Lattice path enumeration on restricted domains

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Details of collaboration and publications:

The contents of Chapter 2 are primarily taken from the paper *A Bijection on Bilateral Dyck Paths*, Australasian Journal of Combinatorics, **59**(1) (2014), 72-80, which is a joint paper with my PhD supervisor, Prof. Thomas Prellberg. For this paper, the initial idea, for the bijection on Dyck paths, was Prellberg's idea and formulation. It was my observations which allowed us to extend the bijection to bilateral Dyck paths.

The contents of Chapter 3 are largely taken from the paper *On the Number of Walks in a Triangular Domain*, Electronic Journal of Combinatorics, **22**(1) (2015), paper 64, which is also a joint paper with Prof. Prellberg. The initial idea to study the triangle model was Prellberg's, after noticing patterns in some preliminary calculations on Maple. Through a joint effort, we were able to derive the explicit generating function formula for walks with no weights on endpoints, and create the formal Kernel method proof. The idea to specify to walks starting in a corner, and the observation of the links to Motzkin paths, were mine. Finally, it was my observation to use the same method to derive the general result for the line model.

Abstract

This thesis concerns the enumeration and structural properties of lattice paths.

The study of Dyck paths and their characteristics is a classical combinatorial subject. In particular, it is well-known that many of their characteristics are counted by the Narayana numbers. We begin by presenting an explicit bijection between Dyck paths with two such characteristics, peaks and up-steps at odd height, and extend this bijection to bilateral Dyck paths.

We then move on to an enumeration problem in which we utilise the Kernel method, which is a cutting-edge tool in algebraic combinatorics. However, while it has proven extremely useful for finding generating functions when used with one or two catalytic variables, there have been few examples where a Kernel method has been successfully used in a general multivariate setting. Here we provide one such example.

We consider walks on a triangular domain that is a subset of the triangular lattice. We then specialise this by dividing the lattice into two directed sublattices with different weights. Our central result on this model is an explicit formula for the generating function of walks starting at a fixed point in this domain and

ending anywhere within the domain. We derive this via use of the algebraic Kernel method with three catalytic variables.

Intriguingly, the specialisation of this formula to walks starting in a fixed corner of the triangle shows that these are equinumerous to bicoloured Motzkin paths, and bicoloured three-candidate Ballot paths, in a strip of finite height. We complete this thesis by providing bijective proofs for small cases of this result.

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Acknowledgements

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This thesis is in no small part down to the love and support of my parents, and sister Sarah, without whom I would not be where I am today. I must also thank David for being a constant source of love, support and humour. Finally, I would like to thank my friends, both those from outside work and those who I have met during my time at Queen Mary; Aylin, Becky, Chloe, Emily, Jack, Joe, Liron, Louise, Mark, Matt, Phil, Rodrigo, Shankar, Tom, Zak and so many more; for all of the late nights, laughs, and intense discussions. Your friendship has made this process a lot of fun.

Introduction

Lattice walks have been extensively studied in the field of enumerative combinatorics, and more widely both within classical combinatorics and outside, in areas such as algebraic combinatorics, asymptotic analysis, and the behaviour of polymers [18, 29, 31].

Counting lattice walks in unrestricted domains is a relatively simple problem, that can be easily solved using elementary combinatorial methods. However, once restrictions are placed on the lattice, this is no longer the case.

Recently there has been significant development of the Kernel method [1, 2, 27], a technique first introduced in [15]. This method can be used to solve linear combinatorial functional equations in so-called catalytic variables, in particular to find the generating functions of lattice paths.

While the Kernel method has been reasonably well understood when only one or two catalytic variables are involved, once there are more the situation is far from clear. There are indications that the structure of the solution, such as whether the generating function is algebraic or even differentiably finite, depends on the group of symmetries of the kernel of the functional equation [4, 20]. Only recently has there been some progress using a multivariate Kernel method in a special case [3].

Based on the Kernel method, it is possible to derive generating functions for counting problems in previously inaccessible situations. As an example, the exact solution of a lattice model of partially directed walks in a wedge has only been possible using an iterative version of the Kernel method [14], leading to a generating function that is not differentiably finite, as its singularities accumulate at limit points. This example also shows that as a by-product of enumerative combinatorics, deep combinatorial insight into connections between seemingly unrelated systems can be uncovered, leading to spin-off research in bijective combinatorics [26, 28].

Dyck paths are one of the most well-studied random walk models, and have links to many other combinatorial objects. In particular, they are enumerated by the Catalan numbers, and so are in bijection with many other Catalan objects [32].

It is an established fact that many patterns in Dyck paths, such as peaks, valleys and double ascents, are enumerated by the Narayana numbers $N(n, k)$, a refinement of the Catalan numbers [16, 17, 33]. These are most easily proven by generating function techniques [8]. Equinumeracy then follows from showing that different counting problems have the same generating function. However, many of these results can be proven bijectively, which clearly provides more structural insight than a proof using generating function techniques. As a consequence, there are many bijections on Dyck paths in the literature [19, 25, 30, 32].

In this thesis we take a dual approach to tackling problems on lattice paths, splitting our focus between utilising the Kernel method to gain enumerative results, and bijections to gain understanding of structure.

In Chapter 1 we introduce all the key definitions necessary for this thesis, including Dyck paths, Motzkin paths and the triangular domain. We also give an overview of the Kernel method.

In Chapter 2, taken from [21], we present a bijection between two characteristics of Dyck paths. It is known that both the number of Dyck paths with $2n$ steps and k peaks, and the number of Dyck paths with $2n$ steps and k up-steps at odd height, follow the Narayana distribution; we give a bijection which explicitly illustrates this equinumeracy. Moreover, we extend this bijection to bilateral Dyck paths.

In Chapter 3, taken from [22], we use a multivariate Kernel method to give a solution to an enumerative lattice path problem that is expressed in terms of a functional equation with three catalytic variables. We do so by exploiting the high symmetry of the Kernel. The resulting generating function solution allows us to prove a result linking walks on a triangular domain to Motzkin paths.

The equinumeracy result in Chapter 3 naturally leads to the question of finding a bijective proof. In Chapter 4 we examine this problem, giving solutions for the three smallest cases.

Chapter 1

Background

1.1 Dyck paths

In Chapter 2, our results are on the well-established notions of Dyck paths and bilateral Dyck paths, so we begin by defining these.

Definition 1.1. *A bilateral Dyck path is a directed walk on* $\mathbb{Z}^2$ *starting at* $(0,0)$ *in the* (x,y)*-plane and ending on the line* $y = 0$*, which has steps in the* $(1,1)$ *(up-step) and* $(1,-1)$ *(down-step) directions.*

In the literature these are also referred to as free Dyck paths [7] or Grand-Dyck paths [19]. Let $\mathcal{B}$ denote the set of all bilateral Dyck paths.

Definition 1.2. *A Dyck path is a bilateral Dyck path which has no vertices with negative* y*-coordinates.*

We will use $\mathcal{D}$ to denote the set of all Dyck paths.

We will also need the following definition.

Definition 1.3. *A negative Dyck path is a bilateral Dyck path of nonzero length which has no vertices with positive y-coordinates.*

We now move on to define some characteristics of bilateral Dyck paths (and therefore also of Dyck paths and negative Dyck paths). Given a path $\pi \in \mathcal{B}$, we define the *semilength* $n(\pi)$ to be half the number of its steps. We say that an up-step is at *height* j if it starts at a vertex $(i-1, j-1)$ and ends at a vertex (i, j); it is at odd height if and only if j is odd. A down-step is at *height* j if it starts at a vertex (i, j) and ends at a vertex $(i+1, j-1)$; it is at odd height if and only if j is odd. Thus the height of a step is the larger y-coordinate of the two vertices appending it. Therefore the first and last steps in any Dyck path are at height 1, and an up-step and its matching down-step have the same height. This definition of height is consistent with that in [23].

We define a *peak* as an up-step followed immediately by a down-step, and the *height* of a peak as the height of the steps which form it, or equivalently as the y-coordinate of the vertex common to both steps of the peak. We define a *valley* to be a down-step followed immediately by an up-step. We define a *contact* as a down-step at height 1 or an up-step at height 0, and is used to denote a vertex at which the path touches the x-axis. We define a *crossing* to be either a down-step at height 1 followed immediately by a down-step at height 0, or an up-step at height 0 followed immediately by an up-step at height 1. We define a *prime* Dyck path to be a Dyck path with exactly one contact.

In Chapter 2, we use some conventions to concisely describe our maps, which we now introduce. Let $\mathcal{W}$ be the set of words with the symbol set $\{U, D\}$. A *bilateral Dyck word* of semilength n is an element of $\mathcal{W}$ such that the letters U and D each appear n times. A *Dyck word* of semilength n is a bilateral Dyck word such that no initial segment has more Ds than Us. There is an obvious bijection between paths and words, mapping up-steps to Us and down-steps to Ds, and we will use words and paths interchangeably in Chapter 2. For example, we say that a Dyck word is prime if it corresponds to a prime Dyck path.

We conclude this section by defining the types of numbers that count Dyck paths and some of their characteristics respectively.

Definition 1.4. *The Catalan numbers C_n are defined by*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Definition 1.5. *The Narayana numbers $N(n, k)$ are defined by*

$$N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

1.2 Motzkin and Ballot paths

We now move on to the concepts needed for Chapters 3 and 4, and begin with the also well-established notions of a Motzkin path, and that of a bicoloured Motzkin path.

Definition 1.6. *A Motzkin path is a directed walk on $\mathbb{Z}^2$ starting at $(0,0)$ in the (x,y)-plane and ending on the line $y=0$, which has steps in the $(1,1)$ (up-step), $(1,-1)$ (down-step) and $(1,0)$ (horizontal step) directions, and has no vertices with negative y-coordinates.*

Definition 1.7. *A bicoloured Motzkin path is a Motzkin path where each of the steps can be one of two colours.*

A Motzkin path can alternatively be viewed as a Dyck path with horizontal steps inserted. Note that in our second definition, all steps are bicoloured, not just the horizontal steps as in some definitions in the literature [24].

We now introduce some notation which we will need in Chapter 4. In line with the notation used for Dyck paths, in a Motzkin path we say that an up-step is at height j if it starts at a vertex $(i-1, j-1)$ and ends at a vertex (i, j), a down-step is at height j if it starts at a vertex (i, j) and ends at a vertex $(i+1, j-1)$ and a horizontal step is at height j if it starts at a vertex $(i-1, j)$ and ends at a vertex (i, j). As in Chapter 2, to concisely describe our maps, we will use words and paths interchangeably. When referring to Motzkin paths, we will use the letters U_j, D_j and H_j to denote up-, down- and horizontal steps at height j, respectively. When we are referring to steps of a bicoloured Motzkin path, we will use the colours white and black, and so denote the steps as $U_j^{\{w\}}, U_j^{\{b\}}, D_j^{\{w\}}, D_j^{\{b\}}, H_j^{\{w\}}$ and $H_j^{\{b\}}$.

We also need to define the notions of Motzkin paths in a strip, and in a restricted strip. We will only define it for bicoloured Motzkin paths, but the equivalent definition holds for monocoloured Motzkin paths.

Definition 1.8. *A bicoloured Motzkin path in the strip of width H is a bicoloured Motzkin path with no vertices with y-coordinates greater than H.*

Definition 1.9. *A bicoloured Motzkin path in the restricted strip of width H is a bicoloured Motzkin path with no vertices with y-coordinates greater than H, and no horizontal steps at height H.*

Let $\mathcal{M}_H$ denote the set of bicoloured Motzkin paths in the strip of width H, $\overline{\mathcal{M}}_H$ denote the set of bicoloured Motzkin paths in the restricted strip of width H (no horizontal steps at height H), $\mathcal{M}'_H$ denote the set of Motzkin paths in the strip of width H, and $\overline{\mathcal{M}}'_H$ denote the set of Motzkin paths in the restricted strip of width H.

We denote the general step-sets for Motzkin and bicoloured Motzkin paths in a strip by the sets

$$\mathcal{L}'_i = \{U_j, D_j, H_k : 1 \le j \le \left\lfloor \frac{i}{2} \right\rfloor,\ 0 \le k \le \left\lfloor \frac{i-1}{2} \right\rfloor\}$$

and

$$\mathcal{L}_i = \{U_j^{\{w\}}, U_j^{\{b\}}, D_j^{\{w\}}, D_j^{\{b\}}, H_k^{\{w\}}, H_k^{\{b\}} : 1 \le j \le \left\lfloor \frac{i}{2} \right\rfloor,\ 0 \le k \le \left\lfloor \frac{i-1}{2} \right\rfloor\},$$

where for i odd we have the step-set for the strip of width $(i-1)/2$, and for i even we have the step-set for the restricted strip of width $i/2$.

We define a Ballot path as a Dyck path which does not have the constraint of ending on the x-axis, in line with [6]. The idea behind the name is that these paths can be thought of as a ballot, where votes are cast for one of two candidates (A or B), with up-steps being votes for candidate A and down-steps votes for B. The count then has the property that candidate A is never behind candidate B (similarly the Ballot path never drops below the x-axis.) Expanding this concept from two to three candidates, we have the following definition.

Definition 1.10. *A three-candidate Ballot path is a walk on* $\mathbb{Z}^2$ *starting at* $(0,0)$ *with the steps* $(1,1),(1,-1)$ *and* $(1,0)$, *such that after* r *steps the number of* $(1,1)$ *steps is greater than or equal to the number of* $(1,-1)$ *steps, which is greater than or equal to the number of* $(1,0)$ *steps, for all* $0 \leq r \leq n$.

These can also be thought of as a coding for Yamanouchi words with three letters [10, p. 2], where a Yamanouchi word with n letters is a word on the alphabet $\mathbb{Z}_n = \{1,2,\ldots,n\}$ such that any initial segment contains the letter i at least as many times as the letter $i+1$, for $1 \leq i \leq n-1$. We further define the *excess* L of a three-candidate Ballot path as $max(L_i)$, where L_i is the difference between the number of $(1,1)$ steps and $(1,0)$ steps after i steps. As with Motzkin paths we have the notion of bicolouring.

Definition 1.11. *A bicoloured three-candidate Ballot path is a three-candidate Ballot path where each of the steps can be one of two colours.*

1.3 The Kernel method

In this section, we will present a brief introduction to the Kernel method. We will give Knuth's original example [15], also given in Prodinger's survey paper [27], in which he finds the generating function for Dyck paths.

We begin by examining the wider class of Ballot paths. Let $f_i = \sum_{n\geq 0} a_{n,i}\, z^n$ denote the generating functions for Ballot paths ending at height i, at the vertices (n, i). The generating variable z is used to weight a step in a path; a path of n steps is weighted by z^n. Thus the coefficients $a_{n,i}$ of z_n are the number of Ballot paths ending at (n, i). We can then see the following recursions:

$$f_i(z) = zf_{i-1}(z) + zf_{i+1}(z), \qquad i \geq 1\,, \tag{1.1a}$$

$$f_0(z) = 1 + zf_1(z)\,. \tag{1.1b}$$

We now introduce our *catalytic variable* x, which we use to denote the height of the path. Clearly this is not a variable we are interested in directly, as we are only interested in paths that end on the x-axis. However, it is the use of this extra variable which gives us the extra machinery we need to solve this problem.

We examine the bivariate generating function $F(z, x) = \sum_{n\geq 0} f_n(z)x^n$, which uses two generating variables to track both the length and height of paths. Multiplying the above recursions (1.1) by x^i and summing, we find that

$$F(z, x) - f_0(z) = zxF(z, x) + \frac{z}{x}\big[F(z, x) - f_0(z) - xf_1(z)\big]\,. \tag{1.2}$$

Notice that the term zx corresponds to an up-step, and $\frac{z}{x}$ corresponds to a down-step. The term corresponding to down-steps is more complex as we are only counting paths that finish at height 1 or above. Thus our down-step must be taken from at least height 2, so we subtract the terms corresponding to paths ending at heights 0 and 1.

Rearranging and substituting in equation (1.1b), we find that

$$F(z,x) = zxF(z,x) + \frac{z}{x}\big[F(z,x) - f_0(z)\big] + 1\,, \tag{1.3}$$

and so

$$F(z,x) = \frac{zF(z,0) - x}{zx^2 - x + z}\,. \tag{1.4}$$

Note that for equation (1.4) we have used the identity $F(z,0) = f_0(z)$.

Substituting in $x = 0$ gives us nothing, however the denominator factors as $z(x - r_1(z))(x - r_2(z))$, where

$$r_{1,2}(z) = \frac{1 \mp \sqrt{1 - 4z^2}}{2z}\,. \tag{1.5}$$

Note that $x - r_1(z) \sim x - z$ as $x, z \to 0$, so the factor $1/(x - r_1(z))$ has no power series expansion around $(0,0)$. However $F(z,x)$ does have one, so this "bad" factor must disappear, i.e. $(x - r_1(z))$ must also be a factor of the numerator. This implies that $zF(z,0) = r_1(z)$, and so

$$F(z,0) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2}\,, \tag{1.6}$$

which is recognisable as the generating function for the Catalan numbers.

The solution presented above is an example of the normal Kernel method. In this solution, our Kernel was the denominator $zx^2 - x + z$; it was manipulations and observations on this term that allowed us to reach our solution. In the algebraic Kernel method, which is the method we use in Chapter 3, we use a slightly different formulation of the functional equation. In particular, we have that $zx^2 - x + z = -x\big[1 - z\big(x + \frac{1}{x}\big)\big]$, so equation (1.4) can be rewritten as

$$\left[1 - z\left(x + \frac{1}{x}\right)\right] F(z, x) = 1 - \frac{z}{x} F(z, 0)\,. \tag{1.7}$$

We now use the factor on the left side of the equation as our Kernel. We proceed by exploiting symmetries of this Kernel to remove the boundary term, which allows us to solve the equation. In particular, we can easily see that substituting $\frac{1}{x}$ for x leaves the Kernel unchanged, and we have that

$$\left[1 - z\left(x + \frac{1}{x}\right)\right] F(z, \frac{1}{x}) = 1 - zxF(z, 0)\,. \tag{1.8}$$

Using the summation $x(1.7) - \frac{1}{x}(1.8)$ we can cancel the $F(z, 0)$ terms and find that

$$xF(z, x) - \frac{1}{x} F(z, \frac{1}{x}) = \frac{x - \frac{1}{x}}{1 - z\left(x + \frac{1}{x}\right)}\,. \tag{1.9}$$

Notice that the term $xF(z, x)$ has only positive powers in x , and $\frac{1}{x}F(z, \frac{1}{x})$ has only negative powers, so the left side of (1.9) splits in terms of powers of x . Using this observation, and reading the right side of the equation as a formal power series in z whose coefficients are Laurent polynomials in x, we can then extract terms from the right side to find $F(z, x)$.

1.4 The triangular domain

In this section we introduce the model which is the primary source of interest in Chapter 3. Consider walks $(\omega_0, \omega_1, \ldots, \omega_n)$ on $\mathbb{Z}^3$ with steps $\omega_i - \omega_{i-1}$ in a step-set Ω_2 such that with each step exactly one coordinate increases by one and exactly one coordinate decreases by one. More precisely,

$$\Omega_2 = \{(1, -1, 0), (-1, 1, 0), (1, 0, -1), (-1, 0, 1), (0, 1, -1), (0, -1, 1)\}.$$

The step-set Ω_2 ensures that walks lie in planes $\{(n_x, n_y, n_z) \in \mathbb{Z}^3 \mid n_x + n_y + n_z = L\}$ determined by the starting point $\omega_0 = (u_1, u_2, u_3)$ of the walk, where $L = u_1 + u_2 + u_3$. In this thesis, we will study walks on domains given by finite subsets of these planes by restricting the walks to the non-negative orthant $(\mathbb{N}_0)^3$ (cf Figure 1.1). In particular, the walks lie on a bounded triangular domain; henceforth this model will be referred to as the *triangle model*.

We will also consider walks on the 1-dimensional analogue of the triangle model, i.e. walks $(\omega_0, \omega_1, \ldots, \omega_n)$ on $\mathbb{Z}^2$ with steps $\omega_i - \omega_{i-1}$ in a step-set

$$\Omega_1 = \{(1, -1), (-1, 1)\}.$$

Ω_1 ensures that walks lie on lines $\{(n_x, n_y) \in \mathbb{Z}^2 \mid n_x + n_y = L\}$ determined by the starting point $\omega_0 = (u_1, u_2)$ of the walk, where $L = u_1 + u_2$. As with the triangle model, we will study walks on domains given by finite subsets of these lines, by restricting the walks to the non-negative quadrant, in this case $(\mathbb{N}_0)^2$ (cf Figure 1.2). From here on, this model will be referred to as the *line model*.

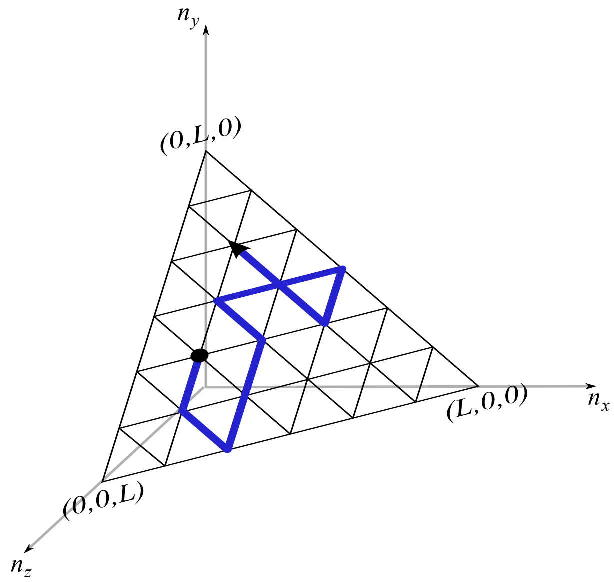

Figure 1.1: A 10-step walk on the bounded triangular domain intersecting the axes at $(L, 0, 0), (0, L, 0)$ and $(0, 0, L)$, where $L = 6$. Points on the domain are given by (n_x, n_y, n_z) with $n_x, n_y, n_z \geq 0$ and $n_x + n_y + n_z = L$.

This model has been studied before [5, pp. 7-8], and has obvious connections to Chebyshev polynomials.

Note that the line and triangle model have strong links to directed walks; in particular the triangle model can also be formulated as two directed walks in a strip (cf Figure 1.3).

We conclude this section by noting that these models can be framed as the low-dimensional cases of a larger class of models. Consider walks $(\omega_0, \omega_1, \dots, \omega_n)$ on $\mathbb{Z}^{d+1}$ with steps $\omega_i - \omega_{i-1}$ in a step-set Ω_d such that with each step exactly one coordinate increases by one and exactly one coordinate decreases by one. More

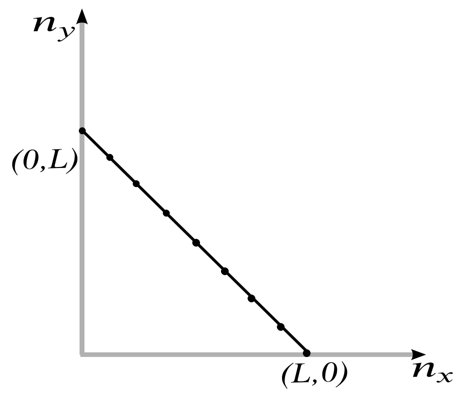

Figure 1.2: The line model, intersecting the axes at $(L, 0)$ and $(0, L)$.

precisely, Ω_d is the set of steps with coordinates $(e_1, e_2, \ldots, e_{d+1})$ such that for all ordered pairs (i, j) with $1 \leq i, j \leq d+1$ and $i \neq j$, $e_i = 1$, $e_j = -1$ and $e_k = 0$ for all $1 \leq k \leq d+1$ and $k \notin \{i, j\}$.

The step-set Ω_d ensures that walks lie in a d-dimensional hyperplane $\{(n_{x_1}, \ldots, n_{x_{d+1}}) \in \mathbb{Z}^{d+1} \mid n_{x_1} + \ldots + n_{x_{d+1}} = L\}$ determined by the starting point $\omega_0 = (u_1, \ldots, u_{d+1})$ of the walk, where $L = \sum_{j=1}^{d+1} u_j$. Walks on domains given by finite subsets of these hyperplanes may be studied by restricting the walks to the non-negative orthant $(\mathbb{N}_0)^{d+1}$. Fixing the dimension d, this class of walks is referred to as the d-dimensional case. The 1-dimensional case is the line model, and the 2-dimensional case is the triangle model. In the 3-dimensional case, the domains would be tetrahedra of side-length L.

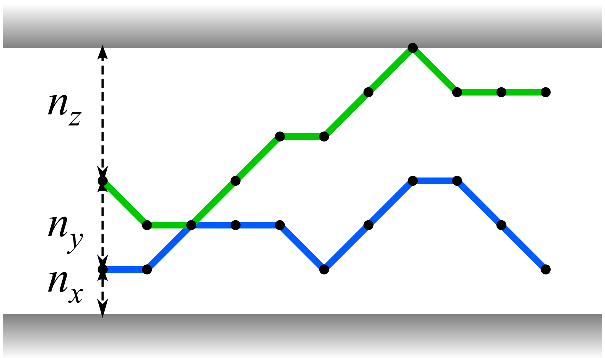

Figure 1.3: The image of the 10-step walk in Figure 1.1. The k-th point (n_x, n_y, n_z) of the walk in Figure 1.1 maps to two points with coordinates (k, n_x) and $(k, n_x + n_y)$, respectively. This mapping generates two non-crossing directed walks that are confined to the strip $0 \le y \le L$. In this example, the two walks touch after 2 steps and the top walk hits the top of the strip after 7 steps, corresponding to the points where the walk in Figure 1.1 touches the sides of the domain.

1.5 Continued fractions

In this section, we discuss the continued fraction expansion for Motzkin paths, as given by Flajolet [12, pp. 6-11]. For a good exposition of this, see Flajolet and Sedgewick [13, pp. 319-323]. We have the result that the following infinite continued fraction counts weighted Motzkin paths with weight a_i on up-steps at height i, weight b_i on down-steps at height i and weight c_i on horizontal steps at height i:

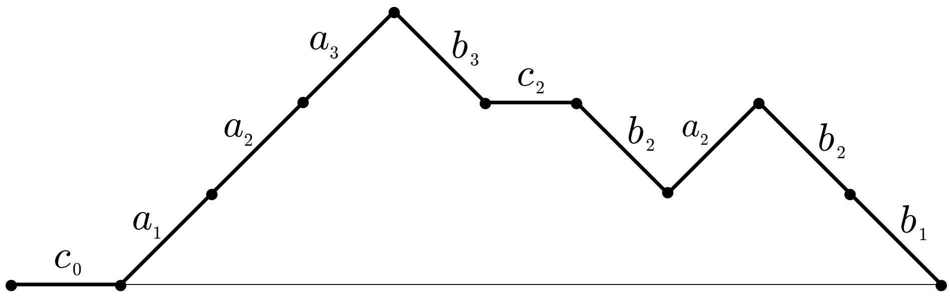

Figure 1.4: An example of a Motzkin path with the step weights a_i, b_i and c_i applied to up-, down- and horizontal steps respectively at height i.

$$\cfrac{1}{1-c_0t-\cfrac{a_1b_1t^2}{1-c_1t-\cfrac{a_2b_2t^2}{1-c_2t-\cfrac{a_3b_3t^2}{\ddots}}}}\,. \tag{1.10}$$

A visual interpretation of this is given in Figure 1.4. Note that if $a_i = b_j = c_k = 1$ for all i, j, k then we have the interpretation for normal Motzkin paths, and if $a_i = b_j = c_k = 2$ for all i, j, k then we have the interpretation for bicoloured Motzkin paths. Setting $c_i = 0$ for $i \geq 0$ gives the interpretation for Dyck paths.

It is easy to see that we get a truncated continued fraction by setting $a_k = 0$ for some $k \in \mathbb{N}$. From Figure 1.4 we can see that such continued fractions would correspond to weighted Motzkin paths in a strip of width $k-1$. If we also set $c_{k-1} = 0$, the continued fraction would correspond to weighted Motzkin paths in a restricted strip of width $k-1$.

Chapter 2

A Bijection on Bilateral Dyck Paths

In this chapter we give a direct bijection between bilateral Dyck paths of semilength n with k up-steps at odd height and bilateral Dyck paths of semilength n with k peaks. This map restricts to normal Dyck paths, where the paths are counted by the Narayana numbers $N(n,k)$. We begin by defining the bijection on $\mathcal{D}$ before extending it to bilateral paths in Section 2.3.

2.1 The Bijection on Dyck Paths

If P is a Dyck path of nonzero length with $s \geq 0$ down steps at height 2 before the first contact, the corresponding Dyck word W_P can be uniquely decomposed into a word of the form $UUW_1DUW_2DU \ldots W_sDDW_{s+1} = U\Big(\prod_{i=1}^{s} UW_iD\Big)DW_{s+1}$, where each W_i is a Dyck word. Using this decomposition, we define the map $\phi : \mathcal{D} \to \mathcal{D}$ recursively by setting

$$\phi(W_P) = \phi\Big(U\Big(\prod_{i=1}^{s} UW_iD\Big)DW_{s+1}\Big) = \Big(\prod_{i=1}^{s} U\phi(W_i)\Big)(UD)D^s\phi(W_{s+1}) \ ,$$

and $\phi(\epsilon) = \epsilon$, where ϵ denotes the empty word.

As the decomposition expresses a Dyck word in terms of strictly smaller Dyck words, the recursion terminates. Therefore ϕ is well-defined on all Dyck words.

Theorem 2.1. *ϕ gives an explicit bijection from the set of Dyck paths of semilength n with m contacts and k up-steps at odd height to the set of Dyck paths of semilength n with m contacts and k peaks.*

Proof. The decomposition implies that the parity of a step in the path corresponding to the sub-word W_i is the same as its parity in the original path P. As the first step of each sub-path corresponding to nonempty W_i is at odd height, each recursive iteration either maps the empty word to itself or maps exactly one up-step at odd height (the first step of the path P) to a peak. Thus ϕ maps a path with k up-steps at odd height to a path with k peaks. Clearly, ϕ does not change the number of contacts.

We now show that ϕ is indeed a bijection by giving its inverse. If P is a Dyck path whose right-most peak before the first contact is at height $s+1$ with $s \geq 0$, the corresponding Dyck word W_P can be uniquely decomposed into a word of the form $UW_1UW_2U \ldots UW_s(UD)D^sW_{s+1} = \Big(\prod_{i=1}^{s} UW_i\Big)UD^{s+1}W_{s+1}$, where each

W_i is a Dyck word. Let $\psi : \mathcal{D} \to \mathcal{D}$ be defined recursively as

$$\psi(W_P) = \psi\Big(\Big(\prod_{i=1}^{s} UW_i\Big)UD^{s+1}W_{s+1}\Big) = U\Big(\prod_{i=1}^{s} U\psi(W_i)D\Big)D\,\psi(W_{s+1})\,,$$

with $\phi(\epsilon) = \epsilon$, where ϵ denotes the empty word. We note in passing that each recursive iteration of ψ to a nonempty path maps exactly one peak to an up-step at odd height.

To show that ψ is the inverse of ϕ, we proceed by induction on n, the semilength of the word. For $n = 0$,

$$\psi(\phi(\epsilon)) = \psi(\epsilon) = \epsilon \quad \text{and} \quad \phi(\psi(\epsilon)) = \phi(\epsilon) = \epsilon\ .$$

Now take a Dyck word W of semilength $n > 0$ and assume that $\psi(\phi(W')) = \phi(\psi(W')) = W'$ for all words W' of semilength less than n. W can be decomposed as either $W = U\Big(\prod_{i=1}^{s} UW_iD\Big)DW_{s+1}$ or $W = \Big(\prod_{i=1}^{s} UW_i\Big)UD^{s+1}W_{s+1}$, where the W_i are Dyck words of semilength strictly less than n. Hence, by the inductive hypothesis,

$$\begin{aligned}
\psi(\phi(W)) &= \psi\Big(\phi\Big(U\Big(\prod_{i=1}^{s} UW_iD\Big)DW_{s+1}\Big)\Big) = \psi\Big(\Big(\prod_{i=1}^{s} U\phi(W_i)\Big)UD^{s+1}\phi(W_{s+1})\Big) \\
&= U\Big(\prod_{i=1}^{s} U\psi(\phi(W_i))D\Big)D\,\psi(\phi(W_{s+1})) = U\Big(\prod_{i=1}^{s} UW_iD\Big)DW_{s+1} = W_P\,; \\
\phi(\psi(W)) &= \phi\Big(\psi\Big(\Big(\prod_{i=1}^{s} UW_i\Big)UD^{s+1}W_{s+1}\Big)\Big) = \phi\Big(U\Big(\prod_{i=1}^{s} U\psi(W_i)D\Big)D\,\psi(W_{s+1})\Big) \\
&= \Big(\prod_{i=1}^{s} U\phi(\psi(W_i))\Big)UD^{s+1}\phi(\psi(W_{s+1})) = \Big(\prod_{i=1}^{s} UW_i\Big)UD^{s+1}W_{s+1} = W_P.
\end{aligned}$$

$\square$

Corollary 2.2. *The number of Dyck paths of semilength n with m contacts and k up-steps at odd height is equal to the number of Dyck paths of semilength n with m contacts and k peaks.*

2.2 An Example

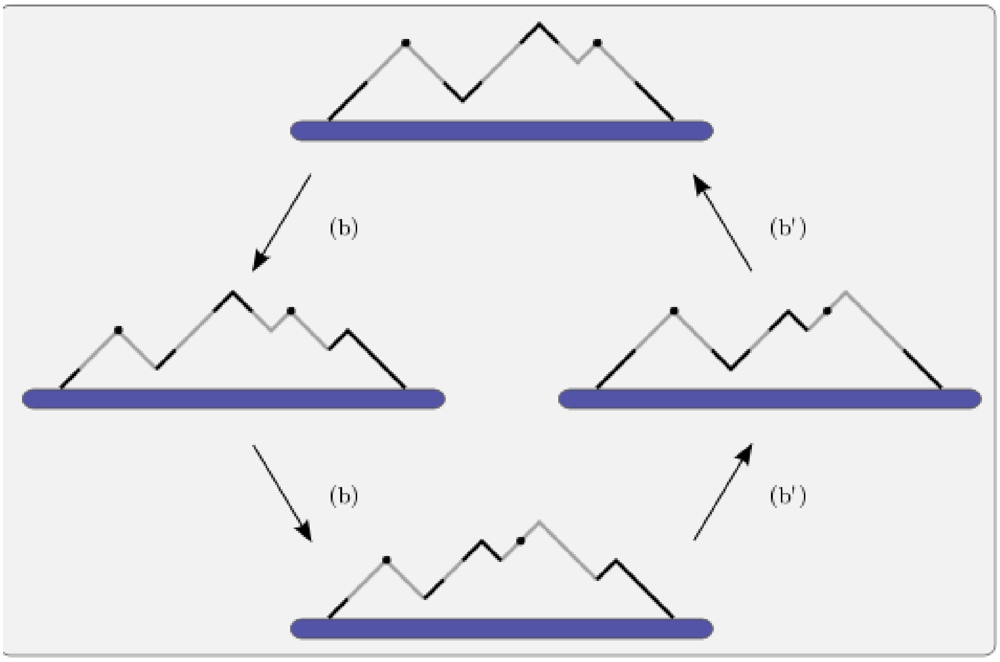

Figure 2.1: ϕ mapping a Dyck path W (top) with semilength 9, 4 up-steps at odd height and 1 contact to a Dyck path $\phi(W)$ (bottom) with semilength 9, 4 peaks and 1 contact, and ψ performing the inverse mapping. Intermediate stages after each recursive iteration of ϕ and ψ are shown on the left and right respectively. Black dots correspond to the occurrence of empty words in the decomposition of the corresponding words.

An example of the bijection ϕ and its inverse ψ is given in Figure 2.1, explicitly showing the intermediate stages after each recursive iteration. When writing the corresponding words below, we insert bracketing corresponding to the decompositions, e.g. we write

$$UUUUDDDUUUUDDUDDDD = UU\Big(UU()DD\Big)DU\Big(UU(UD)DU()DD\Big)DD\ .$$

Note in particular that the double bracket () signifies an empty word in the decomposition. For the sake of legibility, we have avoided the occurrence of non-prime words in this example.

The mapping ϕ corresponding to the left arrows in Figure 2.1 is now performed as follows:

$$\begin{aligned}
&\phi\Big(UU\Big(UU()DD\Big)DU\Big(UU(UD)DU()DD\Big)DD\Big) && \text{top path}\\
&\qquad = U\phi\Big(UU()DD\Big)U\phi\Big(UU(UD)DU()DD\Big)UDDD && \text{left path}\\
&\qquad = U\Big(U\phi()UDD\Big)U\Big(U\phi(UD)U\phi()UDDD\Big)UDDD && \text{bottom path}\\
&\qquad = U\Big(U()UDD\Big)U\Big(U(UD)U()UDDD\Big)UDDD\,. && \text{bottom path}
\end{aligned}$$

The first recursive iteration of ϕ gives

$$\phi(UUW_1DUW_2DD) = U\phi(W_1)U\phi(W_2)UDDD\ ,$$

and the second step applies the recursion to the sub-words W_1 and W_2 respectively. In the third step the recursion is applied to empty subwords, and hence the path no longer changes; the corresponding figures look identical. Note in particular that while $\phi(UU()DD) = U\phi()UDD = U()UDD$ does not change the word $UUDD$, the position of the empty word (), and correspondingly the position of the black dot in Figure 2.1 changes.

The inverse mapping ψ corresponding to the right arrows in Figure 2.1 is performed similarly, with different intermediate stages:

$$\begin{aligned}
&\psi\Big(U\Big(U()UDD\Big)U\Big(U(UD)U()UDDD\Big)UDDD\Big) && \text{bottom path}\\
&\qquad = UU\psi\Big(U()UDD\Big)DU\psi\Big(U(UD)U()UDDD\Big)DD && \text{right path}\\
&\qquad = UU\Big(UU\psi()DD\Big)DU\Big(UU\psi(UD)DU\psi()DD\Big)DD && \text{top path}\\
&\qquad = UU\Big(UU()DD\Big)DU\Big(UU(UD)DU()DD\Big)DD\,. && \text{top path}
\end{aligned}$$

2.3 Extension to Bilateral Dyck Paths

We now give an extension of ϕ to bilateral Dyck paths. Before we do this, we must define the following two maps.

We define the map $\alpha : \mathcal{B} \to \mathcal{B}$ as follows. If P is a bilateral Dyck path of semilength n with corresponding bilateral Dyck word $W_P = A_1A_2\ldots A_{2n}$, where $A_i \in \{U, D\}$, then (i) $\alpha(U) = D$, (ii) $\alpha(D) = U$ and (iii) $\alpha(W_P) = \alpha\Big(\prod_{i=1}^{2n} A_i\Big) = \prod_{i=1}^{2n} \alpha(A_i)$.

The map α reflects bilateral Dyck paths in the $x-$axis. In particular, it maps nonempty Dyck paths to negative Dyck paths and vice versa. It maps steps at odd height to steps at even height, and vice versa. It also maps peaks to valleys, and maps valleys to peaks.

We define $\beta : \mathcal{D} \to \mathcal{D}$ as follows. If P is a Dyck path of nonzero length, the corresponding Dyck word W_P can be uniquely decomposed into a word of the form UW_1DW_2 where W_1 and W_2 are Dyck words. Then

$$\beta(W_P) = \beta(UW_1DW_2) = UW_2DW_1 \ .$$

Note that β illustrates the equinumeracy between paths with $k+1$ up-steps at odd height and k up-steps at even height; excepting the first up- and down-steps at height 1, β maps steps at odd height to steps at even height, and vice versa (surprisingly, we have not been able to find this simple argument in the literature). It preserves the number of peaks or valleys, but does not preserve the number of contacts. We also note that both α and β are involutions. We are now in a position to state the extended bijection.

Let ϵ denote the empty word. We define $\phi' : \mathcal{B} \to \mathcal{B}$ in the following way:

(a) if P is a Dyck path with corresponding Dyck word W_P, then

$$\phi'(W_P) = \phi(W_P) \, ;$$

(b) if P is a negative Dyck path with corresponding negative Dyck word W_P, then

$$\phi'(W_P) = \alpha(\phi(\beta(\alpha(W_P)))) \, ;$$

(c) if P is a bilateral Dyck path with $l > 0$ crossings then we can uniquely decompose the corresponding bilateral Dyck word W_P into $W_1W_2 \ldots W_{l+1} = \prod_{i=1}^{l+1} W_i$, where each W_i is alternately either a Dyck word of nonzero length or a negative Dyck word. Then

$$\phi'(W_P) = \phi'\Big(\prod_{i=1}^{l+1} W_i\Big) = \prod_{i=1}^{l+1} \phi'(W_i)\,.$$

Theorem 2.3. *ϕ' gives an explicit bijection from the set of bilateral Dyck paths of semilength n with k up-steps at odd height to the set of bilateral Dyck paths of semilength n with k peaks.*

Proof. By Theorem 2.1, (a) maps Dyck paths with k up-steps at odd height to Dyck paths with k peaks.

If P is a negative Dyck path with k up-steps at odd height and corresponding negative Dyck word W_P, then $\alpha(W_P)$ corresponds to a Dyck path with k up-steps at even height. Applying β to this new path gives a Dyck path with $k+1$ up-steps at odd height. Then $\phi(\beta(\alpha(W_P)))$ corresponds to a Dyck path with $k+1$ peaks (and so k valleys). Finally, applying α again will give a negative Dyck path with k peaks. Thus (b) maps negative Dyck paths with k up-steps at odd height to negative Dyck paths with k peaks.

Application of (c) does not alter the parity of steps or the number of peaks. Thus ϕ' maps bilateral Dyck paths with k up-steps at odd height to bilateral Dyck paths with k peaks.

We now show that ϕ' is a bijection by giving its inverse. Let $\psi' : \mathcal{B} \to \mathcal{B}$, be defined as follows:

(a') if P is a Dyck path with corresponding Dyck word W_P, then

$$\psi'(W_P) = \psi(W_P)\,;$$

(b') if P is a negative Dyck path with corresponding negative Dyck word W_P, then

$$\psi'(W_P) = \alpha(\beta(\psi(\alpha(W_P))))\,;$$

(c') if P is a bilateral Dyck path with $l > 0$ crossings then we can uniquely decompose the corresponding bilateral Dyck word W_P into $W_1 W_2 \ldots W_{l+1} = \prod_{i=1}^{l+1} W_i$, where each W_i is alternately either a Dyck word of nonzero length or a negative Dyck word. Then

$$\psi'(W_P) = \psi'\Big(\prod_{i=1}^{l+1} W_i\Big) = \prod_{i=1}^{l+1} \psi'(W_i)\,.$$

To show that ψ' is the inverse of ϕ', we proceed as follows. Let W be a bilateral Dyck word. If the path associated to W has $l > 0$ crossings then we can uniquely decompose W into $W_1 W_2 \ldots W_{l+1} = \prod_{i=1}^{l+1} W_i$, where each W_i is alternately either a Dyck word of nonzero length or a negative Dyck word. Then

$$\psi'(\phi'(W)) = \psi'\Big(\phi'\Big(\prod_{i=1}^{l+1} W_i\Big)\Big) = \psi'\Big(\prod_{i=1}^{l+1} \phi'(W_i)\Big) = \prod_{i=1}^{l+1} \psi'(\phi'(W_i)), \text{ and}$$
$$\phi'(\psi'(W)) = \phi'\Big(\psi'\Big(\prod_{i=1}^{l+1} W_i\Big)\Big) = \phi'\Big(\prod_{i=1}^{l+1} \psi'(W_i)\Big) = \prod_{i=1}^{l+1} \phi'(\psi'(W_i))\,.$$

If W is a Dyck word then $\phi' = \phi$ and $\psi' = \psi$ so

$$\psi'(\phi'(W)) = \psi(\phi(W)) = W \text{ and}$$
$$\phi'(\psi'(W)) = \phi(\psi(W)) = W\,.$$

Using the fact that ϕ and ψ are inverses of each other, together with the involutive properties of α and β, it is easy to show that for negative Dyck words $\psi'(\phi'(W)) = W = \phi'(\psi'(W))$ also holds:

$$\begin{aligned}
\psi'(\phi'(W)) &= \psi'(\alpha(\phi(\beta(\alpha(W)))) = \alpha(\beta(\psi(\alpha(\alpha(\phi(\beta(\alpha(W)))))))) \\
&= \alpha(\beta(\psi(\phi(\beta(\alpha(W)))))) = \alpha(\beta(\beta(\alpha(W)))) = \alpha(\alpha(W)) = W, \text{ and} \\
\phi'(\psi'(W)) &= \phi'(\alpha(\beta(\psi(\alpha(W)))) = \alpha(\phi(\beta(\alpha(\alpha(\beta(\psi(\alpha(W)))))))) \\
&= \alpha(\phi(\beta(\beta(\psi(\alpha(W)))))) = \alpha(\phi(\psi(\alpha(W)))) = \alpha(\alpha(W)) = W\,.
\end{aligned}$$

□

Corollary 2.4. *The number of bilateral Dyck paths of semilength n with k up-steps at odd height is equal to the number of bilateral Dyck paths of semilength n with k peaks.*

2.3.1 Remarks

In [23] a bijection between Dyck paths of semilength n with $k-1$ up-steps at even height and Dyck paths of semilength n with k peaks is given, using the machinery of checkmark sequences. That bijection is given as a restriction of a bijection between bilateral Dyck paths.

To make a connection of our result with the work in [23], we recall that any Dyck path with $k-1$ up-steps at even height can be mapped bijectively to a Dyck path with k up-steps at odd height by the involution β defined in Section 2.3. By mapping Dyck paths via β, followed by application of the map ϕ given in Section 2.1, we also have a bijection between Dyck paths of semilength n with $k-1$ up-steps at even height and Dyck paths of semilength n with k peaks. Already from inspecting the five Dyck paths of semilength $n = 3$ one can see that this bijection is different from the one presented in [23].

We note that the extension of the bijection in [23] to bilateral Dyck paths needs a slightly modified definition of peaks. More precisely, in [23] an initial down-step or final up-step in a bilateral Dyck path is also counted as a peak, and the bijection given is between bilateral Dyck paths of semilength n with $k-1$ up-steps at even height and bilateral Dyck paths of semilength n with k (modified) peaks.

In contrast, our bijection extends to a bijection between bilateral Dyck paths of semilength n with k up-steps at odd height and bilateral Dyck paths of semilength n with k peaks.

We also note that the restriction of our bijection to Dyck paths is essentially identical to a bijection given in [30]. There, the authors used two length-preserving bijections on Dyck paths to show the equidistribution of statistics of certain strings occurring at odd height, at even height, and anywhere. However, the problem discussed here, regarding the equidistribution of up-steps at odd height and peaks anywhere, was not addressed in [30].

Chapter 3

Walks on a triangular domain

Recall from Chapter 1 the definitions of the triangle and line models. In this chapter we will give a number of results on these models. We present a complete solution for the line model, and a solution for the triangle model in the case where endpoints are not weighted, along with a high-symmetry case. We also prove an equinumeracy with Motzkin paths, for a special case of walks on the triangle model.

3.1 Statement of Results

We begin by considering the line model. Given a fixed starting point ω_0, we denote the number of n-step walks starting at ω_0 and ending at $\omega_n = (i_1, i_2)$ by $C_n(i_1, i_2)$ and consider the generating function

$$G(x, y; t) = \sum_{n=0}^{\infty} t^n \sum_{\omega_n \in (\mathbb{N}_0)^2} C_n(\omega_n) x^{i_1} y^{i_2} \ , \tag{3.1}$$

where t is the generating variable conjugate to the length of the walk. Due to the choice of the step-set Ω_1, when counting walks starting at (u_1, u_2), $G(x, y; t)$ is homogeneous of degree $L = u_1 + u_2$ in x, y, i.e.

$$G(\gamma x, \gamma y; t) = \gamma^L G(x, y; t)\ . \tag{3.2}$$

It is easy to solve this model, and we get the following result.

Proposition 3.1. *The generating function $G(x, y; t)$, which counts n-step walks starting at fixed $\omega_0 = (u, v)$, is given by*

$$G(x, y; t) = \frac{1}{1 - \dfrac{\frac{x}{y} + \frac{y}{x}}{p + \frac{1}{p}}} \left(x^u y^v - \frac{x^{u+v+1} p^{v+1} (1 - p^{2u+2})}{y(1 - p^{2u+2v+4})} - \frac{y^{u+v+1} p^{u+1} (1 - p^{2v+2})}{x(1 - p^{2u+2v+4})} \right), \tag{3.3}$$

where

$$p = \frac{1 - \sqrt{1 - 4t^2}}{2t} \tag{3.4}$$

is the generating function of Dyck paths.

This simplifies considerably when specifying $x = y = 1$.

Corollary 3.2. *The generating function $G(1, 1; t)$, which counts n-step walks starting at fixed $\omega_0 = (u, v)$ with no restrictions on the endpoint, is given by*

$$G(1, 1; t) = \frac{(1 + p^2)(1 - p^{u+1})(1 - p^{v+1})}{(1 - p)^2 (1 + p^{u+v+2})}, \tag{3.5}$$

where

$$p = \frac{1 - \sqrt{1 - 4t^2}}{2t}\ . \tag{3.6}$$

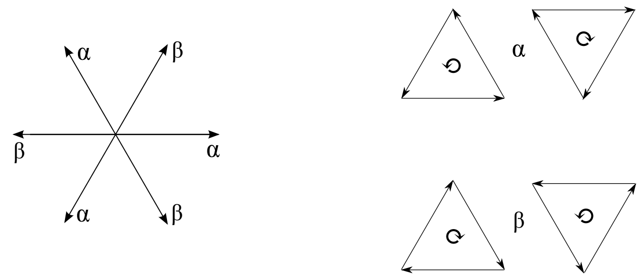

Figure 3.1: The step-set for the triangle partitioned into two smaller step-sets Ω_2' and Ω_2'', with associated weights α and β respectively. Using only steps from Ω_2' restricts to allowing steps in only one orientation (clockwise or anti-clockwise) around each triangle in the domain, with neighbouring triangles (those sharing an edge) permitting opposite orientations. Using only steps from Ω_2'' reverses these orientations.

The main results in this chapter concern the triangle model, or more precisely, a weighted generalisation of this model. We can partition Ω_2 into

$$\begin{aligned} \Omega_2' &= \{(1,0,-1),(-1,1,0),(0,-1,1)\} \quad \text{and} \\ \Omega_2'' &= \{(1,-1,0),(-1,0,1),(0,1,-1)\}\ , \end{aligned} \tag{3.7}$$

with steps in Ω_2' and Ω_2'' given the weights α and β, respectively (cf Figure 3.1).

Given a fixed starting point ω_0, we denote the number of n-step walks starting at ω_0 and ending at $\omega_n = (i_1, i_2, i_3)$ by $C_n(i_1, i_2, i_3)$ and consider the generating function

$$G(x,y,z;t) = \sum_{n=0}^{\infty} t^n \sum_{\omega_n \in (\mathbb{N}_0)^{d+1}} C_n(\omega_n) x^{i_1} y^{i_2} z^{i_3}\ , \tag{3.8}$$

where t is the generating variable conjugate to the length of the walk. Due to the choice of the step-set Ω_2, when counting walks starting at (u_1, u_2, u_3), $G(x,y,z;t)$

is homogeneous of degree $L = u_1 + u_2 + u_3$ in x, y, z, i.e.

$$G(\gamma x, \gamma y, \gamma z; t) = \gamma^L G(x, y, z; t) \,. \tag{3.9}$$

The main result of this chapter is as follows.

Theorem 3.3. *The generating function $G(t) \equiv G(1, 1, 1; t)$, which counts n-step walks starting at fixed $\omega_0 = (u, v, w)$ with no restrictions on the endpoint, is given by*

$$G(t) = \frac{(1-p^3)(1-p^{u+1})(1-p^{v+1})(1-p^{w+1})}{(1-p)^3(1-p^{u+v+w+3})} \tag{3.10}$$

with

$$p = (\alpha+\beta)tM((\alpha+\beta)t)\,, \tag{3.11}$$

where

$$M(t) = \frac{1-t-\sqrt{(1+t)(1-3t)}}{2t^2} \tag{3.12}$$

is the generating function of Motzkin paths.

For walks starting in a corner of a triangle of side-length L we find the following intriguing equinumeracy result.

Corollary 3.4.

(a) Walks starting at a corner of a triangle of side-length $L = 2H + 1$ with arbitrary endpoint and taking p steps on Ω_2' and q steps on Ω_2'' are in bijection with bicoloured Motzkin paths in the strip of width H which have p steps coloured with colour A and q steps coloured with colour B.

(b) *Walks starting at a corner of a triangle of side-length $L = 2H$ with arbitrary endpoint and taking p steps on Ω_2' and q steps on Ω_2'' are in bijection with bicoloured Motzkin paths in the restricted strip of width H which have p steps coloured with colour A and q steps coloured with colour B.*

In particular, this immediately implies that walks starting in a corner of a triangle of side-length $L = 2H + 1$ (resp. $L = 2H$) with arbitrary endpoint are in bijection with bicoloured Motzkin paths in the strip of width H (resp. the restricted strip of width H). Additionally, setting $q = 0$ implies that walks starting in a corner of a triangle of side-length $L = 2H + 1$ (resp. $L = 2H$) with arbitrary endpoint, which only take steps on Ω_2', are in bijection with Motzkin paths in the strip of width H (resp. the restricted strip of width H). Henceforth we will call these the *undirected* and *directed* cases respectively.

For walks starting in the centre of a triangle of side-length $L = 3u$, there is a further result.

Proposition 3.5. *The generating function $g(t) \equiv G(1, 1, 0; t)$, which counts walks starting at $\omega_0 = (u, u, u)$ and ending at a fixed side of the triangle, is given by*

$$g(t) = p^u \frac{(1-p^3)(1-p^{u+1})}{(1-p)(1-p^{3u+3})}\,, \tag{3.13}$$

with p as in Theorem 3.3.

For our final result in this chapter, we draw a link between walks on the triangle and Ballot paths.

Proposition 3.6. *Walks starting in a corner of a triangle of side-length L with arbitrary endpoint, restricted to the sublattice Ω_2', are in bijection with three-candidate Ballot paths with excess L.*

3.2 Proofs

3.2.1 Line Model

We will examine the line model first, and then apply the same techniques to the triangle model.

Functional Equation

An n-step walk is uniquely constructed by appending a step from the step-set Ω_1 to an $(n-1)$-step walk, provided $n > 0$. This leads to the following functional equation for the generating function $G(x, y; t)$:

$$\begin{aligned} G(x, y; t) =& x^u y^v + G(x, y; t) t \left(\frac{x}{y} + \frac{y}{x} \right) \\ & - G(x, 0; t) t \left(\frac{x}{y} \right) - G(0, y; t) t \left(\frac{y}{x} \right). \end{aligned} \tag{3.14}$$

Here, the monomial $x^u y^v$ corresponds to a zero-step walk starting (and ending) at $\omega_0 = (u, v)$. The term $G(x, y; t)t\left(\frac{x}{y} + \frac{y}{x}\right)$ corresponds to appending any of the steps in Ω_1 irrespective of whether the resulting walk steps violates the boundary conditions and leaves the domain. This overcounting is adjusted by the remaining terms. For example, $G(x, 0; t)$ corresponds to walks which end at $(i_1, 0; t)$, and therefore $G(x,0;t)\frac{x}{y}$ corresponds precisely to walks stepping across that boundary.

As this is a functional equation for the generating function $G(x,y;t)$ in the catalytic variables x,y only, the t-dependence is dropped by writing $G(x,y;t) \equiv G(x,y)$. We rewrite the functional equation (3.14) as

$$G(x,y)\left[1-t\left(\frac{x}{y}+\frac{y}{x}\right)\right] = x^u y^v - G(x,0)\,t\left(\frac{x}{y}\right) - G(0,y)\,t\left(\frac{y}{x}\right) . \qquad (3.15)$$

The Kernel

This gives us the Kernel $K(x,y;t) \equiv K(x,y)$ of the functional equation,

$$K(x,y) = 1 - t\left(\frac{x}{y}+\frac{y}{x}\right). \qquad (3.16)$$

Symmetry properties of this Kernel are central to the arguments. Note that the Kernel is homogeneous of degree zero, *i.e.* it is invariant under rescaling of all the variables. This trivial symmetry will be implicitly assumed in the considerations below.

We now introduce $\mathrm{G}(S)$, the group of transformations which leaves the Kernel of the functional equation invariant for the step-set S. This is in line with the notation introduced by Fayolle *et al.* [11]. For the line model, the step-set is

$$S_1 = \left\{\frac{x}{y}, \frac{y}{x}\right\},$$

where here and henceforth, steps are identified with their associated combinatorial weights.

Lemma 3.7. *The Kernel $K(x,y)$ is invariant under action of the group of transformations*

$$\mathrm{G}(S_1) = \langle (y,x) \rangle \cong C_2 \ .$$

Here (y,x) is shorthand notation for the map that sends (x,y) to (y,x). In particular, we arrive at the following result.

Lemma 3.8. *The Kernel $K(x,y)$ is invariant under the following 1-parameter substitutions.*

$$K(p,1) = K(1,p) = 1 - t\left(p + \frac{1}{p}\right) \ .$$

The dependence between p and t is henceforth fixed such that

$$1 - t(p + 1/p) = 0, \tag{3.17}$$

which when solved for p gives $p = tD(t)$, where $D(t)$ is the generating function for Dyck paths

$$D(t) = \frac{1 - \sqrt{1-4t^2}}{2t^2} \ , \tag{3.18}$$

as proven in Section 1.3. In particular, p is a well-defined power series with zero constant.

Using this dependency and substituting the two choices from Lemma 3.8 into the functional equation (3.15) implies

$$tpG(p,0) + \frac{t}{p}G(0,1) = p^u, \tag{3.19a}$$

$$\frac{t}{p}G(1,0) + tpG(0,p) = p^v. \tag{3.19b}$$

Using homogeneity of the generating function, we replace

$$G(p,0) = p^{u+v}G(1,0)\ , \quad G(0,p) = p^{u+v}G(0,1)\ , \tag{3.20}$$

and solve the two equations (3.19a) and (3.19b) in the two variables $G(1,0)$ and $G(0,1)$ to find that

$$G(1,0) = \frac{p^{v+1}(1-p^{2u+2})}{t(1-p^{2u+2v+4})}\ , \quad G(0,1) = \frac{p^{u+1}(1-p^{2v+2})}{t(1-p^{2u+2v+4})}\ . \tag{3.21}$$

Applying the homogeneity argument of (3.20) to (3.21), we determine that

$$G(x,0) = \frac{x^{u+v}p^{v+1}(1-p^{2u+2})}{t(1-p^{2u+2v+4})}\ , \quad G(0,y) = \frac{y^{u+v}p^{u+1}(1-p^{2v+2})}{t(1-p^{2u+2v+4})}\ . \tag{3.22}$$

Substituting these into (3.15) and eliminating t via (3.17) gives the result stated in Proposition 3.1,

$$G(x,y;t) = \frac{1}{1-\dfrac{\frac{x}{y}+\frac{y}{x}}{p+\frac{1}{p}}}\left(x^u y^v - \frac{x^{u+v+1}p^{v+1}(1-p^{2u+2})}{y(1-p^{2u+2v+4})} - \frac{y^{u+v+1}p^{u+1}(1-p^{2v+2})}{x(1-p^{2u+2v+4})}\right), \tag{3.23}$$

and substituting $x = y = 1$ into (3.23) then gives Corollary 3.2,

$$G(1,1;t) = \frac{(1+p^2)(1-p^{u+1})(1-p^{v+1})}{(1-p)^2(1+p^{u+v+2})}\ . \tag{3.24}$$

3.2.2 Triangle Model

We now apply the same method to the triangle model, including the weights α and β corresponding to the directed sublattices (equation (3.7).

Functional Equation

Again, an n-step walk is uniquely constructed by appending a step from the step-set Ω_2 to an $(n-1)$-step walk, provided $n > 0$. This leads to the following functional equation for the generating function $G(x, y, z; t)$:

$$\begin{aligned} G(x,y,z;t) =& x^u y^v z^w + G(x,y,z;t)t\left(\frac{\beta x}{y}+\frac{\alpha y}{x}+\frac{\alpha x}{z}+\frac{\beta z}{x}+\frac{\beta y}{z}+\frac{\alpha z}{y}\right) \\ & - G(0,y,z)\,t\left(\frac{\alpha y}{x}+\frac{\beta z}{x}\right) - G(x,0,z)\,t\left(\frac{\beta x}{y}+\frac{\alpha z}{y}\right) \\ & - G(x,y,0)\,t\left(\frac{\alpha x}{z}+\frac{\beta y}{z}\right). \end{aligned} \tag{3.25}$$

Similarly to equation (3.14), the monomial $x^u y^v z^w$ corresponds to a zero-step walk starting (and ending) at $\omega_0 = (u, v, w)$, the second term corresponds to appending any of the steps in Ω_2 (irrespective of whether the resulting walk steps violates the boundary condition and leaves the domain), and any overcounting is adjusted by the remaining three terms, each of which accounts for stepping over one of the three boundary edges of the triangle.

Again, as this is a functional equation for the generating function $G(x, y, z; t)$ in the variables x, y, z only, the t-dependence is dropped by writing $G(x, y, z; t) \equiv G(x, y, z)$. We rewrite the functional equation (3.25) as

$$\begin{aligned} & G(x,y,z)\left[1-t\left(\frac{\beta x}{y}+\frac{\alpha y}{x}+\frac{\alpha x}{z}+\frac{\beta z}{x}+\frac{\beta y}{z}+\frac{\alpha z}{y}\right)\right] = x^u y^v z^w \\ & -G(0,y,z)\,t\left(\frac{\alpha y}{x}+\frac{\beta z}{x}\right)-G(x,0,z)\,t\left(\frac{\beta x}{y}+\frac{\alpha z}{y}\right)-G(x,y,0)\,t\left(\frac{\alpha x}{z}+\frac{\beta y}{z}\right). \end{aligned} \tag{3.26}$$

The Kernel

Again, the Kernel $K(x, y, z; t) \equiv K(x, y, z)$ of the functional equation,

$$K(x, y, z) = 1 - t\left(\frac{\beta x}{y} + \frac{\alpha y}{x} + \frac{\alpha x}{z} + \frac{\beta z}{x} + \frac{\beta y}{z} + \frac{\alpha z}{y}\right) , \tag{3.27}$$

is needed, and symmetry properties of this Kernel are central to our arguments. As before, note that the Kernel is homogeneous of degree zero, *i.e.* it is invariant under rescaling of all the variables, and this trivial symmetry will be implicitly assumed in the considerations below.

Looking again at $\mathrm{G}(S)$, the group of transformations which leaves the Kernel of the functional equation invariant for the step-set S, the step-set for the triangle model is

$$S_2 = \left\{\frac{\beta x}{y}, \frac{\alpha y}{x}, \frac{\alpha x}{z}, \frac{\beta z}{x}, \frac{\beta y}{z}, \frac{\alpha z}{y}\right\} ,$$

and $\mathrm{G}(S_2)$ is generated by a rotation and an inversion.

Lemma 3.9. *The Kernel $K(x, y, z)$ is invariant under action of the group of transformations*

$$\mathrm{G}(S_2) = \left\langle (y, z, x), \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) \right\rangle \cong C_3 \times C_2 .$$

Moreover, there is a one-variable sub-set which has useful consequences.

Lemma 3.10. *The Kernel $K(x, y, z)$ is invariant under the following 1-parameter substitutions.*

$$\begin{aligned} K(1, 1, p) = K(1, p, 1) = K(p, 1, 1) = K(1, p, p) = K(p, 1, p) = K(p, p, 1) \\ = 1 - t(\alpha + \beta)\left(p + 1 + \frac{1}{p}\right) . \end{aligned} \tag{3.28}$$

Fixing the dependence between p and t such that

$$1 - t(\alpha + \beta)(p + 1 + 1/p) = 0 \tag{3.29}$$

gives $p = (\alpha + \beta)tM((\alpha + \beta)t)$, where

$$M(t) = \frac{1 - t - \sqrt{(1+t)(1-3t)}}{2t^2} \tag{3.30}$$

is the generating function of Motzkin paths [9, p. 6]. In particular, p is a well-defined power series in t with zero constant.

Using this dependency and substituting the six choices from Lemma 3.10 into the functional equation (3.26) then implies

$$\frac{(\alpha+\beta)t}{p}G(0,1,1) + t(\alpha + \beta p)G(p,0,1) + t(\alpha p + \beta)G(p,1,0) = p^u\,, \tag{3.31a}$$

$$\frac{(\alpha+\beta)t}{p}G(1,0,1) + t(\alpha p + \beta)G(0,p,1) + t(\alpha + \beta p)G(1,p,0) = p^v\,, \tag{3.31b}$$

$$\frac{(\alpha+\beta)t}{p}G(1,1,0) + t(\alpha + \beta p)G(0,1,p) + t(\alpha p + \beta)G(1,0,p) = p^w\,, \tag{3.31c}$$

$$(\alpha+\beta)tpG(0,p,p) + t\left(\alpha + \frac{\beta}{p}\right)G(1,0,p) + t\left(\frac{\alpha}{p} + \beta\right)G(1,p,0) = p^v p^w\,, \tag{3.31d}$$

$$(\alpha+\beta)tpG(p,0,p) + t\left(\frac{\alpha}{p} + \beta\right)G(0,1,p) + t\left(\alpha + \frac{\beta}{p}\right)G(p,1,0) = p^u p^w\,, \tag{3.31e}$$

$$(\alpha+\beta)tpG(p,p,0) + t\left(\alpha + \frac{\beta}{p}\right)G(0,p,1) + t\left(\frac{\alpha}{p} + \beta\right)G(p,0,1) = p^u p^v\,. \tag{3.31f}$$

Using homogeneity of the generating function, we replace

$$G(p,p,0) = p^L G(1,1,0)\,, \quad G(p,0,p) = p^L G(1,0,1)\,, \quad G(0,p,p) = p^L G(0,1,1)\,, \tag{3.32}$$

and from the linear combination $[(3.31a)+(3.31b)+(3.31c)]-p[(3.31d)+(3.31e)+(3.31f)]$ it is easily found that

$$(\alpha+\beta)t[G(0,1,1)+G(1,0,1)+G(1,1,0)] = \frac{p^{u+1}+p^{v+1}+p^{w+1}-p^{2+L}(p^{-u}+p^{-v}+p^{-w})}{1-p^{3+L}} \; . \tag{3.33}$$

Substituting $(x,y,z)=(1,1,1)$ into (3.26) shows that $G(1,1,1)$ can now be computed explicitly, as

$$(1-3(\alpha+\beta)t)G(1,1,1) = 1-(\alpha+\beta)t[G(0,1,1)+G(1,0,1)+G(1,1,0)] \; . \tag{3.34}$$

Substituting (3.33) into (3.34) and eliminating t via (3.29) gives the desired final result,

$$G(1,1,1) = \frac{(1-p^3)(1-p^{u+1})(1-p^{v+1})(1-p^{w+1})}{(1-p)^3(1-p^{3+L})} \; . \tag{3.35}$$

Finally, note that substituting $p(t)=(\alpha+\beta)tM((\alpha+\beta)t)$ followed by $(\alpha+\beta)t=s$ into (3.29) implies that

$$M(s) = 1+sM(s)+s^2M(s)^2 \; , \tag{3.36}$$

whence $M(s)$ is the Motzkin path generating function. This completes the proof of Theorem 3.3.

Letting $(u,v,w)=(L,0,0)$ in (3.10) implies that the generating function for walks starting in a corner is given by

$$G(1,1,1) = \frac{(1-p^3)(1-p^{1+L})}{(1-p)(1-p^{3+L})} \; . \tag{3.37}$$

Continued Fractions

Equation (3.37) is intimately related to the convergents of the continued fraction expansion of the Motzkin path generating function. We can show by mathematical induction that in this case $G(1,1,1)$ can be written as a continued fraction. More precisely, for $L = 2H$ even, there is a continued fraction of length H,

$$\underbrace{\cfrac{1}{1-(\alpha+\beta)t-\cfrac{(\alpha+\beta)^2t^2}{1-(\alpha+\beta)t-\cfrac{(\alpha+\beta)^2t^2}{\ddots-\cfrac{(\alpha+\beta)^2t^2}{1-(\alpha+\beta)t-(\alpha+\beta)^2t^2}}}}}_{\text{length } H} = \frac{(1-p^3)(1-p^{1+2H})}{(1-p)(1-p^{3+2H})}\,, \tag{3.38}$$

and for $L = 2H+1$ odd, there is a continued fraction of length $H+1$,

$$\underbrace{\cfrac{1}{1-(\alpha+\beta)t-\cfrac{(\alpha+\beta)^2t^2}{1-(\alpha+\beta)t-\cfrac{(\alpha+\beta)^2t^2}{\ddots-\cfrac{(\alpha+\beta)^2t^2}{1-(\alpha+\beta)t}}}}}_{\text{length } H+1} = \frac{(1-p^3)(1-p^{2+2H})}{(1-p)(1-p^{4+2H})}\,. \tag{3.39}$$

Using equation (3.29) it is easy to show that equations (3.38) and (3.39) hold for the base case $H = 0$,

$$1 = \frac{(1-p^3)(1-p^{1+0})}{(1-p^{1+0})(1-p^{3+0})}\,, \tag{3.40a}$$

$$\frac{1}{1-(\alpha+\beta)t} = \frac{(1-p^3)(1-p^{2+0})}{(1-p)(1-p^{4+0})}\,, \tag{3.40b}$$

and the inductive step follows from showing that

$$\frac{(1-p^3)(1-p^{1+(L+2)})}{(1-p)(1-p^{3+(L+2)})} = \frac{1}{1-(\alpha+\beta)t-(\alpha+\beta)^2t^2\dfrac{(1-p^3)(1-p^{1+L})}{(1-p)(1-p^{3+L})}} \,. \tag{3.41}$$

From the combinatorial theory of continued fractions discussed in Section 1.5, the combinatorial interpretation in terms of Motzkin paths follows easily. This immediately implies Corollary 3.4. Substituting $\alpha = 1,\ \beta = 1$ into (3.38) and (3.39) gives coefficients 2 and 4, of t and t^2 respectively, following from the fact that the relevant Motzkin paths are bicoloured, and substituting in $\alpha = 1,\ \beta = 0$ gives the interpretation in terms of normal Motzkin paths.

Further Results

Attempting to solve the triangle model in full generality proved beyond the reach of the techniques used here, as the system of equations (3.31) is underdetermined, linking nine quantities with six equations. The only case in which we can extract further information from it is one of high symmetry, namely when the starting point is chosen to be in the centre of the triangle, i.e. $\omega_0 = (u, u, u)$, in which the triangle has size $L = 3u$. The equations (3.31) then reduce to two equations in two unknowns,

$$\frac{(\alpha+\beta)t}{p}G(1,1,0) + (\alpha+\beta)t(1+p)G(p,1,0) = p^u \,, \tag{3.42a}$$

$$(\alpha+\beta)tp^{1+3u}G(1,1,0) + (\alpha+\beta)t\left(1+\frac{1}{p}\right)G(p,1,0) = p^{2u} \,. \tag{3.42b}$$

This can readily be solved, and

$$(\alpha+\beta)tG(1,1,0) = \frac{p^{1+u}(1-p^{1+u})}{1-p^{3+3u}}\ . \tag{3.43}$$

Eliminating t by using (3.29) proves Proposition 3.5.

It now remains to prove our final result. Without loss of generality, starting in the corner marked by coordinates $(L,0,0)$, the steps $(-1,0,1)$, $(0,1,-1)$, and $(1,-1,0)$ can be mapped to $(1,1)$, $(1,-1)$, and $(1,0)$, respectively. This maps steps in Ω_2'' to steps in three-candidate Ballot paths and the restrictions imposed by the boundaries of the triangle clearly transfer to the restrictions on a Ballot path with excess L. This proves Proposition 3.6.

3.3 Remark

We conclude this chapter with a remark on the general problem for the triangle model.

Generating functions for walks in finite domains are rational; this is a direct result of the fact that the adjacency matrix for such systems has finite dimension. In particular, in the triangle model, a triangle of side-length L contains $\binom{L+2}{2}$ vertices, or states, and therefore we would expect the degree of the numerator and denominator of the generating function to grow quadratically in L. However, for the cases where we are able to prove results, there is some cancellation such that the growth is linear in L.

The process of finding our results began with some initial series generation which allowed us to predict the form of the generating functions. Using this for the general problem for the triangle model, of walks with arbitrary fixed start and end points, we have numerical evidence that in general the degrees of the numerator and denominator grow quadratically in L, and it may be this extra complexity that has prevented us from solving this problem with our method.

Chapter 4

Bijections between the triangular domain and Motzkin paths

In the previous chapter, we proved an intriguing equinumeracy result in Corollary 3.4. Of particular interest are the two special cases noted beneath it; the equinumeracy between walks on the undirected triangular domain and bicoloured Motzkin paths, and that between walks on one of the directed sublattices and normal Motzkin paths. Taking only steps on the directed sublattice Ω_2' halves the out-degree of every vertex in the domain, and so it is clear that the result for bicoloured Motzkin paths implies the result for normal Motzkin paths. This leads us to the problem of finding a purely bijective proof of Corollary 3.4. Were one to be found, this might elucidate the connections between the triangle model and continued fractions, and thus open avenues towards solving other models. Eu [10] gives a bijective proof of the directed case for triangular domains of infinite side-length via standard Young tableaux (which are a coding of Yamanouchi words), and Yeats [34] gives a bijective proof of the undirected case for domains of infinite side-length using intermediate markings. This leaves open the problem for all finite side-lengths.

In this chapter we present bijective proofs for the directed and undirected cases for side-lengths $L = 1, 2$ and 3. These bijections are heavily based on symmetry and restrictive properties of the small domains the paths lie on. Unfortunately, as the domains become larger, these properties fade, and thus we have been unable to expand the ideas behind these bijections to larger domains. We are therefore forced to leave open the general problem. We note that Proposition 3.6 provides a possible alternative route to a bijective proof, via three-candidate Ballot paths.

In this chapter we will use the word walks when we are referring to walks on a triangular domain, and paths when referring to Motzkin paths. Let $\mathcal{T}_L$ denote the set of walks, starting in the corner, on the undirected triangle of side-length L and $\mathcal{T}_L'$ denote the set of walks, starting in the corner, on the directed triangle of side-length L. We note that the result for side-length $L = 0$ is trivial, and begin by proving the result for $L = 1$.

4.1 Bijections for side-length 1

In this case we will begin by giving the bijection for the undirected domain. The result to prove here is a bijection between walks on a triangular domain of side-length 1 and bicoloured Motzkin paths in the strip of width 0. As expected, this is a trivial result. We divide the steps on T_1 into two classes; those which move in a clockwise direction, which we denote by a, and those which move in an anticlockwise direction, denoted by b (see Figure 4.1). Then we define the map ϕ as follows.

Let a walk of n steps be denoted by $\omega_1\omega_2\ldots\omega_n$. We define the map $\phi : \mathcal{T}_1 \to \mathcal{M}_0$ as follows:

- $\phi(\epsilon) = \xi$,
- $\phi(a) = H_0^{\{w\}}$,
- $\phi(b) = H_0^{\{b\}}$,
- $\phi(\omega_1\omega_2\ldots\omega_n) = \phi\Big(\prod_{i=1}^{s}\omega_i\Big) = \Big(\prod_{i=1}^{s}\phi(\omega_i)\Big)$,

where $\omega_i \in \{a, b\}$ and ϵ and ξ denote the empty walk on $\mathcal{T}_1$ and the empty path on $\mathcal{M}_0$ respectively.

The inverse to this map is clear, and it is clear that it performs the required mapping. Note that the map easily restricts to one between the directed triangle and the monocoloured Motzkin strip. This restricted map similarly has an obvious inverse, so is a bijection.

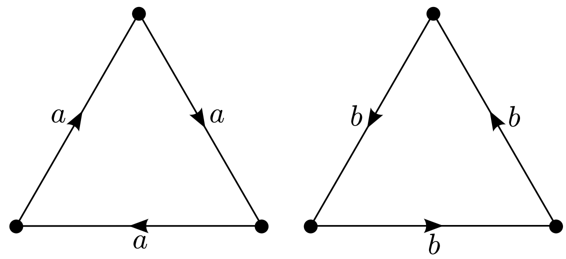

Figure 4.1: Triangle of side-length 1 with associated steps.

4.2 Bijections for side-length 2

In this section we will give bijections between the triangular domain of side-length 2 and the restricted Motzkin strip of width 1.

4.2.1 The directed case

We start by giving a bijection for directed walks and normal Motzkin paths. This bijection works on the heuristic idea that a walk on the directed triangular domain of side-length 2 which starts and ends at middle vertices (vertices of degree 4) looks like a Motzkin path in a restricted strip of width 1 (Figure 4.2). That is, a path at a middle vertex has two options - it can either take one step and end back at another middle vertex, or it can take a step to a corner vertex and then another step back to a middle vertex. In a similar way, if a Motzkin path is at height zero, it has two options - it can take one horizontal step and remain at height zero, or it can take an up-step followed immediately by a down-step, ending back at height zero. However, we must apply some corrective steps to take into account the first,

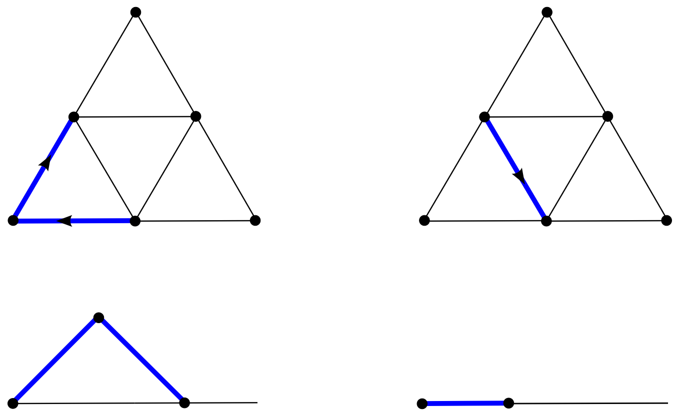

Figure 4.2: The two short triangular walks that are the building blocks for all walks starting and ending at middle vertices compared against the two short Motzkin paths that are the building blocks for all Motzkin paths in the restricted strip of width 1.

and possibly the last, step of the walk.

As in Section 4.1, let a walk of n steps be denoted by $\omega_1\omega_2\dots\omega_n$. We label the steps of the domain as in Figure 4.3. Note that we only consider walks that start in the corner of the triangle, so walks where $\omega_1 = a$.

Defining $\mathcal{S}'$ to be the set of symbols $\{a, b, c\}$, we can define the map $\psi' : \mathcal{S}' \to \mathcal{L}_2'$ as follows:

- $\psi'(a) = D_1\,,$
- $\psi'(b) = H_0\,,$
- $\psi'(c) = U_1\,.$

We can then define our bijection $\phi' : \mathcal{T}_2' \to \overline{\mathcal{M}}_1'$ as follows:

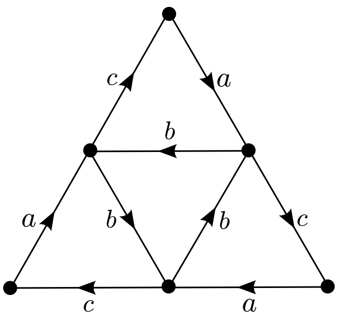

Figure 4.3: The step-set for the directed triangle of side-length 2

- if $n = 0$ then $\phi'(\epsilon) = \xi$;
- if $n \geq 1$, $\omega_1 = a$ and $\omega_n = c$, then

$$\phi'(\omega_1 \dots \omega_n) = U_1 D_1 \Big(\prod_{i=2}^{n-1} \psi'(\omega_i) \Big) ;$$

- if $n \geq 1$, $\omega_1 = a$ and $\omega_n \neq c$, then

$$\phi'(\omega_1 \dots \omega_n) = H_0 \Big(\prod_{i=2}^{n} \psi'(\omega_i) \Big) ;$$

where ϵ and ξ denote the empty paths on $\mathcal{T}_2'$ and $\overline{\mathcal{M}}_1'$ respectively.

Two examples of ϕ' acting on walks are given in Figure 4.4. The two mappings shown illustrate the fundamental property of the mapping, which alters the initial steps of the Motzkin path in accordance with where the triangular walk ends, but preserves the structure of the main body of the path.

4.2.2 The undirected case

We now generalise ϕ' to the undirected case by giving the triangular domain two orientations, with each orientation linked to one colouring of the corresponding

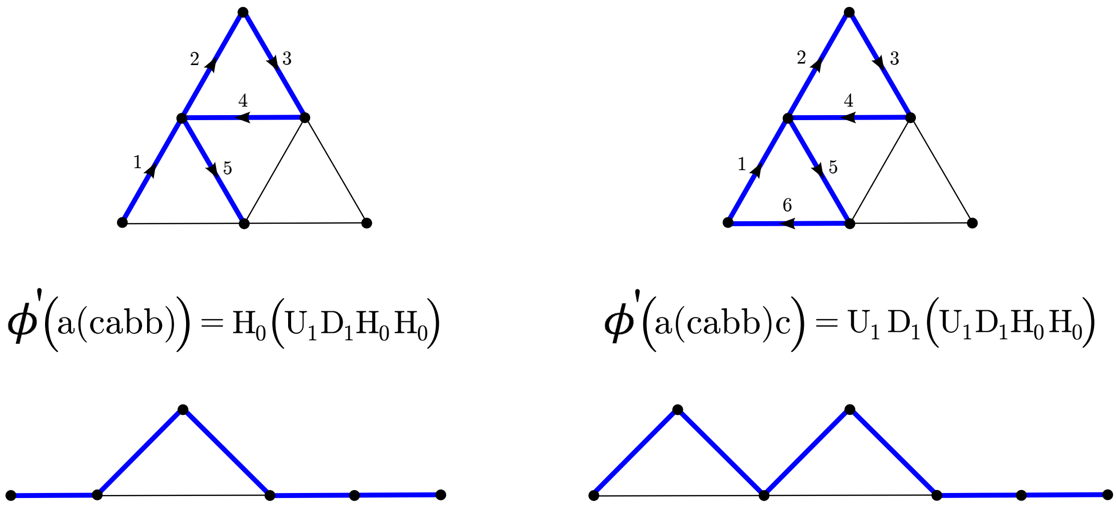

Figure 4.4: Examples of paths mapped under ϕ, showing how the mapping changes depending on whether the path ends at a middle or at a corner vertex.

Motzkin path. Here, we label the steps of the domain as in Figure 4.5. Note that now $\omega_1 \in \{a, a'\}$.

We define $\mathcal{S}$ to be the set of symbols $\{a, b, c, a', b', c'\}$, and then define the map $\psi : \mathcal{S} \to \mathcal{L}_2$ as follows:

- $\psi(a) = D_1^{\{w\}}$,
- $\psi(b) = H_0^{\{w\}}$,
- $\psi(c) = U_1^{\{w\}}$,
- $\psi(a') = D_1^{\{b\}}$,
- $\psi(b') = H_0^{\{b\}}$,
- $\psi(c') = U_1^{\{b\}}$.

We can then define our bijection $\phi : \mathcal{T}_2 \to \overline{\mathcal{M}}_1$ as follows:

- if $n = 0$ then $\phi(\epsilon) = \xi$;

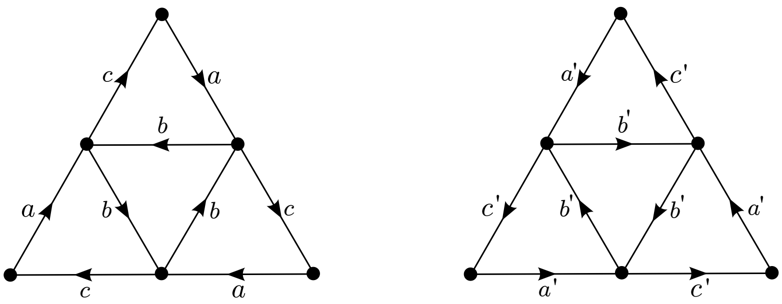

Figure 4.5: The step-set for the undirected triangle of side-length 2.

- if $n \geq 1$, $\omega_1 \in \{a, a'\}$ and $\omega_n \in \{c, c'\}$, then

$$\phi(\omega_1 \dots \omega_n) = U_1^{\{w\}} D_1^{\{w\}} \Big(\prod_{i=2}^{n-1} \psi(\omega_i) \Big) \qquad \text{if } \omega_1 = a \text{ and } \omega_n = c\,,$$

$$\phi(\omega_1 \dots \omega_n) = U_1^{\{w\}} D_1^{\{b\}} \Big(\prod_{i=2}^{n-1} \psi(\omega_i) \Big) \qquad \text{if } \omega_1 = a \text{ and } \omega_n = c'\,,$$

$$\phi(\omega_1 \dots \omega_n) = U_1^{\{b\}} D_1^{\{w\}} \Big(\prod_{i=2}^{n-1} \psi(\omega_i) \Big) \qquad \text{if } \omega_1 = a' \text{ and } \omega_n = c\,,$$

$$\phi(\omega_1 \dots \omega_n) = U_1^{\{b\}} D_1^{\{b\}} \Big(\prod_{i=2}^{n-1} \psi(\omega_i) \Big) \qquad \text{if } \omega_1 = a' \text{ and } \omega_n = c'\,;$$

- if $n \geq 1$, $\omega_1 \in \{a, a'\}$ and $\omega_n \notin \{c, c'\}$, then

$$\phi(\omega_1 \dots \omega_n) = H_0^{\{w\}} \Big(\prod_{i=2}^{n} \psi(\omega_i) \Big) \qquad \text{if } \omega_1 = a\,,$$

$$\phi(\omega_1 \dots \omega_n) = H_0^{\{b\}} \Big(\prod_{i=2}^{n} \psi(\omega_i) \Big) \qquad \text{if } \omega_1 = a'\,;$$

where ϵ and ξ denote the empty walk on $\mathcal{T}_2$ and the empty path on $\overline{\mathcal{M}}_1$ respectively.

4.2.3 Proof

We now proceed to prove that ϕ is our required bijection. It should be noted that this proof easily restricts to ϕ'.

Proposition 4.1. *ϕ gives an explicit bijection from the set of walks of length n on the triangle of side-length* 2 *to the set of bicoloured Motzkin paths of length n in the restricted strip of width* 1*.*

Proof. We begin by noting that ϕ maps walks in $\mathcal{T}_2$ to Motzkin paths of the same length. We must now show that walks on the triangle map to paths in $\overline{\mathcal{M}}_1$; that is, bicoloured Motzkin paths in the restricted strip of width 1. For a walk with last step $\omega_n \in \{c, c'\}$, this maps to a path where the first two steps $U_1^{\{x\}} D_1^{\{x\}}$ (where $x \in \{w, b\}$) form a path in $\overline{\mathcal{M}}_1$. As the concatenation of two paths in $\overline{\mathcal{M}}_1$ is also a path in $\overline{\mathcal{M}}_1$, it remains to show that $\prod_{i=2}^{n-1} \psi(\omega_i)$ is a path in $\overline{\mathcal{M}}_1$. To show this, we note that ω_1 ends at a middle vertex, and ω_n starts at a middle vertex, so the walk $\omega_2 \ldots \omega_{n-1}$ both starts and ends at middle vertices. Similarly, for a walk where the last step $\omega_n \notin \{c, c'\}$, we just need to show that $\prod_{i=2}^{n} \psi(\omega_i)$ is a path in $\overline{\mathcal{M}}_1$, and as $\omega_1 \in \{a, a'\}$ and $\omega_n \notin \{c, c'\}$, the walk $\omega_2 \ldots \omega_n$ both starts and ends at middle vertices. Thus in both cases, we are finished once we show that, if $\omega_j \ldots \omega_k$ is a walk that both starts and ends at middle vertices, then $\prod_{i=j}^{k} \psi(\omega_i)$ is a path in $\overline{\mathcal{M}}_1$.

As is clear from Figure 4.5, if any $\omega_i \in \{c, c'\}$, then $\omega_{i+1} \in \{a, a'\}$ (note that $\omega_k \notin \{c, c'\}$). As c and c' are the only steps which map to up-steps under ψ, both map to up-steps at height 1, and a and a' both map to down-steps under ψ, this means that, in any path in the image, the only up-steps are at height 1 and must be immediately followed by a down-step. Thus the path never goes above height 1, and any horizontal step must be at height ≤ 0. Similarly, if any $\omega_i \in \{a, a'\}$, then $\omega_{i-1} \in \{c, c'\}$ (note that $\omega_j \notin \{a, a'\}$), and as a and a' are the only steps which map to down-steps under ψ, any down-step must be immediately preceded by an up-step. As the only up-steps are at height 1, this means that the only down-steps are at height 1, and the path never goes below height 0. Similarly, if any $\omega_i \in \{b, b'\}$ for $j + 1 \leq i \leq k$, then $\omega_{i-1} \in \{a, a', b, b'\}$. As b and b' are the only steps which map to horizontal steps under ψ, any horizontal step must be preceded by either a horizontal step or a down-step at height 1. Thus the only horizontal steps are at height 0. Finally, as the path starts and ends at middle vertices, $\omega_j \in \{b, b', c, c'\}$ and $\omega_k \in \{a, a', b, b'\}$, so ω_j maps to a step starting at height 0, and ω_k maps to a step ending at height 0. Thus, all walks starting and ending at middle vertices are mapped to paths in $\overline{\mathcal{M}}_1$, so ϕ maps paths in $\mathcal{T}_2$ to Motzkin paths in the restricted strip of width 1.

We now proceed to showing that ϕ is a bijection by giving its inverse $f : \overline{\mathcal{M}}_1 \to \mathcal{T}_2$. We first define $g : \mathcal{L}_2 \to \mathcal{S}$:

- $g\big(D_1^{\{w\}}\big) = a\,,$
- $g\big(D_1^{\{b\}}\big) = a'\,,$
- $g\big(H_0^{\{w\}}\big) = b\,,$
- $g\big(H_0^{\{b\}}\big) = b'\,,$
- $g\big(U_1^{\{w\}}\big) = c\,,$
- $g\big(U_1^{\{b\}}\big) = c'\,.$

Let a path of n steps be denoted by $\kappa_1\kappa_2\ldots\kappa_n$. We then define $f : \overline{\mathcal{M}}_1 \to \mathcal{T}_2$ as follows:

- if $n = 0$ then $f(\xi) = \epsilon\,;$
- if $n \geq 1,\ \kappa_1 \in \{U_1^{\{w\}}, U_1^{\{b\}}\}$ and $\kappa_2 \in \{D_1^{\{w\}}, D_1^{\{b\}}\}$, then

$$f(\kappa_1\ldots\kappa_n) = a\Big(\prod_{i=3}^{n} g(\omega_i)\Big)c \qquad \text{if } \kappa_1 = U_1^{\{w\}} \text{ and } \kappa_2 = D_1^{\{w\}}\,,$$

$$f(\kappa_1\ldots\kappa_n) = a\Big(\prod_{i=3}^{n} g(\omega_i)\Big)c' \qquad \text{if } \kappa_1 = U_1^{\{w\}} \text{ and } \kappa_2 = D_1^{\{b\}}\,,$$

$$f(\kappa_1\ldots\kappa_n) = a'\Big(\prod_{i=3}^{n} g(\omega_i)\Big)c \qquad \text{if } \kappa_1 = U_1^{\{b\}} \text{ and } \kappa_2 = D_1^{\{w\}}\,,$$

$$f(\kappa_1\ldots\kappa_n) = a'\Big(\prod_{i=3}^{n} g(\omega_i)\Big)c' \qquad \text{if } \kappa_1 = U_1^{\{b\}} \text{ and } \kappa_2 = D_1^{\{b\}}\,;$$

- if $n \geq 1,\ \kappa_1 \in \{H_0^{\{w\}}, H_0^{\{b\}}\}$, then

$$f(\kappa_1\ldots\kappa_n) = a\Big(\prod_{i=2}^{n} g(\omega_i)\Big) \qquad \text{if } \kappa_1 = H_0^{\{w\}}\,,$$

$$f(\kappa_1\ldots\kappa_n) = a'\Big(\prod_{i=2}^{n} g(\omega_i)\Big) \qquad \text{if } \kappa_1 = H_0^{\{b\}}\,;$$

where ϵ and ξ denote the empty walk/path on $\mathcal{T}_2$ and $\overline{\mathcal{M}}_1$ respectively.

Note that g is an inverse to ψ:

$$
\begin{aligned}
g(\psi(a)) &= g(D_1^{\{w\}}) = a\,, & g(\psi(a')) &= g(D_1^{\{b\}}) = a'\,,\\
g(\psi(b)) &= g(H_0^{\{w\}}) = b\,, & g(\psi(b')) &= g(H_0^{\{b\}}) = b'\,,\\
g(\psi(c)) &= g(U_1^{\{w\}}) = c\,, & g(\psi(c')) &= g(U_1^{\{b\}}) = c',\\
\psi(g(D_1^{\{w\}})) &= \psi(a) = D_1^{\{w\}}\,, & \psi(g(D_1^{\{b\}})) &= \psi(a') = D_1^{\{b\}}\,,\\
\psi(g(H_0^{\{w\}})) &= \psi(b) = H_0^{\{w\}}\,, & \psi(g(H_0^{\{b\}})) &= \psi(b') = H_0^{\{b\}}\,,\\
\psi(g(U_1^{\{w\}})) &= \psi(c) = U_1^{\{w\}}\,, & \psi(g(U_1^{\{b\}})) &= \psi(c') = U_1^{\{b\}}\,.
\end{aligned}
$$

It remains to show that f is an inverse to ϕ. We first deal with the trivial cases. If $n = 0$ then $f(\phi(\epsilon)) = f(\xi) = \epsilon$ and $\phi(f(\xi)) = \phi(\epsilon) = \xi$. Now if a path of $n \geq 1$ steps $\omega_1\omega_2\ldots\omega_n$ is in $\mathcal{T}_2$, we have that:

- if $\omega_1 \in \{a, a'\}$ and $\omega_n \in \{c, c'\}$, then

$$
\begin{aligned}
f(\phi(\omega_1 \ldots \omega_n)) &= f\Big(U_1^{\{x\}} D_1^{\{y\}} \Big(\prod_{i=2}^{n-1} \psi(\omega_i)\Big)\Big)\\
&= p\Big(\prod_{i=2}^{n-1} g(\psi(\omega_i))\Big)q\\
&= p\Big(\prod_{i=2}^{n-1} \omega_i\Big)q\\
&= \omega_1 \ldots \omega_n
\end{aligned}
$$

  where

  $x = w$ and $p = a$ if $\omega_1 = a$,

  $x = b$ and $p = a'$ if $\omega_1 = a'$,

$y = w$ and $q = c$ if $\omega_1 = c$,

$y = b$ and $q = c'$ if $\omega_1 = c'$;

- if $\omega_1 \in \{a, a'\}$ and $\omega_n \notin \{c, c'\}$, then

$$\begin{aligned} f(\phi(\omega_1 \dots \omega_n)) &= f\Big(H_0^{\{x\}}\Big(\prod_{i=2}^{n} \psi(\omega_i)\Big)\Big) \\ &= p\Big(\prod_{i=2}^{n} g(\psi(\omega_i))\Big) \\ &= p\Big(\prod_{i=2}^{n} \omega_i\Big) \\ &= \omega_1 \dots \omega_n \end{aligned}$$

where

$x = w$ and $p = a$ if $\omega_1 = a$,

$x = b$ and $p = a'$ if $\omega_1 = a'$.

If a path of $n \geq 1$ steps $\kappa_1 \kappa_2 \dots \kappa_n$ is in $\overline{\mathcal{M}}_1$, we similarly see that:

- if $\kappa_1 \in \{U_1^{\{w\}}, U_1^{\{b\}}\}$ and $\kappa_2 \in \{D_1^{\{w\}}, D_1^{\{b\}}\}$, then

$$\begin{aligned} \phi(f(\kappa_1 \dots \kappa_n)) &= \phi\Big(p\Big(\prod_{i=3}^{n} g(\kappa_i)\Big)q\Big) \\ &= U_1^{\{x\}} D_1^{\{y\}}\Big(\prod_{i=3}^{n} \psi(g(\kappa_i))\Big) \\ &= U_1^{\{x\}} D_1^{\{y\}}\Big(\prod_{i=3}^{n} \kappa_i\Big) \\ &= \kappa_1 \dots \kappa_n \end{aligned}$$

where

$x = w$ and $p = a$ if $\kappa_1 = U_1^{\{w\}}$,

$$x = b \text{ and } p = a' \text{ if } \kappa_1 = U_1^{\{b\}}\,,$$

$$y = w \text{ and } q = c \text{ if } \kappa_1 = D_1^{\{w\}}\,,$$

$$y = b \text{ and } q = c' \text{ if } \kappa_1 = D_1^{\{b\}}\,;$$

- if $\kappa_1 \in \{H_0^{\{w\}}, H_0^{\{b\}}\}$, then

$$\begin{aligned}
\phi(f(\kappa_1 \dots \kappa_n)) &= \phi\Big(p\Big(\prod_{i=2}^{n} g(\kappa_i)\Big)\Big) \\
&= H_0^{\{x\}}\Big(\prod_{i=2}^{n} \psi(g(\kappa_i))\Big) \quad (\star) \\
&= H_0^{\{x\}}\Big(\prod_{i=2}^{n} \kappa_i\Big) \\
&= \kappa_1 \dots \kappa_n
\end{aligned}$$

  where

$$x = w \text{ and } p = a \text{ if } \kappa_1 = H_0^{\{w\}}\,,$$

$$x = b \text{ and } p = a' \text{ if } \kappa_1 = H_0^{\{b\}}\,.$$

$(\star)$ This equality is valid as $\kappa_n \notin \{U_1^{\{w\}}, U_1^{\{b\}}\}$, so $g(\kappa_n) \notin \{c, c'\}\big)$.

Thus f is an inverse to ϕ and so ϕ is a bijection.

$\square$

4.3 Bijections for side-length 3

In this section we will give bijections between the triangular domain of side-length 3 and the Motzkin strip of width 1. As in Section 4.2, we will begin by giving the bijection for the directed triangular domain, and monocoloured Motzkin paths.

4.3.1 The directed case

The heuristic idea behind this bijection is that on the directed triangular domain, the vertices can be divided into three subsets (labelled $0, 1, 2$), such that any walk of length 3 which starts at a vertex in one subset will end at a vertex in the same subset. Using this property, we can cut a walk into segments of length 3, which we then individually map to segments of a Motzkin path, and concatenate to form a Motzkin path. As we shall see below, this process requires 'flipping' some of the segments of the Motzkin path to ensure that the last vertex of one segment is the same as the first vertex of the next segment. It also requires that the first step acts as a corrective step, and a corrective segment at the end of the walk when the length of the walk $n \not\equiv 1 \bmod 3$. We will label the steps of the directed domain of side-length 3 as in Figure 4.6, with the condition that all walks start at the bottom left corner.

Mapping individual segments of a walk

We begin by defining the three maps α', β' and γ' which respectively map segments of lengths $1, 2$ and 3 of triangular walks to segments of Motzkin paths. Let $\mathcal{L}_3'$ be

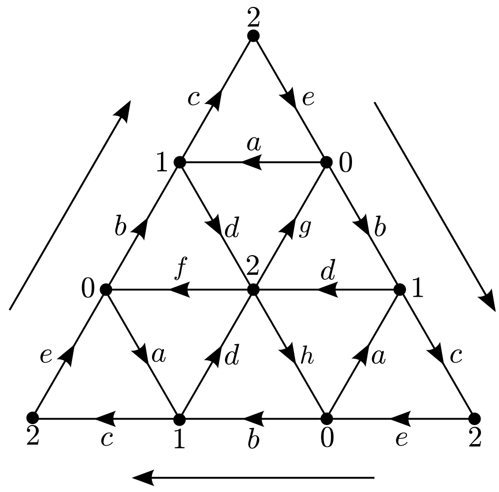

Figure 4.6: The step-set for the directed triangle of side-length 3, with walks starting at the bottom left corner. The vertices are labelled according to the subset they lie in.

as defined in Section 1.2. Defining $\mathcal{W}'$ to be the set of symbols $\{a, b\}$, we define the map $\alpha' : \mathcal{W}' \to \mathcal{L}'_3$ as follows:

- $\alpha'(a) = H_0$,
- $\alpha'(b) = D_1$.

Defining $\mathcal{W}''$ to be the set of symbols $\{ac, ad, bc, bd\}$, we define the map $\beta' : \mathcal{W}'' \to (\mathcal{L}'_3)^2$ as follows:

- $\beta'(ac) = H_0H_0$,
- $\beta'(ad) = H_1D_1$,
- $\beta'(bc) = D_1H_0$,

- $\beta'(bd) = U_1 D_1$.

Defining $\mathcal{W}'''$ to be the set of symbols $\{ace, adf, adg, adh, bce, bdf, bdg, bdh\}$, we define the map $\gamma' : \mathcal{W}''' \to (\mathcal{L}_3')^3$ as follows:

- $\gamma'(ace) = H_0 H_0 H_0$,
- $\gamma'(bce) = D_1 H_0 H_0$,
- $\gamma'(adf) = H_1 H_1 D_1$,
- $\gamma'(bdf) = U_1 H_1 D_1$,
- $\gamma'(adg) = H_1 D_1 H_0$,
- $\gamma'(bdg) = U_1 D_1 H_0$,
- $\gamma'(adh) = H_0 U_1 D_1$,
- $\gamma'(bdh) = D_1 U_1 D_1$.

Flipping and concatenating the segments

We now proceed to define the map δ' which flips the segments of the Motzkin path to ensure that the vertices in consecutive segments match. Once the segments have been mapped under α', β' and γ', we have an ordered list of k segments of Motzkin paths, where the first segment is of length 1, segments 2 to $k-1$ are of length 3, and the final segment is of length $1, 2$ or 3. We need to match the ends of these segments up to ensure that they form a Motzkin path, which we do by flipping them if the vertices do not match. The final segment ends at height 0 (on the x-axis), so we leave that unchanged. We then proceed backwards through the segments. All end at height 0, so we flip a segment precisely when the next segment starts at height 1. More rigorously, this is defined as follows.

Let $\mathcal{M} = \{H_0H_0H_0, H_1H_1D_1, H_1D_1H_0, H_0U_1D_1, D_1H_0H_0, U_1H_1D_1, U_1D_1H_0,$

$D_1U_1D_1\}$

and $N = \{H_0H_0H_0, H_1H_1D_1, H_1D_1H_0, H_0U_1D_1, D_1H_0H_0, U_1H_1D_1, U_1D_1H_0,$

$D_1U_1D_1, H_0H_0, H_1D_1, D_1H_0, U_1D_1, H_0, D_1\}$.

Let $\mathcal{M}^{<\infty} \times \mathcal{N}$ denote the set of words with finitely many letters in the set $\mathcal{M}$ followed by exactly one letter in the set $\mathcal{N}$. Let $(\mathcal{L}_3')^{<\infty}$ denote the set of words of finite length with all letters in $\mathcal{L}_3'$.

We denote by $\lambda_1 \dots \lambda_n$ a word of n letters in $\mathcal{M}^{<\infty} \times \mathcal{N}$. Each letter $\lambda_i \in \mathcal{M}$ can be written as a word in $(\mathcal{L}_3')^3$, where $\lambda_i = \lambda_i^{(1)}\lambda_i^{(2)}\lambda_i^{(3)}$. Its image under δ' can also be written as a word in $(\mathcal{L}_3')^3$, $\delta(\lambda_i) = \overline{\lambda}_i^{(1)}\overline{\lambda}_i^{(2)}\overline{\lambda}_i^{(3)}$. Similarly, the letter $\lambda_n \in \mathcal{N}$, and its image $\delta'(\lambda_n)$, can be written as a word in either $\mathcal{L}_3'$, $(\mathcal{L}_3')^2$ or $(\mathcal{L}_3')^3$. We now have the necessary notation to define $\delta' : \mathcal{M}^{<\infty} \times \mathcal{N} \rightarrow (\mathcal{L}_3')^{<\infty}$, which we do as follows:

- $\delta'(\lambda_n) = \lambda_n$;
- for $1 \leq i \leq n-1$, if $\overline{\lambda}_{i+1}^{(1)} \in \{H_0, U_1\}$ then $\delta'(\lambda_i) = \lambda_i$;
- for $1 \leq i \leq n-1$, if $\overline{\lambda}_{i+1}^{(1)} \in \{H_1, D_1\}$ then

- ∘ if $\lambda_i = H_0H_0H_0$, $\delta'(\lambda_i) = H_1H_1H_1$,
- ∘ if $\lambda_i = D_1H_0H_0$, $\delta'(\lambda_i) = U_1H_1H_1$,
- ∘ if $\lambda_i = H_1H_1D_1$, $\delta'(\lambda_i) = H_0H_0U_1$,
- ∘ if $\lambda_i = U_1H_1D_1$, $\delta'(\lambda_i) = D_1H_0U_1$,
- ∘ if $\lambda_i = H_1D_1H_0$, $\delta'(\lambda_i) = H_0U_1H_1$,
- ∘ if $\lambda_i = U_1D_1H_0$, $\delta'(\lambda_i) = D_1U_1H_1$,
- ∘ if $\lambda_i = H_0U_1D_1$, $\delta'(\lambda_i) = H_1D_1U_1$,
- ∘ if $\lambda_i = D_1U_1D_1$, $\delta'(\lambda_i) = U_1D_1U_1$;

- $$\delta'(\lambda_1\lambda_2\ldots\lambda_n) = \prod_{i=1}^{n} \delta'\big(\lambda_i\big) \, .$$

Mapping the first step

We now define the map which uses the first step as a corrective step, to ensure the path in the image starts on the x-axis. As in Section 4.2, from here on let $\omega_1\omega_2\ldots\omega_n$ denote a walk of n steps on the triangular domain, starting in a corner. Using the notation introduced for δ', we have that $\overline{\lambda}_1^{(1)}$ is the first step of the path $\delta'\Big(\Big(\prod_{\frac{i}{3}=1}^{k/3} \gamma'(\omega_{i-1}\omega_i\omega_{i+1})\Big)\chi'(\omega_{k+2}\ldots\omega_n)\Big)$, where $k = 3\lfloor\frac{n-2}{3}\rfloor$ and $\chi' = \alpha'$, β' or γ' depending on the value of $n-k$. Thus we can define $\rho' : \{e\} \to \{H_0, U_1\}$ as follows:

- if $n = 1$ then $\rho(\omega_1) = H_0$;
- if $n \geq 2$ and $\overline{\lambda}_1^{(1)} \in \{H_0, U_1\}$ then $\rho(\omega_1) = H_0$;
- if $n \geq 2$ and $\overline{\lambda}_1^{(1)} \in \{H_1, D_1\}$ then $\rho(\omega_1) = U_1$.

The bijection

We can now define the bijection $\sigma' : \mathcal{T}_3' \to \mathcal{M}_1'$, which we do as follows:

- if $n = 0$ then $\phi(\epsilon) = \xi$;
- if $n \geq 1$ and $n \equiv 1 \bmod 3$ then
$$\sigma'(\omega_1 \ldots \omega_n) = \rho'(\omega_1)\, \delta'\left(\prod_{\frac{i}{3}=1}^{\frac{n-1}{3}} \gamma'(\omega_{i-1}\, \omega_i\, \omega_{i+1}) \right);$$
- if $n \geq 1$ and $n \equiv 2 \bmod 3$ then
$$\sigma'(\omega_1 \ldots \omega_n) = \rho'(\omega_1)\, \delta'\left(\left(\prod_{\frac{i}{3}=1}^{\frac{n-2}{3}} \gamma'(\omega_{i-1}\, \omega_i\, \omega_{i+1}) \right) \alpha'(\omega_n) \right);$$
- if $n \geq 1$ and $n \equiv 0 \bmod 3$ then
$$\sigma'(\omega_1 \ldots \omega_n) = \rho'(\omega_1)\, \delta'\left(\left(\prod_{\frac{i}{3}=1}^{\frac{n-3}{3}} \gamma'(\omega_{i-1}\, \omega_i\, \omega_{i+1}) \right) \beta'(\omega_{n-1}\omega_n) \right);$$

where ϵ and ξ denote the empty walk on $\mathcal{T}_3'$ and the empty path on $\mathcal{M}_1'$ respectively.

Figure 4.7 shows an example of the map σ' acting on a path in $\mathcal{T}_3'$.

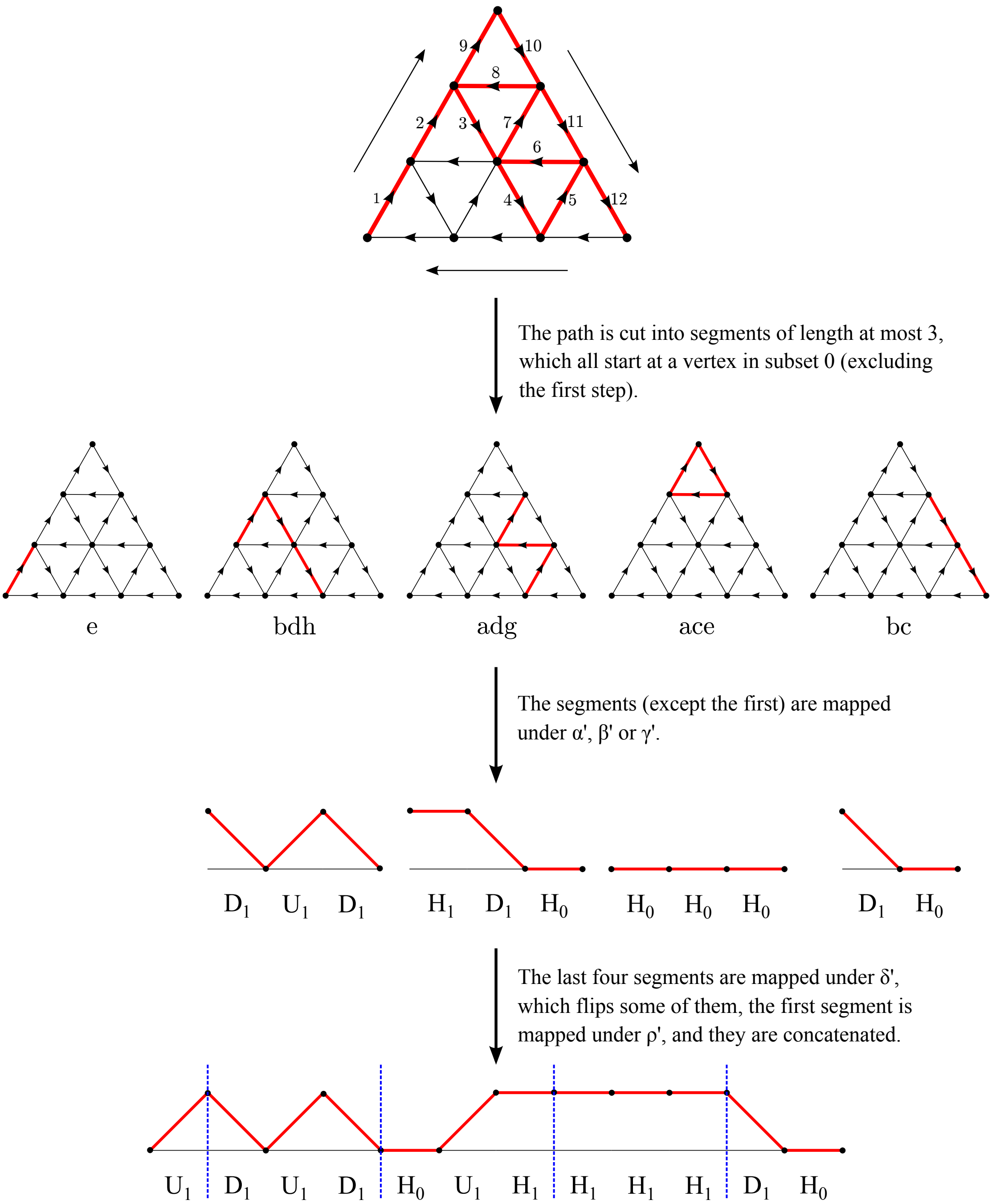

Figure 4.7: An example of σ', mapping a 12-step triangular walk to a 12-step Motzkin path.

4.3.2 The undirected case

We now generalise σ' by giving the triangular domain two orientations, with each orientation linked to one colouring of the corresponding Motzkin path. We will label the steps of the undirected domain of side-length 3 as in Figure 4.8, with the condition that all walks start at the bottom left corner. When defining the bijection for the undirected case, we must proceed somewhat carefully. From hereon we will define a *return to 0* as a step which ends at a vertex in subset 0, and an *exit* as a step which starts at a vertex in subset 0. We will also use the notions of *clockwise* and *anticlockwise* steps to refer respectively to the step-sets shown on the left and right triangles in Figure 4.8, and will label these as C and A respectively.

As before, with the directed case, we split a path on the triangular domain into segments, divided by exits/returns to 0. However here, the segments can be arbitrarily long. They can be divided into three subsets. The first subset consists of just the first segment, which begins with the first step of the path and ends with the first return to 0 (or ends with the last step of the path if it never returns to 0). The third subset consists solely of the last segment, which begins with the last exit (this segment does not exist if the path never returns to 0). The second subset consists of all the intermediate segments, which start with an exit and end with the next return to 0.

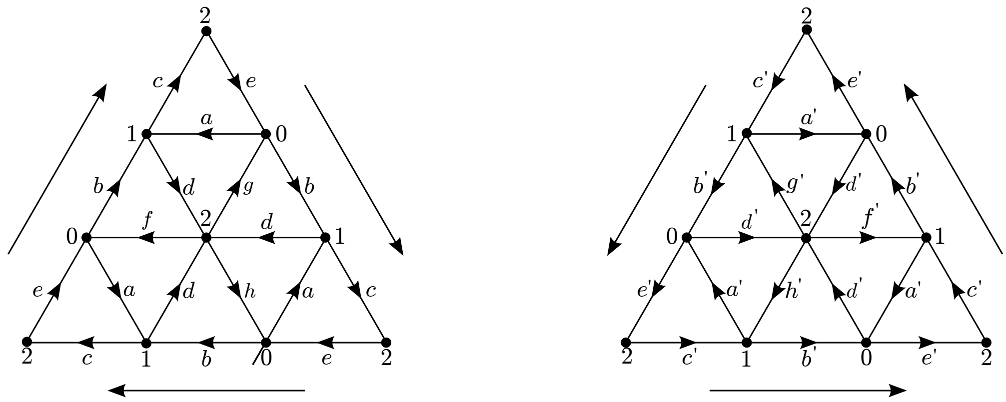

Figure 4.8: The step-set for the triangle of side-length 3, with walks starting at the bottom left corner. The vertices are labelled according to the subset they lie in. Note that if two consecutive steps are aa', bb', cc' and ee', the start and end vertices are the same, but the same is not true for edges going into or away from the central vertex, where dd', ff', gg' and hh' are not valid subpaths.

In the directed case there were a very limited number of types of path in each subset. The first subset consisted of only one path of length 1, the second subset consisted of one type of path, of length 3, and the third subset consisted of three types of path, of lengths $1, 2$ or 3. However, the addition of anticlockwise steps introduces many more variations, and each segment can be of almost any arbitrary length, as the path can move between vertices in subsets 1 and 2 for an unlimited number of steps without ever reaching a vertex in subset 0. Using the labelling introduced above, we can categorise these types as follows.

The first subset can be any of the following 5 types: C, $A(CA)^nA$, $A(CA)^nCC$, $A(CA)^n$, $A(CA)^nC$. Note that the first three are initial segments which end with the first return to 0, but the latter two never return to 0, so are necessarily the whole path.

The second subset can be any of the following 5 types: $C(CA)^nCC$, $C(CA)^nA$, $AA(CA)^nCC$, $AA(CA)^nA$, AC.

The third subset can be any of the following 10 types: $C(CA)^nCC$, $C(CA)^nA$, $AA(CA)^nCC$, $AA(CA)^nA$, AC, $C(CA)^n$, $C(CA)^nC$, $AA(CA)^n$, $AA(CA)^nC$, A. Note that the first five types listed in the third subset are the same as those in the second subset.

Furthermore, the segments can be broken down more finely by splitting them whenever the path reaches a vertex in subset 1. From hereon we will define a *return to 1* as a step which ends at a vertex in subset 1. For example, the segment $bdg'cc'df$, which is of the type $C(CA)^2CC$, can be split into four segments as $(b)(dg')(cc')(df)$, split uniquely by returns to 1. As such, we only need to map certain segments of length 1 or 2.

Mapping individual segments of a walk

We first define the map α which maps segments of lengths 1 and 2 of triangular walks to segments of bicoloured Motzkin paths. Let us define the set of symbols $\mathcal{T} = \{a, b, c, d, a', b', d', e', ce, df, dg, dh, cc', df', dg', dh', e'e, d'f, d'g, d'h, e'c', d'f', d'g', d'h'\}$, and let $\mathcal{L}_3$ be as defined in Section 1.2. We define the map

$\alpha : \mathcal{T} \to \mathcal{L}_3 \cup {\mathcal{L}_3}^2$ as follows:

- $\alpha(a) = H_0^{\{w\}}$
- $\alpha(b) = D_1^{\{w\}}$
- $\alpha(c) = H_0^{\{w\}}$
- $\alpha(d) = D_1^{\{w\}}$
- $\alpha(ce) = H_0^{\{w\}} H_0^{\{w\}}$
- $\alpha(df) = H_1^{\{w\}} D_1^{\{w\}}$
- $\alpha(dg) = D_1^{\{w\}} H_0^{\{w\}}$
- $\alpha(dh) = U_1^{\{w\}} D_1^{\{w\}}$
- $\alpha(cc') = H_0^{\{w\}} H_0^{\{b\}}$
- $\alpha(df') = H_1^{\{w\}} D_1^{\{b\}}$
- $\alpha(dg') = D_1^{\{w\}} H_0^{\{b\}}$
- $\alpha(dh') = U_1^{\{w\}} D_1^{\{b\}}$
- $\alpha(a') = H_0^{\{b\}}$
- $\alpha(b') = D_1^{\{b\}}$
- $\alpha(e') = H_0^{\{b\}}$
- $\alpha(d') = D_1^{\{b\}}$
- $\alpha(e'e) = H_0^{\{b\}} H_0^{\{w\}}$
- $\alpha(d'f) = H_1^{\{b\}} D_1^{\{w\}}$
- $\alpha(d'g) = D_1^{\{b\}} H_0^{\{w\}}$
- $\alpha(d'h) = U_1^{\{b\}} D_1^{\{w\}}$
- $\alpha(e'c') = H_0^{\{b\}} H_0^{\{b\}}$
- $\alpha(d'f') = H_1^{\{b\}} D_1^{\{b\}}$
- $\alpha(d'g') = D_1^{\{b\}} H_0^{\{b\}}$
- $\alpha(d'h') = U_1^{\{b\}} D_1^{\{b\}}$.

Note that the set $\mathcal{T}$ has been chosen as it contains exactly the segments which correspond to the finer segments in the elements of the subsets listed directly above. For example, cc', df', dg', dh' correspond to the segments CA which appear as $(CA)^n$ in most elements of the subsets, while $e'e, d'f, d'g, d'h$ correspond only to the segment AC in the element AC, in the second and third subsets. The only segments missing are the first step of segments in the first subset (i.e. the first steps of walks). The reason for this is that these will be mapped separately later on, when we define our corrective map ρ.

As an aside, we note that we have made a number of choices in defining α, and making different choices would result in a different, but equally valid, map. In fact, there are $(2!)^4(4!)^4 = 48^4$ ways of defining α which have a consistent correspondence between triangular step direction and Motzkin step colour.

Flipping and concatenating the segments

We now proceed to define the map that flips segments, as in the directed case. We begin by defining the set P, which consists of all 20 elements in the image of α.

$$\text{Let } \mathcal{P} = \left\{ \begin{array}{c} H_0^{\{w\}}, H_0^{\{b\}}, D_1^{\{w\}}, D_1^{\{b\}}, \\ H_0^{\{w\}}H_0^{\{w\}}, H_1^{\{w\}}D_1^{\{w\}}, D_1^{\{w\}}H_0^{\{w\}}, U_1^{\{w\}}D_1^{\{w\}}, \\ H_0^{\{w\}}H_0^{\{b\}}, H_1^{\{w\}}D_1^{\{b\}}, D_1^{\{w\}}H_0^{\{b\}}, U_1^{\{w\}}D_1^{\{b\}}, \\ H_0^{\{b\}}H_0^{\{w\}}, H_1^{\{b\}}D_1^{\{w\}}, D_1^{\{b\}}H_0^{\{w\}}, U_1^{\{b\}}D_1^{\{w\}} \\ H_0^{\{b\}}H_0^{\{b\}}, H_1^{\{b\}}D_1^{\{b\}}, D_1^{\{b\}}H_0^{\{b\}}, U_1^{\{b\}}D_1^{\{b\}} \end{array} \right\}.$$

Let $\mathcal{P}^{<\infty}$ denote the set of words of finite length with all letters in $\mathcal{P}$, and let $(\mathcal{L}_3)^{<\infty}$ denote the set of words of finite length with all letters in $\mathcal{L}_3$.

Let us denote by $\lambda_1\lambda_2\ldots\lambda_n$ a word of n letters in $\mathcal{P}^{<\infty}$. Each letter $\lambda_i \in \mathcal{P}$ can be written as a word in $(\mathcal{L}_3)$ or $(\mathcal{L}_3)^2$, where $\lambda_i = \lambda_i^{(1)}$ or $\lambda_i = \lambda_i^{(1)}\lambda_i^{(2)}$. Its image under δ can be written as a word in the same set, as either $\delta(\lambda_i) = \overline{\lambda}_i^{(1)}$ or $\delta(\lambda_i) = \overline{\lambda}_i^{(1)}\overline{\lambda}_i^{(2)}$. We now have the necessary notation to define $\delta : \mathcal{P}^{<\infty} \to (\mathcal{L}_3)^{<\infty}$, which we do as follows:

- $\delta(\lambda_n) = \lambda_n$;
- for $1 \leq i \leq n-1$, if $\overline{\lambda}_{i+1}^{(1)} \in \{H_0, U_1\}$ then $\delta(\lambda_i) = \lambda_i$;
- for $1 \leq i \leq n-1$, if $\overline{\lambda}_{i+1}^{(1)} \in \{H_1, D_1\}$ then

  - if $\lambda_i = H_0^{\{w\}}$, $\delta(\lambda_i) = H_1^{\{w\}}$,
  - if $\lambda_i = D_1^{\{w\}}$, $\delta(\lambda_i) = U_1^{\{w\}}$,
  - if $\lambda_i = H_0^{\{b\}}$, $\delta(\lambda_i) = H_1^{\{b\}}$,
  - if $\lambda_i = D_1^{\{b\}}$, $\delta(\lambda_i) = U_1^{\{b\}}$,
  - if $\lambda_i = H_0^{\{w\}}H_0^{\{w\}}$, $\delta(\lambda_i) = H_1^{\{w\}}H_1^{\{w\}}$,
  - if $\lambda_i = H_0^{\{b\}}H_0^{\{w\}}$, $\delta(\lambda_i) = H_1^{\{b\}}H_1^{\{w\}}$,
  - if $\lambda_i = H_1^{\{w\}}D_1^{\{w\}}$, $\delta(\lambda_i) = H_0^{\{w\}}U_1^{\{w\}}$,
  - if $\lambda_i = H_1^{\{b\}}D_1^{\{w\}}$, $\delta(\lambda_i) = H_0^{\{b\}}U_1^{\{w\}}$,
  - if $\lambda_i = D_1^{\{w\}}H_0^{\{w\}}$, $\delta(\lambda_i) = U_1^{\{w\}}H_1^{\{w\}}$,
  - if $\lambda_i = D_1^{\{b\}}H_0^{\{w\}}$, $\delta(\lambda_i) = U_1^{\{b\}}H_1^{\{w\}}$,
  - if $\lambda_i = U_1^{\{w\}}D_1^{\{w\}}$, $\delta(\lambda_i) = D_1^{\{w\}}U_1^{\{w\}}$,
  - if $\lambda_i = U_1^{\{b\}}D_1^{\{w\}}$, $\delta(\lambda_i) = D_1^{\{b\}}U_1^{\{w\}}$,
  - if $\lambda_i = H_0^{\{w\}}H_0^{\{b\}}$, $\delta(\lambda_i) = H_1^{\{w\}}H_1^{\{b\}}$,
  - if $\lambda_i = H_0^{\{b\}}H_0^{\{b\}}$, $\delta(\lambda_i) = H_1^{\{b\}}H_1^{\{b\}}$,
  - if $\lambda_i = H_1^{\{w\}}D_1^{\{b\}}$, $\delta(\lambda_i) = H_0^{\{w\}}U_1^{\{b\}}$,
  - if $\lambda_i = H_1^{\{b\}}D_1^{\{b\}}$, $\delta(\lambda_i) = H_0^{\{b\}}U_1^{\{b\}}$,
  - if $\lambda_i = D_1^{\{w\}}H_0^{\{b\}}$, $\delta(\lambda_i) = U_1^{\{w\}}H_1^{\{b\}}$,
  - if $\lambda_i = D_1^{\{b\}}H_0^{\{b\}}$, $\delta(\lambda_i) = U_1^{\{b\}}H_1^{\{b\}}$,
  - if $\lambda_i = U_1^{\{w\}}D_1^{\{b\}}$, $\delta(\lambda_i) = D_1^{\{w\}}U_1^{\{b\}}$,
  - if $\lambda_i = U_1^{\{b\}}D_1^{\{b\}}$, $\delta(\lambda_i) = D_1^{\{b\}}U_1^{\{b\}}$;

- $$\delta(\lambda_1\lambda_2\ldots\lambda_n) = \prod_{i=1}^{n} \delta\big(\lambda_i\big).$$

Mapping the first step

We now define the map which uses the first step as a corrective step, to ensure the path in the image starts on the x-axis. From here on let $\upsilon_1\upsilon_2\ldots\upsilon_k$ denote a walk on the triangular domain, starting in a corner, where $\upsilon_i \in \mathcal{T}$ for $2 \leq i \leq k$ and $\upsilon_1 \in \{e, c'\}$. Using the notation introduced for δ, we have that $\overline{\lambda}_1^{(1)}$ is the first step of the path $\delta\Big(\prod_{i=2}^{k}\alpha(\upsilon_i)\Big)$. Thus we can define $\rho : \{e, c'\} \to \{H_0, U_1\}$ as follows:

- if $k = 1$ then $\rho(e) = H_0^{\{w\}}$ and $\rho(c') = H_0^{\{b\}}$;
- if $k \geq 2$ and $\overline{\lambda}_1^{(1)} \in \{H_0^{\{w\}}, H_0^{\{b\}}, U_1^{\{w\}}, U_1^{\{b\}}\}$ then
$$\rho(e) = H_0^{\{w\}} \text{ and } \rho(c') = H_0^{\{b\}}\,;$$
- if $k \geq 2$ and $\overline{\lambda}_1^{(1)} \in \{H_1^{\{w\}}, H_1^{\{b\}}, D_1^{\{w\}}, D_1^{\{b\}}\}$ then
$$\rho(e) = U_1^{\{w\}} \text{ and } \rho(c') = U_1^{\{b\}}.$$

The bijection

We can now define the bijection $\sigma : \mathcal{T}_3 \to \mathcal{M}_1$:

- if $k = 0$ then $\phi(\epsilon) = \xi$;
- if $k \geq 1$ then
$$\sigma(\upsilon_1\ldots\upsilon_k) = \rho(\upsilon_1)\,\delta\Big(\prod_{i=2}^{k}\alpha(\upsilon_i)\Big)\,;$$

where ϵ and ξ denote the empty walk on $\mathcal{T}_3$ and the empty path on $\mathcal{M}_1$ respectively.

Note that we have defined the maps for the directed and undirected cases, σ' and σ, slightly differently. In both maps we cut the triangular walk into segments, map these segments to segments of a Motzkin path, then concatenate these new segments. The difference is that while for the directed case, we cut the walk at vertices in subset 0, for the undirected case we cut at vertices in subsets 0 and 1. These extra cuts are necessary in the undirected case to account for arbitrarily long sections where the path does not reach a node in subset 0. It would be possible to define σ' in the same way, however we feel that doing this would lose the intuition behind why the map works.

Despite this difference, we note that the map σ does in fact restrict to σ' when we restrict walks to $\mathcal{T}_L'$, the clockwise orientation. This equates to restricting α to acting only on the set $\{a, b, c, d, ce, df, dg, dh\}$ and ρ to acting only on $\{e\}$. This can be illustrated by applying σ to the triangular path in Figure 4.7:

$$\begin{aligned}
&\sigma(ebdhadgacebc) = \\
&= \rho(e)\ \delta\Big(\alpha(b)\alpha(dh)\alpha(a)\alpha(dg)\alpha(a)\alpha(ce)\alpha(b)\alpha(c)\Big) \\
&= \rho(e)\ \delta\Big((D_1^{\{w\}})(U_1^{\{w\}}D_1^{\{w\}})(H_0^{\{w\}})(D_1^{\{w\}}H_0^{\{w\}})(H_0^{\{w\}})(H_0^{\{w\}}H_0^{\{w\}})(D_1^{\{w\}})(H_0^{\{w\}})\Big) \\
&= U_1^{\{w\}}D_1^{\{w\}}U_1^{\{w\}}D_1^{\{w\}}H_0^{\{w\}}U_1^{\{w\}}H_1^{\{w\}}H_1^{\{w\}}H_1^{\{w\}}H_1^{\{w\}}D_1^{\{w\}}H_0^{\{w\}} \\
&= \sigma'(ebdhadgacebc).
\end{aligned}$$

4.3.3 Proof

We now proceed to prove that σ is our required bijection. It should be noted that this proof easily restricts to σ'.

Proposition 4.2. *σ gives an explicit bijection from the set of walks of length n on the triangle of side-length 3 to the set of bicoloured Motzkin paths of length n in the strip of width 1.*

Proof. We must first show that walks on the triangle map to paths in $\mathcal{M}_1$; that is, bicoloured Motzkin paths in the strip of width 1. Firstly, we should note that the map δ ensures that disjointed segments are connected, so σ does indeed map to a valid path. As α only maps to segments which end at height 0, it is clear that the path ends at height 0, and ρ uses the first step as a corrective step to ensure the path starts at height 0. α, δ and ρ only map to the steps $H_0^{\{\chi\}}, H_1^{\{\chi\}}, U_1^{\{\chi\}}, D_1^{\{\chi\}}$ with $\chi \in \{w, b\}$, so all steps lie in the strip of width 1. Thus all paths in the image of σ lie in $\mathcal{M}_1$. We note that walks in $\mathcal{T}_3$ map to Motzkin paths of the same length. This is clear as α and ρ only map between segments of the same length.

We must also show that σ is defined on all walks in $\mathcal{T}_3$. To do this we must show that all segments are mapped by ρ and α. Recall that the segments of a walk in $\mathcal{T}_3$ are uniquely defined by cutting the walk whenever it reaches a vertex in subset 0 or 1. Thus it is clear that all segments are of length 1 or 2. For the purposes of ease of reading, we will now introduce the notation that vertices in subsets $0, 1,$ or 2 are written as $v_0, v_1,$ or v_2 respectively. As all walks in $\mathcal{T}_3$ start at a corner vertex v_2, the first segment must be a one-step path, either $v_2 \to v_3$ or $v_2 \to v_1$. Thus it must be either e or c', which is exactly the set on which ρ is defined.

A middle segment (one which is not the first or last segment) must both start and end at vertices in $\{v_0, v_1\}$. Thus it is either a one-step path of the form $v_0 \to v_1$ or $v_1 \to v_0$, or it is a two-step path of the form $v_1 \to v_2 \to v_1$, $v_1 \to v_2 \to v_0$, $v_0 \to v_2 \to v_1$ or $v_0 \to v_2 \to v_0$. The last segment can be any of the middle or first segments. However, it may also end at v_2, which allows for the one-step paths $v_1 \to v_2$ and $v_0 \to v_2$. This list corresponds precisely to the set $\mathcal{T}$ on which α is defined. Thus σ is defined on all walks in $\mathcal{T}_3$.

It remains to show that σ is indeed a bijection, which we do by giving its inverse $t : \mathcal{M}_1 \to \mathcal{T}_3$. As with σ, we must first define some other maps and variables. For a path in $\mathcal{M}_1$, we define x_i to be the number of white steps $(H_0^{\{w\}}, H_1^{\{w\}}, U_1^{\{w\}}, D_1^{\{w\}})$ in the first i steps of the path, and y_i to be the number of black steps $(H_0^{\{b\}}, H_1^{\{b\}}, U_1^{\{b\}}, D_1^{\{b\}})$ in the first i steps of the path. We then define $z_i = x_i - y_i + 2$. We proceed to cut the path whenever $z_i = 0$ or $z_i = 1$, creating k segments $\zeta_1 \zeta_2 \ldots \zeta_k$, where each segment ζ_i consists of either 1 or 2 steps (it is in either $\mathcal{L}_3$ or ${\mathcal{L}_3}^2$).

We now define the set Q, which consists of the 40 possible segments of a path in $\mathcal{M}_1$ of length 1 or 2. It consists of all 8 elements of $\mathcal{L}_3$ plus the 32 connecting elements of ${\mathcal{L}_3}^2$ (for example, $H_0^{\{w\}} H_0^{\{w\}}$ connects but $H_0^{\{w\}} H_1^{\{w\}}$ does not, as $H_0^{\{w\}}$ ends at height 0 and $H_1^{\{w\}}$ begins at height 1). The list of elements could also be viewed as the 20 elements in the set $\mathcal{P}$ (defined above immediately before the definition of δ), plus their flips, or more precisely, their images under δ.

$$\text{Let } \mathcal{Q} = \left\{ \begin{array}{c} H_0^{\{w\}}, H_1^{\{w\}}, D_1^{\{w\}}, U_1^{\{w\}}, H_0^{\{b\}}, H_1^{\{b\}}, D_1^{\{b\}}, U_1^{\{b\}}, \\ H_0^{\{w\}}H_0^{\{w\}}, H_1^{\{w\}}D_1^{\{w\}}, D_1^{\{w\}}H_0^{\{w\}}, U_1^{\{w\}}D_1^{\{w\}}, \\ H_1^{\{w\}}H_1^{\{w\}}, H_0^{\{w\}}U_1^{\{w\}}, U_1^{\{w\}}H_1^{\{w\}}, D_1^{\{w\}}U_1^{\{w\}}, \\ H_0^{\{w\}}H_0^{\{b\}}, H_1^{\{w\}}D_1^{\{b\}}, D_1^{\{w\}}H_0^{\{b\}}, U_1^{\{w\}}D_1^{\{b\}}, \\ H_1^{\{w\}}H_1^{\{b\}}, H_0^{\{w\}}U_1^{\{b\}}, U_1^{\{w\}}H_1^{\{b\}}, D_1^{\{w\}}U_1^{\{b\}}, \\ H_0^{\{b\}}H_0^{\{w\}}, H_1^{\{b\}}D_1^{\{w\}}, D_1^{\{b\}}H_0^{\{w\}}, U_1^{\{b\}}D_1^{\{w\}}, \\ H_1^{\{b\}}H_1^{\{w\}}, H_0^{\{b\}}U_1^{\{w\}}, U_1^{\{b\}}H_1^{\{w\}}, D_1^{\{b\}}U_1^{\{w\}}, \\ H_0^{\{b\}}H_0^{\{b\}}, H_1^{\{b\}}D_1^{\{b\}}, D_1^{\{b\}}H_0^{\{b\}}, U_1^{\{b\}}D_1^{\{b\}}, \\ H_1^{\{b\}}H_1^{\{b\}}, H_0^{\{b\}}U_1^{\{b\}}, U_1^{\{b\}}H_1^{\{b\}}, D_1^{\{b\}}U_1^{\{b\}} \end{array} \right\}.$$

Recall that earlier (immediately preceding the definition of α) we defined the set of symbols $\mathcal{T} = \{a, b, c, d, a', b', d', e', ce, df, dg, dh, cc', df', dg', dh', e'e, d'f, d'g, d'h, e'c', d'f', d'g', d'h'\}$. We now define the map $r : \mathcal{Q} \to \mathcal{T}$, which maps segments of Motzkin paths ζ_i, for $2 \le i \le k$, to segments of triangular walks, as follows:

- $r(H_0^{\{w\}}) = r(H_1^{\{w\}}) = c$ when $i = k$ and $z_k \equiv 2 \bmod 3$

  $r(H_0^{\{w\}}) = r(H_1^{\{w\}}) = a$ otherwise,
- $r(D_1^{\{w\}}) = r(U_1^{\{w\}}) = d$ when $i = k$ and $z_k \equiv 2 \bmod 3$

  $r(D_1^{\{w\}}) = r(U_1^{\{w\}}) = b$ otherwise,

- $r(H_0^{\{b\}}) = r(H_1^{\{b\}}) = e'$ when $i = k$ and $z_k \equiv 2 \bmod 3$
  $r(H_0^{\{b\}}) = r(H_1^{\{b\}}) = a'$ otherwise,
- $r(D_1^{\{b\}}) = r(U_1^{\{b\}}) = d'$ when $i = k$ and $z_k \equiv 2 \bmod 3$
  $r(D_1^{\{b\}}) = r(U_1^{\{b\}}) = b'$ otherwise,
- $r(H_0^{\{w\}}H_0^{\{w\}}) = r(H_1^{\{w\}}H_1^{\{w\}}) = ce$,
- $r(H_0^{\{b\}}H_0^{\{w\}}) = r(H_1^{\{w\}}H_1^{\{b\}}) = e'e$,
- $r(H_1^{\{w\}}D_1^{\{w\}}) = r(H_0^{\{w\}}U_1^{\{w\}}) = df$,
- $r(H_1^{\{b\}}D_1^{\{w\}}) = r(H_0^{\{w\}}U_1^{\{b\}}) = d'f$,
- $r(D_1^{\{w\}}H_0^{\{w\}}) = r(U_1^{\{w\}}H_1^{\{w\}}) = dg$,
- $r(D_1^{\{b\}}H_0^{\{w\}}) = r(U_1^{\{w\}}H_1^{\{b\}}) = d'g$,
- $r(U_1^{\{w\}}D_1^{\{w\}}) = r(D_1^{\{w\}}U_1^{\{w\}}) = dh$,
- $r(U_1^{\{b\}}D_1^{\{w\}}) = r(D_1^{\{w\}}U_1^{\{b\}}) = d'h$,
- $r(H_0^{\{w\}}H_0^{\{b\}}) = r(H_1^{\{b\}}H_1^{\{w\}}) = cc'$,
- $r(H_0^{\{b\}}H_0^{\{b\}}) = r(H_1^{\{b\}}H_1^{\{b\}}) = e'c'$,
- $r(H_1^{\{w\}}D_1^{\{b\}}) = r(H_0^{\{b\}}U_1^{\{w\}}) = df'$,
- $r(H_1^{\{b\}}D_1^{\{b\}}) = r(H_0^{\{b\}}U_1^{\{b\}}) = d'f'$,
- $r(D_1^{\{w\}}H_0^{\{b\}}) = r(U_1^{\{b\}}H_1^{\{w\}}) = dg'$,
- $r(D_1^{\{b\}}H_0^{\{b\}}) = r(U_1^{\{b\}}H_1^{\{b\}}) = d'g'$,
- $r(U_1^{\{w\}}D_1^{\{b\}}) = r(D_1^{\{b\}}U_1^{\{w\}}) = dh'$,
- $r(U_1^{\{b\}}D_1^{\{b\}}) = r(D_1^{\{b\}}U_1^{\{b\}}) = d'h'$.

We now define the map $s : \{H_0^{\{w\}}, U_1^{\{w\}}, H_0^{\{b\}}, U_1^{\{b\}}\} \to \{e, c'\}$, which will be used to map the first segment ζ_1, as follows:

- $s(H_0^{\{w\}}) = s(U_1^{\{w\}}) = e$,
- $s(H_0^{\{b\}}) = s(U_1^{\{b\}}) = c'$.

We can now define our inverse map $t : \mathcal{M}_1 \to \mathcal{T}_3$:

- if $k = 0$ then $\phi(\xi) = \epsilon$;

- if $k \geq 1$ then

$$t(\zeta_1 \ldots \zeta_n) = s(\zeta_1) \prod_{i=2}^{k} r(\zeta_i);$$

where ξ and ϵ denote the empty path on $\mathcal{M}_1$ and the empty walk on $\mathcal{T}_3$ respectively.

It remains to show that t is an inverse to σ. We first deal with the trivial cases; if $n = 0$ then $t(\sigma(\epsilon)) = t(\xi) = \epsilon$ and $\sigma(t(\xi)) = \sigma(\epsilon) = \xi$. We now note that $s(\rho(\upsilon_1)) = \upsilon_1$ for $\upsilon_1 \in \{e, c'\}$, and $r(\delta(\alpha(\upsilon_i))) = r(\alpha(\upsilon_i)) = \upsilon_i$ for $\upsilon_i \in \mathcal{T}$, $i \geq 2$. We also have that $\delta(\alpha(r(\zeta_i))) = \zeta_i$ when $\zeta_i \in \mathcal{Q}$, $i \geq 2$ and $\zeta_1\zeta_2 \ldots \zeta_k \in \mathcal{M}_1$. This last equality can easily be shown using an induction-style argument, working from the last step backwards; by inspection, $\alpha(r(\zeta_i))$ equals either ζ_i or the flip of ζ_i. δ leaves the last step ζ_n unchanged, and ζ_n must end on the x-axis, so $\delta(\alpha(r(\zeta_n))) = \alpha(r(\zeta_n))) = \zeta_n$, and then working backwards, we see that $\delta(\alpha(r(\zeta_i)))$ must end at the same height as ζ_i for all i. Following this reasoning, we also get that $\rho(s(\zeta_1)) = \zeta_1$.

We also note that the cuts which split up a walk $P \in \mathcal{T}_3$ occur at exactly the same vertices (after the same numbers of steps) as the cuts in the Motzkin path $\sigma(P)$, and vice versa (for $P \in \mathcal{M}_1$ and $t(P)$). This is because of the similar parity arguments used to make the cuts, and the link between clockwise steps and white steps, and anticlockwise and black steps. Thus when considering $t(\sigma(P))$ and $\sigma(t(N))$, for walks $P \in \mathcal{T}_3$ and paths $N \in \mathcal{M}_1$ respectively, we do not need to be concerned about re-cutting the path midway through the mapping, and we are justified in using the same value k to denote the number of cuts in each path/walk.

Now, using the observations above, if a path of $n \geq 1$ steps in $\mathcal{T}_3$ is written as $\upsilon_1\upsilon_2 \dots \upsilon_k$, where $\upsilon_i \in \mathcal{T}$ for $2 \leq i \leq k$ and $\upsilon_1 \in \{e, c'\}$, we have that

$$\begin{aligned} t(\sigma(\upsilon_1 \dots \upsilon_k)) &= t\Big(\rho(\upsilon_1)\, \delta\Big(\prod_{i=2}^{k} \alpha(\upsilon_i)\Big)\Big) \\ &= t\Big(\rho(\upsilon_1) \prod_{i=2}^{k} \delta(\alpha(\upsilon_i))\Big) \\ &= s(\rho(\upsilon_1)) \prod_{i=2}^{k} r(\delta(\alpha(\upsilon_i))) \\ &= \upsilon_1 \prod_{i=2}^{k} \upsilon_i \\ &= \upsilon_1 \dots \upsilon_k\,. \end{aligned}$$

If a path of $n \geq 1$ steps in $\mathcal{M}_1$ is written as $\zeta_1\zeta_2 \dots \zeta_k$, where $\zeta_i \in \mathcal{Q}$ for $2 \leq i \leq k$ and $\upsilon_1 \in \{H_0^{\{w\}}, U_1^{\{w\}}, H_0^{\{b\}}, U_1^{\{b\}}\}$, we have that

$$\begin{aligned} \sigma(t(\zeta_1 \dots \zeta_n)) &= \sigma\Big(s(\zeta_1) \prod_{i=2}^{k} r(\zeta_i)\Big) \\ &= \rho(s(\zeta_1))\, \delta\Big(\prod_{i=2}^{k} \alpha(r(\zeta_i)\Big) \\ &= \rho(s(\zeta_1)) \prod_{i=2}^{k} \delta(\alpha(r(\zeta_i)) \\ &= \zeta_1 \prod_{i=2}^{k} \zeta_i \\ &= \zeta_1 \dots \zeta_n\,. \end{aligned}$$

Thus t is an inverse to σ and so σ is a bijection. $\square$

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