GENERALISED APÉRY NUMBERS MODULO 9

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Abstract. We characterise the behaviour of (generalised) Apéry numbers modulo 9, thereby in particular establishing two conjectures in *"A method for determining the mod-3^k behaviour of recursive sequences"* [`arχiv:1308.2856`].

1. Introduction

For non-negative integers r, s and n, the (generalised) Apéry numbers $a_n(r,s)$ are defined by

$$a_n(r,s) = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s. \tag{1.1}$$

The (classical) Apéry numbers $a_n(2,1)$ and $a_n(2,2)$ appear in Apéry's proof [2] of the irrationality of $\zeta(2)$ and $\zeta(3)$ (cf. also [7]) as the leading coefficients in certain linear forms in $\zeta(2)$ and 1, and in $\zeta(3)$ and 1, respectively. Similarly, the Apéry numbers $a_n(r,1)$ appear as leading coefficients in linear forms in $\zeta(r)$, $\zeta(r-2)$, ..., 1 in [3], where it is shown that infinitely many values among $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, ... are irrational. In the past, the general Apéry numbers $a_n(r,s)$ have been the object of numerous arithmetic investigations, see, for instance, [1, 4, 6, 8, 14] for a highly non-exhaustive selection.

We point out that the Apéry numbers with second parameter $s = 0$ have received special attention in the literature. It is simple to see that $a_n(1,0) = 2^n$ and $a_n(2,0) = \binom{2n}{n}$. The numbers $a_n(3,0)$ are also known under the name of "Franel numbers," and the more general numbers $a_n(r,0)$ for $r \ge 4$ as "extended" or "generalised Franel numbers."

Given a prime number p and the p-adic expansion of n, $n = n_0 + n_1 p + n_2 p^2 + \cdots + n_m p^m$, it follows from more general theorems of McIntosh [13, Theorems 3 and 6] that the factorisation

$$a_n(r,s) \equiv \prod_{i=0}^{m} a_{n_i}(r,s) \pmod p \tag{1.2}$$

holds. Deutsch and Sagan [6, Theorem 5.9] rediscover this factorisation, and they use it to characterise the congruence classes of $a_n(r,s)$ modulo 3 in terms of precise conditions which the 3-adic expansion of n must satisfy. Gessel [8] obtains the above factorisation in the special case where $r = s = 2$, and furthermore, in Theorem 3(iii) of that paper, proves that the factorisation (1.2) remains valid for $r = s = 2$ with 9 in place of the modulus p.

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The goal of our paper is to determine the behaviour of the Apéry numbers $a_n(r,s)$ modulo 9 for arbitrary positive integers r and non-negative integers s. This will be achieved by first deriving a Lucas-type result for binomial coefficients modulo 9 (see Lemma 1 in the next section), and using this to establish an analogue of formula (1.2) for the modulus 9; see Theorems 2 and 3 in Section 3. From these theorems, one easily derives explicit congruences for $a_n(r,s)$ depending on the congruence classes of r and s modulo 6; see Section 4 for a sample of such results. In the long version [11] of this paper, all possible cases are worked out (there are 32 of them). In particular, Corollaries 8 and 9 confirm Conjectures 65 and 66 from [10] concerning the explicit description of the classical Apéry numbers $a_n(2,1)$ and $a_n(2,2)$ modulo 9. As a side result, we obtain generalisations of Gessel's factorisation result for $a_n(2,2)$ modulo 9 mentioned above; see the remark after (4.6).

We point out that Rowland and Yassawi [15] have also provided proofs for Conjectures 65 and 66 from [10] (among many other things), using however a completely different approach, based on extracting diagonals from rational power series and construction of automata. It is conceivable that their approach would also achieve proofs of the other results in this paper.

It should be clear that our approach could also be used to obtain explicit descriptions of the congruence classes of Apéry numbers modulo higher powers of 3, and the same remark applies to the approach in [15]. The analysis and the results would be more complex than the ones in this paper, though.

2. A Lucas-type theorem modulo 9

The classical result of Lucas in [12, p. 230, Eq. (137)] says that, if p is a prime, $n = n_0+n_1p+n_2p^2+\cdots+n_mp^m$, and $k = k_0+k_1p+k_2p^2+\cdots+k_mp^m$, $0 \le n_i, k_i \le p-1$, then

$$\binom{n}{k} \equiv \prod_{i=0}^{m} \binom{n_i}{k_i} \pmod p.$$

Using the generating function approach proposed in [9, Sec. 6], we find an analogue of this formula for the modulus 9, which is different from the mod-9 cases of the generalisations of Lucas' formula to prime powers given in [5] and [9].

Lemma 1. *For all non-negative integers $n = n_0 + 3n_1 + 9n_2 + \cdots + 3^m n_m$ and $k = k_0 + 3k_1 + 9k_2 + \cdots + 3^m n_m$, where $0 \le n_i, k_i \le 2$ for all i, we have*

$$\begin{aligned}\binom{n}{k} \equiv{} & \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_m}{k_m} \\ & + \sum_{\nu=1}^{m} 3n_\nu \chi(n_{\nu-1}=0)\binom{n_0}{k_0}\cdots\binom{n_{\nu-2}}{k_{\nu-2}}\binom{1}{k_{\nu-1}-1}\binom{n_\nu-1}{k_\nu}\binom{n_{\nu+1}}{k_{\nu+1}}\cdots\binom{n_m}{k_m} \\ & + \sum_{\nu=1}^{m} 3n_\nu \chi(n_{\nu-1}=1,\ k_{\nu-1}=0)\binom{n_0}{k_0}\cdots\binom{n_{\nu-2}}{k_{\nu-2}}\binom{n_\nu}{k_\nu}\cdots\binom{n_m}{k_m} \\ & - \sum_{\nu=1}^{m} 3n_\nu \chi(n_{\nu-1}=1)\binom{n_0}{k_0}\cdots\binom{n_{\nu-2}}{k_{\nu-2}}\binom{2}{k_{\nu-1}}\binom{n_\nu-1}{k_\nu}\binom{n_{\nu+1}}{k_{\nu+1}}\cdots\binom{n_m}{k_m} \\ & + \sum_{\nu=1}^{m} 3n_\nu \chi(n_{\nu-1}=2,\ k_{\nu-1}=1)\binom{n_0}{k_0}\cdots\binom{n_{\nu-2}}{k_{\nu-2}}\binom{n_\nu}{k_\nu}\cdots\binom{n_m}{k_m} \pmod 9,\end{aligned} \tag{2.1}$$

where $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ otherwise.

Proof. During this proof, given polynomials $f(z)$ and $g(z)$ with integer coefficients, we write

$$f(z) = g(z) \text{ modulo } 9$$

to mean that the coefficients of z^i in $f(z)$ and $g(z)$ agree modulo 9 for all i. We use an analogous notation for the modulus 3.

An easy induction shows that

$$(1+z)^{3^\nu} = 1 + z^{3^\nu} + 3z^{3^{\nu-1}}\left(1 + z^{3^{\nu-1}}\right) \quad \text{modulo } 9.$$

This implies the expansion

$$\begin{aligned}(1+z)^n &= (1+z)^{n_0}\big((1+z)^3\big)^{n_1}\big((1+z)^9\big)^{n_2}\cdots\big((1+z)^{3^m}\big)^{n_m} \\ &= (1+z)^{n_0}\prod_{\nu=1}^{m}\left(\left(1+z^{3^\nu}\right) + 3z^{3^{\nu-1}}\left(1+z^{3^{\nu-1}}\right)\right)^{n_\nu} \\ &= \prod_{\nu=0}^{m}\left(1+z^{3^\nu}\right)^{n_\nu} + \sum_{\nu=1}^{m} 3n_\nu (1+z)^{n_0}\cdots\left(1+z^{3^{\nu-2}}\right)^{n_{\nu-2}} \\ &\qquad \cdot z^{3^{\nu-1}}\left(1+z^{3^{\nu-1}}\right)^{n_{\nu-1}+1}\left(1+z^{3^\nu}\right)^{n_\nu-1}\cdots\left(1+z^{3^m}\right)^{n_m} \\ &\qquad\qquad \text{modulo } 9.\end{aligned}$$

Now, in case $n_{\nu-1} \ge 1$, one applies the formulae

$$\begin{aligned} z^{3^{\nu-1}}\left(1+z^{3^{\nu-1}}\right)^2 &= \left(1+z^{3^\nu}\right) - \left(1+z^{3^{\nu-1}}\right)^2 \quad \text{modulo } 3 \\ z^{3^{\nu-1}}\left(1+z^{3^{\nu-1}}\right)^3 &= z^{3^{\nu-1}}\left(1+z^{3^\nu}\right) \quad \text{modulo } 3\end{aligned}$$

to the terms involving $3^{\nu-1}$ in the sum, depending on whether $n_{\nu-1}+1 = 2$ or $n_{\nu-1}+1 = 3$. Finally, every binomial term $(1+z^{3^r})^{n'_r}$ is expanded using the binomial theorem, and subsequently the coefficient of

$$z^k = z^{k_0+3k_1+9k_2+\cdots+3^m k_m}$$

is read off in all the terms. This leads directly to (2.1). $\square$

3. The main theorems

Here we prove the actual main results of our paper, namely Theorems 2 and 3 below, which provide a refinement of (1.2) in the case where $p = 3$ to modulus 9. The explicit congruences for the Apéry numbers $a_n(r, s)$ given in the next section are then simple consequences. We state the results for $s \ge 1$ and $s = 0$ separately in order to keep expressions at a moderate size.

Theorem 2. *For all positive integers r and s, and non-negative integers $n = n_0+3n_1+9n_2+\cdots+3^m n_m$, where $0 \le n_i \le 2$ for all i, we have*

$$a_n(r,s) \equiv \prod_{i=0}^{m} a_{n_i}(r,s) + 3\sum_{\nu=1}^{m}\left(\prod_{\substack{i=0\\ i\ne\nu-1,\nu}}^{m} a_{n_i}(r,s)\right) f(n_{\nu-1},n_\nu;r,s) \pmod 9, \tag{3.1}$$

where

$$\begin{aligned} f(n_{\nu-1},n_\nu;r,s) = n_{\nu-1}(n_{\nu-1}+1)n_\nu\Big(s(n_\nu+1)+(-1)^s\big(s(n_\nu-1)+rn_\nu\big)\Big)\\ +\chi(r=1)(n_{\nu-1}+2)(n_{\nu-1}+1)n_\nu\\ +\chi(s=1)((-1)^r-1)(n_{\nu-1}-1)n_{\nu-1}n_\nu^2. \end{aligned} \tag{3.2}$$

Proof. By the definition (1.1) of the Apéry numbers and the Chu–Vandermonde summation formula, we may write

$$a_n(r,s) = \sum_{k=0}^{n}\binom{n}{k}^r\binom{n+k}{k}^s = \sum_{k=0}^{n}\binom{n}{k}^r\left(\sum_{\ell=0}^{n}\binom{n}{\ell}\binom{k}{\ell}\right)^s.$$

Now we apply the Lucas-type congruence from Lemma 1 to all three binomial coefficients on the right-hand side of the above formula. Using the trivial congruences

$$(a+3b)^s \equiv a^s + 3a^{s-1}b \pmod 9$$

and

$$(a+3b)(c+3d) \equiv ac + 3bc + 3ad \pmod 9,$$

we expand the resulting expression. In this way, we obtain a large number of terms. Writing $\ell = \ell_0 + 3\ell_1 + 9\ell_2 + \cdots + 3^m\ell_m$, the leading term is

$$\sum_{0\le k_1,\dots,k_m\le 2}\left(\prod_{i=0}^{m}\binom{n_i}{k_i}^r\right)\left(\sum_{0\le \ell_1,\dots,\ell_m\le 2}\prod_{i=0}^{m}\binom{n_i}{\ell_i}\binom{k_i}{\ell_i}\right)^s,$$

while one obtains 12 more terms which are similar. At this point, the summation over the ℓ_i's can be carried out easily, in all of the arising terms. This leads to the congruence

$$\begin{aligned}
a_n(r,s) \equiv{} & \prod_{i=0}^{m} a_{n_i}(r,s) \\
&+ 3r \sum_{\nu=1}^{m} \left(\prod_{\substack{i=0 \\ i\neq \nu-1,\nu}}^{m} a_{n_i}(r,s) \right) \left(\sum_{k_{\nu-1}=0}^{2} \binom{1}{k_{\nu-1}-1} \binom{n_{\nu-1}}{k_{\nu-1}}^{r-1} \binom{n_{\nu-1}+k_{\nu-1}}{k_{\nu-1}}^{s} \right) \\
&\qquad \cdot \left(\sum_{k_\nu=0}^{2} \binom{n_\nu - 1}{k_\nu} \binom{n_\nu}{k_\nu}^{r-1} \binom{n_\nu + k_\nu}{k_\nu}^{s} \right) \chi(n_{\nu-1}=0) n_\nu + \cdots \\
&+ 3s \sum_{\nu=1}^{m} \left(\prod_{\substack{i=0 \\ i\neq \nu-1,\nu}}^{m} a_{n_i}(r,s) \right) \left(\sum_{k_{\nu-1}=0}^{2} \binom{k_{\nu-1}+1}{k_{\nu-1}-1} \binom{n_{\nu-1}}{k_{\nu-1}}^{r} \binom{n_{\nu-1}+k_{\nu-1}}{k_{\nu-1}}^{s-1} \right) \\
&\qquad \cdot \left(\sum_{k_\nu=0}^{2} \binom{n_\nu + k_\nu - 1}{k_\nu} \binom{n_\nu}{k_\nu}^{r} \binom{n_\nu + k_\nu}{k_\nu}^{s-1} \right) \chi(n_{\nu-1}=0) n_\nu + \cdots \\
&+ 3s \sum_{\nu=1}^{m} \left(\prod_{\substack{i=0 \\ i\neq \nu-1,\nu}}^{m} a_{n_i}(r,s) \right) \left(\sum_{k_{\nu-1}=0}^{0} \binom{n_{\nu-1}+1}{n_{\nu-1}-1} \binom{n_{\nu-1}}{k_{\nu-1}}^{r} \binom{n_{\nu-1}+k_{\nu-1}}{k_{\nu-1}}^{s-1} \right) \\
&\qquad \cdot \left(\sum_{k_\nu=0}^{2} k_\nu \binom{n_\nu + k_\nu - 1}{n_\nu} \binom{n_\nu}{k_\nu}^{r} \binom{n_\nu + k_\nu}{k_\nu}^{s-1} \right) + \cdots \pmod 9,
\end{aligned} \tag{3.3}$$

where each of the dots $\cdots$ represents three similar terms. (All the sums above over ν come from the second line in (2.1).) The sums over $k_{\nu-1}$ and k_ν are written out explicitly. The resulting formula is then simplified using elementary congruences modulo 3. Namely, to simplify powers, we use

$$n^a(n+1)^b \equiv (-1)^{b-1} n(n+1) \pmod 3, \quad \text{for } a,b \geq 1.$$

Furthermore, by

$$\chi(n \equiv a \pmod 3) \equiv -(n-a+1)(n-a+2) \pmod 3, \tag{3.4}$$

we may write

$$n^a \equiv \chi(a=0) + (1-\chi(a=0))(-n(n+1) - n(n-1)(-1)^a) \pmod 3, \quad \text{for } a \geq 0.$$

Finally, higher powers of $n_{\nu-1}$ and n_ν are lowered by use of the (Fermat) congruence

$$n^3 \equiv n \pmod 3.$$

After these steps, one collects terms, to obtain a congruence of the form

$$a_n(r,s) \equiv \prod_{i=0}^{m} a_{n_i}(r,s) + 3 \sum_{\nu=1}^{m} \left(\prod_{\substack{i=0 \\ i\neq \nu-1,\nu}}^{m} a_{n_i}(r,s) \right) f(n_{\nu-1}, n_\nu; r, s).$$

Upon considerable simplification, one sees that the term $f(n_{\nu-1}, n_\nu; r, s)$ can be given by the right-hand side of (3.2). $\square$

Remark. By applying (3.4) again several times (namely "backwards"), the term $f(n_{\nu-1}, n_\nu; r, s)$ can alternatively be rewritten as

$$\begin{aligned} f(n_{\nu-1}, n_\nu; r, s) \equiv \chi(n_{\nu-1} = 1)\Big(s\chi(n_\nu = 1) + (-1)^s s\chi(n_\nu = 2)&\\ + (-1)^s r\big(\chi(n_\nu = 0) - 1\big)\Big)&\\ - \chi(r = 1)\chi(n_{\nu-1} = 0)n_\nu&\\ + \chi(s = 1)((-1)^r - 1)\chi(n_{\nu-1} = 2)\big(\chi(n_\nu = 0) - 1\big) \pmod 3.& \end{aligned} \tag{3.5}$$

Following the same approach, we may establish an analogous result for the (generalised) Franel numbers; that is, for the case where $s = 0$.

Theorem 3. *For all positive integers r and non-negative integers $n = n_0 + 3n_1 + 9n_2 + \cdots + 3^m n_m$, where $0 \le n_i \le 2$ for all i, we have*

$$a_n(r, 0) \equiv \prod_{i=0}^{m} a_{n_i}(r, 0) + 3\sum_{\nu=1}^{m}\left(\prod_{\substack{i=0\\ i\ne\nu-1,\nu}}^{m} a_{n_i}(r, 0)\right) f(n_{\nu-1}, n_\nu; r, 0) \pmod 9, \tag{3.6}$$

where

$$\begin{aligned} f(n_{\nu-1}, n_\nu; r, 0) = \chi(r = 1)\big(n_\nu - \chi(n_\nu = 2)\big)\big(\chi(n_{\nu-1} = 2) - 1\big)&\\ - \big(n_\nu + \chi(n_\nu = 2)(-1)^r\big)\big(\chi(n_{\nu-1} = 1) - (-1)^r\chi(n_{\nu-1} = 2)\big).& \end{aligned} \tag{3.7}$$

4. Explicit description of the Apéry numbers modulo 9

We are now going to exploit Theorems 2 and 3 to obtain explicit congruences modulo 9 for the Apéry numbers $a_n(r, s)$, depending on the congruence classes of r and s modulo 6. In view of (3.1) and (3.6), we have to analyse the congruence behaviour modulo 9 of $a_n(r, s)$ for $n = 0, 1, 2$, as well as the behaviour modulo 3 of the terms $f(n_{\nu-1}, n_\nu; r, s)$ given by (3.2) (or (3.5)) and (3.7).

We begin with the Apéry numbers for small indices. We have

$$a_0(r, s) = 1, \tag{4.1}$$

$$a_1(r, s) = 1 + 2^s \equiv \begin{cases} 2 \pmod 9, & \text{if } s \equiv 0 \pmod 6,\\ 3 \pmod 9, & \text{if } s \equiv 1 \pmod 6,\\ 5 \pmod 9, & \text{if } s \equiv 2 \pmod 6,\\ 0 \pmod 9, & \text{if } s \equiv 3 \pmod 6,\\ 8 \pmod 9, & \text{if } s \equiv 4 \pmod 6,\\ 6 \pmod 9, & \text{if } s \equiv 5 \pmod 6, \end{cases} \tag{4.2}$$

and

$$a_2(r,s)=1+2^r3^s+6^s\equiv\begin{cases}3 \pmod 9, & \text{if } r\equiv 0 \pmod 6 \text{ and } s=0,\\ 4 \pmod 9, & \text{if } r\equiv 1 \pmod 6 \text{ and } s=0,\\ 6 \pmod 9, & \text{if } r\equiv 2 \pmod 6 \text{ and } s=0,\\ 1 \pmod 9, & \text{if } r\equiv 3 \pmod 6 \text{ and } s=0,\\ 0 \pmod 9, & \text{if } r\equiv 4 \pmod 6 \text{ and } s=0,\\ 7 \pmod 9, & \text{if } r\equiv 5 \pmod 6 \text{ and } s=0,\\ 1 \pmod 9, & \text{if } r\equiv 0,2 \pmod 3 \text{ and } s=1,\\ 4 \pmod 9, & \text{if } r\equiv 1 \pmod 3 \text{ and } s=1,\\ 1 \pmod 9, & \text{if } s\ge 2.\end{cases} \tag{4.3}$$

Distinguishing between the various cases which arise when r and s run through the congruence classes modulo 6, we obtain

$$f(n_{\nu-1},n_\nu;r,s)$$
$$\equiv\begin{cases}0 \pmod 3, & \text{if } r\equiv 0 \pmod 3 \text{ and } s\equiv 0 \pmod 3,\\ \chi(n_{\nu-1}=1)n_\nu \pmod 3, & \text{if } r\equiv 0 \pmod 6 \text{ and } s\equiv 1 \pmod 6,\\ \chi(n_{\nu-1}=1)\big(\chi(n_\nu=0)-1\big) \pmod 3, & \text{if } r\equiv 0 \pmod 3 \text{ and } s\equiv 2 \pmod 6,\\ \chi(n_{\nu-1}=1)\big(1-\chi(n_\nu=0)\big) \pmod 3, & \text{if } r\equiv 0 \pmod 3 \text{ and } s\equiv 4 \pmod 6,\\ -\chi(n_{\nu-1}=1)n_\nu \pmod 3, & \text{if } r\equiv 0 \pmod 3 \text{ and } s\equiv 5 \pmod 6,\\ \chi(n_{\nu-1}=1)\big(\chi(n_\nu=0)-1\big)-\chi(n_{\nu-1}=0)n_\nu \pmod 3, & \\ & \text{if } r=1 \text{ and } s\equiv 0 \pmod 6,\\ -n_{\nu-1}\chi(n_\nu=2)-n_\nu \pmod 3, & \text{if } r=s=1,\end{cases} \tag{4.4}$$

and

$$
f(n_{\nu-1},n_\nu;r,s)
\equiv
\begin{cases}
\chi(n_\nu=1)n_{\nu-1}-n_\nu-\big(1-\chi(n_{\nu-1}=0)\big)\big(1-\chi(n_\nu=0)\big)\pmod 3,\\
\qquad\qquad \text{if } r=1,\ s\equiv 1 \pmod 6,\ \text{and } s\ge 7,\\
\chi(n_{\nu-1}=1)\big(1-\chi(n_\nu=0)\big)-\chi(n_{\nu-1}=0\big)n_\nu\pmod 3,\\
\qquad\qquad \text{if } r=1 \text{ and } s\equiv 2 \pmod 6,\\
\chi(n_{\nu-1}=1)\big(1-\chi(n_\nu=0)\big)-\chi(n_{\nu-1}=0)n_\nu\pmod 3,\\
\qquad\qquad \text{if } r=1 \text{ and } s\equiv 3 \pmod 6,\\
-\chi(n_{\nu-1}=0)n_\nu\pmod 3, & \text{if } r=1 \text{ and } s\equiv 4 \pmod 6,\\
\big(\chi(n_{\nu-1}=2)-1\big)n_\nu+\chi(n_{\nu-1}=1)\big(1-\chi(n_\nu=0)\big)\pmod 3,\\
\qquad\qquad \text{if } r=1 \text{ and } s\equiv 5 \pmod 6,\\
\chi(n_{\nu-1}=1)\big(\chi(n_\nu=0)-1\big)\pmod 3,\\
\qquad\qquad \text{if } r\equiv 1 \pmod 3,\ r\ge 4, \text{ and } s\equiv 0 \pmod 6,\\
\chi(n_{\nu-1}=1)n_\nu+\big(1-\chi(n_\nu=0)\big)n_{\nu-1}\pmod 3,\\
\qquad\qquad \text{if } r\equiv 1 \pmod 6\big),\ r\ge 7, \text{ and } s=1,\\
-\chi(n_{\nu-1}=1)\chi(n_\nu=1)\pmod 3, & \text{if } r,s\equiv 1 \pmod 6 \text{ and } r,s\ge 7,\\
& \text{and if } r\equiv 4 \pmod 6 \text{ and } s\equiv 1 \pmod 6,\\
\chi(n_{\nu-1}=1)\big(1-\chi(n_\nu=0)\big)\pmod 3,\\
\qquad\qquad \text{if } r\equiv 1 \pmod 3,\ r\ge 4,\\
\qquad\qquad \text{and } s\equiv 2,3 \pmod 6,\\
0\pmod 3, & \text{if } r\equiv 1 \pmod 3,\ r\ge 4, \text{ and } s\equiv 4 \pmod 6,\\
-\chi(n_{\nu-1}=1)\chi(n_\nu=2)\pmod 3,\\
\qquad\qquad \text{if } r\equiv 1 \pmod 3,\ r\ge 4, \text{ and } s\equiv 5 \pmod 6,\\
\chi(n_{\nu-1}=1)\big(1-\chi(n_\nu=0)\big)\pmod 3,\\
\qquad\qquad \text{if } r\equiv 2 \pmod 3 \text{ and } s\equiv 0 \pmod 6,\\
\chi(n_{\nu-1}=1)\chi(n_\nu=2)\pmod 3, & \text{if } r\equiv 2 \pmod 6 \text{ and } s\equiv 1 \pmod 6,\\
0\pmod 3, & \text{if } r\equiv 2 \pmod 3 \text{ and } s\equiv 2 \pmod 6,\\
\chi(n_{\nu-1}=1)\big(\chi(n_\nu=0)-1\big)\pmod 3,\\
\qquad\qquad \text{if } r\equiv 2 \pmod 3 \text{ and } s\equiv 3 \pmod 6,\\
\chi(n_{\nu-1}=1)\big(\chi(n_\nu=0)-1\big)\pmod 3,\\
\qquad\qquad \text{if } r\equiv 2 \pmod 3 \text{ and } s\equiv 4 \pmod 6,\\
\chi(n_{\nu-1}=1)\chi(n_\nu=1)\pmod 3, & \text{if } r\equiv 2 \pmod 3 \text{ and } s\equiv 5 \pmod 6,\\
\chi(n_{\nu-1}=2)\big(\chi(n_\nu=0)-1\big)+\chi(n_{\nu-1}=1)n_\nu\pmod 3,\\
\qquad\qquad \text{if } r\equiv 3 \pmod 6 \text{ and } s=1,\\
\chi(n_{\nu-1}=1)n_\nu\pmod 3, & \text{if } r\equiv 3 \pmod 6,\ s\equiv 1 \pmod 6, \text{ and } s\ge 7,\\
\chi(n_{\nu-1}=1)n_\nu-\big(1-\chi(n_{\nu-1}=0)\big)\big(1-\chi(n_\nu=0)\big)\pmod 3,\\
\qquad\qquad \text{if } r\equiv 5 \pmod 6 \text{ and } s=1,\\
\chi(n_{\nu-1}=1)\chi(n_\nu=2)\pmod 3, & \text{if } r\equiv 5 \pmod 6,\ s\equiv 1 \pmod 6, \text{ and } s\ge 7,
\end{cases}
\tag{4.5}
$$

and

$$
f(n_{\nu-1},n_\nu;r,s)
$$
$$
\equiv \begin{cases}
\big(\chi(n_{\nu-1}=2)-\chi(n_{\nu-1}=1)\big)\big(\chi(n_\nu=2)+n_\nu\big) \pmod 3,\\
\hfill \text{if } r\equiv 0 \pmod 6,\ r\ge 6,\ \text{and } s=0,\\
\big(1+\chi(n_{\nu-1}=1)\big)\big(\chi(n_\nu=2)-n_\nu\big) \pmod 3,\\
\hfill \text{if } r=1 \text{ and } s=0,\\
\big(\chi(n_{\nu-1}=1)+\chi(n_{\nu-1}=2)\big)\big(\chi(n_\nu=2)-n_\nu\big) \pmod 3,\\
\hfill \text{if } r\equiv 1,3,5 \pmod 6,\ r\ge 3,\ \text{and } s=0,\\
\big(\chi(n_{\nu-1}=2)-\chi(n_{\nu-1}=1)\big)\big(\chi(n_\nu=2)+n_\nu\big) \pmod 3,\\
\hfill \text{if } r\equiv 2,4 \pmod 6 \text{ and } s=0.
\end{cases}
\tag{4.6}
$$

Remark. By examining (4.4)–(4.6), one sees that Gessel's result [8, Theorem 3(iii)], namely that

$$
a_n(r,s)\equiv\prod_{i=0}^m a_{n_i}(r,s) \pmod 9 \tag{4.7}
$$

for $r=s=2$, does not only hold in that case, but more generally for $r\equiv 2 \pmod 3$ and $s\equiv 2 \pmod 6$, and also for $r\equiv s\equiv 0 \pmod 3$, and for $r\equiv 1 \pmod 3$, $r\ge 4$, and $s\equiv 4 \pmod 6$.

If we combine Theorems 2 and 3 with (4.1)–(4.6), we obtain detailed results concerning the behaviour of $a_n(r,s)$ modulo 9. We confine ourselves here to providing six results representative for the total of 27 listed in [11].

Corollary 4. *If r and s are positive integers with $r\equiv 0 \pmod 3$ and $s\equiv 0 \pmod 6$, then the Apéry numbers $a_n(r,s)$ obey the following congruences modulo 9:*

(i) $a_n(r,s)\equiv 1 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k$ digits 1, for some k, and otherwise only 0's and 2's;*
(ii) $a_n(r,s)\equiv 2 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+1$ digits 1, for some k, and otherwise only 0's and 2's;*
(iii) $a_n(r,s)\equiv 4 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+2$ digits 1, for some k, and otherwise only 0's and 2's;*
(iv) $a_n(r,s)\equiv 5 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+5$ digits 1, for some k, and otherwise only 0's and 2's;*
(v) $a_n(r,s)\equiv 7 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+4$ digits 1, for some k, and otherwise only 0's and 2's;*
(vi) $a_n(r,s)\equiv 8 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+3$ digits 1, for some k, and otherwise only 0's and 2's;*
(vii) *in the cases not covered by Items (i)–(vi), $a_n(r,s)$ is divisible by 9; in particular, $a_n(r,s)\not\equiv 3,6 \pmod 9$ for all n.*

Corollary 5. *If r and s are positive integers with $r\equiv 0 \pmod 6$ and $s\equiv 1 \pmod 6$, or with $r\equiv 3 \pmod 6$, $s\equiv 1 \pmod 6$, and $s\ge 7$, then the Apéry numbers $a_n(r,s)$ obey the following congruences modulo 9:*

(i) $a_n(r,s)\equiv 1 \pmod 9$ *if, and only if, the 3-adic expansion of n contains 0's and 2's only;*

(ii) $a_n(r,s) \equiv 3 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *has exactly one occurrence of the string* 01 (*including an occurrence of a* 1 *at the beginning*) *or of the string* 11 *— but not both — and otherwise contains only* 0*'s and* 2*'s;*

(iii) *in the cases not covered by Items* (i)–(ii), $a_n(r,s)$ *is divisible by* 9*; in particular,* $a_n(r,s) \not\equiv 2,4,5,6,7,8 \pmod 9$ *for all* n.

Corollary 6. *If* r *and* s *are positive integers with* $r \equiv 0 \pmod 3$ *and* $s \equiv 3 \pmod 6$, *then the Apéry numbers* $a_n(r,s)$ *obey the following congruences modulo* 9*:*

(i) $a_n(r,s) \equiv 1 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *contains* 0*'s and* 2*'s only;*

(ii) *in all other cases* $a_n(r,s)$ *is divisible by* 9*; in particular,* $a_n(r,s) \not\equiv 2,3,4,5,6,7, 8 \pmod 9$ *for all* n.

Corollary 7. *If* r *and* s *are positive integers with* $r \equiv 2 \pmod 3$ *and* $s \equiv 0 \pmod 6$, *then the Apéry numbers* $a_n(r,s)$ *obey the following congruences modulo* 9*:*

(i) $a_n(r,s) \equiv 1 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *has* $2d$ *digits* 1, o_1 *occurrences of the string* 11, o_2 *occurrences of the string* 21, *and* $d + o_1 - o_2 \equiv 0 \pmod 3$*;*

(ii) $a_n(r,s) \equiv 2 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *has* $2d+1$ *digits* 1, o_1 *occurrences of the string* 11, o_2 *occurrences of the string* 21, *and* $d + o_1 - o_2 \equiv 0 \pmod 3$*;*

(iii) $a_n(r,s) \equiv 4 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *has* $2d$ *digits* 1, o_1 *occurrences of the string* 11, o_2 *occurrences of the string* 21, *and* $d + o_1 - o_2 \equiv 1 \pmod 3$*;*

(iv) $a_n(r,s) \equiv 5 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *has* $2d+1$ *digits* 1, o_1 *occurrences of the string* 11, o_2 *occurrences of the string* 21, *and* $d + o_1 - o_2 \equiv 2 \pmod 3$*;*

(v) $a_n(r,s) \equiv 7 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *has* $2d$ *digits* 1, o_1 *occurrences of the string* 11, o_2 *occurrences of the string* 21, *and* $d + o_1 - o_2 \equiv 2 \pmod 3$*;*

(vi) $a_n(r,s) \equiv 8 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *has* $2d+1$ *digits* 1, o_1 *occurrences of the string* 11, o_2 *occurrences of the string* 21, *and* $d + o_1 - o_2 \equiv 1 \pmod 3$*;*

(vii) *in the cases not covered by Items* (i)–(vi), $a_n(r,s)$ *is divisible by* 9*; in particular,* $a_n(r,s) \not\equiv 3,6 \pmod 9$ *for all* n.

Corollary 8. *If* r *and* s *are positive integers with* $r \equiv 2 \pmod 6$ *and* $s \equiv 1 \pmod 6$, *or with* $r \equiv 5 \pmod 6$, $s \equiv 1 \pmod 6$, *and* $s \ge 7$, *then the Apéry numbers* $a_n(r,s)$ *obey the following congruences modulo* 9*:*

(i) $a_n(r,s) \equiv 1 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *contains* 0*'s and* 2*'s only;*

(ii) $a_n(r,s) \equiv 3 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *has exactly one occurrence of the string* 01 (*including an occurrence of a* 1 *at the beginning*) *and otherwise contains only* 0*'s and* 2*'s;*

(iii) $a_n(r,s) \equiv 6 \pmod 9$ *if, and only if, the* 3*-adic expansion of* n *has exactly one occurrence of the string* 21 *and otherwise contains only* 0*'s and* 2*'s;*

(iv) *in the cases not covered by Items* (i)–(iii), $a_n(r,s)$ *is divisible by* 9*; in particular,* $a_n(r,s) \not\equiv 2,4,5,7,8 \pmod 9$ *for all* n.

For $r = 2$ and $s = 1$, this corollary establishes Conjecture 65 in [10].

Corollary 9. *If r and s are positive integers with $r \equiv 2 \pmod 3$ and $s \equiv 2 \pmod 6$, then the Apéry numbers $a_n(r,s)$ obey the following congruences modulo 9:*

(i) $a_n(r,s) \equiv 1 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k$ digits 1, for some k, and otherwise only 0's and 2's;*
(ii) $a_n(r,s) \equiv 2 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+5$ digits 1, for some k, and otherwise only 0's and 2's;*
(iii) $a_n(r,s) \equiv 4 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+4$ digits 1, for some k, and otherwise only 0's and 2's;*
(iv) $a_n(r,s) \equiv 5 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+1$ digits 1, for some k, and otherwise only 0's and 2's;*
(v) $a_n(r,s) \equiv 7 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+2$ digits 1, for some k, and otherwise only 0's and 2's;*
(vi) $a_n(r,s) \equiv 8 \pmod 9$ *if, and only if, the 3-adic expansion of n contains $6k+3$ digits 1, for some k, and otherwise only 0's and 2's;*
(vii) *in the cases not covered by Items* (i)–(vi), $a_n(r,s)$ *is divisible by 9; in particular,* $a_n(r,s) \not\equiv 3, 6 \pmod 9$ *for all n.*

For $r = s = 2$, this corollary establishes Conjecture 66 in [10].

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