Describing Quaternary Codes Using Binary Codes

Fatma Salim Ali Al Kharoosi

School of Mathematical Sciences

Queen Mary

University of London

Thesis submitted to University of London

for the degree of Doctor of Philosophy in Mathematics

London, January 2011

Abstract

For a quaternary code $\mathbf{C}$ of length n, define a pair of binary codes $\{C_1, C_2\}$ as:

-$C_1 = \mathbf{C} \bmod 2$

-$C_2 = h(\mathbf{C} \cap 2\mathbb{Z}_4^n)$

where h is a bijection from $2\mathbb{Z}_4$ to $\mathbb{Z}_2$ mapping 0 to 0 and 2 to 1 and for the extension to a map acting coordinatewise. Here $C_1 \leq C_2$.

For a pair of binary codes $\{C_1, C_2\}$ with $C_1 \leq C_2$, let $\mathcal{C}(C_1, C_2)$ be the set of $\mathbb{Z}_4$-codes giving rise to this binary pair as defined above. Our main goal is to describe the set $\mathcal{C}(C_1, C_2)$ using the binary pair of codes $\{C_1, C_2\}$.

In Chapter 1, we give some preliminaries. In Chapter 2, we start with a general description of codes $\{C_1, C_2\}$ which give cardinality of $\mathcal{C}(C_1, C_2)$. Then we show that $\mathcal{C}(C_1, C_2) \simeq C_1^* \otimes \mathbb{Z}_2^n / C_2$. The cohomology of $\mathcal{C}(C_1, C_2)$ is given in Section (2.2). Then we end chapter 2 with a description of dual codes of $\mathcal{C}(C_1, C_2)$. Chapter 3 is about weight enumerators of codes in $\mathcal{C}(C_1, C_2)$. The average swe is given in terms of weight enumerators of C_1 and C_2 in Section(3.1) as

$$\overline{\text{swe}}(x, y, z) = \frac{|C_2|}{2^n}(\text{we}_{C_1}(x+z, 2y) - (x+z)^n) + \text{we}_{C_2}(x, z)$$

Detailed computations of swe's of codes in $\mathcal{C}(C_1, C_2)$ using codes $\{C_1, C_2\}$ is then given. Information about different weight enumerators of codes in $\mathcal{C}(C_1, C_2)$ is given in Section (3.2). These weight enumerators are included in an affine space of polynomials. Then we end chapter 3 with a description of weight enumerators of self dual codes. Chapter 4 deals with actions of

the automorphism group $G = \mathrm{Aut}(C_1) \cap \mathrm{Aut}(C_2) \leq S_n$ on $\mathcal{C}(C_1, C_2)$ which preserves cwe of codes. Corresponding action on $C_1^* \otimes \mathbb{Z}_2^n / C_2$ is explained in this chapter. Changing signs of coordinates can be defined as an action of $\mathbb{Z}_2^n$ on $\mathcal{C}(C_1, C_2)$. This action preserves swe of codes. Corresponding action on $C_1^* \otimes \mathbb{Z}_2^n / C_2$ is provided in this chapter.

In the appendix, we give a complete description of $\mathbb{Z}_4$-codes in $\mathcal{C}(C_1, C_2)$ with $C_1 = C_2 =$ Extended Hamming Code of length 8. A programming code in GAP for computing derivations is given. And a description of the affine space containing the swe's of $\mathbb{Z}_4$-codes is given with examples of different $C_1 = C_2$ having same weight enumerator.

To my first teacher; Grandmother.

To you my precious parents;

Salim Ali Al Kharousi and

Jamila Rashid Al Ma'wali.

Acknowledgments

I would like to thank my supervisor, Prof. Peter Cameron for his great support and patience with my lack of knowledge, impatience and being slow. Many thanks to my second supervisor, Leonard Soicher, my colleagues and the support stuff for helping me when I needed. I am grateful to Sultan Qaboos University and the government of the Sultanate of Oman for providing the scholarship.

Glad to have a chance to thank people who had a positive impact in my life. Without their support, love, care and help reaching this level of study did not seem to be possible.

My family for their support, help and prayers throughout my life. My elementary teacher, Fatma bint Muhammad, the first person who encouraged me to do math. My secondary teachers, Muna Al Harami, Sausan Hilmi and Ms Grace who believed in my abilities. My great BSc teachers of Sultan Qaboos University from whom I have learned a lot. People who helped me a lot and I keep going back to them for help, Prof. Ibrahim Elteyeb, Dr. Magda Taleb, Dr. Cengiz Ercill, Dr Yasher, Dr Ali Benmerzouga, Dr. Saif Al Bahri, Dr. Mohd S. Khan, Dr. B. Yallaoui.

My friends and MSc teachers at Michigan State University. My friend Haniza Yon and East Lansing people for giving me a great support I have never dreamed of, being away from home.

Last but not least, many thanks to my sweet friend Khairia Ahmed for the help, support, love and care I have never dreamed of. You supported me when I most needed help.

Contents

Chapter 1

Introduction

The basic goal of **coding theory** is transferring data in an efficient and reliable way. Normally, data is transmitted using **binary codes**; a set of sequences of 0's and 1's called codewords. A code is said to be **linear** iff the sum of any two codewords is a codeword (sum is taken coordinatewise). An $[n, k]$ linear binary code is an additive subgroup of $\mathbb{Z}_2^n$ of dimension k. A **quaternary code C** of length n is an additive subgroup of $\mathbb{Z}_4^n$. Throughout, $\mathbb{Z}_k$ stands for the integers modulo k.

1.1 Weight Distribution of Codes

The weight distribution of a code is given by **weight enumerators**. Several weight enumerators are associated with a quaternary code **C**. The **complete weight enumerator** of **C** is

$$\mathrm{cwe}_{\mathbf{C}}(x, y, z, w) = \sum_{a \in \mathbf{C}} x^{n_0(a)} y^{n_1(a)} z^{n_2(a)} w^{n_3(a)} \tag{1.1}$$

where $n_j(a)$ is the number of components congruent to j mod 4. Permuting coordinates of a code fixes number of 0's, 1's, 2's and 3's in codewords and so fixing its complete weight enumerator.

Two quaternary codes are said to be **permutation equivalent** if one can be obtained from the other by permuting its coordinates. For many applications there is no need to distinguish between $+1$ and -1 components and so we say that two codes are **equivalent** if one can be obtained from the other by permuting or changing signs of certain coordinates (this is the same as multiplying some coordinates by -1). Equivalent codes might have distinct cwe's. Since multiplying by -1 only affects odd coordinates, we need a weight enumerator that treats odd entries the same. This would be the **symmetrized weight enumerator**, swe, obtained by identifying y and w in (1.1)

$$\mathrm{swe}_{\mathbf{C}}(x,y,z) = \mathrm{cwe}_{\mathbf{C}}(x,y,z,y) \tag{1.2}$$

For a code C of length n over $\mathbb{Z}_k$, define the **Lee distance** between codewords $a, b \in C$ to be

$$d_L(a,b) = \sum_{i=1}^{n} \min(|\{a\}_i - \{b\}_i|, |\{b\}_i - \{a\}_i|) \bmod k$$

The **Lee weight** of a codeword, denoted by $\mathrm{wt}_L(x)$, is its Lee distance from the zero word. That is $\mathrm{wt}_L(x) = d_L(x,0)$. Lee weights of 0, 1, 2 and $3 \in \mathbb{Z}_4$ are 0, 1, 2 and 1 respectively. The Lee weight of a word is the rational sum of Lee weights of its components. The **Lee weight enumerator** of a code describes Lee weight distribution of words in a code. It is defined to be

$$\mathrm{Lee}_{\mathbf{C}}(x,y) = \sum_{a \in \mathbf{C}} x^{2n-\mathrm{wt}_L(a)} y^{\mathrm{wt}_L(a)} = \mathrm{swe}_C(x^2, xy, y^2) \tag{1.3}$$

For binary codes define the weight enumerator, $\mathrm{we}_C(x,y)$, to be

$$\mathrm{we}_C(x,y) = \sum_{a \in C} x^{n_0(a)} y^{n_1(a)} \tag{1.4}$$

where $n_j(a)$ is the number of components congruent to j mod 2.

Define a map $\phi : \mathbb{Z}_4^n \to \mathbb{Z}_2^{2n}$ called the **Gray map**[1] sending:

$$0 \to 00$$
$$1 \to 01$$
$$2 \to 11$$
$$3 \to 10$$

and coordinatewise mapping $x \in \mathbb{Z}_4^n$ to $\mathbb{Z}_2^{2n}$.

The **Hamming distance** between two words x and y of $\mathbb{Z}_k^n$ is the number of places they differ denoted by $d(x, y)$. Accordingly **Hamming weight** of a codeword, denoted $\text{wt}(x)$, is defined to be number of nonzero entries. That is $\text{wt}(x) = d(x, 0)$. Note that $d(x, y) \neq d_L(x, y)$ in general. They are equal for binary codes. A map $\theta : \mathbb{Z}_k^n \to \mathbb{Z}_r^m$ is said to be **distance preserving** if $d_L(a, b) = d_L(\theta(a), \theta(b))$ for all $a, b \in \mathbb{Z}_k^n$.

Theorem 1.1 *The Gray map is a distance preserving map from $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^{2n}$.*

Proof This can be deduced directly from the definition of the Gray map since $d_L(i, j) = d(\phi(i), \phi(j))$ for $i, j \in \{0, 1, 2, 3\}$ mod 4 and so $d_L(a, b) = d(\phi(a), \phi(b))$ for $a, b \in \mathbb{Z}_4^n$.

From definitions above, if $C = \phi(\mathbf{C})$ is the gray map image of a $\mathbb{Z}_4$-code $\mathbf{C}$, then

$$\text{we}_C(x, y) = \text{Lee}_{\mathbf{C}}(x, y) \tag{1.5}$$

Hence, the Gray map preserves weight distribution of codes.

A binary code is said to be **distance invariant** if the Hamming weight distributions of its translates $C + u$ are the same for all $u \in C$. In general, for codes over $\mathbb{Z}_k$, define distance invariant codes C to have fixed Lee weight distributions for all translations of the form $C - u$ for $u \in C$. This is satisfied for linear codes since $C - u = C$ but not always satisfied for nonlinear codes.

Theorem 1.2 *If* $\mathbf{C}$ *is a quaternary code then* $C = \phi(\mathbf{C})$ *(its Gray map image) is distance invariant.*

Proof $\mathbf{C}$ is linear so distance invariant with respect to the Lee distance. And the result follows since $d_L(a, b) = d(\phi(a), \phi(b))$.

A generator matrix is a matrix whose rows form a basis for the linear code.

Proposition 1.1 *Any quaternary code is permutation equivalent to a code with generator matrix of the form:*

$$\begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix} \tag{1.6}$$

where A *and* C *are matrices over* $\mathbb{Z}_2$ *and* B *is a matrix over* $\mathbb{Z}_4$*. The code is then an abelian group of order* $|\mathbf{C}| = 4^{k_1}2^{k_2}$*. [1]*

Proof [9] $\mathbb{Z}_4$-codes are finite subgroups of $\mathbb{Z}_4^n$. So we can have a finite set of generators $\{v_1, \ldots, v_k\}$ for any $\mathbb{Z}_4$-code, $\mathbf{C}$. Let M be a matrix with rows $\{v_i\}$. A generator matrix of the form (1.6) can be obtained from M.

If $\mathbf{C} = \langle\{v_1, \ldots, v_k\}\rangle$ then if $y = a_1v_1 + \ldots + a_kv_k$ with a_1 a unit then replacing v_1 by y give another generating set $\{y, v_2, \ldots, v_k\}$ since

$$v_1 = a_1^{-1}y - a_1^{-1}a_2v_2 - \ldots - a_1^{-1}a_kv_k$$

Hence,

- replacing a generating vector by its unit multiple give another generating set.
- replacing a vector v_i in the generating set by $v_i\pm$ a multiple of another generating vector give another generating set.

These operations are called elementary row operations. Permuting coordinates of words of a code give a permutation equivalent code. Performing a sequence of row operations on M along with permutation of coordinates give a generator matrix of the form (1.6) as follows.

Let $\phi(M)$ be an entry with the minimum of the absolute value of nonzero entries of M (this is an entry with minimum Lee weight). With $\mathbb{Z}_4$-codes $\phi(M) = 1, 2$, or 3. If $M = 0$ then there is nothing to be done. Let $M \neq 0$ and A be a matrix obtained from M by performing row operations and permutations as in the algorithm below.

1. If M consists of 0's and 2's only then go to step (7) otherwise permute rows and columns of M to get A with $\phi(A)$ in the upper left corner (i.e in a_{11}). If $\phi(A) = 3$ then multiply the first row by -1 to get $a_{11} = 1$.
2. Subtract the appropriate multiple of the first row from rows $\{2, \ldots, k\}$ to get $a_{i1} = 0$ for all $2 \leq i \leq k$. Let A be the matrix obtained.
3. Let A_1 be the matrix obtained by deleting the first row and column from A. If $A_1 = 0$ go to step (10). If A_1 consists of 0's and 2's only then go to step (7). Otherwise, permute rows $\{2, \ldots, k\}$ and columns $\{2, \ldots, n\}$ of A to get $a_{22} = \phi(A_1)$. If $a_{22} = 3$ then multiply the 2nd row by -1 to change a_{22} to 1.
4. Subtract the appropriate multiple of the second row from rows $j \neq 2$ to get $a_{j2} = 0$ for all $j \neq 2$.

5. Let $A_{1,\ldots,i}$ be the matrix obtained by deleting rows and columns $1,\ldots,i$ form matrix A. If $A_{1,\ldots,i}=0$ go to step (10). Else if it consists of 0's and 2's only then go to step (7). Otherwise, repeat the process above: finding $\phi(A_{1,\ldots,i})$, permuting rows $\{i+1,\ldots,k\}$ and columns $\{i+1,\ldots,n\}$ to have $a_{i+1,i+1}=\phi(A_{1,\ldots,i})$, then subtracting an appropriate multiple of the $(i+1)$th row from other rows to have a column $(0,\ldots,0,1,0,\ldots,0)^T$ where the 1 is located in the $i+1$st row. The process continues as long as $A_{1,\ldots,i}$ have an entry ± 1.

6. When we reach a stage where $A_{1,\ldots,k_1}=0$ or $\phi(A_{1,\ldots,k_1})=2$, matrix A would have first k_1 coumns of the form $\begin{pmatrix} I_{k_1} \\ 0 \end{pmatrix}$. We still need to refurbish the rest of the matrix.

7. If $A_{1,\ldots,k_1}=0$ then go to step 10 otherwise permute rows $\{k_1+1,\ldots,k\}$ and columns $\{k_1+1,\ldots,n\}$ to have $a_{k_1+1,k_1+1}=2$.

8. Subtract the appropriate multiple of the (k_1+1)th row from the jth rows for $j\neq k_1+1$ to get $a_{jk_1+1}=0$ for all $k_1+2\leq j\leq k$ and $a_{jk_1+1}=0$ or 1 for $1\leq i\leq k_1$.

9. Repeat the process (steps 7 and 8) for $A_{1,\ldots,i}$ with $j_i>k_1+1$ until we get $A_{1,\ldots,i}=0$ or until $j=k$.

10. Ignoring zero rows if any, by the end of this stage we get a generator matrix:
$$\begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix}$$
with matrices A and C having entries 0's and 1's and B with entries from $\mathbb{Z}_4$.

Define **an inner (dot) product** on $\mathbb{Z}_k^n$ to be $a \cdot b = a_1 b_1 + \ldots + a_n b_n (\bmod k)$. The **dual code**, $C^\perp$, is defined to be the subgroup of $\mathbb{Z}_k^n$ annihilating words of C. That is $C^\perp = \{x \in \mathbb{Z}_k^n$ such that $x \cdot a = 0$ for all $a \in C\}$. A code is said to be **self orthogonal** if $C \subseteq C^\perp$. It is called **self dual** if $C = C^\perp$. For $a, b \in \mathbb{Z}_k^n$, define $a * b$ to be the vector in $\mathbb{Z}_k^n$ obtained by multiplying entries of vectors a and b coordinatewise.

Theorem 1.3 *The dual code of a $\mathbb{Z}_4$-code with generator* (1.6) *has generator*

$$G_\perp = \begin{pmatrix} -B^{tr} - C^{tr}A^{tr} & C^{tr} & I_{n-k_1-k_2} \\ 2A^{tr} & 2I_{k_2} & 0 \end{pmatrix} \tag{1.7}$$

where M^{tr} is the transposed matrix of the matrix M.

Proof Let G be the general generator matrix of a $\mathbb{Z}_4$-code:

$$G = \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix}$$

then since $(M_1 M_2)^{tr} = M_2^{tr} \cdot M_1^{tr}$ and $(M^{tr})^{tr} = M$, we have

$$\begin{aligned} G \cdot G_\perp^{tr} &= \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix} \cdot \begin{pmatrix} -B - AC & 2A \\ C & 2I_{k_2} \\ I_{n-k_1-k_2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -B - AC + AC + B & 2A + 2A \\ 2C + 2C & 4I_{k_2} \end{pmatrix} \\ &= 0 \bmod 4 \end{aligned}$$

and

$$\begin{aligned} G_\perp \cdot G^{tr} &= \begin{pmatrix} -B^{tr} - C^{tr}A^{tr} & C^{tr} & I_{n-k_1-k_2} \\ 2A^{tr} & 2I_{k_2} & 0 \end{pmatrix} \cdot \begin{pmatrix} I_{k_1} & 0 \\ A^{tr} & 2I_{k_2} \\ B^{tr} & 2C^{tr} \end{pmatrix} \\ &= \begin{pmatrix} -B^{tr} - C^{tr}A^{tr} + C^{tr}A^{tr} + B^{tr} & 2C^{tr} + 2C^{tr} \\ 2A^{tr} + 2A^{tr} & 4I_{k_2} \end{pmatrix} \\ &= 0 \bmod 4 \end{aligned}$$

So row space of $G_\perp$ is in the dual space of the code. But cardinality of the dual space is

$$\begin{aligned} \frac{4^n}{4^{k_1} \cdot 2^{k_2}} &= \frac{4^n \cdot 2^{k_2}}{4^{k_1} \cdot 2^{k_2} \cdot 2^{k_2}} \\ &= 4^{(n-k_1-k_2)} \cdot 2^{k_2} \\ &= \text{cardinality of the row space of } G_\perp \end{aligned}$$

The **MacWilliams** Theorems give weight enumerators of dual codes $\mathbf{C}^\perp$ in terms of weight enumerators of the codes[1].

For $\mathbb{Z}_4$-codes:

$$\begin{aligned} \text{cwe}_{\mathbf{C}^\perp}(x, y, z, w) &= \frac{1}{|\mathbf{C}|}\text{cwe}_{\mathbf{C}}(x + y + z + w, x + iy - z - iw, \\ &\qquad x - y + z - w, x - iy - z + iw) \\ \text{swe}_{\mathbf{C}^\perp}(x, y, z) &= \frac{1}{|\mathbf{C}|}\text{swe}_{\mathbf{C}}(x + 2y + z, x - z, x - 2y + z) \\ \text{Lee}_{\mathbf{C}^\perp}(x, y) &= \frac{1}{|\mathbf{C}|}\text{Lee}_{\mathbf{C}}(x + y, x - y) \end{aligned}$$

and for binary codes:

$$\text{we}_{C^\perp}(x, y) = \frac{1}{|C|}\text{we}_C(x + y, x - y)$$

Two binary codes are said to be **formal duals** if they are nonlinear but the weight enumerator of one is the MacWilliam transform of the weight enumerator of the other code. A code is said to be **formally self dual** if its weights enumerator is equal to its MacWilliams transform.

Theorem 1.4 *For a quaternary code* $\mathbf{C}$*, if* ϕ *is the Gray map then* $\phi(\mathbf{C})$ *and* $\phi(\mathbf{C}^{\perp})$ *are formal duals. For a self dual quaternary code,* $\phi(\mathbf{C})$ *is formally self dual.*

Proof Result can be obtained as follows:

$$\begin{aligned}
\mathrm{we}_{\phi(\mathbf{C}^{\perp})}(x,y) &= \mathrm{Lee}_{\mathbf{C}^{\perp}}(x,y) \\
&= \frac{1}{|\mathbf{C}|}\mathrm{Lee}_{\mathbf{C}}(x+y,x-y) \\
&= \frac{1}{|\mathbf{C}|}\mathrm{we}_{\phi(\mathbf{C})}(x+y,x-y)
\end{aligned}$$

Linear codes are easier to describe, encode and decode compared to nonlinear codes. Certain nonlinear binary codes though contain more codewords than any known linear codes with the same length and minimum distance. These include the Nordstrom-Robinson code, Kerdock, Preparata, Goethals and Delsarte-Goethals codes. These codes have excellent error correcting capabilities. The Kerdock and Preparata codes are formal duals. These codes exist for all lengths $n = 4^m \geq 16$. At length 16 they coincide giving the Nordstrom-Robinson code making it formally self dual. There are many versions of the Nordstrom-Robinson code, Kerdock and Preparata codes and it was not clear if these codes are duals in some more algebraic sense. It was then shown that when the Kerdock and Preparata codes are properly defined, they can be simply constructed as binary images under the Gray map of quaternary codes. So they are distance invariant. The $\mathbb{Z}_4$-codes mapped to Kerdock and Preparata codes using the Gray map are shown to be duals. Decoding codes mentioned is greatly simplified by working in the

$\mathbb{Z}_4$-domain, where they are linear. Decoding the Nordstrom-Robinson code and Preparata codes is especially simple[1].

1.2 What can we say about codewords

Let a, a' be two words of a binary code, then we can rearrange their coordinates to have sets of coordinates with relations:

$$\begin{array}{ccccc} a & 0\ldots0 & 1\ldots1 & 1\ldots1 & 0\ldots0 \\ a' & 0\ldots0 & 1\ldots1 & 0\ldots0 & 1\ldots1 \\ & S_0 & S_1 & S_2 & S_3 \end{array}$$

So we would have,

- Sum of two words of even weights or two words of odd weights is an even weight word since $\mathrm{wt}(a+a') = \mathrm{wt}(a) + \mathrm{wt}(a') - 2\,\mathrm{wt}(a * a')$.
- $|S_1|$, $|S_2|$ and $|S_3|$ have the same parity when looking at pairs of words of even weights. (Given that $|S_1|+|S_2|$ is even and $|S_1|+|S_3|$ is even. If $|S_1|$ is even(odd) then $|S_2|$ and $|S_3|$ are even(odd). If $|S_2|$ is even(odd) then $|S_1|$ is even(odd) so is $|S_3|$).
- $a \cdot a' = |S_1| \bmod 2 = \mathrm{wt}(a * a')$.
- In a self orthogonal code C, all words have even weights since $a \cdot a = 0 \bmod 2$ for all $a \in C$. But $a \cdot a' = |S_1| \bmod 2$. So cardinalities of $|S_1|$, $|S_2|$ and $|S_3|$ are even for all pairs of words $a, a' \in C$ since $a \cdot a' = 0 \bmod 2$.
- $\mathrm{wt}(a+a') = |S_2| + |S_3| = |d(a, a')|$.

Also note that in quaternary codes,

- If $\{t_1, t_2\} \in 2\mathbb{Z}_4^n$ then $t_1 \cdot t_2 = (0, \ldots, 0) \bmod 4$ since $0.0 = 0.2 = 2.0 = 2.2 = 0 \bmod 4$.
- $x = 0 \bmod 4 \Rightarrow x = 0 \bmod 2$ but the converse is not always true.
- In general for any even integer k, in a code over the ring $\mathbb{Z}_k$, sum of words of even weights or odd weights is a word of even weight. We can show this similarly as above. We can rearrange coordinates of a and a' in this case to have sets of coordinates with relations:

a	even	odd	even	odd
a'	odd	odd	even	even
	S_0	S_1	S_2	S_3

  if $\mathrm{wt}(a)$ and $\mathrm{wt}(a')$ are even then $|S_1| + |S_3|$ is even so is $|S_0| + |S_1|$. Giving us $|S_0| + 2|S_1| + |S_3|$ even or $|S_0| + |S_3|$ even. But these are the coordinates $a + a'$ of odd parity. Hence, $a + a'$ have even weight.

Proposition 1.2 *In a binary linear code C, either all codewords begin with* 0 *or exactly $\frac{1}{2}$ of them begin with* 0.*(This can be shown to be true for any code over a field $\mathbb{F}_q$ with $\frac{1}{2}$ replaced by $\frac{1}{q}$). This is also true for any position, $1 \leq i \leq n$[3].*

Proof Consider the map $\theta : C \mapsto \mathbb{Z}_2$ given by $\theta(v) = \{v\}_1$, mapping codewords to their first coordinate. This is linear and the image have dimension 0 or 1. So the kernel has codimension 0 or 1. That is all codewords get mapped to 0 or half of them get mapped to 0.

Proposition 1.3 *In a quaternary code* $\mathbf{C}$*, either all codewords begin with* 0*, exactly half of them begin with* 0 *(the other half beginning with* 2*) or exactly* $\frac{1}{4}th$ *of the codewords begin with* 0 *(and* $\frac{1}{4}$ *begin with* ± 1 *and* 2*). This is also true for all other coordinate position.*

Proof Similar to proof of proposition (2.1). Consider the map $\theta : \mathbf{C} \mapsto \mathbb{Z}_4$ given by $\theta(v) = \{v\}_1$ mapping codewords to their first coordinate. This is linear and the image has order 1, 2 or 4. So the kernel has index 1, 2 or 4. That is all codewords get mapped to 0, half of them get mapped to 0 or $\frac{1}{4}th$ get mapped to 0 (respectively $\frac{1}{4}th$ get mapped to ± 1 and 2).

Hence, if for all positions, $1 \leq i \leq n$, some codewords have 1 in the ith position then sum of Lee weights of all codewords in a $\mathbb{Z}_4$-code of length n having C_1 of dimension k_1 and C_2 of dimension $k_1 + k_2$ is

$$\frac{n}{4}(4^{k_1} \cdot 2^{k_2} + 2 \times 4^{k_1} \cdot 2^{k_2} + 4^{k_1} \cdot 2^{k_2}) = n4^{k_1} \cdot 2^{k_2} \tag{1.8}$$

Minimum weight is less than the average weight of non zero words. Therefore, Minimum weight $\leq \frac{n4^{k_1}.2^{k_2}}{4^{k_1}.2^{k_2}-1} = n + \frac{n}{|\mathbf{C}|-1}$. So, if $n < |\mathbf{C}| - 1$, then the minimum weight is less than n.

1.3 Group Extensions

A sequence of groups and group homomorphisms: $\ldots \to A_1 \to^{\phi_1} A_2 \to^{\phi_2} A_3 \to \ldots$ is said to be **exact** at A_2 if $\mathrm{Im}(\phi_1) = \ker(\phi_2)$. A sequence is said to be exact if it is exact at every point of the sequence.

If $0 \to^{\phi_0} A_1 \to^{\phi_1} A_2 \to^{\phi_2} A_3 \to^{\phi_3} 0$ is exact then ϕ_1 is one to one since $\mathrm{Im}(\phi_0) = 0 = \ker(\phi_1)$ and ϕ_2 is onto since $\ker(\phi_3) = A_3 = \mathrm{Im}(\phi_2)$. This sequence is called a **short exact sequence**.

Let E be a group, A be an abelian normal subgroup of E and $G = E/A$.

Definition **An extension** of A by G is a short exact sequence

$$0 \to A \to E \to G \to 1$$

Definition Let $0 \to A \to E \to^{\pi} G \to 1$ be an extension. **A lifting** is a function; $\lambda : G \to E$ with $\pi\lambda = I_G$ and $\lambda 1 = 0$.

Definition An Extension, $0 \to A \to E \to^{\pi} G \to 1$, is **split** if there is a homomorphism $\lambda : G \mapsto E$ such that $\pi\lambda = I_G$. The middle group is then called a **semidirect product** denoted by $A \rtimes G$. This is isomorphic to the direct product $A \times G$.

An extension is split iff E contains a subgroup $C \simeq G$ with $A + C = E$ and $A \cap C = 0$. Here, C is called a complement of A in E. An extension of A by G determines an action of G on A. Since A is normal in E. We can let G act on A as

$$x \cdot a = \lambda x + a - \lambda x \tag{1.9}$$

Given an extension, $0 \to A \to E \to^{\pi} G \to 1$. Choose a lifting, λ. Then $\mathrm{Im}(\lambda)$ is transversal (a complete set of coset representatives) of A in E. Since $\lambda 1 = 0$, every element of E can be uniquely expressed as $a + \lambda x$. Both λxy and $\lambda x + \lambda y$ represent the same coset of A then

$$\lambda x + \lambda y = [x, y] + \lambda xy \tag{1.10}$$

for some $[x, y] \in A$. With G's conjugation action on A,

$$x \cdot a + \lambda x = \lambda x + a \tag{1.11}$$

Definition The function $[\ ,\ ] : G \times G \mapsto A$ defined by (1.10) is called a **factor set**. The set of all factor sets is a group under pointwise addition denoted by $Z^2(G, A)$.

A function $[\ ,\ ] : G \times G \mapsto A$ for a G-module A is a factor set if and only if

$$[x, 1] = [1, x] = 0 \quad \text{and} \tag{1.12}$$

$$x[y, z] - [xy, z] + [x, yz] - [x, y] = 0 \tag{1.13}$$

(1.12) can be deduced from equation(1.10), by letting $y = I_G$. Equation (1.11) and associativity of E give (1.13) as follows.

$$\begin{aligned}
(\lambda x + \lambda y) + \lambda z &= \lambda x + (\lambda y + \lambda z) \\
[x, y] + \lambda xy + \lambda z &= \lambda x + [y, z] + \lambda yz \\
[x, y] + [xy, z] + \lambda xyz &= x[y, z] + \lambda x + \lambda yz \\
&= x[y, z] + [x, yz] + \lambda xyz
\end{aligned}$$

For the converse, let $[\ ,\ ] : G \times G \mapsto A$ satisfy (1.12) and (1.13). Define E to be the set of all ordered pairs $(a, x) \in A \times G$ with addition:

$$(a, x) + (b, y) = (a + xb + [x, y], xy)$$

identity $(0, I)$ and $-(a, x) = (-x^{-1}a - x^{-1}[x, x^{-1}], x^{-1})$ then E is a group. Define $\pi : E \mapsto G$ by $(a, x) \mapsto x$ then π is onto with kernel (a, I_G) that can be identified by A. A lifting can be defined as $\lambda x = (0, x)$.

For an extension of A by G, we can produce two different factor sets by choosing different liftings. The resulting factor sets would be considered equivalent. Following is a way to determine equivalent factor sets.

Theorem 1.5 *Let* $0 \to A \to E \to^{\pi} G \to 1$ *be an extension. Let,* λ *and* λ' *be different liftings. If* $[\ ,\ ]$ *and* $(\ ,\ )$ *are their corresponding factor sets then there is a function* $\varphi : G \mapsto A$ *satisfying:*

- $\varphi(1) = 0$ *and*
- $[x, y] - (x, y) = x\varphi(y) - \varphi(xy) + \varphi(x)$

Definition Set of all functions $f : G \times G \mapsto A$ for which there is a function $\varphi : G \mapsto A$ with $\varphi(1) = 0$ such that $f(x, y) = x\varphi(y) - \varphi(xy) + \varphi(x)$ is called the **coboundary set** of G with coefficient set A denoted by $B^2(G, A)$.

The coboundary set of a group G with coefficient set A is a subgroup of the group of factor sets of G with coefficient set A. Two factor sets differing by a coboundary are said to be equivalent.

Define the **second cohomology group** of G with coefficient group A to be $\mathbf{H^2(G, A)} = Z^2(G, A)/B^2(G, A)$.

Definition A **derivation** is a function $\phi : G \mapsto A$ with $\phi(xy) = x\phi(y) + \phi(x)$. The set of all derivations, $\mathrm{Der}(G, A)$ is an abelian group under pointwise addition.

Definition A **principal derivation**(or inner derivation) is a function $f : G \mapsto A$ of the form $f(x) = a_0 - xa_0$ for some $a_0 \in A$. The set of principal derivations, $\mathrm{IDer}(G, A)$ is a subgroup of $\mathrm{Der}(G, A)$.

Let $\mathbf{H^1(G, A)}$ be the set of cosets of $\mathrm{IDer}(G, A)$ in $\mathrm{Der}(G, A)$. That is $H^1(G, A) = \mathrm{Der}(G, A)/\,\mathrm{IDer}(G, A)$. This is the **first cohomology group** of G with coefficient set A. It gives conjugate classes of complements of A in $A \rtimes G$.

Two extensions of A by G are equivalent if and only if they fit in a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & E & \to & G & \to & 0 \\ & & Id\downarrow & & \downarrow\varphi & & \downarrow Id & & \\ 0 & \to & A & \to & E & \to & G & \to & 0 \end{array}$$

φ is called a stabilizing automorphism of E. The set of stabilizing automorphisms is a subgroup of $\mathrm{Aut}(E)$ denoted by $s(E)$. Here $s(E) \simeq Der(G, A)$ and $\varphi \in s(E)$ can be taken to be of the form $\varphi(a + \lambda x) = a + \langle x\rangle + \lambda x$ for an element $a + \lambda x \in E$, a derivation $\langle\ \rangle$ and a lifting λ. There is a one to one correspondence between $H^2(G, A)$ and the set of equivalent classes of extensions of A by G.

We can write affine actions as

$$g \cdot a = ga + \delta(g)$$

where ga is the linear part of the action and $\delta(g) \in A$ is the affine part depending on g. For the action to be well defined, we need $\delta(I) = 0$ and $(g_1g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$. Since,

$$g_1 \cdot (g_2 \cdot a) = g_1 \cdot (g_2a + \delta(g_2)) = g_1g_2a + g_1\delta(g_2) + \delta(g_1)$$

We have, $\delta(g_1g_2) = g_1\delta(g_2) + \delta(g_1)$. That is the affine part need to be a derivation. If the derivation is inner then $\delta(g) = a_0 - ga_0$ for some $a_0 \in A$ and

$$g \cdot a_0 = ga_0 + a_0 - ga_0 = a_0$$

On the other hand if a_0 is a fixed point of the affine action then

$$g \cdot a_0 = ga_0 + \delta(g) = a_0$$

giving us an inner derivation $\delta(g) = a_0 - ga_0$. Hence the derivation is inner if and only if G has a fixed point. We can make the action linear by choosing the fixed point to be an origin.

Define $\mathbf{H^0(G, A)}$ to be the set of fixed points of G's linear action on A.

With the help of Cohomology theory, from a short exact sequence of G modules, $0 \to A' \to A \to A'' \to 0$, we get a long exact sequence:

$$\begin{aligned} 0 &\to H^0(G, A') \to H^0(G, A) \to H^0(G, A'') \\ &\to H^1(G, A') \to H^1(G, A) \to H^1(G, A'') \\ &\to H^2(G, A') \to H^2(G, A) \to H^2(G, A'') \quad \to \dots \end{aligned}$$

See [5] for further details.

Chapter 2

A Classification of $\mathbb{Z}_4$-Codes

Let $\mathbf{C}$ be a quaternary code of length n. Define a pair of binary codes $\{C_1, C_2\}$ by:

- $C_1 = \mathbf{C} \pmod 2$
- $C_2 = h(\mathbf{C} \cap 2\mathbb{Z}_4^n)$

where h is a bijection from $2\mathbb{Z}_4^n$ to $\mathbb{Z}_2^n$ mapping 0 to 0 and 2 to 1 and for the extension to a map acting coordinatewise. h here stands for half. Let d (taken from double) be the inverse of h. The map $x \mapsto x \bmod 2$ is a homomorphism from $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^n$ with kernel $2\mathbb{Z}_4^n$. So its restriction to $\mathbf{C}$ is a homomorphism from $\mathbf{C}$ to C_1 with a kernel $d(C_2)$. By the First Isomorphism Theorem, $|\mathbf{C}| = |C_1| \cdot |C_2|$.

2.1 The class $\mathcal{C}(C_1, C_2)$

For a $\mathbb{Z}_4$-code $\mathbf{C}$, let $\{C_1, C_2\}$ be the binary pair of $\mathbb{Z}_2$-codes defined as above. Then for a word $a \in C_1$, there is an $x \in \mathbf{C}$ with $a = x \bmod 2$ and $h(2x) = a \in C_2$. So C_1 is a subgroup of C_2. For $\mathbf{C}$ with generator matrix (1.6), the binary code C_1 is an $[n, k_1]$ code with generator

$$\begin{pmatrix} I_{k_1} & A & \alpha(B) \end{pmatrix} \tag{2.1}$$

where $\alpha(B) = B \mod 2$. While C_2 is an $[n, k_1 + k_2]$ code with generator

$$\begin{pmatrix} I_{k_1} & A & \alpha(B) \\ 0 & I_{k_2} & C \end{pmatrix} \tag{2.2}$$

For a pair of binary codes $\{C_1, C_2\}$ with $C_1 \leq C_2$, let $\mathcal{C}(C_1, C_2)$ be the set of $\mathbb{Z}_4$-codes giving rise to this binary pair as defined above throughout. For example when $C_1 = C_2 =$ Extended Hamming code of length 8, the class $\mathcal{C}(C_1, C_2)$ contains the Nordstrom-Robinson code which has a generator:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 & 3 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 3 & 3 & 2 \end{pmatrix}$$

We want to describe the class $\mathcal{C}(C_1, C_2)$ using a fixed pair of binary codes $\{C_1, C_2\}$ defined as above.

Proposition 2.1 *Number of $\mathbb{Z}_4$-codes in $\mathcal{C}(C_1, C_2)$ is* $(\frac{2^n}{|C_2|})^{\dim C_1}$

Proof We get different codes of $\mathcal{C}(C_1, C_2)$ when adding different even vectors to a basis of C_1. From proposition(1.1), the difference in generator matrices for codes of $\mathcal{C}(C_1, C_2)$ lies in matrix B in the form (1.6). These are determined mod 2 by C_1 and C_2. In other words, codes in $\mathcal{C}(C_1, C_2)$ correspond to maps;

$$\{\text{basis of } C_1\} \mapsto \mathbb{Z}_2^n / C_2$$

extended linearly. If we take coset representatives of C_2 to be vectors of the form $[0, 0, *]$. Number of these cosets is $2^{n-\dim(C_2)}$. We can write a function, $f : C_1 \mapsto \mathbb{Z}_2^n / C_2$ as an $\dim(C_1) \times (n - \dim(C_2))$ matrix X. For a vector v of length $\dim(C_1)$, we have $f : v \mapsto [0, 0, vX]$ mapping basis of C_1 to rows of X. So,

$$\{\text{set of codes}\} \simeq \{\text{set of all } \dim(C_1) \times (n - \dim(C_2)) \text{ matrices } X \text{ over } \mathbb{Z}_2\}$$

and we can write the general generator matrix for a code $\mathbf{C} \in \mathcal{C}(C_1, C_2)$ in the form:

$$\begin{pmatrix} I_{k_1} & A & \alpha(B) + d(X) \\ 0 & 2I_{k_2} & 2C \end{pmatrix} \tag{2.3}$$

where $\alpha(B) = B \bmod 2$ and X a matrix in $\mathbb{Z}_2$. Number of these matrices is $|\frac{\mathbb{Z}_2^n}{C_2}|^{\dim C_1} = 2^{(n-m_2)m_1}$ where $m_i = \dim(C_i)$.

We have a class $\mathcal{C}(C_1, C_2)$ of cardinality 2^N for $N = (n - m_2)m_1$. It is natural to look for a bijection between this set and some vector space of dimension N over $\mathbb{Z}_2$.

Definition Let A and B be abelian groups. Let F be the free abelian group on the set $A \times B$. That is F is $\mathbb{Z}$-linear combinations of all ordered pairs (a, b). Let K be the subgroup of F generated by all elements of the forms:

- $(a + a', b) - (a, b) - (a', b)$;
- $(a, b + b') - (a, b) - (a, b')$;

- $(ar, b) - (a, rb)$.

for all $a, a' \in A$, $b, b' \in B$, $r \in \mathbb{Z}$. The quotient group F/K is called the **tensor product** of A and B denoted by $A \otimes B$[6]. The coset $(a, b) + K$ of the element (a, b) in F is denoted by $a \otimes b$. Elements of $A \otimes B$ can be written as finite sums of the form: $\sum_i n_i(a_i \otimes b_i)$.

In other words, the tensor product of groups A and B is the largest abelian group generated by symbols $a \otimes b$ for $a \in A$ and $b \in B$ such that the map from ordered pairs (a, b) to $a \otimes b$ is linear in each argument.

Let $C_1^* = Hom(C_1, \mathbb{Z}_2)$ be the dual space of C_1. Define a linear action of $C_1^* \otimes \mathbb{Z}_2^n$ on $\mathcal{C}(C_1, C_2)$ by the rule

$$(f \otimes w)(c) = c + d(f(c \bmod 2)w) \tag{2.4}$$

for $f \in C_1^*$ and $w \in \mathbb{Z}_2^n$. This is an additive homomorphism so $(f \otimes w)(\mathbf{C}) = \mathbf{C}'$ is a $\mathbb{Z}_4$-code. Since $d(x)$ is even, $(f \otimes w)(c) \equiv c \bmod 2$. Hence $C_1' = C_1$ and if $c \in 2\mathbb{Z}_4^n$ then $c \bmod 2 = 0$. So $(f \otimes w)(c) = c$ and $C_2' = C_2$. Hence, $\mathbf{C}' \in \mathcal{C}(C_1, C_2)$ and the action is well defined.

For a basis $\{a_1, \ldots, a_{k_1}\}$ of C_1, let us fix a basis for C_1^* throughout to be $\{f_1, \ldots, f_{k_1}\}$, where $f_i(a_j) = 1$ if $i = j$ and 0 otherwise. Let $\{w_1, \ldots, w_{k_3}\}$ be a basis for $\mathbb{Z}_2^n/C_2$ where $k_3 = n - (k_1 + k_2)$ (here k_1 and k_2 are the dimensions of the identity matrices in the general generator matrix as in (1.6)) and $\{w_i\}$ are vectors of the form $[0, 0, *]$ taken from the standard basis of $\mathbb{Z}_2^n$. That is $\{w_i\}$ has a 1 in the $(k_1 + k_2 + i)$th position and 0 elsewhere.

$C_1^* \otimes \mathbb{Z}_2^n$ is an abelian group acting transitively on $\mathcal{C}(C_1, C_2)$. That is, for all codes $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C}(C_1, C_2)$, there is an $x \in C_1^* \otimes \mathbb{Z}_2^n$ such that $x \cdot \mathbf{C}_1 = \mathbf{C}_2$. This is true since if $\mathbf{C}_1$ is a $\mathbb{Z}_4$-code corresponding to matrix X_1 and $\mathbf{C}_2$

is a $\mathbb{Z}_4$-code corresponding to matrix X_2, then we can act on $\mathbf{C_1}$ by $x = \sum(n_{ij} + m_{ij})f_i \otimes w_j$ where $n_{ij} = \{X_1\}_{ij}$ and $m_{ij} = \{X_2\}_{ij}$ to get $\mathbf{C_2}$.

Note that, if $w \in C_2$ then $d(w) \in d(C_2)$ and $d(f(c \bmod 2)w) \in \mathbf{C}$ for all $c \in \mathbf{C}$. Thus elements of $C_1^* \otimes C_2$ fix $\mathbf{C}$ and form the kernel of this action. Hence, $A = C_1^* \otimes \mathbb{Z}_2^n/C_2$ acts regularly on $\mathcal{C}(C_1, C_2)$. Since for all $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C}(C_1, C_2)$, there is a unique $x \in A$ such that $x \cdot \mathbf{C}_1 = \mathbf{C}_2$. So if W is the set of matrices X then $W \simeq A = C_1^* \otimes \mathbb{Z}_2^n/C_2 \simeq \mathcal{C}(C_1, C_2)$.

For binary codes, $\{C_1, C_2, D_1, D_2\}$, suppose that $C_1 \leq C_2$ and $D_1 \leq D_2$. Then $\mathcal{C}(C_1, C_2) \simeq C_1^* \otimes \mathbb{Z}_2^n/C_2$ and $\mathcal{C}(D_1, D_2) \simeq D_1^* \otimes \mathbb{Z}_2^m/D_2$. So,

$$\begin{aligned}
\mathcal{C}(C_1 \oplus D_1, C_2 \oplus D_2) &\simeq (C_1 \oplus D_1) \otimes \mathbb{Z}_2^{(n+m)}/(C_2 + D_2) \\
&= (C_1 \oplus D_1) \otimes (\mathbb{Z}_2^n/C_2 \oplus \mathbb{Z}_2^m/D_2) \\
&= (C_1 \otimes \mathbb{Z}_2^n/C_2) \oplus (C_1 \otimes \mathbb{Z}_2^m/D_2) \\
&\quad \oplus (D_1 \otimes \mathbb{Z}_2^n/C_2) \oplus (D_1 \otimes \mathbb{Z}_2^m/D_2) \\
&\simeq \mathcal{C}(C_1, C_2) \oplus \mathcal{C}(D_1, D_2) \oplus (C_1 \otimes \mathbb{Z}_2^m/D_2) \oplus (D_1 \otimes \mathbb{Z}_2^n/C_2)
\end{aligned}$$

Proposition 2.2 *All $\mathbb{Z}_4$-codes of $\mathcal{C}(C_1, C_2)$ have same number of words of even weight.*

Proof As mentioned in the proof of proposition(2.1), codes in $\mathcal{C}(C_1, C_2)$ differ by changing the function from a basis of C_1 to coset representatives of C_2. This changes $0 \leftrightarrow 2$ or $1 \leftrightarrow 3$. Hence, parity of coordinates of codewords is the same for different $\mathbb{Z}_4$-codes. As all words of $d(C_2)$ have even weights, number of words of even weight in a $\mathbb{Z}_4$-code is determined by that of its corresponding C_1. In fact,

$$\begin{array}{ccc}
\text{number of words} & = & \text{number of words} \quad \times |C_2| \\
\text{of even weight in} & & \text{of even weight} \\
\mathbf{C} \in \mathcal{C}(C_1, C_2) & & \text{in } C_1
\end{array}$$

2.2 Cohomology of $\mathcal{C}(C_1, C_2)$

Consider extensions of $\mathbb{Z}_2^n$ by C_1:

$$0 \to \mathbb{Z}_2^n \to E \to^{\pi} C_1 \to 0 \tag{2.5}$$

C_1 acts trivially on $\mathbb{Z}_2^n$. (This is a conjugation action but $C_1 \leq \mathbb{Z}_2^n$ and $\mathbb{Z}_2^n$ is an abelian group which makes the action trivial).

Set of extensions is defined by different addition tables in E. These are defined by factor sets which are functions $[\ ,\ ] : C_1 \times C_1 \to \mathbb{Z}_2^n$ defined by:

$$\lambda x + \lambda y = [x, y] + \lambda xy$$

where λ is a lifting, $\lambda(x) = (a, x)$ for some $a \in \mathbb{Z}_2^n$. Addition in $\mathbb{Z}_4$-codes of $\mathcal{C}(C_1, C_2)$ is given by one of the addition tables (a factor set) defining an extension: $0 \to \mathbb{Z}_2^n \to E \to C_1 \to 0$. For elements,$\{(a, x), (b, y)\} \in E$, addition is defined as:

$$(a, x) + (b, y) = (a + b + [x, y], x + y) \tag{2.6}$$

In a $\mathbb{Z}_4$-code $\mathbf{C} \in \mathcal{C}(C_1, C_2)$, the even words are represented by $d(C_2) \leq d(\mathbb{Z}_2^n)$. If $a_1, a_2 \in C_2$ then $d(a_1), d(a_2) \in \mathbf{C}$ and $d(a_1)+d(a_2) = d(a_1+a_2)$, defined by addition in C_2. These can be represented by elements $(a_1, 0), (a_2, 0) \in E$ and so $(a_1, 0) + (a_2, 0) = (a_1 + a_2, 0)$ for $a_1, a_2 \in C_2 \leq \mathbb{Z}_2^n$. On the other hand since $C_1 = \mathbf{C} \bmod 2$, addition of non even words in $\mathbf{C}$ is not defined by addition in C_1 since addition in $\mathbb{Z}_2$ is different from the one in $\mathbb{Z}_4$. These can be represented by elements $(a_1, x_1), (a_2, x_2) \in E$ so addition is given by (2.6) above for $x_1, x_2 \in C_1$. A factor set corresponding to a code in the class of codes of the Hamming example is defined by the addition table below:

	1000 0111	0100 1011	0010 1101	0001 1110	1100 1100	...
1000 0111	1000 0111	0000 0011	0000 0101	0000 0110	1000 0100	...
0100 1011	0000 0011	0100 1011	0000 1001	0000 1010	0100 1000	...
0010 1101	0000 0101	0000 1001	0010 1101	0000 1100	0000 1100	...
...	...	...	...	...	...	$\ddots$

We may choose the lifting:

$$\begin{aligned}
1000\ 0111 &\to (0\ldots0,\ 1000\ 0111)\\
0100\ 1011 &\to (0\ldots0,\ 0100\ 1011)\\
0010\ 1101 &\to (0\ldots0,\ 0010\ 1101)\\
0001\ 1110 &\to (0\ldots0,\ 0001\ 1110)\\
1100\ 1100 &\to (0000\ 0011,\ 1100\ 1100)\\
1010\ 1010 &\to (0000\ 0101,\ 1010\ 1010)\\
1001\ 1001 &\to (0000\ 0110,\ 1001\ 1001)\\
0110\ 0110 &\to (0000\ 1001,\ 0110\ 0110)\\
0101\ 0101 &\to (0000\ 1010,\ 0101\ 0101)\\
0011\ 0011 &\to (0000\ 1100,\ 0011\ 0011)\\
1110\ 0001 &\to (0000\ 1111,\ 1110\ 0001)\\
1101\ 0010 &\to (0000\ 1111,\ 1101\ 0010)\\
1011\ 0100 &\to (0000\ 1111,\ 1101\ 0010)\\
0111\ 1000 &\to (0000\ 1111,\ 0111\ 1000)\\
1111\ 1111 &\to (0000\ 1111,\ 1111\ 1111)
\end{aligned}$$

with the addition table above to get the $\mathbb{Z}_4$-code corresponding to $X = 0$. A lifting differing from this by a homomorphism $\delta : C_1 \to \mathbb{Z}_2^n/C_2$ will give us another code in the class $\mathcal{C}(C_1, C_2)$. In general, we may fix a factor set defining addition in an extension corresponding to a class of codes $\mathcal{C}(C_1, C_2)$ to be

$$[x, y] = x * y \quad \text{for } x, y \in C_1$$

We get different codes by choosing different liftings so different codes are obtained from a set of equivalent extensions. But two extensions of $\mathbb{Z}_2^n$ by C_1 are equivalent if and only if they fit in a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}_2^n & \to & E & \to & C_1 & \to & 0 \\ & & Id \downarrow & & \downarrow \varphi & & \downarrow Id & & \\ 0 & \to & \mathbb{Z}_2^n & \to & E & \to & C_1 & \to & 0 \end{array}$$

where φ is a stabilizing automorphism of E of the form $\varphi(a + \lambda x) = a + \langle x \rangle + \lambda x$ for an element $a + \lambda x \in E$, a derivation $\langle\ \rangle \in \mathrm{Der}(C_1, \mathbb{Z}_2^n)$ and a lifting $\lambda : C_1 \to E$. If the images of a derivation are chosen to be in C_2, we would be mapping E to the same corresponding code.

Consider extensions of C_2 by C_1:

$$0 \to C_2 \to E \to C_1 \to 0$$

set of factor sets defined by these extensions is a subgroup of the set of the factor sets defined by the earlier extension. Since adding words in $d(C_2)$ to a $\mathbb{Z}_4$-code stays within the $\mathbb{Z}_4$-code, we may regard factor sets defined by this extensions as zero factor sets when looking for our addition table. And define our addition table in codes of $\mathcal{C}(C_1, C_2)$ from factor sets of extensions:

$$0 \to \mathbb{Z}_2^n / C_2 \to E \to C_1 \to 0$$

With the lifting used in the example above $Im(\lambda)$ is a subgroup of E and so the extensions are split as extensions of $\mathbb{Z}_2^n / C_2$ by C_1.

Now we have a set of equivalent codes ($\equiv$ set of codes in $\mathcal{C}(C_1, C_2)$) defined by an extension and its equivalent extensions defined by the derivation set $Der(C_1, \mathbb{Z}_2^n/C_2)$. Since C_1 acts trivially on $\mathbb{Z}_2^n$ then $Der(C_1, \mathbb{Z}_2^n/C_2) = Hom(C_1, \mathbb{Z}_2^n/C_2)$. We get these homomorphisms by fixing images of a basis of C_1 and extending linearly. Hence, we get the right number of $\mathbb{Z}_4$-codes in $\mathcal{C}(C_1, C_2)$. Note that $InDer(C_1, \mathbb{Z}_2^n/C_2) = \{0\}$ so $H^1(C_1, \mathbb{Z}_2^n/C_2) = Der(C_1, \mathbb{Z}_2^n/C_2)$ and $\mathcal{C}(C_1, C_2) \simeq H^1(C_1, \mathbb{Z}_2^n/C_2) \simeq C_1^* \otimes \mathbb{Z}_2^n/C_2$ as shown in Section(2.1).

$\mathbb{Z}_4$-codes can be precisely represented by subgroups of the extensions (2.5) above generated by $\{(b, I) : b \in C_2\}$ and $\{(f(a), a) :$ for $a \in C_1$ and a homomorphism $f : C_1 \to \mathbb{Z}_2^n\}$.

2.3 Dual codes of $\mathcal{C}(C_1, C_2)$

For a given class $\mathcal{C}(C_1, C_2)$, we can work out number of self orthogonal or self dual codes. To do this, we need the following.

Proposition 2.3 *If A is an $m \times n$ matrix over a field $\mathbb{F}$ and y is a given vector in $\mathbb{F}^m$ then set of solutions $x \in \mathbb{F}^n$ of the equation $Ax = y$ is either empty or form an affine subspace in $\mathbb{F}^n$.*

Proof From linear algebra tools, we can get solutions of $Ax = y$ by computing Reduced Gaussian elimination for the system $[A : y]$. If we end up with a system with a zero row to the left of the colon facing a nonzero entry to the right of the colon, the set of solutions is empty. Otherwise solutions are of the form: $\sum_{i_j \in [n]} x_{i_j} v_{i_j} + s$ for some $v_{i_j}, s \in \mathbb{F}^m$.

Theorem 2.1 *Let $\mathcal{C}(C_1, C_2)$ be a set of $\mathbb{Z}_4$-codes giving rise to $\{C_1, C_2\}$ and*

W be its isomorphic $\mathbb{Z}_2$ vector space (set of $m_1 \times (n - m_2)$ matrices over $\mathbb{Z}_2$ where $m_i = \dim(C_i)$). Then, $\mathcal{C}(C_1, C_2)$ can only contain self orthogonal codes if $C_2 \leq (C_1)^\perp$ and words of C_1 are doubly even. In this case, the set of self orthogonal codes in $\mathcal{C}(C_1, C_2)$ is either empty or form an affine space of dimension at least $(n - m_2)m_1 - \binom{m_1}{2}$. For $\mathcal{C}(C_1, C_2)$ to have self dual codes, n needs to be even as well.

Proof A word in a $\mathbb{Z}_4$-code can be expressed in the form: $a_1 + d(e + c_2)$ where $a_1 \in C_1$, $e \in \mathbb{Z}_2^n$ and $c_2 \in C_2$. For $\mathbf{C} \in \mathcal{C}(C_1, C_2)$ to be self orthogonal, we need $(a_1 + d(e + c_2)) \cdot d(c_2') = 0 \bmod 4$ for $c_2' \in C_2$. But

$$(a_1 + d(e + c_2)) \cdot d(c_2') = a_1 \cdot d(c_2') = 0 \bmod 4 \Leftrightarrow a_1 \cdot c_2' = 0 \bmod 2$$

That is $C_2 \leq C_1^\perp$. We should also have

$$(a_1 + d(e + c_2)) \cdot (a_1 + d(e + c_2)) = 0 \bmod 4$$

but

$$\begin{aligned}(a_1 + d(e + c_2)) \cdot (a_1 + d(e + c_2)) &= a_1 \cdot a_1 + 2(a_1 \cdot d(e)) \bmod 4 \\ &= a_1 \cdot a_1 \bmod 4\end{aligned}$$

since $a_1 \cdot d(e) = 2(a_1 \cdot e)$. Hence, words of C_1 need to be doubly even to have a self orthogonal code in $\mathcal{C}(C_1, C_2)$.

Now if $\mathbf{C}$ is a self orthogonal code and $(a_1 + d(e + c_2))$, $(a_1' + d(e' + c_2')) \in \mathbf{C}$, then

$$\begin{aligned}(a_1 + d(e + c_2)) \cdot (a_1' + d(e' + c_2')) &= a_1 \cdot a_1' + (a_1 \cdot d(e') + a_1' \cdot d(e)) \\ &= 0 \bmod 4\end{aligned}$$

But $a_1 \cdot d(e') = 0 \bmod 2$ and $a_1 \cdot a_1' = 0 \bmod 2$ since $C_1 \leq C_2 \leq C_1{}^\perp$. Hence, for C to be a self orthogonal code either

$$\begin{aligned} & a_1 \cdot a_1' = 2 \bmod 4 \quad \text{and} \quad (a_1 \cdot e' + a_1' \cdot e) = 1 \bmod 2 \\ \text{or} \quad & a_1 \cdot a_1' = 0 \bmod 4 \quad \text{and} \quad (a_1 \cdot e' + a_1' \cdot e) = 0 \bmod 2 \end{aligned}$$

If this is satisfied for a basis of C_1, it will be satisfied for all words. So every pair in the basis of C_1 give us a relation to be satisfied by variables in matrix X. Number of relations is $\binom{m_1}{2}$ and number of variables is $(n-m_2)m_1$. So we can write this as a system of linear equations to be solved. If solvable, we get a solution set isomorphic to a vector space of dimension at least $(n-m_2)m_1 - \binom{m_1}{2}$ depending on linear dependence of the relations. We can say more about the linear dependence of the relations above in the case where $C_1 = C_2$ using the following proposition.

Proposition 2.4 *In a set $\mathcal{C}(C_1, C_2)$ of $\mathbb{Z}_4$-codes rising from binary codes $\{C_1, C_2\}$ with $C_1 = C_2 \leq C_1^\perp$, if $\{b_i\}$ are rows of matrix $\alpha(B)$ then $\{b_i\}$ are linearly independent. Here,* $\alpha(B) = B \bmod 2$.

Proof When $C_1 = C_2$, the general generator matrix for a $\mathbb{Z}_4$-code in $\mathcal{C}(C_1, C_2)$ is of the form:

$$\begin{pmatrix} I_{k_1} & B \end{pmatrix}$$

where B is a matrix in $\mathbb{Z}_4$. Let $\{r_i\}$ be rows of the generator matrix of C_1. The vectors b_i satisfy $b_i \cdot b_i = 1$ and $b_i \cdot b_j = 0$ for $i \neq j$, since r_i has just one more entry 1 than b_i and these 1's occur in different positions. Suppose that we have a linear combination equal to zero, say $\sum a_i b_i = 0$ for some $a_i \in \mathbb{Z}_2$ and i running from 1 to k_1. Now for any fixed j, taking the dot product of this linear combination with b_j gives $a_j = 0$. So the b_i are linearly independent.

Looking back at proof of theorem(2.1), for each pair of rows $\{r_i, r_j\}$, if $\{e_i, e_j\}$ are the corresponding rows of X then the relation we get for such a pair is of the form $(b_i \cdot e_j + b_j \cdot e_i) = k_{ij}$ where $k_{ij} = 0$ or 1. We can arrange these relations in a matrix

$$\begin{pmatrix} b_2 & b_1 & 0\dots0 & \dots & 0\dots0 \\ b_3 & 0\dots0 & b_1 & \dots & 0\dots0 \\ \dots & & & & \\ b_k & 0\dots0 & 0\dots0 & \dots & b_1 \\ --- & --- & --- & --- & --- \\ 0\dots0 & b_3 & b_2 & \dots & 0\dots0 \\ \dots & & & & \\ 0\dots0 & b_k & 0\dots0 & \dots & b_2 \\ --- & --- & --- & --- & --- \\ \dots & \dots & \dots & \dots & \dots \\ --- & --- & --- & --- & --- \\ 0\dots0 & 0\dots0 & \dots & b_k & b_{k-1} \end{pmatrix}$$

Every block has linearly independent rows and different blocks are linearly independent. Here also, $m_1 = m_2$. Hence if $(n-m_1)m_1 \geq \binom{m_1}{2}$ the system is solvable and the solution set of such a system is an affine space of dimension $(n-m_1)m_1 - \binom{m_1}{2}$. But $C_1 \leq C_2 \leq C_1^{\perp}$. So $m_1 \leq m_2 \leq (n-m_1)$. Hence, $(n-m_1)m_1 \geq m_2 \cdot m_1 \geq m_1 \cdot m_1 \geq m_1 \frac{(m_1-1)}{2}$. Therefore, in a set $\mathcal{C}(C_1, C_2)$ of $\mathbb{Z}_4$-codes rising from binary codes $\{C_1, C_2\}$ with $C_1 = C_2 \leq C_1^{\perp}$, if words of C_1 are doubly even then the dimension of the affine space of self orthogonal codes in this class is $(n-m_1)m_1 - \binom{m_1}{2}$.

For a self dual code, $\mathbf{C} = \mathbf{C}^{\perp}$. But $\dim(\mathbf{C}^{\perp}) = n - \dim(\mathbf{C})$. Hence n needs to be even. This theorem gives a method for finding self dual codes in a class $\mathcal{C}(C_1, C_2)$.

Corollary 2.1 *Let C_1 be a binary self dual code of length n. If words of*

C_1 are doubly even then the class $\mathcal{C}(C_1, C_1)$ has $2^{\frac{n(n+2)}{8}}$ self dual $\mathbb{Z}_4$-codes. Otherwise the class $\mathcal{C}(C_1, C_1)$ has no self dual codes.

Proof C_1 is self dual so $\dim(C_1) = \frac{n}{2}$. From Theorem(2.1) and Proposition(2.4), if $\dim(C_i) = m_i$ then set of self dual $\mathbb{Z}_4$-codes in $\mathcal{C}(C_1, C_2)$ with $C_1 = C_2$ form an affine space with dimension:

$$\begin{aligned}(n - m_2)m_1 - \binom{m_1}{2} &= (n - \frac{n}{2})\frac{n}{2} - \frac{n}{4}(\frac{n}{2} - 1) \\ &= \frac{n}{2}(\frac{n}{2} - \frac{n}{4} + \frac{1}{2}) \\ &= \frac{n(n+2)}{8}\end{aligned}$$

If $C_1 \leq C_2$ then $C_2^{\perp} \leq C_1^{\perp}$. This is true since having $C_1 \leq C_2$ and $v \in C_2^{\perp}$ gives $v \cdot x = 0$ for all $x \in C_2$. But $C_1 \leq C_2$. So $v \cdot x = 0$ for all $x \in C_1$. That is $v \in C_1^{\perp}$ and so $C_2^{\perp} \leq C_1^{\perp}$. Hence, we can define a class $\mathcal{C}(C_2^{\perp}, C_1^{\perp})$ of $\mathbb{Z}_4$-codes.

Proposition 2.5 *$\mathcal{C}(C_2^{\perp}, C_1^{\perp})$ consists of duals of the codes in $\mathcal{C}(C_1, C_2)$.*

Proof Take $\mathbf{C} \in \mathcal{C}(C_1, C_2)$, so that

$$\mathbf{C} \bmod 2 = C_1, \qquad h(\mathbf{C} \cap 2\mathbb{Z}_4^n) = C_2.$$

We have to show that $\mathbf{C}^{\perp} \in \mathcal{C}(C_2^{\perp}, C_1^{\perp})$. That is,

$$\mathbf{C}^{\perp} \bmod 2 = C_2^{\perp}, \qquad h(\mathbf{C}^{\perp} \cap 2\mathbb{Z}_4^n) = C_1^{\perp}.$$

For $v \in \mathbf{C}^{\perp}$, we have $v \cdot c = 0$ in $\mathbb{Z}_4$ for all $c \in \mathbf{C}$.

(i) If $c \in d(C_2) = \mathbf{C} \cap 2\mathbb{Z}_4^n$, let $c = 2c'$, for some $c' \in C_2$. Then $v \cdot c' = 0$ in $\mathbb{Z}_2$ (if we read $v \bmod 2$, it doesn't make a difference since $2 \cdot 1 =$

$2 \cdot 3 = 2 \bmod 4$ and $2 \cdot 0 = 2 \cdot 2 \bmod 4$). So $(v \bmod 2) \in C_2^{\perp}$ and $\mathbf{C}^{\perp} \bmod 2 \leq C_2^{\perp}$. Conversely if $(v \bmod 2) \in C_2^{\perp}$ then $(v \bmod 2) \cdot c = 0$ in $\mathbb{Z}_2$ for all $c \in C_2$ and $v \cdot d(c) = 0$ in $\mathbb{Z}_4$. So $\mathbf{C}^{\perp} \bmod 2 = C_2^{\perp}$.

(ii) Part (i) says that

$$\mathbf{C}^{\perp} \bmod 2 = (h(\mathbf{C} \cap 2\mathbb{Z}_4^n))^{\perp}$$

for any $\mathbb{Z}_4$-code $\mathbf{C}$. Applying this to $\mathbf{C}^{\perp}$, we have

$$\begin{aligned} C_1 &= \mathbf{C} \bmod 2 \\ &= \mathbf{C}^{\perp\perp} \bmod 2 \\ &= (h(\mathbf{C}^{\perp} \cap 2\mathbb{Z}_4^n))^{\perp}. \end{aligned}$$

Taking the dual of both sides gives $C_1^{\perp} = h(\mathbf{C}^{\perp} \cap 2\mathbb{Z}_4^n)$, as required.

We can prove this result by some matrix multiplications by showing that if $\mathbf{C} \in \mathcal{C}(C_1, C_2)$ then $\mathbf{C}^{\perp} \in \mathcal{C}(C_2^{\perp}, C_1^{\perp})$ using their generator matrices. We can use MacWilliams transforms defined in (1.3) to get weight enumerators of a class $\mathcal{C}(C_2^{\perp}, C_1^{\perp})$ from those of $\mathcal{C}(C_1, C_2)$.

Hence if $C_1 = C_2 = C_1^{\perp}$, a $\mathbb{Z}_4$-code $\mathbf{C} \in \mathcal{C}(C_1, C_1)$ if and only if $\mathbf{C}^{\perp} \in \mathcal{C}(C_1, C_1)$. Having a code with certain weight enumerator tells us that there is a code in the same class with a weight enumerator equals to its MacWilliams transform. The class $\mathcal{C}(C_1, C_1)$ with a self dual C_1 can be split into a set of self dual codes and a set of pairs $\{\mathbf{C}, \mathbf{C}^{\perp}\}$.

Chapter 3

Computing Symmetrized Weight Enumerators

As defined earlier, the symmetrized weight enumerator of a $\mathbb{Z}_4$-code is a three variable homogeneous polynomial defined as:

$$\mathrm{swe}_{\mathbf{C}}(x, y, z) = \sum_{c \in \mathbf{C}} x^{n_0(c)} y^{n_1(c)+n_3(c)} z^{n_2(c)}$$

We can obtain weight enumerators of C_1 and C_2 from the symmetrized weight enumerator of $\mathbf{C}$ as:

$$\begin{aligned}
\mathrm{we}_{C_1}(x, y) &= \frac{1}{|C_2|} \cdot \mathrm{swe}_{\mathbf{C}}(x, y, x) \\
\mathrm{we}_{C_2}(x, y) &= \mathrm{swe}_{\mathbf{C}}(x, 0, y)
\end{aligned}$$

We can also obtain the Lee weight enumerator which is the same as the weight enumerator of the Gray map image of $\mathbf{C}$ as:

$$\mathrm{Lee}_{\mathbf{C}}(x, y) = \mathrm{swe}_{\mathbf{C}}(x^2, xy, y^2)$$

Now fix C_1 and C_2. Let us try to get information about swe's of codes in $\mathcal{C}(C_1, C_2)$ from C_1 and C_2.

3.1 Average Weight Enumerators

Theorem 3.1 *The average symmetrized weight enumerator of the codes in $\mathcal{C}(C_1, C_2)$ is given by*

$$\overline{\mathrm{swe}}(x,y,z) = \frac{|C_2|}{2^n}(\mathrm{we}_{C_1}(x+z, 2y) - (x+z)^n) + \mathrm{we}_{C_2}(x,z)$$

Proof Let $m_i = \dim C_i$, for $i = 1, 2$. For any $v \in C_1$ there are 2^{m_2} words of each $\mathbf{C} \in \mathcal{C}(C_1, C_2)$ that gives v when read modulo 2. If $v \neq 0$ then each vector in $d(\mathbb{Z}_2^n)$ can be added to v to give words of the same parity pattern. If v has weight j, the corresponding term in we_{C_1} is $x^{n-j}y^j$. The words of the same parity pattern are counted by $(x+z)^{n-j}(2y)^j$ in the sum of symmetrized weight enumerators of codes in $\mathcal{C}(C_1, C_2)$. But this expression is too much by a factor of $2^n/|C_2|$ for the average, since it counts 2^n words for each $v \in C_1$ whereas there are only $|C_2|$ of them in each code $\mathbf{C} \in \mathcal{C}(C_1, C_2)$. So the contribution to the average from nonzero words of C_1 is

$$\frac{|C_2|}{2^n}(\mathrm{we}_{C_1}(x+z, 2y) - (x+z)^n)$$

If $v = 0$ then the corresponding words of all $\mathbf{C} \in \mathcal{C}(C_1, C_2)$ are the same. They are the words of $d(C_2)$, so they are counted by $\mathrm{we}_{C_2}(x, z)$.

For $C_1 = C_2 =$ Extended Hamming code of length 8.

$$\mathrm{we}_{C_1} = x^8 + 14x^4y^4 + y^8$$

Average symmetrized weight enumerator is

$$\overline{\mathrm{swe}} = 14(x+z)^4y^4 + 16y^8 + (x^8 + 14x^4z^4 + z^8)$$

Corollary 3.1 *If all codes of a class $\mathcal{C}(C_1, C_2)$ have same* swe *then this should be equal to* $\overline{\mathrm{swe}}$.

Using the same principle used above (replacing $2y$ by $y+w$), we can get the average complete weight enumerator in $\mathcal{C}(C_1, C_2)$ to be

$$\overline{\text{cwe}}(x, y, z, w) = \frac{|C_2|}{2^n}(\text{we}_{C_1}(x+z, y+w) - (x+z)^n) + \text{we}_{C_2}(x, z)$$

Proposition 3.1 *The average* cwe *of codes in* $\mathcal{C}(C_2^{\perp}, C_1^{\perp})$ *is equal to the MacWilliams transform of the average* cwe *of codes in* $\mathcal{C}(C_1, C_2)$*. All other weight enumerators satisfy this as well since they are defined in terms of* cwe*.*

Proof Result follows since the MacWilliams transform is linear. We can verify this directly as shown below

The average cwe of $\mathcal{C}(C_1, C_2)$ is

$$\overline{\text{cwe}}(x, y, z) = \frac{|C_2|}{2^n}(\text{we}_{C_1}(x+z, y+w) - (x+z)^n) + \text{we}_{C_2}(x, z)$$

Average cwe for $\mathcal{C}(C_2^{\perp}, C_1^{\perp})$ is

$$\begin{aligned}
& \frac{|C_1^{\perp}|}{2^n}(\text{we}_{C_2^{\perp}}(x+z, y+w) - (x+z)^n) + \text{we}_{C_1^{\perp}}(x, z) \\
= & \frac{|C_1^{\perp}|}{2^n}(\frac{1}{|C_2|}\text{we}_{C_2}(x+y+z+w, x-y+z-w) - (x+z)^n) + \frac{1}{|C_1|}\text{we}_{C_1}(x+z, x-z) \\
= & \frac{1}{2^{2k_1+k_2}}\text{we}_{C_2}(x+y+z+w, x-y+z-w) - \frac{1}{2^{k_1}}(x+z)^n + \frac{1}{2^{k_1}}\text{we}_{C_1}(x+z, x-z)
\end{aligned}$$

MacWilliams transform of the average of $\mathcal{C}(C_1, C_2)$ is the following:

$$\begin{aligned}
& \frac{1}{|\mathbf{C}|}[\frac{|C_2|}{2^n}(\text{we}_{C_1}(2x+2z, 2x-2z) - (2x+2z)^n) + \text{we}_{C_2}(x+y+z+w, x-y+z-w)] \\
= & \frac{1}{2^{2k_1+k_2}}[\frac{2^{k_1+k_2}}{2^n}(2^n\text{we}_{C_1}(x+z, x-z) - 2^n(x+z)^n) + \text{we}_{C_2}(x+2y+z, x-2y+z)] \\
= & \frac{1}{2^{k_1}}(\text{we}_{C_1}(x+z, x-z) - (x+z)^n) + \frac{1}{2^{2k_1+k_2}}\text{we}_{C_2}(x+y+z+w, x-y+z-w)
\end{aligned}$$

same as above result.

Corollary 3.2 *If* $C_1 = C_1^{\perp}$ *then* $\overline{\text{swe}}$ *(also* $\overline{\text{cwe}}$*) of the class of codes* $\mathcal{C}(C_1, C_1)$ *is equal to its MacWilliams transform.*

3.2 Variation in Symmetrized Weight Enumerators of $\mathcal{C}(C_1, C_2)$

Define an affine subspace of polynomials to be a coset of a finitely generated space of polynomials. We can use codes $\{C_1, C_2\}$ to get information about variations of swe's in $\mathcal{C}(C_1, C_2)$. Here is some of what we can get supported with what happens in the Hamming example where $C_1 = C_2$ with a generator matrix:

$$G_{C_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

- Each word $x \in \mathbf{C}$ has a unique expression as $x = c_1 + c_2$ for $c_2 \in d(C_2)$ and c_1 a sum of rows of the generator matrix.

- $|d(C_2)| = |C_2| = 2^{m_2}$ and $|C_1| = 2^{m_1}$ so $|\mathbf{C}| = 2^{m_1+m_2}$ where $m_i = \dim(C_i)$. In the Hamming example, we would have $|C_1| = |C_2| = 16$ and $|\mathbf{C}| = 16 \times 16 = 256$.

- Let C^* be the set of words including the ith row of the generator matrix and C^{**} be the set of words not including the ith row as a summand.

- A change in the ith row affects words of C^* and so affecting their monomial representations in a weight enumerator. But words of C^{**} will not be affected. $|C^*| = 2^{m_1-1} \times 2^{m_2} = |C^{**}|$ since we need to count different sums of the rows excluding the ith row. So in the Hamming case this would be $|C^*| = 8 \times 16 = 128 = |C^{**}|$.

- If C_2 is a code where there is a word which is nonzero for every coordinate position, $\frac{1}{2}$ of the words in $d(C_2)$ have a 0 in a position and the other $\frac{1}{2}$ have 2 in that position, as deduced in Proposition(1.2).

- If c_1 with $(c_1 \bmod 2) \in C_1$ is even in a certain position, then $\frac{1}{2}$ of its variations with words of $d(C_2)$ will have a 0 in that position and the other half will have 2 in that position. If the position carries an odd then $\frac{1}{2}$ of its variations with words of $d(C_2)$ will have 1 and the other $\frac{1}{2}$ will have 3 there.

- C^{**} is a subgroup of $\mathbf{C}$ whose words are even in the ith coordinate position. For all other coordinate positions, we have equally many 0's, 1's, 2's and 3's given that conditions of Proposition(1.3) are satisfied. All words of C^* will be odd in the ith position and in all other positions it will have equally many 0's, 1's, 2's and 3's. So this number will be $\frac{|C^*|}{4} = 2^{m_2+m_1-1}/4 = 2^{m_2+m_1-3}$. In the Hamming case, number of words having 0 (also $2, 1$ or 3) in a position other than the ith position is $4 \times 8 = 32$. Let $\{a_j\}$ be rows of the generator matrix G_{C_1}. Taking $i = 1$, words of C^* are $\{a_1,\ a_1 + a_j$ for $j \neq 1,\ a_1 + a_l + a_j$ for $l, j \neq 1,\ a_1 + a_2 + a_3 + a_4\}$.

- When a 1 is changed to 3 or a 3 is changed to 1 in a word, its monomial representation in the swe is not affected. So there is a possible change when we exchange 0's and 2's. So we should look at words having even entry in the position changed.

- If $\mathrm{we}_{C_1} = \sum_{c \in C_1} x^{n_0(c)} y^{n_1(c)}$ then for a code $\mathbf{C} \in \mathcal{C}(C_1, C_2)$ we would have,

$$\mathrm{swe}_{\mathbf{C}}(x, y, z) = \sum_{c \in C_1} P_{n_0(c)}(x, z) y^{n_1(c)}$$

where $P_{n_0(c)}$ are homogeneous polynomials of degree $n_0(c)$ with coefficients summing to $|C_2|$. Changes made in the generator matrix affect these polynomials. If $C_1 = C_2 = C_1^{\perp}$ then all words of C_2 have even weights since $a \cdot a = 0$ for all $a \in C_2$ and $1 \ldots 1 \in C_2$. With this, we get all powers of x even or all odd in polynomials $P_{n_0(c)}$. Also, coefficients of $x^i z^{n_0(c)-i}$ and $x^{n_0(c)-i} z^i$ is the same in a polynomial $P_{n_0(c)}$. In the Hamming example,

$$\mathrm{we}_{C_1}(x, y) = x^8 + 14x^4y^4 + y^8$$

Hence all swe's of codes $\mathbf{C} \in \mathcal{C}(C_1, C_2)$ will be of the form:

$$\mathrm{swe}_{\mathbf{C}}(x, y, z) = \mathrm{we}_{C_1}(x, z) + \sum_{\{c \in C_1, \mathrm{wt}(c)=4\}} P_4(x, z)y^4 + 16y^8$$

So here we would have $P_4(x, z)$ of the form $(a_0(x^4 + z^4) + a_2x^2z^2)$ or $a_1(x^3z + xz^3)$ where $2a_0 + a_2 = 16$ and $2a_1 = 16 \Rightarrow a_1 = 8$.

- If a 0 is changed to 2 in a word, its monomial representation in the swe would change from $x^e y^f z^g$ to $x^{e-1} y^f z^{g+1}$ whereas when a 2 is changed to 0, the corresponding polynomial representation will change to $x^{e+1} y^f z^{g-1}$. In the Hamming example, if a word and its transversals with $d(C_2)$ words have a polynomial representation $8y^4(x^3z + xz^3)$ then

coord. change	monomial rep. change	Reasoning
$0 \to 2$	$2xy^4z^3 \to 2y^4z^4$	Equally likely to have any of
$2 \to 0$	$6xy^4z^3 \to 6x^2y^4z^2$	$\{2220, 2202, 2022, 0222\}$
$0 \to 2$	$6x^3y^4z \to 6x^2y^4z^2$	Equally likely to have any of
$2 \to 0$	$2x^3y^4z \to 2x^4y^4$	$\{2000, 0200, 0020, 0002\}$

So $a_0 = 2$ and $a_2 = 12$ gives a possible polynomial representation $P_4 = 2x^4 + 12x^2z^2 + 2z^4$. If a word and its transversals with words of

$d(C_2)$ have a polynomial representation $2x^4y^4 + 12x^2y^4z^2 + 2y^4z^4$ then a change in a position give the following

coord. change	monomial rep. change	Reasoning
$0 \to 2$	$2x^4y^4 \to 2x^3y^4z$	0000 is the only option
$0 \to 2$	$6x^2y^4z^2 \to 6xy^4z^3$	Equally likely to have {2200,
$2 \to 0$	$6x^2y^4z^2 \to 6x^3y^4z$	2020, 2002, 0220, 0202, 0022}
$2 \to 0$	$2y^4z^4 \to 2xy^4z^3$	2222 is the only option

giving $P_4(x, z) = 8(x^3z + xz^3)$. So these are the only options we may have.

- Considering the first row of the generator, words of C^* and the possible changes in swe's are listed below:

1000 0111	$\Rightarrow$ 1000 2111	$2y^4(x^4 + z^4 + 6x^2z^2) \Rightarrow 8y^4(x^3z + xz^3)$
1100 1122 $+ v_2$	$\Rightarrow$ 1100 3122 $+ v_2$	$\Rightarrow$ no change
1010 1212 $+ v_3$	$\Rightarrow$ 1010 3212 $+ v_3$	$\Rightarrow$ no change
1001 1221 $+ v_4$	$\Rightarrow$ 1001 3221 $+ v_4$	$\Rightarrow$ no change
1110 2223 $+ v_{23}$	$\Rightarrow$ 1110 0223 $+ v_{23}$	$8y^4(x^3z + xz^3) \leftrightarrow 2y^4(x^4 + z^4 + 6x^2z^2)$
1101 2232 $+ v_{24}$	$\Rightarrow$ 1101 0232 $+ v_{24}$	$8y^4(x^3z + xz^3) \leftrightarrow 2y^4(x^4 + z^4 + 6x^2z^2)$
1011 2322 $+ v_{34}$	$\Rightarrow$ 1011 0322 $+ v_{34}$	$8y^4(x^3z + xz^3) \leftrightarrow 2y^4(x^4 + z^4 + 6x^2z^2)$
1111 3333 $+ v_{234}$	$\Rightarrow$ 1111 1333 $+ v_{234}$	$\Rightarrow$ no change
Net change	a multiple of	$2y^4[2(x^4 + z^4 + 6x^2z^2) - 8(x^3z + xz^3)]$

1000 0111	$\Rightarrow$ 1000 0311	$\Rightarrow$ no change
$1100\ 1122 + v_2$	$\Rightarrow 1100\ 1322 + v_2$	$\Rightarrow$ no change
$1010\ 1212 + v_3$	$\Rightarrow 1010\ 1012 + v_3$	$2y^4(x^4+z^4+6x^2z^2) \leftrightarrow 8y^4(x^3z+xz^3)$
$1001\ 1221 + v_4$	$\Rightarrow 1001\ 1021 + v_4$	$2y^4(x^4+z^4+6x^2z^2) \leftrightarrow 8y^4(x^3z+xz^3)$
$1110\ 2223 + v_{23}$	$\Rightarrow 1110\ 2023 + v_{23}$	$8y^4(x^3z+xz^3) \leftrightarrow 2y^4(x^4+z^4+6x^2z^2)$
$1101\ 2232 + v_{24}$	$\Rightarrow 1101\ 2032 + v_{24}$	$8y^4(x^3z+xz^3) \leftrightarrow 2y^4(x^4+z^4+6x^2z^2)$
$1011\ 2322 + v_{34}$	$\Rightarrow 1011\ 2122 + v_{34}$	$\Rightarrow$ no change
$1111\ 3333 + v_{234}$	$\Rightarrow 1111\ 3133 + v_{234}$	$\Rightarrow$ no change
Net change	a multiple of	$2y^4[2(x^4+z^4+6x^2z^2) - 8(x^3z+xz^3)]$

- Adding other even vectors to 1000 0111 will always result in adding or subtracting the expression:

$$Exp = 2y^4[2x^4 + 2z^4 + 12x^2z^2 - 8(x^3z + xz^3)] = 4y^4(x-z)^4$$

  Changes made to different rows will affect swe as above. Hence any change in the generator matrix result in adding a number of Exp and set of different swe's fall in a one dimensional affine subspace in this example.

- Rows of the generator matrix that are $G = Aut(C_1) \cap Aut(C_2)$ permutation equivalent give same polynomial representations.

- Changes in the polynomial representations above are computed regardless any code. So we may choose any code $\mathbf{C_0} \in \mathcal{C}(C_1, C_2)$ to be a reference code, gather different expressions we may have from different matrices X in (2.3) to have for $\mathbf{C} \in \mathcal{C}(C_1, C_2)$,

$$\text{swe}_{\mathbf{C}}(x,y,z) = \text{swe}_{\mathbf{C_0}} + a_1 Exp_1 + \ldots + a_k Exp_k \qquad (3.1)$$

for some integers $a_1, \ldots, a_k$. In the Hamming example choosing $\mathbf{C_0}$ to be the code corresponding to $X = 0$ in generator matrix form (2.3), we get

$$\begin{aligned}\mathrm{swe}_{C_0}(x,y,z) &= (x^8 + 14x^4z^4 + z^8) + 16y^8 + 20y^4(x^4 + 6x^2z^2 + z^4) \\ &\quad +8y^4(4x^3z + 4xz^3)\end{aligned}$$

So possible swe's would have coefficients of x^4y^4 (respectively y^4z^4), x^3y^4z (respectively xy^4z^3) and $x^2y^4z^2$ in the range:

	x^4y^4	x^3y^4z	$x^2y^4z^2$
$\mathrm{swe}_{NR} \rightarrow$	0	112	0
	.	.	.
$\mathrm{swe}_{C_0} \rightarrow$	20	32	120
	.	.	.
	28	0	168

Other monomials have fixed coefficients in all swe's. Without choosing a reference code C_0, we can deduce that swe's are of the form:

$$\mathrm{swe}_{\mathbf{C}}(x,y,z) = \overline{\mathrm{swe}} + a_1 Exp_1 + \ldots + a_k Exp_k \tag{3.2}$$

but to choose proper integer constants $\{a_1, \ldots, a_k\}$ we better use swe of a code in $\mathcal{C}(C_1, C_2)$. For instance, computing weight enumerators for all codes in the class $\mathcal{C}(C_1, C_1)$ with C_1 = Extended Hamming code of length 8, we get only the weight enumerators listed below:

$$\mathrm{swe}_1 = x^8 + 14x^4z^4 + z^8 + 16y^8 + \ 112x^3y^4z + 112xy^4z^3$$

$$\mathrm{swe}_2 = x^8+14x^4z^4+z^8+16y^8+ 8x^4y^4+80x^3y^4z+48x^2y^4z^2+80xy^4z^3+8y^4z^4$$

$$\mathrm{swe}_3 = x^8+14x^4z^4+z^8+16y^8+12x^4y^4+64x^3y^4z+72x^2y^4z^2+64xy^4z^3+12y^4z^4$$

$$\mathrm{swe}_4 = x^8+14x^4z^4+z^8+16y^8+16x^4y^4+48x^3y^4z+96x^2y^4z^2+48xy^4z^3+16y^4z^4$$

$$\mathrm{swe}_5 = x^8+14x^4z^4+z^8+16y^8+20x^4y^4+32x^3y^4z+120x^2y^4z^2+32xy^4z^3+20y^4z^4$$

$$\mathrm{swe}_6 = x^8+14x^4z^4+z^8+16y^8+24x^4y^4+16x^3y^4z+144x^2y^4z^2+16xy^4z^3+24y^4z^4$$

Proposition 3.2 *The set of weight enumerators of a set $\mathcal{C}(C_1, C_2)$ is contained in an affine subspace of polynomials.*

Proof There are only finitely many codes in $\mathcal{C}(C_1, C_2)$, and so there are only finitely many swe's, say $p_0, p_1, \ldots, p_s$, that occur. These swe's are contained in the affine subspace:

$$\{p_0 + a_1(p_1 - p_0) + \ldots + a_s(p_s - p_0) | a_i \in \mathbb{Q}\}$$

We comment that the dimension of the smallest affine subspace containing the swe's of $\mathcal{C}(C_1, C_2)$ is often very much smaller than the Proof of Proposition (3.2) seems to suggest.

Corollary 3.3 *If all expressions defining the affine subspace of weight enumerators as in* (3.1) *are multiples of linear combinations of formally self dual codes and* $\overline{\mathrm{swe}}$ *is equal to its MacWilliams transform then all codes of* $\mathcal{C}(C_1, C_2)$ *are formally self dual. For instance, when* $C_1 = C_1^{\perp}$ *then all codes of* $\mathcal{C}(C_1, C_1)$ *are formally self dual if expressions defining the affine subspace of weight enumerators are multiples of linear combinations of formally self dual codes.*

From Corollary (2.1) for $C_1 = C_1^{\perp}$, the class $\mathcal{C}(C_1, C_1)$ contain self dual codes. From Corollary (3.1), in this class $\overline{\mathrm{swe}}$ is equal to its MacWilliams transform.

So all linear combinations of $\overline{\text{swe}}$ and swe's of self dual cades are equal to their MacWilliams transform. Expressions made from these linear combinations are equal to their MacWilliams transform. Hence, codes with swe's made from these linear combinations are formally self dual.

Following are more examples showing the variation of swe's in a class of $\mathbb{Z}_4$ codes. The first example show a class with a fixed swe for all codes in $\mathcal{C}(C_1, C_2)$. The second example is an interesting example like the Hamming example.

Example Let $\mathcal{C}(C_1, C_2)$ be the set of $\mathbb{Z}_4$-codes having generator matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & d(X) \\ 0 & 0 & 0 \end{pmatrix}$$

where X is a 3×4 matrix over $\mathbb{Z}_2$, then all codes have:

$$\text{swe} = (x^7 + 7x^4z^3 + 7x^3z^4 + z^7) + \ 14x^3y^4 + 42x^2y^4z + 42xy^4z^2 + 14y^4z^3$$

Example If $C_1 = C_2 =$ extended Golay Code, a generator matrix for C_1

can be taken to be $[I_{12}A]$ where I_{12} is the 12×12 identity matrix and

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

[4]. Here C_1 have words of weights $0, 8, 12, 16$ and 24. For words of weights 16 and 12, the case is similar to the extended hamming code example, all even patterns are included and no odd pattern so an odd change would give us one expression in each case. For words of weight 8, we have 2^{11} even patterns included in the set of even positions but these are not all the even patterns (total number is $\frac{2^{16}}{2} = 2^{15}$) and an even change may give us another expression. From computations, we got an expression for even changes and another one for odd changes as shown below. The dimensions of the affine

space in this case at most 4 with

$$\begin{aligned}
Ex_1 &= 256x^7y^{16}z + 1792x^5y^{16}z^3 + 1792x^3y^{16}z^5 + 256xy^{16}z^7 \\
&\quad -(32x^8y^{16} + 896x^6y^{16}z^2 + 2240x^4y^{16}z^4 + 896x^2y^{16}z^6 + 32y^{16}z^8) \\
&\quad \text{for weight 16 words} \\
Ex_2 &= 24x^{11}y^{12}z + 440x^9y^{12}z^3 + 1584x^7y^{12}z^5 + 1584x^5y^{12}z^7 + 440x^3y^{12}z^9 + 24xy^{12}z^{11} \\
&\quad -(2x^{12}y^{12} + 110x^{10}y^{12}z^2 + 660x^8y^{12}z^4 + 924x^6y^{12}z^6 + 330x^4y^{12}z^8 + 22x^2y^{12}z^{10}) \\
&\quad \text{for weight 12 words} \\
Ex_3 &= 2(8x^{14}y^8z^2 + 112x^{12}y^8z^4 + 504x^{10}y^8z^6 + 800x^8y^8z^8 + 504x^6y^8z^{10} + 112x^4y^8z^{12} \\
&\quad +8x^2y^8z^{14}) - 2(x^{16}y^8 + 140x^{12}y^8z^4 + 448x^{10}y^8z^6 + 870x^8y^8z^8 + 448x^6y^8z^{10} \\
&\quad +140x^4y^8z^{12} + y^8z^{16}) \\
&\quad \text{ and} \\
Ex_4 &= 2(x^{15}y^8z + 35x^{13}y^8z^3 + 273x^{11}y^8z^5 + 715x^9y^8z^7 + 715x^7y^8z^9 + 273x^5y^8z^{11} \\
&\quad +35x^3y^8z^{13} + xy^8z^{15}) - 2(x^{16}y^8 + 140x^{12}y^8z^4 + 448x^{10}y^8z^6 + 870x^8y^8z^8 \\
&\quad +448x^6y^8z^{10} + 140x^4y^8z^{12} + y^8z^{16}) \\
&\quad \text{for weight 8 words}
\end{aligned}$$

Computations are shown below starting with different expressions we get for words of weight 16 and 12 then ending with expressions of weight 8 words.

diff.weight C_1 words	trans. for even x_e	poly. rep
011111111111001000111010 $\mathrm{wt}(w) = 16$	$\{$011111111111001000111010, 211111111111221222111212, 031111111111203220113032, $\ldots\}$	$32x^8y^{16} + 896x^6y^{16}z^2$ $+2240x^4y^{16}z^4 + 896x^2y^{16}z^6$ $+32y^{16}z^8$
	trans. for odd x_e	poly. rep
	$\{$211111111111001000111010, 011111111111221222111212, $\ldots\}$	$256x^7y^{16}z + 1792x^5y^{16}z^3$ $+1792x^3y^{16}z^5 + 256xy^{16}z^7$
	trans. for even x_e	poly. rep
000000000001111111111110 $\mathrm{wt}(w) = 12$	$\{$000000000001111111111110, 200000000001331333111312, 020000000001313331113132, $\ldots\}$	$2x^{12}y^{12} + 110x^{10}y^{12}z^2$ $+660x^8y^{12}z^4 + 924x^6y^{12}z^6$ $+330x^4y^{12}z^8 + 22x^2y^{12}z^{10}$
	trans. for odd x_e	poly. rep
	$\{$200000000001111111111110, 000000000001331333111312, 220000000001313331113132, $\ldots\}$	$24x^{11}y^{12}z + 440x^9y^{12}z^3$ $+1584x^7y^{12}z^5 + 1584x^5y^{12}z^7$ $+440x^3y^{12}z^9 + 24xy^{12}z^{11}$

	trans. for $x_e = 0$ or equiv	poly. rep
1000000000000110111000101 $\mathrm{wt}(w) = 8$	$\{$1000000000000110111000101, 1200000000000312331002123, 1020000000000132311020323, $\ldots\}$	$2(x^{16}y^8 + 140x^{12}y^8z^4$ $+448x^{10}y^8z^6 + 870x^8y^8z^8$ $+448x^6y^8z^{10} + 140x^4y^8z^{12}$ $+y^8z^{16})$
	trans. for even $x_e \simeq 0$	poly. rep
	$\{$1220000000000110111000101, 1020000000000312331002123, 1200000000000132311020323, $\ldots\}$	$2(8x^{14}y^8z^2 + 112x^{12}y^8z^4$ $+504x^{10}y^8z^6 + 800x^8y^8z^8$ $+504x^6y^8z^{10} + 112x^4y^8z^{12}$ $+8x^2y^8z^{14})$
	trans. for odd x_e	poly. rep
	$\{$1200000000000110111000101, 1000000000000312331002123, 1220000000000132311020323, $\ldots\}$	$2(x^{15}y^8z + 35x^{13}y^8z^3$ $+273x^{11}y^8z^5 + 715x^9y^8z^7$ $+715x^7y^8z^9 + 273x^5y^8z^{11}$ $+35x^3y^8z^{13} + xy^8z^{15})$

More computation on these polynomials on different examples is supplied in Appendix (A.4). In those examples we can see that choosing different binary codes having same weight enumerator may give corresponding classes of $\mathbb{Z}_4$-codes, $\mathcal{C}_i(C_1, C_1)$, with swe's contained in different affine spaces.

3.3 Possible swe's of self dual codes

Let $C_1 = C_2 = C_1^{\perp}$. We can use Gleason's technique [8] to find possible swe's of self dual $\mathbb{Z}_4$-codes in $\mathcal{C}(C_1, C_2)$. Since C_1 is self dual, all words of C_1 have even weights. So swe's of $\mathbb{Z}_4$-codes of $\mathcal{C}(C_1, C_2)$ satisfy:

$$\text{swe}(x, y, z) = \text{swe}(x, -y, z)$$

Here also self dual codes satisfy $|\mathbf{C}| = 4^{\frac{n}{2}}$ and the MacWilliam's transform.

$$\begin{aligned} \text{swe}(x, y, z) &= \frac{1}{|\mathbf{C}|} \text{swe}(x + 2y + z, x - z, x - 2y + z) \\ &= \frac{1}{4^{\frac{n}{2}}} \text{swe}(x + 2y + z, x - z, x - 2y + z) \\ &= \frac{1}{2^n} \text{swe}(x + 2y + z, x - z, x - 2y + z) \\ &= \text{swe}(\frac{x + 2y + z}{2}, \frac{x - z}{2}, \frac{x - 2y + z}{2}) \end{aligned}$$

So swe's of self dual codes are invariant polynomials of the group $G = \langle A_1, A_2 \rangle$ where

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Here, $A_1^2 = A_2^2 = (A_1A_2)^4 = I$. So G has order 8. Molien's Theorem and some computations show that the Hilbert series of $\mathbb{C}[x, y, z]$ is $\frac{1}{(1-t)(1-t^2)(1-t^4)}$. Hence the dimension of the nth homogeneous component is equal to the number of ways of writing n as a sum of 1's, 2's and 4's.

- $F_1 = x + z$ is a weight enumerator of a self dual $\mathbb{Z}_4$-code $\{0, 2\}$.
- $F_2 = x^2 + z^2 + 2y^2$ does not represent a self dual code but it represent a code which is invariant under G. For $\mathbf{C} = \{00, 11, 22, 33\}$, the dual $\mathbf{C}^{\perp} = \{00, 13, 22, 31\}$ which has same swe.
- $F_4 = x^4 + 6x^2z^2 + z^4 + 8y^4$ represents a weight enumerator of a self dual code. $\mathbf{C} = \{0000, 0202, 0022, 0220, 2020, 2200, 2002, 2222,\ 1111, 1313, 1133, 1331, 3131, 3311, 3113, 3333\}$.

These polynomials are independent. Applying Gleason's technique, we get the following.

Theorem 3.2 *The symmetrized weight enumerator of a self dual $\mathbb{Z}_4$-code of length n in a class $\mathcal{C}(C_1, C_2)$ with $C_1 = C_2 = C_1^{\perp}$ has the form:*

$$\sum_{4j_1+2j_2+j_3} a_1(x+z)^{j_3}(x^2+z^2+2y^2)^{j_2}(x^4+6x^2y^2+z^4+8y^4)^{j_1}$$

Result is true also for formally self dual codes.

For instance, a self dual $\mathbb{Z}_4$-code of length 8 has a weight enumerator of the form

$$a_1F_1^8+a_2F_1^6F_2+a_3F_1^4F_2^2+a_4F_1^2F_2^3+a_5F_2^4+a_6F_1^4F_4+a_7F_4^2+a_8F_2^2F_4+a_9F_1^2F_2F_4$$

which give the following

$$\begin{aligned}
& (a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9)x^8 \\
+ \quad & (8a_1+6a_2+4a_3+2a_4+4a_6+2a_9)x^7z \\
+ \quad & (2a_2+4a_3+6a_4+8a_5+4a_8+2a_9)x^6y^2 \\
+ \quad & (28a_1+16a_2+8a_3+4a_4+4a_5+12a_6+12a_7+8a_8+8a_9)x^6z^2 \\
+ \quad & (12a_2+16a_3+12a_4+4a_9)x^5y^2z \\
+ \quad & (56a_1+26a_2+12a_3+6a_4+28a_6+14a_9)x^5z^3 \\
+ \quad & (4a_3+12a_4+24a_5+8a_6+16a_7+12a_8+8a_9)x^4y^4 \\
+ \quad & (30a_2+28a_3+18a_4+24a_5+28a_8+14a_9)x^4y^2z^2 \\
+ \quad & (70a_1+30a_2+14a_3+6a_4+6a_5+38a_6+38a_7+14a_8+14a_9)x^4z^4 \\
+ \quad & (16a_3+24a_4+32a_6+16a_9)x^3y^4z \\
+ \quad & (40a_2+32a_3+24a_4+24a_9)x^3y^2z^3 \\
+ \quad & (56a_1+26a_2+12a_3+6a_4+28a_6+14a_9)x^3z^5
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (8a_4 + 32a_5 + 32a_8 + 16a_9)x^2y^6 \\
&+ \quad (24a_3 + 24a_4 + 48a_5 + 48a_6 + 96a_7 + 40a_8 + 16a_9)x^2y^4z^2 \\
&+ \quad (30a_2 + 28a_3 + 18a_4 + 24a_5 + 28a_8 + 14a_9)x^2y^2z^4 \\
&+ \quad (28a_1 + 16a_2 + 8a_3 + 4a_4 + 4a_5 + 12a_6 + 12a_7 + 8a_8 + 8a_9)x^2z^6 \\
&+ \quad (16a_4 + 32a_9)xy^6z \\
&+ \quad (16a_3 + 24a_4 + 32a_6 + 16a_9)xy^4z^3 \\
&+ \quad (12a_2 + 16a_3 + 12a_4 + 4a_9)xy^2z^5 \\
&+ \quad (8a_1 + 6a_2 + 4a_3 + 2a_4 + 4a_6 + 2a_9)xz^7 \\
&+ \quad (16a_5 + 64a_7 + 32a_8)y^8 \\
&+ \quad (8a_4 + 32a_5 + 32a_8 + 16a_9)y^6z^2 \\
&+ \quad (4a_3 + 12a_4 + 24a_5 + 8a_6 + 16a_7 + 12a_8 + 8a_9)y^4z^4 \\
&+ \quad (2a_2 + 4a_3 + 6a_4 + 8a_5 + 4a_8 + 2a_9)y^2z^6 \\
&+ \quad (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9)z^8
\end{aligned}
$$

Using this form to get possible weight enumerators of self dual codes for the Hamming example, we will have the following system to be solved

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & : & 1 \\
8 & 6 & 4 & 2 & 0 & 4 & 0 & 0 & 2 & : & 0 \\
0 & 2 & 4 & 6 & 8 & 0 & 0 & 4 & 2 & : & 0 \\
28 & 16 & 8 & 4 & 4 & 12 & 12 & 8 & 8 & : & 0 \\
0 & 12 & 16 & 12 & 0 & 0 & 0 & 0 & 4 & : & 0 \\
56 & 26 & 12 & 6 & 0 & 28 & 0 & 0 & 14 & : & 0 \\
0 & 30 & 28 & 18 & 24 & 0 & 0 & 28 & 14 & : & 0 \\
70 & 30 & 14 & 6 & 6 & 38 & 38 & 14 & 14 & : & 14 \\
0 & 40 & 32 & 24 & 0 & 0 & 0 & 0 & 24 & : & 0 \\
0 & 0 & 0 & 8 & 32 & 0 & 0 & 32 & 16 & : & 0 \\
0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 32 & : & 0 \\
0 & 0 & 0 & 0 & 16 & 0 & 64 & 32 & 0 & : & 16
\end{pmatrix}$$

letting a_9 be a free variable, solution of the linear system need to satisfy:

$$a_1 = -\frac{3}{4} + \frac{1}{4}a_9, \quad a_2 = -a_9, \quad a_3 = 2a_9, \quad a_4 = -2a_9,$$

$$a_5 = a_9, \quad a_6 = \frac{3}{2} - \frac{1}{2}a_9, \quad a_7 = \frac{1}{4} - \frac{1}{4}a_9, \quad a_8 = -a_9$$

for $a_9 = 0$ we get

$$a_1 = -\frac{3}{4}, \quad a_6 = \frac{3}{2}, \quad a_7 = \frac{1}{4}, \text{ and } a_i = 0 \text{ for all } i \neq 1, 6, 7$$

giving

$$F = x^8 + 14x^4z^4 + z^8 + 16x^4y^4 + 48x^3y^4z + 96x^2y^4z^2 + 48xy^4z^3 + 16y^4z^4 + 16y^8$$

for $a_9 = 1$ we get $F + Exp$. Hence even though not all codes are self dual in this class, all swe's of this class are invariant under G. This can be deduced also from Corollary (3.3), since here Exp is a multiple of the difference between self dual codes. With Theorem (2.1), we found self dual codes having the weight enumerators:

$$\begin{aligned} \text{swe} &= x^8 + 14x^4z^4 + z^8 + 16x^4y^4 + 48x^3y^4z + 96x^2y^4z^2 + 48xy^4z^3 + 16y^4z^4 + 16y^8 \\ \text{swe} &= x^8 + 14x^4z^4 + z^8 + 112x^3y^4z + 112xy^4z^3 + 16y^8 \end{aligned}$$

Chapter 4

Actions on $\mathcal{C}(C_1, C_2)$

Let the semidirect product $\mathbb{Z}_2^n \rtimes G$ where $G = \text{Aut}(C_1) \cap \text{Aut}(C_2) \leq S_n$ act on $\mathcal{C}(C_1, C_2)$. Elements of $\mathbb{Z}_2^n$ act coordinatewise. If $\nu \in \mathbb{Z}_2^n$ has an ith coordinate 1, acting by ν on a code $\mathbf{C} \in \mathcal{C}(C_1, C_2)$ changes signs of entries in the ith coordinate of $\mathbf{C}$. This action preserves the class $\mathcal{C}(C_1, C_2)$. It exchanges 1's and 3's and does not affect the evens. Elements of G permute coordinates. This automorphism group is chosen to preserve the class $\mathcal{C}(C_1, C_2)$.

4.1 Defining action of $\mathbb{Z}_2^n$ on W

For $a, b \in \mathbb{Z}_2^n$, define $a * b \in \mathbb{Z}_2^n$ to be the coordinatewise multiple of coordinates of a and b. Acting by $(1, \ldots, 1)$ fixes any $\mathbb{Z}_4$-code since changing signs of all coordinates fixes a linear code. So $\mathbb{Z}_2^n$'s action is not faithful. The kernel of this action in the case where $C_2 = C_1$ can be described as follows.

Theorem 4.1 *Let C_1 be a binary code, $\mathcal{C}(C_1, C_1)$ be the set of $\mathbb{Z}_4$-codes corresponding to $\{C_1, C_1\}$ and $\mathcal{A} = \{a_1, \ldots, a_{k_1}\}$ be a basis of C_1. If there is* *no coordinate in which all words of* C_1 *is zero and* $\mathcal{A}$ *can not be partitioned into two sets;* $\{\mathcal{A}_1, \mathcal{A}_2\}$ *(that is* $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$ *and* $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$*) such that* $a_i * a_l = (0, \ldots, 0)$ *for all* $a_i \in \mathcal{A}_1$ *and* $a_l \in \mathcal{A}_2$ *then the kernel of the* $\mathbb{Z}_2^n$*'s action is the subgroup* $\langle(1 \ldots 1)\rangle$.

Proof For $\nu \in \mathbb{Z}_2^n$ and $c \in \mathbf{C}$, we can describe $\mathbb{Z}_2^n$'s action on $\mathcal{C}(C_1, C_2)$ as follows:

$$\nu \circ c = c + d(\nu * (c \bmod 2)) \tag{4.1}$$

for $\nu \in \mathbb{Z}_2^n$. Codes of $\mathcal{C}(C_1, C_2)$ are linear. So if $c \in \mathbf{C} \in \mathcal{C}(C_1, C_2)$ then $-c \in \mathbf{C}$. But $-c = (1, \ldots, 1) \circ c$. Hence $(1, \ldots, 1)$ is in the kernel of $\mathbb{Z}_2^n$'s action.

Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, having 0 for all positions $j \neq i$ and 1 for $j = i$. If the ith position of all words of C_1 is 0, then $e_i * (c \bmod 2) = (0, \ldots, 0)$ for all codewords $c \in \mathbf{C} \in \mathcal{C}(C_1, C_2)$ and so $e_i \cdot \mathbf{C} = \mathbf{C}$ for all $\mathbb{Z}_4$-codes in the class $\mathcal{C}(C_1, C_2)$. So the kernel of $\mathbb{Z}_2^n$'s action contains the subgroup $\langle(1 \ldots 1),\ \{e_i : \{c\}_i = 0,\ \forall\, c \in C_1\}\rangle$.

Let ν be in the kernel of $\mathbb{Z}_2^n$'s action on $\mathcal{C}(C_1, C_1)$. Here $C_1 = C_2$ so the general generator matrix of a code in $\mathcal{C}(C_1, C_2)$ has the form:

$$\begin{pmatrix} I_{k_1} & \alpha(B) + d(X) \end{pmatrix}$$

Notice that

- If $\nu * a = (0, \ldots, 0)$ for some $a \in \mathcal{A}$ then $\{\nu\}_i = 0$ whenever $\{a\}_i$ is odd and $\nu \circ a = a$.
- There is no coordinate for which all C_1 codewords is zero so there is at least one basis vector which is 1 in a specific coordinate. If $\nu * a = (0, \ldots, 0)$ for all basis vectors $a \in \mathcal{A}$ then $\nu = (0, \ldots, 0)$.

- If $\nu * a \neq (0, \ldots, 0)$ for some $a \in \mathcal{A}$ then $\nu \circ a = -a$ and $\{\nu\}_i = 1$ whenever $\{a\}_i = 1$.

- If $\nu * a \neq (0, \ldots, 0)$ for all $a \in \mathcal{A}$ then $\nu = (1, \ldots, 1)$.

- If $\nu \neq (0, \ldots, 0)$ or $(1, \ldots, 1)$ then for some $a_i \in \mathcal{A}$, we have $\nu * a_i = (0, \ldots, 0)$ and for other $a_j \in \mathcal{A}$, we have $\nu * a_j \neq (0, \ldots, 0)$. Let $\mathcal{A}_1 = \{a \in \mathcal{A}$ such that $\nu * a = (0, \ldots, 0)\}$ and $\mathcal{A}_2 = \{a \in \mathcal{A}$ such that $\nu * a \neq (0, \ldots, 0)\}$ then ν has an ith coordinate 0 in every position where there is an element in $\mathcal{A}_1$ with 1 in its ith coordinate. ν has an ith coordinate 1 in every position where there is an element in $\mathcal{A}_2$ with 1 its ith coordinate. ν fixes the codes so $\mathcal{A}$ is partitioned into two sets $\{\mathcal{A}_1, \mathcal{A}_2\}$ such that $a_i * a_j = (0, \ldots, 0)$ for all $a_i \in \mathcal{A}_1$ and $a_j \in \mathcal{A}_2$.

If $C_1 = \langle a_1, \ldots, a_{k_1} \rangle$ such that for some a_i in the basis, $a_i \circ a_l = \pm a_l$ for all a_l in the set of basis vectors then such an a_i is in kernel of $\mathbb{Z}_2^n$'s action also.

In general, if $\mathcal{A}$ is partitioned into $\{\mathcal{A}_1, \ldots \mathcal{A}_k\}$ with $a_s \cdot a_t = 0$ for all $a_s \in \mathcal{A}_s$ and $a_t \in \mathcal{A}_t$ then the set of vectors ν with coordinates $\{\nu\}_i = 1$ for all coordinates such that $\mathcal{A}_j$ for some j has a word with nonzero ith coordinate and $\{\nu\}_i = 0$ otherwise are in the kernel of $\mathbb{Z}_2^n$'s action.

Let Z be the kernel of $\mathbb{Z}_2^n$'s action. Then $\mathbb{Z}_2^n/Z$ acts faithfully on $\mathcal{C}(C_1, C_2)$. Corresponding $\mathbb{Z}_2^n$ action on W (set of all matrices X in different generator matrices of codes in $\mathcal{C}(C_1, C_2)$) can be described by watching behavior of actions of a basis of $\mathbb{Z}_2^n$ as follows.

Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ have an ith coordinate 1 and 0 elsewhere. For a $\mathbb{Z}_4$-code with a generator matrix of the form:

$$G = \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix}$$

Let $\{a_i\}$ be rows of matrix A above, $\{b_i\}$ be rows of matrix B and $\{c_i\}$ be rows of matrix C. Let e_i act on a code with the above generator matrix. Then,

- for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$,

$$\begin{aligned} e_i \circ G &= \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix} + \begin{pmatrix} & 0\ldots0 & 0\ldots0 \\ 0 & d(a_i) & d(b_i) \\ & 0\ldots0 & 0\ldots0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix} + \begin{pmatrix} & & 0\ldots0 \\ 0 & 0 & d(b_i) + \sum_j d(a_{ij})c_j \\ & & 0\ldots0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

- For $k_1 < i \leq k_1 + k_2$:

$$\begin{aligned} e_i \circ G &= \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix} + \begin{pmatrix} & 0\ldots d(a_{1i})\ldots0 & \\ 0 & \ldots & 0 \\ & 0\ldots d(a_{k_1 i})\ldots0 & \\ 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix} + \begin{pmatrix} & & d(a_{1i})c_i \\ 0 & 0 & \ldots \\ & & d(a_{k_1 i})c_i \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

- For $k_1 + k_2 < i \leq n$:

$$e_i \circ G = \begin{pmatrix} I_{k_1} & A & B \\ 0 & 2I_{k_2} & 2C \end{pmatrix} + \begin{pmatrix} & & 0\ldots d(b_{1i})\ldots0 \\ 0 & 0 & \ldots \\ & & 0\ldots d(b_{k_1 i})\ldots0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence we can define $\mathbb{Z}_2^n$'s action on W as translation by matrices M_i as:

$$\nu \cdot W = W + \sum_{1 \leq i \leq n} d(\{\nu\}_i) \cdot M_i \tag{4.2}$$

for $\nu \in \mathbb{Z}_2^n$ and

$$M_i = \begin{cases} \begin{pmatrix} 0 \dots 0 \\ \alpha(b_i) + \sum_j a_{ij} c_j \\ 0 \dots 0 \end{pmatrix} & \text{for } 1 \leq i \leq k_1 \\ \\ \begin{pmatrix} (a_{1i})c_i \\ \dots \\ (a_{k_1 i})c_i \end{pmatrix} & \text{for } k_1 < i \leq k_1 + k_2 \\ \\ \begin{pmatrix} 0 \dots \alpha(b_{1i}) \dots 0 \\ \dots \\ 0 \dots \alpha(b_{k_1 i}) \dots 0 \end{pmatrix} & \text{for } k_1 + k_2 < i \leq n \end{cases}$$

$\mathbb{Z}_2^n/Z$ acts faithfully by translation with $\frac{2^{k_1(n-(k_1+k_2))}}{2^{n-\dim(Z)}} = 2^{k_1(n-(k_1+k_2))-n+\dim(Z)}$ orbits each of size $2^{n-\dim(Z)}$.

If W is the set of all matrices X as in the generator matrix. Let $W_0 \leq W$ be the set of matrices spanned by matrices M_i defined above. Then adding a matrix from W_0 to a matrix X gives a corresponding $\mathbb{Z}_4$-code with same swe as the swe of the code corresponding to X. If the kernel of $\mathbb{Z}_2^n$'s action is $\langle 1 \dots 1 \rangle$ then $W_0 \simeq \mathbb{Z}_2^n/\langle 1 \dots 1 \rangle$. We can also describe action of $\mathbb{Z}_2^n/\langle 1 \dots 1 \rangle$ as an affine action having a trivial linear action and a derivation obtained from the split extension of W by $\mathbb{Z}_2^n/\langle 1 \dots 1 \rangle$

$$0 \to W \to E \to \mathbb{Z}_2^n/\langle 1 \dots 1 \rangle \to 0$$

The factor set we are considering here would be the zero factor set so addition in this extension would be coordinatewise as in the direct product. Action of sign changing of coordinates then would be an affine action with a trivial linear action and a derivation in $Der(\mathbb{Z}_2^n/\langle 1\ldots 1\rangle, W) = Hom(\mathbb{Z}_2^n/\langle 1\ldots 1\rangle, W)$ defined by $\delta(0) = 0,\ \delta(e_1) = M_1, \ldots,\ \delta(e_n) = M_n$, and $W_0 = \langle M_1, \ldots, M_{n-1}\rangle$ since $M_n = M_1 + \cdots + M_{n-1}$. Hence orbit of 0 would be W_0 and orbit of $a \in W$ would be $a + W_0$.

4.2 Derivations Computation

Derivations are functions $\phi : G \mapsto A$ satisfying $\phi(xy) = x\phi(y)+\phi(x)$. We can think of derivations as different complement of A in the semidirect product $A \rtimes G$. When $G \leq S_n$ is generated by odd order automorphisms, computing derivations can be made easier as described below.

4.2.1 Describing G's linear action

G acts on $\mathbb{Z}_2^n$ by permuting coordinates. Actions of $g \in G$ are represented by permutation matrices. These are matrices that have exactly one nonzero coordinate in every row and every column. If an ith coordinate is fixed by g then we would have $a_{ii} = 1$ in M_g and so $(1 - \lambda)$ in that position of matrix $M_g - \lambda I$. This gives a factor $(1 - \lambda)^m$ in the characteristic polynomial with $m \geq$ number of fixed coordinates by g. If a permutation has order k then $M_g^k = I$ and the characteristic polynomial will divide $\lambda^k - 1$. Precisely, for every cycle of length k in a permutation, we have a corresponding factor $\lambda^k - 1$ in the characteristic polynomial. Hence, the characteristic polynomial of a permutation is a product of factors $(\lambda^{k_i} - 1)$ one for each cycle with corresponding cycle length k_i. So, this polynomial will be of the form

$$(\lambda - 1)^m(\lambda^{k_1-1} + \ldots + \lambda + 1) \ldots (\lambda^{k_m-1} + \ldots + \lambda + 1)$$

where m is the number of cycles of the permutation.

If g has odd order then all cycles have odd lengths and number of entries in $\lambda^{k_i-1} + \cdots + \lambda + 1$ is odd for every cycle. So $(\lambda - 1)$ is not a factor for any of these polynomials. Also, these polynomials would be a product of irreducible polynomials with distinct roots since they divide $\lambda^{ord(g)} - 1$. So the action is diagonalizable in this case. When n (code's length) is even, power of the factor $(\lambda - 1)$ in the characteristic polynomial is then ≥ 2.

Let k be the order of a permutation then $(A_g^k - 1)v = 0$ for any v. Note that,

$$(x^k - 1) = (x - 1)(x^{k-1} + \ldots + 1)$$

For odd k, the factors $(x-1)$ and $(x^{k-1}+\ldots+1)$ are coprime. So $V = V_1 + V_2$ where $V_1 = (A_g - I)V$ and $V_2 = (A_g^{k-1} + \ldots + A_g + I)V$ and any v can be expressed as $v = (A_g - I)v_1 + (A_g^{k-1} + \ldots + A_g + I)v_2$. Here, V_1 and V_2 are disjoint since if $v \in V_1 \cap V_2$ then $v = (A_g - I)v_1 = (A_g^{k-1} + \ldots + A_g + I)v_2$ for some $v_1, v_2 \in V$. So we would have $(A_g - I)^2 v_1 = 0 \Rightarrow (A_g - I)v_1 = 0 \Rightarrow v_1 = 0$. Hence, $V = V_1 \oplus V_2$.

When $G \leq S_n$ act on $\mathbb{Z}_2^n$, a permutation $g \in G$ with m cycles has the characteristic polynomial $p = (x^{k_1} - 1)(x^{k_2} - 1) \ldots (x^{k_m} - 1)$ where k_i are cycle lengths for different cycles of g.

$g \in Aut(C_1)$ for a subspace $C_1 \leq \mathbb{Z}_2^n$ act on C_1. Its characteristic polynomial, say p_1 divides p. This g act on $\mathbb{Z}_2^n / C_1$ with characteristic polynomial p_2 and $p_1 \cdot p_2 = p$. For $p_1 = a_m x^m + a_{m-1}x^{m-1} + \ldots a_1 x + a_0$, let $\tilde{p_1} = a_0 x^m + a_1 x^{m-1} + \ldots a_{m-1} x + a_m$. If p_1 is a characteristic polynomial of g's action on C_1. Then $\tilde{p_1}$ is the characteristic polynomial for the action on C_1^*. If $C_1 \leq C_2 \leq C_1^\perp$. Then p_1 divides p (the characteristic polynomial on $\mathbb{Z}_2^n$). Characteristic polynomial on $\mathbb{Z}_2^n/C_1$ is p/p_1 and characteristic polynomial on $\mathbb{Z}_2^n/C_2$ then divides p/p_1.

An automorphism of odd order acts on $\mathbb{Z}_2^n$ with a diagonalizable action. When it acts on a subspace, V_1, the action on V_1 is diagonalizable also.

Let V_1 and V_2 be subspaces of $\mathbb{Z}_2^n$. If G acts on these spaces. We can let it act on $V_1 \otimes V_2$ as $g \cdot (V_1 \otimes V_2) = g \cdot V_1 \otimes g \cdot V_2$.

If $g \in G$ acts on V_1 with diagonalizable action. Let A_g be the corresponding matrix of action. Then A_g has a basis of eigenvectors say $\{\nu_i\}$ with a corresponding eigenvalues $\{\lambda_i\}$ and $g \cdot \nu_i = \lambda_i \nu_i$. Similarly if g's action on V_2 is diagonalizable and B_g is its matrix of action. Then B_g has a basis of eigenvectors say $\{\varphi_i\}$ with a corresponding eigenvalues $\{\mu_i\}$ and $g \cdot \varphi_i = \mu_i \varphi_i$. So $\{\nu_i \otimes \varphi_j\}$ is a basis for $V_1 \otimes V_2$ and $g \cdot (\nu_i \otimes \varphi_j) = \lambda_i \mu_j (\nu_i \otimes \varphi_j)$.

4.2.2 First Cohomology Group with Coefficient Set $\mathbb{Z}_2^n$

As defined earlier, a derivation is a function, $\phi : G \to W$, satisfying:

$$\phi(g_1 g_2) = \phi(g_1)^{g_2} + \phi(g_2)$$

Derivations are determined by their values on a set of generators. A derivation $\delta \in Der(\langle g \rangle, \mathbb{Z}_2^n)$ is determined by $\delta(g)$ since

$$\delta(g^2) = \delta(g)^g + \delta(g) = (g + I) \cdot \delta(g)$$

and inductively

$$\delta(g^k) = (g^{k-1} + \ldots + g + I) \cdot \delta(g)$$

If g has order $ord(g)$ then a derivation defined by $\delta(g) = x$ need to satisfy

$$(g^{ord(g)-1} + \ldots + g + I) \cdot x = 0 \tag{4.3}$$

This is the space of vectors satisfying

$$(A_g^{ord(g)-1} + \ldots + A_g + I) \cdot x = 0$$

If g has odd order then as given above in the characteristic polynomial computation,

$$\begin{aligned} Der(\langle g \rangle, \mathbb{Z}_2^n) &= \{v : (A_g^{ord(g)-1} + \ldots + A_g + I)v = 0\} \\ &= \{(A_g - I)v\} \end{aligned}$$

This is the Inner derivation set. Hence for an automorphism g of odd order, all derivations $Der(\langle g \rangle, \mathbb{Z}_2^n)$ are inner. Also note that:

- $(A_g - I)a = (A_g - I)(1 \ldots 1 + a)$ since $(A_g - I)(1 \ldots 1) = 0 \ldots 0$.
- Dimension of the Inner derivation set, $\mathrm{IDer}(\langle g \rangle, \mathbb{Z}_2^n)$ is equal to $n-$ number of cycles. Since $\delta(g) = a - a^g$ is zero only if $a^g = a$. But dimension of fixed points space for $g \in G$ on $\mathbb{Z}_2^n$ is equal to number of cycles.
- If G is generated by a set $\{g_1, g_2, \ldots, g_k\}$. Then

$$\begin{aligned} Dim(\mathrm{IDer}(G, V)) &= n - Dim(\text{intersection of fixed point spaces of all } g_i) \\ &= n - Dim(\cap\{v : (A_{g_i} - I)v = 0 \text{ for all } g \in G\}) \end{aligned}$$

$H^1(G, \mathbb{Z}_2^n)$ can be described as the set of conjugate classes of complements of $\mathbb{Z}_2^n$ in $\mathbb{Z}_2^n \rtimes G$. To compute $H^1(G, \mathbb{Z}_2^n)$ we need to:

1. Check all possible complement.

2. Divide them into conjugate classes.

3. Assign a representative for each conjugate class.

4. Check derivation conditions for the set of conjugate class representatives.

$|H^1(G, \mathbb{Z}_2^n)|$ is then the number of conjugate class representatives satisfying the derivation conditions.

If G is generated by a set $\{g_1, g_2, \ldots, g_k\}$ of odd order permutations then for every g_i, all derivations are inner in $Der(\langle g_i \rangle, \mathbb{Z}_2^n)$. So derivations in $Der(\langle g_i \rangle, \mathbb{Z}_2^n)$ have images of the form $a - a^g$ for some $a \in \mathbb{Z}_2^n$. The set $Der(G, \mathbb{Z}_2^n)$ is then a subset of $Der(\langle g_1 \rangle, \mathbb{Z}_2^n) \times \ldots \times Der(\langle g_k \rangle, \mathbb{Z}_2^n)$.

Let us start with $G = \langle g_1, g_2 \rangle$. Since all derivation images are given by their images on a set of generators of G, we shall define complements by their values on the generators. Complements conjugate to the zero complement will be of the form $\{(a_0 - a_0^{g_1}, g_1), (a_0 - a_0^{g_2}, g_2)\}$. All possible complements here will be of the form $\{(a_1 - a_1^{g_1}, g_1), (a_2 - a_2^{g_2}, g_2)\}$. So a complement represents an inner derivation if and only if $a_1 = a_2$.

We can choose $\{(0, g_1), (0, g_2)\}$ as a representative for the zero complement. To get representatives for other conjugate classes, note that if $\{(a_1 - a_1^{g_1}, g_1), (a_2 - a_2^{g_2}, g_2)\}$ is a complement then $\{(0, g_1), ((a_2 - a_2^{g_2}) - (a_1 - a_1^{g_2}), g_2)\} = \{(0, g_1), ((a_2 - a_1) - (a_2 - a_1)^{g_2}, g_2)\}$ is a complement in the same conjugate class. So complements of the form $\{(0, g_1), (a_0 - a_0^{g_2}, g_2)\}$ will cover the whole set of conjugate class representatives. If $a_0^{g_1} = a_0$ then we will have a complement conjugate to the zero complement. So a set of representatives with no 2 elements in the same conjugate class would be the set $\{(0, g_1), (a_0 - a_0^{g_2}, g_2)\}$ with a_0 chosen to be in $\mathbb{Z}_2^n$ not fixed by g_1 or g_2.

Number of these elements is $\frac{2^n \times |H^0(G,\mathbb{Z}_2^n)|}{|H^0(\langle g_1\rangle,\mathbb{Z}_2^n)| \times |H^0(\langle g_2\rangle,\mathbb{Z}_2^n)|}$.

Let $\delta(g_1) = 0$ and $\delta(g_2) = x$. Computation of $H^1(G, \mathbb{Z}_2^n)$ might be done by hand. Checking conditions here is done easily by summing coordinates of a cycle for a certain permutation. We can make this process even simpler using the following:

- $\delta(g_1^i g_2^j) = \delta(g_2^j)$. The automorphisms $\{g_1^i g_2^j\}$ have different orders. We need to check that (4.3) is satisfied for all automorphisms $\{g_1^i g_2^j\}$ with $x = \delta(g_2^j)$. No need to check the conditions for automorphisms $\{g_2^i g_1^j\}$ since this is a set of inverses of the earlier set and so it will not add further constrains. If $\delta(g_1^i g_2^j) = \delta(g_2^j) = x$ satisfy derivation order condition so does $\delta(g_2^{-j} g_1^{-i})$. Since $ord(g_1^i g_2^j) = ord(g_2^{-j} g_1^{-i})$ and if

$$((g_1^i g_2^j)^{ord-1} + \ldots + (g_1^i g_2^j) + I)x = 0$$

  then multiplying by $(g_2^{-j} g_1^{-i})^{ord-1}$, we get

$$(I + (g_2^{-j} g_1^{-i}) + \ldots + (g_2^{-j} g_1^{-i})^{ord-1})x = 0$$

- Once the previous check is done. We don't need to check conditions for $\{(g_1^i g_2^j)^m\}$ or $\{(g_2^i g_1^j)^m\}$. Since conditions on this will be the same obtained earlier.

- Still need to check conditions for automorphisms $\{g_1^{i_1} g_2^{i_2} g_1^{i_3}\}$, $\{g_1^{i_1} g_2^{i_2} g_1^{i_3} g_2^{i_4}\}, \ldots$

Once we get a set of derivation class representatives for $G = \langle g_1, g_2\rangle$. Let derivation values be $\{0, rep_1, rep_2, \ldots\}$ on $\langle g_1, g_2\rangle$ and compute derivations of $G = \langle g_1, g_2, g_3\rangle$ similarly. And keep on extending G until we get the derivation modulo inner derivations set for $G = \langle g_1, g_2, \ldots, g_k\rangle$.

4.2.3 $H^1(G, \mathbb{Z}_2^n)$ for the Automorphism Group of the Hamming Example

The method we have so far simplifies computing first cohomology when G is generated by odd order elements. In fact, G can be generated by a set of odd order elements if and only if it has no subgroup of index 2. The automorphism group of the hamming example, $G = \langle(2843567), (1234)(5678)\rangle$ satisfies this. Where we can write

$$\begin{aligned}(1234)(5678) &= (12345)(45678)\\ &= (45)(35)(25)(15)(78)(68)(58)(48)\\ &= (45)(35)(78)(68)(25)(15)(58)(48)\\ &= (345)(678)(125)(458)\\ &= (345)(687)(687)(125)(458)\\ &= (345)(687)(1257684)\end{aligned}$$

and $\{(345)(687), (1257684)\} \in G$. So $G = \langle(2843567), (345)(687), (1257684)\rangle$.

Let $h_1 = (1257684)$, $h_2 = (345)(687)$ and $h_3 = (2843567)$. Since all h_i have odd orders, all derivations are inner in $Der(\langle h_i\rangle, \mathbb{Z}_2^n)$. Let us assume that the derivation value is zero on $\langle h_1\rangle$ and compute $H^1(\langle h_1, h_2\rangle, \mathbb{Z}_2^n)$. Nonzero complements of $\mathbb{Z}_2^n$ in $\mathbb{Z}_2^n \rtimes \langle h_1, h_2\rangle$ would be of the form:

$$\{(0, h_1), (x - x^{h_2}, h_2)\}$$

Here x is chosen to be an element in $\mathbb{Z}_2^n$ not fixed by h_1 or h_2. Let $x = (a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8)$. Then x is fixed by h_1 if $x^{h_1} = x$. That is if

$$a_1 = a_2 = a_4 = a_5 = a_6 = a_7 = a_8$$

Hence fixed space by h_1 is spanned by

$$\{(1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 1,\ 1), (0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0)\}$$

We need to pick x to be not in the subspace spanned by this set.

$$\begin{aligned}\delta(h_2) &= x - x^{h_2}\\ &= (0,\ 0,\ a_3+a_5,\ a_3+a_4,\ a_4+a_5,\ a_6+a_7,\ a_7+a_8,\ a_6+a_8)\end{aligned}$$

For $x = (1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 1,\ 1)$ or $(0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0)$ we get $\delta(h_2) = (0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0)$. This gives a complement conjugate to the zero complement. So we need to exclude this derivation value from the set of complements defining derivations at the end of computations.

Now we need to start checking derivation conditions for $\delta(h_1) = 0$ and $\delta(h_2) = (0,\ 0,\ a_3+a_5,\ a_3+a_4,\ a_4+a_5,\ a_6+a_7,\ a_7+a_8,\ a_6+a_8)$. Note that $\delta(h_1^i h_2) = \delta(h_2)$ and so

- For $g = h_1h_2 = (1234)(5678)$, we need $a_4 + a_5 = 0$ and so $\delta(h_2) = (0,\ 0,\ a_3+a_4,\ a_3+a_4,\ 0,\ a_6+a_7,\ a_7+a_8,\ a_6+a_8)$.
- For $g = h_1^2h_2 = (1342658)$, we need $a_7 + a_8 = 0$. So $\delta(h_2) = (0,\ 0,\ a_3+a_4,\ a_3+a_4,\ 0,\ a_6+a_7,\ 0,\ a_6+a_7)$.
- For $g = h_1^3h_2 = (16)(28)(34)(57)$, we get $a_6 + a_7 = 0$. So $\delta(h_2) = (0,\ 0,\ a_3+a_4,\ a_3+a_4,\ 0,\ 0,\ 0,\ 0)$. This is an inner derivation given by choosing x fixed points of h_1 mentioned above.

Hence, $H^1(\langle h_1, h_2\rangle, \mathbb{Z}_2^n) = \{0\}$.

Now assume that derivation value is zero on $\langle h_1, h_2\rangle$ and try computing $H^1(G, \mathbb{Z}_2^n)$. Complements of $\mathbb{Z}_2^n$ on $\mathbb{Z}_2^n \rtimes G$ would be of the form:

$$\{(0, h_1),\ (0, h_2),\ (x - x^{h_3}, h_3)\}$$

and x should not be chosen to be fixed by $\langle h_1, h_2\rangle$. The subspace fixed by h_1 is

$$(\mathbb{Z}_2^n)^{h_1} = \langle(1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 1,\ 1), (0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0)\rangle$$

as obtained above. $x = (a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8)$ is fixed by h_2 if

$$a_3 = a_5 = a_4 \text{ and } a_6 = a_7 = a_8$$

So subspace fixed by h_2 is

$$\begin{aligned}(\mathbb{Z}_2^n)^{h_2} &= \langle (1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0), (0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0), \\ &\quad (0,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0), (0,\ 0,\ 0,\ 0,\ 0\ ,\ 1,\ 1,\ 1)\rangle\end{aligned}$$

Intersection of these two subspaces is $\langle (1, \ldots, 1)\rangle$

$$\begin{aligned}\delta(h_3) &= x - x^{h_3} \\ &= (0,\ a_2 + a_7,\ a_3 + a_4,\ a_4 + a_8,\ a_3 + a_5,\ a_5 + a_6,\ a_6 + a_7,\ a_2 + a_8)\end{aligned}$$

For $x = (1, \ldots, 1)$, we get $\delta(h_3) = (0, \ldots, 0)$, so nothing to exclude from the solution set at the end.

Now we need to start checking derivation conditions for $\delta(h_1) = \delta(h_2) = 0$ and $\delta(h_3)$ defined above. Here again $\delta(hh_3) = \delta(h_3)$ for all $h \in \langle h_1, h_2\rangle$.

- For $g = h_2h_3 = (28)(46)$, we need $a_7 = a_8$ and $a_4 + a_8 = a_5 + a_6$ or $a_7 = a_8 = a_4 + a_5 + a_6$. So $\delta(h_3) = (0,\ a_2 + a_4 + a_5 + a_6,\ a_3 + a_4,\ a_5 + a_6,\ a_3 + a_5,\ a_5 + a_6,\ a_4 + a_5,\ a_2 + a_4 + a_5 + a_6)$
- For $g = h_1h_2h_3 = (1862574)$, we need $a_3 + a_4 = 0$ so $\delta(h_3) = (0,\ a_2 + a_3 + a_5 + a_6,\ 0,\ a_5 + a_6,\ a_3 + a_5,\ a_5 + a_6,\ a_3 + a_5,\ a_2 + a_3 + a_5 + a_6)$.
- For $g = h_1^2h_2h_3 = (1548)(27)$, we need $a_2 + a_5 = 0$ and $a_2 + a_6 = 0$ so $\delta(h_3) = (0,\ a_2 + a_3,\ 0,\ 0,\ a_2 + a_3,\ 0,\ a_2 + a_3,\ a_2 + a_3)$.
- For $g = h_1h_3 = (1835264)$, we need $a_2 + a_3 = 0$ so $\delta(h_3) = (0, \ldots, 0)$.

Hence, $H^1(G, \mathbb{Z}_2^n) = \{0\}$ for $G = \langle h_1, h_2, h_3\rangle$.

4.2.4 First Cohomology Group with Coefficient Set $V_1 \otimes V_2$

For $\delta(g) = x = \sum a_{ij}(\nu_i \otimes \varphi_j)$ to define a derivation in $Der(\langle g\rangle, V_1 \otimes V_2)$ it needs to satisfy $(g^{ord(g)-1} + \ldots + g + I) \cdot x = 0$. But

$$\begin{aligned}(g^{ord(g)-1} + \ldots + g + I) \cdot x &= \sum (g^{ord(g)-1} + \ldots + g + I) \cdot a_{ij}(\nu_i \otimes \varphi_j) \\ &= \sum ((\lambda_i\mu_j)^{ord(g)-1} + \ldots + \lambda_i\mu_j + 1)a_{ij}(\nu_i \otimes \varphi_j)\end{aligned}$$

So $\delta(g)$ defines a derivation in $Der(\langle g\rangle, V_1 \otimes V_2)$ if and only if

$$((\lambda_i\mu_j)^{ord(g)-1} + \ldots + \lambda_i\mu_j + 1) = 0 \text{ or } a_{ij} = 0$$

since $\{\nu_i \otimes \varphi_j\}$ are linearly independent. Hence

$$Dim(Der(\langle g\rangle, V_1 \otimes V_2)) = |\{\{\lambda_i, \mu_j\} : ((\lambda_i\mu_j)^{ord(g)-1} + \ldots + \lambda_i\mu_j + 1) = 0\}|$$

If g has odd order then $((\lambda_i\mu_j)^{ord(g)-1} + \ldots + \lambda_i\mu_j + 1)$ has odd number of terms. Thus $\lambda_i\mu_j = 1$ does not give a derivation.

For $\mu_j = 1$, all $\lambda_i \neq 1$ satisfy the equation since these eigenvalues are chosen to be zeros of the characteristic polynomial on V_1. And these polynomials divide $x^{ord(g)} - 1$. Similarly for $\lambda_i = 1$ all $\mu_j \neq 1$ satisfy the above equation. What about $\lambda_i \neq 1, \mu_j \neq 1$ and $\lambda_i\mu_j \neq 1$. Characteristic polynomial on $V_1 \otimes V_2$ need to divide $(x^{k_1} - 1)(x^{k_2} - 1)\ldots(x^{k_m} - 1)$. If g has order k characteristic polynomial divides $x^k - 1 = (x-1)(x^{k-1} + \ldots + x + 1)$. Here $x = 1$ is not a zero for $(x^{k-1} + \ldots + x + 1)$. Eigenvalues of $V_1 \otimes V_2$ are $\{\lambda_i\mu_j\}$ so zero's of $(x^{k-1} + \ldots + x + 1)$ are precisely all $\lambda_i\mu_j \neq 1$. We have also,

$$Inder(\langle g\rangle, V_1 \otimes V_2) = \{f : f(g) = a_0 - a_0^g \text{ for } a_0 \in V_1 \otimes V_2\}$$

but

$$\nu_i \otimes \varphi_j - (\nu_i \otimes \varphi_j)^g = (1 - \lambda_i\mu_j)(\nu_i \otimes \varphi_j)$$

which is nonzero when $\lambda_i\mu_j \neq 1$. Hence,

$$\begin{aligned} Dim(Inder(\langle g \rangle, V_1 \otimes V_2) &= |\{\{\lambda_i, \mu_j\} : \lambda_i\mu_j \neq 1\}| \\ &= Dim(Der(\langle g \rangle, V_1 \otimes V_2)) \end{aligned}$$

Giving, $H^1(\langle g \rangle, V_1 \otimes V_2) = \{0\}$.

For $G = \langle g_1, g_2 \rangle$ with $\{g_1, g_2\}$ of odd orders. All derivations are inner in $Der(\langle g_i \rangle, V_1 \otimes V_2)$ and $Der(G, V_1 \otimes V_2)$ is a subset of $Der(\langle g_1 \rangle, V_1 \otimes V_2) \times Der(\langle g_2 \rangle, V_1 \otimes V_2)$. So $\delta \in Der(G, V_1 \otimes V_2)$ is defined by $\delta(g_1) = a_1 - a_1^{g_1}$ and $\delta(g_2) = a_2 - a_2^{g_2}$ and δ is inner if and only if $a_1 = a_2$. So here also, we can compute derivations just like we did earlier with coefficient set $\mathbb{Z}_2^n$. Then we would have number of elements in $H^1(G, V_1 \otimes V_2)$ is number of maps δ defined by $\delta(g_1) = 0$ and $\delta(g_2) = a_0 - a_0^{g_2}$ satisfying derivation conditions for a_0 chosen to be elements in $V_1 \otimes V_2$ not fixed by g_1 or g_2. Then computation of $H^1(G, V_1 \otimes V_2)$ is extended as before for $G = \langle \{g_1, g_2, \ldots, g_k\} \rangle$.

4.2.5 $H^1(G, C_1^* \otimes \mathbb{Z}_2^n / C_2)$ for the Hamming Example

Here $G = \langle h_1, h_2, h_3 \rangle$ for $h_1 = (1257684)$, $h_2 = (345)(687)$ and $h_3 = (2843567)$. All h_i have odd orders so all derivations are inner in $Der(\langle h_i \rangle, C_1^* \otimes \mathbb{Z}_2^n / C_2)$. We may follow the method above to compute first cohomology. $g \in G$ act on C_1^* and $\mathbb{Z}_2^n / C_2$ so we can let it act on $C_1^* \otimes \mathbb{Z}_2^n / C_2$ as described above. An element $x \in C_1^* \otimes \mathbb{Z}_2^n / C_2$ can be represented as a sum:

$$\begin{aligned} x &= (1,0,0,0) \otimes (a_1, a_2, a_3, a_4) + (0,1,0,0) \otimes (a_5, a_6, a_7, a_8) \\ &\quad + (0,0,1,0) \otimes (a_9, a_{10}, a_{11}, a_{12}) + (0,0,0,1) \otimes (a_{13}, a_{14}, a_{15}, a_{16}) \end{aligned}$$

for $a_i \in \mathbb{Z}_2$. We can act on these elements by the corresponding 4×4 matrices of action on C_1^* and $\mathbb{Z}_2^n / C_2$.

Just like computations of derivations with coefficient set $\mathbb{Z}_2^n$. Let us assume that derivation value is zero on $\langle h_1 \rangle$ and compute $H^1(\langle h_1, h_2 \rangle, C_1^* \otimes \mathbb{Z}_2^n/C_2)$. Complements of $W = C_1^* \otimes \mathbb{Z}_2^n/C_2$ in $W \rtimes \langle h_1, h_2 \rangle$ would be of the form $\{(0, h_1), (x - x^{h_2}, h_2)\}$, where x is chosen to be elements in W not fixed by h_1 or h_2. Elements of $W = C_1^* \otimes \mathbb{Z}_2^n/C_1$ fixed by $h_1 = (1257684)$ can be obtained from actions on C_1^* and $\mathbb{Z}_2^n/C_1$. Matrix of action on C_1 is

$$M_{(h_1,C_1)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M_{(h_1,C_1^*)} = M^{-1}_{(h_1,C_1^*)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Characteristic polynomial on C_1^* is $(x-1)(x^3+x^2+1)$. So characteristtic polynomial on C_1 is $(x-1)(x^3+x+1)$. Since characteristic polynomial on $\mathbb{Z}_2^n$ is $(x-1)^2(x^6+x^5+x^4+x^3+x^2+x+1) = (x-1)^2(x^3+x+1)(x^3+x^2+1)$ then characteristic polynomial on $\mathbb{Z}_2^n/C_1$ is $(x-1)(x^3+x^2+1)$. Eigenvalues on C_1^* are $1, \beta$ (zero of (x^3+x^2+1)), β^2 and β^4. These are same eigenvalues on $\mathbb{Z}_2^n/C_1$. So we get an eigenvalue 1 on the tensor product only from eigenvalue 1 on C_1^* and 1 on $\mathbb{Z}_2^n/C_1$. Corresponding eigenspace on C_1^* is spanned by vector $(0,0,1,0)$ and on $\mathbb{Z}_2^n/C_1$ is spanned by vector $(1,1,0,1)$. Hence fixed point of h_1 on the tensor product is $(0,0,1,0)^T \otimes (1,1,0,1)$. So we need to pick $x \neq (0,0,1,0)^T \otimes (1,1,0,1)$ to get the derivation set modulo inner derivations. This is the fixed point of h_1 obtained above.

In general if $C_1 \leq \mathbb{Z}_2^8$ (or $\mathbb{Z}_2^n$ for even n), the eigenspace of eigenvalue 1 (over $\mathbb{Z}_2^n$) have dimension 1 for an automorphisms of order 7 (or $n-1$ for the general case). All other eigenvalues are distinct. If p is the characteristic polynomial of g over $\mathbb{Z}_2^8$ (or $\mathbb{Z}_2^n$) and p_1 is the characteristic polynomial of g over C_1. Then $\tilde{p}_1 = p/p_1$ is the characteristic polynomial over C_1^* which is the same as the characteristic polynomial over $\mathbb{Z}_2^8/C_1$ (in general $\mathbb{Z}_2^n/C_1$).

For eigenvalues $\lambda \neq 1$ we have $\lambda^7 = 1$ (or $\lambda^{n-1} = 1$) and $\lambda^k \neq 1$ for $k < 7$ (or $k < n-1$). Hence, contribution to fixed point space of $C_1^* \otimes \mathbb{Z}_2^8/C_1$ comes only from eigenvalues 1 on C_1^* and $\mathbb{Z}_2^8/C_1$. So it is of dimension one.

For an automorphisms h of order 3, the eigenspace of eigenvalue 1 over $\mathbb{Z}_2^n$ have dimension 4. It has 3 other eigenvalues each one gives a 2-dimensional space. If p is the characteristic polynomial over $\mathbb{Z}_2^8$ and p_1 is the characteristic polynomial over C_1. Then $\tilde{p}_1 = p/p_1$ is the characteristic polynomial over C_1^* which is the same as the characteristic polynomial over $\mathbb{Z}_2^8/C_1$. But in this case $\tilde{p_1} = p_1$. So each eigenvalue on C_1^* contribute to the dimension of the fixed point space on the tensor product by 1 and dimension of fixed point space on the tensor product is 4.

Matrices of action of h_2 on the two spaces are:

$$M_{(h_2, C_1^*)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad M_{(h_2, \mathbb{Z}_2^n/C_2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Then we get,

$$\begin{aligned}
\delta(h_2) &= x - x^{h_2} \\
&= (1,0,0,0)^T \otimes (0,\ a_1 + a_2 + a_3,\ a_3 + a_4,\ a_1 + a_2 + a_4) \\
&\quad +(0,1,0,0)^T \otimes (a_{13},\ a_5 + a_6 + a_7 + a_{13} + a_{15}, \\
&\quad a_7 + a_8 + a_{16},\ a_5 + a_6 + a_8 + a_{13} + a_{14}) \\
&\quad +(0,0,1,0)^T \otimes (a_9 + a_{13},\ a_{10} + a_{13} + a_{15},\ a_{11} + a_{16},\ a_{12} + a_{13} + a_{14}) \\
&\quad +(0,0,0,1)^T \otimes (a_9,\ a_9 + a_{11} + a_{13} + a_{14} + a_{15}, \\
&\quad a_{12} + a_{15} + a_{16},\ a_9 + a_{10} + a_{13} + a_{14} + a_{16})
\end{aligned}$$

For $x = (0,0,1,0)^T \otimes (1,1,0,1)$, we get $\delta(h_2) = (0,0,1,0)^T \otimes (1,1,0,1) +$

$(0,0,0,1)^T \otimes (1,1,1,0)$. This is an inner derivation to be excluded from the solution set at the end of derivation modulo inner derivation computation.

Now we need to start checking derivation conditions for $\delta(h_1) = 0$ and $\delta(h_2)$ defined above to get $H^1(\langle h_1, h_2\rangle, W)$. When checking relations we get for all permutation pairs $\{f_1, f_2\}$ using a program in GAP, we get relation set spanned by a set:

$$\begin{aligned}
a_5 + a_6 + a_7 + a_9 + a_{10} + a_{11} + a_{14} + a_{15} &= 0 \\
a_1 + a_2 + a_4 + a_{10} + a_{12} + a_{16} &= 0 \\
a_3 + a_4 + a_7 + a_8 + a_9 + a_{10} + a_{13} + a_{14} &= 0 \\
a_7 + a_8 + a_9 + a_{11} + a_{12} + a_{13} + a_{15} + a_{16} &= 0 \\
a_{11} + a_{13} + a_{14} + a_{15} &= 0 \\
a_{10} + a_{12} + a_{14} + a_{15} &= 0 \\
a_9 + a_{12} + a_{14} &= 0 \\
a_{13} + a_{14} + a_{15} + a_{16} &= 0
\end{aligned}$$

These relations reduce $\delta(h_2)$ to

$$\begin{aligned}
\delta(h_2) &= (1,0,0,0)^T \otimes (0,\ a_{13},\ 0,\ a_{13}) \\
&\quad +(0,1,0,0)^T \otimes (a_{13},\ 0,\ 0,\ a_{13}) \\
&\quad +(0,0,1,0)^T \otimes (a_9 + a_{13},\ a_9 + a_{13},\ 0,\ a_9 + a_{13}) \\
&\quad +(0,0,0,1)^T \otimes (a_9,\ a_9,\ a_9 + a_{13},\ 0)
\end{aligned}$$

For $a_9 = 1,\ a_{13} = 0$, we get:

$$\delta(h_2) = (0,0,1,0)^T \otimes (1,1,0,1) + (0,0,0,1)^T \otimes (1,1,1,0)$$

which is the inner derivation to be excluded. For $a_9 = 0,\ a_{13} = 1$ we get:

$$\begin{aligned}
\delta(h_2) &= (1,0,0,0)^T \otimes (0,1,0,1) + (0,1,0,0)^T \otimes (1,0,0,1) \\
&\quad +(0,0,1,0)^T \otimes (1,1,0,1) + (0,0,0,1)^T \otimes (0,0,1,0) = d
\end{aligned}$$

Hence $H^1(\langle h_1 h_2\rangle, W) = \{0, \delta_1\}$ where 0 is the 0 derivation and δ_1 is defined by $\delta_1(h_1) = 0$ and $\delta_1(h_2) = d$ above.

Now to compute $H^1(G, W)$, we check derivation conditions satisfied by,

$$\begin{aligned}
\delta(h_1) &= 0 \\
\delta(h_2) &= (1,0,0,0)^T \otimes (0,y,0,y) + (0,1,0,0)^T \otimes (y,0,0,y) \\
&\quad +(0,0,1,0)^T \otimes (y,y,0,y) + (0,0,0,1)^T \otimes (0,0,y,0) \\
\delta(h_3) &= x - x^{h_3}
\end{aligned}$$

for some $x \in W$ not fixed by $\langle h_1, h_2\rangle$. As checked above, the fixed point by h_1 is not fixed by h_2. So no nonzero value of x define an inner derivation in the definition $x - x^{h_3}$. Checking relations we get by going through a double loop using GAP, we get a relation set spanned by the following relations:

$$\begin{aligned}
a_1 + a_6 + a_7 + a_8 + a_{10} + a_{11} + a_{12} + a_{14} + a_{15} + a_{16} &= 0 \\
a_5 + a_6 + a_7 + a_8 &= 0 \\
a_3 + a_4 + a_7 + a_8 + a_9 + a_{10} + a_{13} + a_{14} &= 0 \\
a_2 + a_4 + a_6 + a_8 + a_{10} + a_{12} + a_{13} + a_{15} &= 0 \\
a_{13} + a_{14} + a_{15} + a_{16} &= 0 \\
a_9 + a_{10} + a_{11} + a_{15} + a_{16} &= 0 \\
a_7 + a_{14} &= 0 \\
a_8 + a_{14} + a_{16} &= 0 \\
a_{10} + a_{12} + a_{14} + a_{15} &= 0 \\
a_{16} &= 0 \\
a_6 + a_{11} + a_{12} + a_{15} &= 0 \\
a_{11} + a_{12} + a_{15} &= 0 \\
a_{12} + a_{14} + a_{15} &= 0
\end{aligned}$$

and $a_{15} = 0$ These relations reduce $\delta(h_3)$ to

$$\begin{aligned}\delta(h_3) &= (1,0,0,0)^T \otimes (0,\ a_1,\ a_1,\ 0) + (0,1,0,0)^T \otimes (0,\ a_1,\ a_1,\ a_1) \\ &\quad +(0,0,1,0)^T \otimes (a_1,\ 0,\ 0,\ a_1) + (0,0,0,1)^T \otimes (a_1,\ a_1,\ 0,\ a_1) = b\end{aligned}$$

Hence, $H^1(G, W)$ is two dimensional. $H^1(G, W) = \{0, \delta_1, \delta_2\}$ where 0 is the zero derivation, δ_1 is defined by $\delta_1(h_2) = d$ and $\delta_1(h_1) = \delta_1(h_3) = 0$,and δ_2 is defined by $\delta_2(h_3) = b$ and $\delta_2(h_1) = \delta_2(h_2) = 0$. A complete GAP program for this example is given in the Appendix.

4.3 Defining action of G on $C_1^* \otimes \mathbb{Z}_2^n / C_2$

$\mathbb{Z}_4$-codes are additive subgroups of $\mathbb{Z}_4^n$. A permutation group $G \leq S_n$ acts linearly on $\mathbb{Z}_4^n$ preserving subgroup structures. So we can define an action of G on the set of subgroups of $\mathbb{Z}_4^n$. For $g \in G$, we have $g \cdot A_1 = A_2$ for subgroups A_1 and A_2 with permutation equivalent vectors. Here we are after a special class of subgroups, $\mathbb{Z}_4$-codes with common $\{C_1, C_2\}$. So we choose the automorphism group to be $G = Aut(C_1) \cap Aut(C_2)$. The action of this automorphism groups is still linear on the set of subgroups of $\mathbb{Z}_4^n$ preserving the class $\mathcal{C}(C_1, C_2)$. Action of G need to fix the trivial subgroups 0 and $\mathbb{Z}_4^n$ itself. Action of G fixes $d(C_2)$ and acts on cosets of $d(C_2)$ in $2\mathbb{Z}_4^n$. For $g \in G$ to fix a code $\mathbf{C} \in \mathcal{C}(C_1, C_2)$, we need to have $g \cdot c \in \mathbf{C}$ for all $c \in \mathbf{C}$. Checking this condition on a basis of $\mathbf{C}$ is sufficient. $\mathbf{C}$ is fixed by an automorphism group $G = \langle\{g_1, g_2, \ldots g_k\}\rangle$ if $g_i \cdot a_j \in \mathbf{C}$ for a basis $\{a_j\}$ of $\mathbf{C}$ and all $1 \leq i \leq k$.

Codes of $\mathcal{C}(C_1, C_2)$ correspond to a set of subgroups of an extension of $\mathbb{Z}_2^n$ by C_1 generated by the subgroup of the extension isomorphic to C_2 and images of liftings; $\lambda : C_1 \to \mathbb{Z}_2^n$. This extension is not the split extension in general as we have seen in the hamming example in Section(2.2).

We can view this set of codes as a fixed coset of a subspace W of a bigger vector space, say V, where G's action is linear. The coset $W + v$ is fixed by G. So,

$$g \cdot (w + v) = w' + v \text{ for all } w \in W$$

The mapping $w \to w'$ is affine since:

$$g \cdot (w + v) = w^g + v^g = w^g + (v^g - v) + v$$

so $w \to w^g + v_0$ for $v_0 = v^g - v$ and $v \in V$. We have a bijection $W \leftrightarrow W + v$ where G's linear action on W is transformed to an affine action on $W + v$.

$$\begin{array}{ccc} & g \text{ linear} & \\ W & \to & W \\ \downarrow & & \downarrow \\ W+v & \to & W+v \\ & g \text{ affine} & \end{array}$$

$g \to v_0$ defines a derivation, $\delta : G \to W$ with $\delta(g) = v_0 = v^g - v$. This derivation depends on the choice of the coset representative v. If we choose a different coset representative, we change δ by an inner derivation. For $W + v' = W + v$, considering actin on $W + v'$ instead, we get, $\delta'(g) = v'^g - v'$ and $\delta'(g) - \delta(g) = (v' - v)^g - (v' - v)$ for $v' - v \in W$. Action of G on a fixed coset of W is set by action of G on W and an element of $\mathrm{Der}(G, W)$.

The derivation of an action is inner if the factor set defining addition in the extension corresponding to $\mathcal{C}(C_1, C_2)$ is zero, $C_2 = \mathbb{Z}_2^n$ since then we would have only one code in $\mathcal{C}(C_1, C_2)$ this is E itself in the extension $0 \to \mathbb{Z}_2^n \to E \to C_1 \to 0$, when $a_i * a_j = 0 \ldots 0$ for all different basis vectors of C_1, or when image of 0 is 0 for all elements of a basis of G. These cases can be considered trivial.

$G = Aut(C_1) \cap Aut(C_2)$ is a group with linear action on $W = C_1^* \otimes \mathbb{Z}_2^n / C_2$. The semidirect product $W \rtimes G$ acts on W by

$$(w, g) : x \to g \cdot x + w$$

$\{(w, I) : w \in W\} \simeq W$ is a normal subgroup of $W \rtimes G$ acting on W by translation.

$$(-v_g^{g^{-1}}, g^{-1}) + (w, I) + (v_g, g) = (-v_g^{g^{-1}}, g^{-1}) + (w^g + v_g, g) = (w^g, I) \quad (4.4)$$

A complement G^* of W in $W \rtimes G$ is a subgroup with elements (v_g, g) where $\delta(g) = v_g$ is a derivation. Conversely any derivation give a complement this way. Two complements are conjugate in $W \rtimes G$ if the difference defines an inner derivation. {Set of derivations/ Set of inner derivations} $\simeq$ conjugancy classes of complements. Stabilizer of $w \in W$ is $\{(v, g) : w^g + v = w\} = \{(w - w^g, g) : g \in G\}$. For a fixed w, this is a complement subgroup conjugate to $\{(0, g) : g \in G\} \simeq G$, the stabilizer of zero. The derivation of action is inner if there is an element $a_0 \in W$ such that $G_{a_0} = G$.

4.3.1 The Derivation of action for G generated by odd order automorphisms

From Section(4.3) when G is generated by odd order automorphisms; $G = \langle \{g_1, g_2, \ldots, g_k\} \rangle$. Then all derivations in $\mathrm{Der}(G, W)$ are defined by images on the generators of the form:

$$\begin{aligned}
\delta(g_1) &= a_0 - a_0^{g_1} \\
\delta(g_2) &= (a_0 + a_2) - (a_0 + a_2)^{g_2} \\
&\ldots \\
\delta(g_k) &= (a_0 + a_k) - (a_0 + a_k)^{g_k}
\end{aligned}$$

where $\{a_2, \ldots, a_k\}$ are computed in the derivation modulo inner derivations as in (4.3.4) and a_0 varying in W. So the derivation of action can be chosen from this set using the fact that permutations on $\mathbb{Z}_4$-codes fixes complete weight enumerators. Following is one way of doing this:

- compute $H^1(G, W)$ as in (4.3.4). This would be a set with representatives $\{0, rep_1, rep_2, \ldots\}$.
- For $a_0 \in W$ find the set of derivations defined by images having cwe same as $\text{cwe}_{X=0}$. Let S be the set of derivations having cwe $= \text{cwe}_{X=0}$.
- Pick the appropriate element in S matching our action by looking at corresponding fixed codes for different derivation definitions.

4.3.2 Orbit Decomposition

The orbit of an element $x \in W$ is obtained by acting on x by the complement subgroup corresponding to the derivation of action:

$$(\delta(g), g) \cdot x = x^g + \delta(g)$$

Two elements $\{a_0, a_1\} \in W$ belong to the same orbit if $a_1 = a_0^g + \delta(g)$ for some $g \in G$. So $a_1 - a_0 = a_0^g - a_0 + \delta(g)$ for some $g \in G$.

Orbit of $0 \in W$ in an affine action of G on W defined by a derivation δ is $\{\delta(g) : g \in G\}$. Cardinality of this orbit is $|G|/|G_0|$ where $G_0 = \{g \in G : \delta(g) = 0\}$, this is the stabilizer of zero. Orbit of $a_0 \in W$ is the set $\{a_0^g + \delta(g) : g \in G\}$ and its cardinality is $|G|/|G_{a_0}|$ where $G_{a_0} = \{\delta(g) : a_0 = a_0^g + \delta(g)\} = \{\delta(g) : \delta(g) = a_0 - a_0^g\}$, this is the stabilizer of a_0.

If the derivation happen to be inner of the form $\delta(g) = a_0 - a_0^g$ for some $a_0 \in W$. Then orbits of G's action are the orbits of the linear action shifted by a fixed point of the affine action of G. As $b^{g_i} + (a_0 - a_0^{g_i}) = a_0 + (b - a_0)^{g_i}$. So in this case, orbits of G would be of the form:

$$| \; a_0 \; | \; a_0 + a_1^G \; | \; a_0 + a_2^G \; | \ldots$$

where a_0 is a fixed point for G and $\{a_0, a_1, a_2\}$ are elements of W. If the derivation of action is not linear then orbits would be of the form:

$$| \; \{\delta(g_i)\} \; | \; \{a_1^{g_i} + \delta(g_i)\} \; | \; \{a_2^{g_i} + \delta(g_i)\} \; | \; \ldots$$

All elements of $W \rtimes G$ can be expressed as a linear combination of the form $(a, I) + (\delta(g), g)$ for some $a \in W$ and $g \in G$ so the problem of decomposing W into G-orbits is the same as splitting the semidirect product into cosets $(a, I) + \langle(\delta(g_i), g_i)\rangle$.

Orbit decomposition can be made easier when the derivation of action is not inner using a corrected version of G's action on W making it inner. Let $g \in G$ act on $[W \; \mathbb{Z}_2]$ instead by

$$\begin{pmatrix} g & \delta(g) \\ 0 & 1 \end{pmatrix}$$

where $\delta(g)$ is the derivation of our action. Here,

$$\begin{pmatrix} g & \delta(g) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} g \cdot a \\ 0 \end{pmatrix}$$

represens the linear action on W and

$$\begin{pmatrix} g & \delta(g) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} g \cdot a + \delta(g) \\ 1 \end{pmatrix}$$

represents an affine action on W with derivation value $\delta(g)$. Orbits for the example where $C_1 = C_2 =$ Extended Hamming Code of length 8 are listed in Appendix $(A.1)$.

4.3.3 Counting Orbits

Orbits of action can be counted using the orbit counting lemma[7] which can be deduced as follows. For $a \in W$, the orbit Ga satisfies:

$$\sum_{a_i \in Ga} \frac{1}{|Ga_i|} = |Ga| \times \frac{1}{|Ga|} = 1$$

and so

$$\begin{aligned}
\text{no. of orbits} &= \underbrace{1 + \ldots + 1}_{\text{no.of orbits times}} \\
&= \sum_{a_i \in Ga_1} \frac{1}{|Ga_i|} + \ldots + \sum_{a_i \in Ga_m} \frac{1}{|Ga_i|} \\
\{\text{orbits are disjoint so }\} &= \sum_{a \in W} \frac{1}{|Ga|} \\
&= \sum_{a \in W} \frac{|G_a|}{|G|} \\
&= \frac{1}{|G|} \sum_{a \in W} |G_a| \\
\{\text{but} \sum_{a \in W} |G_a| = \sum_{g \in G} |W^g| \text{ so }\} &= \frac{1}{|G|} \sum_{g \in G} |W^g|
\end{aligned}$$

here W^g is the set of fixed points of G in W. Hence if the action is affine defined by a derivation δ, we would have

$$\text{no.of orbits} = \frac{1}{|G|} \sum_{g \in G} |a \in W : \delta(g) = a - a^g| \tag{4.5}$$

Number of fixed points of an affine action is 0 or equal to the number of fixed points in the corresponding linear action depending on whether the derivation is inner or not. If every automorphism fix an element of W then number of orbits of the affine action is equal to number of orbits of the linear action. Otherwise, the affine action have less orbits. In general, groups are classified into three types:

- G fixes an element of W. The derivation in this case is inner.
- Each $g \in G$ fixes an element of W but there is no global fixed point. The derivation is not inner in this case. It is inner when considering action of every cyclic subgroup, $\langle g \rangle$, on W.
- There is an element of G not fixing any element of W.

In the first two cases, number of fixed points of the affine action is equal to number of fixed points of the corresponding linear action. So we can consider the linear action to count the orbits. Let I be a set of indexes of a conjugate class representatives of G. If $g_1 = h^{-1}g_2h$, for some $h \in G$ and g_2 fixes an element $a_0 \in W$, then

$$(a_0^h)^{g_1} = (a_0^h)^{h^{-1}g_2h} = a_0^{g_2h} = a_0^h$$

That is g_1 fixes a_0^h. Hence, Conjugate classes of G have same number of fixed points. Fixed points are eigenvectors corresponding to eigenvalue 1 as seen earlier. So if $\{\lambda_i\}$ is the set of eigenvalues of an automorphism g on C_1^* then dimension of fixed point space on W would be $d = \sum_{\lambda_i} mult_{C_1^*}(\lambda_i) \cdot mult_{\mathbb{Z}_2^n/C_2}(\lambda_i^{-1})$ and $|W^g| = 2^d$. So

$$\text{no.of orbits} = \frac{1}{|G|}\sum_{i\in I}\frac{|G|}{|N_{g_i}|}\cdot|W^{g_i}| = \sum_{i\in I}\frac{1}{|N_{g_i}|}\cdot 2^d$$

Example $\mathcal{C}(C_1, C_2)$ for $C_1 = C_2 =$ Extended Hamming Code of length 8 is an example of the third case. In this example, generators of the automorphism group $\{(2843567), (1234)(5678)\}$ have no common fixed point and for conjugate classes of G, one conjugate class does not fix any point. Computation results are shown below. The following tables show conjugate class representatives of G and their cardinalities, number of their fixed points with examples of fixed points and derivations of the conjugate class representative:

Conj. Class rep.	Card.	no.of f.p.($\mathcal{C}/W$)	eg.	Derivation
(2843567)	192	2	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(2374685)	192	2	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$
(345)(687)	224	64	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$
(1234)(5678)	168	16	$\begin{pmatrix} 0 & \dots & 0 \\ . & . & . \\ 0 & \dots & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \dots & 0 \\ . & . & . \\ 0 & \dots & 0 \end{pmatrix}$

Conj. Class rep.	Card.	no.of f.p.($\mathcal{C}/W$)	eg.	Derivation
(13)(24)(57)(68)	42	256	$\begin{pmatrix} 0 & \dots & 0 \\ . & . & . \\ 0 & \dots & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \dots & 0 \\ . & . & . \\ 0 & \dots & 0 \end{pmatrix}$
(14)(27)(36)(58)	7	1024	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$
(45)(67)	42	1024	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(1485)(2736)	84	64	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

Conj. Class rep.	Card.	no.of f.p.$(\mathcal{C}/W)$	eg.	Derivation
(148273)(56)	224	16	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$
Identity	1	all codes	$\begin{pmatrix} 0 & \dots & 0 \\ . & . & . \\ 0 & \dots & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \dots & 0 \\ . & . & . \\ 0 & \dots & 0 \end{pmatrix}$
(3487)(56)	168	0/64	–	$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Acting by $S = \langle (2843567), (13)(24)(57)(68) \rangle \leq G$ instead we get a fixed point for every conjugate class but there is still no common fixed point for the whole group. This group has order 168 with every conjugate class fixing some points.

4.4 Action of $\mathbb{Z}_2^n \rtimes G$ on W

Action of $\mathbb{Z}_2^n \rtimes G$ on W can be treated as action of G on W/W_0 where W_0 is the set of matrices corresponding to sign changes of coordinates. $\mathbb{Z}_2^n$ act on $\mathcal{C}(C_1, C_2)$ (and so on W) and splits it into W_0 cosets:

W_0	$a_1 + W_0$	$a_2 + W_0$	$\dots$

where $a_1, a_2 \notin W_0$ and $a_1 \neq a_2 \bmod W_0$. When acting by G on W, if $a_1 = b_1 \pmod{W_0}$ then $g \cdot a_1 = g \cdot b_1 \pmod{W_0}$. So G permute W_0 cosets and normalizes action of $\mathbb{Z}_2^n$ as in equation (4.4). The action of G on W_0 cosets is linear if at least one of these cosets is fixed. That is $g \cdot a_0 + W_0 = a_0 + W_0$ for some $a_0 \in W$ and all $g \in G$. If this is satisfied for a basis of G it will be satisfied for the automorphism group. Also checking this for a set of coset representatives is enough.

If we already have G orbits in hand. We might check orbit cardinalities. If there were a number of G orbits with cardinalities summing up to $|W_0|$, check if the union of these orbits is a W_0 coset. If so then this is the fixed W_0 coset we are after.

Action of $\mathbb{Z}_2^n \rtimes G$ on W is given by actions of $\mathbb{Z}_2^n$ and G as described previously. So we act by an element $v \rtimes g$ affinely with the linear action taken to be g's linear action and the affine part given by derivations of g and v.

$$(v, g) \circ a = a^g + \delta(g) + \delta(v)$$

If $\mathbb{Z}_2^n$ is a principal $\mathbb{Z}_2[G]$-module and $\delta(g) = a_0 - a_0^g$ for some $a_0 \in W$ and all $g \in G$ then $a_0 + W_0$ is split into permutation orbits as follows:

$$\mid a_0 \mid a_0 + M_i \mid a_0 + M_i + M_{j \neq i} \mid a_0 + M_i + M_{j \neq i} + M_{k \notin \{i,j\}} \mid$$

$$| \ a_0 + M_i + M_{j\neq i} + M_{k\notin\{i,j\}} + M_{l\notin\{i,j,k\}} \ |$$

Other cosets $a_0 + a + W_0$ will also be decomposed to permutation orbits as:

$$| \ a_0 + a^G \ | \ a_0 + a^G + M_i \ | \ a_0 + a^G + M_i + M_{j\neq i} \ | \ a_0 + a^G + M_i + M_{j\neq i} + M_{k\notin\{i,j\}} \ |$$

$$| \ a_0 + a^G + M_i + M_{j\neq i} + M_{k\notin\{i,j\}} + M_{l\notin\{i,j,k\}} \ |$$

If δ is not inner but $\delta(g) \in a_0 - a_0^g + W_0$ for some $a_0 \in W$ and all $g \in G$ then G fixes $a_0 + W_0$ and permutes cosets of W_0. The decomposition of $a_0 + W_0$ above will be mixed up within the coset. So we can define an action of G on the set of cosets of W_0 as:

$$g \circ (a + W_0) = a^g + \delta(g) + W_0$$

So the derivation of action is inner on W_0 cosets if for some $a \in W$,

$$a^g + \delta(g) + W_0 = a + W_0$$

That is $\delta(g) \in a - a^g + W_0$ for some $a \in W$ and all $g \in G$. This is what happen in the Hamming example. In this example, $G = \mathrm{Aut}(C_1)$ acts on $\mathbb{Z}_2^8$ with orbits:

$0\ldots0$	$10\ldots0$	$110\ldots0$	11100000	11110000	11111000	$1\ldots100$	$1\ldots10$	$1\ldots1$
	$01\ldots0$	$101\ldots0$	11010000	11101000	11110100	$1\ldots010$	$1\ldots01$	
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
	$0\ldots01$	$0\ldots011$	00000111	00001111	00011111	$01\ldots11$	$01\ldots1$	
wt 0	wt 1	wt 2	wt 3	wt 4	wt 5	wt 6	wt 7	wt 8
1	8	28	56	70	56	28	8	1

Orbits of G's action on $\mathbb{Z}_2^8/\langle 1\ldots1\rangle \simeq W_0$ are:

$0\dots0$	$100\dots0$	$110\dots0$	11100000	11110000
	$010\dots0$	$101\dots0$	11010000	11101000
	$\dots$	$\dots$		
1	8	28	56	35

The W_0 coset including the Nordstrom-Robinson code is fixed by action of $W_0 \rtimes G$. This orbit is divided into permutation orbits of cardinalities $64, 56$ and 8.

Let $W_{even} \leq W_0$ be a subspace corresponding to changing signs of even number of coordinates. So this is a 6 dimensional subspace of W spanned by $\{M_i + M_j, \text{ for } i \neq j\}$ where M_i are matrices defined in Section (4.2) spanning W_0. Since G does not keep the orbit decomposition of even vectors in this orbit as in the linear action on even vectors of $\mathbb{Z}_2^8/\langle 1\dots1\rangle$ then the action is not inner if we act by $W_{even} \rtimes G$ but inner with action of $W_0 \rtimes G$ since the orbit decomposition of odd vectors is the same as on the linear one.

In general for an automorphism group G acting affinely with a derivation δ and $a \in W$, orbits would be of the form:

$$\left|\begin{array}{c} 0 \\ \delta(g_i) \end{array}\right|\begin{array}{c} M_j \\ M_j^{g_i} + \delta(g_i) \end{array}\left|\begin{array}{c} \dots \\ \dots \end{array}\right|\begin{array}{c} a \\ a^{g_i} + \delta(g_i) \end{array}\left|\begin{array}{c} a + M_j \\ a_{g_i} + M_j^{g_i} + \delta(g_i) \end{array}\right|\begin{array}{c} \dots \\ \end{array}$$

Here also we can simplify decomposition of W into $\mathbb{Z}_2^n$-orbits using same idea as in (4.3.2) and let $\mathbb{Z}_2^n$ act instead on $[W\ 1]$ by

$$\begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix}$$

where I is the identity mapping on W and $x \in W_0$ written in a column. And we get orbits of $\mathbb{Z}_2^n \rtimes G$ by acting on $[W\ 1]$ by matrices in the multiplication group: $\langle \begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix}$ for all $x \in W_0$, $\begin{pmatrix} g & \delta(g) \\ 0 & 1 \end{pmatrix}$ for $g \in G\rangle$. Orbits of $\mathbb{Z}_2^n \rtimes G$ for the Hamming example are supplied in Appendix $(A.1)$.

If $G = \langle g_1, \ldots, g_k \rangle$ and all basis automorphisms are of odd orders then $H^1(\langle g_i \rangle, W) = \{0\}$ for all $i = 1, \ldots, k$. So all complements of W in $W \rtimes G$ have the form $\{(a_1 - a_1^{g_1}, g_1), \ldots, (a_k - a_k^{g_k}, g_k)\}$. Derivation of G's action of W/W_0 is inner if $a_i - a_i^{g_i} \in a_1 - a_1^{g_i} + W_0$ for all i.

Theorem 4.2 *Let $G = \mathrm{Aut}(C_1) \cap \mathrm{Aut}(C_2) \leq S_n$. If $\mathbb{Z}_2^n \rtimes G$ act on $\mathcal{C}(C_1, C_2)$ with a corresponding affine action on $W = C_1^* \otimes \mathbb{Z}_2^n / C_2$ then number of orbits is:*

$$\begin{aligned} \textit{no. orbits} &= \frac{1}{|G|} \sum_{g \in G} (W/W_0)^g \\ &= \frac{1}{|G|} \sum_{g \in G} |\{a + W_0 : \delta(g) \in a - a^g + W_0\}| \end{aligned}$$

Proposition 4.1 *Let $G = \mathrm{Aut}(C_1) \cap \mathrm{Aut}(C_2)$. Then, $\mathbb{Z}_2^n \rtimes G$ has equally many orbits on $\mathcal{C}(C_1, C_2)$ and $\mathcal{C}(C_2^\perp, C_1^\perp)$.*

Proof Observe first that $H = \mathbb{Z}_2^n \rtimes G$ is generated by scalar multiplications of coordinates by -1 and the group $\mathrm{Aut}(C_2^\perp) \cap \mathrm{Aut}(C_1^\perp)$, since a code and its dual have the same automorphism group.

We have to show that the linear actions of the group on these two sets of codes have the same numbers of fixed points. This holds because the actions are dual:

$$(C_1 \otimes (\mathbb{Z}_2^n / C_2))^* \cong (\mathbb{Z}_2^n / C_1^\perp) \otimes C_2^\perp \cong C_2^\perp \otimes (\mathbb{Z}_2^n / C_1^\perp).$$

We get this isomorphism from:

Claim: If $W \leq V$ then $W^* \simeq V/W^\perp$

Proof Define a map $\theta : V \mapsto W^*$ sending $v \mapsto \varphi_v$ where $\varphi_v(w) = v \cdot w$. Here $\varphi_v \in W^*$ and $\varphi_v = 0$ if and only if $v \in W^\perp$. So $\theta : V/W^\perp \mapsto W^*$ is an isomorphism since it is one to one and $|V/W^\perp| = |W| = |W^*|$.

We then get the above isomorphism by letting $W = C_1 \otimes \mathbb{Z}_2^n/C_2$.

We also need to show that a subgroup fixes a code in one class if and only if it fixes a code in the other. This is true because a subgroup fixing a code C also fixes $C^\perp$, and $C \in \mathcal{C}(C_1, C_2)$ if and only if $C^\perp \in \mathcal{C}(C_2^\perp, C_1^\perp)$, as we have seen. Number of elements in the two sets is the same as $|\mathcal{C}(C_2^\perp, C_1^\perp)| = 2^{(n-(n-m_1))(n-m_2)} = 2^{m_1(n-m_2)}$ where $m_i = \dim(C_i)$.

Corollary 4.1 *If C is a self orthogonal $\mathbb{Z}_4$-code then all its equivalent codes are self orthogonal.*

Proof Permutation equivalent codes of a self orthogonal code are automatically self orthogonal. Changing sign of a coordinate of a self orthogonal code C still give a self orthogonal code since for C to be self dual $v \cdot w = \sum \{v\}_i\{w\}_i = 0 \bmod 4$ but

$$\sum_i \{v\}_i\{w\}_i = \sum_{i \neq j} \{v\}_i\{w\}_i + v_j \cdot w_j = \sum_{i \neq j} \{v\}_i\{w\}_i + (-v_j) \cdot (-w_j) = 0$$

In the Hamming Example the self dual codes are the ones in $\mathbb{Z}_2^n \rtimes G$ orbits (1) and (7) as given in Appendix (A.1).

4.5 The long exact sequence

Let $G = \mathrm{Aut}(C_1) \cap \mathrm{Aut}(C_2)$. The short exact sequence of G modules:

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2^n \to \mathbb{Z}_2^n/\langle 1 \dots 1 \rangle \to 0$$

give the long exact sequence of cohomology groups:

$$0 \to H^0(G, \mathbb{Z}_2) \to H^0(G, \mathbb{Z}_2^n) \to H^0(G, \mathbb{Z}_2^n/\langle 1 \dots 1\rangle) \to H^1(G, \mathbb{Z}_2) \to$$

$$H^1(G, \mathbb{Z}_2^n) \to H^1(G, \mathbb{Z}_2^n/\langle 1 \dots 1\rangle) \to H^2(G, \mathbb{Z}_2) \to \dots$$

G acts trivially on $\mathbb{Z}_2$ and permute coordinates of $\mathbb{Z}_2^n$. If $\mathbb{Z}_2^n$ is a principal $\mathbb{Z}_2[G]$ module then some of the cohomology groups appearing in the above long exact sequence are determined completely. For instance,

- $H^0(G, \mathbb{Z}_2) = \mathbb{Z}_2$ since action is trivial here.
- $H^0(G, \mathbb{Z}_2^n) = \{0 \dots 0, 1 \dots 1\}$. For a vector $v \in \mathbb{Z}_2^n$ to be fixed by a permutation $g \in G$, all coordinates of v corresponding to same cycles of g need to be equal. This need to be satisfied for every automorphism of G in order to have v fixed by the whole automorphism group. Because $\mathbb{Z}_2^n$ is a principal $\mathbb{Z}_2[G]$ module, all entries of a fixed point of G need to be equal.
- $H^1(G, \mathbb{Z}_2) = Hom(G, \mathbb{Z}_2)$. Let $\phi : G \to \mathbb{Z}_2$ be a homomorphism then if it is not the zero homomorphism $Im(\phi) = \mathbb{Z}_2$ and $G/ker(\phi) \simeq \mathbb{Z}_2$. So, $ker(\phi)$ is a subgroup of index 2. Hence $H^1(G, \mathbb{Z}_2) = \{0\}$ if and only if G has no subgroup of index 2. This is satisfied if and only if G is generated by odd order automorphisms. If G has a subgroup of index 2 then $H^1(G, \mathbb{Z}_2)$ is nonzero. In fact, $|H^1(G, \mathbb{Z}_2)|$ = number of subgroups of index 2.
- $H^1(G, \mathbb{Z}_2^n)$ can be computed as in the method supplied in Section(4.2.2) if G is generated by odd order automorphisms and using the original definitions if it is not.

- $H^2(G, \mathbb{Z}_2)$ is zero if there is no group $\hat{G}$ with a normal subgroup N of order 2 such that $\hat{G}/N \simeq G$. Also, $H^2(G, \mathbb{Z}_2) = 0 \Leftrightarrow$ Schur multiplier of G has odd order.

- $|H^0(G, \mathbb{Z}_2^n/\langle 1 \dots 1\rangle)| = |H^0(G, W_0)|$ is bounded by $H^0(G, \mathbb{Z}_2)$ from the long exact sequence. So, if G has no subgroup of index 2 then $H^1(G, \mathbb{Z}_2^n/\langle 1 \dots 1\rangle) = \{0\}$. Otherwise we might just check vectors of weight $\frac{n}{2}$ if there is a vector $v \in \mathbb{Z}_2^n$ such that $v^g = v$ or $(1 \dots 1) + v$ for all $g \in G$ then $|H^0(G, \mathbb{Z}_2^n/\langle 1 \dots 1\rangle)| =$ half of number of those vectors.

- $H^1(G, \mathbb{Z}_2^n/\langle 1 \dots 1\rangle) \simeq H^1(G, W_0)$ is what we are after. We get its cardinality from the long exact sequence after obtaining other cohomology groups.

Now consider the short exact sequence:

$$0 \to W_0 \to W \to W/W_0 \to 0$$

This sequence of G modules give the long exact sequence:

$$0 \to H^0(G, W_0) \to H^0(G, W) \to H^0(G, W/W_0) \to H^1(G, W_0) \to$$

$$H^1(G, W) \to H^1(G, W/W_0) \to H^2(G, W_0) \to \dots$$

Here we have

- $H^0(G, W_0)$ and $H^1(G, W_0)$ are obtained in the previous exact sequence.

- $H^0(G, W) =$ space of fixed points of G on W. This is the intersection of fixed point spaces for a basis set $\{g_i\}$ of G. Here $W = C_1^* \otimes \mathbb{Z}_2^n/C_2$. Action of G on the tensor product can be defined from the actions on C_1^* and $\mathbb{Z}_2^n/C_2$ as done in Section (4.2.5). Space of fixed points for an automorphism $g \in G$ is the eigenspace of eigenvalue 1. So fixed points on the tensor product are elements of the form $\nu \otimes \mu$, where ν is an eigenvector corresponding to eigenvalue λ on C_1^* and μ is an eigenvector corresponding to eigenvalue λ^{-1} on $\mathbb{Z}_2^n/C_2$.

- $H^1(G, W)$ is computed as in Section (4.2.5)
- With the help of the long exact sequence, we obtain information about $H^0(G, W/W_0)$ and a lower bound for $H^1(G, W/W_0)$.

A nonzero element $\delta \in H^1(G, W)$ is mapped to a zero element in $H^1(G, W/W_0)$. That is δ is mapped to an element which is equivalent to an element in the kernel of $H^1(G, W) \to H^1(G, W/W_0)$, if and only if $H^1(G, W_0)$ is nonzero and this element is equivalent to an element in image of $H^1(G, W_0) \to H^1(G, W)$.

Theorem 4.3 *A derivation in Der$(G, W/W_0)$ is nonzero if and only if it is not conjugate to any element in Der(G, W_0).*

In the Hamming example, the derivation of action is not inner on W but inner in W/W_0. So it is equivalent to an element in the kernel of the mapping $H^1(G, W) \to H^1(G, W/W_0)$. From exactness of the sequence, the derivation is in the image of the mapping $H^1(G, W_0) \to H^1(G, W)$.

Summary and Conclusion

For a $\mathbb{Z}_4$-code $\mathbf{C}$, we can define a pair of binary codes $\{C_1, C_2\}$ as:

- $C_1 = \mathbf{C}$ and
- $C_2 = h(\mathbf{C} \cap 2\mathbb{Z}_4^n)$

where h coordinatewise sends 0 to 0 and 2 to 1. Fixing a pair of binary codes $C_1 \leq C_2$ let $\mathcal{C}(C_1, C_2)$ be the set of $\mathbb{Z}_4$-codes giving rise to $\{C_1, C_2\}$ as above. Then

- The cardinality of this set of codes is $2^{\dim(C_1)(n-\dim(C_2))}$.
- Codes of $\mathcal{C}(C_1, C_2)$ have generator matrices of the form

$$G = \begin{pmatrix} I_{k_1} & A & B + 2X \\ 0 & 2I_{k_2} & 2C \end{pmatrix}$$

  where I_{k_1} and I_{k_2} are identity matrices of dimensions $k_1 = \dim(C_1)$ and $k_2 = \dim(C_2) - \dim(C_1)$ respectively. A, B and C are fixed matrices over $\mathbb{Z}_2$ for a fixed pair $\{C_1, C_2\}$. And X is a matrix with entries varying over $\mathbb{Z}_2$.

- $\mathcal{C}(C_1, C_2) \simeq Hom(C_1, \mathbb{Z}_2^n/C_2) \simeq C_1^* \otimes \mathbb{Z}_2^n/C_2$.

- We get this set of codes from an extension of $\mathbb{Z}_2^n$ by C_1.
- Average symmetrized weight enumerator of codes in $\mathcal{C}(C_1, C_2)$ can be computed from weight enumerators of C_1 and C_2 as:
$$\overline{\text{swe}}(x, y, z) = \frac{|C_2|}{2^n}(\text{we}_{C_1}(x + z, 2y) - (x + z)^n) + \text{we}_{C_2}(x, z)$$
- $G = \text{Aut}(C_1) \cap \text{Aut}(C_2)$ acts on $\mathcal{C}(C_1, C_2)$ fixing complete weight enumerators of codes. Corresponding action on $C_1^* \otimes \mathbb{Z}_2^n/C_2$ is affine with the affine element given by an element of $H^1(G, C_1^* \otimes \mathbb{Z}_2^n/C_2)$.
- $H^1(G, C_1^* \otimes \mathbb{Z}_2^n)$ for a group G generated with odd order automorphisms can be computed using corresponding actions of G on C_1^* and $\mathbb{Z}_2^n/C_2$.
- If G act on a space W affinely with a derivation δ then number of orbits is
$$\frac{1}{G}\sum_{g \in G} |\{a \in W : \delta(g) = a - a^g\}|$$
- We might define changing signs of coordinates for codes in $\mathcal{C}(C_1, C_2)$ by an action of $\mathbb{Z}_2^n$.
- The combined action of $\mathbb{Z}_2^n \rtimes G$ on $\mathcal{C}(C_1, C_2)$ (permuting coordinates and changing signs of coordinates) is described as an affine action on $C_1^* \otimes \mathbb{Z}_2^n/C_2$ with the affine element given by an element of $H^1(\mathbb{Z}_2^n \rtimes G, C_1^* \otimes \mathbb{Z}_2^n/C_2)$.
- Number of orbits of $\mathbb{Z}_2^n \rtimes G$ on W is
$$\frac{1}{|G|}\sum_{g \in G} |\{a + W_0 : \delta(g) \in a - a^g + W_0\}|$$
- Action of $\mathbb{Z}_2^n \rtimes G$ on W can be described as action of G on W/W_0 where W_0 is the subspace of $W = C_1^* \otimes \mathbb{Z}_2^n/C_2$ generated to matrices corresponding to sign changes in coordinates of codes.

- $\mathbb{Z}_2^n \rtimes G$ act on W with a zero element of $H^1(\mathbb{Z}_2^n \rtimes G, C_1^* \otimes \mathbb{Z}_2^n/C_2)$ if the derivation of action is conjugate to an element of $\mathrm{Der}(G, W_0)$.

Appendix A

Orbits and Programming Codes

A.1 Orbits of the Hamming Example

Let $\mathcal{C}(C_1, C_1)$ be the set of all $\mathbb{Z}_4$-codes corresponding to the pair $\{C_1, C_1\}$ of $\mathbb{Z}_2$-codes, where C_1 is the Extended Hamming code of length 8. Let W be its isomorphic set of matrices of dimension $m_1 \times (n - m_2)$ where $m_i = \dim(C_i)$. This set has $2^{16} = 65536$ elements.

Let G be the automorphism group of C_1. This group has order 1344 generated by permutations $(1, 2, 3, 4)(5, 6, 7, 8)$ and $(2, 8, 4, 3, 5, 6, 7)$. Then G act on $\mathcal{C}(C_1, C_1)$ by coordinate permutations. This action preserve complete weight enumerators of codes. Let H be the group generated by G and the operations of changing signs of coordinates. Then H has order $2^7|G|$. Elements of H act on $\mathcal{C}(C_1, C_1)$ preserving symmetrized and Lee weight enumerators.

I have computed H orbits of $\mathcal{C}(C_1, C_1)$ and their divisions into G orbits. Number of H orbits is 9 falling into 114 orbits of G. For each orbit, I give below the size of the orbit, an orbit representative, cwe and swe's. Orbit representatives are displayed below as 4×4 matrices X with the property that $M + [0_4\ 2X]$ is a generator matrix for the relevant code where 0_4 is the 4×4 zero matrix and

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Here is the list of orbits:

Orbit number (1) **of size** 128 **with symmetrized weight enumerator** swe $= x^8 + 14x^4z^4 + 112x^3y^4z + 112xy^4z^3 + 16y^8 + z^8$ **and orbit representative:**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Number of permutation orbits= 3

- Permutation orbit number (1) of size 64 and representative: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} \quad = \quad & x^8 + 14x^4z^4 + 7x^3y^4z + 28x^3y^3zw + 42x^3y^2zw^2 + 28x^3yzw^3 + 7x^3zw^4 \\ & +7xy^4z^3 + 28xy^3z^3w + 42xy^2z^3w^2 + 28xyz^3w^3 + 7xz^3w^4 + y^7w + 7y^5w^3 \\ & +7y^3w^5 + yw^7 + z^8 \end{aligned}$$

- Permutation orbit number (2) of size 56 and representative: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 14x^4z^4 + 8x^3y^4z + 24x^3y^3zw + 48x^3y^2zw^2 + 24x^3yzw^3 + 8x^3zw^4 \\ &+8xy^4z^3 + 24xy^3z^3w + 48xy^2z^3w^2 + 24xyz^3w^3 + 8xz^3w^4 + 4y^6w^2 \\ &+8y^4w^4 + 4y^2w^6 + z^8 \end{aligned}$$

- Permutation orbit number(3) of size 8 and representative: $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\text{cwe} = x^8+14x^4z^4+56x^3y^3zw+56x^3yzw^3+56xy^3z^3w+56xyz^3w^3+y^8+14y^4w^4+z^8+w^8$$

  This orbit includes the Nordstrom-Robinson Code.

Orbit number (2) **of size** 7168 **with symmetrized weight enumerator**

$\text{swe} = x^8 + 8x^4y^4 + 14x^4z^4 + 80x^3y^4z + 48x^2y^4z^2 + 80xy^4z^3 + 16y^8 + 8y^4z^4 + z^8$

and orbit representative:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Number of permutation orbits= 8

- Permutation orbit number (1) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 2x^4y^3w + 2x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + x^4w^4 \\ &+6x^3y^4z + 16x^3y^3zw + 36x^3y^2zw^2 + 16x^3yzw^3 + 6x^3zw^4 \\ &+2x^2y^4z^2 + 12x^2y^3z^2w + 20x^2y^2z^2w^2 + 12x^2yz^2w^3 \\ &+2x^2z^2w^4 + 6xy^4z^3 + 16xy^3z^3w + 36xy^2z^3w^2 + 16xyz^3w^3 \\ &+6xz^3w^4 + 4y^6w^2 + y^4z^4 + 8y^4w^4 + 2y^3z^4w + 2y^2z^4w^2 \\ &+4y^2w^6 + 2yz^4w^3 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number (2) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 2x^4y^3w + 2x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + x^4w^4 \\ &+5x^3y^4z + 20x^3y^3zw + 30x^3y^2zw^2 + 20x^3yzw^3 + 5x^3zw^4 \\ &+2x^2y^4z^2 + 12x^2y^3z^2w + 20x^2y^2z^2w^2 + 12x^2yz^2w^3 \\ &+2x^2z^2w^4 + 5xy^4z^3 + 20xy^3z^3w + 30xy^2z^3w^2 + 20xyz^3w^3 \\ &+5xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 + 2y^3z^4w + 7y^3w^5 + 2y^2z^4w^2 \\ &+2yz^4w^3 + yw^7 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number(3) of size 448 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 6x^4y^2w^2 + 14x^4z^4 + x^4w^4 + 40x^3y^3zw + 40x^3yzw^3 \\ &+6x^2y^4z^2 + 36x^2y^2z^2w^2 + 6x^2z^2w^4 + 40xy^3z^3w + 40xyz^3w^3 \\ &+4y^6w^2 + y^4z^4 + 8y^4w^4 + 6y^2z^4w^2 + 4y^2w^6 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number(4) of size 448 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 6x^4y^2w^2 + 14x^4z^4 + x^4w^4 + 7x^3y^4z + 12x^3y^3zw \\ &+42x^3y^2zw^2 + 12x^3yzw^3 + 7x^3zw^4 + 6x^2y^4z^2 + 36x^2y^2z^2w^2 \\ &+6x^2z^2w^4 + 7xy^4z^3 + 12xy^3z^3w + 42xy^2z^3w^2 + 12xyz^3w^3 \\ &+7xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 + 7y^3w^5 + 6y^2z^4w^2 + yw^7 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number(5) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+2x^4y^3w+4x^4y^2w^2+2x^4yw^3+14x^4z^4+5x^3y^4z+20x^3y^3zw\\&+30x^3y^2zw^2+20x^3yzw^3+5x^3zw^4+4x^2y^4z^2+12x^2y^3z^2w\\&+16x^2y^2z^2w^2+12x^2yz^2w^3+4x^2z^2w^4+5xy^4z^3+20xy^3z^3w\\&+30xy^2z^3w^2+20xyz^3w^3+5xz^3w^4+y^7w+7y^5w^3+2y^3z^4w+7y^3w^5\\&+4y^2z^4w^2+2yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number(6) of size 1344 and representative: $\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&0&1&0\\1&0&1&0\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+2x^4y^3w+4x^4y^2w^2+2x^4yw^3+14x^4z^4+6x^3y^4z+16x^3y^3zw\\&+36x^3y^2zw^2+16x^3yzw^3+6x^3zw^4+4x^2y^4z^2+12x^2y^3z^2w\\&+16x^2y^2z^2w^2+12x^2yz^2w^3+4x^2z^2w^4+6xy^4z^3+16xy^3z^3w\\&+36xy^2z^3w^2+16xyz^3w^3+6xz^3w^4+4y^6w^2+8y^4w^4+2y^3z^4w\\&+4y^2z^4w^2+4y^2w^6+2yz^4w^3+z^8\end{aligned}$$

- Permutation orbit number(7) of size 448 and representative: $\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&1\\0&0&1&0\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+4x^4y^3w+4x^4yw^3+14x^4z^4+3x^3y^4z+28x^3y^3zw+18x^3y^2zw^2\\&+28x^3yzw^3+3x^3zw^4+24x^2y^3z^2w+24x^2yz^2w^3+3xy^4z^3+28xy^3z^3w\\&+18xy^2z^3w^2+28xyz^3w^3+3xz^3w^4+y^7w+7y^5w^3+4y^3z^4w+7y^3w^5\\&+4yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number(8) of size 448 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^3w + 4x^4yw^3 + 14x^4z^4 + 4x^3y^4z + 24x^3y^3zw + 24x^3y^2zw^2 \\ &+24x^3yzw^3 + 4x^3zw^4 + 24x^2y^3z^2w + 24x^2yz^2w^3 + 4xy^4z^3 + 24xy^3z^3w \\ &+24xy^2z^3w^2 + 24xyz^3w^3 + 4xz^3w^4 + y^8 + 14y^4w^4 + 4y^3z^4w + 4yz^4w^3 \\ &+z^8 + w^8 \end{aligned}$$

Orbit number(3) **of size** 2688 **with symmetrized weight enumerator**
swe $= x^8 + 8x^4y^4 + 14x^4z^4 + 80x^3y^4z + 48x^2y^4z^2 + 80xy^4z^3 + 16y^8 + 8y^4z^4 + z^8$
and orbit representative:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Number of permutation orbits= 8.

- Permutation orbit number (1) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + x^4y^4 + 2x^4y^3w + 2x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + x^4w^4 + 5x^3y^4z \\ &+20x^3y^3zw + 30x^3y^2zw^2 + 20x^3yzw^3 + 5x^3zw^4 + 2x^2y^4z^2 \\ &+12x^2y^3z^2w + 20x^2y^2z^2w^2 + 12x^2yz^2w^3 + 2x^2z^2w^4 + 5xy^4z^3 \\ &+20xy^3z^3w + 30xy^2z^3w^2 + 20xyz^3w^3 + 5xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 \\ &+2y^3z^4w + 7y^3w^5 + 2y^2z^4w^2 + 2yz^4w^3 + yw^7 + z^8 + z^4w^4\end{aligned}$$

- Permutation orbit number (2) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + x^4y^4 + 2x^4y^3w + 2x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + x^4w^4 + 6x^3y^4z \\ &+16x^3y^3zw + 36x^3y^2zw^2 + 16x^3yzw^3 + 6x^3zw^4 + 2x^2y^4z^2 \\ &+12x^2y^3z^2w + 20x^2y^2z^2w^2 + 12x^2yz^2w^3 + 2x^2z^2w^4 + 6xy^4z^3 \\ &+16xy^3z^3w + 36xy^2z^3w^2 + 16xyz^3w^3 + 6xz^3w^4 + 4y^6w^2 + y^4z^4 \\ &+8y^4w^4 + 2y^3z^4w + 2y^2z^4w^2 + 4y^2w^6 + 2yz^4w^3 + z^8 + z^4w^4\end{aligned}$$

- Permutation orbit number(3) of size 84 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + 2x^4y^4 + 4x^4y^2w^2 + 14x^4z^4 + 2x^4w^4 + 40x^3y^3zw + 40x^3yzw^3 \\ &+4x^2y^4z^2 + 40x^2y^2z^2w^2 + 4x^2z^2w^4 + 40xy^3z^3w + 40xyz^3w^3 + y^8 \\ &+2y^4z^4 + 14y^4w^4 + 4y^2z^4w^2 + z^8 + 2z^4w^4 + w^8\end{aligned}$$

- Permutation orbit number(4) of size 168 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 6x^4y^2w^2 + 14x^4z^4 + x^4w^4 + 8x^3y^4z + 8x^3y^3zw + 48x^3y^2zw^2 \\ &+8x^3yzw^3 + 8x^3zw^4 + 6x^2y^4z^2 + 36x^2y^2z^2w^2 + 6x^2z^2w^4 + 8xy^4z^3 \\ &+8xy^3z^3w + 48xy^2z^3w^2 + 8xyz^3w^3 + 8xz^3w^4 + 4y^6w^2 + y^4z^4 + 8y^4w^4 \\ &+6y^2z^4w^2 + 4y^2w^6 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number(5) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^3w + 4x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 6x^3y^4z + 16x^3y^3zw \\ &+36x^3y^2zw^2 + 16x^3yzw^3 + 6x^3zw^4 + 4x^2y^4z^2 + 12x^2y^3z^2w \\ &+16x^2y^2z^2w^2 + 12x^2yz^2w^3 + 4x^2z^2w^4 + 6xy^4z^3 + 16xy^3z^3w \\ &+36xy^2z^3w^2 + 16xyz^3w^3 + 6xz^3w^4 + 4y^6w^2 + 8y^4w^4 + 2y^3z^4w \\ &+4y^2z^4w^2 + 4y^2w^6 + 2yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number(6) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+2x^4y^3w+4x^4y^2w^2+2x^4yw^3+14x^4z^4+5x^3y^4z+20x^3y^3zw\\&+30x^3y^2zw^2+20x^3yzw^3+5x^3zw^4+4x^2y^4z^2+12x^2y^3z^2w\\&+16x^2y^2z^2w^2+12x^2yz^2w^3+4x^2z^2w^4+5xy^4z^3+20xy^3z^3w\\&+30xy^2z^3w^2+20xyz^3w^3+5xz^3w^4+y^7w+7y^5w^3+2y^3z^4w+7y^3w^5\\&+4y^2z^4w^2+2yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number(7) of size 336 and representative: $\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&1\\0&1&0&1\end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+4x^4y^3w+4x^4yw^3+14x^4z^4+4x^3y^4z+24x^3y^3zw+24x^3y^2zw^2\\&+24x^3yzw^3+4x^3zw^4+24x^2y^3z^2w+24x^2yz^2w^3+4xy^4z^3+24xy^3z^3w\\&+24xy^2z^3w^2+24xyz^3w^3+4xz^3w^4+4y^6w^2+8y^4w^4+4y^3z^4w+4y^2w^6\\&+4yz^4w^3+z^8\end{aligned}$$

- Permutation orbit number(8) of size 84 and representative: $\begin{pmatrix}0&0&1&1\\0&1&0&1\\1&0&1&0\\0&0&1&1\end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+8x^4y^2w^2+14x^4z^4+40x^3y^3zw+40x^3yzw^3+8x^2y^4z^2+32x^2y^2z^2w^2\\&+8x^2z^2w^4+40xy^3z^3w+40xyz^3w^3+y^8+14y^4w^4+8y^2z^4w^2+z^8+w^8\end{aligned}$$

Orbit number(4) of size 21504 with symmetrized weight enumerator

swe $= x^8+12x^4y^4+14x^4z^4+64x^3y^4z+72x^2y^4z^2+64xy^4z^3+16y^8+12y^4z^4+z^8$

and orbit representative:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Number of permutation orbits= 27

- Permutation orbit number (1) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^4 + 2x^4y^3w + 4x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 2x^4w^4 + 6x^3y^4z \\ &+8x^3y^3zw + 36x^3y^2zw^2 + 8x^3yzw^3 + 6x^3zw^4 + 4x^2y^4z^2 + 12x^2y^3z^2w \\ &+40x^2y^2z^2w^2 + 12x^2yz^2w^3 + 4x^2z^2w^4 + 6xy^4z^3 + 8xy^3z^3w + 36xy^2z^3w^2 \\ &+8xyz^3w^3 + 6xz^3w^4 + 4y^6w^2 + 2y^4z^4 + 8y^4w^4 + 2y^3z^4w + 4y^2z^4w^2 \\ &+4y^2w^6 + 2yz^4w^3 + z^8 + 2z^4w^4 \end{aligned}$$

- Permutation orbit number (2) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^4 + 2x^4y^3w + 4x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 2x^4w^4 + 5x^3y^4z \\ &+12x^3y^3zw + 30x^3y^2zw^2 + 12x^3yzw^3 + 5x^3zw^4 + 4x^2y^4z^2 + 12x^2y^3z^2w \\ &+40x^2y^2z^2w^2 + 12x^2yz^2w^3 + 4x^2z^2w^4 + 5xy^4z^3 + 12xy^3z^3w \\ &+30xy^2z^3w^2 + 12xyz^3w^3 + 5xz^3w^4 + y^7w + 7y^5w^3 + 2y^4z^4 + 2y^3z^4w \\ &+7y^3w^5 + 4y^2z^4w^2 + 2yz^4w^3 + yw^7 + z^8 + 2z^4w^4 \end{aligned}$$

- Permutation orbit number(3) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^4 + 2x^4y^3w + 4x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 2x^4w^4 + 6x^3y^4z \\ &+8x^3y^3zw + 36x^3y^2zw^2 + 8x^3yzw^3 + 6x^3zw^4 + 4x^2y^4z^2 + 12x^2y^3z^2w \\ &+40x^2y^2z^2w^2 + 12x^2yz^2w^3 + 4x^2z^2w^4 + 6xy^4z^3 + 8xy^3z^3w + 36xy^2z^3w^2 \\ &+8xyz^3w^3 + 6xz^3w^4 + 4y^6w^2 + 2y^4z^4 + 8y^4w^4 + 2y^3z^4w + 4y^2z^4w^2 \\ &+4y^2w^6 + 2yz^4w^3 + z^8 + 2z^4w^4 \end{aligned}$$

- Permutation orbit number(4) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^4 + 2x^4y^3w + 4x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 2x^4w^4 + 5x^3y^4z \\ &+12x^3y^3zw + 30x^3y^2zw^2 + 12x^3yzw^3 + 5x^3zw^4 + 4x^2y^4z^2 + 12x^2y^3z^2w \\ &+40x^2y^2z^2w^2 + 12x^2yz^2w^3 + 4x^2z^2w^4 + 5xy^4z^3 + 12xy^3z^3w + 30xy^2z^3w^2 \\ &+12xyz^3w^3 + 5xz^3w^4 + y^7w + 7y^5w^3 + 2y^4z^4 + 2y^3z^4w + 7y^3w^5 \\ &+4y^2z^4w^2 + 2yz^4w^3 + yw^7 + z^8 + 2z^4w^4 \end{aligned}$$

- Permutation orbit number (5) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+2x^4y^4+8x^4y^2w^2+14x^4z^4+2x^4w^4+32x^3y^3zw+32x^3yzw^3\\ &+8x^2y^4z^2+56x^2y^2z^2w^2+8x^2z^2w^4+32xy^3z^3w+32xyz^3w^3+4y^6w^2\\ &+2y^4z^4+8y^4w^4+8y^2z^4w^2+4y^2w^6+z^8+2z^4w^4\end{aligned}$$

- Permutation orbit number(6) of size 672 and representative: $\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&1\\ 0&1&0&1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+2x^4y^4+8x^4y^2w^2+14x^4z^4+2x^4w^4+32x^3y^3zw+32x^3yzw^3\\ &+8x^2y^4z^2+56x^2y^2z^2w^2+8x^2z^2w^4+32xy^3z^3w+32xyz^3w^3+4y^6w^2\\ &+2y^4z^4+8y^4w^4+8y^2z^4w^2+4y^2w^6+z^8+2z^4w^4\end{aligned}$$

- Permutation orbit number (7) of size 1344 and representative: $\begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&1\\ 0&0&1&1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+4x^4y^3w+2x^4y^2w^2+4x^4yw^3+14x^4z^4+x^4w^4+4x^3y^4z\\ &+16x^3y^3zw+24x^3y^2zw^2+16x^3yzw^3+4x^3zw^4+2x^2y^4z^2+24x^2y^3z^2w\\ &+20x^2y^2z^2w^2+24x^2yz^2w^3+2x^2z^2w^4+4xy^4z^3+16xy^3z^3w\\ &+24xy^2z^3w^2+16xyz^3w^3+4xz^3w^4+4y^6w^2+y^4z^4+8y^4w^4+4y^3z^4w\\ &+2y^2z^4w^2+4y^2w^6+4yz^4w^3+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (8) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 4x^4y^3w + 2x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + x^4w^4 + 3x^3y^4z \\ &+20x^3y^3zw + 18x^3y^2zw^2 + 20x^3yzw^3 + 3x^3zw^4 + 2x^2y^4z^2 + 24x^2y^3z^2w \\ &+20x^2y^2z^2w^2 + 24x^2yz^2w^3 + 2x^2z^2w^4 + 3xy^4z^3 + 20xy^3z^3w + 18xy^2z^3w^2 \\ &+20xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 + 4y^3z^4w + 7y^3w^5 + 2y^2z^4w^2 \\ &+4yz^4w^3 + yw^7 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number(9) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 4x^4y^3w + 2x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + x^4w^4 + 4x^3y^4z \\ &+16x^3y^3zw + 24x^3y^2zw^2 + 16x^3yzw^3 + 4x^3zw^4 + 2x^2y^4z^2 + 24x^2y^3z^2w \\ &+20x^2y^2z^2w^2 + 24x^2yz^2w^3 + 2x^2z^2w^4 + 4xy^4z^3 + 16xy^3z^3w + 24xy^2z^3w^2 \\ &+16xyz^3w^3 + 4xz^3w^4 + 4y^6w^2 + y^4z^4 + 8y^4w^4 + 4y^3z^4w + 2y^2z^4w^2 + 4y^2w^6 \\ &+4yz^4w^3 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number (10) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+4x^4y^3w+2x^4y^2w^2+4x^4yw^3+14x^4z^4+x^4w^4+3x^3y^4z\\ &+20x^3y^3zw+18x^3y^2zw^2+20x^3yzw^3+3x^3zw^4+2x^2y^4z^2+24x^2y^3z^2w\\ &+20x^2y^2z^2w^2+24x^2yz^2w^3+2x^2z^2w^4+3xy^4z^3+20xy^3z^3w+18xy^2z^3w^2\\ &+20xyz^3w^3+3xz^3w^4+y^7w+7y^5w^3+y^4z^4+4y^3z^4w+7y^3w^5+2y^2z^4w^2\\ &+4yz^4w^3+yw^7+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (11) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+4x^4y^3w+2x^4y^2w^2+4x^4yw^3+14x^4z^4+x^4w^4+3x^3y^4z\\ &+20x^3y^3zw+18x^3y^2zw^2+20x^3yzw^3+3x^3zw^4+2x^2y^4z^2+24x^2y^3z^2w\\ &+20x^2y^2z^2w^2+24x^2yz^2w^3+2x^2z^2w^4+3xy^4z^3+20xy^3z^3w+18xy^2z^3w^2\\ &+20xyz^3w^3+3xz^3w^4+y^7w+7y^5w^3+y^4z^4+4y^3z^4w+7y^3w^5+2y^2z^4w^2\\ &+4yz^4w^3+yw^7+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (12) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+2x^4y^4+4x^4y^3w+4x^4yw^3+14x^4z^4+2x^4w^4+4x^3y^4z+16x^3y^3zw\\ &+24x^3y^2zw^2+16x^3yzw^3+4x^3zw^4+24x^2y^3z^2w+24x^2y^2z^2w^2\\ &+24x^2yz^2w^3+4xy^4z^3+16xy^3z^3w+24xy^2z^3w^2+16xyz^3w^3+4xz^3w^4+y^8\\ &+2y^4z^4+14y^4w^4+4y^3z^4w+4yz^4w^3+z^8+2z^4w^4+w^8\end{aligned}$$

- Permutation orbit number (13) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 2x^4y^3w + 6x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + x^4w^4 + 5x^3y^4z \\ &+12x^3y^3zw + 30x^3y^2zw^2 + 12x^3yzw^3 + 5x^3zw^4 + 6x^2y^4z^2 + 12x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 12x^2yz^2w^3 + 6x^2z^2w^4 + 5xy^4z^3 + 12xy^3z^3w + 30xy^2z^3w^2 \\ &+12xyz^3w^3 + 5xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 + 2y^3z^4w + 7y^3w^5 + 6y^2z^4w^2 \\ &+2yz^4w^3 + yw^7 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number (14) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 2x^4y^3w + 6x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + x^4w^4 + 6x^3y^4z \\ &+8x^3y^3zw + 36x^3y^2zw^2 + 8x^3yzw^3 + 6x^3zw^4 + 6x^2y^4z^2 + 12x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 12x^2yz^2w^3 + 6x^2z^2w^4 + 6xy^4z^3 + 8xy^3z^3w + 36xy^2z^3w^2 \\ &+8xyz^3w^3 + 6xz^3w^4 + 4y^6w^2 + y^4z^4 + 8y^4w^4 + 2y^3z^4w + 6y^2z^4w^2 + 4y^2w^6 \\ &+2yz^4w^3 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number (15) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} \;=\;& x^8+x^4y^4+2x^4y^3w+6x^4y^2w^2+2x^4yw^3+14x^4z^4+x^4w^4+5x^3y^4z\\ &+12x^3y^3zw+30x^3y^2zw^2+12x^3yzw^3+5x^3zw^4+6x^2y^4z^2+12x^2y^3z^2w\\ &+36x^2y^2z^2w^2+12x^2yz^2w^3+6x^2z^2w^4+5xy^4z^3+12xy^3z^3w+30xy^2z^3w^2\\ &+12xyz^3w^3+5xz^3w^4+y^7w+7y^5w^3+y^4z^4+2y^3z^4w+7y^3w^5+6y^2z^4w^2\\ &+2yz^4w^3+yw^7+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (16) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} \;=\;& x^8+4x^4y^3w+4x^4y^2w^2+4x^4yw^3+14x^4z^4+3x^3y^4z+20x^3y^3zw\\ &+18x^3y^2zw^2+20x^3yzw^3+3x^3zw^4+4x^2y^4z^2+24x^2y^3z^2w\\ &+16x^2y^2z^2w^2+24x^2yz^2w^3+4x^2z^2w^4+3xy^4z^3+20xy^3z^3w\\ &+18xy^2z^3w^2+20xyz^3w^3+3xz^3w^4+y^7w+7y^5w^3+4y^3z^4w+7y^3w^5\\ &+4y^2z^4w^2+4yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number (17) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} \;=\;& x^8+4x^4y^3w+4x^4y^2w^2+4x^4yw^3+14x^4z^4+3x^3y^4z+20x^3y^3zw+18x^3y^2zw^2\\ &+20x^3yzw^3+3x^3zw^4+4x^2y^4z^2+24x^2y^3z^2w+16x^2y^2z^2w^2+24x^2yz^2w^3\\ &+4x^2z^2w^4+3xy^4z^3+20xy^3z^3w+18xy^2z^3w^2+20xyz^3w^3+3xz^3w^4+y^7w\\ &+7y^5w^3+4y^3z^4w+7y^3w^5+4y^2z^4w^2+4yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number (18) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^3w + 4x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 4x^3y^4z + 16x^3y^3zw + 24x^3y^2zw^2 \\ &+16x^3yzw^3 + 4x^3zw^4 + 4x^2y^4z^2 + 24x^2y^3z^2w + 16x^2y^2z^2w^2 + 24x^2yz^2w^3 \\ &+4x^2z^2w^4 + 4xy^4z^3 + 16xy^3z^3w + 24xy^2z^3w^2 + 16xyz^3w^3 + 4xz^3w^4 + y^8 \\ &+14y^4w^4 + 4y^3z^4w + 4y^2z^4w^2 + 4yz^4w^3 + z^8 + w^8 \end{aligned}$$

- Permutation orbit number (19) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^3w + 4x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 4x^3y^4z + 16x^3y^3zw + 24x^3y^2zw^2 \\ &+16x^3yzw^3 + 4x^3zw^4 + 4x^2y^4z^2 + 24x^2y^3z^2w + 16x^2y^2z^2w^2 + 24x^2yz^2w^3 \\ &+4x^2z^2w^4 + 4xy^4z^3 + 16xy^3z^3w + 24xy^2z^3w^2 + 16xyz^3w^3 + 4xz^3w^4 + 4y^6w^2 \\ &+8y^4w^4 + 4y^3z^4w + 4y^2z^4w^2 + 4y^2w^6 + 4yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number (20) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} \;=\;& x^8+2x^4y^3w+8x^4y^2w^2+2x^4yw^3+14x^4z^4+5x^3y^4z+12x^3y^3zw+30x^3y^2zw^2\\ &+12x^3yzw^3+5x^3zw^4+8x^2y^4z^2+12x^2y^3z^2w+32x^2y^2z^2w^2+12x^2yz^2w^3\\ &+8x^2z^2w^4+5xy^4z^3+12xy^3z^3w+30xy^2z^3w^2+12xyz^3w^3+5xz^3w^4+y^7w\\ &+7y^5w^3+2y^3z^4w+7y^3w^5+8y^2z^4w^2+2yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number (21) of size 1344 and representative: $\begin{pmatrix} 0&0&0&1\\ 0&0&1&1\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} \;=\;& x^8+2x^4y^3w+8x^4y^2w^2+2x^4yw^3+14x^4z^4+5x^3y^4z+12x^3y^3zw+30x^3y^2zw^2\\ &+12x^3yzw^3+5x^3zw^4+8x^2y^4z^2+12x^2y^3z^2w+32x^2y^2z^2w^2+12x^2yz^2w^3\\ &+8x^2z^2w^4+5xy^4z^3+12xy^3z^3w+30xy^2z^3w^2+12xyz^3w^3+5xz^3w^4+y^7w\\ &+7y^5w^3+2y^3z^4w+7y^3w^5+8y^2z^4w^2+2yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number (22) of size 1344 and representative: $\begin{pmatrix} 0&0&0&1\\ 0&0&1&1\\ 0&0&1&0\\ 0&0&1&0\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} \;=\;& x^8+4x^4y^3w+4x^4y^2w^2+4x^4yw^3+14x^4z^4+3x^3y^4z+20x^3y^3zw+18x^3y^2zw^2\\ &+20x^3yzw^3+3x^3zw^4+4x^2y^4z^2+24x^2y^3z^2w+16x^2y^2z^2w^2+24x^2yz^2w^3\\ &+4x^2z^2w^4+3xy^4z^3+20xy^3z^3w+18xy^2z^3w^2+20xyz^3w^3+3xz^3w^4+y^7w\\ &+7y^5w^3+4y^3z^4w+7y^3w^5+4y^2z^4w^2+4yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number (23) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^3w + 8x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 6x^3y^4z + 8x^3y^3zw + 36x^3y^2zw^2 \\ &+8x^3yzw^3 + 6x^3zw^4 + 8x^2y^4z^2 + 12x^2y^3z^2w + 32x^2y^2z^2w^2 + 12x^2yz^2w^3 \\ &+8x^2z^2w^4 + 6xy^4z^3 + 8xy^3z^3w + 36xy^2z^3w^2 + 8xyz^3w^3 + 6xz^3w^4 + 4y^6w^2 \\ &+8y^4w^4 + 2y^3z^4w + 8y^2z^4w^2 + 4y^2w^6 + 2yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number (24) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^3w + 4x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 4x^3y^4z + 16x^3y^3zw + 24x^3y^2zw^2 \\ &+16x^3yzw^3 + 4x^3zw^4 + 4x^2y^4z^2 + 24x^2y^3z^2w + 16x^2y^2z^2w^2 + 24x^2yz^2w^3 \\ &+4x^2z^2w^4 + 4xy^4z^3 + 16xy^3z^3w + 24xy^2z^3w^2 + 16xyz^3w^3 + 4xz^3w^4 + 4y^6w^2 \\ &+8y^4w^4 + 4y^3z^4w + 4y^2z^4w^2 + 4y^2w^6 + 4yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number (25) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+4x^4y^3w+4x^4y^2w^2+4x^4yw^3+14x^4z^4+4x^3y^4z+16x^3y^3zw+24x^3y^2zw^2\\&+16x^3yzw^3+4x^3zw^4+4x^2y^4z^2+24x^2y^3z^2w+16x^2y^2z^2w^2+24x^2yz^2w^3\\&+4x^2z^2w^4+4xy^4z^3+16xy^3z^3w+24xy^2z^3w^2+16xyz^3w^3+4xz^3w^4+y^8\\&+14y^4w^4+4y^3z^4w+4y^2z^4w^2+4yz^4w^3+z^8+w^8\end{aligned}$$

- Permutation orbit number (26) of size 336 and representative: $\begin{pmatrix}0&0&0&1\\1&0&0&1\\1&1&1&0\\1&0&1&0\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+2x^4y^3w+8x^4y^2w^2+2x^4yw^3+14x^4z^4+6x^3y^4z+8x^3y^3zw+36x^3y^2zw^2\\&+8x^3yzw^3+6x^3zw^4+8x^2y^4z^2+12x^2y^3z^2w+32x^2y^2z^2w^2+12x^2yz^2w^3\\&+8x^2z^2w^4+6xy^4z^3+8xy^3z^3w+36xy^2z^3w^2+8xyz^3w^3+6xz^3w^4+4y^6w^2\\&+8y^4w^4+2y^3z^4w+8y^2z^4w^2+4y^2w^6+2yz^4w^3+z^8\end{aligned}$$

- Permutation orbit number (27) of size 336 and representative: $\begin{pmatrix}0&0&1&1\\0&1&0&1\\0&0&1&1\\0&0&1&1\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+12x^4y^2w^2+14x^4z^4+32x^3y^3zw+32x^3yzw^3+12x^2y^4z^2+48x^2y^2z^2w^2\\&+12x^2z^2w^4+32xy^3z^3w+32xyz^3w^3+4y^6w^2+8y^4w^4+12y^2z^4w^2+4y^2w^6+z^8\end{aligned}$$

Orbit number(5) **of size** 21504 **with symmetrized weight enumerator**

swe $= x^8+16x^4y^4+14x^4z^4+48x^3y^4z+96x^2y^4z^2+48xy^4z^3+16y^8+16y^4z^4+z^8$

and orbit representative:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Number of permutation orbits $= 24$

- Permutation orbit number (1) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 3x^4y^4 + 10x^4y^2w^2 + 14x^4z^4 + 3x^4w^4 + 24x^3y^3zw + 24x^3yzw^3 + 10x^2y^4z^2 \\ &+76x^2y^2z^2w^2 + 10x^2z^2w^4 + 24xy^3z^3w + 24xyz^3w^3 + 4y^6w^2 + 3y^4z^4 + 8y^4w^4 \\ &+10y^2z^4w^2 + 4y^2w^6 + z^8 + 3z^4w^4 \end{aligned}$$

- Permutation orbit number (2) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 3x^4y^4 + 2x^4y^3w + 6x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 3x^4w^4 + 5x^3y^4z \\ &+4x^3y^3zw + 30x^3y^2zw^2 + 4x^3yzw^3 + 5x^3zw^4 + 6x^2y^4z^2 + 12x^2y^3z^2w \\ &+60x^2y^2z^2w^2 + 12x^2yz^2w^3 + 6x^2z^2w^4 + 5xy^4z^3 + 4xy^3z^3w + 30xy^2z^3w^2 \\ &+4xyz^3w^3 + 5xz^3w^4 + y^7w + 7y^5w^3 + 3y^4z^4 + 2y^3z^4w + 7y^3w^5 + 6y^2z^4w^2 \\ &+2yz^4w^3 + yw^7 + z^8 + 3z^4w^4 \end{aligned}$$

- Permutation orbit number(3) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8+2x^4y^4+4x^4y^3w+4x^4y^2w^2+4x^4yw^3+14x^4z^4+2x^4w^4+4x^3y^4z \\ &+8x^3y^3zw+24x^3y^2zw^2+8x^3yzw^3+4x^3zw^4+4x^2y^4z^2+24x^2y^3z^2w \\ &+40x^2y^2z^2w^2+24x^2yz^2w^3+4x^2z^2w^4+4xy^4z^3+8xy^3z^3w \\ &+24xy^2z^3w^2+8xyz^3w^3+4xz^3w^4+4y^6w^2+2y^4z^4+8y^4w^4 \\ &+4y^3z^4w+4y^2z^4w^2+4y^2w^6+4yz^4w^3+z^8+2z^4w^4 \end{aligned}$$

- Permutation orbit number (4) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8+2x^4y^4+4x^4y^3w+4x^4y^2w^2+4x^4yw^3+14x^4z^4+2x^4w^4+4x^3y^4z \\ &+8x^3y^3zw+24x^3y^2zw^2+8x^3yzw^3+4x^3zw^4+4x^2y^4z^2+24x^2y^3z^2w \\ &+40x^2y^2z^2w^2+24x^2yz^2w^3+4x^2z^2w^4+4xy^4z^3+8xy^3z^3w \\ &+24xy^2z^3w^2+8xyz^3w^3+4xz^3w^4+4y^6w^2+2y^4z^4+8y^4w^4+4y^3z^4w \\ &+4y^2z^4w^2+4y^2w^6+4yz^4w^3+z^8+2z^4w^4 \end{aligned}$$

- Permutation orbit number (5) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^4 + 4x^4y^3w + 4x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 2x^4w^4 + 3x^3y^4z \\ &+12x^3y^3zw + 18x^3y^2zw^2 + 12x^3yzw^3 + 3x^3zw^4 + 4x^2y^4z^2 + 24x^2y^3z^2w \\ &+40x^2y^2z^2w^2 + 24x^2yz^2w^3 + 4x^2z^2w^4 + 3xy^4z^3 + 12xy^3z^3w \\ &+18xy^2z^3w^2 + 12xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + 2y^4z^4 + 4y^3z^4w \\ &+7y^3w^5 + 4y^2z^4w^2 + 4yz^4w^3 + yw^7 + z^8 + 2z^4w^4 \end{aligned}$$

- Permutation orbit number (6) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^4 + 4x^4y^3w + 4x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 2x^4w^4 + 3x^3y^4z \\ &+12x^3y^3zw + 18x^3y^2zw^2 + 12x^3yzw^3 + 3x^3zw^4 + 4x^2y^4z^2 + 24x^2y^3z^2w \\ &+40x^2y^2z^2w^2 + 24x^2yz^2w^3 + 4x^2z^2w^4 + 3xy^4z^3 + 12xy^3z^3w \\ &+18xy^2z^3w^2 + 12xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + 2y^4z^4 + 4y^3z^4w \\ &+7y^3w^5 + 4y^2z^4w^2 + 4yz^4w^3 + yw^7 + z^8 + 2z^4w^4 \end{aligned}$$

- Permutation orbit number (7) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+4x^4y^3w+6x^4y^2w^2+4x^4yw^3+14x^4z^4+x^4w^4+3x^3y^4z\\ &+12x^3y^3zw+18x^3y^2zw^2+12x^3yzw^3+3x^3zw^4+6x^2y^4z^2+24x^2y^3z^2w\\ &+36x^2y^2z^2w^2+24x^2yz^2w^3+6x^2z^2w^4+3xy^4z^3+12xy^3z^3w\\ &+18xy^2z^3w^2+12xyz^3w^3+3xz^3w^4+y^7w+7y^5w^3+y^4z^4+4y^3z^4w\\ &+7y^3w^5+6y^2z^4w^2+4yz^4w^3+yw^7+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (8) of size 1344 and representative: $\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\0&0&1&0\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+4x^4y^3w+6x^4y^2w^2+4x^4yw^3+14x^4z^4+x^4w^4+3x^3y^4z\\ &+12x^3y^3zw+18x^3y^2zw^2+12x^3yzw^3+3x^3zw^4+6x^2y^4z^2+24x^2y^3z^2w\\ &+36x^2y^2z^2w^2+24x^2yz^2w^3+6x^2z^2w^4+3xy^4z^3+12xy^3z^3w+18xy^2z^3w^2\\ &+12xyz^3w^3+3xz^3w^4+y^7w+7y^5w^3+y^4z^4+4y^3z^4w+7y^3w^5+6y^2z^4w^2\\ &+4yz^4w^3+yw^7+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (9) of size 672 and representative: $\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\1&0&1&0\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+4x^4y^3w+6x^4y^2w^2+4x^4yw^3+14x^4z^4+x^4w^4+4x^3y^4z\\ &+8x^3y^3zw+24x^3y^2zw^2+8x^3yzw^3+4x^3zw^4+6x^2y^4z^2+24x^2y^3z^2w\\ &+36x^2y^2z^2w^2+24x^2yz^2w^3+6x^2z^2w^4+4xy^4z^3+8xy^3z^3w+24xy^2z^3w^2\\ &+8xyz^3w^3+4xz^3w^4+4y^6w^2+y^4z^4+8y^4w^4+4y^3z^4w+6y^2z^4w^2+4y^2w^6\\ &+4yz^4w^3+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (10) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+2x^4y^4+4x^4y^3w+4x^4y^2w^2+4x^4yw^3+14x^4z^4+2x^4w^4+4x^3y^4z\\ &+8x^3y^3zw+24x^3y^2zw^2+8x^3yzw^3+4x^3zw^4+4x^2y^4z^2+24x^2y^3z^2w\\ &+40x^2y^2z^2w^2+24x^2yz^2w^3+4x^2z^2w^4+4xy^4z^3+8xy^3z^3w+24xy^2z^3w^2\\ &+8xyz^3w^3+4xz^3w^4+y^8+2y^4z^4+14y^4w^4+4y^3z^4w+4y^2z^4w^2+4yz^4w^3\\ &+z^8+2z^4w^4+w^8\end{aligned}$$

- Permutation orbit number (11) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + x^4y^4 + 4x^4y^3w + 6x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + x^4w^4 + 3x^3y^4z \\ &+12x^3y^3zw + 18x^3y^2zw^2 + 12x^3yzw^3 + 3x^3zw^4 + 6x^2y^4z^2 + 24x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 24x^2yz^2w^3 + 6x^2z^2w^4 + 3xy^4z^3 + 12xy^3z^3w + 18xy^2z^3w^2 \\ &+12xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 + 4y^3z^4w + 7y^3w^5 + 6y^2z^4w^2 \\ &+4yz^4w^3 + yw^7 + z^8 + z^4w^4\end{aligned}$$

- Permutation orbit number (12) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + x^4y^4 + 4x^4y^3w + 6x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + x^4w^4 + 4x^3y^4z \\ &+8x^3y^3zw + 24x^3y^2zw^2 + 8x^3yzw^3 + 4x^3zw^4 + 6x^2y^4z^2 + 24x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 24x^2yz^2w^3 + 6x^2z^2w^4 + 4xy^4z^3 + 8xy^3z^3w + 24xy^2z^3w^2 \\ &+8xyz^3w^3 + 4xz^3w^4 + 4y^6w^2 + y^4z^4 + 8y^4w^4 + 4y^3z^4w + 6y^2z^4w^2 + 4y^2w^6 \\ &+4yz^4w^3 + z^8 + z^4w^4\end{aligned}$$

- Permutation orbit number (13) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+6x^4y^3w+2x^4y^2w^2+6x^4yw^3+14x^4z^4+x^4w^4+2x^3y^4z \\ &+16x^3y^3zw+12x^3y^2zw^2+16x^3yzw^3+2x^3zw^4+2x^2y^4z^2+36x^2y^3z^2w \\ &+20x^2y^2z^2w^2+36x^2yz^2w^3+2x^2z^2w^4+2xy^4z^3+16xy^3z^3w+12xy^2z^3w^2 \\ &+16xyz^3w^3+2xz^3w^4+4y^6w^2+y^4z^4+8y^4w^4+6y^3z^4w+2y^2z^4w^2 \\ &+4y^2w^6+6yz^4w^3+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (14) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+6x^4y^3w+2x^4y^2w^2+6x^4yw^3+14x^4z^4+x^4w^4+x^3y^4z \\ &+20x^3y^3zw+6x^3y^2zw^2+20x^3yzw^3+x^3zw^4+2x^2y^4z^2+36x^2y^3z^2w \\ &+20x^2y^2z^2w^2+36x^2yz^2w^3+2x^2z^2w^4+xy^4z^3+20xy^3z^3w+6xy^2z^3w^2 \\ &+20xyz^3w^3+xz^3w^4+y^7w+7y^5w^3+y^4z^4+6y^3z^4w+7y^3w^5+2y^2z^4w^2 \\ &+6yz^4w^3+yw^7+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (15) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+4x^4y^3w+6x^4y^2w^2+4x^4yw^3+14x^4z^4+x^4w^4+3x^3y^4z\\ &+12x^3y^3zw+18x^3y^2zw^2+12x^3yzw^3+3x^3zw^4+6x^2y^4z^2+24x^2y^3z^2w\\ &+36x^2y^2z^2w^2+24x^2yz^2w^3+6x^2z^2w^4+3xy^4z^3+12xy^3z^3w+18xy^2z^3w^2\\ &+12xyz^3w^3+3xz^3w^4+y^7w+7y^5w^3+y^4z^4+4y^3z^4w+7y^3w^5+6y^2z^4w^2\\ &+4yz^4w^3+yw^7+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (16) of size 672 and representative: $\begin{pmatrix}0&0&0&0\\1&0&0&1\\1&0&0&1\\1&0&0&1\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+14x^4y^2w^2+14x^4z^4+x^4w^4+24x^3y^3zw+24x^3yzw^3+14x^2y^4z^2\\ &+68x^2y^2z^2w^2+14x^2z^2w^4+24xy^3z^3w+24xyz^3w^3+4y^6w^2+y^4z^4+8y^4w^4\\ &+14y^2z^4w^2+4y^2w^6+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (17) of size 672 and representative: $\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&1&1\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+4x^4y^3w+8x^4y^2w^2+4x^4yw^3+14x^4z^4+3x^3y^4z+12x^3y^3zw\\ &+18x^3y^2zw^2+12x^3yzw^3+3x^3zw^4+8x^2y^4z^2+24x^2y^3z^2w+32x^2y^2z^2w^2\\ &+24x^2yz^2w^3+8x^2z^2w^4+3xy^4z^3+12xy^3z^3w+18xy^2z^3w^2+12xyz^3w^3\\ &+3xz^3w^4+y^7w+7y^5w^3+4y^3z^4w+7y^3w^5+8y^2z^4w^2+4yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number (18) of size 1344 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 6x^4y^3w + 4x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + 2x^3y^4z + 16x^3y^3zw + 12x^3y^2zw^2 \\ &+16x^3yzw^3 + 2x^3zw^4 + 4x^2y^4z^2 + 36x^2y^3z^2w + 16x^2y^2z^2w^2 + 36x^2yz^2w^3 \\ &+4x^2z^2w^4 + 2xy^4z^3 + 16xy^3z^3w + 12xy^2z^3w^2 + 16xyz^3w^3 + 2xz^3w^4 + 4y^6w^2 \\ &+8y^4w^4 + 6y^3z^4w + 4y^2z^4w^2 + 4y^2w^6 + 6yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number (19) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 6x^4y^3w + 4x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + x^3y^4z + 20x^3y^3zw + 6x^3y^2zw^2 \\ &+20x^3yzw^3 + x^3zw^4 + 4x^2y^4z^2 + 36x^2y^3z^2w + 16x^2y^2z^2w^2 + 36x^2yz^2w^3 \\ &+4x^2z^2w^4 + xy^4z^3 + 20xy^3z^3w + 6xy^2z^3w^2 + 20xyz^3w^3 + xz^3w^4 + y^7w \\ &+7y^5w^3 + 6y^3z^4w + 7y^3w^5 + 4y^2z^4w^2 + 6yz^4w^3 + yw^7 + z^8 \end{aligned}$$

- Permutation orbit number (20) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+4x^4y^3w+8x^4y^2w^2+4x^4yw^3+14x^4z^4+4x^3y^4z+8x^3y^3zw+24x^3y^2zw^2\\ &+8x^3yzw^3+4x^3zw^4+8x^2y^4z^2+24x^2y^3z^2w+32x^2y^2z^2w^2+24x^2yz^2w^3\\ &+8x^2z^2w^4+4xy^4z^3+8xy^3z^3w+24xy^2z^3w^2+8xyz^3w^3+4xz^3w^4+y^8\\ &+14y^4w^4+4y^3z^4w+8y^2z^4w^2+4yz^4w^3+z^8+w^8\end{aligned}$$

- Permutation orbit number (21) of size 1344 and representative: $\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&1&0&1\\0&0&1&0\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+4x^4y^3w+8x^4y^2w^2+4x^4yw^3+14x^4z^4+3x^3y^4z+12x^3y^3zw\\ &+18x^3y^2zw^2+12x^3yzw^3+3x^3zw^4+8x^2y^4z^2+24x^2y^3z^2w\\ &+32x^2y^2z^2w^2+24x^2yz^2w^3+8x^2z^2w^4+3xy^4z^3+12xy^3z^3w\\ &+18xy^2z^3w^2+12xyz^3w^3+3xz^3w^4+y^7w+7y^5w^3+4y^3z^4w+7y^3w^5\\ &+8y^2z^4w^2+4yz^4w^3+yw^7+z^8\end{aligned}$$

- Permutation orbit number (22) of size 1344 and representative: $\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&1&0&1\\0&0&1&1\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+4x^4y^3w+8x^4y^2w^2+4x^4yw^3+14x^4z^4+4x^3y^4z+8x^3y^3zw\\ &+24x^3y^2zw^2+8x^3yzw^3+4x^3zw^4+8x^2y^4z^2+24x^2y^3z^2w\\ &+32x^2y^2z^2w^2+24x^2yz^2w^3+8x^2z^2w^4+4xy^4z^3+8xy^3z^3w\\ &+24xy^2z^3w^2+8xyz^3w^3+4xz^3w^4+4y^6w^2+8y^4w^4+4y^3z^4w\\ &+8y^2z^4w^2+4y^2w^6+4yz^4w^3+z^8\end{aligned}$$

- Permutation orbit number (23) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^3w + 8x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 4x^3y^4z + 8x^3y^3zw \\ &+24x^3y^2zw^2 + 8x^3yzw^3 + 4x^3zw^4 + 8x^2y^4z^2 + 24x^2y^3z^2w \\ &+32x^2y^2z^2w^2 + 24x^2yz^2w^3 + 8x^2z^2w^4 + 4xy^4z^3 + 8xy^3z^3w \\ &+24xy^2z^3w^2 + 8xyz^3w^3 + 4xz^3w^4 + 4y^6w^2 + 8y^4w^4 + 4y^3z^4w \\ &+8y^2z^4w^2 + 4y^2w^6 + 4yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number (24) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^3w + 12x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 5x^3y^4z + 4x^3y^3zw + 30x^3y^2zw^2 \\ &+4x^3yzw^3 + 5x^3zw^4 + 12x^2y^4z^2 + 12x^2y^3z^2w + 48x^2y^2z^2w^2 + 12x^2yz^2w^3 \\ &+12x^2z^2w^4 + 5xy^4z^3 + 4xy^3z^3w + 30xy^2z^3w^2 + 4xyz^3w^3 + 5xz^3w^4 + y^7w \\ &+7y^5w^3 + 2y^3z^4w + 7y^3w^5 + 12y^2z^4w^2 + 2yz^4w^3 + yw^7 + z^8 \end{aligned}$$

Orbit number(6) **of size** 3584 **with symmetrized weight enumerator**

swe $= x^8+16x^4y^4+14x^4z^4+48x^3y^4z+96x^2y^4z^2+48xy^4z^3+16y^8+16y^4z^4+z^8$

and orbit representative:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Number of permutation orbits= 14.

- Permutation orbit number (1) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^4 + 4x^4y^3w + 4x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 2x^4w^4 + 3x^3y^4z \\ &+12x^3y^3zw + 18x^3y^2zw^2 + 12x^3yzw^3 + 3x^3zw^4 + 4x^2y^4z^2 + 24x^2y^3z^2w \\ &+40x^2y^2z^2w^2 + 24x^2yz^2w^3 + 4x^2z^2w^4 + 3xy^4z^3 + 12xy^3z^3w + 18xy^2z^3w^2 \\ &+12xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + 2y^4z^4 + 4y^3z^4w + 7y^3w^5 \\ &+4y^2z^4w^2 + 4yz^4w^3 + yw^7 + z^8 + 2z^4w^4 \end{aligned}$$

- Permutation orbit number (2) of size 112 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 3x^4y^4 + 2x^4y^3w + 6x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 3x^4w^4 + 6x^3y^4z \\ &+36x^3y^2zw^2 + 6x^3zw^4 + 6x^2y^4z^2 + 12x^2y^3z^2w + 60x^2y^2z^2w^2 \\ &+12x^2yz^2w^3 + 6x^2z^2w^4 + 6xy^4z^3 + 36xy^2z^3w^2 + 6xz^3w^4 + 4y^6w^2 \\ &+3y^4z^4 + 8y^4w^4 + 2y^3z^4w + 6y^2z^4w^2 + 4y^2w^6 + 2yz^4w^3 + z^8 + 3z^4w^4 \end{aligned}$$

- Permutation orbit number (3) of size 84 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^4 + 8x^4y^2w^2 + 14x^4z^4 + 4x^4w^4 + 24x^3y^3zw + 24x^3yzw^3 + 8x^2y^4z^2 \\ &\quad +80x^2y^2z^2w^2 + 8x^2z^2w^4 + 24xy^3z^3w + 24xyz^3w^3 + y^8 + 4y^4z^4 + 14y^4w^4 \\ &\quad +8y^2z^4w^2 + z^8 + 4z^4w^4 + w^8 \end{aligned}$$

- Permutation orbit number (4) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 4x^4y^3w + 6x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + x^4w^4 + 3x^3y^4z \\ &\quad +12x^3y^3zw + 18x^3y^2zw^2 + 12x^3yzw^3 + 3x^3zw^4 + 6x^2y^4z^2 + 24x^2y^3z^2w \\ &\quad +36x^2y^2z^2w^2 + 24x^2yz^2w^3 + 6x^2z^2w^4 + 3xy^4z^3 + 12xy^3z^3w + 18xy^2z^3w^2 \\ &\quad +12xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 + 4y^3z^4w + 7y^3w^5 + 6y^2z^4w^2 \\ &\quad +4yz^4w^3 + yw^7 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number (5) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+4x^4y^3w+6x^4y^2w^2+4x^4yw^3+14x^4z^4+x^4w^4+4x^3y^4z\\ &+8x^3y^3zw+24x^3y^2zw^2+8x^3yzw^3+4x^3zw^4+6x^2y^4z^2+24x^2y^3z^2w\\ &+36x^2y^2z^2w^2+24x^2yz^2w^3+6x^2z^2w^4+4xy^4z^3+8xy^3z^3w+24xy^2z^3w^2\\ &+8xyz^3w^3+4xz^3w^4+4y^6w^2+y^4z^4+8y^4w^4+4y^3z^4w+6y^2z^4w^2+4y^2w^6\\ &+4yz^4w^3+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (6) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+x^4y^4+4x^4y^3w+6x^4y^2w^2+4x^4yw^3+14x^4z^4+x^4w^4+3x^3y^4z\\ &+12x^3y^3zw+18x^3y^2zw^2+12x^3yzw^3+3x^3zw^4+6x^2y^4z^2+24x^2y^3z^2w\\ &+36x^2y^2z^2w^2+24x^2yz^2w^3+6x^2z^2w^4+3xy^4z^3+12xy^3z^3w+18xy^2z^3w^2\\ &+12xyz^3w^3+3xz^3w^4+y^7w+7y^5w^3+y^4z^4+4y^3z^4w+7y^3w^5+6y^2z^4w^2\\ &+4yz^4w^3+yw^7+z^8+z^4w^4\end{aligned}$$

- Permutation orbit number (7) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + x^4y^4 + 6x^4y^3w + 2x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + x^4w^4 + 2x^3y^4z \\ &+16x^3y^3zw + 12x^3y^2zw^2 + 16x^3yzw^3 + 2x^3zw^4 + 2x^2y^4z^2 + 36x^2y^3z^2w \\ &+20x^2y^2z^2w^2 + 36x^2yz^2w^3 + 2x^2z^2w^4 + 2xy^4z^3 + 16xy^3z^3w + 12xy^2z^3w^2 \\ &+16xyz^3w^3 + 2xz^3w^4 + 4y^6w^2 + y^4z^4 + 8y^4w^4 + 6y^3z^4w + 2y^2z^4w^2 + 4y^2w^6 \\ &+6yz^4w^3 + z^8 + z^4w^4\end{aligned}$$

- Permutation orbit number (8) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + x^4y^4 + 4x^4y^3w + 6x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + x^4w^4 + 4x^3y^4z \\ &+8x^3y^3zw + 24x^3y^2zw^2 + 8x^3yzw^3 + 4x^3zw^4 + 6x^2y^4z^2 + 24x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 24x^2yz^2w^3 + 6x^2z^2w^4 + 4xy^4z^3 + 8xy^3z^3w + 24xy^2z^3w^2 \\ &+8xyz^3w^3 + 4xz^3w^4 + 4y^6w^2 + y^4z^4 + 8y^4w^4 + 4y^3z^4w + 6y^2z^4w^2 + 4y^2w^6 \\ &+4yz^4w^3 + z^8 + z^4w^4\end{aligned}$$

- Permutation orbit number (9) of size 28 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + 2x^4y^4 + 12x^4y^2w^2 + 14x^4z^4 + 2x^4w^4 + 24x^3y^3zw + 24x^3yzw^3 + 12x^2y^4z^2 \\ &+72x^2y^2z^2w^2 + 12x^2z^2w^4 + 24xy^3z^3w + 24xyz^3w^3 + y^8 + 2y^4z^4 + 14y^4w^4 \\ &+12y^2z^4w^2 + z^8 + 2z^4w^4 + w^8\end{aligned}$$

- Permutation orbit number (10) of size 28 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^4 + 12x^4y^2w^2 + 14x^4z^4 + 2x^4w^4 + 24x^3y^3zw + 24x^3yzw^3 + 12x^2y^4z^2 \\ &\quad +72x^2y^2z^2w^2 + 12x^2z^2w^4 + 24xy^3z^3w + 24xyz^3w^3 + y^8 + 2y^4z^4 + 14y^4w^4 \\ &\quad +12y^2z^4w^2 + z^8 + 2z^4w^4 + w^8 \end{aligned}$$

- Permutation orbit number (11) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 6x^4y^3w + 4x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + 2x^3y^4z + 16x^3y^3zw + 12x^3y^2zw^2 \\ &\quad +16x^3yzw^3 + 2x^3zw^4 + 4x^2y^4z^2 + 36x^2y^3z^2w + 16x^2y^2z^2w^2 + 36x^2yz^2w^3 \\ &\quad +4x^2z^2w^4 + 2xy^4z^3 + 16xy^3z^3w + 12xy^2z^3w^2 + 16xyz^3w^3 + 2xz^3w^4 + 4y^6w^2 \\ &\quad +8y^4w^4 + 6y^3z^4w + 4y^2z^4w^2 + 4y^2w^6 + 6yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number (12) of size 112 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 2x^4y^3w + 12x^4y^2w^2 + 2x^4yw^3 + 14x^4z^4 + 6x^3y^4z + 36x^3y^2zw^2 + 6x^3zw^4 \\ &\quad +12x^2y^4z^2 + 12x^2y^3z^2w + 48x^2y^2z^2w^2 + 12x^2yz^2w^3 + 12x^2z^2w^4 + 6xy^4z^3 \\ &\quad +36xy^2z^3w^2 + 6xz^3w^4 + 4y^6w^2 + 8y^4w^4 + 2y^3z^4w + 12y^2z^4w^2 + 4y^2w^6 \\ &\quad +2yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number (13) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^3w + 8x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 3x^3y^4z + 12x^3y^3zw \\ &+18x^3y^2zw^2 + 12x^3yzw^3 + 3x^3zw^4 + 8x^2y^4z^2 + 24x^2y^3z^2w \\ &+32x^2y^2z^2w^2 + 24x^2yz^2w^3 + 8x^2z^2w^4 + 3xy^4z^3 + 12xy^3z^3w \\ &+18xy^2z^3w^2 + 12xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + 4y^3z^4w + 7y^3w^5 \\ &+8y^2z^4w^2 + 4yz^4w^3 + yw^7 + z^8 \end{aligned}$$

- Permutation orbit number (14) of size 84 and representative: $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 16x^4y^2w^2 + 14x^4z^4 + 24x^3y^3zw + 24x^3yzw^3 + 16x^2y^4z^2 + 64x^2y^2z^2w^2 \\ &+16x^2z^2w^4 + 24xy^3z^3w + 24xyz^3w^3 + y^8 + 14y^4w^4 + 16y^2z^4w^2 + z^8 + w^8 \end{aligned}$$

Orbit number(7) **of size** 896 **with symmetrized weight enumerator**

swe $= x^8+16x^4y^4+14x^4z^4+48x^3y^4z+96x^2y^4z^2+48xy^4z^3+16y^8+16y^4z^4+z^8$

and orbit representative:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Number of permutation orbits$= 4$.

- Permutation orbit number (1) of size 448 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 4x^4y^3w + 6x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + x^4w^4 + 3x^3y^4z \\ &+12x^3y^3zw + 18x^3y^2zw^2 + 12x^3yzw^3 + 3x^3zw^4 + 6x^2y^4z^2 + 24x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 24x^2yz^2w^3 + 6x^2z^2w^4 + 3xy^4z^3 + 12xy^3z^3w + 18xy^2z^3w^2 \\ &+12xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 + 4y^3z^4w + 7y^3w^5 + 6y^2z^4w^2 \\ &+4yz^4w^3 + yw^7 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number (2) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 4x^4y^3w + 6x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + x^4w^4 + 4x^3y^4z \\ &+8x^3y^3zw + 24x^3y^2zw^2 + 8x^3yzw^3 + 4x^3zw^4 + 6x^2y^4z^2 + 24x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 24x^2yz^2w^3 + 6x^2z^2w^4 + 4xy^4z^3 + 8xy^3z^3w + 24xy^2z^3w^2 \\ &+8xyz^3w^3 + 4xz^3w^4 + 4y^6w^2 + y^4z^4 + 8y^4w^4 + 4y^3z^4w + 6y^2z^4w^2 + 4y^2w^6 \\ &+4yz^4w^3 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number (3) of size 56 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+2x^4y^4+12x^4y^2w^2+14x^4z^4+2x^4w^4+24x^3y^3zw+24x^3yzw^3\\ &+12x^2y^4z^2+72x^2y^2z^2w^2+12x^2z^2w^4+24xy^3z^3w+24xyz^3w^3+y^8\\ &+2y^4z^4+14y^4w^4+12y^2z^4w^2+z^8+2z^4w^4+w^8\end{aligned}$$

- Permutation orbit number (4) of size 56 and representative: $\begin{pmatrix}0&0&0&1\\0&1&0&1\\0&1&1&0\\0&1&0&0\end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+8x^4y^3w+8x^4yw^3+14x^4z^4+24x^3y^3zw+24x^3yzw^3+48x^2y^3z^2w\\ &+48x^2yz^2w^3+24xy^3z^3w+24xyz^3w^3+4y^6w^2+8y^4w^4+8y^3z^4w\\ &+4y^2w^6+8yz^4w^3+z^8\end{aligned}$$

Orbit number (8) **of size** 7168 **with symmetrized weight enumerator**
swe $= x^8+20x^4y^4+14x^4z^4+32x^3y^4z+120x^2y^4z^2+32xy^4z^3+16y^8+20y^4z^4+z^8$
and orbit representative:

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

Number of permutation orbits= 19

- Permutation orbit number (1) of size 56 and representative: $\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^4 + 12x^4y^2w^2 + 14x^4z^4 + 4x^4w^4 + 16x^3y^3zw + 16x^3yzw^3 \\ &+12x^2y^4z^2 + 96x^2y^2z^2w^2 + 12x^2z^2w^4 + 16xy^3z^3w + 16xyz^3w^3 \\ &+4y^6w^2 + 4y^4z^4 + 8y^4w^4 + 12y^2z^4w^2 + 4y^2w^6 + z^8 + 4z^4w^4 \end{aligned}$$

- Permutation orbit number (2) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 3x^4y^4 + 4x^4y^3w + 6x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 3x^4w^4 + 3x^3y^4z \\ &+4x^3y^3zw + 18x^3y^2zw^2 + 4x^3yzw^3 + 3x^3zw^4 + 6x^2y^4z^2 + 24x^2y^3z^2w \\ &+60x^2y^2z^2w^2 + 24x^2yz^2w^3 + 6x^2z^2w^4 + 3xy^4z^3 + 4xy^3z^3w \\ &+18xy^2z^3w^2 + 4xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + 3y^4z^4 + 4y^3z^4w \\ &+7y^3w^5 + 6y^2z^4w^2 + 4yz^4w^3 + yw^7 + z^8 + 3z^4w^4 \end{aligned}$$

- Permutation orbit number (3) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^4 + 12x^4y^2w^2 + 14x^4z^4 + 4x^4w^4 + 16x^3y^3zw + 16x^3yzw^3 + 12x^2y^4z^2 \\ &+96x^2y^2z^2w^2 + 12x^2z^2w^4 + 16xy^3z^3w + 16xyz^3w^3 + 4y^6w^2 + 4y^4z^4 + 8y^4w^4 \\ &+12y^2z^4w^2 + 4y^2w^6 + z^8 + 4z^4w^4 \end{aligned}$$

- Permutation orbit number (4) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 3x^4y^4 + 4x^4y^3w + 6x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 3x^4w^4 + 3x^3y^4z \\ &+4x^3y^3zw + 18x^3y^2zw^2 + 4x^3yzw^3 + 3x^3zw^4 + 6x^2y^4z^2 + 24x^2y^3z^2w \\ &+60x^2y^2z^2w^2 + 24x^2yz^2w^3 + 6x^2z^2w^4 + 3xy^4z^3 + 4xy^3z^3w + 18xy^2z^3w^2 \\ &+4xyz^3w^3 + 3xz^3w^4 + y^7w + 7y^5w^3 + 3y^4z^4 + 4y^3z^4w + 7y^3w^5 + 6y^2z^4w^2 \\ &+4yz^4w^3 + yw^7 + z^8 + 3z^4w^4 \end{aligned}$$

- Permutation orbit number (5) of size 168 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^4 + 4x^4y^3w + 4x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 4x^4w^4 + 4x^3y^4z \\ &+24x^3y^2zw^2 + 4x^3zw^4 + 4x^2y^4z^2 + 24x^2y^3z^2w + 64x^2y^2z^2w^2 \\ &+24x^2yz^2w^3 + 4x^2z^2w^4 + 4xy^4z^3 + 24xy^2z^3w^2 + 4xz^3w^4 + y^8 + 4y^4z^4 \\ &+14y^4w^4 + 4y^3z^4w + 4y^2z^4w^2 + 4yz^4w^3 + z^8 + 4z^4w^4 + w^8 \end{aligned}$$

- Permutation orbit number (6) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8+2x^4y^4+6x^4y^3w+4x^4y^2w^2+6x^4yw^3+14x^4z^4+2x^4w^4+2x^3y^4z \\ &+8x^3y^3zw+12x^3y^2zw^2+8x^3yzw^3+2x^3zw^4+4x^2y^4z^2+36x^2y^3z^2w \\ &+40x^2y^2z^2w^2+36x^2yz^2w^3+4x^2z^2w^4+2xy^4z^3+8xy^3z^3w \\ &+12xy^2z^3w^2+8xyz^3w^3+2xz^3w^4+4y^6w^2+2y^4z^4+8y^4w^4+6y^3z^4w \\ &+4y^2z^4w^2+4y^2w^6+6yz^4w^3+z^8+2z^4w^4 \end{aligned}$$

- Permutation orbit number (7) of size 672 and representative: $\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&1&1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8+2x^4y^4+6x^4y^3w+4x^4y^2w^2+6x^4yw^3+14x^4z^4+2x^4w^4+x^3y^4z \\ &+12x^3y^3zw+6x^3y^2zw^2+12x^3yzw^3+x^3zw^4+4x^2y^4z^2+36x^2y^3z^2w \\ &+40x^2y^2z^2w^2+36x^2yz^2w^3+4x^2z^2w^4+xy^4z^3+12xy^3z^3w+6xy^2z^3w^2 \\ &+12xyz^3w^3+xz^3w^4+y^7w+7y^5w^3+2y^4z^4+6y^3z^4w+7y^3w^5 \\ &+4y^2z^4w^2+6yz^4w^3+yw^7+z^8+2z^4w^4 \end{aligned}$$

- Permutation orbit number (8) of size 672 and representative: $\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&1&0&0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + 2x^4y^4 + 6x^4y^3w + 4x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + 2x^4w^4 + 2x^3y^4z \\ &+8x^3y^3zw + 12x^3y^2zw^2 + 8x^3yzw^3 + 2x^3zw^4 + 4x^2y^4z^2 + 36x^2y^3z^2w \\ &+40x^2y^2z^2w^2 + 36x^2yz^2w^3 + 4x^2z^2w^4 + 2xy^4z^3 + 8xy^3z^3w + 12xy^2z^3w^2 \\ &+8xyz^3w^3 + 2xz^3w^4 + 4y^6w^2 + 2y^4z^4 + 8y^4w^4 + 6y^3z^4w + 4y^2z^4w^2 \\ &+4y^2w^6 + 6yz^4w^3 + z^8 + 2z^4w^4\end{aligned}$$

- Permutation orbit number (9) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + x^4y^4 + 6x^4y^3w + 6x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + x^4w^4 + x^3y^4z \\ &+12x^3y^3zw + 6x^3y^2zw^2 + 12x^3yzw^3 + x^3zw^4 + 6x^2y^4z^2 + 36x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 36x^2yz^2w^3 + 6x^2z^2w^4 + xy^4z^3 + 12xy^3z^3w + 6xy^2z^3w^2 \\ &+12xyz^3w^3 + xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 + 6y^3z^4w + 7y^3w^5 + 6y^2z^4w^2 \\ &+6yz^4w^3 + yw^7 + z^8 + z^4w^4\end{aligned}$$

- Permutation orbit number (10) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + x^4y^4 + 6x^4y^3w + 6x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + x^4w^4 + 2x^3y^4z \\ &+8x^3y^3zw + 12x^3y^2zw^2 + 8x^3yzw^3 + 2x^3zw^4 + 6x^2y^4z^2 + 36x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 36x^2yz^2w^3 + 6x^2z^2w^4 + 2xy^4z^3 + 8xy^3z^3w + 12xy^2z^3w^2 \\ &+8xyz^3w^3 + 2xz^3w^4 + 4y^6w^2 + y^4z^4 + 8y^4w^4 + 6y^3z^4w + 6y^2z^4w^2 + 4y^2w^6 \\ &+6yz^4w^3 + z^8 + z^4w^4\end{aligned}$$

- Permutation orbit number (11) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + x^4y^4 + 6x^4y^3w + 6x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + x^4w^4 + x^3y^4z \\ &+12x^3y^3zw + 6x^3y^2zw^2 + 12x^3yzw^3 + x^3zw^4 + 6x^2y^4z^2 + 36x^2y^3z^2w \\ &+36x^2y^2z^2w^2 + 36x^2yz^2w^3 + 6x^2z^2w^4 + xy^4z^3 + 12xy^3z^3w + 6xy^2z^3w^2 \\ &+12xyz^3w^3 + xz^3w^4 + y^7w + 7y^5w^3 + y^4z^4 + 6y^3z^4w + 7y^3w^5 + 6y^2z^4w^2 \\ &+6yz^4w^3 + yw^7 + z^8 + z^4w^4\end{aligned}$$

- Permutation orbit number (12) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8 + 6x^4y^3w + 8x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + 2x^3y^4z + 8x^3y^3zw \\ &+12x^3y^2zw^2 + 8x^3yzw^3 + 2x^3zw^4 + 8x^2y^4z^2 + 36x^2y^3z^2w + 32x^2y^2z^2w^2 \\ &+36x^2yz^2w^3 + 8x^2z^2w^4 + 2xy^4z^3 + 8xy^3z^3w + 12xy^2z^3w^2 + 8xyz^3w^3 \\ &+2xz^3w^4 + 4y^6w^2 + 8y^4w^4 + 6y^3z^4w + 8y^2z^4w^2 + 4y^2w^6 + 6yz^4w^3 + z^8\end{aligned}$$

- Permutation orbit number (13) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^3w + 12x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 3x^3y^4z + 4x^3y^3zw \\ &+18x^3y^2zw^2 + 4x^3yzw^3 + 3x^3zw^4 + 12x^2y^4z^2 + 24x^2y^3z^2w + 48x^2y^2z^2w^2 \\ &+24x^2yz^2w^3 + 12x^2z^2w^4 + 3xy^4z^3 + 4xy^3z^3w + 18xy^2z^3w^2 + 4xyz^3w^3 \\ &+3xz^3w^4 + y^7w + 7y^5w^3 + 4y^3z^4w + 7y^3w^5 + 12y^2z^4w^2 + 4yz^4w^3 + yw^7 + z^8 \end{aligned}$$

- Permutation orbit number (14) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 4x^4y^3w + 12x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 4x^3y^4z + 24x^3y^2zw^2 + 4x^3zw^4 \\ &+12x^2y^4z^2 + 24x^2y^3z^2w + 48x^2y^2z^2w^2 + 24x^2yz^2w^3 + 12x^2z^2w^4 + 4xy^4z^3 \\ &+24xy^2z^3w^2 + 4xz^3w^4 + y^8 + 14y^4w^4 + 4y^3z^4w + 12y^2z^4w^2 + 4yz^4w^3 + z^8 + w^8 \end{aligned}$$

- Permutation orbit number (15) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 6x^4y^3w + 8x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + x^3y^4z + 12x^3y^3zw + 6x^3y^2zw^2 \\ &+12x^3yzw^3 + x^3zw^4 + 8x^2y^4z^2 + 36x^2y^3z^2w + 32x^2y^2z^2w^2 + 36x^2yz^2w^3 \\ &+8x^2z^2w^4 + xy^4z^3 + 12xy^3z^3w + 6xy^2z^3w^2 + 12xyz^3w^3 + xz^3w^4 + y^7w \\ &+7y^5w^3 + 6y^3z^4w + 7y^3w^5 + 8y^2z^4w^2 + 6yz^4w^3 + yw^7 + z^8 \end{aligned}$$

- Permutation orbit number (16) of size 336 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} \quad = \quad & x^8 + 6x^4y^3w + 8x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + 2x^3y^4z + 8x^3y^3zw + 12x^3y^2zw^2 \\ & +8x^3yzw^3 + 2x^3zw^4 + 8x^2y^4z^2 + 36x^2y^3z^2w + 32x^2y^2z^2w^2 + 36x^2yz^2w^3 \\ & +8x^2z^2w^4 + 2xy^4z^3 + 8xy^3z^3w + 12xy^2z^3w^2 + 8xyz^3w^3 + 2xz^3w^4 + 4y^6w^2 \\ & +8y^4w^4 + 6y^3z^4w + 8y^2z^4w^2 + 4y^2w^6 + 6yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number (17) of size 672 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} \quad = \quad & x^8 + 4x^4y^3w + 12x^4y^2w^2 + 4x^4yw^3 + 14x^4z^4 + 3x^3y^4z + 4x^3y^3zw + 18x^3y^2zw^2 \\ & +4x^3yzw^3 + 3x^3zw^4 + 12x^2y^4z^2 + 24x^2y^3z^2w + 48x^2y^2z^2w^2 + 24x^2yz^2w^3 \\ & +12x^2z^2w^4 + 3xy^4z^3 + 4xy^3z^3w + 18xy^2z^3w^2 + 4xyz^3w^3 + 3xz^3w^4 + y^7w \\ & +7y^5w^3 + 4y^3z^4w + 7y^3w^5 + 12y^2z^4w^2 + 4yz^4w^3 + yw^7 + z^8 \end{aligned}$$

- Permutation orbit number (18) of size 56 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+4x^4y^3w+12x^4y^2w^2+4x^4yw^3+14x^4z^4+4x^3y^4z+24x^3y^2zw^2\\ &+4x^3zw^4+12x^2y^4z^2+24x^2y^3z^2w+48x^2y^2z^2w^2+24x^2yz^2w^3\\ &+12x^2z^2w^4+4xy^4z^3+24xy^2z^3w^2+4xz^3w^4+y^8+14y^4w^4+4y^3z^4w\\ &+12y^2z^4w^2+4yz^4w^3+z^8+w^8\end{aligned}$$

- Permutation orbit number (19) of size 168 and representative: $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+20x^4y^2w^2+14x^4z^4+16x^3y^3zw+16x^3yzw^3+20x^2y^4z^2+80x^2y^2z^2w^2\\ &+20x^2z^2w^4+16xy^3z^3w+16xyz^3w^3+4y^6w^2+8y^4w^4+20y^2z^4w^2+4y^2w^6\\ &+z^8\end{aligned}$$

Orbit number (9) **of size** 896 **with symmetrized weight enumerator**

swe $= x^8+24x^4y^4+14x^4z^4+16x^3y^4z+144x^2y^4z^2+16xy^4z^3+16y^8+24y^4z^4+z^8$

and orbit representative:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Number of permutation orbits= 7

- Permutation orbit number (1) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+3x^4y^4+6x^4y^3w+6x^4y^2w^2+6x^4yw^3+14x^4z^4+3x^4w^4+x^3y^4z\\ &+4x^3y^3zw+6x^3y^2zw^2+4x^3yzw^3+x^3zw^4+6x^2y^4z^2+36x^2y^3z^2w\\ &+60x^2y^2z^2w^2+36x^2yz^2w^3+6x^2z^2w^4+xy^4z^3+4xy^3z^3w+6xy^2z^3w^2\\ &+4xyz^3w^3+xz^3w^4+y^7w+7y^5w^3+3y^4z^4+6y^3z^4w+7y^3w^5+6y^2z^4w^2\\ &+6yz^4w^3+yw^7+z^8+3z^4w^4\end{aligned}$$

- Permutation orbit number (2) of size 28 and representative: $\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&1&1&1\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+6x^4y^4+12x^4y^2w^2+14x^4z^4+6x^4w^4+8x^3y^3zw+8x^3yzw^3+12x^2y^4z^2\\ &+120x^2y^2z^2w^2+12x^2z^2w^4+8xy^3z^3w+8xyz^3w^3+y^8+6y^4z^4+14y^4w^4\\ &+12y^2z^4w^2+z^8+6z^4w^4+w^8\end{aligned}$$

- Permutation orbit number (3) of size 112 and representative: $\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\1&1&1&0\end{pmatrix}$

with complete weight enumerator

$$\begin{aligned}\text{cwe} &= x^8+3x^4y^4+6x^4y^3w+6x^4y^2w^2+6x^4yw^3+14x^4z^4+3x^4w^4+2x^3y^4z\\ &+12x^3y^2zw^2+2x^3zw^4+6x^2y^4z^2+36x^2y^3z^2w+60x^2y^2z^2w^2\\ &+36x^2yz^2w^3+6x^2z^2w^4+2xy^4z^3+12xy^2z^3w^2+2xz^3w^4+4y^6w^2\\ &+3y^4z^4+8y^4w^4+6y^3z^4w+6y^2z^4w^2+4y^2w^6+6yz^4w^3+z^8+3z^4w^4\end{aligned}$$

- Permutation orbit number (4) of size 168 and representative: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + x^4y^4 + 8x^4y^3w + 6x^4y^2w^2 + 8x^4yw^3 + 14x^4z^4 + x^4w^4 + 8x^3y^3zw \\ &+8x^3yzw^3 + 6x^2y^4z^2 + 48x^2y^3z^2w + 36x^2y^2z^2w^2 + 48x^2yz^2w^3 \\ &+6x^2z^2w^4 + 8xy^3z^3w + 8xyz^3w^3 + 4y^6w^2 + y^4z^4 + 8y^4w^4 + 8y^3z^4w \\ &+6y^2z^4w^2 + 4y^2w^6 + 8yz^4w^3 + z^8 + z^4w^4 \end{aligned}$$

- Permutation orbit number (5) of size 112 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 6x^4y^3w + 12x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + 2x^3y^4z + 12x^3y^2zw^2 \\ &+2x^3zw^4 + 12x^2y^4z^2 + 36x^2y^3z^2w + 48x^2y^2z^2w^2 + 36x^2yz^2w^3 \\ &+12x^2z^2w^4 + 2xy^4z^3 + 12xy^2z^3w^2 + 2xz^3w^4 + 4y^6w^2 + 8y^4w^4 + 6y^3z^4w \\ &+12y^2z^4w^2 + 4y^2w^6 + 6yz^4w^3 + z^8 \end{aligned}$$

- Permutation orbit number (6) of size 224 and representative: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 6x^4y^3w + 12x^4y^2w^2 + 6x^4yw^3 + 14x^4z^4 + x^3y^4z + 4x^3y^3zw \\ &+6x^3y^2zw^2 + 4x^3yzw^3 + x^3zw^4 + 12x^2y^4z^2 + 36x^2y^3z^2w + 48x^2y^2z^2w^2 \\ &+36x^2yz^2w^3 + 12x^2z^2w^4 + xy^4z^3 + 4xy^3z^3w + 6xy^2z^3w^2 + 4xyz^3w^3 \\ &+xz^3w^4 + y^7w + 7y^5w^3 + 6y^3z^4w + 7y^3w^5 + 12y^2z^4w^2 + 6yz^4w^3 + yw^7 \\ &+z^8 \end{aligned}$$

- Permutation orbit number (7) of size 28 and representative: $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

with complete weight enumerator

$$\begin{aligned} \text{cwe} &= x^8 + 24x^4y^2w^2 + 14x^4z^4 + 8x^3y^3zw + 8x^3yzw^3 + 24x^2y^4z^2 + 96x^2y^2z^2w^2 \\ &+24x^2z^2w^4 + 8xy^3z^3w + 8xyz^3w^3 + y^8 + 14y^4w^4 + 24y^2z^4w^2 + z^8 + w^8 \end{aligned}$$

A.2 Programming Code for Computing Derivations of G with a Coefficent Set W

```
one:=One(ZmodnZ(2));
C1:=[[1,0,0,0, 0,1,1,1],
     [0,1,0,0, 1,0,1,1],
     [0,0,1,0, 1,1,0,1],
     [0,0,0,1, 1,1,1,0]];
Cdim:=Size(C1); Id:=IdentityMat(Cdim,1);
n:=[]; for i in[1..Cdim]do
 Add(n,[Id[i],ListWithIdenticalEntries(Cdim,0)*one]); od;
```

```
GenList:=[(1,2,5,7,6,8,4),(3,4,5)(6,8,7),(2,8,4,3,5,6,7)]; #<<---

DnGen:=[GenList[1]]; H:=Group(GenList[1]); G:=Group(DnGen);

MatGC1:=[]; oGMatC1:=[]; MatGC2Cos:=[]; oGMatC2Cos:=[]; obtlist:=[];
comp:=[];

a:=[]; for j in[1..(Size(GenList)-1)*Cdim^2]do
 Add( a,Indeterminate(GF(2),j) ); od;
dim:=(Size(GenList)-1)*Cdim^2;

ComputeB:=function(g) # action from the left M*v
 local t,l,i,j,M,k;
 if(g=())then return Id;
 elif(g in oGMatC2Cos)then return MatGC2Cos[Position(oGMatC2Cos,g)];
 else
  t:=Permuted([0,0,0,0,1,2,3,4],g);
  l:=[[],[],[],[]];
  for i in[1..4]do
   if(t[i]=1)then
    for k in[1..4]do if (k<>i)then Add(l[k],1);fi;od;
   elif(t[i]=2)then
    for k in[1..4]do if (k<>i)then Add(l[k],2);fi;od;
   elif(t[i]=3)then
    for k in[1..4]do if (k<>i)then Add(l[k],3);fi;od;
   elif(t[i]=4)then
    for k in[1..4]do if (k<>i)then Add(l[k],4);fi;od;
```

```
    fi;
   od;
   for i in[5..8]do
    if(t[i]=1)then Add(l[i-4],1);
    elif(t[i]=2)then Add(l[i-4],2);
    elif(t[i]=3)then Add(l[i-4],3);
    elif(t[i]=4)then Add(l[i-4],4);fi;
   od;
   M:=[[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0]];
   for i in[1..4]do
   for j in[1..4]do
    if (j in l[i])then M[i][j]:=1;fi;
   od;od;
   Add(oGMatC2Cos,g); Add(MatGC2Cos,M);
  fi;
  return M;
end;#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ ComputeMatOnC2Cosets

MofActionC1:=function(g)
 local t,i,A;
 if(g=())then return Id;
 elif(g in oGMatC1)then return MatGC1[Position(oGMatC1,g)];
 else
  t:=[]; A:=[];
  for i in[1..Cdim]do
   Add(t,Permuted(C1[i],g));
   Add(A,t[i]{[1..Cdim]});
```

```
  od;                                                        ### altered
  A:=InverseMatMod(A,2);
  Add(oGMatC1,g); Add(MatGC1,A);
 fi;
 return A;
end;#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ MofActionC1

GetMat:=function(v)
 local i,j,M;
 M:=NullMat(Cdim,Cdim,Integers);
 for i in[1..Size(v)]do
 for j in[1..Cdim]do
  M[j]:=M[j]+v[i][1][j]*v[i][2];
 od;od;
 return M;
end;#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ GetMat

delta:=function(g)
 local c,var, list, j,i,l,t1,t2,Mat;
 if(g in H)then return n;
 elif(g in obtlist)then
   return comp[Position(obtlist,g)];
 else list:=[];
  for t1 in G do if t1=() then continue;fi;      ### delta(x) or delta(y)
   if( ((g*t1 in obtlist)or(g*t1 in H))and((t1 in obtlist)or(t1 in H)) )then
    for j in[1..Cdim]do
     Add(list,[MofActionC1(t1^(-1))*delta(g*t1)[j][1],
```

```
            ComputeB(t1^(-1))*delta(g*t1)[j][2]]);
  Add(list,[MofActionC1(t1^(-1))*delta(t1)[j][1],
            ComputeB(t1^(-1))*delta(t1)[j][2]]);
 od;
 break;
elif( ((t1*g in obtlist)or(t1*g in H))and((t1 in obtlist)or(t1 in H)) )then
 for j in[1..Cdim]do
  Add(list,delta(t1*g)[j]);
  Add(list,[MofActionC1(g)*delta(t1)[j][1],ComputeB(g)*delta(t1)[j][2]]);
 od;
 break;
fi;

for t2 in G do if t2=() then continue;fi;      # delta(xy)
 if((t1*t2=g)and((t1 in obtlist)or(t1 in H))and((t2 in obtlist)or(t2 in H)))
  for j in[1..Cdim]do
   Add(list,delta(t2)[j]);
   Add(list,[MofActionC1(t2)*delta(t1)[j][1],ComputeB(t2)*delta(t1)[j][2]]);
  od;
 break;
 elif((t2*t1=g)and((t1 in obtlist)or(t1 in H))and((t2 in obtlist)or(t2 in H))
  for j in[1..Cdim]do
   Add(list,delta(t1)[j]);
   Add(list,[MofActionC1(t1)*delta(t2)[j][1],ComputeB(t1)*delta(t2)[j][2]]);
  od;
  break;
 fi;
```

```
  od;
  if(Size(list)>0)then break;fi;
 od;

 if(Size(list)>0)then
  Mat:=GetMat(list);
  Add(obtlist,g); list:=[];
  for j in[1..Cdim]do
   Add(list,[one*Id[j],Mat[j]]);od;
  Add(comp,list);
  return list;
 else

  for j in[dim+1..dim+(Cdim)^2]do
   Add(a, Indeterminate(GF(2),j) ); od;
  list:=[];
  for j in[1..Cdim]do
   l:=[];
   for i in[1..Cdim]do
    Add(l,a[dim+(j-1)*Cdim+i]);od;
   Add(list,[one*Id[j],l]);
  od;
  dim:=dim+Cdim^2;
  Add(obtlist,g); Add(comp,list);
  return list;
 fi;
fi;
```

```
end;#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ delta

OrdRelations:=function(R)
 local ORel, i1, i2;
 ORel:=ShallowCopy(ListWithIdenticalEntries(dim,a[1]-a[1]));
 for i1 in[1..Size(R)]do
  for i2 in[1..dim]do
   if(LeadingCoefficient(R[i1],i2)<>(a[1]-a[1]))then break;fi;
  od;
  ORel[i2]:=R[i1];
 od;
 return ORel;
end;#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ OrderRelations

ReduceRelations:=function(R)
 local mat,row,j1,j2,nR,p;
 mat:=[];
 for j1 in[1..Size(R)]do
  row:=[];
  for j2 in[1..dim]do
   Add(row,LeadingCoefficient(R[j1],j2));od;
  Add(mat,row);
 od;
 mat:=BaseMat(mat);

 nR:=[];
 for j1 in[1..Size(mat)]do
```

```
  p:=0*one;
  for j2 in[1..dim]do
   p:=p+mat[j1][j2]*a[j2];od;
  Add(nR,p);
 od;
 return nR;
end;#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ Reduce Relations

ReduceDelta:=function(d,R,Dng,g,t)
 # d: delta to be reduced
 # R: ordered relations
 # Dng: generators of earlier group to reduce delta from fixed points
 # g: the aut of delta d, used to define the delta for the fixed point set.
 # t: used to make sure diff var's are used for diff gen's when redefining
 local j1,j2,relcount,dSp,fRel,fSp,e,j3,r,mm,de,v, f,l;

 for j1 in[1..Size(d)]do      # Here write delta interms of free variables
 for j2 in[1..Size(d[1][2])]do
  relcount:=1;
  while((d[j1][2][j2]<>(a[1]-a[1]))and(relcount<=dim))do
   d[j1][2][j2]:=d[j1][2][j2]
              +LeadingCoefficient(d[j1][2][j2],relcount)*R[relcount];
   relcount:=relcount+1;
  od;
 od;od;

# get the delta resulting from fixed points
```

```
fSp:=[]; v:=[NullMat(Cdim,Cdim,GF(2))*a[1]];
e:=[];
for j1 in[1..Cdim]do
 l:=[];
 for j2 in[1..Cdim]do
  Add(l,ShallowCopy(a[Cdim*(j1-1)+j2]));od;
 Add(e,[one*Id[j1],l]);
od;

fRel:=[];
for f in Dng do
 r:=[];
 for j1 in[1..Cdim]do
  Add(r,[MofActionC1(f)*e[j1][1],ComputeB(f)*e[j1][2]]);
  Add(r,e[j1]);
 od;
 mm:=GetMat(r);
 for j1 in[1..Size(mm)]do
 for j2 in[1..Size(mm[1])]do
  if(not(mm[j1][j2] in fRel)and(mm[j1][j2]<>(a[1]-a[1])))then
   Add(fRel,ShallowCopy(mm[j1][j2]));fi;
 od;od;
od;
if(Size(fRel)>0)then
 fRel:=ReduceRelations(fRel);
 fRel:=OrdRelations(fRel);
 for j1 in[1..Size(e)]do
```

```
for j2 in[1..Size(e[1][2])]do
 relcount:=1;
 while((e[j1][2][j2]<>(a[1]-a[1])) and(relcount<=dim))do
  e[j1][2][j2]:=e[j1][2][j2]
              +LeadingCoefficient(e[j1][2][j2],relcount)*fRel[relcount];
  relcount:=relcount+1;
 od;
od;od;

de:=[];
for j1 in[1..Cdim]do
 Add(de,[MofActionC1(g)*e[j1][1],ComputeB(g)*e[j1][2]]);
 Add(de,e[j1]);
od;
mm:=GetMat(de); de:=[];
for j1 in[1..Cdim]do
 Add(de,[Id[j1],mm[j1]]);od;

for j3 in[1..dim]do
 e:=ShallowCopy(n);
 for j1 in[1..Size(de)]do
 for j2 in[1..Size(de[1][2])]do
  e[j1][2][j2]:=LeadingCoefficient(de[j1][2][j2],j3);
 od;od;
 mm:=ShallowCopy(GetMat(e));
 if not(mm in v)then
  Add(fSp,mm);
```

```
    for j1 in[1..Size(v)]do Add(v,v[j1]+mm);od;
   fi;
  od;
 fi;

#  Here get the subspace spanning delta value
 if(GetMat(d)<>NullMat(Cdim,Cdim,Integers)*(a[1]-a[1]))then
  dSp:=[];
  for j3 in[1..dim]do
   e:=ShallowCopy(n);
   for j1 in[1..Size(d)]do
   for j2 in[1..Size(d[1][2])]do
    e[j1][2][j2]:=LeadingCoefficient(d[j1][2][j2],j3);
   od;od;
   mm:=ShallowCopy(GetMat(e));
   if not(mm in v)then
    Add(dSp,mm);
    for j1 in[1..Size(v)]do Add(v,v[j1]+mm);od;
   fi;
  od;

  mm:=ShallowCopy(NullMat(Cdim,Cdim,GF(2)));
  for j3 in[1..Size(dSp)]do
   mm:=mm+a[t+j3]*dSp[j3];od;

  e:=[];
  for j3 in[1..Cdim]do
```

```
   Add(e,[Id[j3],mm[j3]]);od;
  return e;
 else return d;fi;
end;#~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ Reduce Delta

DefSet:=[n]; newgen:=ShallowCopy(DnGen);

for k in[2..Size(GenList)]do
 H1:=Group(DnGen);
 Add(newgen,GenList[k]);
 G:=Group(newgen);
 var:=[];
 for j in[1..Cdim]do
  l:=[];
  for i in[1..Cdim]do
   Add(l,a[(k-2)*Cdim^2+(j-1)*Cdim+i]);od;
  Add(var,[one*Id[j],l]);
 od;

 dh:=[];
 for j in[1..Cdim]do
  Add(dh,[MofActionC1(GenList[k])*var[j][1],ComputeB(GenList[k])*var[j][2]]);
  Add(dh,var[j]);
 od;
 M:=GetMat(dh); dh:=[];
 for i in[1..Cdim]do
  Add(dh,[one*Id[i],M[i]]);od;
```

```
 Add(obtlist,GenList[k]);
 Add(comp,ShallowCopy(dh));

 Rel:=[];
 for f1 in G do if f1=() then continue;fi;
 for f2 in G do if f2=() or (f1 in H1 and f2 in H1)then continue;fi;
  list:=[];
  for i1 in[1..Cdim]do
   Add(list,delta(f1*f2)[i1]);
   Add(list,[MofActionC1(f2)*delta(f1)[i1][1],ComputeB(f2)*delta(f1)[i1][2]]);
   Add(list,delta(f2)[i1]);
  od;

  rm:=GetMat(list);
  for i1 in[1..Size(rm)]do
  for i2 in[1..Size(rm[1])]do
   if (not(rm[i1][i2] in Rel) and rm[i1][i2]<>one*0 and rm[i1][i2]<>(a[1]-a[1]))
    then Add(Rel,rm[i1][i2]);fi;
  od;od;
 od; od;
 Print("\n",k," Done with Relations, no= ",Size(Rel),"\n");

 Rel1:=ReduceRelations(Rel);

Print("\nDone with the rel bases, no=",Size(Rel1),"\n",Rel1);
 Rel2:=OrdRelations(Rel1);
 Add(DefSet,ShallowCopy(ReduceDelta(dh,Rel2,DnGen,GenList[k],(k-2)*Cdim^2)));
```

```
 for k2 in[2..k-1]do
  DefSet[k2]:=
    ShallowCopy(ReduceDelta(DefSet[k2],Rel2,[],GenList[k2],(k2-2)*Cdim^2));
 od;

 comp:=[]; obtlist:=[];
 for k2 in[2..Size(DefSet)]do
  Add(comp,DefSet[k2]);
  Add(obtlist,GenList[k2]);
 od;

 DnGen:=ShallowCopy(newgen);
Print("Deltas found",DefSet,"\n");
od;

PrintTo("derHTd.output","\n number of variable",dim,
        "\n number of Relations obtained=",Size(Rel),
        "\n Size of a bases set is",Size(Rel1),
        "\n Bases Relations: ",Rel1,
        "\n Delta of generators: ",DefSet);
```

A.3 Different Binary Codes with same Weight Enumerators but Different Corresponding $\mathbb{Z}_4$-codes

For a word $w \in \mathbf{C}$ define the even subword w_e to be the word consisting of even positions of w. So w_e has length equal to $k = n - \text{wt}(w \bmod 2)$.

Proposition A.1 *The set of subwords $(w + d(C_2))_e$ of $(w + d(C_2))$ form a subspace of $2\mathbb{Z}_4^m$.*

Proof C_2 is a subspace of $\mathbb{Z}_2^n$. So $d(C_2)$ form a subspace of $\mathbb{Z}_2^n$. Also, for $0 \leq m \leq n$, any set of subwords of fixed m positions in $d(C_2)$ form a subspace of $2\mathbb{Z}_4^m$ since we are dealing with linear codes. Here we might get repeated copies of the subspace. But $(w + d(C_2))_e$ is a set of subwords of $d(C_2)$ with the m positions chosen according to w. Hence, result follows.

There are two doubly even self dual codes of length 16:

- $C = H + H$ where H is the Extended hamming code of length 8. Then

$$\begin{aligned} \underset{C}{\text{we}} &= (x^8 + 14x^4y^4 + z^8)^2 \\ &= x^{16} + y^{16} + 28x^4y^{12} + 198x^8y^8 + 28x^{12}y^4 \end{aligned}$$

- C' spanned by all words of the form:

$$\begin{array}{ccccccccccc} 1 & & \ldots & & 8 & 9 & & \ldots & & & 16 \\ & 1 & & 1 & & & 1 & & 1 & & \\ & \uparrow & & \uparrow & & & \uparrow & & \uparrow & \text{0 elsewhere} & \\ & i & & j & & & i+8 & & j+8 & & \end{array}$$

And 1 $\ldots$ 1 0 $\ldots$ 0

Here all generators have weight 4 and any two are orthogonal. Here also,

$$\underset{C'}{\mathrm{we}} = (x^8 + 14x^4y^4 + z^8)^2$$

Let us compute possible Symmetrized Weight Enumerators for codes in $\mathcal{C}(C, C)$ and in $\mathcal{C}(C', C')$. We start with the common details we get from the common weight enumerator of C and C'.

All Symmetrized Weight Enumerators in the two classes have the form:

$$\underset{\mathbf{C}}{\mathrm{swe}} = we_C(x,z) + 256y^{16} + y^{12}\sum_{i=1}^{28} P_{4,j}(x,z) + y^8\sum_{i=1}^{198} P_{8,j}(x,z) + y^4\sum_{i=1}^{28} P_{12,j}(x,z)$$

where $P_{k,j}$ might be different for the two classes. From earlier discussion, we get same $P_{4,j}$ for the two classes. These are listed below:

$$\begin{aligned} P_{4,0}(x,z) &= \frac{2^8}{2^3}\binom{4}{0}(x^4+z^4) + \frac{2^8}{2^3}\binom{4}{2}x^2z^2 \\ &= 32(x^4+z^4) + 192x^2z^2 \\ P_{4,1}(x,z) &= \frac{2^8}{2^3}\binom{4}{1}(x^3z+xz^3) \\ &= 128(x^3z+xz^3) \end{aligned}$$

$P_{8,0}(x, z)$ is of the form:

$$P_{8,0}(x,z) = 2a_0(x^8+z^8) + 2a_1(x^6z^2+x^2z^6) + 4a_2x^4z^4 \qquad \text{(A.1)}$$

Here $\{S_i\}_e$ for the different corresponding words would have one of the following options:

- 2 copies of the biggest 2^7 even subspace.
- 4 copies of a 2^6 subspace.
- 8 copies of a 2^5 subspace.

- 16 copies of a 2^4 subspace.

Then other $P_{8,j}$'s depend on the subspace we got.

For $P_{12,j}(x,z)$ the set of even subwords $\{S_i\}_e$ will form 2 copies of a 2^7 subspace (cannot be the biggest even, since this has cardinality 2^{11}, more than what we have, 2^8). So this polynomial will be of the form:

$$P_{12,0}(x,z) = 2(x^{12}+z^{12}) + 2a_1(x^{10}z^2+x^2z^{10}) + 2a_2(x^8z^4+x^4z^8) + 4a_3x^6z^6$$

Now lets see what computation in the above cases give.

- In the first case where $C_1 = C_2 = H + H$, we have:

  A word of weight 8 will have its weight distributed in one of the forms $(8-0)$, $(0-8)$ or $(4-4)$. For the first 2 forms, we will have:

$$\begin{aligned} P_{8,0a}(x,z) &= 16(x^8+14x^4z^4+z^8) \\ &= 16(x^8+z^8)+224x^4z^4 \end{aligned}$$

  which is the forth option in the list above and so

$$\begin{aligned} P_{8,1a}(x,z) &= 16(x^7z+xz^7)+112(x^5z^3+x^3z^5) \\ P_{8,2a}(x,z) &= 64(x^6z^2+x^2z^6)+128x^4z^4 \end{aligned}$$

  For words of the form $(4-4)$, we will have:

$$\begin{aligned} P_{8,0b}(x,z) &= [2(x^4+z^4)+12x^2z^2]\times[2(x^4+z^4)+12x^2z^2] \\ &= 4(x^8+z^8)+48(x^6z^2+x^2z^6)+152x^4z^4 \end{aligned}$$

  which is the second option in the list above and so

$$\begin{aligned} P_{8,1b}(x,z) &= [2(x^4+z^4)+12x^2z^2]\times[8(x^3z+xz^3)] \\ &= 16(x^7z+xz^7)+112(x^5z^3+x^3z^5) = P_{8,1a} \\ P_{8,2b}(x,z) &= [8(x^3z+xz^3)\times[8(x^3z+xz^3) \\ &= 64(x^6z^2+x^2z^6)+128x^4z^4 = P_{8,2a} \end{aligned}$$

A word of weight 4 would have weight distribution $(0-4)$ or $(4-0)$ giving:

$$\begin{aligned}
P_{12,0}(x,z) &= [2(x^4+z^4)+12x^2z^2]\times[x^8+14x^4z^4+z^8] \\
&= 2(x^{12}+z^{12})+12(x^{10}z^2+x^2z^{10})+30(x^8z^4+x^4z^8)+168x^6z^6 \\
P_{12,1a}(x,z) &= [8(x^3z+xz^3)\times[x^8+14x^4z^4+z^8] \\
&= 8(x^{11}z+xz^{11})+8(x^9z^3+x^3z^9)+112(x^7z^5+x^5z^7) \\
P_{12,1b}(x,z) &= [2(x^4+z^4)+12x^2z^2]\times[(x^7z+xz^7)+7(x^5z^3+x^3z^5)] \\
&= 2(x^{11}z+xz^{11})+26(x^9z^3+x^3z^9)+100(x^7z^5+x^5z^7) \\
P_{12,2a}(x,z) &= [8(x^3z+xz^3)]\times[(x^7z+xz^7)+7(x^5z^3+x^3z^5)] \\
&= 8(x^{10}z^2+x^2z^{10})+64(x^8z^4+x^4z^8)+112x^6z^6 \\
P_{12,2b}(x,z) &= [2(x^4+z^4)+12x^2z^2]\times[4(x^6z^2+x^2z^6)+8x^4z^4] \\
&= 8(x^{10}z^2+x^2z^{10})+64(x^8z^4+x^4z^8)+112x^6z^6 = P_{12,2a} \\
P_{12,3}(x,z) &= [8(x^3z+xz^3)\times[4(x^6z^2+x^2z^6)+8x^4z^4] \\
&= 32(x^9z^3+x^3z^9)+96(x^7z^5+x^5z^7)
\end{aligned}$$

- In the second case where $C_1 = C_2 = C'$ as described above, we have:

  A word of weight 8 will have the weight distribution $(8-0)$, $(0-8)$ or $(4-4)$. In the first 2 forms, we will have:

$$\begin{aligned}
P_{8,0a}(x,z) &= 8(x^8+z^8)+32(x^6z^2+x^2z^6)+176x^4z^4 \\
P_{8,1a}(x,z) &= 64(x^6z^2+x^2z^6)+128x^4z^4
\end{aligned}$$

  In words of the form $(4-4)$ we have:

$$\begin{aligned}
P_{8,0b}(x,z) &= 2(x^8+z^8)+56(x^6z^2+x^2z^6)+140x^4z^4 \\
P_{8,1b}(x,z) &= 16(x^7z+xz^7)+112(x^5z^3+x^3z^5)
\end{aligned}$$

For words of weight 4, we have:

$$
\begin{aligned}
P_{12,0}(x,z) &= 2(x^{12}+z^{12})+12(x^{10}z^2+x^2z^{10})+30(x^8z^4+x^4z^8)+168x^6z^6\\
P_{12,1}(x,z) &= 4(x^{11}z+xz^{11})+20(x^9z^3+x^3z^9)+104(x^7z^5+x^5z^7)\\
P_{12,2}(x,z) &= 8(x^{10}z^2+x^2z^{10})+64(x^8z^4+x^4z^8)+112x^6z^6\\
P_{12,3}(x,z) &= 32(x^9z^3+x^3z^9)+96(x^7z^5+x^5z^7)
\end{aligned}
$$

A.4 Describing Action of an Automorphism Group on $C_1^* \otimes \mathbb{Z}_2^n/C_2$

Set of matrices defining the set of codes, $W \simeq A = C_1^* \otimes \mathbb{Z}_2^n/C_2$. An $X \in W$ can be represented by the sum $\sum\{X\}_{ij} f_i \otimes w_j$. Where $\{f_i\}$ and $\{w_j\}$ are basis of C_1^* and $\mathbb{Z}_2^n/C_2$ respectively as described in Section (2.1).

When G act on two spaces V_1 and V_2, we can define an action on $V_1 \otimes V_2$ by:

$$g(V_1 \otimes V_2) = gV_1 \otimes gV_2$$

Let V be a vector space with dimension k. Then, if G acts linearly on V, we can describe G's action on V as a matrix multiplication. That is $gv = A_g v$ where A_g is an $k \times k$ matrix illustrating g's action. For instance, if $\{v_1, \ldots, v_k\}$ is a basis for V and $g : v_i \mapsto \sum_j l_{ij} v_j$ then $\{A\}_{ij} = l_{ji}$.

Proposition A.2 *If A_g is g's matrix of action on V then ${A_g}^{-1}$ is g's matrix of action on the dual space of V. This is $V^* = Hom(V, \mathbb{Z}_2)$.*

Proof Let $\{v_1, \ldots, v_k\}$ be a basis for the vector space V. Let $\{f_1, \ldots, f_k\}$ be a basis for V^*, where $f_i(v_j) = 1$ if $i = j$ and 0 otherwise. Let $v_i^g = \sum_j l_{ij} v_j$.

Then

$$(v_i \mapsto t_i)^g \Rightarrow \sum_j l_{ij} v_j \mapsto t_i$$

where $t_i = 0$ or 1. So

$$\begin{pmatrix} l_{11} & \dots & l_{1k} \\ . & & . \\ . & & . \\ l_{k1} & \dots & l_{kk} \end{pmatrix} \begin{pmatrix} v_1 \\ . \\ . \\ v_k \end{pmatrix} \mapsto \begin{pmatrix} t_1 \\ . \\ . \\ t_k \end{pmatrix}$$

Let B be a $k \times k$ matrix with entries $\{B\}_{ij} = l_{ij}$ then

$$\begin{pmatrix} v_1 \\ . \\ . \\ v_k \end{pmatrix} \mapsto B^{-1} \begin{pmatrix} t_1 \\ . \\ . \\ t_k \end{pmatrix}$$

If $\{B^{-1}\}_{ij} = b_{ij}$ then we have

$$\begin{pmatrix} v_1 \\ . \\ . \\ v_k \end{pmatrix} \mapsto \begin{pmatrix} b_{11} & \dots & b_{1k} \\ . & & . \\ . & & . \\ b_{k1} & \dots & b_{kk} \end{pmatrix} \begin{pmatrix} t_1 \\ . \\ . \\ t_k \end{pmatrix}$$

and $v_i \mapsto \sum_j b_{ij} t_j$. That is $f_i \mapsto \sum_j b_{ij} f_j$. Giving matrix, M, of linear action on V^* defined by, $\{M\}_{ij} = b_{ji}$. So $M = (B^{-1})^T$. But $B = A^T$. Since $(M^T)^{-1} = (M^{-1})^T$, we have $M = A^{-1}$.

Action on $\mathbb{Z}_2^n/C_2$ is just the normal permutation action on $\mathbb{Z}_2^n$ taking the result modulo C_2. With this we can compute matrices of action of $g \in G$ on C_1^*, say A, and on $\mathbb{Z}_2^n/C_2$, say B. If we take $\{f_1 \otimes v_1, \dots, f_1 \otimes v_{k_3}, f_2 \otimes v_1, \dots, f_{k_1} \otimes v_{k_3}\}$ to be a basis for $C_1^* \otimes \mathbb{Z}_2^n/C_2$ (here $k_3 = n - (k_1 + k_2)$) we have $g : f_i \otimes v_j \mapsto A_g f_i \otimes B_g v_j$.

A.5 Fixed points on $C_1^* \otimes \mathbb{Z}_2^n / C_2$

As mentioned above, a linear action of g on a vector space V can be expressed as a matrix multiplication. For $g \in G$, let the matrix representation for g's action on V be A_g, then $gv = A_g v$ for $v \in V$. Matrix representation for g^{-1} is then $A_{g^{-1}} = (A_g)^{-1}$. If $v \in V$ is a fixed point of g then $gv = A_g v = v$. So space of fixed points of g's action is precisely the space spanned by eigenvectors corresponding to the eigenvalue 1. To simplify computing fixed points note the following:

- If α is an eigenvalue of a matrix A then α^{-1} is an eigenvalue for A^{-1} since
$$A_g v = \alpha v \Leftrightarrow (A_g)^{-1} v = \alpha^{-1} v$$

- Here we are working with vector spaces over $\mathbb{Z}_2$. So all calculations will be in $\mathbb{Z}_2$. If t is a root of the irreducible polynomial:
$$f(x) = x^s + a_{s-1}x^{s-1} + \ldots + a_1 x + 1$$
then t^{-1} is a root for
$$\tilde{f}(x) = x^s + a_1 x^{s-1} + \ldots + a_{s-1} x + 1$$
We get this by plugging t in $f(x)$, giving us $f(t) = 0$ then multiplying the result by t^{-s}.

- If r is a positive integer not divisible by the characteristic of the field (this is not applicable for $char = 0$) then $x^r - 1$ has no repeated roots. In our case $char = 2$. If g has odd order r, its minimal polynomial divides $x^r - 1$. Hence, g is diagonalizable over some extension field. Here the dimension of the fixed point space still equals multiplicity of 1 as an eigenvalue of g.

Now, Let g act on vector spaces V_1 and V_2 and let the action be diagonalizable on both. Then there is a basis $\{f_i\}$ of V_1 and $\{e_i\}$ of V_2 such that $g : f_i \mapsto \mu_i f_i$ and $g : e_i \mapsto \lambda_i e_i$. So $g : f_i \otimes e_j \mapsto \mu_i \lambda_j f_i \otimes e_j$. With this, action of g on $V_1 \otimes V_2$ is also diagonalizable. Dimension of the fixed point space equals number of pairs $\{i, j\}$ such that $\mu_i \lambda_j = 1$. That is, dimension of the fixed point space is

$$\sum_{\mu,\lambda,\ \mu.\lambda=1} mult_{V_1}(\mu).mult_{V_2}(\lambda)$$

If F_1 and F_2 are characteristic polynomials of g on V_1 and V_2 respectively. Each irreducible of degree s has a contribution of $s.mult_{F_1}(f).mult_{F_2}(\tilde{f})$ to the dimension of the fixed point space in the extension field and $mult_{F_1}(f).mult_{F_2}(\tilde{f})$ in the ground field.

In our case, we are trying to compute fixed points of an automorphism group G acting on a vector space over $\mathbb{Z}_2$, this is $A = C_1^* \otimes \mathbb{Z}_2^n / C_2$. A way of computing fixed points of an automorphism g on A would be to check if g's action is diagonalizable on C_1^* and $\mathbb{Z}_2^n/C_2$. If g has an odd order for example, we know it is. If g is diagonalizable on C_1^* and on $\mathbb{Z}_2^n/C_2$ then we proceed as follows:

- Compute matrix of action on C_1, call it X.
- Find characteristic equation of A, say F. Then characteristic equation of A^{-1} is $F_1 = \tilde{F}$.
- Factorize F_1. The roots are the eigenvalues on C_1^*.
- Compute matrix of action on $\mathbb{Z}_2^n/C_2$. Call it B.
- Find characteristic equation of B, say F_2 and factorize it.

- If f is a factor of F_1 and $\tilde{f}$ is a factor F_2. This factor contributes to the dimension of the fixed point space by $mult_{F_1}(f).mult_{F_2}(\tilde{f})$ giving us a dimension for the fixed point space equal to $\sum_{factors\ f} mult_{F_1}(f).mult_{F_2}(\tilde{f})$.

- If $\{\lambda_1, \ldots, \lambda_s\}$ are zeros of f with corresponding basis eigenvectors $\{v_{\lambda_1}, \ldots, v_{\lambda_s}\}$ and $\{\lambda_1^{-1}, \ldots, \lambda_s^{-1}\}$ are zeros of $\tilde{f}$ with corresponding basis eigenvectors $\{\bar{v}_{\lambda_1}, \ldots, \bar{v}_{\lambda_s}\}$ then $\sum_i v_{\lambda_i} \otimes \bar{v}_{\lambda_i}$ is a fixed element in $V_1 \otimes V_2$.

If g is not diagonalizable on C_1^* or on $\mathbb{Z}_2^n/C_2$ then write down the matrix of action on W and check the dimension of its eigenspace corresponding to eigenvalue 1. If A_g is a permutation matrix (a matrix in which every row and column have exactly one none zero entry equal to 1), then it has only 1 as an eigenvalue so its corresponding eigenspace is nonempty.

Bibliography

[1] A. R. Hammons, Jr. P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, The $\mathbb{Z}_4$-linearity of kerdock, Preparata, Goethals and Related codes, *IEEE Trans. Inform. Theory* **40** (1994), 301-319.

[2] Raymond Hill, *A First Course in Coding Theory*, Oxford Applied Mathematics and Computing Science Series, Oxford University Press Inc., New York, 1986.

[3] J.I Hall, Notes on Coding Theory, Michigan State University, East Lansing, USA, 3 January 2003,
`http://www.mth.msu.edu/~ jhall/classes/codenotes/coding-notes.html`

[4] Melissa Kanemasu, Golay Codes, *MIT Undergraduate Journel of Mathematics*, 95-99.

[5] Joseph J. Rotman, *An Introduction to Homological Algebra*, Academic Press Limited, London, 1979.

[6] Thomas W.Hungerford, *Algebra*, Springer, New York, 1979.

[7] Serge Lange, *Algebra, 3rd ed*, Addison-Wesley, New York, 1965.

[8] Peter Cameron, Finite geometry and coding theory, School of Mathematical Sciences, Queen Mary and Westfield College, Socrates Intensive Programm Finite Geometries and Their Automorphisms, Potenza, Italy, June 1999.

[9] E.S. Lander, *Symmetric Designs: An Algebraic Approach*, Cambridge University Press, New York, 1983.