_{A'} + \tau\iota_A\bar{o}_{A'} - \rho\iota_A\bar{\iota}_{A'}\right).$$

In the electrovacuum case, assuming an algebraically general Killing spinor and that the Maxwell spinor and the Killing spinor satisfy the matter alignment condition (9), the characterisation of the Kerr-Newman spacetime given in Theorem 3 suggests an expression for ς in terms of the scalars $\varkappa$, ψ and φ —namely $\mathfrak{c}' - \bar{\varkappa}\bar{\psi}/2\bar{\varphi}$ with $\mathfrak{c}'$ a constant. Combining the above expressions one concludes that

$$\nabla_{AA'}\left(\mathfrak{c}' - \frac{\bar{\varkappa}\bar{\psi}}{2\bar{\varphi}}\right) = \varsigma_{AA'} \tag{30}$$

so that one can set

$$\varsigma = \mathfrak{c}' - \frac{1}{2}\frac{\bar{\varkappa}\bar{\psi}}{\bar{\varphi}} \qquad \text{for some} \quad \mathfrak{c}' \in \mathbb{C}. \tag{31}$$

Moreover, combining expression (23) for $\nabla_{AA'}\varkappa$ with (29) one concludes that

$$\nabla_{AA'}\left(\varkappa^2\varphi\right) = 0$$

meaning the combination $\varkappa^2\varphi$ is constant. Thus, there exists a (possibly complex) constant $\mathfrak{Q}$ such that

$$\varkappa^2\varphi = \mathfrak{Q}. \tag{32}$$

In the electrovacuum case the relation between the scalars $\varkappa$ and ψ takes a more complicated form than in vacuum —cf. equation (27). Given a complex constant $\mathfrak{C}'$, a calculation using expressions (23), (29) and relation (32) shows that

$$\nabla_{AA'}\left(\frac{\mathfrak{C}}{\bar{\varkappa}} + \varkappa^3\psi\right) = -\left(\frac{6|\mathfrak{Q}|^2\varkappa\mu}{\bar{\varkappa}^2} + \frac{\mathfrak{C}\bar{\mu}}{\bar{\varkappa}}\right)o_A\bar{o}_{A'} - \left(\frac{6|\mathfrak{Q}|^2\varkappa\pi}{\bar{\varkappa}^2} + \frac{\mathfrak{C}\bar{\tau}}{\bar{\varkappa}}\right)o_A\bar{\iota}_{A'}$$
$$+ \left(\frac{6|\mathfrak{Q}|^2\varkappa\tau}{\bar{\varkappa}^2} + \frac{\mathfrak{C}\bar{\pi}}{\bar{\varkappa}}\right)\iota_A\bar{o}_{A'} + \left(\frac{6|\mathfrak{Q}|^2\varkappa\rho}{\bar{\varkappa}^2} + \frac{\mathfrak{C}\bar{\rho}}{\bar{\varkappa}}\right)\iota_A\bar{\iota}_{A'}.$$

If the spinor $\xi_{AA'}$ is assumed to be Hermitian, then the previous expression reduces to

$$\nabla_{AA'}\left(\frac{\mathfrak{C}}{\bar{\varkappa}} + \varkappa^3\psi\right) = -\frac{\varkappa(\mathfrak{C} + 6|\mathfrak{Q}|^2)}{\bar{\varkappa}^2}\left(\mu o_A\bar{o}_{A'} + \pi o_A\bar{\iota}_{A'} - \tau\iota_A\bar{o}_{A'} - \rho\iota_A\bar{\iota}_{A'}\right).$$

Thus, if one chooses $\mathfrak{C} = -6|\mathfrak{Q}|^2$, then the combination $\mathfrak{C}/\bar{\varkappa} + \varkappa^3\psi$ is a constant —that is, there exists $\mathfrak{M}' \in \mathbb{C}$ such that

$$\varkappa^3\psi - \frac{6|\mathfrak{Q}|^2}{\bar{\varkappa}} = \mathfrak{M}'. \tag{33}$$

Thus, the scalar ψ can be expressed solely in terms of $\varkappa$ as

$$\psi = \frac{1}{\varkappa^3}\left(\mathfrak{M}' + \frac{6|\mathfrak{Q}|^2}{\bar{\varkappa}}\right). \tag{34}$$

Note that when the Maxwell field vanishes, then the constant $\mathfrak{Q}$ also vanishes and this equation reduces to the vacuum case given by (27).

Finally, it is observed that expanding expression (11) in terms of the spinor basis $\{o, \iota\}$ and using expressions (15) and (29) one concludes, after a calculation, that

$$\mathcal{L}_\xi \varphi_{AB} = 0$$

—so that φ_{AB} inherits the symmetry generated by the Killing spinor κ_{AB}.

2.6 Spacetimes with an algebraically special Killing spinor

In this section we briefly consider electrovacuum spacetimes with an algebraically special Killing spinor. These spacetimes will not play a role in the remainder of this article. The reason for this is the following result:

Lemma 1. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth electrovacuum spacetime with a matter content satisfying the matter alignment condition and admitting a valence-2 Killing spinor κ_{AB} such that the associated field $\xi^{AA'}$ is a Hermitian spinor. If κ_{AB} is algebraically special (i.e. $\kappa_{AB} = \alpha_A \alpha_B$ for some non-vanishing spinor α_A) then $\xi^a = 0$.*

Proof. It follows directly from the existence of a non-vanishing algebraically special Killing spinor that the spacetime $(\mathcal{M}, \boldsymbol{g})$ must be of Petrov type N —see equation (6). That is, one has that

$$\Psi_{ABCD} = \psi \alpha_A \alpha_B \alpha_C \alpha_D \tag{35}$$

for some function ψ. As the matter alignment condition holds by assumption, the Hermitian spinor $\xi_{AA'}$ is the spinorial counterpart of a real Killing vector ξ^a. The content of the Killing form of ξ^a is encoded in the symmetric spinor η_{AB}. Substituting the expansions (35) and $\kappa_{AB} = \alpha_A \alpha_B$ into equation (14), it follows directly that $\eta_{AB} = 0$. Thus, the Killing form H_{ab} of ξ^a vanishes. Accordingly, ξ^a is a covariantly constant vector on $(\mathcal{M}, \boldsymbol{g})$ —i.e. one has

$$\nabla_a \xi^b = 0. \tag{36}$$

In order to further investigate the consequences of this observation we introduce a normalised spin dyad $\{o^A, \iota^A\}$ with $o_A = \alpha_A$ and $o_A \iota^A = 1$. One can then write

$$\kappa_{AB} = o_A o_B, \qquad \phi_{AB} = \varphi o_A o_B.$$

Substituting the first of the above expressions into the Killing spinor equation $\nabla_{A'(A}\kappa_{BC)} = 0$ one finds that

$$\gamma = \alpha = \sigma = \kappa = 0, \qquad \rho + \epsilon = 0, \qquad \tau + \beta = 0. \tag{37}$$

Moreover, one finds that the Hermitian spinor $\xi_{AA'}$ can be expressed as

$$\xi_{AA'} = -3\beta o_A \bar{o}_{A'} + 3\epsilon o_A \bar{\iota}_{A'}.$$

The spinorial version of equation (36) implies $D\xi_{AA'} = 0$, $\Delta\xi_{AA'} = 0$, $\delta\xi_{AA'} = 0$ and $\bar{\delta}\xi_{AA'} = 0$. In particular, from $\Delta\xi_{AA'} = 0$ and $\bar{\delta}\xi_{AA'} = 0$, expanding in terms of the basis one finds that $\beta\tau = 0$ and $\epsilon\rho = 0$. Combining this expression with the third and fourth conditions in (37) one concludes that

$$\tau = \beta = \epsilon = \rho = 0.$$

It follows then that

$$\xi_{AA'} = 0.$$

$\square$

As we want to use the asymptotics of the Killing vector $\xi_{AA'}$ in the characterisation of the Kerr and Kerr-Newman spacetime, we will rule out the algebraically special case and assume that the Killing spinor is algebraically general —i.e. $\kappa_{AB}\kappa^{AB} \neq 0$.

Remark. Note that as we have $\Psi_{ABCD} \propto \kappa_{(AB}\kappa_{CD)}$, then the conditions $\Psi_{ABCD}\Psi^{ABCD} \neq 0, \Psi_{ABCD} \neq 0$ imply that the Killing spinor is algebraically general and non-zero, i.e. $\kappa_{AB}\kappa^{AB} \neq 0, \kappa_{AB} \neq 0$. These two conditions on the curvature are precisely the ones assumed in Theorem 6 of [3], and so the characterisation of Kerr in terms of Killing spinors presented in that article is essentially the same as the one presented here. Despite this, we reproduce the result in this paper for completeness and ease of comparison with the electrovacuum case. We do this using the local result of Mars given in [15], whereas the proof in [3] uses the global result from [16]. In the absence of a generalisation to the electrovacuum case of the characterisation of [16], our analysis of the Kerr-Newman spacetime must make use of the local result by Wong [22].

3   Boundary conditions

This section provides a brief discussion of the boundary conditions which will be used in conjunction with the properties of Killing spinors to characterise the Kerr and Kerr-Newman spacetimes.

3.1   Stationary asymptotically flat ends

In the remainder of this article we will be particularly interested in spacetimes with a *stationary asymptotically flat 4-end* —see e.g. [22].

Definition 1. *A stationary asymptotically flat 4-end in an electrovacuum spacetime* $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ *is an open submanifold* $\mathcal{M}_\infty \subset \mathcal{M}$ *diffeomorphic to* $I \times (\mathbb{R}^3 \setminus \mathcal{B}_R)$ *where* $I \subset \mathbb{R}$ *is an open interval and* $\mathcal{B}_R$ *is a closed ball of radius R. In the local coordinates* (t, x^α) *defined by the diffeomorphism the components* $g_{\mu\nu}$ *and* $F_{\mu\nu}$ *of the metric* $\boldsymbol{g}$ *and the Faraday tensor* $\boldsymbol{F}$ *satisfy*

$$|g_{\mu\nu} - \eta_{\mu\nu}| + |r\partial_\alpha g_{\mu\nu}| \leq Cr^{-1}, \tag{38a}$$

$$|F_{\mu\nu}| + |r\partial_\alpha F_{\mu\nu}| \leq C'r^{-2}, \tag{38b}$$

$$\partial_t g_{\mu\nu} = 0, \tag{38c}$$

$$\partial_t F_{\mu\nu} = 0, \tag{38d}$$

where C and C' are positive constants, $r \equiv (x^1)^2 + (x^2)^2 + (x^3)^2$, *and* $\eta_{\mu\nu}$ *denote the components of the Minkowski metric in Cartesian coordinates.*

Remark 1. It follows from condition (38c) in Definition 1 that the stationary asymptotically flat end $\mathcal{M}_\infty$ is endowed with a Killing vector ξ^a which takes the form $\boldsymbol{\partial}_t$ —a so-called *time translation*. From condition (38d) one has that the electromagnetic field inherits the symmetry of the spacetime —that is $\mathcal{L}_\xi \boldsymbol{F} = 0$, with $\mathcal{L}_\xi$ the Lie derivative along ξ^a.

Of particular interest will be those stationary asymptotically flat ends *generated by a Killing spinor*:

Definition 2. *A stationary asymptotically flat end* $\mathcal{M}_\infty \subset \mathcal{M}$ *in an electrovacuum spacetime* $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ *endowed with a Killing spinor* κ_{AB} *is said to be generated by a Killing spinor if the spinor* $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ *is the spinorial counterpart of the Killing vector* ξ^a.

Remark 2. Stationary spacetimes have a natural definition of mass in terms of the Killing vector ξ^a that generates the isometry —the so-called *Komar mass m* defined as

$$m \equiv -\frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \epsilon_{abcd}\nabla^c\xi^d \mathrm{d}S^{ab}$$

where S_r is the sphere of radius r centred at $r = 0$ and $\mathrm{d}S^{ab}$ is the binormal vector to S_r. Similarly, one can define the *total electromagnetic charge* of the spacetime via the integral

$$q = -\frac{1}{4\pi} \lim_{r\to\infty} \int_{S_r} F_{ab}\mathrm{d}S^{ab}.$$

Remark 3. In the asymptotic region the components of the metric can be written in the form

$$g_{00} = 1 - \frac{2m}{r} + O(r^{-2}),$$

$$g_{0\alpha} = \frac{4\epsilon_{\alpha\beta\gamma}S^\beta x^\gamma}{r^3} + O(r^{-3}),$$

$$g_{\alpha\beta} = -\delta_{\alpha\beta} + O(r^{-1}),$$

where m is the Komar mass of ξ^a in the end $\mathcal{M}_\infty$, $\epsilon_{\alpha\beta\gamma}$ is the flat rank 3 totally antisymmetric tensor and S^β denotes a 3-dimensional tensor with constant entries. For the components of the Faraday tensor one has that

$$F_{0\alpha} = \frac{q}{r^2} + O(r^{-3}),$$

$$F_{\alpha\beta} = O(r^{-3})$$

—see e.g. [20]. Thus, to leading order any stationary electrovacuum spacetime is asymptotically a Kerr-Newman spacetime.

Remark 4. In the case of the exact Kerr-Newman spacetime with mass m, angular momentum a and charge q a NP frame $\{l^a, n^a, m^a, \bar{m}^a\}$ with associated spin dyad $\{o^A, \iota^A\}$ such that the scalars $\varkappa$, φ and ψ introduced in equations (5), (10) and (6), respectively, take the form

$$\varkappa = \frac{2}{3}(r - \mathrm{i}a\cos\theta),$$

$$\varphi = \frac{q}{(r - \mathrm{i}a\cos\theta)^2},$$

$$\psi = \frac{6}{(r - \mathrm{i}a\cos\theta)^3}\left(\frac{q^2}{r + \mathrm{i}a\cos\theta} - m\right),$$

where r denotes the standard *Boyer-Lindquist* radial coordinate —see [1] for more details.

3.2   Killing spinor and Killing vector asymptotics

In general, the spinor $\xi_{AA'}$ obtained from a Killing spinor κ_{AB} using formula (7) is not Hermitian. It is, however, well know that for the Kerr-Newman spacetime $\xi_{AA'}$ is indeed the spinorial counterpart of a real Killing vector ξ^a —see e.g. [1]. More generally, this observation applies to any electrovacuum spacetime with a stationary asymptotically flat end. To see this, we first notice the following:

Lemma 2. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ be a smooth electrovacuum spacetime with a stationary asymptotically flat end $\mathcal{M}_\infty$, admitting a complex Killing vector field ξ^a. If ξ^a tends to a time translation at infinity in $\mathcal{M}_\infty$, then ξ^a is in fact a real vector in $\mathcal{M}_\infty$.*

Proof. The complex Killing vector can be written $\xi^a = \xi_1^a + \mathrm{i}\xi_2^a$ for two real vectors ξ_1^a, ξ_2^a, which are also Killing vectors by linearity of the Killing vector equation. As a time translation $(\partial_t)^a$ is a real vector, we have $\xi_1^a \to (\partial_t)^a$ and $\xi_2^a \to 0$ as $r \to \infty$ in the asymptotically flat end $\mathcal{M}_\infty$. However, it is well known that there are no non-trivial real Killing vectors which vanish at infinity —see e.g. [6, 7]. Therefore, $\xi_2^a = 0$ on $\mathcal{M}_\infty$, and $\xi^a = \xi_1^a$ is a real Killing vector. □

Therefore, by assuming that the Killing vector ξ^a is asymptotically a time translation, then we can drop the assumption requiring its spinorial equivalent $\xi_{AA'}$ to be a Hermitian spinor. In fact, it is possible to replace this condition on the Killing vector with an asymptotic condition on the Killing spinor, as described in the following proposition:

Proposition 1. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ denote an electrovacuum spacetime with a stationary asymptotically flat end $\mathcal{M}_\infty$ generated by a Killing spinor κ_{AB}. Then the functions $\varkappa$, φ and ψ as defined*

by equations (5), (6) *and* (10) *satisfy*

$$\varkappa = \frac{2}{3}r + O(1),$$

$$\varphi = \frac{q}{r^2} + O(r^{-3}),$$

$$\psi = -\frac{6m}{r^3} + O(r^{-4}).$$

Moreover, one has that

$$\xi^2 \equiv \xi_{AA'}\xi^{AA'} = 1 + O(r^{-1}).$$

Proof. The analysis in [20] shows that to leading order the electrovacuum spacetime $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ coincides on $\mathcal{M}_\infty$ with the Kerr-Newman spacetime. Thus, the expansions for the fields $\varkappa$, φ and φ must coincide, to leading order with their expressions for the Kerr-Newman spacetime —see [1]. □

4   Characterisations of the Kerr spacetime

The motivation behind our analysis is the following theorem by M. Mars —see [16]:

Theorem 1. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth, vacuum spacetime admitting a Killing vector ξ^a with selfdual Killing form $\mathcal{H}_{ab}$. Let $\mathcal{M}$ satisfy:*

(i) *there exists a non-empty region $\mathcal{M}_\bullet \subset \mathcal{M}$ where*

$$\mathcal{H}^2 \equiv \mathcal{H}_{ab}\mathcal{H}^{ab} \neq 0;$$

(ii) *The selfdual Killing form and the Weyl tensor are related by*

$$\mathcal{C}_{abcd} = H\left(\mathcal{H}_{ab}\mathcal{H}_{cd} - \frac{1}{3}\mathcal{H}^2\mathcal{I}_{abcd}\right) \tag{39}$$

where

$$\mathcal{I}_{abcd} \equiv \frac{1}{4}(g_{ac}g_{bd} - g_{ad}g_{bc} + \mathrm{i}\epsilon_{abcd})$$

and H is a complex scalar function.

Then there exist two complex constants $\mathfrak{l}$ and $\mathfrak{c}$ such that

$$H = \frac{6}{\mathfrak{c} - \chi}, \qquad \mathcal{H}^2 = -\mathfrak{l}(\mathfrak{c} - \chi)^4.$$

If, in addition, $\mathfrak{c} = 1$ and $\mathfrak{l}$ is real positive, then $(\mathcal{M}, \boldsymbol{g})$ is locally isometric to the Kerr spacetime.

Remark 1. It is important to emphasise that in the above Theorem the existence of the constants $\mathfrak{c}$ and $\mathfrak{l}$ and the functional dependence of H and $\mathcal{H}^2$ with respect to χ are a consequence of the hypotheses of the theorem —this should be contrasted with the electrovacuum case in which the existence of the analogue constants needs to be assumed.

Remark 2. A particular case of Theorem 1 occurs when $(\mathcal{M}, \boldsymbol{g})$ is *a priori* assumed to have an stationary asymptotically flat end $\mathcal{M}_\infty$ with the Killing vector ξ^a tending asymptotically to a time translation at infinity and such that the Komar mass associated to ξ^a is non-zero. These assumptions ensure that $\mathcal{H}^2 \neq 0$ in a region of the spacetime —namely, in $\mathcal{M}_\infty$. Thus, one only needs to verify condition (39) to conclude that

$$H = \frac{6}{1 - \chi}$$

and that the spacetime is locally isometric to the Kerr spacetime —see Theorem 2 in [15].

Remark 3. In the subsequent discussion we will make use of the spinorial transcription of the conditions in the previous Theorem. In particular, we notice that the content of the combination $\mathcal{H}_{ab}\mathcal{H}_{cd} - \frac{1}{3}\mathcal{H}^2\mathcal{I}_{abcd}$ can be encoded in terms of the spinor η_{AB} as defined in equation (13) as

$$\left(4\eta_{AB}\eta_{CD} - \frac{2}{3}\eta_{EF}\eta^{EF}(\epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{BD})\right)\epsilon_{A'B'}\epsilon_{C'D'} = 4\eta_{(AB}\eta_{CD)}\epsilon_{A'B'}\epsilon_{C'D'}$$

where the last expression follows from a decomposition in irreducible terms. Thus, condition (39) can be concisely expressed in terms of spinors as

$$\Psi_{ABCD} = 2H\eta_{(AB}\eta_{CD)}. \tag{40}$$

Finally, it is noticed that the condition $\mathcal{H}^2 \neq 0$ can be expressed as

$$8\eta_{AB}\eta^{AB} \neq 0.$$

4.1 Killing spinors and the Mars characterisation

In what follows we analyse the extent to which existence of a Killing spinor on a vacuum space-times implies the hypotheses of the characterisation of Kerr given in Theorem 1. For bookkeeping purposes we explicitly state the assumptions to be made in the remainder of this section:

Assumption 1. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth vacuum spacetime and let $\mathcal{K} \subset \mathcal{M}$ such that:*

  (i) on $\mathcal{K}$ there exists an algebraically general Killing spinor κ_{AB};

  (ii) the spinor $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ is on $\mathcal{K}$ the spinorial counterpart of a real Killing spinor ξ^a —i.e. $\xi_{AA'}$ is Hermitian.

Under the above assumptions, it follows from combining the basis expansion for Ψ_{ABCD} and η_{AB}, equations (6) and (15), respectively, that

$$\Psi_{ABCD} = \frac{16}{\varkappa^2\psi}\eta_{(AB}\eta_{CD)}.$$

Thus, hypothesis *(ii)* of Theorem 1 is satisfied with

$$H = \frac{8}{\varkappa^2\psi}$$

—cf. equation (40). Using the expression for the Ernst potential predicted by the theory of Killing spinors, equation (28), one obtains that

$$H = \frac{6}{\mathfrak{c} - \chi}$$

which is precisely the form for H predicted by Theorem 1. From this expression one further concludes that

$$\mathcal{H}^2 = -\frac{\mathfrak{M}}{3}\left(\frac{4}{3\mathfrak{M}}\right)^3(\mathfrak{c} - \chi)^4.$$

This, again, is the form predicted by Theorem 1.

The above observations allow us to formulate the following *Killing spinor version* of Theorem 1:

Proposition 2. *Let $(\mathcal{M}, \boldsymbol{g})$ denote a smooth vacuum spacetime endowed with a Killing spinor κ_{AB} with $\kappa_{AB}\kappa^{AB} \neq 0$ such that the spinor $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ is Hermitian. Then there exist two complex constants $\mathfrak{l}$ and $\mathfrak{c}$ such that*

$$\mathcal{H}^2 = -\mathfrak{l}(\mathfrak{c} - \chi)^4.$$

If, in addition, $\mathfrak{c} = 1$ and $\mathfrak{l}$ is real positive, then $(\mathcal{M}, \boldsymbol{g})$ is locally isometric to the Kerr spacetime.

4.1.1 A characterisation using asymptotic flatness

In this subsection we simplify the previous discussion by assuming that the set $\mathcal{K} \subset \mathcal{M}$ contains a stationary asymptotically flat end with the Killing spinor κ_{AB} generating the time translation Killing vector.

From Proposition 1 it readily follows that

$$(\mathfrak{c} - \chi)^4 = \frac{16m^4}{r^4} + O(r^{-5}).$$

Similarly, one has, using equation (15), that

$$\mathcal{H}^2 = -4\eta^2 = -\frac{4m^2}{r^4} + O(r^{-5}).$$

Thus, by consistency with the required asymptotic behaviour of the Ernst potential, one has to set $\mathfrak{c} = 1$ and the constant $\mathfrak{l}$ in Proposition 2 is given by $\mathfrak{l} = 1/4m^2$.

We can summarise the discussion of the previous section in the following:

Theorem 2. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth vacuum spacetime containing a stationary asymptotically flat end $\mathcal{M}_\infty$ generated by a Killing spinor κ_{AB}. Let the Komar mass associated to the Killing vector $\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}$ in $\mathcal{M}_\infty$ be non-zero. Then, $(\mathcal{M}, \boldsymbol{g})$ is locally isometric to the Kerr spacetime.*

Remark. As observed in [1] the requirement on the non-vanishing of the Komar mass can be replaced by an assumption on the existence of a horizon.

5 Characterisations of the Kerr-Newman spacetime

In this section we discuss characterisations of the Kerr-Newman spacetime through Killing spinors. Our starting point is the following result —see [22]:

Theorem 3. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ be a smooth, electrovacuum spacetime admitting a real Killing vector ξ^a. Assume further that ξ^a is timelike somewhere in $\mathcal{M}$ and that F_{ab} is non-null on $\mathcal{M}$ (i.e. $\mathcal{F}^2 \equiv \mathcal{F}_{ab}\mathcal{F}^{ab} \neq 0$) and that it inherits the symmetry of the spacetime —i.e.*

$$\mathcal{L}_\xi \mathcal{F}_{ab} = 0. \tag{41}$$

Assume, furthermore, that there exists a complex scalar P, a normalisation for ς and a complex constant $\mathfrak{c}_1$ such that:

$$P^{-4} = -\mathfrak{c}_1^2 \mathcal{F}^2, \tag{42a}$$

$$\mathcal{H}_{ab} = -\frac{1}{2}\bar{\varsigma}\mathcal{F}_{ab}, \tag{42b}$$

$$\mathcal{C}_{abcd} = 3P\left(\frac{1}{2}\mathcal{F}_{ab}\mathcal{H}_{cd} + \frac{1}{2}\mathcal{F}_{ab}\mathcal{H}_{cd} - \frac{1}{3}\mathcal{I}_{abcd}\mathcal{F}_{ef}\mathcal{H}^{ef}\right). \tag{42c}$$

Then there exist complex constants $\mathfrak{c}_2$ and $\mathfrak{c}_3$ such that:

$$P = \frac{2}{\mathfrak{c}_2 - \varsigma}, \tag{43a}$$

$$4\xi^2 - |\varsigma|^2 = \mathfrak{c}_3. \tag{43b}$$

If, further, $\mathfrak{c}_2$ is such that $\mathfrak{c}_1\bar{\mathfrak{c}}_2$ is real and $\mathfrak{c}_3$ is such that $|\mathfrak{c}_2|^2 + \mathfrak{c}_3 = 4$, then $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is locally isometric to a Kerr-Newman spacetime.

Remark 1. As in Section 4, we will make use of a reformulation of the conditions in Theorem 3 in spinorial formalism. A direct computation shows that (42a) can be rewritten as

$$P^{-4} = -8\mathfrak{c}_1^2 \phi_{AB} \phi^{AB}.$$

Similarly, condition (42b) can be readily expressed in terms of the spinors η_{AB} and φ_{AB} as

$$\eta_{AB} = -\frac{1}{2} \bar{\varsigma} \phi_{AB},$$

while, finally, equation (42c) is equivalent to

$$\Psi_{ABCD} = 6P \eta_{(AB} \phi_{CD)}.$$

5.1 Killing spinors and Wong's characterisation

In this section we investigate some further consequences of the existence of Killing spinors on electrovacuum spacetimes. For easy reference we state the assumptions to be made in what follows:

Assumption 2. *Let $(\mathcal{M}, \boldsymbol{g})$ be a smooth electrovacuum spacetime and let $\mathcal{K} \subset \mathcal{M}$ such that:*

(i) on $\mathcal{K}$ there exists an algebraically general Killing spinor κ_{AB};

(ii) the spinor $\xi_{AA'} \equiv \nabla^B{}_{A'} \kappa_{AB}$ is on $\mathcal{K}$ the spinorial counterpart of a real Killing spinor ξ^a —i.e. $\xi_{AA'}$ is Hermitian;

(iii) the Killing spinor κ_{AB} and the Maxwell spinor ϕ_{AB} satisfy the alignment condition $\kappa_{(A}{}^Q \phi_{B)Q} = 0$ —that is, they are proportional.

As already discussed in Section 2.5.3, under the above assumptions it follows that $\mathcal{L}_\xi \phi_{AB} = 0$ which, in turn, implies that $\mathcal{L}_\xi F_{ab} = 0$. Thus, the electromagnetic field inherits the symmetry generated by the Killing spinor κ_{AB}.

From the discussion in Sections 2.1 and 2.4 it follows that

$$\Psi_{ABCD} = \psi o_{(A} o_B \iota_C \iota_{D)}, \qquad \eta_{(AB} \phi_{CD)} = \eta \varphi o_{(A} o_B \iota_C \iota_{D)}.$$

Thus, the spinorial version of condition (42c) in Theorem 3 is satisfied with a proportionality function P given by

$$P = \frac{2}{3 \varkappa \varphi}.$$

Now, making use of expressions (31), (32) and (34) to rewrite P in terms of the electromagnetic Ernst potential one finds that

$$P = \frac{2}{\mathfrak{c}_2 - \varsigma}, \qquad \mathfrak{c}_2 \equiv \mathfrak{c}' - \frac{\bar{\mathfrak{M}}'}{2\bar{\mathfrak{Q}}}.$$

Thus, under the Assumptions 2, hypothesis (42c) and conclusion (43a) in Theorem 3 are satisfied.

Moreover, from the discussion in Section 2.5.3 it follows that the spinors η_{AB} and ϕ_{AB} are proportional to each other with a proportionality function $\bar{\varsigma}$ given by

$$\bar{\varsigma} = -\frac{\varkappa \psi}{2\varphi}.$$

The calculations of Section 2.5, cf. equation (30) in particular, show that ς satisfies the properties to be expected from the electromagnetic Ernst potential. Therefore, by setting the constant $\mathfrak{c}'$ in the definition of ς given by (31) to zero (and thereby fixing the normalisation of the potential), condition (42b) is satisfied. A similar remark holds for condition (42a) with the constant $\mathfrak{c}_1$ given by

$$\mathfrak{c}_1^2 = \frac{81}{64} \mathfrak{Q}^2.$$

15

In the presence of a Killing spinor, the norm $\xi^2 \equiv \xi_a \xi^a$ of the associated Killing vector is related to the electromagnetic form ς. To see this consider

$$
\begin{aligned}
\nabla_{AA'}\xi^2 &= 2\xi^{BB'}\nabla_{AA'}\xi_{BB'} \\
&= -2\eta_{AB}\xi^B{}_{A'} - 2\bar{\eta}_{A'B'}\xi_A{}^{B'}
\end{aligned}
$$

where in the second line it has been used that

$$
\nabla_{AA'}\xi_{BB'} = \eta_{AB}\epsilon_{A'B'} + \bar{\eta}_{A'B'}\epsilon_{AB}.
$$

As the spinors η_{AB} and ϕ_{AB} are proportional to each other, cf. the previous paragraph, one can write

$$
\begin{aligned}
\nabla_{AA'}\xi^2 &= \bar{\varsigma}\xi^A{}_{B'}\phi_{AB} + \varsigma\xi_B{}^{A'}\bar{\phi}_{A'B'} \\
&= \frac{1}{4}\left(\bar{\varsigma}\nabla_{BB'}\varsigma + \varsigma\nabla_{BB'}\bar{\varsigma}\right) \\
&= \frac{1}{4}\nabla_{BB'}|\varsigma|^2.
\end{aligned}
$$

Thus, locally there exists a constant $\mathfrak{c}_3$ such that

$$
4\xi^2 - |\varsigma|^2 = \mathfrak{c}_3.
$$

Thus, conclusion (43b) in Theorem 3 is also a consequence of the existence of a Killing spinor.

We can summarise the discussion of this section with the following Killing spinor version Theorem (3):

Proposition 3. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ denote a smooth electrovacuum spacetime satisfying the matter alignment condition, endowed with a Killing spinor κ_{AB} with $\kappa_{AB}\kappa^{AB} \neq 0$ such that the spinor $\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$ is Hermitian. Then there exist two constants $\mathfrak{c}_2$ and $\mathfrak{c}_3$ such that*

$$
(\mathfrak{c}_2 - \varsigma)^4 = -\left(\frac{9}{8}\mathfrak{Q}\right)^2 \mathcal{F}^2, \qquad 4\xi^2 - |\varsigma|^2 = \mathfrak{c}_3.
$$

If, further, $\mathfrak{c}_2$ is such that $\bar{\mathfrak{c}}_2\mathfrak{Q}$ is real and $\mathfrak{c}_3$ is such that $|\mathfrak{c}_2|^2 + \mathfrak{c}_3 = 4$, then $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is locally isometric to a Kerr-Newman spacetime.

5.1.1 A characterisation using asymptotic flatness

In this section we assume that the domain $\mathcal{K} \subset \mathcal{M}$ considered in the Assumptions 3 contains an stationary asymptotic flat end with the Killing spinor κ_{AB} generating the time translation Killing vector. We use this further assumption to determine the values of the constants in Proposition 3.

Combining the asymptotic expansions of Proposition 1 with the relation (32) one readily concludes that

$$
\mathfrak{Q} = \frac{4}{9}q \in \mathbb{R}.
$$

Similarly, using equation (33) one concludes that

$$
\mathfrak{M}' = -\frac{16}{9}m.
$$

A further computation using equation (31) and (32), respectively, show that

$$
(\mathfrak{c}_2 - \varsigma)^4 = \left(\mathfrak{c}_2 - \frac{2m}{q} + O(r^{-1})\right)^4, \qquad \left(\frac{9}{8}\mathfrak{Q}\right)^2 \mathcal{F}^2 = -\frac{q^4}{r^4} + O(r^{-5}).
$$

Thus, for consistency, one has to set

$$
\mathfrak{c}_2 = \frac{2m}{q}
$$

—thus, one has that $\bar{\mathfrak{c}}_2 \mathfrak{Q} \in \mathbb{R}$. From the previous discussion it follows that $\varsigma = 2m/q + O(r^{-1})$ so that, together with $\xi^2 = 1 + O(r^{-1})$ one concludes that $\mathfrak{c}_3$ as defined by equation (43b) is given by

$$\mathfrak{c}_3 = 4\left(1 - \frac{m^2}{q^2}\right).$$

Accordingly one has that $|\mathfrak{c}_2|^2 + \mathfrak{c}_3 = 4$ as required.

One can summarise the discussion of the previous paragraphs in the following:

Theorem 4. *Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ be a smooth, electrovacuum spacetime satisfying the matter alignment condition, with a stationary asymptotically flat end $\mathcal{M}_\infty$ generated by a Killing spinor κ_{AB}. Let both the Komar mass associated to the Killing vector $\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}$ and the total electromagnetic charge in $\mathcal{M}_\infty$ be non-zero. Then $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F})$ is locally isometric to the Kerr-Newman spacetime.*

6 Applications

The advantage of the Killing spinor characterisation of the Kerr and Kerr-Newman solutions is that the existence of such a spinor is a geometric condition, with only reasonable asymptotic conditions needing to be further assumed for the results presented above. This geometric condition can be converted into initial data for a spacelike Cauchy surface, in a way compatible with the constraint equations. This can then be exploited to construct a geometric invariant for the initial data set, which parametrises the deviation of the resulting global development of the initial data set from the exact Kerr or Kerr-Newman solution. Various versions of this construction analysis have been considered in [2, 3, 4, 5] for the vacuum case. A generalisation of these constructions to the electrovacuum case is given in [9].

Finally we point out that the results of this article suggest that the characterisations of the Kerr-Newman spacetime given by Wong in [22] can be improved to an *optimal* theorem in which condition (42a) in Theorem 3 is a consequence of the other hypothesis. An optimal result of this type is desirable if one is to attempt to use this type of characterisations to construct an alternative approach to the uniqueness of black holes.

References

[1] L. Andersson, T. Bäckdahl, & P. Blue, *Spin geometry and conservation laws in the Kerr spacetime*, in *One hundred years of General Relativity*, edited by L. Bieri & S.-T. Yau, volume 20, page 183, International Press, 2015.

[2] T. Bäckdahl & J. A. Valiente Kroon, *Geometric invariant measuring the deviation from Kerr data*, Phys. Rev. Lett. **104**, 231102 (2010).

[3] T. Bäckdahl & J. A. Valiente Kroon, *On the construction of a geometric invariant measuring the deviation from Kerr data*, Ann. Henri Poincaré **11**, 1225 (2010).

[4] T. Bäckdahl & J. A. Valiente Kroon, *The "non-Kerrness" of domains of outer communication of black holes and exteriors of stars*, Proc. Roy. Soc. Lond. A **467**, 1701 (2011).

[5] T. Bäckdahl & J. A. Valiente Kroon, *Constructing "non-Kerrness" on compact domains*, J. Math. Phys. **53**, 04503 (2012).

[6] R. Beig & P. T. Chruściel, *Killing vectors in asymptotically flat spacetimes. I. Asymptotically translational Killing vectors and rigid positive energy theorem*, J. Math. Phys. **37**, 1939 (1996).

[7] D. Christodoulou & N. O'Murchadha, *The boost problem in general relativity*, Comm. Math. Phys. **80**, 271 (1981).

[8] P. T. Chruściel, J. L. Costa, & M. Heusler, *Stationary black holes: uniqueness and beyond*, Living. Rev. Relativity **15** (2012).

[9] M. J. Cole & J. A. Valiente Kroon, *Geometric invariant measuring the deviation from Kerr-Newman data*, In preparation.

[10] J. J. Ferrando & J. A. Sáez, *An intrinsic characterization of the Kerr metric*, Class. Quantum Grav. **26**, 075013 (2009).

[11] L. P. Hughston & P. Sommers, *Spacetimes with Killing tensors*, Comm. Math. Phys. **32**, 147 (1973).

[12] L. P. Hughston & P. Sommers, *The symmetries of Kerr black holes*, Comm. Math. Phys. **33**, 129 (1973).

[13] L. Hughston, R. Penrose, P. Sommers, & M. Walker, *On a quadratic first integral for the charged partice orbits in the charged Kerr solutions*, Comm. Math. Phys. **27**, 303 (1972).

[14] N. Kamran, *Killing-Yano tensors and their role in the separation of variables*, in *Proceedings of the secon Canadian conference on general relativity and relativistic astrophysics*, edited by A. Coley, C. Dyer, & B. O. J. Tupper, World Scientific, 1988.

[15] M. Mars, *A spacetime characterization of the Kerr metric*, Class. Quantum Grav. **16**, 2507 (1999).

[16] M. Mars, *Uniqueness properties of the Kerr metric*, Class. Quantum Grav. **17**, 3353 (2000).

[17] R. G. McLenaghan & N. van den Bergh, *Spacetimes admitting Killing 2-spinors*, Class. Quantum Grav. **10**, 2179 (1993).

[18] R. Penrose & W. Rindler, *Spinors and space-time. Volume 1. Two-spinor calculus and relativistic fields*, Cambridge University Press, 1984.

[19] R. Penrose & W. Rindler, *Spinors and space-time. Volume 2. Spinor and twistor methods in space-time geometry*, Cambridge University Press, 1986.

[20] W. Simon, *The multipole expansion of stationary Einstein–Maxwell fields*, J. Math. Phys. **25**, 1035 (1984).

[21] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, & E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 2003, Second edition.

[22] W. W. Wong, *A space-time characterization of the Kerr-Newman metric*, Ann. Henri Poincaré **10**, 453 (2009).