Noname manuscript No.
(will be inserted by the editor)

On the locally rotationally symmetric Einstein-Maxwell perfect fluid

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Received: date / Accepted: date

Abstract We examine the stability of Einstein-Maxwell perfect fluid configurations with a privileged radial direction by means of a $1+1+2$-tetrad formalism. We use this formalism to cast in a quasilinear symmetric hyperbolic form the equations describing the evolution of the system. This hyperbolic reduction is used to discuss the stability of linear perturbations in some special cases. By restricting the analysis to isotropic fluid configurations, we assume a constant electrical conductivity coefficient for the fluid. As a result of this analysis we provide a complete classification and characterization of various stable and unstable configurations. We find, in particular, that in many cases the stability conditions are strongly determined by the constitutive equations and the electric conductivity. A threshold for the emergence of the instability appears in both contracting and expanding systems.

Keywords locally rotationally symmetric solutions · $1+1+2$-formalism · linear perturbations · stability · magnetohydrodynamics

1 Introduction

The stability problem of plasma configurations is an important issue in a variety of astrophysical scenarios involving, for example, stellar objects and accretion disks. Various phenomena of high-energy astrophysics are related to accretion disks with unstable processes such as accretion or jet emission. In this article we consider the situation where gravity plays a decisive role in determining both the equilibrium states of the configurations and the dynamical phases associated instability so that a full general relativistic analysis is required —see [1,2,3,4,5,6,7,8,9]. These systems are described by the coupled Einstein-Maxwell-Euler equations. A notable example of this type of configurations is given by general rel-

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ativistic (GR) magnetohydrodynamic (MHD) systems. Given the complexity of the equations, the analysis of perturbations in this type of systems must be conducted with suitable assumptions on the symmetries for the system (for example in toroidal accretion disks). In order to discuss dynamic situations numerical approaches are often required. A major challenge in dealing with these systems is to find an appropriate formulation of the problem: on the one hand one requires a formulation adapted to the symmetries of the configuration and, on the other hand, a suitable formulation of an initial value problem is necessary for the construction of the numerical solutions so as to ensure local and global existence and the analysis of the stability of certain reference solutions [10,11,12,13,14,6,15,8,9,7].

In this work we explore the stability properties of systems with a preferred radial direction (from now on, for simplicity, the "radial direction") governed by the Einstein-Maxwell-Euler equations casted in a quasi-symmetric hyperbolic form. The procedure of hyperbolic reduction used in the present article follows the presentation given in [16]. Our analysis is based on an adapted $1+1+2$-*tetrad formalism* [17]. This formalism is adapted to the description of a general locally rotationally symmetric system (LRS), a remarkable example of which is given by the simple case of spherically symmetric configurations [17] —for a more specific discussion on the stability of spherically symmetric plasma see, for example, [18,19,20,21,22,23]. This formalism is an extension of the usual $1+3$-formalism in which the existence of a privileged timelike vector field u^a is assumed —in applications involving the description of fluids, it is natural to let u^a to follow the congruence generated by the fluid [25]. In the $1+1+2$-formalism the presence of a further (spatial) vector field n^a is assumed. Further discussion on the decomposition of Einstein-Maxwell-perfect fluid equations can be found in [17,24,36]. In [21], the same formalism has been used to analyze self gravitating spherically symmetric charged perfect fluid configurations in hydrostatic equilibrium. Details on the $1+3$ and $1+1+2$-decompositions of the Einstein-Maxwell (EM) equations in LRS spacetimes can be found in [26,27,28].

In this article we consider perturbations of the metric tensor and of the matter and EM fields. All the quantities share the same preferred radial direction, as in the locally rotationally symmetric classes II space-times as described in [17] [1]. We focus our attention to the special case of an infinitely conducting plasma which describes an adiabatic flow so that the entropy per particle is conserved along the flow lines. In particular, we investigate stability for systems with a magnetic field but without dissipative effects. A central aspect of the stability analysis undertaken in this article is the construction of a quasilinear symmetric hyperbolic evolution system for the variables describing the configuration. This, in turn, ensures the well-posedness of the Cauchy problem for the system —in other words, the local existence and uniqueness of solutions for the Einstein-Maxwell-Euler equations. By prescribing suitable initial data on an initial hypersurface, a unique solution exists in a neighborhood of that hypersurface. This solution depends continuously on the initial data. Accordingly, we first write the evolution equations for the independent components of n variables collected in a n-dimensional vector $\mathbf{v}$ used to obtain a suitable symmetric hyperbolic evolution system of the form $\mathbf{A}^t\partial_t\mathbf{v}-\mathbf{A}^j\partial_j\mathbf{v}=\mathbf{B}\mathbf{v}$, where $\mathbf{A}^t$ and $\mathbf{A}^j$ and $\mathbf{B}$ are smooth matrix valued functions of the coordinates (t,x) and the variables $\mathbf{v}$ with the index j asso-

[1] We refer to [21,17,24,36,26,29,30] for a general discussion concerning the configurations with a unique radial direction described by the Einstein-Maxwell-Euler system. We mention also the well-known solution of a toroidal magnetic field widely used in the axes-symmetric accretion configurations [32], and [31] for a discussion on the case of a poloidal magnetic field where a metric representation is adapted to the direction of the field, as proved by Bekenstein & Oron [33,34]. For a deeper and more general discussion about MHD configurations in spherical symmetry, see for example [18,19,20,21,22,23].

ciated to some spatial coordinates x. The system is symmetric hyperbolic if the matrices $\mathbf{A}^t$ and $\mathbf{A}^j$ are symmetric and if $\mathbf{A}^t$ is a positive-definite matrix.

In our analysis we take into account gravito-electromagnetic (GEM) effects and we provide a complete classification of the solutions in terms of the scalars of the conformal Weyl tensor. We consider electromagnetic fields described by real vector functions —notice, however, that in the discussion of LSR spacetimes in [24,36,26,29,30] it is often natural to make use of the complex variables $\psi = E + iB$ and $\psi^* = E - iB$ to decouple the equations with the appropriate symmetries and obtain linear equations in the fields. The resulting evolution system is then used to analyze the stability problem for small non-linear perturbations of a background solution. More precisely, we perform a first order perturbation to $\mathbf{v}$ of the form $\mathbf{v} \mapsto \varepsilon\mathring{\mathbf{v}} + \check{\mathbf{v}}$, where the parameter ε sets the order of the perturbation while $\check{\mathbf{v}}$ describes the (linear) perturbation of the background solution. Now, assuming the background variables $\mathring{\mathbf{v}}$ to satisfy the evolutions we end up with an evolution system for the perturbations of the form $\mathring{\mathbf{A}}^t\partial_t\check{\mathbf{v}} - \mathring{\mathbf{A}}^j\partial_j\check{\mathbf{v}} = \mathring{\mathbf{B}}\check{\mathbf{v}}$. The core of the stability analysis consists of the study of the term $\mathring{\mathbf{B}}$ using some relaxed stability eigenvalue conditions. The procedure to analyse stability used here is adapted from [38] —see also discussion in [39] and cited references. Under suitable circumstances it can be regarded as a first step toward the analysis of *non-linear* stability. In our case, the elements of the matrix $\mathring{\mathbf{B}}$ are, in general, functions of the space and time coordinates. For a general discussion on the time dependent case and the case of non-constant matrix coefficients (depending on both time and space) we refer to [38]. The case of a linearized system where the coefficients are constant matrices is discussed in [40]. Finally, the case of systems with vanishing eigenvalues has been discussed in [41,42,43]. In our case, a full analysis of the stability properties of the system turns out extremely cumbersome because of the form of the matrix $\mathring{\mathbf{B}}$ associated to the present problem. Thus, we analyse of the signs of the eigenvalues using indirect methods based on the inspection of the characteristic polynomial. In order to keep the problem manageable, we analyze a number of simplified systems obtained by making some assumptions about the configuration. We provide a detailed classification by considering systems with particular kinematic configurations.

The present article is structured as follows: variables and equations are introduced in Section 2. Section 3 discusses the first order perturbations of the variables. Section 3.1 provides some general remarks on the set of perturbed equations and the stability of the system. Section 4 contains the main results concerning the linear stability of the symmetric hyperbolic system. Some concluding remarks are given in Section 5. Further details on the subclasses of the solutions are in Appendix A. Finally in Appendix B we provide some remarks concerning the thermodynamic quantities.

2 Variables and equations

In the sequel we consider the stability problem for a configuration described by the Einstein-Maxwell-Euler equations for a perfect fluid with a preferred radial direction. By applying a $1+1+2$ decomposition we can take full advantage of the symmetries of the LRS system. This procedure allows to construct in an easy and relatively immediate manner a quasi-linear hyperbolic system. This property ensures also the consistency of possible numerical approaches without requiring the introduction of any auxiliary variable to handle both the propagation equations along the timelike direction and the constraint part of the system. Moreover, the covariant and gauge-invariant perturbation formalisms are especially suitable for dealing with spacetimes with some preferred spatial direction by introducing a radial unit vector and decomposing all covariant quantities with respect to the latter [17,24,44,12]. All the equations and the quantities related to the fields and the curvature tensors are decomposed according to the $1+1+2$ procedure.

The implementation of the $1+3$-formalism used in the present article follows, as much as possible, the notation and conventions of [25] while for the $1+1+2$-formalism we refer to [17]. We consider 4-dimensional metrics g_{ab} with signature $(-,+,+,+)$. The Latin indices $a,b,c...$ will denote spacetime tensorial indices taking the values $(0,1,2,3)$ while $i,j,k...$ will correspond to spatial frame indices ranging over $(1,2,3)$. The Levi-Civita covariant derivative of g_{ab} will be denoted by ∇_a where $\nabla_a g_{bc} = g_{bc;a} = 0$. The timelike vector field (*flow vector*) u^a will describe the normalised future directed 4-velocity of the fluid, $u^a u_a = -1$. Indices are raised and lowered with g_{ab}. The tensor $h^{ab} \equiv g^{ab} + u^a u^b$ is the projector onto the three dimensional subspace orthogonal to u^a, and n_a denotes a spacelike normalized vector chosen along a preferred direction of the spacetime. Therefore, one has that $n_a n^a = 1$, $u_a n^a = 0$ and a further projection tensor $N_a^b \equiv h_a^b - n_a n^b = \delta_a^b + u_a u^b - n_a n^b$. In general, $\dot{\mathbf{Q}} \equiv u^b \nabla_b \mathbf{Q}$ for any quantity $\mathbf{Q}$, while $\hat{\phi}_{a\cdots b}{}^{c\cdots d} \equiv n^e D_e \phi_{a\cdots b}{}^{c\cdots d}$ for a tensor ϕ, where the operator D_a corresponds to the 3-dimensional covariant derivative obtained from projecting the spacetime covariant derivative in the distribution orthogonal to u_b. Thus, for a generic 2-rank tensor T_{bc} one has that $D_a T_{bc} = h^s{}_a h^t{}_b h^p{}_c T_{st;p}$. We perform the projection of any quantity along the radial direction n^a. Our analysis leads to a symmetric hyperbolic system consisting of eleven evolution equations for eleven scalar variables of the form $\mathbf{A}^t \partial_t \mathbf{v} - \mathbf{A}^j \partial_j \mathbf{v} = \mathbf{B}\mathbf{v}$, for a vector-valued unknown with components given by

$$\mathbf{v} \equiv \big(\rho, E, B, \mathscr{E}, \mathscr{B}, Q, T, \xi, \Phi, \Omega, \mathscr{A}\big), \tag{1}$$

where $E^a n_a = E$ and $B^a n_a = B$ with (E^a, B^b) the electric and magnetic fields, J^a is the 4–current and $j = J^a n_a$. The scalars $\mathscr{E}$ and $\mathscr{B}$ denote, respectively, the totally projected electric and magnetic parts of the Weyl curvature tensor of the metric g_{ab}. We introduce the variables Q and T via the relations $Q \equiv T + 3\Sigma$, $T \equiv \frac{2}{3}\Theta - \Sigma$ so that $\Sigma = \frac{1}{3}(Q-T)$, $\Theta = \frac{3}{2}(T+\Sigma)$, where the scalar Σ is the totally projected part of the *shear of the 3-sheet* and Θ is the *volume expansion scalar*. The scalar Φ denotes is the *expansion* of the 2–sheet generated by n_a, and ξ is the *rotation* of n^a (i.e. the *twist* of the 2–sheet). Then, $\mathscr{A} \equiv n^a u^b \nabla_b u_a$ and Ω describe the components of the acceleration and the vorticity in the direction of n^a, respectively. Finally, ρ is the matter density and p is the pressure. The evolution equations are

$$\dot{\rho} - Ej + \frac{3}{2}(p+\rho)\Big(\frac{1}{3}(Q-T) + T\Big) = 0, \tag{2}$$

$$\dot{E} - 2B\xi + ET + j = 0, \tag{3}$$

$$\dot{B} + BT + 2E\xi = 0, \tag{4}$$

$$\dot{\mathscr{E}} - 3\mathscr{B}\xi + \frac{1}{2}(B^2+E^2)T + \frac{3}{2}\mathscr{E}T + \frac{2}{3}Ej + \frac{1}{6}(p+\rho)(Q-T) = 0,$$

$$\dot{\mathscr{B}} + \frac{3}{2}\mathscr{B}T + 3\mathscr{E}\xi + (B^2+E^2)\xi = 0, \tag{5}$$

$$\dot{\Phi} - \mathscr{A}T - 2\xi\Omega + \frac{1}{2}T\Phi = 0, \tag{6}$$

$$\dot{\xi} - \mathscr{A}\Omega + \frac{1}{2}(\xi T + \Phi\Omega - \mathscr{B}) = 0, \tag{7}$$

$$\dot{\Omega} - \mathscr{A}\xi + T\Omega = 0. \tag{8}$$

Now, assuming $p = p(\rho)$ and introducing the square of the velocity of sound $v_s \equiv \partial p/\partial\rho$ and the charge density $\tilde{\rho}_C$, the evolution equations for T, Q and $\mathscr{A}$ are

$$\dot{T} - \mathscr{A}\Phi - \mathscr{E} + p + \frac{1}{3}\rho + \frac{1}{2}T^2 - 2\Omega^2 = 0, \tag{9}$$

$$\dot{Q} - 2\hat{\mathscr{A}} - 2\mathscr{A}^2 + B^2 + E^2 + 2\mathscr{E} + p + \frac{1}{3}\rho + \frac{1}{2}Q^2 = 0, \tag{10}$$

$$\begin{aligned}\frac{4\dot{\mathscr{A}}}{v_s} - 2\hat{Q} + \frac{4E\hat{j}}{p+\rho} - \frac{4E\dot{\tilde{\rho}}_C}{v_s(p+\rho)} - \frac{2}{v_s(p+\rho)}\Big(&\mathscr{A}\left[Q(v_s-1)+2Tv_s\right](p+\rho) \\ &-2\mathscr{A}Ej(1+2v_s) + \tilde{\rho}_C\left[EQ(2+v_s)+2(ETv_s+2B\xi)\right] \\ &+v_s(p+\rho)\left[\Phi(Q-T)+4\xi\Omega\right] - 2j\left[\tilde{\rho}_C(1+v_s)+v_s(2B\Omega-E\Phi)\right]\Big) = 0.\end{aligned} \tag{11}$$

Moreover, one finds that

$$\mathscr{B} - 3\mathscr{A}\xi + (2\mathscr{A}-\Phi)\Omega = 0.$$

In equation (11) the equation of state and the constraint equation have been used to obtain a simpler expression. In fact, the fields p and $\tilde{\rho}_C$ can be regarded as functions of ρ, while the charge current j_c is regarded as a function of the electric field E. Consistent with *Ohm's law*, we assume a linear relation between the conduction current j^a and the electric field. More precisely, we set $j^a = \sigma^{ab}E_b$, where σ^{ab} denotes the *conductivity* of the fluid (plasma). We will restrict our attention to isotropic fluids for which $\sigma^{ab} = \sigma_J g^{ab}$, so that $J^a = \tilde{\rho}_C u^a + \sigma_J E^a$, with σ_J the *electrical conductivity coefficient*. In terms of a $1+1+2$-decomposition one has $j^a = E\sigma_J n^a$. The system (2)-(8) is complemented by the constraint equations for the components of the vector-valued function $\mathbf{v}$. However, as the primary aim of this work is the study of the time evolution and the analysis of stability, the discussion of the constraint equations will remain in the background. In what follows, it is assumed that the constraint equations are satisfied at all time. This assumption can be removed by fairly general arguments –see [37,38].

3 Perturbations

We will perform a perturbation to first order of the variables $\mathbf{v}$ so that $\mathbf{v} \mapsto \mathring{\mathbf{v}} + \varepsilon\check{\mathbf{v}}$ with the parameter ε controlling the size of perturbation and $\check{\mathbf{v}}$ describing the perturbation of the background solution $\mathring{\mathbf{v}}$. Thus, assuming that the background variables $\mathring{\mathbf{v}}$ satisfy the evolution equations, and defining $(\mathrm{d}\mathbf{v}^{(a)}/\mathrm{d}\varepsilon)_{\varepsilon=0} \equiv \check{\mathbf{v}}^{(a)}$, we can obtain from equations (2)-(8) the following equations for the linear perturbations:

$$
\begin{aligned}
&\dot{\check{\rho}}-\left\{\check{j}\mathring{E}+\check{E}\mathring{j}-\frac{1}{2}\left[(\check{p}+\check{\rho})(\mathring{Q}+2\mathring{T})+(\check{Q}+2\check{T})(\mathring{p}+\mathring{\rho})\right]\right\}=0,\\
&\dot{\check{E}}+\check{j}-2\check{\xi}\mathring{B}+\check{T}\mathring{E}+\check{E}\mathring{T}-2\check{B}\mathring{\xi}=0,\quad \dot{\check{B}}+\check{T}\mathring{B}+2\check{\xi}\mathring{E}+\check{B}\mathring{T}+2\check{E}\mathring{\xi}=0,\\
&\dot{\check{\mathscr{E}}}-3\check{\xi}\mathring{\mathscr{B}}+\frac{1}{6}\check{T}\left[3(\mathring{B}^2+\mathring{E}^2+3\mathring{\mathscr{E}})-(\mathring{p}+\mathring{\rho})\right]+\frac{2}{3}\check{j}\mathring{E}+\frac{\check{E}(2\mathring{j}+3\mathring{E}\mathring{T})}{3}+\frac{\check{Q}(\mathring{p}+\mathring{\rho})}{6}\\
&\qquad\qquad+\frac{(\check{p}+\check{\rho})}{6}(\mathring{Q}-\mathring{T})+\frac{3}{2}\check{\mathscr{E}}\mathring{T}-3\check{\mathscr{B}}\mathring{\xi}+\check{B}\mathring{B}\mathring{T}=0, \qquad (12)\\
&\dot{\check{\mathscr{B}}}+\frac{3}{2}\check{T}\mathring{\mathscr{B}}+\check{\xi}(\mathring{B}^2+3\mathring{\mathscr{E}})+\frac{3}{2}\check{\mathscr{B}}\mathring{T}+3\check{\mathscr{E}}\mathring{\xi}+2\check{B}\mathring{B}\mathring{\xi}=0,\\
&\dot{\check{\Phi}}-\check{T}(\mathring{\mathscr{A}}-\frac{1}{2}\mathring{\Phi})-\check{\mathscr{A}}\mathring{T}+\frac{1}{2}\check{\Phi}\mathring{T}-2\check{\Omega}\mathring{\xi}-2\check{\xi}\mathring{\Omega}=0, \qquad (13)\\
&\dot{\check{\xi}}-\frac{1}{2}\check{\mathscr{B}}-\check{\Omega}(\mathring{\mathscr{A}}-\frac{1}{2}\mathring{\Phi})+\frac{1}{2}\check{\xi}\mathring{T}+\frac{1}{2}\check{T}\mathring{\xi}-\check{\mathscr{A}}\mathring{\Omega}+\frac{1}{2}\check{\Phi}\mathring{\Omega}=0,\\
&\dot{\check{\Omega}}-\check{\xi}\mathring{\mathscr{A}}+\check{\Omega}\mathring{T}-\check{\mathscr{A}}\mathring{\xi}+\check{T}\mathring{\Omega}=0. \qquad (14)
\end{aligned}
$$

From the evolution of the variables T and Q we obtain

$$
\begin{aligned}
&\dot{\check{T}}-\check{\mathscr{E}}+\check{p}+\frac{1}{3}\check{\rho}-\check{\Phi}\mathring{\mathscr{A}}+\check{T}\mathring{T}-\check{\mathscr{A}}\mathring{\Phi}-4\check{\Omega}\mathring{\Omega}=0,\\
&\dot{\check{Q}}-2\check{\mathscr{A}}+2\check{\mathscr{E}}+\check{p}+\frac{1}{3}\check{\rho}-4\check{\mathscr{A}}\mathring{\mathscr{A}}+2\check{B}\mathring{B}+2\check{E}\mathring{E}+\check{Q}\mathring{Q}=0.
\end{aligned}
$$

The equation for the radial acceleration can be recovered after some further calculations as a complicated linearized equation for the perturbation of $\mathscr{A}$. Thus, in order to undertake a stability analysis we make some further assumptions about the configuration so as to obtain a simplified system. First, it is worth pointing out that the derivative along the radial direction of the acceleration $\mathscr{A}$ appears in the evolution equations for Θ and Σ in Eq. (9). Equations (2)-(8) do not contain explicit dependence on the derivative of the matter density along the radial direction. Thus, in order to simplify the stability analysis and to be able to extract some information from the rather complicated perturbed equations we will focus on the vanishing radial acceleration case. Moreover, we assume that $\partial_a v_s = 0$–see Section B. Accordingly, we consider the simplified vector-valued unknown $\mathbf{v} \equiv (\rho, E, B, \mathscr{E}, \mathscr{B}, Q, T, \xi, \Phi, \Omega)$.

3.1 Remarks on the set of equations

We consider the set of linearized evolution equations in the form $\mathring{\mathbf{A}}^t\partial_t\check{\mathbf{v}}-\mathring{\mathbf{A}}^j\partial_j\check{\mathbf{v}}=\mathring{\mathbf{B}}\check{\mathbf{v}}$. General theory of linear of partial differential equations shows that systems of this form can either converge (in an oscillating manner or exponentially) to constant values, have an asymptotic exponential (oscillating) decay or are unstable —see e.g. [40]. In the rest of this article, we look at solutions in which the perturbations stabilize on constant values or decay. In order to study the asymptotical behavior of the characteristic solutions of perturbed system it is necessary the study of the eigenvalues of the matrix $\mathring{\mathbf{B}}$ as $t \to \infty$. Notice that the entries in the matrix-valued function $\mathring{\mathbf{B}}$ are, in general, functions of the coordinates. Under these circumstances, the core of the stability analysis consists on an investigation on whether the matrix $\mathring{\mathbf{B}}$ satisfies some appropriate *relaxed stability eigenvalue conditions*. More precisely, we study the eigenvalue problem of $\mathring{\mathbf{B}}$ by looking at the sign of the real parts of the corresponding eigenvalues λ_i (or *modes of the perturbed system*). A similar approach has been used in [38] —see also the discussion in [39]. General theory concerning the case where the coefficients of the linearized system are constant is discussed in [40]. The case of systems with vanishing eigenvalues has been discussed in [41,42,43] while for a general discussion of the time dependent case and the case of non constant matrix coefficient see [38]. Due to the form of the matrix $\mathring{\mathbf{B}}$ associated to the present problem, a full analysis of the stability properties of this system is extremely cumbersome. We proceed by analyzing the sign of the eigenvalues using the fact that a sufficient condition for the *instability* of the system is the existence of at least one eigenvalue with positive real part. For a more extended discussion concerning the requirements of the so-called of the relaxed stability eigenvalue condition see [41,43]. This simple requirement provides an immediate way to obtain conditions where instability occurs —namely, if there is at least one i such that $Re(\lambda_i) > 0$. Crucially, in the discussion of stability one does not need explicitly to compute the eigenvalues of $\mathring{\mathbf{B}}$ —the sign of the real part of the eigenvalues can be determined by inspection the structure of the matrix $\mathring{\mathbf{B}}$. Given the *characteristic polynomial* $\mathfrak{P}(\mathring{\mathbf{B}})(x) = \sum_{i=0}^{n} c_i x^i$ of the matrix $\mathring{\mathbf{B}}$, we will make use of the following well-known results on the roots sign: the trace of a matrix is equal to the sum of its eigenvalues, hence if the trace of the matrix is positive then the system is *unstable*. Now, recall that the determinant of a matrix is the product of the eigenvalues. Accordingly, a *necessary condition* for *stability* is that $\det \mathring{\mathbf{B}} \neq 0$ and $(-1)^n \det \mathring{\mathbf{B}} > 0$ where n is the number of distinct eigenvalues. Other stability criteria based on inspection of the characteristic polynomial are the *Routh-Hurwitz criterion* and the *Liénard-Chipart theorem* —see [39,45,46]. Wherever possible, we will also make use of the Descartes criterion to determine the *maximum* number of positive and negative real roots of the polynomial $\mathfrak{P}(\mathring{\mathbf{B}})(x)$ for $c_i \in \mathbb{R}$. In particularly simple cases one can exploit a generalisation of the Descartes rule considering the Routh-Hurwitz criterion to determine the number of roots with positive and negative real part of $\mathfrak{P}(\mathring{\mathbf{B}})(x)$ by constructing the associated *Routh matrix*. In the case under consideration, the matrix $\mathring{\mathbf{B}}$ has eleven columns and the coefficients of the characteristic polynomial are, unfortunately, large expressions without an obvious structure. Therefore, we approach the analysis of the stability problem by making the following assumptions:

$$\mathring{j} = \mathring{E}\sigma_J, \qquad \breve{j} = \breve{E}\sigma_J, \qquad \mathring{p} = \mathring{\rho} v_s, \qquad \breve{p} = \breve{\rho} v_s.$$

Thus, we take the square of the sound velocity $v_s = v_s^2$ to be constant and assume *Ohm's law* $j = E\sigma_J$ with $\mathring{\sigma}_j \equiv \breve{\sigma}_j$ and restrict our attention to the case of null radial acceleration —i.e. $\mathscr{A} = 0$. This assumption leads to significant simplifications in the analysis. It is, however, physically quite restrictive as such fluids are not normally encountered in astrophysical scenarios, although they are possible, for example, in some early cosmological scenarios —see also discussion in the Appendix B.

4 Discussion on the stability of the symmetric hyperbolic system

In this Section we discuss necessary conditions for the stability of the configuration under consideration. We focus on the reference solution described by the matrix $\mathring{\mathbf{B}}$ in (1). We present the analysis of the stability for some special configurations defined by proper assumptions on the kinematic variables given by the $1+1+2$-decomposition. Each case is then identified according to the assumptions that characterize it —the fields assumed to vanish are indicated in parenthesis. We discuss in detail the *instability* condition for the complete list of cases. Assuming the radial acceleration and other kinematic variables to be zero,

Table 1 The 10×10-matrix $\mathring{\mathbf{B}}$.

$\frac{-(1+v_s)\left(\mathring{Q}+2\mathring{T}\right)}{2}$	$2\sigma_J \mathring{E}$	0	0	0	$\frac{-(1+v_s)\mathring{\rho}}{2}$	$-(1+v_s)\mathring{\rho}$	0	0	0
0	$-\sigma_J - \mathring{T}$	$2\mathring{\xi}$	0	0	0	$-\mathring{E}$	$2\mathring{B}$	0	0
0	$-2\mathring{\xi}$	$-\mathring{T}$	0	0	0	$-\mathring{B}$	$-2\mathring{E}$	0	0
$\frac{(1+v_s)\left(\mathring{T}-\mathring{Q}\right)}{6}$	$\frac{-\mathring{E}\left(4\sigma_J+3\mathring{T}\right)}{3}$	$-\mathring{B}\mathring{T}$	$\frac{-3\mathring{T}}{2}$	$3\mathring{\xi}$	$\frac{-(1+v_s)\mathring{\rho}}{6}$	$\frac{\left[(1+v_s)\mathring{\rho}-3\left(\mathring{B}^2+3\mathring{\mathscr{E}}+\mathring{E}^2\right)\right]}{6}$	$3\mathring{\mathscr{B}}$	0	0
0	$-2\mathring{E}\mathring{\xi}$	$-2\mathring{B}\mathring{\xi}$	$-3\mathring{\xi}$	$\frac{-3\mathring{T}}{2}$	0	$\frac{-3\mathring{\mathscr{B}}}{2}$	$-\mathring{B}^2 - 3\mathring{\mathscr{E}} - \mathring{E}^2$	0	0
$-\frac{1}{3} - v_s$	$-2\mathring{E}$	$-2\mathring{B}$	-2	0	$-\mathring{Q}$	0	0	0	0
$-\frac{1}{3} - v_s$	0	0	1	0	0	$-\mathring{T}$	0	0	$4\mathring{\Omega}$
0	0	0	0	$\frac{1}{2}$	0	$-\frac{\mathring{\xi}}{2}$	$-\frac{\mathring{T}}{2}$	$-\frac{\mathring{\Omega}}{2}$	$-\frac{\mathring{\Phi}}{2}$
0	0	0	0	0	0	$-\frac{\mathring{\Phi}}{2}$	$2\mathring{\Omega}$	$-\frac{\mathring{T}}{2}$	$2\mathring{\xi}$
0	0	0	0	0	0	$-\mathring{\Omega}$	0	0	$-\mathring{T}$

we explore the implications of the assumption on the configuration and its stability. A first insight into the stability of the system can be inferred from the trace of the matrix expressed in terms of Σ and Θ as $\mathrm{Tr}\,\mathring{\mathbf{B}} = 6\mathring{\Sigma} - (7+v_s)\mathring{\Theta} - \sigma_J$. It readily follows that $\mathrm{Tr}\,\mathring{\mathbf{B}} > 0$ so that the system is *not stable* if

$$\Sigma > \frac{(7+v_s)\Theta + \sigma_J}{6}. \tag{15}$$

In the remainder of this article we proceed to a more detailed analysis in which we classify the results according to suitable assumptions on the remaining kinematic variables $(T, Q, \Phi, \xi, \Omega)$. It should be pointed out, for ease of reference, that the condition $T = 0$ is equivalent to the relation $\frac{2}{3}\Theta = \Sigma$ between the expansion Θ of the 3-sheets and the radial part of the shear of the 3-sheet Σ. Similarly, the condition $Q = 0$ is equivalent to $\frac{1}{3}\Theta = -\Sigma$, while the two conditions $T = 0$ and $Q = 0$ imply $\Sigma = 0$ and $\Theta = 0$. We also notice from the Maxwell equations that the evolution of the electric (respectively, magnetic) field is coupled to the magnetic (respectively, electric) field via the twisting of the 2-sheet. Hence, when $\xi = 0$ the two evolution equations decouple and evolve with the only common dependence on T.

4.1 Preliminaries on the symmetries of the system and classes of solutions

The configuration expands or contract along the radial direction according to whether $\Theta > 0$ or $\Theta < 0$, but it cannot accelerate as it is assumed that the parallel acceleration vanishes —i.e. $\mathscr{A} = 0$. The limit situation of vanishing radial acceleration has been adopted to simplify the analysis of the unstable modes. In this scheme we are able to provide a complete classification of the stable and unstable solutions for the Einstein-Maxwell-Euler system, considering appropriate restrictions on the set of the perturbed equations. All the kinematic quantities and fields are reduced to scalars by the projections in the radial direction and on the orthogonal plane by the projector N_{ab}. The set of scalars define the vector variable $\mathbf{v}$ introduced in Eq. (1). Any further restrictions on the system such as the vanishing of other dynamical variables $Q_i \in \overrightarrow{\mathcal{V}}$, $\overrightarrow{\mathcal{V}} \equiv \{T, Q, \Phi, \xi, \Omega\} \subset \mathbf{v}$ leads to particular different solutions of the Einstein-Maxwell-Euler equations with the new symmetry conditions. The self-gravitating systems will be especially constrained by the pair $(\mathscr{E}, \mathscr{B})$. In gravitating systems the background geometry is assumed to be in one of the classes of solutions including the new symmetries. The vector $\overrightarrow{\mathcal{V}}$ is a restriction of the vector variable $\mathbf{v}$, where the fluid four-velocity u_a defines the metric in its $3+1$ form and supplies the projected components of the Weyl tensor along the radial direction (ξ,Φ) and the variation of the radial direction n_a used to construct the metric tensor in its $2+1+1$ form. The vanishing of the elements in Q_i implies further restrictions involving the annihilation of other quantities, or the constancy of $Q_j \neq Q_i$ during the evolution along u_a. We can then refer to the general conditions $\mathfrak{C}$:

$$\mathfrak{C} \quad (Q_i=0): \qquad (E^2,B^2,\mathscr{E},\Omega)\mapsto\left(p+\frac{\rho}{3}\right), \tag{16}$$

$$(Q_j,\mathscr{B})=0, \tag{17}$$

$$\dot{Q}_j=0. \tag{18}$$

The conditions in Eqs. (17), specified in Table 2, define five principal classes of solutions according to the assumption of vanishing $Q_i\in\overrightarrow{V}$. Table 3 shows the sub-classes of solutions constructed by the vanishing of a couple of scalars (Q_i,Q_j). The conditions in (16) are listed in Table 4 and state the relation between the remaining field variables of the vector $\overrightarrow{\mathbf{V}}$ i.e. the pair $(\mathscr{E},\mathscr{B})$, the electromagnetic fields (E,B), and the matter density ρ. Finally, condition (18) is made explicit in Table 5. Tables 2, 3, 4 and 5 characterize entirely the system throughout its subconfigurations, thus providing a complete classification of the solutions. An analysis and a general discussion of the solutions of the Einstein-Maxwell system in terms of the magnetic and electric parts of the Weyl tensor can be found in [35]. The stability analysis is performed on the systems with $\mathscr{A}=0$, on the five classes of configurations and their sub-classes. The $\mathfrak{I}$-class with $(\mathscr{A}\mathbf{T})$ and $\mathfrak{I}\mathfrak{I}$-class with $(\mathscr{A}\mathbf{Q})$ are particularly significant. Systems with $T=0$ are characterized by $\Theta\Sigma>0$, i.e. the sign of the expansion is concordant with that of the radial shear —positive for expanding systems along the radial direction, or negative for contracting systems, see also Figure 1. On the other hand, systems with $Q=0$ correspond to the case $\Theta\Sigma<0$. The relative sign of the scalars of the radial shear and expansion is a significant element affecting the stability of the system and the equilibrium configurations: even in the general case where the only assumption on the system is the vanishing radial acceleration $\mathscr{A}=0$, the balance of the contributions given by the radial shear in the systems in expansion $(\Theta>0)$ or in contraction $(\Theta<0)$ is relevant in determining the states certainly unstable. For contracting systems, or those in expansion but with positive shear, a threshold on the expansion rate Θ or equivalently the radial shear, appears for the emergence of the instability. This is a function of the two parameters (σ_J,v_s). The magnetic field, although not constant in time along the radial direction has no specific role to establish the stability of this model. It is possible to show that for the case $(\mathscr{A}\mathbf{T})$ there is always a threshold for the emergence of unstable phases, while in the case of the expanding systems with negative shear, the difference in sign between the two scalar does not involve any instability threshold. The conditions for the equilibrium of these solutions involve quantities $Q_i\in v$ exclusively related to the fluid dynamics such as $\mathscr{A}T\xi$ or $\mathscr{A}\xi\Omega$ and imply a serious constraint on the background. Table 4 shows the conditions $\mathfrak{C}$ in Eqs. (16): only the electric part of the Weyl tensor is determined by the matter fields and the vorticity. In general, the classes $(\mathscr{A}\mathbf{T})$ and $(\mathscr{A}\mathbf{Q})$ do not imply the vanishing of others variables by conditions $\mathfrak{C}$ in Eqs. (17): that is, the systems do not require any additional symmetry as the initial data $Q_j=0$ for the solutions (16). Remarkably, the $(\mathscr{A}\mathbf{QT})$ class corresponds to the case of null shear and null radial expansion. Tables 2 and 3 show that the pairs (ξ,Ω) and (ξ,Φ) are related. The magnetic field must be constant along the fluid flow for the solution $(\mathscr{A},T,\xi)$. Conversely, one has $\mathscr{B}=0$ whenever $\xi=0$. The solutions $T=0$ have radial vorticity Ω constant during the evolution of the system as shown in Table 5. If the radial vorticity is initially zero, then the expansion Φ of the 2-sheet generated by n_a is constant along u_a and the electric part of the Weyl tensor is entirely determined by the matter field ρ.

Table 2 Vanishing radial acceleration $\mathscr{A}$: the five principal classes of solutions for the linear stability the problem, identified by setting to zero the radial acceleration and a scalar quantity of the set $Q_i \in \nu = \{T, Q, \Phi, \xi, \Omega\}$ and $\mathscr{B}$, the magnetic part of the Weyl tensor. The elements of the table make explicit the conditions $\mathfrak{C}$ in (17). Thus, for example in the $\mathfrak{I}\mathfrak{I}\mathfrak{I}-$class, the conditions of the $(\mathscr{A}\Phi)$ configurations imply a vanishing radial vorticity Ω or the vanishing of the pair ξ and $\mathscr{B}$. Then we assumed that the system has a further constraint represented by a third zero quantity $Q_i \in \nu$ defining the subclasses. The check marks indicate the cases already treated, the table exhausts the five classes $(\mathscr{A}, \mathbf{Q_i})$ and the subclasses $(\mathscr{A}, Q_i, Q_j)$. Particularly the $\mathfrak{I}\mathfrak{I}\mathfrak{I}-$class $(\mathscr{A}\phi)$ shows the symmetry between the solutions at zeros $(\mathscr{A}\mathbf{Q})$, $(\mathscr{A}\mathbf{T})$ and $\{\xi\Omega\mathscr{B}\}$. The subclasses $(\mathscr{A}, Q_i, Q_j, Q_k)$ are summarised in Table 3.

	$\mathscr{A}$	$\mathscr{A}T$	$\mathscr{A}Q$	$\mathscr{A}\xi$
$\mathfrak{I}$-class	$\mathscr{A}T$	✓	✓	✓
$\mathfrak{I}\mathfrak{I}$-class	$\mathscr{A}Q$	$\mathscr{A}TQ$	✓	✓
$\mathfrak{I}\mathfrak{I}\mathfrak{I}$-class	$\mathscr{A}\Phi = \mathscr{A}\Phi\Omega$ $\mathscr{A}\Phi\xi\mathscr{B}$	$\mathscr{A}T\Phi = T + \mathscr{A}\Phi\Omega$ $T + \mathscr{A}\Phi\xi\mathscr{B}$	$\mathscr{A}Q\Phi = Q + \mathscr{A}\Phi\Omega$ $Q + \mathscr{A}\Phi\xi\mathscr{B}$	✓
$\mathfrak{I}\mathfrak{V}$-class	$\mathscr{A}\xi$	$\mathscr{A}T\xi = T + \mathscr{A}\Phi\xi\mathscr{B}$	$\mathscr{A}Q\xi$	✓
$\mathfrak{V}$-class	$\mathscr{A}\Omega$	$\mathscr{A}T\Omega$	$\mathscr{A}Q\Omega$	$\mathscr{A}\xi\Omega = \mathscr{A}\xi\Omega\mathscr{B}$

Table 3 Subclasses $(\mathscr{A}, Q_i, Q_j)$ of the five principal classes $\mathfrak{I} - \mathfrak{V}$ in Table (2). The table highlights the role of the vanishing pairs (TQ) and $\xi\Omega\mathscr{B}$.

$\mathscr{A}TQ$	$\mathscr{A}T\xi$	$\mathscr{A}Q\Phi$	$\mathscr{A}\Phi\xi$
$\mathscr{A}TQ\Phi = Q + T + \mathscr{A}\Phi\xi\mathscr{B}$	✓	✓	✓
$Q + T + \mathscr{A}\Phi\Omega$	✓	✓	✓
$\mathscr{A}TQ\xi$	$\mathscr{A}TQ$	✓	✓
$\mathscr{A}TQ\Omega$	$\mathscr{A}T\xi\Omega = T + \mathscr{A}\xi\Omega\mathscr{B}$	$\mathscr{A}Q\xi\Omega = Q + A\xi\Omega\mathscr{B}$	$\mathscr{A}\Phi\xi\Omega = \Phi + \xi\Omega\mathscr{B}$
$\mathscr{A}TQ\xi\Omega = T + Q + \mathscr{A}\xi\Omega\mathscr{B}$			

If initially $\Phi = 0$ then $\xi = 0$ and therefore $\mathscr{B} = 0$, providing a non-vacuum solution with a vanishing magnetic component of the Weyl tensor —see Table 4.

4.2 Analysis of the system stability, general conditions and results

The symmetries of the system and the adapted $1+1+2$ formalism highlight the emergence of unstable states under linear (and non-linear) perturbations. The unstable phases of the system with $\mathscr{A} = 0$ are primarily regulated by the expansion (or the contraction) and the shear in the radial direction. These scalars are related through the velocity of sound and the conductivity through the relation:

$$\mathfrak{F}: \quad (\Sigma, \Theta) \mapsto (v_s, \sigma_J, \rho), \tag{19}$$

$$\rho \mapsto (v_s, \sigma_J). \tag{20}$$

Conditions $\mathscr{C}$ in Eqs. (16,17,18) describe the symmetries of the system but not its stability. Conversely, the relations $\mathfrak{F}$ in Eqs. (19) relate the only kinematic variables of the radial shear and radial expansion, Σ and Θ,respectively, to the constants of the state and constitutive equations: the velocity of sound v_s and the conductivity σ_J. We emphasize that Eq. (19) does not involve other dynamical variables such as the Weyl scalar or the electromagnetic field. Analogously, for the five classes of solutions and their subclasses according

Table 4 Conditions $\mathfrak{C}$ as in Eq. (16) and Eq. (17).

$\mathfrak{C}$	
$\mathscr{C}(\mathscr{A},T)$:	$\mathscr{E}+2\Omega^2=p+\frac{\rho}{3}>0$
$\mathscr{C}(\mathscr{A},Q)$:	$E^2+B^2+2\mathscr{E}=-(p+\rho/3)<0$
$\mathscr{C}(\mathscr{A},\xi)$:	$\mathscr{B}=\Phi\Omega$
$\mathscr{C}(\mathscr{A},T,\Omega)$:	$\mathscr{E}=p+\rho/3>0$
$\mathscr{C}(\mathscr{A}\,\xi,\Omega)$:	$\mathscr{B}=0$

Table 5 Condition $\mathfrak{C}$ in Eq. (18). Classes of solutions and null evolution of the kinematic quantities.

Class	
$(\mathscr{A},T)$:	$\dot{\Omega}=0$
$(\mathscr{A},T,\xi)$:	$\dot{\Omega}=\dot{\Phi}=\dot{\mathscr{B}}=\dot{B}=0$
$(\mathscr{A},T,\Omega)$:	$\dot{\Phi}=0$

to Eqs. (16,17,18) the relation in (20) provides an upper bound on the matter density as a limiting value determined by the pair (v_s,σ_J). The density, but not its gradient, is in fact a variable in the system and, since this is an iso-entropic and barotropic fluid, one can use this to get information on the hydrostatic pressure the system is subjected to.

4.2.1 Discussion on the stability of the system with vanishing acceleration

A relevant constraint $\mathfrak{F}$ on the stability of the system with vanishing radial acceleration is provided by condition (15) on the trace $\mathrm{Tr}\,\mathring{\mathbf{B}}$. It can be expressed in terms of the expansion along the privileged direction as follows:

$$(\mathbf{u})_{\mathbf{0}}:\quad \Theta<\Theta_u\equiv\frac{6\Sigma-\sigma_J}{\kappa},\quad \kappa\equiv 7+v_s>1. \tag{21}$$

This is a first condition for the instability of the system providing an upper bound —i.e. a threshold for the occurrence of the instability— on the expansion or contraction with respect to the shear projected along the radial direction. For small or zero conductivity the limit Θ_u is a fraction of the positive shear, where the system cannot contract or an instability occurs. For negative shear the contraction is limited by the value $\Theta_u<0$. If the contraction rate is too large, i.e. $\Theta\in(\Theta_u,0)$, the system is unstable according to Eq. (21). We summarize the analysis in the following points:

Systems in contraction $\Theta<0$ and in expansion $\Theta>0$. The *expanding* configurations, described in the **I** and **IV** quadrant of Figure 1, are favored in their stable phases with respect to the systems in contraction. The preference for the stable expansion is especially evident for negative shear, as in the **IV** quadrant which describes also a part of the $(\mathscr{A},Q)$ systems. In the **IV** quadrant the stability conditions are always guaranteed within the condition (21) as $\inf\sigma_J>0$. The condition of the trace in Eq. (21) is not sufficient to establish the emergence of the instability for this system. By contrast, a situation with $\Theta\Sigma<0$ appears to favour the equilibrium. This seems to be confirmed by the situation in the **I** quadrant, with positive shear, describing parts of the $(\mathscr{A},T)$ systems. For slow expansions $\dot{\Theta}\approx 0$ and small conductivity, where the system is unstable, a threshold for the instability appears depending on the conductivity and can be easily seen by Eq. (21). Further restrictions on the symmetries of the system, in the subclasses of Tables 2 and 3 will turn in a deformation of Eq. (21) as shown by Eq. (22). An increase of the shear corresponds in these cases to an increase of the instability regions in the plane $\Theta-\Sigma$ in Figure 1. Considering the instability for the contracting systems, the **II** and **III** quadrants, we can see that the threshold for the emergence of the instability Θ_u decreases in general for positive shear up to the limiting value $\Sigma_0 \equiv \sigma_J v_s$. The contraction is canceled and for higher shear as the system begins an expanding phase. Then the system will necessarily be unstable and the zone of stability will be limited to positive shear in the range $\Sigma \in (0, \Sigma_0)$ and contraction $\Theta \in (\Theta_0, 0)$, where $\Theta_0 \equiv -\sigma_J/\kappa < 0$ is the threshold for the stability of the systems in contraction with non-zero conductivity: for faster contractions $\Theta < \Theta_0$, the system with vanishing shear is unstable. This region, however, increases with increasing conductivity. This means that the conductivity acts to stabilize the contracting configurations with positive shear. Systems $(\mathscr{A}, Q)$ are examples of configurations in the **III** quadrant: the systems are stable for high conductivity, negative shear and sufficiently low contractions, therefore the expanding $(\mathscr{A}\mathbf{Q})$ systems are favored for stability. In general, for systems in contraction, in the **II** and **III** quadrants, a high conductivity is required for stable systems of $(\mathscr{A}\mathbf{T})$ and $(\mathscr{A}\mathbf{Q})$ classes, respectively.

The role of the radial shear. In the **I** and **II** quadrants the radial shear is positive. An increases of Σ generally favors the instability of the configuration. An increase in magnitude of the negative shear in the **III** quadrant ($\Theta\Sigma > 0$), for contracting systems tends to favor the system stability for fast contractions.

The role of the conductivity. In general, an increase of the conductivity parameter particularly in the **II** and **III** quadrants has a stabilizing effect on the system according to the law in Eq. (21). In the **III** quadrant this is the case, for example, for systems of the $\mathfrak{I}-$class or $(\mathscr{A}\mathbf{T})$ configurations, where the sign of the radial shear is equal to that of the contraction. The stability region increases for low contractions and also for small conductivity as the shear is high enough in magnitude; ultimately, for high conductivity the system can remain stable even for high radial contractions. The expansion phase is stable for negative shear, whereas at low expansion the system is unstable. The minimum threshold for the emergency of the instability increases with the shear but decreases with the conductivity, which has then a stabilizing effect on the system for very high conductivity. The stable regions in the $\Theta - \Sigma$ plane extend by increasing the shear and therefore stability is advantaged for high conductivity, for the expansion phase and for shear negative zero or small conductive. In particular, the solutions of the $\mathfrak{I}\mathfrak{I}$-class $(\mathscr{A}\mathbf{Q})$ belong to a wider class of solutions admitting different possible symmetries according to Table 2: stable for any conductivity at negative shear, while for expanding systems as in the **II** quadrant the instability regions are maximum for smaller shear and zero for greater shear. The size of this region increases with the conductivity. However the turning point of the expansion Θ is regulated by the ratio σ_J/κ.

The role of the velocity of sound. Figure 1 emphasizes the role of the velocity of sound in the determining of the unstable states of the system. The shear and expansion scalars have been normalized for the finite conductivity σ_J. For the systems included in the **I**-**II** and **III** quadrants, the velocity of sounds plays a significant role in the determination of the unstable phases, while there is no threshold for the emergence of the instability, crossing with continuity the **III** and **IV** quadrants of the $\Theta - \Sigma$ plane. In the limit of very large v_s the configuration stops the expansion in the **I** quarter or the contraction phase in **III** one i.e. the threshold approaches $\Theta_u = 0$. For sufficiently small velocity of sound one has that $\Theta_u/\sigma_J \approx -1/7 + (6/7)\Sigma$. Then, for sufficiently small Σ the scalar expansion changes sign, and the expansion stage cancels the threshold for the instability, which instead appears for contracting systems. There is a set with positive but small radial shear where the expansion threshold disappears, but for those values of the shear the system is unstable for sufficiently fast contractions. Generally, the upper limit is small and the contracting system is always unstable: the expansion vanishes as the positive shear is $\Sigma/\sigma_J = 1/6$ —thus, for this configuration, the contracting phases are unstable. The solutions $\Theta = 0$ *and* $\Sigma = 0$ belong to $(\mathscr{A}, \mathbf{T}, \mathbf{Q})$ class. For $\Sigma/\sigma_J > 1/6$ the contracting system is always unstable for any pair (σ_J, v_s). Decreasing the velocity of the sound, the region of $\Theta - \Sigma$ plane, for expanding systems corresponds to unstable regions for an increase of the positive shear. As the velocity of sound decreases for positive shear the instability region increases. The instability in the **I** quadrant is carried out for sufficiently high expansions and shear. The increase of the speed of sound stabilizes the expanding system for positive shear and destabilizes the contracting systems at $\Sigma/\sigma_J > 1/6$. In the window of smaller but positive shear, the increase of the speed of sound acts to destabilize the system while a decrease of this corresponds to an increase the stability provided that the contraction is sufficiently small in magnitude or $\Theta/\sigma_J > -1/(v_s + 7)$.

We now focus on the effects the velocity of sound in the case of negative shear: for the expanding system in the **IV** quadrant no threshold exists. For contracting systems in the **III** quadrant, an increase of the speed of sound increases the stability of the system. A decrease of the velocity of sound, for contracting systems with negative shear, makes the system unstable even for very small expansions. The increase in magnitude of the radial shear increases the stability of the system. Ultimately, the equilibrium of the system depends on the velocity of sound in **I** and **III** quadrants —that is for concordant shear and expansion— or in the **IV** quadrant for systems in contraction with positive shear smaller than $1/6\sigma_J$. In the limit of very large velocity of sound $\Theta_u > 0$ for $\Sigma/\sigma_J > 1/6$, for $\Sigma/\sigma_J \in (-\infty, 1/6)$ it is $\Theta_u < 0$.

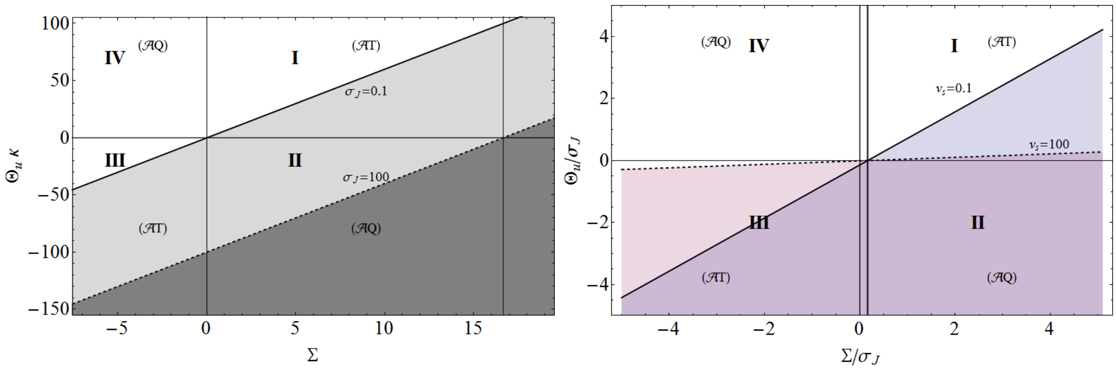

Fig. 1 Left: Plot of the limiting expansion $\Theta_u \kappa$, defined in Eq. (21) as function of the shear in the preferred radial direction where $\kappa \equiv v_s + 7$ for$\sigma_J = 0.1$ (black line) and $\sigma_J = 100$ (dashed black line). A similar plot can be constructed for the expansion Θ_u in terms of σ_J/κ and $\Sigma\kappa$. The instability regions, for $\Theta < \Theta_u$, are colored: for $\sigma_J = 0.1$ this corresponds to the gray region, while the light gray region determines the instability for configurations with conductivity parameter $\sigma_J = 100$. Thus, the gray region is stable for this system. In the white region the system can be stable. Configurations $(\mathscr{A}\mathbf{T})$ are in the **I** and **III** quadrants $(\Sigma\Theta > 0)$ as is the $T = 0$ class of solution for $\Sigma = (2/3)\Theta$, while the $(\mathscr{A}, \mathbf{Q})$ class for $\Sigma = -(1/3)\Theta$ belongs to the **II** and **IV** quadrants $(\Sigma\Theta < 0)$. The stability of a system expanding along the radial direction (but not accelerating, as it is $\mathscr{A} = 0$) is regulated as in the **I** and **IV** quadrants. The contraction occurs in the **II** and **III** quadrants where the radial shear is positive and negative, respectively. The classes are summarized in Table 2. The contraction limit vanishes and the stability limit is only on the radial expansion for high enough values of the shear according to $\Sigma = \sigma_J/6$. Right: Plot of the limiting expansion Θ_u/σ_J, in Eq. (21), versus the radial shear Σ/σ_J for $v_s = 0.01$ (black line) and $v_s = 100$ (dashed black line). The shaded regions mark the sections in the $\Theta - \Sigma$ plane where instability occurs as $\Theta < \Theta_u$. It is $\Theta_u = 0$ for $\Sigma/\sigma_J = 1/6$.

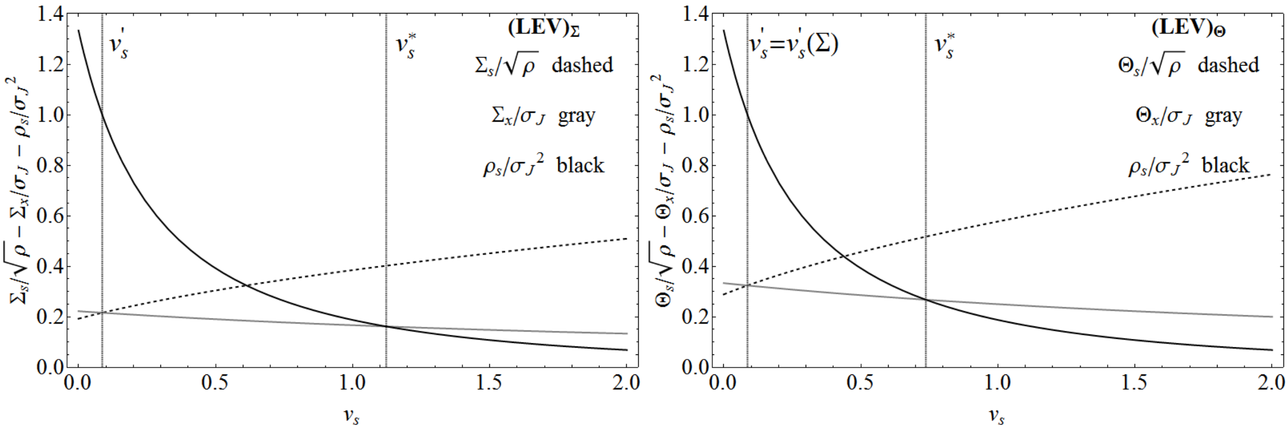

Fig. 2 Ratios of the limiting extreme values $\mathbf{(LEV)}_\Sigma$ (left panel) and $\mathbf{(LEV)}_\Theta$ introduced in Eq. (25), functions of the velocity of sound v_s. At at v_s^* it is $\Sigma_x = \sigma_J^{-1}\rho_s = \frac{2\sigma_J}{4+\sqrt{70}} \approx 0.16\sigma_J$ while $\Theta_x = 0.267592\sigma_J$, in v_s' it is $\Sigma_s = \Sigma_x\sqrt{\rho}/\sigma_J \approx 0.22\sqrt{\rho}$ analogously $\Theta_s = 0.27\sqrt{\rho}$, in the cross point in (v_s', v_s^*) it is $\rho_s = 0.32\sigma_J^2$ where $\Sigma_s = 0.32\sqrt{\rho}$ in $(\mathbf{LEV})_\Sigma$ while the ratio is $\rho_s/\sigma_J^2 \approx 0.44$ in $(\mathbf{LEV})_\Theta$.

4.2.2 Stability of the subclasses with restricted symmetries

Additional restrictions on the system with $\Sigma \neq 0$ and $\Theta \neq 0$, and therefore, on the perturbations modify the condition (21) on the reduced system as follows

$$\Sigma \geq \Sigma_0|_{\mathscr{C}}, \quad \Sigma_0|_{\mathscr{C}} \equiv a[(b+cv_s)\Theta + c\sigma_J], \quad a<1, \tag{22}$$
$$\text{or} \quad a \ll 1, \quad b \gg c, \quad c \geq 1, \quad \{a,b,c\} \in \mathbb{N}. \tag{23}$$

The quantities $\{a,b,c\}$ change depending on the class of solutions. In terms of the expansion Θ, analogously to Eq. (21) one has

$$\Theta \leq \Theta_0|_{\mathscr{C}}, \quad \Theta_0|_{\mathscr{C}} \equiv \frac{\Sigma - ac\sigma_J}{a(b+cv_s)}. \tag{24}$$

It follows that the limit of the expansion vanishes $\Theta_0|_{\mathscr{C}} = 0$. Given $\Sigma_{max} \equiv ac\sigma_J$, for smaller shear $\Sigma < \Sigma_{max}$ the expanding systems are always stable while a threshold for the instability of the contracting systems appears: for $\Theta < \Theta_0|_{\mathscr{C}}$ the system is unstable. It is noteworthy that the limit of the shear $\Sigma_0|_{\mathscr{C}} = 0$ is cancelled for maximal contractions equal to $\Theta_{max} \equiv -c\sigma_J/(b+cv_s)$. More generally, for each subclasses of Table 2 and Table 3 the unstable phase will be regulated by some limiting extreme values **(LEV)** for the systems with $\Theta = 0$ or $\Sigma = 0$ providing the boundary values for the stable regions of the $\Theta - \Sigma$ plane:

$$\mathbf{(LEV)}_\Sigma: \quad \rho_s \equiv \frac{12\sigma_J^2}{(3+v_s)^2(1+3v_s)}, \quad \Sigma_s \equiv \frac{1}{3}\sqrt{\left(\frac{1}{3}+v_s\right)\rho}, \quad \Sigma_x \equiv \frac{2\sigma_J}{3(3+v_s)}. \tag{25}$$
$$\mathbf{(LEV)}_\Theta: \quad \rho_s = \rho_s|_{\mathbf{(LEV)}_\Sigma}, \quad \Theta_s \equiv \frac{3\Sigma_s}{2}, \quad \Theta_x \equiv \frac{3\Sigma_x}{2}, \tag{26}$$

see Figure 2. A threshold exists for the *stability* in the systems with the density in the two regimes $\rho > \rho_s$ and $\rho < \rho_s$, according to $(\mathbf{LEV})_\Sigma$ or $(\mathbf{LEV})_\Theta$, with conditions on the radial shear or, respectively, the expansion or contraction. The details of the analysis are considered in the Appendix A. For stability, in general, one has the two following cases:

$$\rho < \rho_s \quad \Delta \in [-\Delta_x, -\Delta_s] \quad \text{or} \quad (+\Delta_s, +\infty), \quad \rho > \rho_s \quad (-\Delta_s, \Delta_s) \quad \text{or} \quad (+\Delta_s, +\infty),$$
$$\Delta = \Sigma \quad \text{for} \quad (\mathbf{LEV})_\Sigma, \quad \Delta = \Theta \quad \text{for} \quad (\mathbf{LEV})_\Theta. \tag{27}$$

Each of the regions are typically regulated by additional conditions on $Q_i \in \nu$ $(\mathbf{LEV})_\Sigma$. For example, the $\mathfrak{I}-$class ($\mathscr{A}\mathbf{T}$) solution is determined by a set of conditions on $\mathscr{B}$, ξ and E. We note that the limiting value on the density ρ_s increases with the conductivity but decreases with the velocity of sound —there are three ranges of v_s to be considered as detailed in Figure 2. However, the boundary Δ_s depends on the density ρ and, therefore, it is differently regulated in the three regions. For low density regimes $\rho < \rho_s$, the density function regulates the system stability in a reduced region of the $\rho - \Sigma$ plane through the conditions imposed on the shear or the expansion —especially for low values. This situation is more evident for a low velocity of sounds $v < v_s^*$. For larger values, the ranges for low density variations are restricted to $\rho < 0.2\sigma_J$.

4.3 Comments on the five principal classes of solutions

$\mathfrak{I}$-Class ($\mathscr{A}$ **T**)**:** this case is constrained by the condition $\mathscr{C}(\mathscr{A},T)$ obtained from Eq. (9). This relation is a consequence of the condition $T = 0$, and it will occur in the other subcases characterized by this assumption. The evolution equation for the parallel vorticity Ω is $\dot{\Omega} = 0$. This quantity remains constant along the fluid motion, for the perturbed quantity as well as the unperturbed one. Furthermore, the time evolution of the matter density only involves the kinematic variable Q. We can draw some conclusions on the stability of the system on the basis of the trace of the matrix. The system is *linearly unstable* if it has a negative shear on the radial direction which is bounded by $\mathring{\Sigma} \leq \Sigma_x$. It is worth noting that the limiting case is defined only by the constitutive equation, the conductivity σ_J, and the equation of state by v_s —fixed by the reference solution. Furthermore, considering the coefficients $c_9 = -1$ and $c_8 = \mathrm{Tr}\mathring{\mathbf{B}} < 0$ we infer the condition for the *stability* $c_1 < 0$. Using the condition $\mathscr{C}(\mathscr{A},T)$, this condition can be rephrased as the following two alternatives: (i) $\mathring{\rho} < \mathring{\rho}_s$ with $\mathring{\Sigma} \in (-\mathring{\Sigma}_x, -\mathring{\Sigma}_s) \cup (\mathring{\Sigma}_s, +\infty)$ and (ii) $\mathring{\rho} \geq \mathring{\rho}_s$ with $\mathring{\Sigma} \in (\mathring{\Sigma}_s, +\infty)$, where the $(\mathbf{LEV})_\Sigma$ hold.

$\mathfrak{II}$-Class ($\mathscr{A}$ **Q**)**:** the condition $\frac{1}{3}\Theta = -\Sigma$ implies that the matter and field variables are related by the constraint $\mathscr{C}(\mathscr{A},Q)$ —see equation (4). This relation is a consequence of the assumption $Q = 0$, and it will occur in the other subcases characterized by this hypothesis. We observe that the assumption $Q = 0$ implies, in particular, that the expansion of the 3-sheets and the radial component of the shear of the 3-sheet must have opposite sign. Studying the problem for the reduced system of nine variables we find that the system is *linearly unstable* if $\mathrm{Tr}\overset{o}{\mathbf{B}} > 0$ —that is if $\mathring{\Theta}$ is negative and bounded above by $\mathring{\Theta} \leq \Theta_0|_{\mathscr{C}}$ with $(a = c = 1; b = 9)$. Again, as in the case $T = 0$, we obtain a constraint for the radial expansion that depends on the unperturbed constants (σ_J, v_s).

$\mathfrak{III}-$Class ($\mathscr{A}$ Φ)**:** the assumptions $\mathscr{A} = 0$ and $\Phi = 0$ on equations (6) and (7) give rise to two possible subcases: (i) $\Omega = 0$ or (ii) $\xi = 0$ and $\mathscr{B} = 0$. We consider therefore these two subcases separately:

($\mathscr{A}$ Φ Ω)**:** the trace implies the following condition of *instability* $\mathring{\Sigma} \geq \Sigma_0|_{\mathscr{C}}$ with $(a = 2/9, b = 6, c = 1)$. We note that for the reference solution $\mathring{\Theta}$ and $\mathring{\Sigma}$ are related by the square of the velocity of sound and the conductivity through σ_J. However, the sign of the $\Sigma\Theta$ is not constrained by this relation.

$(\mathscr{A}\,\Phi\,\xi\,\mathscr{B})$**:** in this case T is the only kinematical variable involved in the evolution of the electromagnetic field. The criterion on the trace of the matrix $\mathring{\mathbf{B}}$ implies the following condition of *instability* $\mathring{\Sigma} \geq \Sigma_0|_{\mathscr{C}}$ with $(a=2/21, b=16, c=3)$.

Thus, we finally note that the above subcases are characterized by similar constraints on $\mathring{\Sigma}$ and $\mathring{\Theta}$.

$\mathfrak{IV}-$**Class** $(\mathscr{A}\,\xi)$**:** equation (7) leads to the condition $\mathscr{C}(\mathscr{A},\xi)$. On the other hand, the condition of *instability* on the trace of the matrix $\mathring{\mathbf{B}}$ leads to the inequality $\mathring{\Sigma} \geq \Sigma_0|_{\mathscr{C}}$ with $(a=1/33, b=40, c=6)$.

$\mathfrak{V}-$**Class** $(\mathscr{A}\,\Omega)$**:** in this case the system is *unstable* if $\mathring{\Sigma} \geq \Sigma_0|_{\mathscr{C}}$ with $(a=1/15, b=19, c=3)$.

We conclude this Section pointing out that the stability of the configuration under consideration is constrained by similar relations between the unperturbed radial part of the shear of the 3-sheet $\mathring{\Sigma}$ and the radial expansion $\mathring{\Theta}$. These constraints only depend on the square of the sound velocity and the conductivity. In Appendix A we specialize this analysis to consider the subclasses of Tables 2 and 3.

5 Conclusions

In this article we explored the stability properties of an ideal LRS Einstein-Maxwell perfect fluid system. As a first step we have formulated an hyperbolic initial value problem providing a suitable (quasilinear) symmetric hyperbolic system by means of which one can address the non-linear stability of this system. The propagation of the constraints has been assumed to hold at all times. This result can be proven by fairly general arguments as discussed in [37, 38]. Our analysis is based on a $1+1+2$-tetrad formalism. In our calculations certain choices of kinematic properties of the configuration were motivated by the necessity to extract some information from the rather complicated equations governing the perturbations. We studied five principal classes of solutions and different subcases, considering systems with particular kinematic configurations. The assumption of spherical symmetry simplifies the problem and takes advantage from the $1+1+2$-decomposition [17]. More specifically, we consider a barotropic equation of state: when the fluid entropy is a constant in both space and time, an equation of state to link the pressure p to the matter density ρ can be given in the form $p=p(\rho)$ —see e.g. [10]. In the present work we restrict our attention to isotropic fluids. By restricting our attention to isotropic fluid configurations we can make use of a constant electrical conductivity coefficient for the fluid (plasma). These assumptions simplify considerably the analysis of the stability problem of linear perturbations. Thus, in this work we proceeded as follows: we wrote the $1+1+2$-equations for the LRS system using the radial vector. For the linear perturbation analysis it was useful to introduce a re-parameterized set of evolution equations based of a suitable combination of the radially projected shear and expansion. The resulting evolution system was then used to analyse the stability problem for small linear perturbations of the background. We have presented the main results concerning the linear stability of this system in a classification of subcases that constitutes the principal result of this paper. *In this first investigation of the problem ,we address the stability analysis of the particular case of constant velocity of sound, leaving the investigation of the stability problem of more general cases for a future analysis*[2]. In order to close the system of evolution equations it is necessary to specify the form of the conduction current. Accordingly, we

[2] For a more thorough discussion of this case and more general equations of state in relativistic hydrodynamics and with possible relevance to astrophysics see [47,48] –see also Sec. (B)

assumed Ohm's law so that a linear relation between the conduction current and the electric field, involving a constant electrical conductivity coefficient holds. We have assumed the pressure of the fluid p and the charge density $\tilde{\rho}_C$ to be functions of the matter density ρ, and the charge current j_c function of the electric field E. Although viable, the perturbed equation for the radially projected acceleration $\mathscr{A}$ turned out to be a very complicated expression of the other variables and their derivatives. Thus, in order proceed in the stability analysis, we studied a simplified form system by taking up some assumptions on the configuration. Assuming a vanishing radial acceleration for the reference solution, our analysis is particularly focused on some specific cases defined by fixing the expansions of the 3-sheets and 2-sheets, the radial part of the shear of the 3-sheet, the twisting of the 2-sheet and the radial part of the vorticity of the 3-sheet. In this way we are also able to provide results concerning the structure of the associated LRS taking into consideration all the different subcases. This analysis constitutes the main result of the article: **1.** Firstly, we found that in many cases the stability conditions can be strongly determined by the constitutive equations by means of the square of the velocity of sound and the electric conductivity. In particular, this is evident for the contracting and expanding LRS configurations; in fact, **2.** a threshold for the emergence of the instability appears in both cases. The conditions relate mainly the expansion (or contraction) along the preferred direction with respect to different regimes of the radial shear. These results provide, in a quite immediate manner, information regarding the unstable configurations. The results for these configurations are illustrated schematically in Figure 1 in the four fundamental cases considered here, emphasizing the role of the couple of parameters (v_s, σ_J). Moreover, **3.** the sign of $\Sigma\Theta$ plays an essential role in the determination of the unstable phases of the systems —for expanding configurations with $\Sigma\Theta < 0$, appears *no* threshold for the emergence of instability by means of condition (15). The velocity of sound and the conductivity act in a different way for the systems in the three regions of Figure 1, favouring or preventing the instability. **4.** There is always a threshold for the contraction or expansion of the system —for contracting systems with a fast contraction rate, $|\Theta| > |\Theta_u(\sigma_J, v_s)| > 0$, or for expanding systems where $0 < \Theta < \Theta_u(\sigma_J, v_s)$ instability emerges. However, for some cases the density ρ is constrained while the pair (Σ, Θ) varies in different ranges depending on (ρ, v_s, σ_J).

Acknowledgements DP gratefully acknowledges support from the Blanceflor Boncompagni-Ludovisi, née Bildt and wishes to thank the Angelo Della Riccia Foundation for support during the first development of this work, and thanks the Accademia Nazionale dei Lincei for support during 2015 (within the Royal Society fellowship program), and from the Junior GACR grant of the Czech Science Foundation No:16-03564Y. We thank an anonymous referee for her/his careful reading of our work and helpful comments on our analysis which considerably improved the presentation of our results.

A Details on the subclasses of the solutions

Subclasses of the $\mathfrak{I}$-class $(\mathscr{A}, \mathbf{T})$**.** In what follows we analyze in further detail the following subcases of the configuration$(\mathscr{A}\,\mathbf{T})$:

$(\mathscr{A}\,\mathbf{TQ})$**:** in this case the configuration is defined by the conditions $\mathscr{A} = 0$, $\Sigma = 0$ and $\Theta = 0$. The matter density evolution only depends on the electric field and the current density. The evolution of the electromagnetic fields is regulated by the twist ξ of the 2-sheet. Moreover, the radial vorticity is constant during the motion —i.e. $\dot{\Omega} = 0$. From the evolution equations for T and Q we obtain, respectively, the conditions $\mathscr{C}(\mathscr{A}, T)$ and $\mathscr{C}(\mathscr{A}, Q)$. From these relations we find $\mathscr{E} = p + \frac{1}{3}\rho - 2\Omega^2$ and $4\Omega^2 = B^2 + E^2 + 3p + \rho$. This case has at least one zero eigenvalue associated with the vorticity. This eigenvalue has multiplicity 2. The sum of the eigenvalues of the associated 7-rank matrix is $\mathrm{Tr}\mathring{\mathbf{B}} = -\sigma_J < 0$[3].

$(\mathscr{A}\,\mathbf{T}\,\Phi)$**:** in this case one has $\dot{\Omega} = 0$, and $\mathscr{C}(\mathscr{A},T)$. The conditions $\mathscr{A} = 0$ and $\Phi = 0$ imply from equations (6), (7) either (i) $\Omega = 0$ or (ii) $\xi = 0$ and $\mathscr{B} = 0$. We consider these two subcases separately:

(i) For $(\mathscr{A}\,\mathbf{T}\,\Phi\,\Omega)$ the system is characterised by the condition $\mathscr{C}(\mathscr{A},T,\Omega)$. From the criterion on the trace one can deduce that the system is *unstable* if $\mathring{\Theta} \leq -(\sigma_J)/(3+v_s)$. Now, considering the determinant of the reduced 7×7-linearised matrix[4] the necessary condition for *stability* is that $\det\mathring{\mathbf{B}} < 0$. Combining this condition with the criterion of the trace and using the fact that $\mathring{p} = \mathring{\rho}v_s$ we obtain for *stable* configurations the following conditions:

(ia) for $\mathring{\rho} < \mathring{\rho}_s$ then one has that $\mathring{\Sigma} \in (-\mathring{\Sigma}_s,+\mathring{\Sigma}_s)$ and $\mathfrak{Q}(\mathring{\mathscr{B}},\mathring{\xi},\mathring{E}^2) < 0$, *or* $\mathring{\Sigma} \in (-\mathring{\Sigma}_x,-\mathring{\Sigma}_s)\cup(\mathring{\Sigma}_s,\infty)$ and $\mathfrak{Q}(\mathring{\mathscr{B}},\mathring{\xi},\mathring{E}^2) > 0$;

(ib) For $\mathring{\rho} \geq \rho_s$ then one has $\mathring{\Sigma} \in (-\mathring{\Sigma}_x,+\mathring{\Sigma}_s)$ and $\mathfrak{Q}(\mathring{\mathscr{B}},\mathring{\xi},\mathring{E}^2) < 0$, *or* $\mathring{\Sigma} \in (+\mathring{\Sigma}_s,\infty)$ and $\mathfrak{Q}(\mathring{\mathscr{B}},\mathring{\xi},\mathring{E}^2) > 0$. With $\mathring{\xi} \neq 0$, $\mathfrak{Q}(\mathscr{B},\xi,E^2) \equiv 3\mathscr{B}\xi + 2E^2\sigma_J$ and we consider the $(\mathbf{LEV})_\Sigma$.

We note that the reference matter density and shear ρ_s and Σ_x, depend on the constants (v_s,σ_J) and the limit Σ_s depends only on the matter density and the square of the sound velocity.

(ii) $(\mathscr{A}\,\mathbf{T}\,\Phi,\xi,\mathscr{B})$: in this case the Maxwell equations reduce to $\dot{B} = 0$ and $\dot{E} = -j$. Moreover, $\dot{\Omega} = 0$ and the condition $\mathscr{C}(\mathscr{A},T)$ holds. The corresponding reduced system has six unknowns. We notice here the eigenvalues: $\lambda_0 = 0$ with a subspace of dimension 3 and

$$\lambda_1 = -\sigma_J, \quad \lambda_\pm = \tfrac{1}{2}\left[-(3+v_s)\mathring{\Theta} \pm \sqrt{2\mathring{\rho}(v_s+1)^2 + (v_s-1)^2(\mathring{\Theta})^2}\right]. \tag{28}$$

The conditions on the sign of $\lambda_\pm$ imposes severe restrictions on the radial expansion. These depend on the sound velocity and the background density matter: the condition $\lambda_\pm < 0$ (for the system *stability*) is satisfied if $\mathring{\Theta} > \sqrt{(1+v_s)\mathring{\rho}/2}$.

$(\mathscr{A}\,\mathbf{T}\,\xi)$**:** this case implies the equations $\dot{B} = \dot{\mathscr{B}} = \dot{\Omega} = \dot{\Phi} = 0$ while the electric field satisfies $\dot{E} = -j$. Moreover, one has the condition $\mathscr{C}(\mathscr{A},\xi)$ with $\mathscr{C}(\mathscr{A},T)$. We study the associated 8×8 matrix: the eigenvalues of this matrix are $\lambda_\pm$ and λ_i as defined for the case $(\mathscr{A}\,T\,\Phi,\xi,\mathscr{B})$ and a zero eigenvalue with multiplicity 5 associated to the variables $(B,\mathscr{B},\Omega,\Phi)$.

$(\mathscr{A}\,\mathbf{T}\,\Omega)$**:** in this case one has $\dot{\Phi} = 0$. From equation (9) one finds $\mathscr{C}(\mathscr{A},T,\Omega)$. The time evolution of the electric and magnetic fields are regulated by the twisting ξ of the 2-sheet —see equations (3)-(4). The temporal evolution of ξ is fixed explicitly by the magnetic part of the Weyl tensor —see equation (7). On the other hand, from the trace of the associated rank 7 matrix we infer that the system is *unstable* if $\mathring{\Sigma} < -\Sigma_x$. In order to obtain necessary conditions for *stability* one requires $c_7 = -\mathrm{Tr}\mathring{\mathbf{B}} > 0$, and $c_1 > 0$. These inequalities lead to the following cases with $(\mathbf{LEV})_\Theta$:

(i) if $\mathring{\rho} < \mathring{\rho}_s$ then $\mathring{\Theta} \in (-\mathring{\Theta}_x,-\mathring{\Theta}_s)\cup(+\mathring{\Theta}_s,\infty)$ and $\mathring{\xi} \neq 0$, *or* $\mathring{\Theta} \in (-\mathring{\Theta}_x,+\mathring{\Theta}_s)$ and $\mathring{\xi}\mathring{\mathscr{B}} < 0$;

(ii) if $\mathring{\rho} \geq \rho_s$ then $\mathring{\Theta} \in (-\mathring{\Theta}_s,+\mathring{\Theta}_s)$ and $\mathring{\xi}\mathring{\mathscr{B}} < 0$, *or* $\mathring{\Theta} \in (+\mathring{\Theta}_s,\infty)$ and $\mathring{\xi} \neq 0$.

Subclasses of the $\mathfrak{I}\mathfrak{I}$-class $(\mathscr{A},\mathbf{Q})$. In this subsection we focus on the configurations with $\frac{1}{3}\Theta = -\Sigma$. Taking into account the results on the system $(\mathscr{A}\,\mathbf{Q})$, we consider the following subcases.

$(\mathscr{A}\,Q\,\Phi)$**:** the condition $\mathscr{C}(\mathscr{A},Q)$ holds. Moreover, the conditions $\mathscr{A} = 0$ and $\Phi = 0$, imply from equation (6) two subcases, (i) $\Omega = 0$ and (ii) $(\xi = 0, \mathscr{B} = 0)$, respectively.

(i) $(\mathscr{A}\,Q\,\Phi\,\Omega)$: from the trace of the reduced rank 7 matrix the configuration is bound to be *unstable* if $\mathring{\Sigma} \geq \Sigma_s \equiv 2\sigma_J/3(15+2v_s)$. On the other hand, studying the sign of the characteristic polynomial coefficient $c_5 < 0$ we infer that to obtain *stability*: for $\mathring{\mathscr{E}} < \mathring{\mathscr{E}}_p$ and (a) $\mathring{T} > -2\sigma_J/(15+2v_s)$ with $\mathring{\xi} < -\mathring{\xi}_p$ or $\mathring{\xi} > \mathring{\xi}_p$, or

[3] The non-zero coefficients of the characteristic polynomial are: $c_2 = (6\mathring{\xi}^2\mathring{\Omega})^2 > 0$, $c_3 = \mathring{\xi}^2(8\sigma_J\mathring{E}^2 + 18\mathring{\mathscr{B}}\mathring{\xi} + 9\sigma_J\mathring{\Omega}^2)$, $c_4 = \frac{1}{2}\mathring{\xi}\left(3\sigma_J(3\mathring{\mathscr{B}} - 4\mathring{B}\mathring{E}) + 2\mathring{\xi}\left(9\mathring{\Omega}^2 - 4(\mathring{B}^2 + \mathring{E}^2 - 9\mathring{\xi}^2)\right)\right)$, $c_5 = \frac{9}{2}\mathring{\xi}(\mathring{\mathscr{B}} + 2\sigma_J\mathring{\xi})$, $c_6 = 13\mathring{\xi}^2 > 0 \quad c_7 = -\mathrm{Tr}\mathring{\mathbf{B}} > 0$.

[4] $\det\mathring{\mathbf{B}} = (1+v_s)\mathring{\xi}^2(2\sigma_J\mathring{E}^2 + 3\mathring{\mathscr{B}}\mathring{\xi})\left((1+3v_s)\mathring{\rho} - 3\mathring{Q}^2\right)$.

(b) $\mathring{\mathscr{E}} < \mathring{\mathscr{E}}_p$ and $\mathring{T} > \mathring{T}_p$. Otherwise, for $\mathring{\mathscr{E}} \geq \mathring{\mathscr{E}}_p$ with $\mathring{T} > -2\sigma_J/(15+2v_s)$, with the following definitions:

$$\mathring{\mathscr{E}}_p \equiv \left(\tfrac{5}{6}+\tfrac{5}{2}v_s+v_s^2\right)\mathring{\rho} + \frac{2\big(50+v_s(15+2v_s)\big)\sigma_J^2}{(15+2v_s)^2},$$

$$\mathring{T}_p \equiv \frac{1}{285+78v_s}\left[-3(13+2v_s)\sigma_J + \left\{6(95+26v_s)\left(\big[5+3v_s(5+2v_s)\big]\mathring{\rho} - 6\mathring{\mathscr{E}}\right) + 9(13+2v_s)^2\sigma_J^2\right\}^{1/2}\right],$$

$$\mathring{\xi}_p = \frac{1}{78}\left[\sqrt{39}\left\{2\big(5+3v_s(5+2v_s)\big)\mathring{\rho} - 12\mathring{\mathscr{E}} - 3\mathring{T}^2(95+26v_s) - 6\mathring{T}(13+2v_s)\sigma_J\right\}^{1/2}\right], \tag{29}$$

with $T=-3\Sigma$. Once again, the limiting conditions on the radial part of the shear of the 3-sheet, Σ, and the electric part of the Weyl tensor is completely regulated by the reference density and the constants (σ_J, v_s). This *stability* conditions can be alternatively expressed as follows:

$$\rho > 0, \quad \mathring{\Sigma} > \Sigma_s, \quad \text{and} \quad \mathring{\xi}^2 > \mathring{\xi}_s^2 \equiv \tfrac{1}{52}\left\{2(\mathring{B}^2+\mathring{E}^2) + 4\big[1+v_s(3+v_s)\big]\mathring{\rho} + 3\mathring{\Sigma}\big[2\sigma_J(13+2v_s) - 3(95+26v_s)\mathring{\Sigma}\big]\right\} \tag{30}$$

(ii) $(\mathscr{A}\, Q\, \Phi\, \xi\, \mathscr{B})$: with the condition $\mathscr{C}(\mathscr{A}, Q)$, while the time evolution of the radial projected vorticity is entirely regulated by the radial part of the shear —see equation (8). The system, with a rank 6 matrix, is *unstable* if $\mathring{\Sigma} \geq \Sigma_s \equiv 2\sigma_J/3(13+2v_s)$. Otherwise, it is *stable* if (a) $\mathring{\mathscr{E}} < \mathring{\mathscr{E}}_p$ with $\mathring{T} > -(2\sigma_J)/(13+2v_s)$ and $\mathring{\Omega} > \mathring{\Omega}_p \cup \mathring{\Omega} < -\mathring{\Omega}_p$ *or* $\mathring{\Omega} \in [-\mathring{\Omega}_p, \mathring{\Omega}_p]$ with $\mathring{T} > \mathring{T}_p$; (b) $\mathring{\mathscr{E}} \geq \mathring{\mathscr{E}}_p$ and $\mathring{T} > -(2\sigma_J)/(13+2v_s)$, where

$$\mathring{\Omega}_p \equiv \frac{1}{2\sqrt{6}}\left\{-3\mathring{\mathscr{E}} + 2\big[2+3v_s(2+v_s)\big]\mathring{\rho} + \frac{6\left(73+26v_s+4v_s^2\right)\sigma_J^2}{(13+2v_s)^2}\right\}^{1/2}, \tag{31}$$

$$\mathring{T}_p \equiv \frac{-3(11+2v_s)\sigma_J}{210+66v_s} + + \frac{1}{210+66v_s}\left[\sqrt{3}\left\{8(35+11v_s)\big[2+3v_s(2+v_s)\big]\mathring{\rho} + 3(11+2v_s)^2\sigma_J^2 - 12\mathring{\mathscr{E}}(35+11v_s) - 96(35+11v_s)\mathring{\Omega}^2\right\}^{1/2}\right],$$

$$\mathring{\mathscr{E}}_p \equiv \tfrac{2}{3}\big(2+3v_s(2+v_s)\big)\mathring{\rho} + \frac{2\left(73+26v_s+4v_s^2\right)\sigma_J^2}{(13+2v_s)^2}. \tag{32}$$

Alternatively, these conditions can be reexpressed as $\mathring{\Sigma} < \Sigma_s$ and

$$\mathring{\Omega}^2 > \left[\mathring{B}^2(3\mathring{E}^2-2) - 2\mathring{E}^2 + \big[3+v_s(9+4v_s)\big]\mathring{\rho} + 6\mathring{\Sigma}\big[(11+2v_s)\sigma_J - 3(35+11v_s)\mathring{\Sigma}\big]\right]/16.$$

$(\mathscr{A}\, Q\, \xi)$**:** in this case one has $\Sigma = -\Theta/3$ with $\mathscr{C}(\mathscr{A}, \xi)$ from equation (7) and $\mathscr{C}(\mathscr{A}\, Q)$. The radial shear is the only kinematical variable that explicitly regulates the time evolution of the variables $(E, B, \rho, \Phi, \mathscr{E}, \mathscr{B})$. The radial vorticity and the shear are related by the two evolution equations (8) and (9), respectively. This is a rank 8 matrix problem. The system is *unstable* if $\mathring{\Sigma} \geq (2\sigma_J)/(51+6v_s)$. The system is *stable* if either (a) $\mathring{\mathscr{E}} < \mathring{\mathscr{E}}_p$ and $\mathring{T} > \mathring{T}_p$ *or* $-2\sigma_J/(17+2v_s) < \mathring{T} < \mathring{T}_p$ and $\mathring{\Omega} < -\mathring{\Omega}_p \cup \mathring{\Omega} > \mathring{\Omega}_p$; or (b) $\mathring{\mathscr{E}} \geq \mathring{\mathscr{E}}_p$ and $\mathring{T} > -2\sigma_J/(17+2v_s)$. In the above conditions we used the following definitions:

$$\mathring{\Omega}_p \equiv \frac{\sqrt{3}}{12}\left\{4\big[2+3v_s(2+v_s)\big]\mathring{\rho} - 6\mathring{\mathscr{E}} - 15\mathring{T}^2(25+6v_s) - 6\mathring{T}(15+2v_s)\sigma_J\right\}^{1/2}, \tag{33}$$

$$\mathring{\mathscr{E}}_p \equiv \tfrac{2}{3}\big[2+3v_s(2+v_s)\big]\mathring{\rho} + \frac{4\big[65+v_s(17+2v_s)\big]\sigma_J^2}{(17+2v_s)^2}, \tag{34}$$

$$\mathring{T}_p \equiv \frac{1}{375+90v_s}\left\{30(25+6v_s)\left\{2\big[2+3v_s(2+v_s)\big]\mathring{\rho} - 3(15+2v_s)\sigma_J - 3\mathring{\mathscr{E}}\right\} + 9(15+2v_s)^2\sigma_J^2\right\}^{1/2} \tag{35}$$

Alternatively, these conditions can be written as $\mathring{\Sigma} < 2\sigma_J/(51+6v_s)$ and

$$\mathring{\Omega}^2 > \tfrac{1}{16}\left\{\mathring{B}^2 + \mathring{E}^2 + \big[3+v_s(9+4v_s)\big]\mathring{\rho} + 3\mathring{\Sigma}\big[(30+4v_s)\sigma_J - 15(25+6v_s)\mathring{\Sigma}\big]\right\}.$$

$(\mathscr{A}\, Q\, \Omega)$**:** in this case one has that $T=-3\Sigma$ and $\Sigma = -\Theta/3$ and the condition $\mathscr{C}(\mathscr{A}, Q): B^2+E^2+2\mathscr{E} = -(p+\rho/3)$ holds. The system is *unstable* if $\mathring{\Sigma} \geq \Sigma_s \equiv \sigma_J/3(8+v_s)$. A *stable* system meets the following conditions: (a) $\mathring{\mathscr{E}} < \mathring{\mathscr{E}}_p$, with $\mathring{T} > \mathring{T}_p$ *or* $-3\Sigma_s < \mathring{T} \leq \mathring{T}_p$and $(\mathring{\xi} < -\mathring{\xi}_p,\ \mathring{\xi} > \mathring{\xi}_p)$; or (b) $\mathring{\mathscr{E}} \geq \mathring{\mathscr{E}}_p$ and

$\mathring{T} > -3\Sigma_s$. In the above expressions we have used the definitions

$$\mathring{T}_p \equiv \frac{1}{165+42v_s}\left[-3(7+v_s)\sigma_J + \left\{3(55+14v_s)\mathring{\rho}\left([5+3v_s(5+2v_s)]-6\mathring{\mathscr{E}}\right)+9(7+v_s)^2\right\}^{1/2}\sigma_J^2\right], \quad (36)$$

$$\mathring{\xi}_p \equiv \frac{1}{\sqrt{78}}\left[\left\{[5+3v_s(5+2v_s)]\mathring{\rho}-6\mathring{\mathscr{E}}-3\mathring{T}^2(55+14v_s)-6\mathring{T}(7+v_s)\sigma_J\right\}^{1/2}\right], \quad (37)$$

$$\mathring{\mathscr{E}}_p \equiv \frac{(8+v_s)^2[5+3v_s(5+2v_s)]\mathring{\rho}+3[57+2v_s(8+v_s)]\sigma_J^2}{6(8+v_s)^2}. \quad (38)$$

The conditions can be expressed, alternatively, as $\mathring{\Sigma} < \Sigma_s$ and

$$16\mathring{\xi}^2 > \left[\mathring{B}^2+\mathring{E}^2+2[1+v_s(3+v_s)]\mathring{\rho}-9(55+14v_s)\mathring{\Sigma}^2+6(7+v_s)\mathring{\Sigma}\sigma_J\right].$$

Subclasses of the $\mathfrak{IV}$-class $(\mathscr{A},\xi)$

$(\mathscr{A}\,\xi\,\Omega)$: the assumptions $\mathscr{A}=0$, $\xi=0$ and $\Omega=0$, lead from equation (7) to the condition $\mathscr{C}(\mathscr{A}\,\xi,\Omega)$: $\mathscr{B}=0$. Thus, we analyse the case: $(\mathscr{A}\,\xi\,\Omega\,\mathscr{B})$, with a rank 7 matrix. The system is *unstable* if $\mathring{\Theta} \leq -3(2\sigma_J+(v_s-5)\mathring{\Sigma})/4(7+v_s)$.

The case $\mathscr{A}=0$, $T=0$, and $Q=0$.

$(\mathscr{A}\,T\,Q\,\Phi)$: in this special case we assume the radial acceleration to be zero. In addition, the expansion of the 2-sheet Φ vanishes and also $\Sigma=\Theta=0$. It is worth noting that the assumption $\mathscr{A}=\Sigma=\Theta=\Phi=0$ implies from equation (6) that $\xi\Omega=0$ —that is, the twist of the 2-sheet ξ together with the magnetic part of the Weyl tensor vanish. This system can only accelerate in the radial direction or otherwise radially projected vorticity will vanish. The basic assumptions in this case directly lead to $\dot{\Omega}=0$, $\dot{\xi}=\mathscr{B}/2$ and the conditions $\mathscr{C}(\mathscr{A},T)$ and $\mathscr{C}(\mathscr{A},Q)$ hold. Thus, two subcases occur:

(i) $(\mathscr{A}\,T\,Q\,\Phi\,\xi\,\mathscr{B})$: in this case one further has $\dot{B}=\dot{\Omega}=0$, $\dot{E}=-j$. The only non zero eigenvalue is $\lambda=-\sigma_J$.

(ii) $(\mathscr{A}\,T\,Q\,\Phi\,\Omega)$: in this case, it can be shown that the trace of the reduced matrix $\mathrm{Tr}\mathring{\mathbf{B}}=-\sigma_J<0$ and this system has a zero eigenvalue. Nevertheless, in this case the conditions $\mathscr{C}(\mathscr{A},T,\Omega)$ and $\mathscr{C}(\mathscr{A},Q)$ are $B^2+E^2+2\mathscr{E}=-\left(p+\frac{\rho}{3}\right)<0$. These conditions are inconsistent with the kind of equation of state considered in this work. We notice that this situation always occurs when the two conditions $\mathscr{C}(\mathscr{A},T,\Omega)$ and $\mathscr{C}(\mathscr{A},Q)$ must be satisfied simultaneously.

$(\mathscr{A}\,T\,Q\,\xi)$**:** in this case the problem is simplified considerably and we find the relations $\mathscr{C}(\mathscr{A},\xi)$, $\mathscr{C}(\mathscr{A},T)$ and $\mathscr{C}(\mathscr{A},Q)$. Moreover, one has that $\dot{E}=-j$. The evolution equations $\dot{B}=0$, $\dot{\mathscr{B}}=0$, $\dot{\Omega}=0$, and $\dot{\Phi}=0$ give rise to repeated zero eigenvalues. In addition, one has the eigenvalue $\lambda=-\sigma_J<0$.

$(\mathscr{A}\,T\,Q\,\Omega)$**:** in this case one readily has that $\dot{\Phi}=0$. The associated rank 5 matrix has a zero eigenvalue with multiplicity 2. In addition, one has that $c_6=\mathrm{Tr}\mathring{\mathbf{B}}=-\sigma_J<0$. Thus, imposing the condition $c_6c_7>0$ leads to $\mathring{\xi}^2>-(\mathring{B}^2+\mathring{E}^2+3\mathring{\mathscr{E}})/26$. However, in this case the relations $\mathscr{C}(\mathscr{A},T,\Omega)$ and $\mathscr{C}(\mathscr{A},Q)$ hold. These cannot be satisfied for ordinary matter density (i.e. such that $\rho>0$) and the given equation of state.

The case $\mathscr{A}=0$, $T=0$, $\xi=0$.

$(\mathscr{A}\,T\,\xi\,\Omega)$: the assumptions $\mathscr{A}=0$, $\xi=0$ and $\Omega=0$ lead using equation (7) to $\mathscr{B}=0$. From equation (9) one obtains the conditions $\mathscr{C}(\mathscr{A},T,\Omega)$, and $\dot{\Phi}=\dot{B}=0$. Thus, we analyse the system $(\mathscr{A}\,T\,\xi\,\Omega\,\mathscr{B})$. There is a zero eigenvalue with multiplicity 3. In addition, one has $\lambda_1=-\sigma_J<0$ and

$$\lambda_\pm = \frac{1}{4}(-(3+v_s)3\mathring{\Sigma}\pm\sqrt{8(1+v_s)^2\mathring{\rho}+(v_s-1)^2(3\mathring{\Sigma})^2}).$$

The condition $\lambda_\pm<0$ is satisfied for

$$\mathring{\Sigma} > 2(1+v_s)\sqrt{\mathring{\rho}/(3(v_s+5+2\sqrt{3})(v_s+5-2\sqrt{3}))}.$$

The case $\mathscr{A}=0$, $Q=0$, $\xi=0$.

$(\mathscr{A}\,Q\,\xi\,\Omega)$: this case reduces to $(\mathscr{A}\,Q\,\xi\,\Omega\,\mathscr{B})$ with $\mathscr{C}(\mathscr{A},Q)$. From the trace of the associated rank 6 matrix we infer that the system is *unstable* if $\mathring{\Sigma}\geq\Sigma_s\equiv\sigma_J/3(6+v_s)$. On the other hand, if the system is *stable*, then the following conditions must be verified: (a) $\mathring{\mathscr{E}} > \mathring{\mathscr{E}}_p$ with $\mathring{T} > -3\Sigma_s$, or (b) $\mathring{\mathscr{E}} \leq \mathring{\mathscr{E}}_p$ and $\mathring{T} > \mathring{T}_p$ where we introduced the following notation

$$\mathring{\mathscr{E}}_p \equiv \frac{1}{6(6+v_s)^2}\left[4(6+v_s)^2[2+3v_s(2+v_s)]\mathring{\rho}+3[61+4v_s(6+v_s)]\sigma_J^2\right], \tag{39}$$

$$\mathring{T}_p \equiv \frac{1}{177+60v_s}\left[-6(5+v_s)\sigma_J+\sqrt{6}\{(59+20v_s).\ \ [2(2+3v_s(2+v_s)]\mathring{\rho}-3\mathring{\mathscr{E}})+6(5+v_s)^2\sigma_J^2\right\}^{1/2}\Big].$$

This condition corresponds to the requirements $c_4 > 0$ and $\mathrm{Tr}\mathring{\mathbf{B}} < 0$ arising, in turn, from $c_5 = -\mathrm{Tr}\mathring{\mathbf{B}} > 0$ and $c_4 c_5 > 0$.

The case $\mathscr{A} = 0$, $\Phi = 0$ and $\xi = 0$. The case $(\mathscr{A}\ \Phi\ \xi\ \Omega)$ reduces to $(\mathscr{A}\ \Phi\ \xi\ \Omega\ \mathscr{B})$. We can say that the rank 6 system is *unstable* when $\mathring{\Theta} \leq 3(5\mathring{\Sigma}-2\sigma_J)/2(14+3v_s)$.

The case $\mathscr{A} = 0$, $T = 0$, $Q = 0$ and $\xi = 0$ In this case one has $(\mathscr{A}\ T\ Q\ \xi\ \Omega)$ which reduces to $(\mathscr{A}\ T\ Q\ \xi\ \Omega\ \mathscr{B})$. This system is characterized by $\dot{\Phi} = 0$, $\dot{B} = 0$, $\dot{\mathscr{E}} = -j$. There are the eigenvalues $\lambda = 0$ with multiplicity 4 and $\lambda_1 = -\sigma_J < 0$. Nevertheless, one has that $\mathscr{C}(\mathscr{A}, T, \Omega)$ and $\mathscr{C}(\mathscr{A}, Q$. These conditions cannot be satisfied by the assumptions $\rho > 0$ and the given equation of state.

B Remarks on the thermodynamical quantities

In the present article we consider a one species fluid (simple fluid), and denote, respectively, by n, s, T the particle number density, the entropy per particle and the absolute temperature as measured by comoving observers. We also introduce the volume per particle v and the energy per particle e via the relations $v \equiv 1/n$ and $e \equiv \rho/n$. In terms of these variables the first law of Thermodynamics, $\mathrm{d}e = -p\mathrm{d}v + T\mathrm{d}s$, takes the form

$$\mathrm{d}\rho = \frac{p+\rho}{n}\mathrm{d}n + nT\mathrm{d}s. \tag{40}$$

Assuming an equation of state of the form $\rho = f(n,s) \geq 0$, one obtains from equation (40) that

$$p(n,s) = n\left(\frac{\partial\rho}{\partial n}\right)_s - \rho(n,s), \quad T(n,s) = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_n. \tag{41}$$

Assuming that $\partial p/\partial\rho > 0$ we define the speed of sound $\mathrm{v}_s = \mathrm{v}_s(n,s)$ by

$$\mathrm{v}_s^2 \equiv \left(\frac{\partial p}{\partial\rho}\right)_s = \frac{n}{\rho+p}\frac{\partial p}{\partial n} > 0. \tag{42}$$

Since we are not considering particle annihilation or creation processes we consider the equation of conservation of particle number:

$$u^a\nabla_a n + n\nabla_a u^a = 0. \tag{43}$$

Combining this equation with the conservation equations and (40) we obtain

$$u^a\nabla_a s = \frac{1}{nT}u^b F_b{}^c\nabla^a F_{ac}. \tag{44}$$

When n is subject to equation (43) and T is given by equation (41), using the $1+1+2$ decomposition in a LRS one obtains the equations

$$\dot{s} = +\frac{1}{nT}Ej, \qquad \dot{n} + n\Theta = 0. \tag{45}$$

The case of an infinitely conducting plasma describes an adiabatic flow —that is, $u^a\nabla_a s = 0$, so that the entropy per particle is conserved along the flow lines. A particular case of interest is when s is a constant in both space and time. In this case the equation of state can be given in the form $p = p(\rho)$. In the following, for convenience, it is assumed that $\nabla_a p = v_s\nabla_a\rho$, where we have defined $v_s \equiv \mathrm{v}_s^2$. In this article, we have set the evolution equations and constraints for a general system. However, in a first analysis of stability, we have preferred to simplify the system in order to obtain some general intuition. Thus, in the study of the stability problem we have restricted to situations for which $dp/\rho \approx$ constant at each order. Hence, one has $\hat{v}_s = 0$ and $\dot{v}_s = 0$. For the more general case of a non-isentropic fluid, i.e. $s \neq$constant, the equation of state should be written in the form $p = p(\rho, s)$ —see [10]. This means that when considering the derivatives of the pressure p, it should be taken into account that $\nabla_a p = v_s \nabla_a \rho + (\partial p/\partial s)\nabla_a s$. The evolution equations under consideration contain terms involving $\nabla_a v_s$ and $\nabla_a s$. Some general considerations for the case v_s =constant in systems such as accretion disks, neutron stars or some cosmological models can be found, for example, in [49,48,50,51,52]. In the special case of a polytropic equation of state, where $p = \kappa\rho^\gamma$, and the constant γ is the polytropic index while $\kappa > 0$ is a constant, the condition v_s =constant (i.e. $\nabla_a v_s = 0$) can be realized for $\gamma = 1$ or ρ =constant. The case of an isothermal equation of state ($\gamma = 1$) has been discussed for example in [48,49].

References

1. Z. B. Etienne, Y. T. Liu, & S. L. Shapiro, *Relativistic magnetohydrodynamics in dynamical spacetimes: a new AMR implementation*, Phys. Rev. D **82**, 084031, (2010).
2. J. A. Font, *Numerical hydrodynamics in general relativity*, Living Rev. Rel. **6**, (2003).
3. J. A. Font, *Numerical hydrodynamics and magnetohydrodynamics in general relativity*, Living Rev. Rel. **11**, 7 (2007) .
4. M. Alcubierre, *Introduction to* $3+1$ *numerical Relativity*, Oxford University Press, 2008.
5. B. Giacommazo & L. Rezzolla, *WhiskeyMHD: a new numerical code for general relativistic MHD*, Class. Quantum Grav. **24**, S235, (2007).
6. A. Lichnerowicz, *Relativistic hydrodynamics and magnetohydrodynamics*, Benjamin, New York, 1967.
7. L. Anton, O.Zanotti, J.A.Miralles, J. M. Marti, J. M. Ibanez, J. A. Font and J. A. Pons, *Numerical 3+1 general relativistic magnetohydrodynamics: A Local characteristic approach*, Astrophys. J. **637**, 296, (2006).
8. D. Radice, L. Rezzolla and F. Galeazzi, *High-Order Fully General-Relativistic Hydrodynamics: new Approaches and Tests,* Class. Quant. Grav. **31**, 075012 ,(2014).
9. H. Witek, *Numerical Relativity in higher-dimensional space-times,* Int. J. Mod. Phys. A **28**, 1340017, (2013).
10. D. Pugliese and J.A. Valiente Kroon, *On the evolution equations for ideal magnetohydrodynamics in curved spacetime*, Gen. Rel. Grav. **44**, 2785, (2012).
11. M. Shibata & Y. Sekiguchi, *Magnetohydrodynamics in full general relativity: formulations and tests*, Phys. Rev. D **72**, 044014, (2005).
12. T. W. Baumgarte & S. L. Shapiro, *General relativistic magnetohydrodynamics for the numerical construction of dynamical spacetimes*, Astrophys. J. **585**, 921, (2003).
13. C. Palenzuela, D. Garrett, L. Lehner, & S. Liebling, *Magnetospheres of black hole systems in force-free plasma*, Phys. Rev. D **82**, 044045, (2010).
14. A.M. Anile *Relativistic fluids and magneto-fluids: With applications in astrophysics and plasma physics*, Cambridge UniversityPress, Cambridge, U.K.; New York, U.S.A., 1989.
15. M. M. Disconzi, *On the well-posedness of relativistic viscous fluids,* Nonlinearity **27**, 1915, (2014).
16. C. Lubbe and J. A. Valiente Kroon, *A conformal approach for the analysis of the non-linear stability of pure radiation cosmologies*, Annals Phys. **328**, 1, (2013).
17. C. Clarkson, *A Covariant approach for perturbations of rotationally symmetric spacetimes*, Phys. Rev. D **76**, 104034, (2007).
18. E. Horst, *Symmetric Plasmas and Their Decay*, Commun. Math. Phys. **126**, 613-633. (1990).
19. S. C. Hsu, T. J. Awe, S. Brockington, A. Case, J. T. Cassibry, G. Kagan, S. J. Messer, M. Stanic, X. Tang, D. R. Welch, and F. D. Witherspoon, *Spherically Imploding Plasma Liners as a Standoff Driver for Magnetoinertial Fusion*, Ieee transactions on plasma science **40**, 5, (2012).
20. Tai-Ho Tan and Joseph E. Borovsky, *Spherically symmetric high-velocity plasma expansions into background gases*, Journal of Plasma Physics **35**, 02, 239, (1986).
21. P. D. Laskyand A. W. C. Lun *Gravitational collapse of spherically symmetric plasmas in Einstein-Maxwell spacetimes* . Phys. Rev. D **75**, 104010, (2007).
22. R. L. Viana, R. A. Clemente and S. R. Lopes, *Spherically symmetric stationary MHD equilibria with azimuthal rotation*, Plasma Phys. Control. Fusion **39**, 197 (1997).
23. Yan Guo and A. Shadi Tahvildar-Zadeh, *Formation of singularities in relativistic fluid dynamics and in spherically symmetric plasma dynamics*, Contem. Math., **238**, 151-161. (1999).
24. C. A. Clarkson, M. Marklund, G. Betschart, and P. K. S. Dunsby, *The Electromagnetic Signature of Black Hole Ring-Down* , Astrophys. J. **613**, 492, (2004).
25. G. F. R. Ellis & H. van Elst, *Cosmological models: Cargese lectures 1998*, NATO Adv. Study Inst. Ser. C. Math. Phys. Sci. **541**, 1, (1998).
26. R. B. Burston, *1+1+2 gravitational perturbations on LRS class II spacetimes: decoupling gravito-electromagnetic tensor harmonic amplitudes* , Class. Quantum Grav. **25**, 075004 (2008).

27. C. A. Clarkson and R. K. Barrett, *Covariant perturbations of Schwarzschild black holes* , Class. Quantum Grav. **20**, 3855, (2003).
28. H. van Elsty and, G. F. R. Ellis, *The covariant approach to LRS perfect fluid spacetime geometries* ,Class. Quantum Grav. **13**, 1099, (1996).
29. R. B. Burston and A. W. C. Lun, *1+1+2 Electromagnetic perturbations on general LRS space-times: Regge-Wheeler and Bardeen-Press equations*, Class. Quant. Grav. **25**, 075003, (2008).
30. R. B. Burston, *1+1+2 Electromagnetic perturbations on non-vacuum LRS class II space-times: Decoupling scalar and 2-vector harmonic amplitudes*, Class. Quant. Grav. **25**, 075002, (2008).
31. O. Zanotti and D. Pugliese, *Von Zeipel's theorem for a magnetized circular flow around a compact object*, Gen. Rel. Grav. **47**, 4, 44 (2015).
32. S. S. Komissarov, *Magnetized Tori around Kerr Black Holes: Analytic Solutions with a Toroidal Magnetic Field*, Mon. Not. R. Astron. Soc. **368**, 993-1000, (2006).
33. J.D. Bekenstein, E. Oron, *New conservation laws in general-relativistic magnetohydrodynamics*, Phys. Rev. D **18**, 1809, (1978).
34. J.D. Bekenstein, E. Oron, *Interior magnetohydrodynamic structure of a rotating relativistic star*. Phys. Rev. D **19**, 2827,72837 (1979).
35. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt ,*Exact Solutions of Einstein's Field Equations*, Cambridge Monographs on Mathematical Physics, Paperback 2009.
36. G. Betschart, and C. A. Clarkson, *Scalar field and electromagnetic perturbations on locally rotationally symmetric spacetimes* , Class. Quantum Grav. **21**, 5587 (2004).
37. O. Reula, *Hyperbolic methods for Einstein's equations*, Living Rev. Rel. **3**, 1 (1998).
38. O. Reula, *Exponential decay for small nonlinear perturbations of expanding flat homogeneous cosmologies*, Phys. Rev. D **60**, 083507, (1999).
39. A. Alho, F. C. Mena, & J. A. Valiente Kroon, *The Einstein-Friedrich-nonlinear scalar field system and the stability of scalar field Cosmologies*, In `arXiv:1006.3778`, (2010).
40. H. O. Kreiss and J. Lorenz, *Stability for time-dependent differential equations*. Acta Numerica **7**, 203 (1998).
41. H. O. Kreiss, O. E. Ortiz and O. A. Reula, *Stability of quasi-linear hyperbolic dissipative systems*. Journal of Differential Equations **142**, 78, (1998).
42. H. O. Kreiss, G. B. Nagy, O. E. Ortiz and O. A. Reula, *Global existence and exponential decay for hyperbolic dissipative relativistic fluid theories*. Journal of Mathematical Physics **38**, 5272, (1997).
43. O. E. Ortiz, *Stability of nonconservative hyperbolic systems and relativistic dissipative fluids*. Journal of Mathematical Physics **42**, 1426 (2001).
44. M. Marklund and C. Clarkson, *The General relativistic MHD dynamo*, Mon. Not. Roy. Astron. Soc. **358**, (2005) 892.
45. Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials: Critical Points, Zeros and Extremal Properties*. London Mathematical Society Monographs, Clarendon Press, 2002.
46. S. Barnett, *A New Formulation of the Theorems of Hurwitz, Routh and Sturm* J. Inst. Maths Applies **7**, 240, (1971).
47. Y. Choquet-Bruhat, *General Relativity and the Einstein equations*, Oxford University Press, 2008.
48. L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics*, Oxford University Press, 2013.
49. D. Pugliese, G. Montani and M. G. Bernardini, Mon. Not. Roy. Astron. Soc. **428**, (2), 952 (2012)
50. M. Camenzind *Compact Objects in Astrophysics White Dwarfs, Neutron Stars and Black Holes*, Springer-Verlag Berlin Heidelberg, 2007.
51. J. Frank, A. King, D. Raine, *Accretion Power in Astrophysics* Cambridge University Press 2002.
52. G. W. Collins, *The Fundamentals of Stellar Astrophysics* W H Freeman & Co, 1989.