Nonassociative constructions from inverse property quasigroups

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Statement of Originality

This thesis is submitted to the University of London for the degree of Doctor of Philosophy. It is my own work and only contains results with which I have been directly involved. Chapters 3 and 4 are the product of joint research with my supervisor Professor Shahn Majid. Any results of others have been fully acknowledged.

Signed:

Dated:

Abstract

The notion of a Hopf algebra has been generalized many times by weakening certain properties; we introduce Hopf quasigroups which weaken the associativity of the algebra. Hopf quasigroups are coalgebras with a nonassociative product satisfying certain conditions with the antipode reflecting the properties of classical inverse property quasigroups.

The definitions and properties of Hopf quasigroups are dualized to obtain a theory of Hopf coquasigroups, or 'algebraic quasigroups'. In this setting we are able to study the coordinate algebra over a quasigroup, and in particular the 7-sphere.

One particular class of Hopf quasigroups is obtained by taking a bicrossproduct of a subgroup and a set of coset representatives, in much the same way that Hopf algebras are obtained from matched pairs of groups. Through this construction the bicrossproduct can also be given the structure of a quasi-Hopf algebra.

We adapt the theory of Hopf algebras to Hopf (co)quasigroups, defining integrals and Fourier transformations on these objects. This leads to the expected properties of separable and Frobenius Hopf coquasigroups and notions of (co)semisimplicity.

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Chapter 1

Introduction

Many Lie groups have algebraic descriptions as commutative Hopf algebras, and similarly we will be interested in the properties of 'algebraic quasigroups'. This leads us first of all to define a Hopf quasigroup by linearizing the notion of a classical quasigroup, in the same way that a Hopf algebra linearizes the idea of a group.

The structure of this thesis is as follows: Chapter 2 introduces some of the basic theory and terminology that we will use throughout. We start by discussing the definition of a Hopf algebra by first defining an algebra and then dualizing to define a coalgebra. We recall some properties of Hopf algebras and consider how they act on various objects.

Drinfeld [12] defined a 'quantum group' as a noncommutative, noncocommutative Hopf algebra; we mention an important class of quantum groups called quasitriangular Hopf algebras. Finally, we recall some basic points of category theory that we will refer to in the final chapter.

In Chapter 3 we recall some basic definitions and results for classical quasigroups and the twisted group algebra approach to composition algebras due to Albuquerque and Majid [2]. We focus on the twisted quasialgebras $k_F\mathbb{Z}_2^n$ of the group $\mathbb{Z}_2^n$ by the cochain F, and show how the unit spheres $\mathcal{S}^{2^n-1} \subset k_F\mathbb{Z}_2^n$ can be seen as inverse property quasigroups.

We then introduce the notion of a Hopf quasigroup as a linearization of an inverse property quasigroup. This is a generalization of a Hopf algebra in which the associativity constraint has been weakened to reflect the properties of the classical inverse property quasigroup.

As one may expect, this theory can be dualized to that of a Hopf coquasigroup which is an associative algebra, now with a noncoassociative coproduct. It is in this setting that we can study the coordinate algebras of the unit spheres $k[\mathcal{S}^{2^n-1}]$ and in particular the 'algebraic 7-sphere' $k[\mathcal{S}^7]$.

In the final section of Chapter 3 we briefly discuss the notion of a covariant differential calculus using the Hopf algebra methods developed by Woronowicz in [37] but now adapted to Hopf coquasigroups.

Chapter 4 continues the analysis by constructing a family of examples of noncommutative, noncocommutative Hopf quasigroups. We extend the bicrossproduct construction originally used to provide the first noncommutative and noncocommutative Hopf algebras associated to group factorisations [23]. We consider a group X with a subgroup G and a transversal, a set of right coset representatives, M and follow the approach in [3] to obtain the matched pair data. It is well-known that M has the structure of a quasigroup; we extend the analysis to when M is an inverse property quasigroup. We then construct a bicrossproduct $kM{\triangleright\!\!\blacktriangleleft}k(G)$ and investigate when this acquires the structure of a Hopf quasigroup.

In 1969 Larson and Sweedler [22] proved that the space of integrals on a finite dimensional Hopf algebra is 1-dimensional. In Chapter 5 we prove that the same statement holds for Hopf quasigroups. The proof makes use of a version of the well-known Hopf module lemma. This was adapted to Hopf quasigroups by Brzezinski [5] following my work [21].

In the dual setting we have existence and uniqueness of integrals on a finite dimensional Hopf coquasigroup. The proof first requires a proof of a Hopf coquasigroup version of the Hopf module lemma involving an induced coaction, which we provide. We define integrals to be elements invariant under this induced coaction, leading to a slightly different definition to integrals on Hopf quasigroups.

We use this theory of integrals to define Fourier transformations on Hopf (co)quasigroups given in terms of integrals by the same formula as for Hopf algebras, as a Hopf module map from the object to its dual. We give its inverse and investigate the properties of this map.

The final sections of Chapter 5 discuss further properties of Hopf coquasigroups by applying this theory. First we consider Frobenius objects; the algebra structure of a Hopf coquasigroup is associative and so the usual definition of a separable algebra applies in this setting. The definition of a separable Hopf algebra once again extends to Hopf coquasigroups with integrals as we demonstrate. Finally, we use integral theory to prove a version of Maschke's theorem for Hopf coquasigroups, similar to that for Hopf algebras due to Larson and Sweedler [22].

Chapter 6 demonstrates the link between classical quasigroups and quasi-Hopf algebras through the coset construction discussed in Chapter 4. Quasi-Hopf algebras first appeared in the work of Drinfeld [13], motivated by quantum physics; their main difference to Hopf algebras is that the coproduct is no longer coassociative, but coassociative up to conjugation by a 3-cocycle.

We discuss the dual processes of transmutation and bosonization, first introduced for Hopf algebras by Majid in [25] and [27], respectively, and then partly computed in the quasi-Hopf algebra setting by Bulacu and Nauwelaerts [9]. We extend the computations further.

The final section of Chapter 6 demonstrates this theory starting with the octonion quasigroup $\mathcal{G}_{\mathbb{O}} \subset \mathcal{S}^7$. The constructions in Chapter 4 provide a quasi-Hopf algebra $k\mathcal{G}_{\mathbb{O}} \bowtie k[\mathbb{Z}_2^3]$.

The quantum double was introduced by Drinfeld in [12]; the original construction allows us to associate a quasitriangular Hopf algebra to any Hopf algebra. This result was extended to quasi-Hopf algebras by Majid [29] and examples associated to finite groups were given in [34]. Using this theory, we construct the quantum double of this object to obtain a quasitriangular quasi-Hopf algebra.

The results in Chapters 3 and 4 were published in joint work with Majid in [21] and [20], respectively.

Chapter 2

Preliminaries

In this chapter we recall some of the basic definitions and theory that will be referred to throughout.

The idea of a Hopf algebra originated in the work of Heinz Hopf in 1941 while working on homology and cohomology of topological groups. Several fundamental papers were written on the subject over the next two decades and Sweedler published "Hopf algebras" in 1969 which is considered the standard text on the basics of the subject. We introduce Hopf algebras by first defining an algebra and a coalgebra in order to fix our notation and terminology. Bialgebras and Hopf algebras are algebraic objects with the structure of an algebra and a coalgebra interrelated by certain conditions. Proofs of some of the fundamental properties of such constructions have been omitted.

We then consider how Hopf algebras act on these structures, beginning with modules before dualizing to comodules. As with Hopf algebras, these structures can be combined under certain compatibility conditions to form a Hopf module.

Drinfeld [12] defines a quantum group as a noncommutative, noncocommutative Hopf algebra. We mention an important class of quantum groups called quasitriangular Hopf algebras.

We end the chapter with a discussion of category theory which first appeared in a paper by Eilenberg and Mac Lane [14] who regarded it as providing a handy language to be used in topology and other areas. The theory was developed further by Grothendieck [15] and Kan [18] among others. We will mention only the basic definitions of monoidal categories from [14], and braided categories from the work of Joyal and Street [17].

Conventions

Throughout, unless otherwise stated, everything takes place over a fixed ground field k, in particular *space* or *vector space* means a vector space over the field k, and the tensor product $V \otimes W$ of two spaces is understood to be over the field, *i.e.* $V \otimes_k W$. In general, for a space V we will write V^n for $V \otimes \cdots \otimes V$, n times.

2.1 Hopf algebras

Definition 2.1.1. An *(associative unital) algebra* is a vector space A equipped with a multiplication map $m : A \otimes A \to A$, and a unit $\eta : k \to A$, which are k-linear maps satisfying

$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m), \quad m(\eta \otimes \mathrm{id}) = \mathrm{id} = m(\mathrm{id} \otimes \eta).$$

We write $1_A = \eta(1_k)$ for the unit element of A, where 1_k is the unit element of the field, and if unambiguous, we will write only 1 for the unit element of our space.

A linear map $f : A \to B$ between two algebras (A, m_A, η_A) and (B, m_B, η_B) is an *algebra map* if it satisfies

$$m_B(f \otimes f) = f \cdot m_A, \quad f \cdot \eta_A = \eta_B.$$

Let A and B be algebras, then $A \otimes B$ is an algebra with maps $m_{A\otimes B} : A \otimes B \otimes A \otimes B \to A \otimes B$ and $\eta_{A\otimes B} : k \to A \otimes B$ defined as the composites

$$m_{A\otimes B} = (m \otimes m)(\mathrm{id} \otimes \tau \otimes \mathrm{id}),$$

$$\eta_{A\otimes B} = \eta_A \otimes \eta_B,$$

where $\tau : A \otimes B \to B \otimes A$ denotes the twist map defined by $\tau(a \otimes b) = b \otimes a$ for all $a \in A, b \in B$.

One can rewrite the definition of an algebra as commutative diagrams; the advantage of this approach is that it immediately leads to a dual definition by reversing each of the arrows.

Definition 2.1.2. A *(coassociative counital) coalgebra* is a triple (C, Δ, ε) where C is a vector space over k, and $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ are k-linear maps such that

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \quad (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta.$$

We call Δ the coproduct of C and ε the counit of C. Sweedler introduced the *sigma notation* for the coproduct in [36] as a formal sum $\Delta(c) = \sum_i c_{1i} \otimes c_{2i}$ for $c \in C$, with $c_{ji} \in C$. It is usual to suppress the index i and write $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ with $c_{(1)}, c_{(2)} \in C$. We will usually simplify the notation by omitting the summation and write only $\Delta(c) = c_{(1)} \otimes c_{(2)}$. Using this notation, the properties of coassociativity and counity become

$$\sum c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)},$$

$$\sum c_{(1)}\varepsilon(c_{(2)}) = c = \sum \varepsilon(c_{(1)})c_{(2)},$$

for all $c \in C$. This notation is useful when Δ is applied more than once, in particular $\Delta^2(c) = (\Delta \otimes \mathrm{id})\Delta(c) = (\mathrm{id} \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. This process can be iterated to give

$$\Delta^{n-1}(c) = \sum c_{(1)} \otimes \cdots \otimes c_{(n)}.$$

We can also dualize the definition of an algebra map; a map $f : C \to D$ between two coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ is a *coalgebra map* if it satisfies

$$(f \otimes f)\Delta_C = \Delta_D \cdot f, \quad \varepsilon_C = \varepsilon_D \cdot f.$$

Let C and D be coalgebras and define $\Delta_{C\otimes D}$ and $\varepsilon_{C\otimes D}$ as the composites

$$\Delta_{C\otimes D} = (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta_C \otimes \Delta_D),$$

$$\varepsilon_{C\otimes D} = \varepsilon_C \otimes \varepsilon_D,$$

then $(C \otimes D, \Delta_{C\otimes D}, \varepsilon_{C\otimes D})$ is a coalgebra.

We combine the notion of an algebra and a coalgebra to obtain a bialgebra.

Definition 2.1.3. A bialgebra is a vector space H equipped with maps $m : H \otimes H \to H, \eta : k \to H, \Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ such that (H, m, η) is an algebra, (H, Δ, ε) is a coalgebra and Δ and ε are algebra maps.

Note that saying Δ and ε are algebra maps is equivalent to saying that m and η are coalgebra maps.

Definition 2.1.4. A Hopf algebra is a bialgebra H endowed with a map $S : H \to H$ called the *antipode* such that

$$m(S \otimes \mathrm{id})\Delta = \eta \cdot \varepsilon = m(\mathrm{id} \otimes S)\Delta.$$

In sigma notation, this becomes

$$\sum (Sh_{(1)})h_{(2)} = \varepsilon(h) \cdot 1 = \sum h_{(1)}Sh_{(2)},$$

for all $h \in H$.

A map $f : H \to G$ between two Hopf algebras H and G is a *Hopf algebra map* if it is both an algebra map and a coalgebra map, and $f \cdot S_H = S_G \cdot f$. In fact the last condition is not required as an algebra and coalgebra map necessarily commutes with the antipode.

The following are some important properties of the antipode. We refer the reader to [36] for the proof.

Theorem 2.1.5. *Let H be a Hopf algebra with antipode S, then the following properties hold:*

(1) $S(gh) = S(h)S(g)$ *for all* $g, h \in H$*, i.e.* S *is anti-multiplicative,*

(2) $S(1) = 1$*,*

(3) $\tau(S \otimes S)\Delta = \Delta \cdot S$*, i.e.* $\Delta(Sh) = \sum_h Sh_{(2)} \otimes Sh_{(1)}$ *for all* $h \in H$*,*

(4) $\varepsilon \cdot S = \varepsilon$*,*

(5) *If H is commutative or cocommutative then* $S^2 = \mathrm{id}$.

2.1.1 Duality

We can see that there is a close relationship between algebras and coalgebras by looking at their dual spaces.

We say that two Hopf algebras A and H are dually paired if there is a bilinear map $\langle\, ,\, \rangle : A \otimes H \to k$ obeying

$$\langle \varphi\psi, h\rangle = \langle \varphi, h_{(1)}\rangle\langle \psi, h_{(2)}\rangle, \quad \langle 1, h\rangle = \varepsilon(h),$$

$$\langle \Delta\varphi, h \otimes g\rangle = \langle \varphi, hg\rangle, \quad \varepsilon(\varphi) = \langle \varphi, 1\rangle,$$

$$\langle S\varphi, h\rangle = \langle \varphi, Sh\rangle,$$

for all $\varphi, \psi \in A$ and $h, g \in H$. We call the map $\langle\, ,\, \rangle$ the dual pairing.

Let V be a vector space and $V^* = \mathrm{Hom}_k(V, k)$ be the dual space consisting of k-linear maps from V to k. We can also consider the dual of a map; if $L : V \to W$ is a k-linear map, then $L^* : W^* \to V^*$ is the unique k-linear map induced by $\langle L^*(\phi), v\rangle = \langle \phi, L(v)\rangle$ for all $\phi \in W^*, v \in V$, where we take the dual pairing to be evaluation.

Proposition 2.1.6. *Let H be a Hopf algebra then*

(1) $(H^*, \Delta^*, \varepsilon^*)$ *is an algebra,*

(2) *If H is finite dimensional,* $(H^*, m^*, 1^*)$ *is a coalgebra,*

(3) *If H is finite dimensional,* $(H^*, m^*, 1^*, \Delta^*, \varepsilon^*, S^*)$ *is a Hopf algebra.*

Proof. We refer the reader to [28] for the proof. □

Thus we can conclude that for every finite dimensional Hopf algebra H there is a dual Hopf algebra H^* built on the dual space. Further, H^* is the unique Hopf algebra dually paired to H.

2.1.2 Modules, comodules and Hopf modules

An important property of Hopf algebras is that they can act on other structures, be it a vector space, an algebra or a coalgebra. In the following, unless otherwise stated, H is a Hopf algebra.

Definition 2.1.7. A *left H-module* is a vector space V with a linear map $\alpha : H \otimes V \to V$, written as $\alpha(h \otimes v) = h \triangleright v$ for $h \in H, v \in V$, such that

$$h \triangleright (g \triangleright v) = (hg) \triangleright v, \quad 1 \triangleright v = v, \quad \forall\, h, g \in H, v \in V.$$

The linear map α is called an *action* of H and depending on whether we want to emphasize the action or the object V, we will say that V is a left H-module, or that H acts on V on the left.

Definition 2.1.8. A *left H-module algebra* is a left H-module A endowed with an algebra structure such that

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright 1_A = \varepsilon(h) 1_A,$$

for all $h \in H$ and $a, b \in A$. A coalgebra C is a *left H-module coalgebra* if it is an H-module and

$$\Delta(h \triangleright c) = \sum h_{(1)} \triangleright c_{(1)} \otimes h_{(2)} \triangleright c_{(2)}, \quad \varepsilon(h \triangleright c) = \varepsilon(h)\varepsilon(c), \quad \forall\, h \in H, c \in C.$$

The definitions of a module algebra and a module coalgebra state that the action is an algebra map and a coalgebra map respectively.

We will make use of two common actions of a Hopf algebra on itself in later sections; it is straightforward to check that these maps satisfy the definition of an action.

Example 2.1.9.

(1) The left adjoint action is given by $h \triangleright g = Ad_h(g) = \sum h_{(1)} g S h_{(2)}$, $\forall\, h, g \in H$.

(2) The left regular action is given by the multiplication map as $h \triangleright g = L_h(g) = hg$, $\forall\, h, g \in H$.

The action of a Hopf algebra on the tensor product of modules extends the usual diagonal action for groups. If V, W are left H-modules, then $V \otimes W$ is a left H-module with the action given by $h \triangleright (v \otimes w) = \sum h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w$ for all $h \in H, v \in V, w \in W$.

Definition 2.1.10. Let V be a left H module. The *H-invariants* of V are the set

$$V^H = \{v \in V \mid h \triangleright v = \varepsilon(h) v, \quad \forall\, h \in H\}.$$

We have discussed left actions $\triangleright$, but clearly there is also the notion of a right action $\alpha_R : V \otimes H \to V$ written as $v \otimes h \mapsto v \triangleleft h$. If $\triangleright$ is a left action, then $v \triangleleft h := (Sh) \triangleright v$ is a right action.

The dual of an action is a coaction; the definition is obtained by reversing the arrows in the diagrammatic form of the definition of an action.

Definition 2.1.11. A vector space V is a right H-comodule if there is a linear map $\rho : V \to V \otimes H$ written as $v \mapsto \sum v^{(1)} \otimes v^{(2)}$, such that

$$\sum v^{(1)(1)} \otimes v^{(1)(2)} \otimes v^{(2)} = \sum v^{(1)} \otimes {v^{(2)}}_{(1)} \otimes {v^{(2)}}_{(2)}, \quad v^{(1)} \varepsilon(v^{(2)}) = v,$$

for all $v \in V$. An algebra A is a right H-comodule algebra if it is a right H-comodule and

$$\rho(ab) = \rho(a)\rho(b), \quad \rho(1) = 1 \otimes 1, \quad \forall\, a, b \in A.$$

A coalgebra C is a right H-comodule coalgebra if C is a right H-comodule and

$$\sum {c^{(1)}}_{(1)} \otimes {c^{(1)}}_{(2)} \otimes c^{(2)} = \sum {c_{(1)}}^{(1)} \otimes {c_{(2)}}^{(1)} \otimes {c_{(1)}}^{(2)} {c_{(2)}}^{(2)}, \quad \varepsilon(c^{(1)}) c^{(2)} = \varepsilon(c),$$

for all $c \in C$.

The map ρ is called the *coaction*; as with actions, although we have defined only right coactions here, there is an analogous notion of left coactions. The sigma notation for left and right H-comodules preserves the convention that $v^{(i)} \in H$ for $i \neq 1$.

Dual to Example 2.1.9, there are two common coactions making a Hopf algebra H into a right H-comodule.

Example 2.1.12.

(1) The right coadjoint coaction is defined as $h \mapsto h_{(2)} \otimes (Sh_{(1)})h_{(3)}$, $\forall\, h \in H$.

(2) The right coregular coaction is given by the coproduct map, $h \mapsto h_{(1)} \otimes h_{(2)}$, $\forall\, h \in H$.

Definition 2.1.13. Let V be a right H-comodule. The H-*coinvariants* of V are the set

$$V^{coH} = \{v \in V \mid \rho(v) = v \otimes 1\}.$$

As in Section 2.1.1 there is a strong connection between modules and comodules.

Lemma 2.1.14. *Let H be a Hopf algebra.*

(1) *If V is a right H-comodule, then V is a left H^*-module by $\varphi \triangleright v = v^{(1)}\langle \varphi, v^{(2)}\rangle$, for all $\varphi \in H^*$ and $v \in V$.*

(2) *If V is a left H-module, then V^* is a right H-module by $\langle v^* \triangleleft h, v\rangle = \langle v^*, h \triangleright v\rangle$, for all $v \in V$ $v^* \in V^*$ and $h \in H$.*

A Hopf algebra has the structure of both an algebra and a coalgebra with a compatibility condition that the coproduct and counit are algebra maps. Similarly, a Hopf module has the structure of both a module and a comodule with some required compatibility condition between them:

Definition 2.1.15. A vector space V is a *right H-Hopf module* if it is a right H-module, a right H-comodule and

$$\sum (v \triangleleft h)^{(1)} \otimes (v \triangleleft h)^{(2)} = \sum v^{(1)} \triangleleft h_{(1)} \otimes v^{(2)} h_{(2)},$$

for all $v \in V, h \in H$.

Theorem 2.1.16. *Let V be a right H-Hopf module. Then $V \cong V^{coH} \otimes H$ as right H-Hopf modules, where $V^{coH} \otimes H$ is a right H-Hopf module via*

$$(v \otimes h) \triangleleft g = v \otimes hg, \quad \rho(v \otimes h) = v \otimes h_{(1)} \otimes h_{(2)},$$

for all $v \in V$ and $h, g \in H$.

Proof. The full proof is given in [1]; the isomorphism $\sigma : V^{coH} \otimes H \to V$ is given by $v \otimes h \mapsto v \triangleleft h$, with inverse $\sigma^{-1} : V \to V^{coH} \otimes H$ given by $v \mapsto \sum v^{(1)(1)} \triangleleft (S v^{(1)(2)}) \otimes v^{(2)}$. □

2.1.3 Quasitriangular Hopf algebras

Drinfeld defined a *quantum group* as a noncommutative, noncocommutative Hopf algebra. An important class of Hopf algebras consists of those which are cocommutative only up to conjugation by an element $\mathcal{R}$. These were first introduced by Drinfeld in [12].

Definition 2.1.17. A Hopf algebra H is *quasitriangular* if there is an invertible element $\mathcal{R} \in H \otimes H$ such that

$$\tau \circ \Delta(h) = \mathcal{R}\Delta(h)\mathcal{R}^{-1}, \quad \forall\, h \in H,$$

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

where $\tau : H \otimes H \to H \otimes H$ is the usual twist map.

We use a similar notation to the sigma notation, and write $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$, and $\mathcal{R}_{ij} \in H^3$ with $\mathcal{R}^{(1)}$ in the ith factor, $\mathcal{R}^{(2)}$ in the jth factor and 1 in the remaining factor, for example, $\mathcal{R}_{13} = \sum \mathcal{R}^{(1)} \otimes 1 \otimes \mathcal{R}^{(2)}$.

Dually, if a Hopf algebra is commutative up to conjugation in a certain sense it is called a dual quasitriangular Hopf algebra, or coquasitriangular:

Definition 2.1.18. A Hopf algebra H is *coquasitriangular* if there exists a convolution invertible map $\mathcal{R} : H \otimes H \to k$ such that

$$\sum g_{(1)}h_{(1)}\mathcal{R}(h_{(2)} \otimes g_{(2)}) = \sum \mathcal{R}(h_{(1)} \otimes g_{(1)})h_{(2)}g_{(2)},$$

$$\mathcal{R}(gh \otimes f) = \sum \mathcal{R}(g \otimes f_{(1)})\mathcal{R}(h \otimes f_{(2)}), \quad \mathcal{R}(g \otimes hf) = \sum \mathcal{R}(g_{(1)} \otimes h)\mathcal{R}(g_{(2)} \otimes f),$$

for all $g, h, f \in H$.

The definition above is obtained from Definition 2.1.17 by reversing the arrows in the diagrammatic form of the identities, and was originally formally introduced by Majid [28].

2.2 Monoidal Categories

Categories and natural transformations were introduced by Eilenberg and Mac Lane motivated by their work on braids and links in topology. Joyal and Street were motivated to extend the theory to include braided monoidal categories through their work in homotopy theory. We will recall the definitions here adopting the notation of [28].

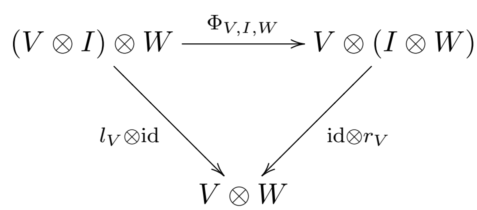
A *category* $\mathcal{C}$ consists of a collection of objects and for each pair U, V of objects of $\mathcal{C}$, a set of morphisms from U to V. These morphisms can be composed in the usual way and the following criteria are satisfied; composition of morphisms is associative and for every object U of $\mathcal{C}$, there exists an identity morphism id_U.

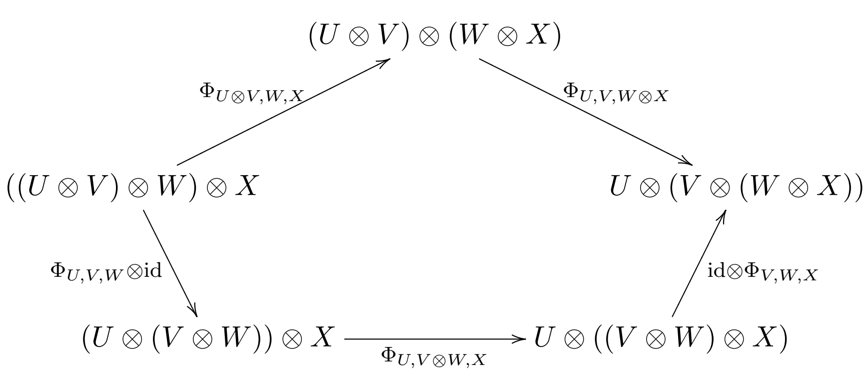
A (covariant) *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories assigns to every object U of $\mathcal{C}$ an object $F(U)$ of $\mathcal{D}$, and to every morphism $f : U \to V$ of $\mathcal{C}$, a morphism $F(f) : F(U) \to F(V)$ of $\mathcal{D}$. Further, $F(\mathrm{id}_U) = \mathrm{id}_{F(U)}$ and $F(f \cdot g) = F(f) \cdot F(g)$.

A *natural transformation* $\theta : F \to G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\{\theta_V : F(V) \to G(V) | V \in \mathcal{C}\}$, in $\mathcal{D}$ such that for any morphism $f : U \to V$ in $\mathcal{C}$, we have $\theta_V \cdot F(f) = G(f) \cdot \theta_U$. We write $\theta \in \mathrm{Nat}(F, G)$ for such a natural transformation. Further, if θ_V is an isomorphism of $\mathcal{C}$ for any object $V \in \mathcal{C}$, we say that $\theta : \mathcal{C} \to \mathcal{D}$ is a *natural isomorphism*.

Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\varepsilon : F \cdot G \to \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to G \cdot F$.

A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$ and natural isomorphisms $\Phi_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $l_V : V \otimes I \to V$ and $r_V : I \otimes V \to V$ such that the following diagrams commute:





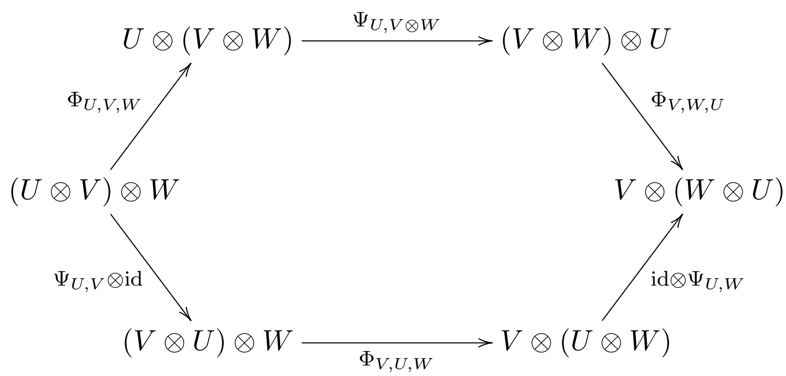
We call Φ the *associativity constraint*, l the *left unit constraint* and r the *right unit constraint*.

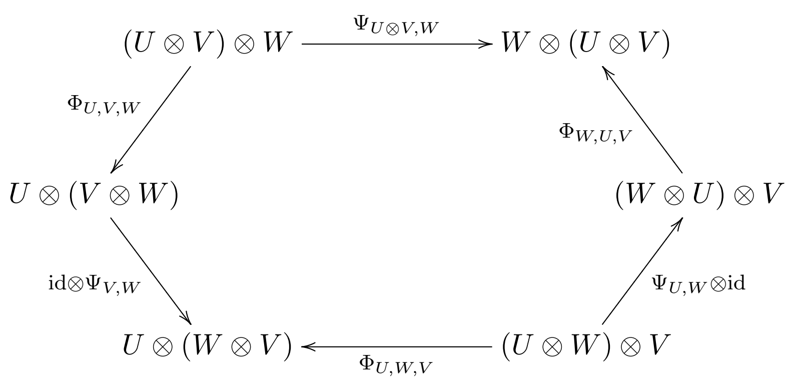
A monoidal category $\mathcal{C}$ is *rigid* if for each object V of $\mathcal{C}$ there exists an object V^* and morphisms $ev_V : V^* \otimes V \to I$ and $coev_V : I \to V \otimes V^*$ of $\mathcal{C}$ such that

$$l_V \cdot (\mathrm{id}_V \otimes ev_V) \cdot \Phi_{V,V^*,V} \cdot (coev_V \otimes \mathrm{id}_V) \cdot r_V^{-1} = \mathrm{id}_V,$$

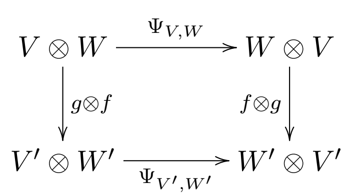
$$r_{V^*} \cdot (ev_V \otimes \mathrm{id}_{V^*}) \cdot \Phi^{-1}_{V^*,V,V^*} \cdot (\mathrm{id}_{V^*} \otimes coev_V) \cdot l^{-1}_{V^*} = \mathrm{id}_{V^*}.$$

A *braided* category [17] is a monoidal category equipped with a natural isomorphism $\Psi_{U,V} : U \otimes V \to V \otimes U$ obeying the hexagon conditions below





and such that the following diagram commutes for all morphisms $f : W \to W'$ and $g : V \to V'$.



The natural isomorphism Ψ is called the *braiding*.

Chapter 3

The algebraic 7-sphere

We recall some basic theory of classical inverse property quasigroups, which we will require throughout the chapter, to set our notation using [35] as a reference. We provide some elementary proofs of properties of quasigroups.

We then detail the quasialgebra approach to composition algebras due to Albuquerque and Majid [2]. This provides a way of working with the octonions as a twisted group algebra $k_F G$ of a group $\mathbb{Z}_2^3$. In particular we discuss how the unit spheres $\mathcal{S}^{2^n-1}$ can be seen as a subgroup of such quasialgebras.

We then linearize these notions, in much the same way that a Hopf algebra linearizes a group, to define a Hopf quasigroup. This is a coalgebra with a nonassociative product and some inversion map. Much of the quasigroup theory extends to this setting, in particular the existence under certain conditions of three equivalent Moufang identities. We define actions of Hopf quasigroups and as a result we obtain a construction for Hopf quasigroups which is an analogue of a cross product for Hopf algebras.

As in Hopf algebra theory, we can dualize these constructions to obtain a Hopf coquasigroup. Here we have an algebra with a linear noncoassociative coproduct. We again demonstrate that the classical theory holds in this setting. We focus on the commutative example of the coordinate algebra of the unit sphere $k[\mathcal{S}^{2^n-1}]$ proving that it is a Moufang Hopf coquasigroup.

As with Hopf algebras, we have a close relationship between Hopf quasigroups and Hopf coquasigroups; we give details of this dual pairing.

Finally, we briefly discuss the notion of a covariant differential calculus using the Hopf algebra methods developed by Woronowicz in [37] extended to Hopf coquasigroups.

The results in this chapter, unless otherwise stated, have been published with Majid in [21].

3.1 Quasigroups

A *quasigroup* is a set Q with a binary multiplication operation $\cdot$, *i.e.* for all $x, y \in Q$ there exist unique $a, b \in Q$ such that $x \cdot a = y$ and $b \cdot x = y$. Equivalently [35], a quasigroup $(Q, \cdot, /, \backslash)$ is a set Q with three binary operations of multiplication, right division $/$ and left division $\backslash$ satisfying

$$y\backslash(y \cdot x) = x, \quad (x \cdot y)/y = x,$$

$$y \cdot (y\backslash x) = x, \quad (x/y) \cdot y = x,$$

for all $x, y \in Q$.

A set Q with a product and only a right division map satisfying the appropriate identities is a *right quasigroup*, and we similarly define a *left quasigroup*.

A quasigroup has the *inverse property* if there exists an identity element $e \in Q$ and for each $x \in Q$ there exists a unique element $x^{-1} \in Q$ such that

$$x^{-1} \cdot (x \cdot y) = y = (y \cdot x) \cdot x^{-1}, \tag{3.1.1}$$

for all $y \in Q$.

Remark 3.1.1. Throughout this section, we will use quasigroup to mean a quasigroup with inverse property. We note that in the literature, a quasigroup with an identity is also referred to as a *loop*.

A quasigroup Q is *flexible* if $x \cdot (y \cdot x) = (x \cdot y) \cdot x$, for all $x, y \in Q$ and *alternative* if $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ and $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ for all $x, y \in Q$. Q is Moufang if it satisfies $x \cdot (y \cdot (x \cdot z)) = ((x \cdot y) \cdot x) \cdot z$ for all $x, y, z \in Q$.

We will discuss and prove some of the well-known properties of quasigroups here as they will serve as templates for some of our proofs in the next section.

In any quasigroup Q, the inverse is unique and anti-multiplicative, that is $(x^{-1})^{-1} = x$ and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ for all $x, y \in Q$. This is easily seen by considering $(x^{-1} \cdot (x \cdot y)) \cdot (x \cdot y)^{-1}$; replacing y in (3.1.1) by x^{-1} and x by $x \cdot y$, this is equal to x^{-1}, but also, by direct application of (3.1.1) we find $(x^{-1} \cdot (x \cdot y)) \cdot (x \cdot y)^{-1} = y \cdot (x \cdot y)^{-1}$. So we have $x^{-1} = y \cdot (x \cdot y)^{-1}$ for all $x, y \in Q$, and by multiplying both sides by y^{-1} on the left and a further application of (3.1.1), we obtain $y^{-1} \cdot x^{-1} = (x \cdot y)^{-1}$.

Theorem 3.1.2. *[4, Lemma 3.1] If a quasigroup Q satisfies any one of the following identities, then it satisfies all three:*

(1) $x \cdot (y \cdot (x \cdot z)) = ((x \cdot y) \cdot x) \cdot z \ \forall \ x, y, z \in Q,$

(2) $((x \cdot y) \cdot z) \cdot y = x \cdot (y \cdot (z \cdot y)) \ \forall \ x, y, z \in Q,$

(3) $(x \cdot y) \cdot (z \cdot x) = (x \cdot (y \cdot z)) \cdot x \ \forall \ x, y, z \in Q.$

Moreover, Q is flexible and alternative.

Proof. Suppose (1) holds. Taking the inverse of both sides, one obtains

$$((z^{-1} \cdot x^{-1}) \cdot y^{-1}) \cdot x^{-1} = z^{-1} \cdot (x^{-1} \cdot (y^{-1} \cdot x^{-1})),$$

for all $x, y, z \in Q$, which is equivalent to (2). Similarly (2) implies (1).

Now suppose (1) holds. Replacing z by $x^{-1} \cdot z$ we obtain $x \cdot (y \cdot z) = ((x \cdot y) \cdot x) \cdot (x^{-1} \cdot z)$ for all $x, y, z \in Q$. Multiplying on the right by $(x^{-1} \cdot z)^{-1}$ gives

$$(x \cdot y) \cdot x = (x \cdot (y \cdot z)) \cdot (x^{-1} \cdot z)^{-1} = (x \cdot (y \cdot z)) \cdot (z^{-1} \cdot x).$$

Then we replace z by z^{-1} and y by $y \cdot z$ to get

$$(x \cdot (y \cdot z)) \cdot x = (x \cdot y) \cdot (z \cdot x),$$

for all $x, y, z \in Q$, which is identity (3).

Suppose (3) holds, then

$$x \cdot y = ((x \cdot (y \cdot z)) \cdot x) \cdot (z \cdot x)^{-1} = ((x \cdot (y \cdot z)) \cdot x) \cdot (x^{-1} \cdot z^{-1}).$$

Replace z by $z^{-1} \cdot x^{-1}$ and y by $y \cdot (x \cdot z)$ to obtain

$$x \cdot (y \cdot (x \cdot z)) = ((x \cdot y) \cdot x) \cdot z,$$

for all $x, y, z \in Q$, which is identity (1).

Finally, if Q is a Moufang quasigroup, then setting $z = e$ in (1) gives $x \cdot (y \cdot x) = (x \cdot y) \cdot x$, *i.e.* Q is flexible. Similarly, we find Q is alternative by setting $y = e$ in (1) and $z = e$ in (2). □

Lemma 3.1.3. *Let Q be a flexible quasigroup, then for all $x, y \in Q$*

$$x \cdot (y \cdot x^{-1}) = (x \cdot y) \cdot x^{-1}.$$

Proof. For all $x, y \in Q$ we have

$$(x \cdot (y \cdot x^{-1})) \cdot x = x \cdot ((y \cdot x^{-1}) \cdot x) = x \cdot y$$

with the first equality coming from flexibility, and the second from the quasigroup property (3.1.1). But also,

$$((x \cdot y) \cdot x^{-1}) \cdot x = x \cdot y.$$

Hence, $(x \cdot (y \cdot x^{-1})) \cdot x = ((x \cdot y) \cdot x^{-1}) \cdot x$ for all $x, y \in Q$, and therefore, by multiplying both sides on the right by x^{-1}, we find $x \cdot (y \cdot x^{-1}) = (x \cdot y) \cdot x^{-1}$ for all $x, y \in Q$. □

For any quasigroup $\mathcal{G}$ we now introduce the *multiplicative associator* $\varphi : \mathcal{G}^3 \to \mathcal{G}$ satisfying

$$(uv)w = \varphi(u, v, w)(u(vw)), \quad \forall\ u, v, w \in \mathcal{G},$$

and in view of (3.1.1), we can also obtain it explicitly as

$$\varphi(u, v, w) = ((uv)w)(u(vw))^{-1} = ((uv)w)((w^{-1}v^{-1})u^{-1}), \tag{3.1.2}$$

for all $u, v, w \in \mathcal{G}$. We define the *group of associative elements* or *nucleus* $N(\mathcal{G})$ by

$$N(\mathcal{G}) = \{a \in \mathcal{G} \,|\, (au)v = a(uv),\ u(av) = (ua)v,\ (uv)a = u(va), \quad \forall\ u, v \in \mathcal{G}\}.$$

It is easy to see that this is indeed closed under the product and inverse operations and hence a group. We say that a quasigroup is *quasi-associative* if φ and all its conjugates $u\varphi u^{-1}$ have their image in $N(\mathcal{G})$, and *central* if the image of φ is in the centre $Z(\mathcal{G})$.

3.2 $k_F G$ approach to composition algebras

In [2] the authors constructed division algebras and higher Cayley algebras as twisted group rings, as follows. Let k be a field, G a finite group and $F : G \times G \to k^*$ a unital 2-cochain, *i.e.* $F(e, a) = F(a, e) = 1$ for all $a \in G$, where e is the group identity. Since our underlying group G is going to be Abelian we will write it additively. Let

$$\phi(a, b, c) = \frac{F(a, b)F(a + b, c)}{F(b, c)F(a, b + c)},$$

be the 3-cocycle coboundary of F. Finally, define $k_F G$ to be a vector space with basis $\{e_a \,|\, a \in G\}$ with product

$$e_a e_b = F(a, b)e_{a+b}.$$

It is easy to see that $(e_a e_b)e_c = \phi(a, b, c)e_a(e_b e_c)$, *i.e.* $k_F G$, while not associative, is *quasi-associative* in the sense that its nonassociativity is strictly controlled by a 3-cocycle. In categorical terms it lives in the symmetric monoidal category of G-graded spaces with associator defined by ϕ and symmetry defined by $\mathcal{R}(a, b) = \frac{F(a,b)}{F(b,a)}$. The choice $G = \mathbb{Z}_2^3$ and a certain F gives the octonions in this form. We note that by $\mathbb{Z}_2$ we mean the integers mod 2.

A unital algebra A over a field k is called a *composition algebra* if there is a nondegenerate multiplicative quadratic form q defined on A. Here we do not need the exact form of F but rather a theorem [2, Proposition 3.8] that $k_F \mathbb{Z}_2^n$ is a composition algebra with respect to the Euclidean norm $q(x) = 1$ for all $x \in G$ iff the following identities hold:

$$F(a,b)^2 = 1, \quad \forall\ a, b, \tag{3.2.1}$$

$$F(a, a+c)F(b, b+c) + F(a, b+c)F(b, a+c) = 0,\ \forall\ a \neq b,\ \forall\ c. \tag{3.2.2}$$

As a consequence, the cochain F will also satisfy

$$F(a, a+b) = -F(a,b), \quad \forall\ a \neq 0,\ \forall\ b, \tag{3.2.3}$$

$$F(a+b, a) = -F(b,a), \quad \forall\ a \neq 0,\ \forall\ b, \tag{3.2.4}$$

$$F(a,b)F(a,c) = -F(a+b,c)F(a+c,b), \quad \forall\ b \neq c,\ \forall\ a, \tag{3.2.5}$$

$$F(a,c)F(b,c) = -F(a,b+c)F(b,a+c), \quad \forall\ a \neq b,\ \forall\ c. \tag{3.2.6}$$

These identities are obtained from (3.2.2), for example, setting $b = 0$ gives (3.2.3). We also know and will use that

$$F(a,a) = -1, \quad \forall\ a \neq 0, \tag{3.2.7}$$

in our examples. This applies to the division algebras of the complex numbers, the quaternions, and the octonions given by $k = \mathbb{R}$ and such F on $\mathbb{Z}_2^n$ for $n = 1, 2, 3$ respectively. Note that the octonions being division algebras have left and right cancellation and obey the three equivalent Moufang loop identities in Theorem 3.1.2.

Lemma 3.2.1. *For any composition algebra $k_F\mathbb{Z}_2^n$,*

$$\phi(a,b,c) = \begin{cases} -1 & a,b,c \text{ linearly independent as vectors over } \mathbb{Z}_2, \\ 1 & \text{otherwise.} \end{cases}$$

$$\mathcal{R}(a,b) = \begin{cases} -1 & a,b \text{ linearly independent as vectors over } \mathbb{Z}_2, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, ϕ and $\mathcal{R}$ are symmetric, and $\phi(a+b,b,c) = \phi(a,b,c)$ and $\mathcal{R}(a+b,b) = \mathcal{R}(a,b)$ for all $a, b \in \mathbb{Z}_2^n$.

Proof. We start with the symmetry $\mathcal{R}$. If $a = b = 0$,

$$\mathcal{R}(a, b) = \mathcal{R}(0, 0) = F(0, 0)F(0, 0) = 1,$$

and if $a = b$ then, by (3.2.1),

$$\mathcal{R}(a, b) = \mathcal{R}(a, a) = F(a, a)F(a, a) = 1.$$

For the case $a \neq b$, one can use the composition identity (3.2.2), by setting $c = 0$ to obtain

$$F(a, a)F(b, b) = -F(a, b)F(b, a) = -\mathcal{R}(a, b).$$

If $a = 0, b \neq 0$ then $F(a, a) = 1$, $F(b, b) = -1$, and hence $\mathcal{R}(a, b) = 1$. Similarly, if $b = 0, a \neq 0$, then $\mathcal{R}(a, b) = 1$. Finally, if $a, b \neq 0$, $F(a, a) = F(b, b) = -1$, and hence $\mathcal{R}(a, b) = -1$. This establishes the stated form of $\mathcal{R}$.

For ϕ we again consider the cases separately. Suppose a, b, c are linearly dependent, say $a = b + c$ with $a, b, c \neq 0$, then,

$$\phi(a, b, c) = \phi(b+c, b, c) = F(b+c, b)F(c, c)F(b, c)F(b+c, b+c) = F(b+c, b)F(b, c),$$

since $F(c, c) = F(b + c, b + c) = -1$. Using identity (3.2.4) we obtain $F(b + c, b) = -F(c, b)$ since $b \neq 0$. Thus,

$$\phi(b + c, b, c) = F(b + c, b)F(b, c) = -F(c, b)F(b, c) = -\mathcal{R}(b, c) = 1.$$

Now suppose, $a = b \neq 0$, then,

$$\phi(a, b, c) = \phi(a, a, c) = F(a, a)F(0, c)F(a, c)F(a, a + c) = -F(a, c)F(a, a + c).$$

Using (3.2.3) and (3.2.1), we get

$$\phi(a, a, c) = F(a, c)F(a, c) = 1.$$

Similarly if $a = c \neq 0$ or $b = c \neq 0$. Next, suppose $a = 0$, then using (3.2.1)

$$\phi(a, b, c) = \phi(0, b, c) = F(0, b)F(b, c)F(b, c)F(0, b + c) = F(b, c)F(b, c) = 1.$$

Similarly for $b = 0$ and $c = 0$. Finally, suppose $a, b, c \neq 0$ are linearly independent, then

$$\phi(a,b,c) = F(a,b)F(a+b,c)F(b,c)F(a,b+c).$$

Using identities (3.2.5) and (3.2.6), we obtain

$$F(a,b)F(a+b,c) = -F(a,c)F(a+c,b),$$

$$F(b,c)F(a,b+c) = -F(a,c)F(b,a+c),$$

hence,

$$\phi(a,b,c) = F(a,c)F(a+c,b)F(a,c)F(b,a+c) = \mathcal{R}(a+c,b) = -1.$$

This establishes the stated form of ϕ.

Finally, we show that ϕ is symmetric. Assume $a, b, c \neq 0$ are linearly independent in $\mathbb{Z}_2^n$. Using first the properties of $\mathcal{R}$ and that $\mathbb{Z}_2^n$ is Abelian, followed by identity (3.2.6), we compute

$$\begin{aligned}
\phi(a,b,c) &= F(a,b)F(a+b,c)F(b,c)F(a,b+c) \\
&= -F(b,a)F(b+a,c)F(b,c)F(a,b+c) \\
&= F(b,a)F(b+a,c)F(a,c)F(b,a+c) \\
&= \phi(b,a,c).
\end{aligned}$$

Similarly, by properties of $\mathcal{R}$,

$$\begin{aligned}
\phi(a,b,c) &= F(a,b)F(a+b,c)F(b,c)F(a,b+c) \\
&= F(b,a)F(c,b+a)F(c,b)F(c+b,a) \\
&= \phi(c,b,a).
\end{aligned}$$

The final conclusions follow immediately as the linear independence of $a+b, b, c$ is equivalent to the linear independence of a, b, c. □

We write a general element $u \in k_F G$ as $\sum_{a\in G} u_a e_a$ where $u_a \in k$ is the coefficient of e_a. The statement that $k_F G$ is a composition algebra means that the norm

$$q(\sum_a u_a e_a) = \sum_a u_a^2,$$

is multiplicative. In particular, it means that the set of elements of unit norm, *i.e.* the (2^n-1)-spheres over k, is closed under multiplication in $k_F\mathbb{Z}_2^n$. We denote this set by

$$\mathcal{S}^{2^n-1} = \{\sum_a u_a e_a \,|\, \sum_a u_a^2 = 1\} \subset k_F\mathbb{Z}_2^n,$$

which becomes a usual sphere if we work over $\mathbb{R}$. The next lemma makes it clear that we have 2-sided inverses by exhibiting them.

Lemma 3.2.2. *In the setting of Lemma 3.2.1, if $u \in k_F\mathbb{Z}_2^n$ has unit norm then*

$$u^{-1} = u_0e_0 - \sum_{a\neq 0} u_a e_a = 2u_0 - u.$$

Proof. We verify that the given element is indeed the inverse of u:

$$(u_0e_0 - \sum_{a\neq 0} u_a e_a)(u_0e_0 + \sum_{b\neq 0} u_b e_b) = \sum u_a^2 e_a - \sum_{a,b,a+b\neq 0} u_a u_b F(a,b) e_{a+b} = q(u),$$

since under the assumptions we know that $F(a,b) = -F(b,a)$ and hence the last summation equals zero. Similarly on the other side. □

We also have $\mathcal{S}^{2^n-1} \subset k_F^{\times}\mathbb{Z}_2^n$, the set of elements of nonzero norm. These are also closed under the product of $k_F\mathbb{Z}_2^n$ and from the proof of Lemma 3.2.2 we deduce that they are invertible with $u^{-1} = (2u_0 - u)/q(u)$. Over $\mathbb{R}$ this larger object is the set of the invertible elements of k_FG. Similarly we define a finite object $\mathcal{G}_n \subset \mathcal{S}^{2^n-1}$ by

$$\mathcal{G}_n = \{\pm e_a \,|\, a \in \mathbb{Z}_2^n\} \subset k_F\mathbb{Z}_2^n.$$

The elements here all have unit norm and since $F(a,b) = \pm 1$ we see that $\mathcal{G}_n$ is closed under multiplication and has identity $1 = e_0$. The inverses are $e_i^{-1} = -e_i$ for $i \neq 0$. In the case $n = 3$ one has $\mathcal{G}_n = \mathcal{G}_{\mathbb{O}}$ the order 16 Moufang loop associated to the octonions.

Clearly the invertible elements of the octonions $\mathbb{O}$ form a quasigroup, as this follows from the fact that they are a Moufang loop. The sphere $\mathcal{S}^7$ of unit octonions and $\mathcal{G}_{\mathbb{O}}$ are therefore subquasigroups. In our theory these facts are easily proven from properties of F. The notion of subquasigroup here is the obvious one and note that the multiplicative associator φ on the subquasigroup is the restriction of φ on the larger one.

Proposition 3.2.3. *In the setting of Lemma 3.2.1, $\mathcal{S}^{2^n-1}$ is a quasigroup and $\mathcal{G}_n$ is a sub-quasigroup. Moreover, $\mathcal{G}_n$ is central and quasi-associative and*

$$\varphi(\pm e_a, \pm e_b, \pm e_c) = \phi(a, b, c),$$

reproducing the coboundary $\phi = \partial F$.

Proof. Working in $k_F\mathbb{Z}_2^n$ we have inverses for $\mathcal{S}^{2^n-1}$ given by $(u^{-1})_a = F(a,a)u_a$ as explained in Lemma 3.2.2. One can directly verify the quasigroup identity (3.1.1) by direct computation inside $k_F\mathbb{Z}_2^n$. Thus

$$u^{-1}(uv) = \sum_{a,b,c} u_a u_b v_c e_{a+b+c} F(a,a)\phi(a,b,c)F(a,b)F(a+b,c).$$

We split the above sum into the partial sums with $a = b$ and $a \neq b$. The partial sum with $a = b$ equals

$$\sum_{a,c} u_a^2 v_c e_c F(a,a)\phi(a,a,c)F(a,a)F(0,c) = \sum_{a,c} u_a^2 v_c e_c = \sum_c v_c e_c = v.$$

Now consider the partial sum with $a \neq b$. We claim that the term with given values for a and b cancels with the term with a and b interchanged. These give, respectively,

$$\sum_c u_a u_b v_c e_{a+b+c} F(a,a)\phi(a,b,c)F(a,b)F(a+b,c), \tag{3.2.8}$$

$$\sum_c u_b u_a v_c e_{b+a+c} F(b,b)\phi(b,a,c)F(b,a)F(b+a,c). \tag{3.2.9}$$

When $a = 0$ and hence $b \neq 0$, these become respectively,

$$\sum_c u_0 u_b v_c e_{b+c} F(b,c),$$

$$\sum_c u_b u_0 v_c e_{b+c} F(b,b)F(b,c) = -\sum_c u_0 u_b v_c e_{b+c} F(b,c),$$

which cancel. Using Lemma 3.2.1, when $a, b \neq 0$, (3.2.8) and (3.2.9) become, respectively,

$$-\sum_c u_a u_b v_c e_{a+b+c}\phi(a,b,c)F(a,b)F(a+b,c),$$

$$\begin{aligned}-\sum_c u_b u_a v_c e_{b+a+c}\phi(b,a,c)F(b,a)F(b+a,c)\\ =\sum_c u_a u_b v_c e_{a+b+c}\phi(a,b,c)F(a,b)F(a+b,c),\end{aligned}$$

which also clearly cancel. Hence,

$$u^{-1}(uv)=v,$$

as required. Similarly, for the other side. We will later need an explicit formula for φ, which is easily computed from (3.1.2) as

$$\begin{aligned}\varphi(u,v,w)=\sum_{a,b,c,a',b',c'} & u_a v_b w_c u_{a'} v_{b'} w_{c'} F(a',a')F(b',b')F(c',c')F(a,b)F(a+b,c)\\ & F(c',b')F(b'+c',a')F(a+b+c,a'+b'+c')e_{a+b+c+a'+b'+c'}.\end{aligned}$$

To prove the restriction of φ to $\mathcal{G}_n$ we compute from this, or obtain it directly from (3.1.2):

$$\begin{aligned}\varphi(e_a,e_b,e_c) &= -((e_a e_b)e_c)((e_c e_b)e_a)\\ &= -F(a,b)F(a+b,c)F(c,b)F(b+c,a)F(a+b+c,a+b+c)\\ &\qquad F(a,a)F(b,b)F(c,c)\\ &= -\phi(a,b,c)\mathcal{R}(a,b+c)\mathcal{R}(b,c)F(a+b+c,a+b+c)\\ &\qquad F(a,a)F(b,b)F(c,c).\end{aligned}$$

Now, we need only consider the case when $a,b,c\neq 0$ as the trivial cases clearly coincide. For the same reason we can assume that a,b,c are distinct. Under these assumptions, $\mathcal{R}(b,c)=-1$ canceling the - sign at the front. If $a+b+c\neq 0$ we have $\mathcal{R}(a,b+c)=-1$ and $F(a+b+c,a+b+c)=-1$, and if $a+b=c$ we have both of these factors $+1$. Hence in all cases the right hand side is $\phi(a,b,c)$. □

The result for $\mathcal{S}^{2^n-1}$ also applies to $k_F^{\times}\mathbb{Z}_2^n$ in the composition case; the proof only requires that we keep track of the norm, $q(u)$. Meanwhile, the result for $\mathcal{G}_n$ can also be obtained 'constructively' as a special case $C = \{\pm 1\} \subset k^*$ of the following general construction (the conditions on F following from (3.2.1)- (3.2.7)). We write all groups multiplicatively here.

Proposition 3.2.4. *Let F be any unital 2-cochain on a group G with values in an Abelian group C and $\mathcal{G}_F = C \times G$ with product $(\lambda, a)(\mu, b) = (\lambda\mu F(a,b), ab)$ for all $a, b \in G$ and $\lambda, \mu \in C$. Then*

(1) *$\mathcal{G}_F$ is a quasigroup* iff $\phi(a^{-1}, a, b) = \phi(b, a^{-1}, a) = 1$ *for all $a, b \in G$, where $\phi = \partial F$,*

(2) *$\mathcal{G}_F$ is a quasigroup* iff $F(a,b)F(a^{-1}, ab) = F(ba^{-1}, a)F(b, a^{-1}) = F(a^{-1}, a)$ *for all $a, b \in G$,*

(3) *The quasigroup $\mathcal{G}_F$ is central, quasi-associative, and $C \hookrightarrow \mathcal{G}_F \twoheadrightarrow G$ as (quasi)-groups.*

Proof. Let 1 and e be the identity elements in C and G respectively. The identity in $\mathcal{G}$ is $(1, e)$ and the inverse if it exists (from the left, say) must then be

$$(\lambda, a)^{-1} = (\lambda^{-1}F^{-1}(a^{-1}, a), a^{-1}).$$

To prove (1) begin by assuming $\mathcal{G}_F$ is a quasigroup, then, since C is an Abelian group and G is a group,

$$\begin{aligned}(\mu, b) &= (\lambda, a)^{-1}((\lambda, a)(\mu, b)) \\ &= (\lambda^{-1}F^{-1}(a^{-1}, a), a^{-1})(\lambda\mu F(a,b), ab) \\ &= (\lambda^{-1}\lambda\mu F^{-1}(a^{-1}, a)F(a,b)F(a^{-1}, ab), a^{-1}ab) \\ &= (\mu\phi^{-1}(a^{-1}, a, b), b),\end{aligned}$$

hence, $\phi(a^{-1}, a, b) = 1$. Similarly, $(\mu, b) = ((\mu, b)(\lambda, a)^{-1})(\lambda, a)$ implies $\phi(b, a^{-1}, a) = 1$ for all $a, b \in G$. For the converse, note that these identities imply $R(a^{-1}, a) = \phi(a^{-1}, a, a^{-1}) = 1$ and $(\lambda, a)^{-1}$ is a right inverse in $\mathcal{G}_F$. Checking the quasigroup identities is straightforward. Thus (1) holds.

The identities in (2) are equivalent to those in (1), so (2) is immediate from (1). Finally, to prove the last part, the multiplicative associator (3.1.2) $\varphi : \mathcal{G}_F^3 \to \mathcal{G}_F$

is given as

$$\varphi((\lambda, a), (\mu, b), (\nu, c)) = (\phi(a, b, c), e) \in C,$$

since $C \hookrightarrow \mathcal{G}_F$ as $\lambda \mapsto (\lambda, e)$. So to show that $\mathcal{G}_F$ is central and quasi-associative, it is sufficient to show that C commutes with every element of $\mathcal{G}_F$ and also associates.

$$(\lambda, e)(\mu, b) = (\lambda\mu F(e, b), b) = (\lambda\mu, b) = (\mu\lambda, b) = (\mu\lambda F(b, e), b) = (\mu, b)(\lambda, e),$$

since C is Abelian and $F(e, b) = F(b, e) = 1$ for all $b \in G$. Hence $\mathcal{G}_F$ is central. Finally,

$$(\lambda, a)((\mu, b)(\nu, c)) = (\lambda\mu\nu F(a, bc)F(b, c), abc),$$

while,

$$((\lambda, a)(\mu, b))(\nu, c) = (\lambda\mu\nu F(a, b)F(ab, c), abc) = (\lambda\mu\nu\phi(a, b, c)F(b, c)F(a, bc), abc).$$

But, $\phi(a, b, c) = 1$ if any of a, b or c equal the identity, *i.e.* these identities are equal if any element lies in C. Thus $\mathcal{G}_F$ is quasi-associative. □

3.3 Hopf quasigroups

We linearize the notion of an inverse property quasigroup and demonstrate that much of the classical theory discussed in the previous section can be extended to this setting.

Definition 3.3.1. A *Hopf quasigroup* is a nonassociative unital algebra H over a field k equipped with algebra morphisms $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ forming a coassociative coalgebra, and a linear map $S : H \to H$ such that

$$m(\mathrm{id} \otimes m)(S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id}) = \varepsilon \otimes \mathrm{id} = m(\mathrm{id} \otimes m)(\mathrm{id} \otimes S \otimes \mathrm{id})(\Delta \otimes \mathrm{id}),$$

$$m(m \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) = \mathrm{id} \otimes \varepsilon = m(m \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes S)(\mathrm{id} \otimes \Delta).$$

The conditions in Definition 3.3.1 are read explicitly on $h, g \in H$ as

$$\sum (Sh_{(1)})(h_{(2)}g) = \varepsilon(h)g = \sum h_{(1)}((Sh_{(2)})g), \tag{3.3.1}$$

$$\sum(hSg_{(1)})g_{(2)} = h\varepsilon(g) = \sum(hg_{(1)})Sg_{(2)}. \tag{3.3.2}$$

The map S in the definition of a Hopf quasigroup is called the *antipode*. Note that by nonassociative, we mean 'not necessarily associative'. We define further properties on Hopf quasigroups by following the theory in the previous section and the Hopf algebra theory in Section 2.1.

Definition 3.3.2. A Hopf quasigroup H is *flexible* if

$$\sum h_{(1)}(gh_{(2)}) = \sum(h_{(1)}g)h_{(2)},$$

for all $h, g \in H$, *alternative* if

$$\sum h_{(1)}(h_{(2)}g) = \sum(h_{(1)}h_{(2)})g, \quad \sum h(g_{(1)}g_{(2)}) = \sum(hg_{(1)})g_{(2)}, \quad \forall\ h, g \in H,$$

and *Moufang* if

$$\sum h_{(1)}(g(h_{(2)}f)) = \sum((h_{(1)}g)h_{(2)})f, \quad \forall\ h, g, f \in H.$$

Proposition 3.3.3. *In any Hopf quasigroup H one has:*

(1) $m(S \otimes \mathrm{id})\Delta = 1 \cdot \varepsilon = m(\mathrm{id} \otimes S)\Delta$,

(2) *S is anti-multiplicative, i.e. $S(hg) = (Sg)(Sh)$ for all $h, g \in H$,*

(3) *S is anti-comultiplicative, i.e. $\Delta(Sh) = \sum Sh_{(2)} \otimes Sh_{(1)}$ for all $h \in H$.*

Hence a Hopf quasigroup is a Hopf algebra iff *it is associative.*

Proof. (1) is obtained from the first quasigroup identity applied to $(h \otimes 1)$. To prove (2) we use the the definition of a Hopf quasigroup:

$$\begin{aligned}
S(hg) &= \varepsilon(g_{(1)})S(hg_{(2)})\\
&= (Sg_{(1)(1)})(g_{(1)(2)}S(hg_{(2)})), \quad \text{by (3.3.1)},\\
&= (Sg_{(1)(1)})(\varepsilon(h_{(1)})g_{(1)(2)}S(h_{(2)}g_{(2)}))\\
&= (Sg_{(1)(1)})(\{(Sh_{(1)(1)})(h_{(1)(2)}g_{(1)(2)})\}S(h_{(2)}g_{(2)})), \quad \text{by (3.3.1)},\\
&= (Sg_{(1)})(\{(Sh_{(1)})(h_{(2)(1)}g_{(2)(1)})\}S(h_{(2)(2)}g_{(2)(2)})), \quad \text{by coassociativity},\\
&= (Sg_{(1)})(((Sh_{(1)})(h_{(2)}g_{(2)})_{(1)})S(h_{(2)}g_{(2)})_{(2)}),
\end{aligned}$$

$$\begin{aligned}&\text{since } \Delta \text{ is an algebra morphism,}\\ &= (Sg_{(1)})(Sh_{(1)})\varepsilon(h_{(2)}g_{(2)}), \quad \text{by (3.3.2)},\\ &= (Sg)(Sh).\end{aligned}$$

Finally, we prove (3) in a similar way:

$$\begin{aligned}\Delta(Sh) &= (Sh)_{(1)} \otimes (Sh)_{(2)}\\ &= (Sh_{(2)})_{(1)} \otimes \varepsilon(h_{(1)})(Sh_{(2)})_{(2)}\\ &= (Sh_{(2)})_{(1)} \otimes (Sh_{(1)(1)})(h_{(1)(2)}(Sh_{(2)})_{(2)}), \quad \text{by (3.3.1) on } h_{(1)},\\ &= \varepsilon(h_{(1)(2)(1)})(Sh_{(2)})_{(1)} \otimes (Sh_{(1)(1)})(h_{(1)(2)(2)}(Sh_{(2)})_{(2)})\\ &= (Sh_{(1)(2)(1)(1)})(h_{(1)(2)(1)(2)}(Sh_{(2)})_{(1)}) \otimes (Sh_{(1)(1)})(h_{(1)(2)(2)}(Sh_{(2)})_{(2)}),\\ &\qquad \text{by (3.3.1) on } h_{(1)(2)(1)},\\ &= (Sh_{(2)(1)})(h_{(2)(2)(1)(1)}(Sh_{(2)(2)(2)})_{(1)}) \otimes (Sh_{(1)})(h_{(2)(2)(1)(2)}(Sh_{(2)(2)(2)})_{(2)}),\\ &\qquad \text{by coassociativity,}\\ &= (Sh_{(2)(1)})(h_{(2)(2)(1)}Sh_{(2)(2)(2)})_{(1)} \otimes (Sh_{(1)})(h_{(2)(2)(1)}Sh_{(2)(2)(2)})_{(2)},\\ &\qquad \text{since } \Delta \text{ is an algebra morphism,}\\ &= Sh_{(2)} \otimes Sh_{(1)}, \quad \text{by (1)}.\end{aligned}$$

□

Proposition 3.3.4. *Let H be a Hopf quasigroup, then $S^2 = \mathrm{id}$ if H is commutative or cocommutative.*

Proof. Let $h \in H$. Since H is coassociative and S is anti-comultiplicative, we have, for all $h \in H$,

$$\begin{aligned}S^2(h) = S^2(h_{(1)})((Sh_{(2)(1)})h_{(2)(2)}) &= S^2(h_{(1)(1)})((Sh_{(1)(2)})h_{(2)})\\ &= S((Sh_{(1)})_{(2)})((Sh_{(1)})_{(1)}h_{(2)}).\end{aligned}$$

If H is commutative, *i.e.* $hg = gh$ for all $h, g \in H$, we find that this

$$= (h_{(2)}(Sh_{(1)})_{(1)})S((Sh_{(1)})_{(2)}) = \varepsilon(Sh_{(1)})h_{(2)} = h,$$

using (3.3.2). However, if H is cocommutative, that is $h_{(1)} \otimes h_{(2)} = h_{(2)} \otimes h_{(1)}$ for all $h \in H$, the expression

$$= S((Sh_{(1)})_{(1)})((Sh_{(1)})_{(2)}h_{(2)}) = \varepsilon(Sh_{(1)})h_{(2)} = h,$$

using (3.3.1). In either case $S^2 = \mathrm{id}$, as required. □

If H is a Moufang Hopf quasigroup with an invertible antipode then, as in the classical case, the Moufang identity is equivalent to two other identities, as in Theorem 3.1.2.

Lemma 3.3.5. *Let H be a Hopf quasigroup such that S^{-1} exists, then the following identities are equivalent:*

(1) $\sum h_{(1)}(g(h_{(2)}f)) = \sum((h_{(1)}g)h_{(2)})f, \quad \forall\, h, g, f \in H,$

(2) $\sum((hg_{(1)})f)g_{(2)} = \sum h(g_{(1)}(fg_{(2)})), \quad \forall\, h, g, f \in H,$

(3) $\sum(h_{(1)}g)(fh_{(2)}) = \sum(h_{(1)}(gf))h_{(2)}, \quad \forall\, h, g, f \in H.$

Proof. Suppose (1) holds. Applying S to both sides and using Proposition 3.3.3, we find

$$(((Sf)(Sh)_{(1)})Sg)(Sh)_{(2)} = (Sf)((Sh)_{(1)}((Sg)(Sh)_{(2)}))$$

for all $h, g, f \in H$, which is equivalent to (2). Similarly, (2) implies (1).

Assume (1) holds. Using (3.3.1) and coassociativity of H,

$$h(gf) = h_{(1)(1)}(g(h_{(1)(2)}((Sh_{(2)})f))) = ((h_{(1)(1)}g)h_{(1)(2)})((Sh_{(2)})f),$$

for all $h, g, f \in H$. By replacing f with $Sf_{(2)}$ and g with $gf_{(1)}$, we obtain

$$hg\varepsilon(f) = h((gf_{(1)})Sf_{(2)}) = ((h_{(1)(1)}(gf_{(1)}))h_{(1)(2)})S(f_{(2)}h_{(2)}),$$

for all $h, g, f \in H$. Now replace h by $h_{(1)}$, f by $f_{(1)}$, and multiply both sides on the right by $f_{(2)}h_{(2)}$ to get

$$\varepsilon(f_{(1)})(h_{(1)}g)(f_{(2)}h_{(2)}) = (((h_{(1)(1)(1)}(gf_{(1)(1)}))h_{(1)(1)(2)})S(f_{(1)(2)}h_{(1)(2)}))(f_{(2)}h_{(2)}).$$

The LHS equals $(h_{(1)}g)(fh_{(2)})$, and by using coassociativity and (3.3.2), we find the RHS

$$= (((h_{(1)(1)}(gf_{(1)}))h_{(1)(2)})S((f_{(2)}h_{(2)})_{(1)}))(f_{(2)}h_{(2)})_{(2)} = (h_{(1)}(gf))h_{(2)}.$$

Thus $(h_{(1)}g)(fh_{(2)}) = (h_{(1)}(gf))h_{(2)}$, which is identity (3).

Finally, assume (3) holds, then

$$((h_{(1)(1)}g)(f_{(1)}h_{(1)(2)}))S(f_{(2)}h_{(2)}) = ((h_{(1)(1)}(gf_{(1)}))h_{(1)(2)})S(f_{(2)}h_{(2)}).$$

Using properties of Δ and (3.3.2), the LHS equals $hg\varepsilon(f)$, so we find

$$hg\varepsilon(f) = ((h_{(1)(1)}(gf_{(1)}))h_{(1)(2)})((Sh_{(2)})Sf_{(2)}),$$

for all $h, g, f \in H$. By replacing f with $f_{(2)}$ and g with $gSf_{(1)}$, we obtain

$$h(gSf_{(1)})\varepsilon(f_{(2)}) = ((h_{(1)(1)}((gSf_{(1)})f_{(2)(1)}))h_{(1)(2)})((Sh_{(2)})Sf_{(2)(2)}).$$

Clearly the LHS simplifies to $h(gSf)$ and the RHS simplifies to

$$((h_{(1)(1)}g)h_{(1)(2)})((Sh_{(2)})Sf).$$

Finally, by replacing h with $h_{(1)}$ and f with $S^{-1}(h_{(2)}f)$, we obtain

$$h_{(1)}(g(h_{(2)}f)) = ((h_{(1)(1)(1)}g)h_{(1)(1)(2)})((Sh_{(1)(2)})(h_{(2)}f)) = ((h_{(1)}g)h_{(2)})f,$$

where the last equality uses coassociativity of H and (3.3.1), which is identity (1). □

If we linearize Lemma 3.1.3, we get a notion of the right (and similarly left) adjoint actions of a Hopf quasigroup on itself under certain conditions.

Lemma 3.3.6. *Let H be a cocommutative, flexible Hopf quasigroup, then*

(1) $((Sh_{(1)})g)h_{(2)} = (Sh_{(1)})(gh_{(2)})$,

(2) $(h_{(1)}g)Sh_{(2)} = h_{(1)}(gSh_{(2)})$,

for all $h, g \in H$.

Proof. By coassociativity and the Hopf quasigroup identity (3.3.1)

$$h_{(1)}((Sh_{(2)(1)})(gh_{(2)(2)})) = h_{(1)(1)}((Sh_{(1)(2)})(gh_{(2)})) = gh,$$

but also, by cocommutativity, coassociativity and flexibility we have

$$\begin{aligned} h_{(1)}(((Sh_{(2)(1)})g)h_{(2)(2)}) &= h_{(1)}(((Sh_{(2)(2)})g)h_{(2)(1)}) \\ &= h_{(1)(1)}(((Sh_{(2)})g)h_{(1)(2)}) \\ &= (h_{(1)(1)}((Sh_{(2)})g))h_{(1)(2)} \\ &= (h_{(1)(2)}((Sh_{(2)})g))h_{(1)(1)} \\ &= (h_{(2)(1)}((Sh_{(2)(2)})g))h_{(1)} \\ &= gh, \quad \text{by (3.3.1)}. \end{aligned}$$

Hence,

$$h_{(1)}(((Sh_{(2)(1)})g)h_{(2)(2)}) = h_{(1)}((Sh_{(2)(1)})(gh_{(2)(2)})),$$

for all $g, h \in H$. If we apply this to $h_{(2)}$ and multiply both sides on the left by $Sh_{(1)}$ we obtain

$$((Sh_{(1)})g)h_{(2)} = (Sh_{(1)})(gh_{(2)}).$$

Proving (2) is similar: by coassociativity and (3.3.2),

$$((h_{(1)(1)}g)Sh_{(1)(2)})h_{(2)} = ((h_{(1)}g)Sh_{(2)(1)})h_{(2)(2)} = hg,$$

but also, by cocommutativity, coassociativity and flexibility we have,

$$\begin{aligned} (h_{(1)(1)}(gSh_{(1)(2)}))h_{(2)} &= (h_{(1)(2)}(gSh_{(1)(1)}))h_{(2)} \\ &= (h_{(2)(1)}(gSh_{(1)}))h_{(2)(2)} \\ &= h_{(2)(1)}((gSh_{(1)})h_{(2)(2)}) \\ &= h_{(2)(2)}((gSh_{(1)})h_{(2)(1)}) \\ &= h_{(2)}((gSh_{(1)(1)})h_{(1)(2)}) \\ &= hg. \end{aligned}$$

So,

$$(h_{(1)(1)}(gSh_{(1)(2)}))h_{(2)} = ((h_{(1)(1)}g)Sh_{(1)(2)})h_{(2)}.$$

Replacing h with $h_{(1)}$ and multiplying on the right by $Sh_{(2)}$ we obtain

$$h_{(1)}(gSh_{(2)}) = (h_{(1)}g)Sh_{(2)},$$

for all $h, g \in H$. □

Definition 3.3.7. Let H be a Hopf quasigroup. The *associator* of H is a linear map $\varphi : H \otimes H \otimes H \to H$ such that

$$(hg)f = \varphi(h_{(1)}, g_{(1)}, f_{(1)})(h_{(2)}(g_{(2)}f_{(2)})), \tag{3.3.3}$$

for all $h, g, f \in H$.

Proposition 3.3.8. *Let H be a Hopf quasigroup, then*

(1) *The associator φ exists and is uniquely determined for all $h, g, f \in H$ by*

$$\varphi(h, g, f) = ((h_{(1)}g_{(1)})f_{(1)})S(h_{(2)}(g_{(2)}f_{(2)})),$$

(2) $\varphi(1, h, g) = \varphi(h, 1, g) = \varphi(h, g, 1) = \varepsilon(h)\varepsilon(g) \cdot 1$, *for all $h, g \in H$,*

(3) $\varphi(h_{(1)}, Sh_{(2)}, g) = \varphi(Sh_{(1)}, h_{(2)}, g) = \varepsilon(h)\varepsilon(g) \cdot 1$,
$\varphi(h, g_{(1)}, Sg_{(2)}) = \varphi(h, Sg_{(1)}, g_{(2)}) = \varepsilon(h)\varepsilon(g) \cdot 1$,
$\varphi(h_{(1)}g_{(1)}, Sg_{(2)}, Sh_{(2)}) = \varphi((Sh_{(1)})Sg_{(1)}, g_{(2)}, h_{(2)}) = \varepsilon(h)\varepsilon(g) \cdot 1$,
$\varphi(Sh_{(1)}, Sg_{(1)}, g_{(2)}h_{(2)}) = \varphi(h_{(1)}, g_{(1)}, (Sg_{(2)})Sh_{(2)}) = \varepsilon(h)\varepsilon(g) \cdot 1$,
$\varphi(Sh_{(1)}, h_{(2)}Sg_{(1)}, g_{(2)}) = \varphi(h_{(1)}, (Sh_{(2)})g_{(1)}, Sg_{(2)}) = \varepsilon(h)\varepsilon(g) \cdot 1$.

Proof. To prove (1), suppose an associator φ exists for H then,

$$\begin{aligned}
&((h_{(1)}g_{(1)})f_{(1)})S(h_{(2)}(g_{(2)}f_{(2)}))\\
&= (\varphi(h_{(1)(1)}, g_{(1)(1)}, f_{(1)(1)})(h_{(1)(2)}(g_{(1)(2)}f_{(1)(2)})))S(h_{(2)}(g_{(2)}f_{(2)})), \quad \text{by (3.3.3)},\\
&= (\varphi(h_{(1)}, g_{(1)}, f_{(1)})(h_{(2)(1)}(g_{(2)(1)}f_{(2)(1)})))S(h_{(2)(2)}(g_{(2)(2)}f_{(2)(2)}))\\
&= (\varphi(h_{(1)}, g_{(1)}, f_{(1)})(h_{(2)}(g_{(2)}f_{(2)}))_{(1)})S((h_{(2)}(g_{(2)}f_{(2)}))_{(2)})\\
&= \varphi(h_{(1)}, g_{(1)}, f_{(1)})\varepsilon(h_{(2)})\varepsilon(g_{(2)})\varepsilon(f_{(2)}), \quad \text{by (3.3.2)},\\
&= \varphi(h, g, f).
\end{aligned}$$

So φ takes the form given in (1). Now we must verify that this φ satisfies (3.3.3). Since Δ is coassociative and an algebra morphism, we find

$$\begin{aligned}
&\varphi(h_{(1)}, g_{(1)}, f_{(1)})(h_{(2)}(g_{(2)}f_{(2)}))\\
&=(((h_{(1)(1)}g_{(1)(1)})f_{(1)(1)})S(h_{(1)(2)}(g_{(1)(2)}f_{(1)(2)})))(h_{(2)}(g_{(2)}f_{(2)}))\\
&=(((h_{(1)}g_{(1)})f_{(1)})S(h_{(2)(1)}(g_{(2)(1)}f_{(2)(1)})))(h_{(2)(2)}(g_{(2)(2)}f_{(2)(2)}))\\
&=(((h_{(1)}g_{(1)})f_{(1)})S((h_{(2)}(g_{(2)}f_{(2)}))_{(1)}))(h_{(2)}(g_{(2)}f_{(2)}))_{(2)}\\
&=((h_{(1)}g_{(1)})f_{(1)})\varepsilon(h_{(2)})\varepsilon(g_{(2)})\varepsilon(f_{(2)})\\
&=(hg)f,
\end{aligned}$$

where the last equality uses (3.3.2). Part (2) is straightforward by using Proposition 3.3.3 (1) and that Δ is an algebra morphism. For example,

$$\varphi(1,h,g) = (h_{(1)}g_{(1)})S(h_{(2)}g_{(2)}) = (hg)_{(1)}S(hg)_{(2)} = \varepsilon(hg)\cdot 1 = \varepsilon(h)\varepsilon(g)\cdot 1.$$

Part (3) requires the Hopf quasigroup identities (3.3.1), and (3.3.2) and that S is anti-comultiplicative. For example,

$$\begin{aligned}
\varphi(h_{(1)}, Sh_{(2)}, g) &= ((h_{(1)(1)}(Sh_{(2)})_{(1)})g_{(1)})S(h_{(1)(2)}((Sh_{(2)})_{(2)}g_{(2)}))\\
&=((h_{(1)(1)}Sh_{(2)(2)})g_{(1)})S(h_{(1)(2)}(Sh_{(2)(1)}g_{(2)}))\\
&=((h_{(1)(1)}Sh_{(2)})g_{(1)})S(h_{(1)(2)(1)}(Sh_{(1)(2)(2)}g_{(2)})),\\
&\qquad\text{by coassociativity,}\\
&=((h_{(1)}Sh_{(2)})g_{(1)})Sg_{(2)}, \quad \text{by (3.3.1) on } h_{(1)(2)},\\
&=\varepsilon(h)g_{(1)}Sg_{(2)}\\
&=\varepsilon(h)\varepsilon(g)\cdot 1.
\end{aligned}$$

The rest is similar. □

These definitions become the more familiar quasigroup identities from the previous section when we consider the group algebra of a quasigroup $\mathcal{G}$:

Lemma 3.3.9. *Let $\mathcal{G}$ be an inverse property quasigroup, then $k\mathcal{G}$ is a Hopf quasigroup with linear extension of the product and $\Delta u = u\otimes u$, $\varepsilon u = 1$, $Su = u^{-1}$ for all basis elements $u \in \mathcal{G}$. Moreover, $k\mathcal{G}$ is Moufang if $\mathcal{G}$ is.*

Proof. We check the required identities on the basis elements $u \in \mathcal{G}$ since the structure maps of $k\mathcal{G}$ are linear. The comultiplication is clearly coassociative and an algebra morphism. Since $\mathcal{G}$ is a quasigroup, we have

$$(Su_{(1)})(u_{(2)}v) = u^{-1}(uv) = v = \varepsilon(u)v,$$

for all $u, v \in \mathcal{G}$ so the LHS of (3.3.1) is satisfied. Similarly, the rest of the Hopf quasigroup identities hold. Clearly, if $\mathcal{G}$ is Moufang, then

$$u_{(1)}(v(u_{(2)}w)) = u(v(uw)) = ((uv)u)w = ((u_{(1)}v)u_{(2)})w,$$

hence $k\mathcal{G}$ is Moufang. □

With Hopf algebras, we have a correspondence between groups and Hopf algebras, similarly, for every quasigroup $\mathcal{G}$ we have an associated Hopf quasigroup $k\mathcal{G}$. Conversely, given any Hopf quasigroup H we set

$$G(H) = \{u \in H \,|\, \Delta u = u \otimes u\},$$

to be the set of group-like elements of H. Then $G(H)$ is a quasigroup with inverse property, and is Moufang if H is Moufang as a Hopf quasigroup.

Example 3.3.10 (*Universal enveloping Hopf quasigroup)***.** Let L be a Mal'tsev algebra, *i.e.* L be a nonassociative algebra such that $x^2 = 0$ for all $x \in L$, and the product satisfies

$$(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y,$$

for all $x, y, z \in L$. Following [33], define $\phi(L)$ to be the free nonassociative algebra with basis L and $I(L)$ to be the ideal generated by the set

$$\{xy - yx - [x, y]\} \cup \{(x, h, g) + (h, x, g)\} \cup \{(h, x, g) + (h, g, x)\}$$

for all $x, y \in L, h, g \in \phi(L)$, where $(a, b, c) = (ab)c - a(bc)$ is the usual associator of an algebra. The enveloping algebra $U(L)$ is the quotient $\phi(L)/I(L)$. In [33] the authors provide a mapping $\Delta : L \to L \otimes L$ defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in L$, which is extended to $U(L)$ as an algebra morphism.

To prove the next proposition, we require two identities stated in [33]:

Lemma 3.3.11. *Let L and $U(L)$ be as above and let $x, y \in L$ and $h \in U(L)$ then,*

(1) $[x, hy] = [x, h]y + h[x, y] + 3(x, y, h)$,

(2) $6(x, y, h) = [[h, x], y] - [[h, y], x] - [h, [x, y]]$.

Proof. Let $x, y \in L$ and $h \in U(L)$ then, by expanding expanding the brackets we find

$$\begin{aligned}
&[x, h]y + h[x, y] + 3(x, y, h) \\
&\qquad = (xh)y - (hx)y + h(xy) - h(yx) + 3(x, y, h) \\
&\qquad = (x, h, y) + x(hy) - (h, x, y) + (h, y, x) - (hy)x + 3(x, y, h) \\
&\qquad = -(x, y, h) + x(hy) - (x, y, h) - (x, y, h) - (hy)x + 3(x, y, h) \\
&\qquad\qquad \text{by properties of the associator,} \\
&\qquad = [x, hy],
\end{aligned}$$

which is identity (1). The proof of the second part is similar:

$$\begin{aligned}
&[[h, x], y] - [[h, y], x] - [h, [x, y]] \\
&\qquad = [h, x]y - y[h, x] - [h, y]x + x[h, y] - h[x, y] + [x, y]h \\
&\qquad = (hx)y - (xh)y - y(hx) + y(xh) - (hy)x + (yh)x + x(hy) \\
&\qquad\qquad -x(yh) - h(xy) + h(yx) + (xy)h - (yx)h \\
&\qquad = ((hx)y - h(xy)) - ((xh)y - x(hy)) + ((yh)x - y(hx)) \\
&\qquad\qquad -((yx)h - y(xh)) - ((hy)x - h(yx)) + ((xy)h - x(yh)) \\
&\qquad = (h, x, y) - (x, h, y) + (y, h, x) - (y, x, h) - (h, y, x) + (x, y, h) \\
&\qquad = 6(x, y, h),
\end{aligned}$$

where the final equality uses properties of the associator. □

Proposition 3.3.12. *For a field k of characteristic not 2 or 3, and $U(L)$ as above, there exist maps $S : U(L) \to U(L)$ and $\varepsilon : U(L) \to k$ defined by $Sx = -x$ and $\varepsilon x = 0$ for all $x \in L$ extended as an anti-algebra morphism and an algebra morphism respectively, making $U(L)$ into a Hopf quasigroup.*

Proof. From Proposition 3.3.3 (1), we know that if S exists then $(Sx_{(1)})x_{(2)} = \varepsilon x$ for all $x \in L$. Then, by definition of Δ on $U(L)$ and ε, we compute

$$0 = (Sx_{(1)})x_{(2)} = (Sx).1 + 1.x = Sx + x, \quad \Rightarrow Sx = -x.$$

Hence, S takes the form stated. To verify that S is well defined, it is easily seen that $S(x,h,g) = (Sg, Sh, x), S(h,x,g) = (Sg, x, Sh)$ and $S(h,g,x) = (x, Sg, Sh)$ for all $x \in L$ and $h, g \in \phi(L)$, and so S respects the relations in $U(L)$.

Finally, we must check that identities (3.3.1) and (3.3.2) are satisfied and we do this by induction. It is clear that the required identities are satisfied for $h \in L$ and $g \in U(L)$. We shall demonstrate the proof of the LHS of (3.3.1), and the rest are similar. Assume that $(Sh_{(1)})(h_{(2)}g) = \varepsilon(h)g$ holds for all h which are sums of products of $\leq n$ elements of L and for all $g \in U(L)$. Let $h \in U(L)$ be a product of n elements of L and $x \in L$. Consider the identity on $hx + xh \in U(L)$:

$$\begin{aligned}
&S((hx)_{(1)})((hx)_{(2)}g) + S((xh)_{(1)})((xh)_{(2)}g) \\
&= S(h_{(1)}x)(h_{(2)}g) + (Sh_{(1)})((h_{(2)}x)g) + S(xh_{(1)})(h_{(2)}g) + (Sh_{(1)})((xh_{(2)})g) \\
&= -(xSh_{(1)})(h_{(2)}g) + (Sh_{(1)})((h_{(2)}x)g) - ((Sh_{(1)})x)(h_{(2)}g) + (Sh_{(1)})((xh_{(2)})g) \\
&\qquad \text{by anti-multiplicity of } S \text{ and the definition of } S \text{ on } L, \\
&= -(x, Sh_{(1)}, h_{(2)}g) - x((Sh_{(1)})(h_{(2)}g)) + (Sh_{(1)})(h_{(2)}, x, g) + (Sh_{(1)})(h_{(2)}(xg)) \\
&\qquad -(Sh_{(1)}, x, h_{(2)}g) - (Sh_{(1)})(x(h_{(2)}g)) + (Sh_{(1)})(x, h_{(2)}, g) + (Sh_{(1)})(x(h_{(2)}g)) \\
&\qquad \text{by definition of the associator,} \\
&= -(x, Sh_{(1)}, h_{(2)}g) - \varepsilon(h)xg + (Sh_{(1)})(h_{(2)}, x, g) + \varepsilon(h)xg \\
&\qquad -(Sh_{(1)}, x, h_{(2)}g) + (Sh_{(1)})(x, h_{(2)}, g) \\
&\qquad \text{by inductive assumption,} \\
&= -((x, Sh_{(1)}, h_{(2)}g) + (Sh_{(1)}, x, h_{(2)}g)) + (Sh_{(1)})((h_{(2)}, x, g) + (x, h_{(2)}, g)) \\
&= 0 \quad \text{by properties of the associator in } U(L), \\
&= \varepsilon(hx + xh).
\end{aligned}$$

Hence, the identity holds on $hx + xh$ if it holds on h. Now $hx + xh = 2hx + [x, h]$, so if $[x, h]$ can be expressed as a sum of products of $\leq n$ generators then by our inductive assumption, the identity holds on this. Therefore, it holds on hx, and hence sums of products of $\leq n + 1$ generators.

It remains to prove that $[x, h]$ can be expressed as a sum of products of $\leq n$ elements of L. This can also be proven by induction. Suppose $h \in U(L)$ can be written as a product of n elements of L, and suppose that $[x, h]$ can be written as a sum of products of $\leq n$ generators. By this assumption, $[x, h]y$ and $h[x, y]$ can both be written as a sum of products of $\leq n+1$ elements of L and so by Lemma 3.3.11 (2), $3(x, y, h)$ can be written as a sum of products of $\leq n+1$ generators. Hence, by Lemma 3.3.11 (1), $[x, hy]$ is also a sum of products of $\leq n+1$ elements whenever hy is. □

Proposition 3.3.13. *$U(L)$ as above is Moufang.*

Proof. Clearly, if $h \in L$ and $g, f \in U(L)$ then

$$\begin{aligned} h_{(1)}(g(h_{(2)}f)) &= h(gf) + g(hf) = (hg)f - (h, g, f) + (gh)f - (g, h, f) \\ &= ((h_{(1)}g)h_{(2)})f - (h, g, f) + (h, g, f) = ((h_{(1)}g)h_{(2)})f. \end{aligned}$$

Now we proceed by induction. Suppose the Moufang identity holds for all $g, f \in U(L)$ and all $h \in U(L)$ that can be expressed as a sum of products of $\leq n$ generators. Let $x \in L$ then,

$$\begin{aligned} &(hx)_{(1)}(g((hx)_{(2)}f)) \\ &\quad = (h_{(1)}x)(g(h_{(2)}f)) + h_{(1)}(g((h_{(2)}x)f)), \\ &\qquad \text{since } \Delta \text{ is an algebra morphism, and by definition of } \Delta \text{ on } L, \\ &\quad = (h_{(1)}, x, g(h_{(2)}f)) + h_{(1)}(x(g(h_{(2)}f))) \\ &\qquad + h_{(1)}(g(h_{(2)}, x, f)) + h_{(1)}(g(h_{(2)}(xf))), \\ &\qquad \text{by definition of the associator,} \\ &\quad = (h_{(1)}, x, g(h_{(2)}f)) + h_{(1)}((xg)(h_{(2)}f)) - h_{(1)}(x, g, h_{(2)}f) \\ &\qquad + h_{(1)}(g(h_{(2)}, x, f)) + h_{(1)}(g(h_{(2)}(xf))) \\ &\quad = (h_{(1)}, x, g(h_{(2)}f)) + ((h_{(1)}(xg))h_{(2)})f - h_{(1)}(x, g, h_{(2)}f) \\ &\qquad + h_{(1)}(g(h_{(2)}, x, f)) + ((h_{(1)}g)h_{(2)})(xf), \\ &\qquad \text{by the Moufang identity under inductive assumption,} \\ &\quad = (h_{(1)}, x, g(h_{(2)}f)) + (((h_{(1)}x)g)h_{(2)})f - ((h_{(1)}, x, g)h_{(2)})f \\ &\qquad - h_{(1)}(x, g, h_{(2)}f) + h_{(1)}(g(h_{(2)}, x, f)) + (((h_{(1)}g)h_{(2)})x)f \\ &\qquad - ((h_{(1)}g)h_{(2)}, x, f), \quad \text{by definition of the associator,} \end{aligned}$$

$$
\begin{aligned}
&= (h_{(1)}, x, g(h_{(2)}f)) + (((h_{(1)}x)g)h_{(2)})f - ((h_{(1)}, x, g)h_{(2)})f \\
&\quad -h_{(1)}(x, g, h_{(2)}f) + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f \\
&\quad +((h_{(1)}g)(h_{(2)}x))f - ((h_{(1)}g)h_{(2)}, x, f) \\
&= (((hx)_{(1)}g)(hx)_{(2)})f + (h_{(1)}, x, g(h_{(2)}f)) - ((h_{(1)}, x, g)h_{(2)})f \\
&\quad -h_{(1)}(x, g, h_{(2)}f) + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f \\
&\quad -((h_{(1)}g)h_{(2)}, x, f), \quad \text{by definition of } \Delta \text{ on } L.
\end{aligned}
$$

It remains to prove the vanishing of

$$
\begin{aligned}
&(h_{(1)}, x, g(h_{(2)}f)) - ((h_{(1)}, x, g)h_{(2)})f - h_{(1)}(x, g, h_{(2)}f) \\
&\qquad +h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f - ((h_{(1)}g)h_{(2)}, x, f).
\end{aligned}
$$

We apply identities on the associator and the Moufang identity under our inductive assumption. The expression

$$
\begin{aligned}
&= -(h_{(1)}, g(h_{(2)}f), x) - h_{(1)}(g, h_{(2)}f, x) + ((h_{(1)}g)h_{(2)}, f, x) \\
&\quad -((h_{(1)}, x, g)h_{(2)})f + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f, \\
&\quad \text{by identities on the associator,} \\
&= -(h_{(1)}(g(h_{(2)}f)))x + h_{(1)}((g(h_{(2)}f))x) - h_{(1)}((g(h_{(2)}f))x) \\
&\quad +h_{(1)}(g((h_{(2)}f)x)) + (((h_{(1)}g)h_{(2)})f)x - ((h_{(1)}g)h_{(2)})(fx) \\
&\quad -((h_{(1)}, x, g)h_{(2)})f + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f, \\
&\quad \text{by definition of the associator,} \\
&= -(h_{(1)}(g(h_{(2)}f)))x + h_{(1)}(g((h_{(2)}f)x)) + (((h_{(1)}g)h_{(2)})f)x - ((h_{(1)}g)h_{(2)})(fx) \\
&\quad -((h_{(1)}, x, g)h_{(2)})f + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f \\
&= -(h_{(1)}(g(h_{(2)}f)))x + h_{(1)}(g((h_{(2)}f)x)) + (h_{(1)}(g(h_{(2)}f)))x - ((h_{(1)}g)h_{(2)})(fx) \\
&\quad -((h_{(1)}, x, g)h_{(2)})f + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f, \\
&\quad \text{by the Moufang identity,} \\
&= h_{(1)}(g((h_{(2)}f)x)) - ((h_{(1)}g)h_{(2)})(fx) \\
&\quad -((h_{(1)}, x, g)h_{(2)})f + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f \\
&= h_{(1)}(g(h_{(2)}, f, x)) + h_{(1)}(g(h_{(2)}(fx))) - ((h_{(1)}g)h_{(2)})(fx) \\
&\quad -((h_{(1)}, x, g)h_{(2)})f + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f,
\end{aligned}
$$

$$
\begin{aligned}
&\quad \text{by definition of the associator,}\\
&= h_{(1)}(g(h_{(2)}, f, x)) + h_{(1)}(g(h_{(2)}(fx))) - h_{(1)}(g(h_{(2)}(fx)))\\
&\quad -((h_{(1)}, x, g)h_{(2)})f + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f,\\
&\quad \text{by the Moufang identity,}\\
&= -h_{(1)}(g(h_{(2)}, x, f)) - ((h_{(1)}, x, g)h_{(2)})f + h_{(1)}(g(h_{(2)}, x, f)) + (h_{(1)}g, h_{(2)}, x)f,\\
&\quad \text{by identities on the associator,}\\
&= ((h_{(1)}, g, x)h_{(2)})f - (h_{(1)}g, x, h_{(2)})f\\
&= (((h_{(1)}g)x)h_{(2)})f - ((h_{(1)}(gx))h_{(2)})f - (((h_{(1)}g)x)h_{(2)})f + ((h_{(1)}g)(xh_{(2)}))f,\\
&\quad \text{by definition of the associator,}\\
&= ((h_{(1)}g)(xh_{(2)}))f - ((h_{(1)}(gx))h_{(2)})f\\
&= ((h_{(1)}(gx))h_{(2)})f - ((h_{(1)}(gx))h_{(2)})f\\
&= 0,
\end{aligned}
$$

where for the penultimate equality we used an equivalent Moufang identity from Lemma 3.3.5. Hence we have shown that the Moufang identity is satisfied for any element which can be expressed as a sum of products of $\leq n+1$ elements of L, and hence $U(L)$ is Moufang. □

Going in the other direction, if H is a Moufang Hopf quasigroup with invertible antipode then the set

$$L(H) = \{x \in H \mid \Delta x = x \otimes 1 + 1 \otimes x\},$$

is a Mal'tsev algebra with the commutator bracket $[x, y] = xy - yx$, *i.e.*

$$[[x, y], [x, z]] = [[[x, y], z], x] + [[[y, z], x], x] + [[[z, x], x], y]$$

for $x, y, z \in L(H)$. Indeed, from Lemma 3.3.5 applied to such elements we see that

$$L(H) \subseteq N_{alt}(H) = \{x \in H \mid (x, h, g) = -(h, x, g) = (h, g, x) \ \forall\ h, g \in H\},$$

(the 'alternative nucleus' used in [33]) as Mal'tsev algebras.

Remark 3.3.14. Independently, Hopf quasigroups were defined as H-bialgebras by means of division maps in 2007 by Perez-Izquierdo [32], who linearized the

Moufang identities to this setting. They also considered the universal enveloping algebra of a Malt'sev algebra and gave an alternative proof that it is a Moufang H-bialgebra. These objects were further studied by Mostovoy and Perez-Izquierdo in [31] and Zhelyabin in [38].

We can also construct examples by cross product methods as follows.

Proposition 3.3.15. *Let H be a Hopf quasigroup equipped with an action of a quasigroup $\mathcal{G}$. Thus there is a linear action $\triangleright$ of $\mathcal{G}$ on H such that $\sigma\triangleright(hg) = (\sigma\triangleright h)(\sigma\triangleright g)$, $\sigma\triangleright 1 = 1$, $(\sigma\otimes\sigma)\triangleright\Delta h = \Delta(\sigma\triangleright h)$, $\varepsilon(\sigma\triangleright h) = \varepsilon(h)$ and $(\sigma\sigma')\triangleright h = \sigma\triangleright(\sigma'\triangleright h)$ for all $\sigma, \sigma' \in \mathcal{G}$ and $h, g \in H$. Then the cross product algebra $H \rtimes k\mathcal{G}$ is again a Hopf quasigroup.*

Proof. The nonassociative algebra product is $(h\otimes\sigma)(g\otimes\sigma') = h(\sigma\triangleright g)\otimes\sigma\sigma'$ and the coproduct is the tensor product one, where $\Delta\sigma = \sigma\otimes\sigma$. This is a coalgebra as required with tensor product counit. It is easy to see that Δ, ε are algebra morphisms.

For the quasigroup identities, we note first a lemma that a morphism of the algebra and coalgebra necessarily commutes with S, so $\sigma\triangleright(Sh) = S(\sigma\triangleright h)$ for all $h \in H$, $\sigma \in \mathcal{G}$, we consider

$$(\sigma\triangleright h_{(1)})(\sigma\triangleright Sh_{(2)}) = \sigma\triangleright(h_{(1)}Sh_{(2)}) = \sigma\triangleright\varepsilon(h) = \varepsilon(h),$$

but also,

$$(\sigma\triangleright h_{(1)})S(\sigma\triangleright h_{(2)}) = (\sigma\triangleright h)_{(1)}S((\sigma\triangleright h)_{(2)}) = \varepsilon(\sigma\triangleright h) = \varepsilon(h).$$

Hence

$$(\sigma\triangleright h_{(1)})S(\sigma\triangleright h_{(2)}) = (\sigma\triangleright h_{(1)})(\sigma\triangleright Sh_{(2)}).$$

By replacing h by $h_{(2)}$ and multiplying on the left by $S(\sigma\triangleright h_{(1)})$, we can use the quasigroup identities to obtain

$$S(\sigma\triangleright h) = \sigma\triangleright Sh,$$

as required.

By the properties of S proven above, we know that S, if it exists on $H \rtimes k\mathcal{G}$, must be given by $S(h \otimes \sigma) = (1 \otimes \sigma^{-1})(Sh \otimes 1) = \sigma^{-1} \triangleright Sh \otimes \sigma^{-1}$. We verify the required identities in a straightforward manner. For example, using properties of a quasigroup and a Hopf quasigroup,

$$\begin{aligned}(h \otimes \sigma)_{(1)}(S((h \otimes \sigma)_{(2)})(g \otimes \sigma')) &= (h_{(1)} \otimes \sigma)((\sigma^{-1} \triangleright Sh_{(2)} \otimes \sigma^{-1})(g \otimes \sigma')) \\ &= (h_{(1)} \otimes \sigma)((\sigma^{-1} \triangleright Sh_{(2)})(\sigma^{-1} \triangleright g) \otimes \sigma^{-1}\sigma') \\ &= (h_{(1)} \otimes \sigma)(\sigma^{-1} \triangleright ((Sh_{(2)})g) \otimes \sigma^{-1}\sigma') \\ &= h_{(1)}(\sigma \triangleright (\sigma^{-1} \triangleright ((Sh_{(2)})g))) \otimes \sigma(\sigma^{-1}\sigma') \\ &= h_{(1)}((\sigma\sigma^{-1}) \triangleright ((Sh_{(2)})g)) \otimes \sigma(\sigma^{-1}\sigma') \\ &= h_{(1)}((Sh_{(2)})g) \otimes \sigma(\sigma^{-1}\sigma') \\ &= \varepsilon(h) g \otimes \sigma' \\ &= \varepsilon(h \otimes \sigma) g \otimes \sigma'.\end{aligned}$$

□

Example 3.3.16. Let $H = k\mathcal{S}^{2^n-1}$ and $G = \mathbb{Z}_2^n$. We label the elements of the latter as σ_a where $a \in \mathbb{Z}_2^n$ and consider the action

$$\sigma_a \triangleright e_b = (-1)^{a \cdot b} e_b,$$

on $k_F G$, where $a \cdot b$ is the vector dot product. Under this action the norm remains invariant and hence it defines an action on the sphere $\mathcal{S}^{2^n-1}$. This action extends linearly to one on H and leads to a cross product $k\mathcal{S}^{2^n-1} \rtimes k\mathbb{Z}_2^n$, which is isomorphic to the Hopf quasigroup associated to the quasigroup cross product $\mathcal{S}^{2^n-1} \rtimes \mathbb{Z}_2^n$.

While this example is not very interesting, we will see in the next section that replacing H by its dual and considering cross coproducts rather than cross products the structures become more so. Also, using our framework one can extend Proposition 3.3.15 to an action by a general cocommutative Hopf quasigroup as we see next. Clearly other Hopf algebra constructions can similarly be extended to the quasigroup case.

Definition 3.3.17. Let H be a Hopf quasigroup. A vector space V is a *left H-module* if there is a linear map $\alpha : H \otimes V \to V$ written as $\alpha(h \otimes v) = h \triangleright v$ such that

$$h \triangleright (g \triangleright v) = (hg) \triangleright v, \quad 1 \triangleright v = v,$$

for all $h, g \in H, v \in V$. A nonassociative algebra A is an *H-module algebra* if further

$$h\triangleright(ab) = (h_{(1)}\triangleright a)(h_{(2)}\triangleright b), \quad h\triangleright 1 = \varepsilon h,$$

for all $h \in H, a, b \in A$. Finally, a coalgebra C is an *H-module coalgebra* if

$$\Delta(h\triangleright c) = h_{(1)}\triangleright c_{(1)} \otimes h_{(2)}\triangleright c_{(2)}, \quad \varepsilon(h\triangleright c) = \varepsilon(h)\varepsilon(c),$$

for all $h \in H, c \in C$. Therefore we have the notion of a *left H-module Hopf quasigroup*; we can similarly define right actions of Hopf quasigroups.

Lemma 3.3.18. *If A is a left H-module Hopf quasigroup with antipode S then*

$$h\triangleright Sa = S(h\triangleright a),$$

for all $h \in H$ and $a \in A$.

Proof. To see this, we use the definition of an action on an algebra to find

$$(h_{(1)}\triangleright a_{(1)})(h_{(2)}\triangleright Sa_{(2)}) = h\triangleright(a_{(1)}Sa_{(2)}) = \varepsilon(h)\varepsilon(a).$$

We also find, using the definition of an action on a coalgebra, that

$$(h_{(1)}\triangleright a_{(1)})S(h_{(2)}\triangleright a_{(2)}) = (h\triangleright a)_{(1)}S(h\triangleright a)_{(2)} = \varepsilon(h\triangleright a) = \varepsilon(h)\varepsilon(a).$$

So we have

$$(h_{(1)}\triangleright a_{(1)})(h_{(2)}\triangleright Sa_{(2)}) = (h_{(1)}\triangleright a_{(1)})S(h_{(2)}\triangleright a_{(2)})$$

Replacing h and a by $h_{(2)}$ and $a_{(2)}$ respectively, and multiplying both sides on the left by $S(h_{(1)}\triangleright a_{(1)})$, we use the Hopf quasigroup identities, to obtain the required identity. □

Proposition 3.3.19. *Let H be a cocommutative Hopf quasigroup and A be a left H-module Hopf quasigroup, then there is a left cross product Hopf quasigroup $A \rtimes H$ built on $A \otimes H$ with tensor product coproduct and unit and*

$$(a \otimes h)(b \otimes g) = a(h_{(1)}\triangleright b) \otimes h_{(2)}g, \quad S(a \otimes h) = (Sh_{(2)})\triangleright Sa \otimes Sh_{(1)},$$

for all $a, b \in A$ and $g, h \in H$.

Proof. To see that Δ is an algebra morphism, we compute directly:

$$\begin{aligned}\Delta((a\otimes h)(b\otimes g)) &= \Delta(a(h_{(1)}\triangleright b)\otimes h_{(2)}g)\\ &= a_{(1)}(h_{(1)}\triangleright b)_{(1)}\otimes h_{(2)(1)}g_{(1)}\otimes a_{(2)}(h_{(1)}\triangleright b)_{(2)}\otimes h_{(2)(2)}g_{(2)}\\ &= a_{(1)}(h_{(1)(1)}\triangleright b_{(1)})\otimes h_{(2)(1)}g_{(1)}\otimes a_{(2)}(h_{(1)(2)}\triangleright b_{(2)})\otimes h_{(2)(2)}g_{(2)},\end{aligned}$$

while,

$$\begin{aligned}\Delta(a\otimes h)\Delta(b\otimes g) &= (a_{(1)}\otimes h_{(1)})(b_{(1)}\otimes g_{(1)})\otimes(a_{(2)}\otimes h_{(2)})(b_{(2)}\otimes g_{(2)})\\ &= a_{(1)}(h_{(1)(1)}\triangleright b_{(1)})\otimes h_{(1)(2)}g_{(1)}\otimes a_{(2)}(h_{(2)(1)}\triangleright b_{(2)})\otimes h_{(2)(2)}g_{(2)}.\end{aligned}$$

These are equal if H is cocommutative. Next we verify the Hopf quasigroup identities:

$$\begin{aligned}&(S(a\otimes h)_{(1)})((a\otimes h)_{(2)}(b\otimes g))\\ &\quad= ((Sh_{(1)})_{(1)}\triangleright Sa_{(1)}\otimes(Sh_{(1)})_{(2)})(a_{(2)}(h_{(2)(1)}\triangleright b)\otimes h_{(2)(2)}g)\\ &\quad= ((Sh_{(1)})_{(1)}\triangleright Sa_{(1)})((Sh_{(1)})_{(2)(1)}\triangleright(a_{(2)}(h_{(2)(1)}\triangleright b)))\otimes(Sh_{(1)})_{(2)(2)}(h_{(2)(2)}g)\\ &\quad= ((Sh_{(1)(1)})_{(1)(1)}\triangleright Sa_{(1)})((Sh_{(1)(1)})_{(1)(2)}\triangleright(a_{(2)}(h_{(1)(2)}\triangleright b)))\otimes(Sh_{(1)(1)})_{(2)}(h_{(2)}g),\\ &\qquad\qquad \text{by coassociativity of } H,\\ &\quad= (Sh_{(1)(1)})_{(1)}\triangleright((Sa_{(1)})(a_{(2)}(h_{(1)(2)}\triangleright b)))\otimes(Sh_{(1)(1)})_{(2)}(h_{(2)}g)\\ &\quad= \varepsilon(a)(Sh_{(1)(2)(1)})\triangleright(h_{(1)(2)(2)}\triangleright b)\otimes(Sh_{(1)(1)})(h_{(2)}g),\\ &\qquad\qquad \text{by coassociativity of } H \text{ and (3.3.1) on } A,\\ &\quad= \varepsilon(a)((Sh_{(1)(2)(1)})h_{(1)(2)(2)}\triangleright b)\otimes(Sh_{(1)(1)})(h_{(2)}g),\\ &\qquad\qquad \text{by the definition of an action of } H \text{ on } A,\\ &\quad= \varepsilon(a)b\otimes(Sh_{(1)})(h_{(2)}g)\\ &\quad= \varepsilon(a)\varepsilon(h)b\otimes g,\quad \text{by (3.3.1) on } H.\end{aligned}$$

Similarly,

$$\begin{aligned}&((b\otimes g)(a\otimes h)_{(1)})S(a\otimes h)_{(2)}\\ &\quad= (b(g_{(1)}\triangleright a_{(1)})\otimes g_{(2)}h_{(1)})(Sh_{(2)(2)}\triangleright Sa_{(2)}\otimes Sh_{(2)(1)})\\ &\quad= (b(g_{(1)}\triangleright a_{(1)}))((g_{(2)}h_{(1)})_{(1)}\triangleright(Sh_{(2)(2)}\triangleright Sa_{(2)}))\otimes(g_{(2)}h_{(1)})_{(2)}Sh_{(2)(1)}\\ &\quad= (b(g_{(1)}\triangleright a_{(1)}))((g_{(2)(1)}h_{(1)(1)})\triangleright(Sh_{(2)(2)}\triangleright Sa_{(2)}))\otimes(g_{(2)(2)}h_{(1)(2)})Sh_{(2)(1)}\end{aligned}$$

$= (b(g_{(1)} \triangleright a_{(1)}))((g_{(2)(1)} h_{(1)}) S h_{(2)(2)} \triangleright S a_{(2)}) \otimes (g_{(2)(2)} h_{(2)(1)(1)}) S h_{(2)(1)(2)}$

by coassociativity of H,

$= (b(g_{(1)} \triangleright a_{(1)}))((g_{(2)(1)} h_{(1)}) S h_{(2)} \triangleright S a_{(2)}) \otimes g_{(2)(2)}$

by the Hopf quasigroup identity on $h_{(2)}$,

$= \varepsilon(h)(b(g_{(1)(1)} \triangleright a_{(1)}))(g_{(1)(2)} \triangleright S a_{(2)}) \otimes g_{(2)}$

$= \varepsilon(h)(b(g_{(1)(1)} \triangleright a_{(1)}))S(g_{(1)(2)} \triangleright a_{(2)}) \otimes g_{(2)}$

by Lemma 3.3.18,

$= \varepsilon(h)(b(g_{(1)} \triangleright a)_{(1)})S(g_{(1)} \triangleright a)_{(2)} \otimes g_{(2)}$

since the action is a coalgebra morphism,

$= \varepsilon(h)\varepsilon(g_{(1)} \triangleright a) b \otimes g_{(2)}$

$= \varepsilon(a \otimes h) b \otimes g$

The other Hopf quasigroup identities are similar. □

Remark 3.3.20. This theory has been extended to nonassociative actions of Hopf quasigroups in [6] and [7] where R-smash products of Hopf quasigroups are considered, noting Proposition 3.3.19 as a special case. These quasimodules of Hopf quasigroups, and their dual notions of quasicomodules of Hopf coquasigroups will be discussed in Chapter 5.

3.4 Hopf coquasigroups

Following the ideas in Section 2.1.1, we can dualize the notion of a Hopf quasigroup. To be consistent with other Hopf terminology, the dual notion is a Hopf coquasigroup. In the finite dimensional case a Hopf quasigroup is equivalent on the dual space to the concept of a Hopf coquasigroup, as will be shown in the next section. The proofs of the general constructions can be obtained from those in Section 3.3 by reversal of arrows, but we include the direct proofs here for completeness.

Definition 3.4.1. A *Hopf coquasigroup* is a unital associative algebra A equipped with counital algebra morphisms $\Delta : A \to A \otimes A$, and $\varepsilon : A \to k$ and a linear map

$S : A \to A$ such that the following identities hold:

$$(m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta = 1 \otimes \mathrm{id} = (m \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta,$$

$$(\mathrm{id} \otimes m)(\mathrm{id} \otimes S \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta = \mathrm{id} \otimes 1 = (\mathrm{id} \otimes m)(\mathrm{id} \otimes \mathrm{id} \otimes S)(\Delta \otimes \mathrm{id})\Delta.$$

Explicitly, the Hopf coquasigroup identities are:

$$\sum (Sa_{(1)})a_{(2)(1)} \otimes a_{(2)(2)} = 1 \otimes a = \sum a_{(1)}Sa_{(2)(1)} \otimes a_{(2)(2)}, \tag{3.4.1}$$

$$\sum a_{(1)(1)} \otimes (Sa_{(1)(2)})a_{(2)} = a \otimes 1 = \sum a_{(1)(1)} \otimes a_{(1)(2)}Sa_{(2)}, \tag{3.4.2}$$

for all $a \in A$. As with Hopf quasigroups, we define further properties on Hopf coquasigroups:

Definition 3.4.2. A Hopf coquasigroup is *flexible* if

$$\sum a_{(1)}a_{(2)(2)} \otimes a_{(2)(1)} = \sum a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)}, \quad \forall\, a \in A,$$

and *alternative* if

$$\sum a_{(1)}a_{(2)(1)} \otimes a_{(2)(2)} = \sum a_{(1)(1)}a_{(1)(2)} \otimes a_{(2)},$$

$$\sum a_{(1)} \otimes a_{(2)(1)}a_{(2)(2)} = \sum a_{(1)(1)} \otimes a_{(1)(2)}a_{(2)},$$

for all $a \in A$. We say A is *Moufang* if

$$\sum a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = \sum a_{(1)(1)(1)}a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)},$$

for all $a \in A$.

Proposition 3.4.3. *Let A be a Hopf coquasigroup, then:*

(1) $m(S \otimes \mathrm{id})\Delta = 1.\varepsilon = m(\mathrm{id} \otimes S)\Delta$,

(2) *S is anti-multiplicative,*

(3) *S is anti-comultiplicative.*

Hence a Hopf coquasigroup is a Hopf algebra iff *it is coassociative.*

Proof. The proof is obtained by reversing the arrows in Proposition 3.3.3. We shall include the full proofs here for completeness. (1) is obtained by applying $(\mathrm{id}\otimes\varepsilon)$ to the first Hopf coquasigroup identity (3.4.1).

To see that S is anti-multiplicative we use (3.4.1) first on $b\in A$ then on $a\in A$ to find:

$$S(ab)=(Sb_{(1)})b_{(2)(1)}S(ab_{(2)(2)})=(Sb_{(1)})(Sa_{(1)})a_{(2)(1)}b_{(2)(1)}S(a_{(2)(2)}b_{(2)(2)}),$$

for all $a,b\in A$. Since Δ is an algebra morphism, this equals

$$(Sb_{(1)})(Sa_{(1)})(a_{(2)}b_{(2)})_{(1)}S(a_{(2)}b_{(2)})_{(2)}=(Sb)(Sa),$$

where the last equality uses (1). Finally, we prove (3); let $a,b\in A$ then

$$\begin{aligned}
\Delta(Sa) &= (Sa)_{(1)}\otimes(Sa)_{(2)}\\
&= (Sa_{(2)(2)})_{(1)}\otimes((Sa_{(1)})a_{(2)(1)})(Sa_{(2)(2)})_{(2)} \quad \text{by (3.4.1) on } a,\\
&= ((Sa_{(2)(1)(1)})a_{(2)(1)(2)(1)})(Sa_{(2)(2)})_{(1)}\otimes((Sa_{(1)})a_{(2)(1)(2)(2)})(Sa_{(2)(2)})_{(2)},\\
&\qquad \text{by (3.4.1) on } a_{(2)(1)},\\
&= (Sa_{(2)(1)(1)})(a_{(2)(1)(2)(1)}(Sa_{(2)(2)})_{(1)})\otimes(Sa_{(1)})(a_{(2)(1)(2)(2)}(Sa_{(2)(2)})_{(2)}),\\
&\qquad \text{by associativity of } A,\\
&= (Sa_{(2)(1)(1)})(a_{(2)(1)(2)}Sa_{(2)(2)})_{(1)}\otimes(Sa_{(1)})(a_{(2)(1)(2)}Sa_{(2)(2)})_{(2)},\\
&\qquad \text{since } \Delta \text{ is an algebra morphism},\\
&= (Sa_{(2)})1_{(1)}\otimes(Sa_{(1)})1_{(2)}, \quad \text{by (3.4.2) on } a_{(2)},\\
&= Sa_{(2)}\otimes Sa_{(1)}, \quad \text{since } \Delta(1)=1\otimes 1.
\end{aligned}$$

This holds for all $a,b\in A$, hence S is anti-comultiplicative. □

Proposition 3.4.4. *Let A be a Hopf coquasigroup, then $S^2=\mathrm{id}$ if A is commutative or cocommutative.*

Proof. This is dual to Proposition 3.3.4, but we give a direct proof here. For all $a\in A$ we have

$$S^2(a)=S^2(a_{(1)})(Sa_{(2)(1)})a_{(2)(2)}=S(a_{(2)(1)}Sa_{(1)})a_{(2)(2)},$$

by Proposition 3.4.3 parts (1) and (2). If A is commutative this equals

$$S((Sa_{(1)})a_{(2)(1)})a_{(2)(2)} = S(1)a = a$$

using (3.4.1). On the other hand, if A is cocommutative the expression becomes

$$S(a_{(1)(1)}Sa_{(2)})a_{(1)(2)} = S(a_{(1)(2)}Sa_{(2)})a_{(1)(1)} = S(1)a = a,$$

by (3.4.2). This holds for all $a \in A$, hence we are done. □

As in the previous section, we find that the Moufang identity for Hopf coquasigroups is equivalent to two other versions if the antipode is invertible:

Lemma 3.4.5. *Let A be a Hopf coquasigroup such that S^{-1} exists, then the following identities are equivalent for all $a \in A$:*

(1) $\sum a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = \sum a_{(1)(1)(1)}a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)}$,

(2) $\sum a_{(1)(1)(1)} \otimes a_{(1)(1)(2)}a_{(2)} \otimes a_{(1)(2)} = \sum a_{(1)} \otimes a_{(2)(1)}a_{(2)(2)(2)} \otimes a_{(2)(2)(1)}$,

(3) $\sum a_{(1)(1)}a_{(2)(2)} \otimes a_{(1)(2)} \otimes a_{(2)(1)} = \sum a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)(1)} \otimes a_{(1)(2)(2)}$.

Proof. Assume (1) holds for all $a \in A$. Applying S to each factor on both sides and using the properties of S in Proposition 3.4.3 one obtains

$$\begin{aligned}(Sa)_{(1)(1)(2)}&(Sa)_{(2)} \otimes (Sa)_{(1)(2)} \otimes (Sa)_{(1)(1)(1)} \\ &= (Sa)_{(2)(1)}(Sa)_{(2)(2)(2)} \otimes (Sa)_{(2)(2)(1)} \otimes (Sa)_{(1)}.\end{aligned}$$

Applying the flip map $\tau_{A\otimes A,A}$ shows this is equivalent to (2). Similarly, (2) implies (1).

Assume (1) holds. By the Hopf coquasigroup identity (3.4.1) we have

$$a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} = a_{(1)}a_{(2)(2)(1)}Sa_{(2)(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)(2)}.$$

By (1) the RHS equals

$$a_{(1)(1)(1)}a_{(1)(2)}Sa_{(2)(1)} \otimes a_{(1)(1)(2)} \otimes a_{(2)(2)}. \tag{3.4.3}$$

Now, by applications of (3.4.1) and (3.4.2) we find

$$\begin{aligned}
&a_{(1)(1)}a_{(2)(2)} \otimes a_{(1)(2)} \otimes a_{(2)(1)} \\
&\quad = a_{(1)(1)}a_{(2)(2)} \otimes a_{(1)(2)(1)(1)} \otimes a_{(1)(2)(1)(2)}(Sa_{(1)(2)(2)})a_{(2)(1)}, \\
&\qquad\quad \text{by (3.4.2) on } a_{(1)(2)}, \\
&\quad = a_{(1)(1)}a_{(1)(2)(2)(1)}(Sa_{(1)(2)(2)(2)(1)})a_{(2)(2)} \otimes a_{(1)(2)(1)(1)} \\
&\qquad \otimes a_{(1)(2)(1)(2)}(Sa_{(1)(2)(2)(2)(2)})a_{(2)(1)}, \quad \text{by (3.4.1) on } a_{(1)(2)(2)}.
\end{aligned}$$

By (3.4.3) on $a_{(1)}$ followed by (3.4.1) on $a_{(1)(2)}$, the RHS is equal to

$$a_{(1)(1)(1)(1)}a_{(1)(1)(2)}(Sa_{(1)(2)(1)})a_{(2)(2)} \otimes a_{(1)(1)(1)(2)(1)} \otimes a_{(1)(1)(1)(2)(2)}(Sa_{(1)(2)(2)})a_{(2)(1)}.$$

Using that S is anti-comultiplicative, Δ is an algebra morphism and (3.4.2) on a, we find that this is equal to $a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)(1)} \otimes a_{(1)(2)(2)}$, which gives us identity (3).

To show (3) implies (1), we assume (3) holds and consider the element

$$a_{(1)(1)(1)}a_{(1)(2)(2)}(Sa_{(2)})_{(2)} \otimes a_{(1)(1)(2)} \otimes a_{(1)(2)(1)}(Sa_{(2)})_{(1)},$$

in $A^{\otimes 3}$. Applying (3) to $a_{(1)}$ and using that S is anti-comultiplicative, this becomes,

$$a_{(1)(1)(1)}a_{(1)(2)}Sa_{(2)(1)} \otimes a_{(1)(1)(2)(1)} \otimes a_{(1)(1)(2)(2)}Sa_{(2)(2)}.$$

But also, since Δ is an algebra morphism, and using (3.4.2), the original element equals

$$a_{(1)(1)(1)}(a_{(1)(2)}Sa_{(2)})_{(2)} \otimes a_{(1)(1)(2)} \otimes (a_{(1)(2)}Sa_{(2)})_{(1)} = a_{(1)} \otimes a_{(2)} \otimes 1.$$

We obtain

$$a_{(1)} \otimes a_{(2)} \otimes 1 = a_{(1)(1)(1)}a_{(1)(2)}Sa_{(2)(1)} \otimes a_{(1)(1)(2)(1)} \otimes a_{(1)(1)(2)(2)}Sa_{(2)(2)},$$

therefore

$$\begin{aligned}
a_{(1)} \otimes a_{(2)(1)} \otimes Sa_{(2)(2)} = {} & a_{(1)(1)(1)}a_{(1)(2)}Sa_{(2)(1)} \otimes a_{(1)(1)(2)(1)(1)} \\
& \otimes (Sa_{(1)(1)(2)(1)(2)})a_{(1)(1)(2)(2)}Sa_{(2)(2)}.
\end{aligned}$$

By applying S^{-1} to the third factor on both sides and using (3.4.2) on $a_{(1)(1)(2)}$ on the RHS, we obtain

$$a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)}Sa_{(2)(1)} \otimes a_{(1)(1)(2)} \otimes a_{(2)(2)}.$$

Finally, applying Δ to the third component and multiplying in the first gives

$$\begin{aligned} &a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} \\ &\qquad = a_{(1)(1)(1)}a_{(1)(2)}(Sa_{(2)(1)})a_{(2)(2)(1)} \otimes a_{(1)(1)(2)} \otimes a_{(2)(2)(2)}, \end{aligned}$$

which becomes (1) by applying (3.4.1) to $a_{(2)}$ on the RHS. □

Recall the right coadjoint coaction in Example 2.1.12 of a Hopf algebra on itself. In the case of Hopf coquasigroups, we do not have coassociativity and hence no notion of a right adjoint coaction. However, in a lemma dual to Lemma 3.3.6, we see that if a Hopf coquasigroup A is commutative and flexible, this coaction does make sense:

Lemma 3.4.6. *Let A be a commutative flexible Hopf coquasigroup, then*

(1) $a_{(1)(2)} \otimes (Sa_{(1)(1)})a_{(2)} = a_{(2)(1)} \otimes (Sa_{(1)})a_{(2)(2)}$,

(2) $a_{(1)(2)} \otimes a_{(1)(1)}Sa_{(2)} = a_{(2)(1)} \otimes a_{(1)}Sa_{(2)(2)}$,

for all $a \in A$.

Proof. By commutativity and flexibility we have

$$\begin{aligned} a_{(2)(1)(2)} \otimes a_{(1)}(Sa_{(2)(1)(1)})a_{(2)(2)} &= a_{(2)(1)(2)} \otimes a_{(1)}a_{(2)(2)}Sa_{(2)(1)(1)} \\ &= a_{(1)(2)(2)} \otimes a_{(1)(1)}a_{(2)}Sa_{(1)(2)(1)} \\ &= a_{(1)(2)(2)} \otimes a_{(1)(1)}(Sa_{(1)(2)(1)})a_{(2)} \\ &= a_{(1)} \otimes a_{(2)}, \quad \text{by (3.4.1) on } a_{(1)}. \end{aligned}$$

Also by (3.4.1) on a,

$$a_{(2)(2)(1)} \otimes a_{(1)}(Sa_{(2)(1)})a_{(2)(2)(2)} = a_{(1)} \otimes a_{(2)}.$$

Hence, for all $a \in A$,

$$a_{(2)(1)(2)} \otimes a_{(1)}(Sa_{(2)(1)(1)})a_{(2)(2)} = a_{(2)(2)(1)} \otimes a_{(1)}(Sa_{(2)(1)})a_{(2)(2)(2)}. \qquad (3.4.4)$$

Finally, for all $a \in A$ we have

$$\begin{aligned} &a_{(1)(2)} \otimes (Sa_{(1)(1)})a_{(2)} \\ &\quad = a_{(2)(2)(1)(2)} \otimes (Sa_{(1)})a_{(2)(1)}(Sa_{(2)(2)(1)(1)})a_{(2)(2)(2)}, \quad \text{by (3.4.1)}, \\ &\quad = a_{(2)(2)(2)(1)} \otimes (Sa_{(1)})a_{(2)(1)}(Sa_{(2)(2)(1)})a_{(2)(2)(2)(2)}, \quad \text{by (3.4.4) on } a_{(2)}, \\ &\quad = a_{(2)(1)} \otimes (Sa_{(1)})a_{(2)(2)}, \quad \text{by (3.4.1)}, \end{aligned}$$

as required. Similarly, for all $a \in A$,

$$\begin{aligned} &a_{(1)(2)(1)} \otimes a_{(1)(1)}(Sa_{(1)(2)(2)})a_{(2)} \\ &\quad = a_{(1)(2)(1)} \otimes a_{(1)(1)}a_{(2)}Sa_{(1)(2)(2)} \\ &\quad = a_{(2)(1)(1)} \otimes a_{(1)}a_{(2)(2)}Sa_{(2)(1)(2)} \\ &\quad = a_{(2)(1)(1)} \otimes a_{(1)}(Sa_{(2)(1)(2)})a_{(2)(2)} \\ &\quad = a_{(2)} \otimes a_{(1)} \quad \text{by (3.4.2) on } a_{(2)}, \\ &\quad = a_{(1)(1)(2)} \otimes a_{(1)(1)(1)}(Sa_{(1)(2)})a_{(2)} \quad \text{by Proposition3.4.3 (1)}. \end{aligned}$$

By applying the equality above to $a_{(1)}$ and multiplying on the right by $Sa_{(2)}$, we obtain identity (2). □

As in the previous theory, we define the *coassociator*, now by a map $\Phi : A \to A \otimes A \otimes A$ such that

$$(\Delta \otimes \mathrm{id})\Delta(a) = \Phi(a_{(1)})(\mathrm{id} \otimes \Delta)\Delta(a_{(2)}).$$

For the next proposition, we will introduce some convenient notation; let A be a Hopf coquasigroup, and $a \in A$. We write $\Phi(a) = \sum {\Phi_a}^{(1)} \otimes {\Phi_a}^{(2)} \otimes {\Phi_a}^{(3)}$.

Proposition 3.4.7. *Let A be a Hopf coquasigroup with coassociator Φ. Then, for all $a \in A$,*

(1) *Φ always exists and is unique. It can be expressed as*

$$\Phi(a) = a_{(1)(1)(1)}(Sa_{(2)})_{(1)} \otimes a_{(1)(1)(2)}(Sa_{(2)})_{(2)(1)} \otimes a_{(1)(2)}(Sa_{(2)})_{(2)(2)},$$

(2) $(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})\Phi_a = (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})\Phi_a = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)\Phi_a = \varepsilon(a) \cdot 1,$

(3) ${\Phi_a}^{(1)}S({\Phi_a}^{(2)}) \otimes {\Phi_a}^{(3)} = S({\Phi_a}^{(1)}){\Phi_a}^{(2)} \otimes {\Phi_a}^{(3)} = \varepsilon(a) \cdot 1$
${\Phi_a}^{(1)} \otimes S({\Phi_a}^{(2)}){\Phi_a}^{(3)} = {\Phi_a}^{(1)} \otimes {\Phi_a}^{(2)}S({\Phi_a}^{(3)}) = \varepsilon(a) \cdot 1$
${\Phi_a}^{(1)}{}_{(1)}S({\Phi_a}^{(3)}) \otimes {\Phi_a}^{(1)}{}_{(2)}S({\Phi_a}^{(2)}) = S({\Phi_a}^{(1)}{}_{(1)}){\Phi_a}^{(3)} \otimes S({\Phi_a}^{(1)}{}_{(2)}){\Phi_a}^{(2)} = \varepsilon(a) \cdot 1$
$S({\Phi_a}^{(1)}){\Phi_a}^{(3)}{}_{(2)} \otimes S({\Phi_a}^{(2)}){\Phi_a}^{(3)}{}_{(1)} = {\Phi_a}^{(1)}S({\Phi_a}^{(3)}{}_{(2)}) \otimes {\Phi_a}^{(2)}S({\Phi_a}^{(3)}{}_{(1)}) = \varepsilon(a) \cdot 1$
$S({\Phi_a}^{(1)}){\Phi_a}^{(2)}{}_{(1)} \otimes S({\Phi_a}^{(2)}{}_{(2)}){\Phi_a}^{(3)} = {\Phi_a}^{(1)}S({\Phi_a}^{(2)}{}_{(1)}) \otimes {\Phi_a}^{(2)}{}_{(2)}S({\Phi_a}^{(3)}) = \varepsilon(a) \cdot 1$

Proof. This proof is dual to that of Proposition 3.3.8 in the previous section. We will prove (1) and give an example of (3). Suppose Φ exists, then we apply it to $a_{(1)}$ and use that Δ is an algebra morphism to compute,

$$\begin{aligned}
&a_{(1)(1)(1)}(Sa_{(2)})_{(1)} \otimes a_{(1)(1)(2)}(Sa_{(2)})_{(2)(1)} \otimes a_{(1)(2)}(Sa_{(2)})_{(2)(2)} \\
&= \Phi^{(1)}_{a_{(1)(1)}} a_{(1)(2)(1)}(Sa_{(2)})_{(1)} \otimes \Phi^{(2)}_{a_{(1)(1)}} a_{(1)(2)(2)(1)}(Sa_{(2)})_{(2)(1)} \\
&\qquad \otimes \Phi^{(3)}_{a_{(1)(1)}} a_{(1)(2)(2)(2)}(Sa_{(2)})_{(2)(2)} \\
&= \Phi^{(1)}_{a_{(1)(1)}}(a_{(1)(2)}Sa_{(2)})_{(1)} \otimes \Phi^{(2)}_{a_{(1)(1)}}(a_{(1)(2)}Sa_{(2)})_{(2)(1)} \otimes \Phi^{(3)}_{a_{(1)(1)}}(a_{(1)(2)}Sa_{(2)})_{(2)(2)} \\
&= \Phi^{(1)}_a \otimes \Phi^{(2)}_a \otimes \Phi^{(3)}_a, \quad \text{by (3.4.2)}, \\
&= \Phi(a).
\end{aligned}$$

Next we verify that Φ satisfies the required identity; since Δ is an algebra morphism and using properties of a Hopf coquasigroup, we have

$$\begin{aligned}
&\Phi(a_{(1)})(\mathrm{id} \otimes \Delta)\Delta(a_{(2)}) \\
&\qquad = a_{(1)(1)(1)(1)}(Sa_{(1)(2)})_{(1)}a_{(2)(1)} \otimes a_{(1)(1)(1)(2)}(Sa_{(1)(2)})_{(2)(1)}a_{(2)(2)(1)} \\
&\qquad\quad \otimes a_{(1)(1)(2)}(Sa_{(1)(2)})_{(2)(2)}a_{(2)(2)(2)}
\end{aligned}$$

$$
\begin{aligned}
&= a_{(1)(1)(1)(1)}((Sa_{(1)(2)})a_{(2)})_{(1)} \otimes a_{(1)(1)(1)(2)}((Sa_{(1)(2)})a_{(2)})_{(2)(1)} \\
&\quad \otimes a_{(1)(1)(2)}((Sa_{(1)(2)})a_{(2)})_{(2)(2)} \\
&= a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)}, \quad \text{by (3.4.2)}, \\
&= (\Delta \otimes \mathrm{id})\Delta(a).
\end{aligned}
$$

Identity (2) is clear. We will prove that ${\Phi_a}^{(1)} S({\Phi_a}^{(2)}) \otimes {\Phi_a}^{(3)} = \varepsilon(a) \cdot 1$:

$$
\begin{aligned}
&{\Phi_a}^{(1)} S({\Phi_a}^{(2)}) \otimes {\Phi_a}^{(3)} \\
&\quad = a_{(1)(1)(1)}(Sa_{(2)})_{(1)} S(a_{(1)(1)(2)}(Sa_{(2)})_{(2)(1)}) \otimes a_{(1)(2)}(Sa_{(2)})_{(2)(2)} \\
&\quad = a_{(1)(1)(1)}(Sa_{(2)})_{(1)} S((Sa_{(2)})_{(2)(1)}) Sa_{(1)(1)(2)} \otimes a_{(1)(2)}(Sa_{(2)})_{(2)(2)}, \\
&\qquad\quad \text{by Proposition 3.4.3 (2)}, \\
&\quad = a_{(1)(1)(1)} Sa_{(1)(1)(2)} \otimes a_{(1)(2)} Sa_{(2)}, \quad \text{by (3.4.1) on } Sa_{(2)}, \\
&\quad = \varepsilon(a_{(1)(1)}) \otimes a_{(1)(2)} Sa_{(2)}, \quad \text{by Proposition 3.4.3 (1) on } a_{(1)(1)}, \\
&\quad = 1 \otimes a_{(1)} S(a_{(2)}) \\
&\quad = \varepsilon(a) \cdot 1.
\end{aligned}
$$

The rest are similar. □

Clearly as $\mathcal{G}_n = \{\pm e_a \,|\, a \in \mathbb{Z}_2^n\} \subset \mathcal{S}^{2^n-1}$ is a finite quasigroup, $k\mathcal{G}_n$ is a finite-dimensional Hopf quasigroup and hence its dual $k[\mathcal{G}_n]$ of functions on $\mathcal{G}_n$ with pointwise multiplication is a Hopf coquasigroup. However, the dual theory is more powerful and also allows 'coordinate algebra' versions of infinite-dimensional quasigroups $\mathcal{G}$ as we now demonstrate. Specifically, we consider the algebra $k[\mathcal{S}^{2^n-1}]$ of functions on the spheres $\mathcal{S}^{2^n-1} \subset k_F\mathbb{Z}_2^n$, as discussed in Section 3.2. This algebra is generated by the functions

$$x_a(u) = u_a,$$

that pick out the value of the a-th coordinate of $u = \sum_a u_a e_a \in \mathcal{S}^{2^n-1}$. More precisely, $k[\mathcal{S}^{2^n-1}]$ is defined to be the (commutative) polynomial algebra $k[x_a \,|\, a \in \mathbb{Z}_2^n]$ with relations $\sum_a x_a^2 = 1$.

Proposition 3.4.8. *$k[\mathcal{S}^{2^n-1}]$ is a Hopf coquasigroup with structure*

$$\Delta x_a = \sum_{b+c=a} x_b \otimes x_c\, F(b,c), \quad \varepsilon x_a = \delta_{a,0}, \quad S x_a = x_a\, F(a,a),$$

for all $a \in \mathbb{Z}_2^n$. The coassociator is

$$\Phi(x_d) = \sum_{a+b+c+a'+b'+c'=d} x_a x_{a'} \otimes x_b x_{b'} \otimes x_c x_{c'}\, F(a',a')F(b',b')F(c',c')$$
$$F(a,b)F(a+b,c)F(c',b')F(b'+c',a')F(a+b+c,a'+b'+c').$$

Proof. By construction, it is clear that Φ satisfies $(\Delta \otimes \mathrm{id})\Delta(x_a) = \Phi(x_{a(1)})(\mathrm{id} \otimes \Delta)\Delta(x_{a(2)})$.

To show that $k[\mathcal{S}^{2^n-1}]$ is a Hopf coquasigroup, we will prove only that $(m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta = 1 \otimes \mathrm{id}$, the others are similar. The LHS on x_d gives us

$$\sum_{a+b+c=d} x_a x_b \otimes x_c\, F(a,a)\phi(a,b,c)F(a+b,c)F(a,b).$$

Consider the partial sum with $a = b$ so $c = d$; this gives

$$\sum_a x_a^2 \otimes x_d\, F(a,a)\phi(a,a,d)F(0,d)F(a,a) = \sum_a x_a^2 \otimes x_d = 1 \otimes x_d.$$

Now consider the remaining partial sum with $a \neq b$. We claim that the term with given values a', b' for a and b cancels with the term with $a = b', b = a'$. These give, respectively,

$$x_{a'} x_{b'} \otimes x_c\, F(a',a')\phi(a',b',c)F(a'+b',c)F(a',b'),$$

$$x_{b'} x_{a'} \otimes x_c\, F(b',b')\phi(b',a',c)F(b'+a',c)F(b',a').$$

When $a' = 0$ and hence $b' \neq 0$, these become

$$x_0 x_{b'} \otimes x_c\, F(b',c),$$

$$x_{b'} x_0 \otimes x_c\, F(b',b')F(b',c) = -x_0 x_{b'} \otimes x_c\, F(b',c),$$

which cancel. When $a', b' \neq 0$, these become

$$-x_{a'}x_{b'} \otimes x_c\, \phi(a', b', c)F(a' + b', c)F(a', b'),$$

$$-x_{b'}x_{a'} \otimes x_c\, \phi(b', a', c)F(b' + a', c)F(b', a') = x_{a'}x_{b'} \otimes x_c\, \phi(a', b', c)F(a' + b', c)F(a', b'),$$

which also clearly cancel. Hence,

$$\sum_{a+b+c=d} x_a x_b \otimes x_c\, F(a, a)\phi(a, b, c)F(a + b, c)F(a, b) = 1 \otimes x_d,$$

as required. □

Proposition 3.4.9. *$k[\mathcal{S}^{2^n-1}]$ is a Moufang Hopf coquasigroup, and hence flexible and alternative.*

Proof. Consider a generator $x_f \in k[\mathcal{S}^{2^n-1}]$ and the two expressions:

$$\begin{aligned}
&(1) \quad x_{f(1)}x_{f(2)(2)(1)} \otimes x_{f(2)(1)} \otimes x_{f(2)(2)(2)} \\
&= \sum_{a+b+c+d=f} x_a x_c \otimes x_b \otimes x_d\, F(c, d)F(b, c + d)F(a, b + c + d),
\end{aligned}$$

$$\begin{aligned}
&(2) \quad x_{f(1)(1)(1)}x_{f(1)(2)} \otimes x_{f(1)(1)(2)} \otimes x_{f(2)} \\
&= \sum_{a+b+c+d=f} x_a x_c \otimes x_b \otimes x_d\, F(a, b)F(a + b, c)F(a + b + c, d).
\end{aligned}$$

Then $k[\mathcal{S}^{2^n-1}]$ is Moufang if (1)=(2). We will consider the possible cases.

Case 1: $f = 0$.
Since $a + b + c + d = 0$, by Lemma 3.2.1, $\phi(a + b, c, d) = \phi(a, b, c + d) = 1$, so we have

$$\begin{aligned}
(1) &= \sum_{a+b+c+d=0} x_a x_c \otimes x_b \otimes x_d\, F(c, d)F(b, c + d)F(a, b + c + d) \\
&= \sum_{a+b+c+d=0} x_a x_c \otimes x_b \otimes x_d \\
&\qquad \phi(a + b, c, d)\phi(a, b, c + d)F(c, d)F(b, c + d)F(a, b + c + d) \\
&= \sum_{a+b+c+d=0} x_a x_c \otimes x_b \otimes x_d\, F(a, b)F(a + b, c)F(a + b + c, d) \\
&= (2).
\end{aligned}$$

Case 2: $f \neq 0$.
We will split the sum into part expressions and show that these partial sums in (1) equal partial sums in (2) by applying identities (3.2.3)-(3.2.7) and Lemma 3.2.1.

Part expression: we look at the terms in (1) where $a = 0$ and b, c, d are linearly independent. We claim that this equals the part expression of (2) when $c = 0$ and a, b, d are linearly independent. Since b, c, d are linearly independent, $R(b, c) = \phi(b, c, d) = -1$ so,

$$\begin{aligned}(1) &= \sum_{b+c+d=f} x_0x_c \otimes x_b \otimes x_d \, F(c, d)F(b, c + d) \\ &= \sum_{b+c+d=f} x_0x_c \otimes x_b \otimes x_d \, \phi(b, c, d)F(b, c)F(b + c, d) \\ &= -\sum_{b+c+d=f} x_cx_0 \otimes x_b \otimes x_d \, F(b, c)F(b + c, d) \\ &= \sum_{c+b+d=f} x_cx_0 \otimes x_b \otimes x_d \, F(c, b)F(c + b, d),\end{aligned}$$

which, up to relabeling, is clearly the part expression of (2) when $c = 0$ and a, b, d are linearly independent.

Part expression: Similarly, the part expression when $c = 0$ with a, b, d linearly independent in (1) is equal to the part sum when $a = 0$ in (2) with b, c, d linearly independent.

Part expression: terms where $a = 0$ and b, c, d are linearly dependent. Under these conditions, $\phi(b, c, d) = 1$ hence,

$$\begin{aligned}(1) &= \sum_{b+c+d=f} x_0x_c \otimes x_b \otimes x_d \, F(c, d)F(b, c + d) \\ &= \sum_{b+c+d=f} x_0x_c \otimes x_b \otimes x_d \, \phi(b, c, d)F(b, c)F(b + c, d) \\ &= \sum_{b+c+d=f} x_0x_c \otimes x_b \otimes x_d \, F(b, c)F(b + c, d) \\ &= (2).\end{aligned}$$

Part expression: now consider terms where $b = 0$. We have

$$(1) = \sum_{a+c+d=f} x_a x_c \otimes x_0 \otimes x_d \, F(c,d)F(a,c+d),$$

$$(2) = \sum_{a+c+d=f} x_a x_c \otimes x_0 \otimes x_d \, F(a,c)F(a+c,d).$$

If a, c, d are linearly dependent, then $\phi(a,c,d) = 1$, so clearly (1)=(2). Suppose a, c, d are linearly independent, then using commutativity of the generators

$$\begin{aligned}(1) &= \sum_{a+c+d=f} x_a x_c \otimes x_0 \otimes x_d \, F(c,d)F(a,c+d) \\ &= \sum_{a+c+d=f} x_a x_c \otimes x_0 \otimes x_d \, \phi(a,c,d)F(a,c)F(a+c,d) \\ &= -\sum_{a+c+d=f} x_a x_c \otimes x_0 \otimes x_d \, F(a,c)F(a+c,d) \\ &= \sum_{c+a+d=f} x_c x_a \otimes x_0 \otimes x_d \, F(c,a)F(c+a,d),\end{aligned}$$

which, after relabeling, is equal to (2) with $b = 0$. The third equality uses linear independence of a, c, d and the final equality uses $R(a,c) = -1$.

Part expression: terms where $d = 0$. We have

$$(1) = \sum_{a+b+c=f} x_a x_c \otimes x_b \otimes x_0 \, F(b,c)F(a,b+c),$$

$$(2) = \sum_{a+b+c=f} x_a x_c \otimes x_b \otimes x_0 \, F(a,b)F(a+b,c).$$

Clearly (1) = (2) for those terms with a, b, c linearly dependent since $\phi(a,b,c) = 1$. When a, b, c are linearly independent we have $R(b,c) = R(a,b+c) = -1$ so,

$$\begin{aligned}(1) &= \sum_{a+b+c=f} x_a x_c \otimes x_b \otimes x_0 \, F(b,c)F(a,b+c) \\ &= \sum_{a+b+c=f} x_c x_a \otimes x_b \otimes x_0 \, R(b,c)F(c,b)R(a,b+c)F(b+c,a) \\ &= \sum_{a+b+c=f} x_c x_a \otimes x_b \otimes x_0 \, F(c,b)F(c+b,a).\end{aligned}$$

Relabeling gives the part expression of (2) when $d = 0$ and a, b, c are linearly independent.

Part expression: terms where any two variables are equal, e.g. $a + b = 0$. We have

$$(1) = \sum_{a+b+c+d=f} x_a x_c \otimes x_b \otimes x_d \, F(c,d)F(b,c+d)F(a,b+c+d),$$

$$(2) = \sum_{a+b+c+d=f} x_a x_c \otimes x_b \otimes x_d \, \phi(a+b,c,d)\phi(a,b,c+d)F(c,d)F(b,c+d)F(a,b+c+d).$$

If $a+b = 0$ then, $\phi(a+b,c,d) = \phi(0,c,d) = 1$ and $\phi(a,b,c+d) = \phi(a,a,c+d) = 1$ by linear dependence of the variables. Similarly if $c+d = 0$. If any two other variables are equal, for example $a+c = 0$ then $\phi(a+b,c,d)\phi(a,b,c+d) = \phi(a,b,d)^2 = 1$ by the symmetric properties of ϕ. Hence (1)=(2) in each of these part expressions.

Part expression: terms where $a + b + c = 0$. Note that this means $d = f$, and we assume $a, b, c \neq 0, f$ as these sums are included in previous part expressions. Since the terms with any two values equal have already been considered, we also have a, c, f linearly independent.

$$(1) = \sum_{a,c} x_a x_c \otimes x_{a+c} \otimes x_f \, F(c,f)F(a+c,c+f)F(a,a+f),$$

$$(2) = \sum_{a,c} x_a x_c \otimes x_{a+c} \otimes x_f \, F(a,a+c)F(c,c) = \sum_{a,c} x_a x_c \otimes x_{a+c} \otimes x_f \, F(a,c),$$

since $F(c,c) = -1$ and $F(a,a+c) = -F(a,c)$. Linear independence of a, c, f means that $\phi(a,c,f) = -1$ and $R(a,c) = -1$, so

$$\begin{aligned}
(1) &= \sum_{a \neq c} x_a x_c \otimes x_{a+c} \otimes x_f \, F(c,f)F(a+c,c+f)F(a,a+f) \\
&= \sum_{a \neq c} x_a x_c \otimes x_{a+c} \otimes x_f \, F(c,c+f)F(a+c,c+f)F(a,f) \quad \text{by (3.2.3)}, \\
&= \sum_{a \neq c} x_a x_c \otimes x_{a+c} \otimes x_f \, \phi(a,c,c+f)F(a,c) \\
&\qquad \text{since } \phi(a,c,c+f) = \phi(a,c,f) = -1, \\
&= -\sum_{a \neq c} x_a x_c \otimes x_{a+c} \otimes x_f \, F(a,c)
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{c\neq a} x_c x_a \otimes x_{c+a} \otimes x_f \, F(a,c) \\
&= -\sum_{a\neq c} x_a x_c \otimes x_{a+c} \otimes x_f \, F(c,a) \quad \text{ by relabeling,} \\
&= \sum_{a\neq c} x_a x_c \otimes x_{a+c} \otimes x_f \, F(a,c) \\
&= (2).
\end{aligned}$$

Part expression: terms where $a+c+d=0$. As in the previous part expression, $b=f\neq 0$, and we can assume $a,c,d\neq 0$ and that a,c,f are linearly independent. Using $R(a,f)=-1$ we find

$$\begin{aligned}
(1) &= \sum_{a\neq c} x_a x_c \otimes x_f \otimes x_{a+c} \, F(c,a+c)F(f,a)F(a,f+a) \\
&= \sum_{a\neq c} x_a x_c \otimes x_f \otimes x_{a+c} \, F(c,a)F(f,a)F(a,f) \quad \text{by (3.2.3),} \\
&= \sum_{a\neq c} x_a x_c \otimes x_f \otimes x_{a+c} \, F(c,a)R(a,f) \\
&= -\sum_{a\neq c} x_a x_c \otimes x_f \otimes x_{a+c} \, F(c,a).
\end{aligned}$$

Now, using $\phi(f,a,c)=-1$ since a,f,c are linearly independent, and relabeling we find

$$\begin{aligned}
(2) &= \sum_{a\neq c} x_a x_c \otimes x_f \otimes x_{a+c} \, F(a,f)F(a+f,c)F(a+f+c,a+c) \\
&= \sum_{a\neq c} x_a x_c \otimes x_f \otimes x_{a+c} \, F(f,a)F(f+a,c)F(f,a+c) \\
&= \sum_{a\neq c} x_a x_c \otimes x_f \otimes x_{a+c} \, \phi(f,a,c)F(a\neq c) \\
&= -\sum_{a\neq c} x_a x_c \otimes x_f \otimes x_{a+c} \, F(a,c) \\
&= -\sum_{c,a} x_c x_a \otimes x_f \otimes x_{c+a} \, F(a,c) \\
&= -\sum_{a\neq c} x_a x_c \otimes x_f \otimes x_{a+c} \, F(c,a), \quad \text{by relabeling,} \\
&= (1).
\end{aligned}$$

Part expression: terms where $a+b+d=0$ and $b+c+d=0$. We claim that the partial sum with $a+b+d=0$ in (1) equals the partial sum with $b+c+d=0$ in (2). Consider the part expression of (1) with $d=a+b$ and $c=f$;

$$\begin{aligned}
(1) &= \sum_{a\neq b} x_a x_f \otimes x_b \otimes x_{a+b}\, F(f,a+b)F(b,a+b+f)F(a,a+f)\\
&= \sum_{a\neq b} x_a x_f \otimes x_b \otimes x_{a+b}\, F(f,b+a)F(b,a+f)F(a,f), \quad \text{by (3.2.3)},\\
&= -\sum_{a\neq b} x_a x_f \otimes x_b \otimes x_{a+b}\, F(b+a,f)F(b,a+f)F(a,f), \quad \text{by Lemma 3.2.1},\\
&= -\sum_{a\neq b} x_a x_f \otimes x_b \otimes x_{a+b}\, \phi(b,a,f)F(b,a)\\
&= -\sum_{a\neq b} x_f x_a \otimes x_b \otimes x_{a+b}\, \phi(f,b,a)F(b,a), \quad \text{by Lemma 3.2.1},\\
&= -\sum_{a\neq b} x_f x_a \otimes x_b \otimes x_{a+b}\, F(f,b)F(f+b,a)F(f,b+a)\\
&= \sum_{a\neq b} x_f x_a \otimes x_b \otimes x_{a+b}\, F(f,b)F(f+b,a)F(f+b+a,b+a), \quad \text{by (3.2.4).}
\end{aligned}$$

It is straightforward to see that, up to relabeling, this is equal to the partial sum in (2) when $d=b+c$, *i.e.* $b+c+d=0$. Similarly, the case with $b+c+d=0$ in (1) equals the case with $a+b+d=0$ in (2).

Part expression: terms where a,b,c,d are linearly independent. In this case $\phi(a+b,c,d)=\phi(a,b,c+d)=-1$, hence

$$\begin{aligned}
&(1)\\
&= \sum_{a+b+c+d=f} x_a x_c \otimes x_b \otimes x_d\, F(a,b)F(a+b,c)F(a+b+c,d)\\
&= \sum_{a+b+c+d=f} x_a x_c \otimes x_b \otimes x_d\, \phi(a+b,c,d)\phi(a,b,c+d)F(c,d)F(b,c+d)F(a,b+c+d)\\
&= \sum_{a+b+c+d=f} x_a x_c \otimes x_b \otimes x_d\, F(c,d)F(b,c+d)F(a,b+c+d)\\
&= (2).
\end{aligned}$$

The partial sums considered above are disjoint and include every term in the full expression. Since we can equate each partial sum in (1) with a unique partial sum in (2), we can conclude that $(1)=(2)$ and $k[\mathcal{S}^{2^n-1}]$ is Moufang. □

Remark 3.4.10. Although Φ has rather a large number of terms we can differentiate it at the identity $(1,0,0,0,0,0,0,0)$ by setting $x_i x_j = 0$ if $i,j > 0$ to obtain

$$\Phi_*(x_d) = \sum_{a+b+c=d} (\phi(a,b,c)-1)F(b,c)F(a,b+c)x_a \otimes x_b \otimes x_c.$$

This is essentially adjoint to the map $(x,y,z) = (xy)z - x(yz)$ on the underlying quasialgebra $k_F G$, which on basis elements takes the form

$$(e_a, e_b, e_c) = (\phi(a,b,c)-1)F(b,c)F(a,b+c)e_{a+b+c}.$$

Proposition 3.4.11. *Let A be a Hopf coquasigroup equipped with a linear action of a group G such that $\sigma\triangleright(ab) = (\sigma\triangleright a)(\sigma\triangleright b)$, $\sigma\triangleright 1 = 1$, $(\sigma\otimes\sigma)\triangleright\Delta a = \Delta(\sigma\triangleright a)$, $\varepsilon(\sigma\triangleright a) = \varepsilon(a)$ and $(\sigma\sigma')\triangleright a = \sigma\triangleright(\sigma'\triangleright a)$ for all $\sigma,\sigma' \in G$ and $a,b \in A$. Then the cross product algebra $A \rtimes G$ is again a Hopf coquasigroup and is Moufang if A is.*

Proof. The algebra is a standard cross product construction to give an associative algebra $A \rtimes kG$. We define the tensor product Δ, ε as in Proposition 3.3.15 and check that these are algebra morphisms just as in the Hopf quasigroup case. Similarly, the antipode is defined as $S(a\otimes\sigma) = \sigma^{-1}\triangleright Sa \otimes \sigma^{-1}$. We then verify the Hopf coquasigroup identities, for example,

$$\begin{aligned}
S((a\otimes\sigma)_{(1)})&(a\otimes\sigma)_{(2)(1)} \otimes (a\otimes\sigma)_{(2)(2)} \\
&= S(a_{(1)}\otimes\sigma)(a_{(2)(1)}\otimes\sigma)\otimes(a_{(2)(2)}\otimes\sigma) \\
&= (\sigma^{-1}\triangleright Sa_{(1)}\otimes\sigma^{-1})(a_{(2)(1)}\otimes\sigma)\otimes(a_{(2)(2)}\otimes\sigma) \\
&= ((\sigma^{-1}\triangleright Sa_{(1)})(\sigma^{-1}\triangleright a_{(2)(1)})\otimes\sigma^{-1}\sigma)\otimes(a_{(2)(2)}\otimes\sigma) \\
&= (\sigma^{-1}\triangleright((Sa_{(1)})a_{(2)(1)})\otimes 1)\otimes(a_{(2)(2)}\otimes\sigma) \\
&= (1\otimes 1)\otimes(a\otimes\sigma), \quad \text{by (3.4.1)}.
\end{aligned}$$

The proofs of the other identities are similar. Finally, let A be a Moufang Hopf coquasigroup, and G be a group acting on A. It is straightforward to see that $A \rtimes kG$ is a Moufang Hopf coquasigroup; we consider

$$(a\otimes\sigma)_{(1)}(a\otimes\sigma)_{(2)(2)(1)} \otimes (a\otimes\sigma)_{(2)(1)} \otimes (a\otimes\sigma)_{(2)(2)(2)}.$$

By definition of the coproduct, this equals

$$(a_{(1)} \otimes \sigma)(a_{(2)(2)(1)} \otimes \sigma) \otimes (a_{(2)(1)} \otimes \sigma) \otimes (a_{(2)(2)(2)} \otimes \sigma).$$

Since A is Moufang, and again using the definition of the coproduct, we see this equals

$$(a \otimes \sigma)_{(1)(1)(1)}(a \otimes \sigma)_{(1)(2)} \otimes (a \otimes \sigma)_{(1)(1)(2)} \otimes (a \otimes \sigma)_{(2)},$$

and hence $A \rtimes kG$ is Moufang. □

Example 3.4.12. The Hopf coquasigroup $k[\mathcal{S}^{2^n-1}]$ has an action of $G = \mathbb{Z}_2^n$ by

$$\sigma_a \triangleright x_b = (-1)^{a \cdot b} x_b$$

and the resulting cross product $k[\mathcal{S}^{2^n-1}] \rtimes k\mathbb{Z}_2^n$ is a noncommutative Hopf coquasigroup.

The action is adjoint to the one in Example 3.3.16 in the last section. One can verify all the required properties. The resulting associative algebra in the case $n = 3$ has generators x_a and $\sigma_{001}, \sigma_{010}$ and σ_{100} with relations

$$x_a\sigma_{001} = (-1)^{a_3}\sigma_{001}x_a, \quad x_a\sigma_{010} = (-1)^{a_2}\sigma_{010}x_a, \quad x_a\sigma_{100} = (-1)^{a_1}\sigma_{100}x_a.$$

For completeness we consider the dual constructions to those for Hopf quasigroups at the end of the last section.

Definition 3.4.13. Let A be a Hopf coquasigroup. A vector space V is a *right A-comodule*, if there is a linear map $\beta : V \to V \otimes A$ written as $\beta(v) = v^{(1)} \otimes v^{(2)}$ such that

$$v^{(1)(1)} \otimes v^{(1)(2)} \otimes v^{(2)} = v^{(1)} \otimes v^{(2)}{}_{(1)} \otimes v^{(2)}{}_{(2)}, \quad v^{(1)}\varepsilon(v^{(2)}) = v,$$

for all $v \in V$. An algebra H is an *A-comodule algebra* if further

$$\beta(hg) = \beta(h)\beta(g), \quad \beta(1) = 1 \otimes 1,$$

for all $h, g \in H$. Finally, a noncoassociative coalgebra C is an *A-comodule coalgebra* if

$$c^{(1)}{}_{(1)} \otimes c^{(1)}{}_{(2)} \otimes c^{(2)} = c_{(1)}{}^{(1)} \otimes c_{(2)}{}^{(1)} \otimes c_{(1)}{}^{(2)} c_{(2)}{}^{(2)}, \quad \varepsilon(c^{(1)})c^{(2)} = \varepsilon(c),$$

for all $c \in C$. Therefore we have the notion of a *right A-comodule Hopf coquasigroup*; we can similarly define left actions of Hopf coquasigroups.

Lemma 3.4.14. *Let A be a Hopf coquasigroup and C be a right A-comodule Hopf coquasigroup, then the coaction commutes with the antipode, that is*

$$Sc^{(1)} \otimes c^{(2)} = (Sc)^{(1)} \otimes (Sc)^{(2)},$$

for all $c \in C$.

Proof. Using the property of the coaction on an algebra we have

$$c_{(1)}{}^{(1)}(Sc_{(2)})^{(1)} \otimes c_{(1)}{}^{(2)}(Sc_{(2)})^{(2)} = (c_{(1)}Sc_{(2)})^{(1)} \otimes (c_{(1)}Sc_{(2)})^{(2)} = \varepsilon(c) \cdot 1.$$

Now using the property of a coaction on a coalgebra we obtain

$$c_{(1)}{}^{(1)}Sc_{(2)}{}^{(1)} \otimes c_{(1)}{}^{(2)}c_{(2)}{}^{(2)} = c^{(1)}{}_{(1)}Sc^{(1)}{}_{(2)} \otimes c^{(2)} = \varepsilon(c^{(1)})c^{(2)} = \varepsilon(c) \cdot 1.$$

So we have

$$c_{(1)}{}^{(1)}(Sc_{(2)})^{(1)} \otimes c_{(1)}{}^{(2)}(Sc_{(2)})^{(2)} = c_{(1)}{}^{(1)}Sc_{(2)}{}^{(1)} \otimes c_{(1)}{}^{(2)}c_{(2)}{}^{(2)},$$

from which we can obtain the required identity. □

Proposition 3.4.15. *Let A be a commutative Hopf coquasigroup and let C be a right A-comodule Hopf coquasigroup. There is a right cross coproduct Hopf coquasigroup $A \blacktriangleright\!\!< C$ built on $A \otimes C$ with tensor product algebra and counit and*

$$\Delta(a \otimes c) = a_{(1)} \otimes c_{(1)}{}^{(1)} \otimes a_{(2)}c_{(1)}{}^{(2)} \otimes c_{(2)},$$

$$S(a \otimes c) = S(ac^{(2)}) \otimes Sc^{(1)},$$

for all $a \in A, c \in C$.

Proof. We check that Δ is an algebra morphism by computing directly:

$$\begin{aligned}
&\Delta((a\otimes c)(b\otimes d))\\
&\qquad\qquad = \Delta(ab\otimes cd)\\
&\qquad\qquad = (ab)_{(1)}\otimes {(cd)_{(1)}}^{(1)}\otimes (ab)_{(2)}{(cd)_{(1)}}^{(2)}\otimes (cd)_{(2)}\\
&\qquad\qquad = a_{(1)}b_{(1)}\otimes {c_{(1)}}^{(1)}{d_{(1)}}^{(1)}\otimes a_{(2)}b_{(2)}{c_{(1)}}^{(2)}{d_{(1)}}^{(2)}\otimes c_{(2)}d_{(2)},
\end{aligned}$$

and,

$$\begin{aligned}
&\Delta(a\otimes c)\Delta(b\otimes d)\\
&\qquad = (a_{(1)}\otimes {c_{(1)}}^{(1)})(b_{(1)}\otimes {d_{(1)}}^{(1)})\otimes (a_{(2)}{c_{(1)}}^{(2)}\otimes c_{(2)})(b_{(2)}{d_{(1)}}^{(2)}\otimes d_{(2)})\\
&\qquad = a_{(1)}b_{(1)}\otimes {c_{(1)}}^{(1)}{d_{(1)}}^{(1)}\otimes a_{(2)}{c_{(1)}}^{(2)}b_{(2)}{d_{(1)}}^{(2)}\otimes c_{(2)}d_{(2)}.
\end{aligned}$$

Clearly these are equal if A is commutative. Next we check the coquasigroup identities:

$$\begin{aligned}
&S((a\otimes c)_{(1)})(a\otimes c)_{(2)(1)}\otimes (a\otimes c)_{(2)(2)}\\
&= (S(a_{(1)}{c_{(1)}}^{(1)(2)})\otimes S{c_{(1)}}^{(1)(1)})(a_{(2)(1)}{{c_{(1)}}^{(2)}}_{(1)}\otimes {c_{(2)(1)}}^{(1)})\\
&\quad\;\otimes(a_{(2)(2)}{{c_{(1)}}^{(2)}}_{(2)}{c_{(2)(1)}}^{(2)}\otimes c_{(2)(2)})\\
&= ((S{c_{(1)}}^{(1)(2)})(Sa_{(1)})a_{(2)(1)}{{c_{(1)}}^{(2)}}_{(1)}\otimes (S{c_{(1)}}^{(1)(1)}){c_{(2)(1)}}^{(1)})\\
&\quad\;\otimes(a_{(2)(2)}{{c_{(1)}}^{(2)}}_{(2)}{c_{(2)(1)}}^{(2)}\otimes c_{(2)(2)})\\
&= ((S{c_{(1)}}^{(1)(2)}){{c_{(1)}}^{(2)}}_{(1)}\otimes (S{c_{(1)}}^{(1)(1)}){c_{(2)(1)}}^{(1)})\otimes (a{{c_{(1)}}^{(2)}}_{(2)}{c_{(2)(1)}}^{(2)}\otimes c_{(2)(2)})\\
&= ((S{{c_{(1)}}^{(2)}}_{(1)}){{c_{(1)}}^{(2)}}_{(2)(1)}\otimes (S{c_{(1)}}^{(1)}){c_{(2)(1)}}^{(1)})\otimes (a{{c_{(1)}}^{(2)}}_{(2)(2)}{c_{(2)(1)}}^{(2)}\otimes c_{(2)(2)})\\
&= (1\otimes (S{c_{(1)}}^{(1)}){c_{(2)(1)}}^{(1)})\otimes (a{c_{(1)}}^{(2)}{c_{(2)(1)}}^{(2)}\otimes c_{(2)(2)})\\
&= (1\otimes (Sc_{(1)})^{(1)}{c_{(2)(1)}}^{(1)})\otimes (a(Sc_{(1)})^{(2)}{c_{(2)(1)}}^{(2)}\otimes c_{(2)(2)})\\
&= (1\otimes ((Sc_{(1)})c_{(2)(1)})^{(1)})\otimes (a((Sc_{(1)})c_{(2)(1)})^{(2)}\otimes c_{(2)(2)})\\
&= (1\otimes 1)\otimes (a\otimes c).
\end{aligned}$$

The third equality uses (3.4.1) on a, the fourth uses the definition of a coaction on a vector space on $c_{(1)}$, and the fifth and seventh equalities use the coquasigroup identities on ${c_{(1)}}^{(2)}$ and c respectively. The sixth equality uses Lemma 3.4.14. The other identities are similar. □

3.5 Duality of finite Hopf (co)quasigroups

Theorem 3.5.1. *Let H be a Hopf quasigroup, then*

(1) *$(H^*, \Delta^*, \varepsilon^*)$ is an associative unital algebra.*

(2) *If H is finite dimensional then, $(H^*, m^*, 1^*)$ is a noncoassociative counital coalgebra.*

(3) *If H is finite dimensional then, $(H^*, \Delta^*, \varepsilon^*, m^*, 1^*, S^*)$ is a Hopf coquasigroup.*

Proof. Let $(H, m, 1, \Delta, \varepsilon, S)$ be a Hopf quasigroup. We note that for a vector space V, we have $V^* \otimes V^* \subseteq (V \otimes V)^*$, with equality if V is finite dimensional. Recall that given a map $L : V \to W$ between spaces, the dual map $L^* : W^* \to V^*$ is uniquely defined by $\langle L^*(\varphi), v\rangle = \langle \varphi, L(v)\rangle$ for all $\varphi \in W^*$ and $v \in V$.

To prove (1) we first note that H^* is a vector space with induced maps Δ^*, ε^*. Since $H^* \otimes H^* \subseteq (H \otimes H)^*$, we can restrict Δ^* to get a map $\Delta^* : H^* \otimes H^* \to H^*$. Explicitly, Δ^* is given by

$$\Delta^*(\varphi \otimes \psi)(h) = \sum \varphi(h_{(1)})\psi(h_{(2)}),$$

for all $\varphi, \psi \in H^*$ and $h \in H$. Using coassociativity in H, we can see that Δ^* is associative:

$$\begin{aligned}\langle \Delta^*(\Delta^* \otimes \mathrm{id})(\varphi \otimes \psi \otimes \xi), h\rangle &= \langle \varphi, h_{(1)(1)}\rangle\langle \psi, h_{(1)(2)}\rangle\langle \xi, h_{(2)}\rangle \\ &= \langle \varphi, h_{(1)}\rangle\langle \psi, h_{(2)(1)}\rangle\langle \xi, h_{(2)(2)}\rangle \\ &= \langle \Delta^*(\mathrm{id} \otimes \Delta^*)(\varphi \otimes \psi \otimes \xi), h\rangle.\end{aligned}$$

Similarly, we can show that $\varepsilon^*(1)$ is a unit, hence $(H^*, \Delta^*, \varepsilon^*)$ is an associative unital algebra.

To prove that $(H^*, m^*, 1^*)$ is an algebra we require the extra condition that H is finite dimensional. Since, if H is not finite dimensional then the inclusion $H^* \otimes H^* \subset (H \otimes H)^*$ is strict, and the image of the induced map $m^* : H^* \to (H \otimes H)^*$ may not lie completely in $H^* \otimes H^*$. However, in the finite dimensional case, we have $H^* \otimes H^* = (H \otimes H)^*$, and $m^* : H^* \to H^* \otimes H^*$. By a similar method used to prove (1) we can show that $(H^*, m^*, 1^*)$ is a counital coalgebra, with the counit defined as $1^*(\varphi) = \varphi(1)$ for all $\varphi \in H^*$.

Finally, to prove (3), we only have to show that $m^*, 1^*$ are algebra morphisms, and that the identities (3.4.1) and (3.4.2) are satisfied. That m^* and 1^* are algebra morphisms follows from the fact that m and 1 are coalgebra morphisms. We shall prove the LHS of (3.4.1) and the rest are similar.

$$\begin{aligned}
&\langle(\Delta^* \otimes \mathrm{id})(S^* \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes m^*)m^*(\varphi), h \otimes g\rangle \\
&\quad = \langle\varphi, m(\mathrm{id} \otimes m)(S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(h \otimes g)\rangle \\
&\quad = \langle\varphi, (\varepsilon \otimes \mathrm{id})(h \otimes g)\rangle \quad \text{since } H \text{ is a Hopf quasigroup,} \\
&\quad = \langle(\varepsilon^* \otimes \mathrm{id})(\varphi), h \otimes g\rangle.
\end{aligned}$$

□

3.6 Differential calculus on Hopf coquasigroups

Let A be an associative algebra. As usual, we define an ideal, an A-module and an A-bimodule in the usual way, with commuting left and right actions written multiplicatively. As in most approaches to noncommutative geometry we define differential structures by specifying the bimodule of 1-forms. We adopt the approach and notation from [37].

Definition 3.6.1. A *first order differential calculus* over A is a pair (Ω^1, d) such that:

(1) Ω^1 is an A-bimodule,

(2) $\mathrm{d} : A \to A$ is a linear map satisfying

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b, \quad \forall\ a, b \in A,$$

(3) $\Omega^1 = \mathrm{span}\{a\,\mathrm{d}b \,|\, \forall\ a, b \in A\}$.

The universal calculus, Ω^1_{univ} is defined in the usual way as the kernel of the multiplication map with $\mathrm{d}a = 1 \otimes a - a \otimes 1$. When A is a Hopf coquasigroup, the algebra structure is an associative algebra and so the above definition makes sense, however we would like some form of 'translation invariance' with respect to the quasigroup multiplication expressed in the coproduct. Left invariance is effected for ordinary Hopf algebras by the 'left Maurer-Cartan form' ω and in the Hopf coquasigroup case we take this as the definition.

Definition 3.6.2. Let A be a Hopf coquasigroup and A^+ denote the ideal $\ker\varepsilon$. A first order differential calculus Ω^1 over A is *left covariant* if it is a free left A-module over $\mathrm{Im}(\omega)$, *i.e.*

$$\Omega^1 = A \cdot \mathrm{Im}(\omega),$$

where $\omega : A^+ \to \Omega^1$ is defined by

$$\omega(a) = (Sa_{(1)})\mathrm{d}a_{(2)}, \quad \forall\, a \in A^+.$$

We similarly define Ω^1 to be *right covariant* if

$$\Omega^1 = \mathrm{Im}(\omega_R) \cdot A, \quad \omega_R(a) = (\mathrm{d}a_{(1)})Sa_{(2)},$$

for all $a \in A^+$, with respect to a right-handed Maurer-Cartan form. The calculus is bicovariant if both of these hold. The universal calculus is bicovariant.

Lemma 3.6.3. *Let A be a Hopf coquasigroup and $A^+ = \ker\varepsilon$ the augmentation ideal. Then $\Omega^1_{univ} \cong A \otimes A^+$ via the left Maurer-Cartan form.*

Proof. Define $r : \Omega^1_{univ} \to A \otimes A^+$ by $a \otimes b \mapsto ab_{(1)} \otimes b_{(2)}$. We can check that the RHS lies in $A \otimes A^+$ by applying $(\mathrm{id} \otimes \varepsilon)$:

$$(\mathrm{id} \otimes \varepsilon)r(a \otimes b) = (\mathrm{id} \otimes \varepsilon)(ab_{(1)} \otimes b_{(2)}) = ab_{(1)}\varepsilon(b_{(2)}) = ab = 0,$$

since $a \otimes b \in \Omega^1_{univ}$. Hence $r(a \otimes b) \in A \otimes A^+$. The inverse map is $r^{-1}(a \otimes b) = aSb_{(1)} \otimes b_{(2)} = a\omega(b)$ provided by the left Maurer-Cartan form for the universal calculus. One can also see directly that it lies in Ω^1_{univ} by

$$mr^{-1}(a \otimes b) = m(aS(b_{(1)}) \otimes b_{(2)}) = aS(b_{(1)})b_{(2)} = a\varepsilon(b) = 0,$$

since $b \in A^+$. To show these maps are mutually inverse we require the defining properties of a Hopf coquasigroup. We find,

$$rr^{-1}(a \otimes b) = r(aSb_{(1)} \otimes b_{(2)}) = a(Sb_{(1)})b_{(2)(1)} \otimes b_{(2)(2)} = a \otimes b,$$

$$r^{-1}r(a \otimes b) = r^{-1}(ab_{(1)} \otimes b_{(2)}) = ab_{(1)}Sb_{(2)(1)} \otimes b_{(2)(2)} = a \otimes b,$$

using the Hopf coquasigroup identity (3.4.1). □

Theorem 3.6.4. *Let A be a Hopf coquasigroup. Left covariant first order calculi over A are in 1-1 correspondence with right ideals $I \subset A^+$ of A.*

Proof. Let Ω^1 be a left covariant first order differential calculus over A, then $\Omega^1 = A \cdot \mathrm{Im}\omega$, where $\omega : A^+ \to \Omega^1$ is as in Definition 3.6.2. Define $I = \ker\omega$, then I is a right ideal of A since if $x \in I$ and $a \in A$, then

$$\begin{aligned}\omega(xa) &= S(x_{(1)}a_{(1)})\mathrm{d}(x_{(2)}a_{(2)})\\ &= (Sa_{(1)})(Sx_{(1)})x_{(2)}\mathrm{d}a_{(2)} + (Sa_{(1)})(Sx_{(1)})(\mathrm{d}x_{(2)})a_{(2)}\\ &= (Sa_{(1)})\varepsilon(x)\mathrm{d}a_{(2)} + (Sa_{(1)})\omega(x)a_{(2)}\\ &= 0,\end{aligned}$$

since $x \in I \subset A^+$. We note that $A \cdot \mathrm{Im}(\omega) \cong A \otimes \mathrm{Im}(\omega)$ by the product, and $\mathrm{Im}(\omega) \cong A^+/I$, hence $\Omega^1 \cong A \otimes A^+/I$.

Conversely, given a right ideal $I' \subset A^+$ of A, define $N = r^{-1}(A \otimes I')$. Then N is a sub-bimodule of Ω^1_{univ}: let $a, x \in A$ and $b \in I'$

$$\begin{aligned}r((r^{-1}(a \otimes b))(1 \otimes x)) &= r(r^{-1}(a \otimes b))\Delta(x)\\ &= (a \otimes b)\Delta(x) \quad \in A \otimes I',\end{aligned}$$

since I' is a right ideal. Therefore $(r^{-1}(a \otimes b))(1 \otimes x) \in N$. Similarly,

$$\begin{aligned}r((x \otimes 1)r^{-1}(a \otimes b)) &= (x \otimes 1)r(r^{-1}(a \otimes b))\\ &= (x \otimes 1)(a \otimes b) \quad \in A \otimes I',\end{aligned}$$

so $(x \otimes 1)r^{-1}(a \otimes b) \in N$, as required. Therefore $\Omega^1 = \Omega^1_{univ}/N$ is a first order differential calculus over A. It remains to check that Ω^1 is left covariant. By

Lemma 3.6.3, $\Omega^1_{univ} \cong A \otimes A^+$, and there are canonical projections $\Omega^1_{univ} \to \Omega^1_{univ}/N$ and $A \otimes A^+ \to A \otimes A^+/I'$. Therefore, we have an isomorphism $A \otimes A^+/I' \cong \Omega^1$ sending $a \otimes b \to a\omega(b)$, where ω is the left Maurer-Cartan form on Ω^1. Thus $\Omega^1 \cong A \otimes A^+/I' \cong A \otimes \mathrm{Im}(\omega)$, and Ω^1 is left covariant.

These processes are mutually inverse; let $\pi : \Omega^1_{univ} \to \Omega^1$ be the canonical projection sending $a \otimes b \to a\mathrm{d}b$, then $\omega(x) = \pi r^{-1}(1 \otimes x)$ and it is clear that $I' = I$. $\square$

Similarly, right covariant first order calculi over A are in 1-1 correspondence with left ideals in A^+. Bicovariant calculi correspond to a compatible pair of ideals or to right ideals (say) with further properties.

Chapter 4

A bicrossproduct Hopf quasigroup

In Hopf algebra theory, a large class of noncommutative and noncocommutative examples is obtained through constructing bicrossproduct Hopf algebras. One approach is by considering a matched pair of groups [28].

Two groups (G, M) form a *right-left matched pair* if there is a right action $\triangleleft$ of G on M and a left action $\triangleright$ of M on G satisfying

$$e_M \triangleleft u = e_M, \quad (st)\triangleleft u = (s\triangleleft(t\triangleright u))(t\triangleleft u),$$

$$s\triangleright e_G = e_G, \quad s\triangleright(uv) = (s\triangleright u)((s\triangleleft u)\triangleright v),$$

for all $s, t \in M$ and $u, v \in G$, where e_M and e_G are the identity elements of M and G respectively.

If (G, M) is a right-left matched pair of groups, then the right action of G on M induces a left action of kG on $k(M)$, and the left action of M on G induces a right coaction of $k(M)$ on kG. This action and coaction gives a left-right bicrossproduct Hopf algebra $k(M)\blacktriangleright\!\!\triangleleft kG$. By a similar method we can obtain a right-left bicrossproduct $kM\triangleright\!\!\blacktriangleleft k(G)$.

In this chapter we follow a similar approach to construct bicrossproduct Hopf quasigroups with a group X factorized into a finite subgroup G and a transversal M containing the group identity. It is well-known that such a transversal M

acquires the structure of a quasigroup [35]. We discuss the matched pair approach to this data developed by Beggs in [3] and extend the analysis of M to when it is an inverse property quasigroup.

Finally, we obtain a bicrossproduct $kM{\triangleright\!\!\blacktriangleleft}k(G)$ as a semidirect product by the action of M and a semidirect coproduct by the action of G, and investigate under which conditions it attains the structure of a Hopf quasigroup.

Unless otherwise stated, the results in this chapter have been published with Majid in [20].

4.1 Group transversals

Let X be a group and $G \subset X$ be a subgroup. Let $M \subset X$ be a set of right coset representatives of G, that is for each $x \in X$ there exists a unique $s \in M$ such that $x \in Gs$. M is not a subgroup as in the matched pairs approach, but it is now a *transversal*. We will assume throughout that $e \in M$, where e is the identity element of X. We follow the approach to this data in [3].

Let $s, t \in M$ and $u \in G$; define a cocycle $\tau : M \times M \to G$ and a product $\cdot$ on M by the unique factorisation of $st \in X$ and $su \in X$ as follows,

$$st = \tau(s,t)(s \cdot t), \quad su = (s{\triangleright}u)(s{\triangleleft}u), \tag{4.1.1}$$

for some action $\triangleleft$ of G on M and some 'quasi-action' $\triangleright$ of M on G.

Proposition 4.1.1. *[3, Proposition 2.4] The following identities between $(M, \cdot)$, $\triangleright$, $\triangleleft$ and τ hold for all $s, t, r \in M$ and $u, v \in G$:*

$$\tau(s,t)\tau(s \cdot t, r) = (s{\triangleright}\tau(t,r))\tau(s{\triangleleft}\tau(t,r), t \cdot r), \tag{4.1.2}$$

$$s{\triangleleft}(uv) = (s{\triangleleft}u){\triangleleft}v, \tag{4.1.3}$$

$$(s \cdot t){\triangleleft}u = (s{\triangleleft}(t{\triangleright}u)) \cdot (t{\triangleleft}u), \tag{4.1.4}$$

$$(s \cdot t) \cdot r = (s{\triangleleft}\tau(t,r)) \cdot (t \cdot r), \tag{4.1.5}$$

$$\tau(s,t)((s \cdot t){\triangleright}u) = (s{\triangleright}(t{\triangleright}u))\tau(s{\triangleleft}(t{\triangleright}u), t{\triangleleft}u), \tag{4.1.6}$$

$$s{\triangleright}(uv) = (s{\triangleright}u)((s{\triangleleft}u){\triangleright}v), \tag{4.1.7}$$

$$\tau(s,e) = e = \tau(e,s), \quad s \cdot e = s = e \cdot s, \tag{4.1.8}$$

$$s\triangleright e = e, \quad s\triangleleft e = s, \quad e\triangleright u = u, \quad e\triangleleft u = e. \tag{4.1.9}$$

Proof. We will demonstrate how to obtain a couple of the identities, the full proof is in [3]. Since X is a group we have associativity in X, in particular, $(st)r = s(tr)$ for all $s,t,r \in M$. We can rewrite the LHS of this equality as

$$(st)r = \tau(s,t)(s \cdot t)r = \tau(s,t)\tau(s \cdot t, r)((s \cdot t) \cdot r),$$

by (4.1.1). Similarly, the RHS can be written as

$$s\tau(t,r)(t\cdot r) = (s\triangleright\tau(t,r))(s\triangleleft\tau(t,r))(t\cdot r) = (s\triangleright\tau(t,r))\tau(s\triangleleft\tau(t,r), t\cdot r)((s\triangleleft\tau(t,r))\cdot(t\cdot r)).$$

Equating the elements of G and the elements of X we obtain identities (4.1.2) and (4.1.5). $\square$

The set M is not a group, however the inverse of some elements with respect to the product in X may also lie in M. Let $s \in M$; if $s^{-1} \in X$ also lies in M then $e = s^{-1}s = \tau(s^{-1},s)s^{-1}\cdot s$ and we conclude $\tau(s^{-1},s) = e$ and $s^{-1}\cdot s = e$. Similarly, we find $\tau(s,s^{-1}) = e$ and $s \cdot s^{-1} = e$. However, since M is not a subgroup of X, in general we have $s^{-1} = ut$ for some $u \in G$ and $t \in M$:

Lemma 4.1.2. *For every $s \in M$ there is a unique $s^{-L} \in M$ such that $s^{-1} = \tau^{-1}(s^{-L},s)s^{-L}$, and satisfying*

$$s^{-L} \cdot s = e, \tag{4.1.10}$$

$$(s\triangleleft\tau^{-1}(s^{-L},s)) \cdot s^{-L} = e, \tag{4.1.11}$$

$$s\triangleright\tau^{-1}(s^{-L},s) = \tau^{-1}(s\triangleleft\tau^{-1}(s^{-L},s), s^{-L}). \tag{4.1.12}$$

Proof. We let $s^{-1} = \chi(s)s^{-}$ where $\chi(s) \in G$ and $s^{-} \in M$. Then

$$e = s^{-1}s = \chi(s)s^{-}s = \chi(s)\tau^{-1}(s^{-},s)s^{-} \cdot s.$$

We conclude $\chi(s) = \tau^{-1}(s^-, s)$ and $s^- \cdot s = e$. Since s^- is a left inverse of s with respect to the product in M, we shall label it s^{-L}. Now, using (4.1.1) we have

$$\begin{aligned} e = ss^{-1} &= s\tau^{-1}(s^{-L}, s)s^{-L} = (s{\triangleright}\tau^{-1}(s^{-L}, s))(s{\triangleleft}\tau^{-1}(s^{-L}, s))s^{-L} \\ &= (s{\triangleright}\tau^{-1}(s^{-L}, s))\tau(s{\triangleleft}\tau^{-1}(s^{-L}, s), s^{-L})(s{\triangleleft}\tau^{-1}(s^{-L}, s)) \cdot s^{-L}, \end{aligned}$$

which implies $s{\triangleright}\tau^{-1}(s^{-L}, s) = \tau^{-1}(s{\triangleleft}\tau^{-1}(s^{-L}, s), s^{-L})$ and $(s{\triangleleft}\tau^{-1}(s^{-L}, s)) \cdot s^{-L} = e$, as required. □

It is a familiar fact [35, Proposition 2.3] that the transversal M under our assumptions has the structure of a right quasigroup (with identity in our case, *i.e.* a right loop). Recall from Section 3.1, this means the existence of a right division map $/ : M \times M \to M$ such that $(t \cdot s)/s = t = (t/s) \cdot s$ for all $s, t \in M$ and means in particular that we have right cancellation: $t \cdot s = t' \cdot s \Rightarrow t = t'$. This is the content of Lemma 4.1.2 in terms of the matched pair data $\triangleright, \triangleleft, \tau$:

Proposition 4.1.3. *M in the setting of Proposition 4.1.1 is a right quasigroup (with identity). The division map is*

$$t/s = (t{\triangleleft}\tau^{-1}(s^{-L}, s)) \cdot s^{-L}.$$

Moreover,

$$(s^{-L})^{-L} = s{\triangleleft}\tau^{-1}(s^{-L}, s).$$

Proof. First we verify that the given form of the division map satisfies the right quasigroup identities. For all $s, t \in M$,

$$(t/s) \cdot s = ((t{\triangleleft}\tau^{-1}(s^{-L}, s)) \cdot s^{-L}) \cdot s = t \cdot (s^{-L} \cdot s) = t,$$

by (4.1.5) and (4.1.10). On the other side

$$\begin{aligned} (t \cdot s)/s &= ((t \cdot s){\triangleleft}\tau^{-1}(s^{-L}, s)) \cdot s^{-L} \\ &= ((t{\triangleleft}(s{\triangleright}\tau^{-1}(s^{-L}, s))) \cdot (s{\triangleleft}\tau^{-1}(s^{-L}, s))) \cdot s^{-L}, \quad \text{by (4.1.4)}, \\ &= ((t{\triangleleft}\tau^{-1}(s{\triangleleft}\tau^{-1}(s^{-L}, s), s^{-L})) \cdot (s{\triangleleft}\tau^{-1}(s^{-L}, s))) \cdot s^{-L}, \quad \text{by (4.1.12)}, \\ &= t \cdot ((s{\triangleleft}\tau^{-1}(s^{-L}, s)) \cdot s^{-L}), \quad \text{by (4.1.5)}, \end{aligned}$$

$$= t, \quad \text{by (4.1.11)}.$$

Once we have right cancellation, we know that s^{-L} is the unique left inverse for each $s \in M$, hence (4.1.11) implies the second part of the proposition. □

Using identities (4.1.7) and (4.1.4) respectively, we also obtain the following useful identities:

$$(t \triangleright v)^{-1} = (t \triangleleft v) \triangleright v^{-1}, \tag{4.1.13}$$

$$(t \triangleleft v)^{-L} = t^{-L} \triangleleft (t \triangleright v). \tag{4.1.14}$$

Finally, we will know that we have captured *all* of the input data of Proposition 4.1.1 if we can rebuild X from $\triangleright, \triangleleft$ and τ and the identities there. In the group factorization case this is the construction of a double cross product group $G \bowtie M$ from a matched pair data. Most of this is in [3].

Proposition 4.1.4. *Suppose that G is a group, $(M, \cdot)$ is a set with a binary operation, and the maps $\triangleright, \triangleleft$ and τ satisfy the identities* (4.1.2)- (4.1.7)*. Then the set $G \times M$ acquires the structure of a group $G \bowtie M$ by*

$$(u, s)(v, t) = (u(s \triangleright v)\tau(s \triangleleft v, t), (s \triangleleft v) \cdot t),$$

$$(u, s)^{-1} = (\tau^{-1}(s^{-L}, s)(s^{-L} \triangleright u^{-1}), s^{-L} \triangleleft u^{-1}),$$

for all $u, v \in G$ and $s, t \in M$, and identity (e, e).

Proof. The required form of the product is easily obtained from (4.1.1) and $(u, s)(v, t) = (u, e)(e, s)(v, e)(e, t)$ after which everything can be verified by direct computation. For $u, v, w \in G$ and $s, t, r \in M$,

$$\begin{aligned}
&((u, s)(v, t))(w, r) \\
&= (u(s \triangleright v)\tau(s \triangleleft v, t), (s \triangleleft v) \cdot t)(w, r) \\
&= (u(s \triangleright v)\tau(s \triangleleft v, t)(((s \triangleleft v) \cdot t) \triangleright w)\tau(((s \triangleleft v) \cdot t) \triangleleft w, r), (((s \triangleleft v) \cdot t) \triangleleft w) \cdot r) \\
&= (u(s \triangleright v)((s \triangleleft v) \triangleright (t \triangleright w))\tau((s \triangleleft v) \triangleleft (t \triangleright w), t \triangleleft w)\tau(((s \triangleleft v) \cdot t) \triangleleft w, r), \\
&\qquad\qquad (((s \triangleleft v(t \triangleright w)) \cdot (t \triangleleft w)) \cdot r) \quad \text{by (4.1.6) and (4.1.4)}, \\
&= (u(s \triangleright v(t \triangleright w))\tau(s \triangleleft v(t \triangleright w), t \triangleleft w)\tau((s \triangleleft v(t \triangleright w)) \cdot (t \triangleleft w), r), \\
&\qquad\qquad (s \triangleleft v(t \triangleright w)\tau(t \triangleleft w, r)) \cdot ((t \triangleleft w) \cdot r)) \quad \text{by (4.1.4), (4.1.7) and (4.1.5)},
\end{aligned}$$

$$
\begin{aligned}
&= (u(s{\triangleright}v(t{\triangleright}w))((s{\triangleleft}v(t{\triangleright}w)){\triangleright}\tau(t{\triangleleft}w,r))\tau(s{\triangleleft}v(t{\triangleright}w)\tau(t{\triangleleft}w,r),(t{\triangleleft}w)\cdot r),\\
&\qquad\qquad (s{\triangleleft}v(t{\triangleright}w)\tau(t{\triangleleft}w,r))\cdot((t{\triangleleft}w)\cdot r)) \quad \text{by (4.1.2)},\\
&= (u(s{\triangleright}v(t{\triangleright}w)\tau(t{\triangleleft}w,r))\tau(s{\triangleleft}v(t{\triangleright}w)\tau(t{\triangleleft}w,r),(t{\triangleleft}w)\cdot r),\\
&\qquad\qquad (s{\triangleleft}v(t{\triangleright}w)\tau(t{\triangleleft}w,r))\cdot((t{\triangleleft}w)\cdot r)) \quad \text{by (4.1.7)},\\
&= (u,s)(v(t{\triangleright}w)\tau(t{\triangleleft}w,r),(t{\triangleleft}w)\cdot r)\\
&= (u,s)((v,t)(w,r)).
\end{aligned}
$$

Hence the product in $G\times M$ is associative. That $(u,s)^{-1}$ is a left inverse is straightforward:

$$
\begin{aligned}
&(u,s)^{-1}(u,s)\\
&= (\tau^{-1}(s^{-L},s)(s^{-L}{\triangleright}u^{-1}),s^{-L}{\triangleleft}u^{-1})(u,s)\\
&= (\tau^{-1}(s^{-L},s)(s^{-L}{\triangleright}u^{-1})((s^{-L}{\triangleleft}u^{-1}){\triangleright}u)\tau((s^{-L}{\triangleleft}u^{-1}){\triangleleft}u,s),((s^{-L}{\triangleleft}u^{-1}){\triangleleft}u)\cdot s)\\
&= (\tau^{-1}(s^{-L},s)(s^{-L}{\triangleright}u^{-1})(s^{-L}{\triangleright}u^{-1})^{-1}\tau(s^{-L},s),s^{-L}\cdot s) \quad \text{by (4.1.13) and (4.1.3)},\\
&= (e,e).
\end{aligned}
$$

To verify that we have a right inverse, we compute

$$
\begin{aligned}
&(u,s)(\tau^{-1}(s^{-L},s)(s^{-L}{\triangleright}u^{-1}),s^{-L}{\triangleleft}u^{-1})\\
&\quad= (u(s{\triangleright}(\tau^{-1}(s^{-L},s)(s^{-L}{\triangleright}u^{-1})))\tau(s{\triangleleft}(\tau^{-1}(s^{-L},s)(s^{-L}{\triangleright}u^{-1})),s^{-L}{\triangleleft}u^{-1}),\\
&\qquad\qquad (s{\triangleleft}(\tau^{-1}(s^{-L},s)(s^{-L}{\triangleright}u^{-1})))\cdot(s^{-L}{\triangleleft}u^{-1}))\\
&\quad= (u(s{\triangleright}\tau^{-1}(s^{-L},s))((s{\triangleleft}\tau^{-1}(s^{-L},s)){\triangleright}(s^{-L}{\triangleright}u^{-1}))\\
&\qquad\quad \tau((s{\triangleleft}\tau^{-1}(s^{-L},s)){\triangleleft}(s^{-L}{\triangleright}u^{-1}),s^{-L}{\triangleleft}u^{-1}),\\
&\qquad\qquad ((s{\triangleleft}\tau^{-1}(s^{-L},s)){\triangleleft}(s^{-L}{\triangleright}u^{-1}))\cdot(s^{-L}{\triangleleft}u^{-1})), \quad \text{by (4.1.3) and (4.1.7)},\\
&\quad= (u(s{\triangleright}\tau^{-1}(s^{-L},s))\tau(s{\triangleleft}\tau^{-1}(s^{-L},s),s^{-L})(((s{\triangleleft}\tau^{-1}(s^{-L},s))\cdot s^{-L}){\triangleright}u^{-1}),\\
&\qquad\qquad ((s{\triangleleft}\tau^{-1}(s^{-L},s))\cdot s^{-L}){\triangleleft}u^{-1}), \quad \text{by (4.1.4) and (4.1.6)},\\
&\quad= (u(s{\triangleright}\tau^{-1}(s^{-L},s))\tau(s{\triangleleft}\tau^{-1}(s^{-L},s),s^{-L})(e{\triangleright}u^{-1}),e{\triangleleft}u^{-1}), \quad \text{by (4.1.11)},\\
&\quad= (u(s{\triangleright}\tau^{-1}(s^{-L},s))\tau(s{\triangleleft}\tau^{-1}(s^{-L},s),s^{-L})u^{-1},e), \quad \text{by (4.1.9)},\\
&\quad= (e,e).
\end{aligned}
$$

The final equality uses (4.1.12) from Lemma 4.1.2. □

The above proposition allows us to remove the dependence on the group X, since finding G and M with maps satisfying the conditions in Proposition 4.1.4, is equivalent to the existence of the group.

Continuing our analysis, depending on M we may also have a right inverse s^{-R} of s in M. If so we have the following compatibility relations

$$s^{-L}\triangleright\tau(s,s^{-R})=\tau(s^{-L},s),$$

$$s^{-L}\triangleleft\tau(s,s^{-R})=s^{-R},$$

which are obtained by considering $(s^{-L}\cdot s)\cdot s^{-R}=(s^{-L}\triangleleft\tau(s,s^{-R}))\cdot(s\cdot s^{-R})$ and similar identities. The two inverses will not generally coincide unless M is an inverse property quasigroup.

Proposition 4.1.5. *In the setting of Proposition 4.1.1, the following are equivalent*

(1) $\forall\ s\in M,\ s\triangleright G=G,$

(2) $\forall\ s\in M$ *there exists a right inverse* $s^{-R}\in M,$

(3) $X=MG.$

Proof. We will show that (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

Suppose $u\in G$, $s\in M$ and that (1) holds. We seek t,v such that $us=tv=(t\triangleright v)(t\triangleleft v)$. If we can find v we will have a unique $t=s\triangleleft v^{-1}$. It remains to find v solving $(s\triangleleft v^{-1})\triangleright v=u$. By (4.1.13) this is equivalent to $(s\triangleright v^{-1})^{-1}=u$, or $s\triangleright v^{-1}=u^{-1}$. Such v exists under our assumption, hence (3) holds. We see also that if $s\triangleright(\ )$ is bijective then the factorisation as MG is unique.

Suppose (3) holds, then for $s\in M$, $s^{-1}=s^{-}\psi(s)$ for some $s^{-}\in M$ and $\psi(s)\in G$. By (4.1.1),

$$e=ss^{-1}=ss^{-}\psi(s)=\tau(s,s^{-})(s\cdot s^{-})\psi(s)=\tau(s,s^{-})((s\cdot s^{-})\triangleright\psi(s))((s\cdot s^{-})\triangleleft\psi(s)).$$

We conclude $(s\cdot s^{-})\triangleleft\psi(s)=e$, which implies $s\cdot s^{-}=e$, hence right inverses s^{-R} exist, and (2) holds. We also learn that $\psi(s)=\tau^{-1}(s,s^{-R})$. Note that if the factorisation as MG is unique we can write $st=s\circ t\sigma(s,t)\in MG$ as a left-right reversal of our previous theory. Then $ss^{-R} = \tau(s, s^{-R})s \cdot s^{-R} = \tau(s, s^{-R})$, but also

$$ss^{-R} = s \circ s^{-R}\sigma(s, s^{-R}) = ((s \circ s^{-R}){\triangleright}\sigma(s, s^{-R}))((s \circ s^{-R}){\triangleleft}\sigma(s, s^{-R})),$$

from which we see that $e = (s \circ s^{-R}){\triangleleft}\sigma(s, s^{-R})$ and hence $s \circ s^{-R} = e$. But as $(M, \circ)$ has left cancellation, s^{-R} is uniquely determined.

Finally, suppose (2) holds and let $s \in M, u \in G$. If there exists $v \in G$ such that $s{\triangleright}v = u$ then

$$\begin{aligned} v &= (s^{-L} \cdot s){\triangleright}v \\ &= \tau^{-1}(s^{-L}, s)(s^{-L}{\triangleright}(s{\triangleright}v))\tau(s^{-L}{\triangleleft}(s{\triangleright}v), s{\triangleleft}v), \quad \text{by (4.1.6)}, \\ &= \tau^{-1}(s^{-L}, s)(s^{-L}{\triangleright}u)\tau(s^{-L}{\triangleleft}u, s{\triangleleft}v), \end{aligned}$$

where the final equality substitutes in $s{\triangleright}v = u$. By (4.1.14), $(s{\triangleleft}v)^{-L} = s^{-L}{\triangleleft}(s{\triangleright}v) = s^{-L}{\triangleleft}u$, which can be solved by assumption as $s{\triangleleft}v = (s^{-L}{\triangleleft}u)^{-R}$. Accordingly, we take

$$v = \tau^{-1}(s^{-L}, s)(s^{-L}{\triangleright}u)\tau(s^{-L}{\triangleleft}u, (s^{-L}{\triangleleft}u)^{-R}), \tag{4.1.15}$$

as a definition and verify,

$$\begin{aligned} & s{\triangleright}(\tau^{-1}(s^{-L}, s)(s^{-L}{\triangleright}u)\tau(s^{-L}{\triangleleft}u, (s^{-L}{\triangleleft}u)^{-R})) \\ &= (s{\triangleright}\tau^{-1}(s^{-L}, s))((s{\triangleleft}\tau^{-1}(s^{-L}, s)){\triangleright}((s^{-L}{\triangleright}u)\tau(s^{-L}{\triangleleft}u, (s^{-L}{\triangleleft}u)^{-R})) \\ &= (s{\triangleright}\tau^{-1}(s^{-L}, s))((s{\triangleleft}\tau^{-1}(s^{-L}, s)){\triangleright}(s^{-L}{\triangleright}u)) \\ &\qquad (((s{\triangleleft}\tau^{-1}(s^{-L}, s)){\triangleleft}(s^{-L}{\triangleright}u)){\triangleright}\tau(s^{-L}{\triangleleft}u, (s^{-L}{\triangleleft}u)^{-R})) \\ &= (s{\triangleright}\tau^{-1}(s^{-L}, s))\tau(s{\triangleleft}\tau^{-1}(s^{-L}, s), s^{-L})(((s{\triangleleft}\tau^{-1}(s^{-L}, s)) \cdot s^{-L}){\triangleright}u) \\ &\qquad \tau^{-1}((s{\triangleleft}\tau^{-1}(s^{-L}, s)){\triangleleft}(s^{-L}{\triangleright}u), s^{-L}{\triangleleft}u) \\ &\qquad\quad (((s{\triangleleft}\tau^{-1}(s^{-L}, s)){\triangleleft}(s^{-L}{\triangleright}u)){\triangleright}\tau(s^{-L}{\triangleleft}u, (s^{-L}{\triangleleft}u)^{-R})) \\ &= (e{\triangleright}u)\tau^{-1}((s{\triangleleft}\tau^{-1}(s^{-L}, s)){\triangleleft}(s^{-L}{\triangleright}u), s^{-L}{\triangleleft}u) \\ &\qquad\quad (((s{\triangleleft}\tau^{-1}(s^{-L}, s)){\triangleleft}(s^{-L}{\triangleright}u)){\triangleright}\tau(s^{-L}{\triangleleft}u, (s^{-L}{\triangleleft}u)^{-R})) \\ &= u\tau(((s{\triangleleft}\tau^{-1}(s^{-L}, s)){\triangleleft}(s^{-L}{\triangleright}u)) \cdot (s^{-L}{\triangleleft}u), (s^{-L}{\triangleleft}u)^{-R}) \\ &\qquad \tau^{-1}(((s{\triangleleft}\tau^{-1}(s^{-L}, s)){\triangleleft}(s^{-L}{\triangleright}u)){\triangleleft}\tau(s^{-L}{\triangleleft}u, (s^{-L}{\triangleleft}u)^{-R}), (s^{-L}{\triangleleft}u) \cdot (s^{-L}{\triangleleft}u)^{-R}) \\ &= u\tau(((s{\triangleleft}\tau^{-1}(s^{-L}, s)) \cdot s^{-L}){\triangleleft}u, (s^{-L}{\triangleleft}u)^{-R}) \end{aligned}$$

$$
\begin{aligned}
&\tau^{-1}(((s{\triangleleft}\tau^{-1}(s^{-L},s)){\triangleleft}(s^{-L}{\triangleright}u)){\triangleleft}\tau(s^{-L}{\triangleleft}u,(s^{-L}{\triangleleft}u)^{-R}),e)\\
&=u\tau(e{\triangleleft}u,(s^{-L}{\triangleleft}u)^{-R})\\
&=u,
\end{aligned}
$$

as required. Here the first and second equalities use (4.1.7), the third uses (4.1.6). The fourth equality uses (4.1.11) and (4.1.12), the fifth uses (4.1.2), the sixth uses (4.1.4), and the seventh uses (4.1.11). We also see that if the right inverse is unique then so is v because $s{\triangleleft}v$ is, and hence v then necessarily has the form given. □

Remark 4.1.6. Note that if G is finite then $s{\triangleright}(\ )$ in condition (1) will be bijective. We have noted in the proof that this is equivalent to the factorisation in (3) being unique and to the the right inverse in (2) being unique.

Proposition 4.1.7. *M in the setting of Proposition 4.1.1 is a (two-sided inverse property) quasigroup* iff

$$t=t{\triangleleft}\tau(s^{-L},s),$$

$$s^{-L}=s^{-L}{\triangleleft}\tau(s,t),$$

for all $s,t\in M$. In this case $(s^{-L})^{-L}=s$, i.e. $s^{-L}=s^{-R}$.

Proof. Suppose M is an inverse property quasigroup, then for all $s,t\in M$,

$$t=(t\cdot s^{-L})\cdot s=(t{\triangleleft}\tau(s^{-L},s))\cdot(s^{-L}\cdot s)=(t{\triangleleft}\tau(s^{-L},s))\cdot e=t{\triangleleft}\tau(s^{-L},s).$$

Similarly, using (4.1.5) we have $t=s^{-L}\cdot(s\cdot t)=((s^{-L}{\triangleleft}\tau^{-1}(s,t))\cdot s)\cdot t$ and can conclude that $s^{-L}{\triangleleft}\tau(s,t)=s^{-L}$. Setting $t=s^{-R}$ in $t=s^{-L}\cdot(s\cdot t)$ confirms that $s^{-L}=s^{-R}$ as it must in an inverse property quasigroup.

Conversely, suppose $t=t{\triangleleft}\tau(s^{-L},s)$ then from Proposition 4.1.3 we see that $t/s=t\cdot s^{-L}$ and hence $t=(t\cdot s)/s=(t\cdot s)\cdot s^{-L}$. For the other side if we suppose that $s^{-L}=s^{-L}{\triangleleft}\tau(s,t)$ then $s^{-L}\cdot(s\cdot t)=(s^{-L}{\triangleleft}\tau(s,t))\cdot(s\cdot t)=(s^{-L}\cdot s)\cdot t=t$ using (4.1.5). □

When M is an inverse property quasigroup, we shall continue to denote the (left and right) inverse of $s\in M$ by s^{-L} to distinguish it from $s^{-1}\in X$. We shall need two further elementary lemmas.

Lemma 4.1.8. *In the setting of Proposition 4.1.1,*

$$(s\triangleleft u)^{-L}\triangleright(s\triangleright u)^{-1} = u^{-1},$$

holds for $s\in M$, $u\in G$ iff $u^{-1}\tau(s^{-L},s)u = \tau(s^{-L}\triangleleft(s\triangleright u), s\triangleleft u)$.

Proof.

$$\begin{aligned}(s\triangleleft u)^{-L}\triangleright(s\triangleright u)^{-1} &= (s^{-L}\triangleleft(s\triangleright u))\triangleright(s\triangleright u)^{-1}\\ &= (s^{-L}\triangleright(s\triangleright u))^{-1}\\ &= \tau(s^{-L}\triangleleft(s\triangleright u), s\triangleleft u)u^{-1}\tau^{-1}(s^{-L},s),\end{aligned}$$

using (4.1.14), (4.1.13) and (4.1.6), respectively. So equality of this to u^{-1} is precisely the stated condition for τ. □

Lemma 4.1.9. *In the setting of Proposition 4.1.1, we have*

$$(s\triangleleft u)^{-L}\triangleleft(s\triangleright u)^{-1} = s^{-L},$$

for all $s\in M$ and $u\in G$.

Proof. Using (4.1.13) and (4.1.14),

$$(s\triangleleft u)^{-L}\triangleleft(s\triangleright u)^{-1} = (s^{-L}\triangleleft(s\triangleright u))\triangleleft(s\triangleright u)^{-1} = s^{-L}\triangleleft((s\triangleright u)(s\triangleright u)^{-1}) = s^{-L},$$

where we used (4.1.3) to arrive at the second equality. □

In summary, we have dissected the usual coset construction in terms of properties of the matched pair data $\triangleright,\triangleleft,\tau$. We will now use this data to construct something different.

4.2 Bicrossproduct $kM{\triangleright\!\blacktriangleleft}k(G)$

With M and G as in Proposition 4.1.1 and G finite, we consider the bicrossproduct $kM{\triangleright\!\blacktriangleleft}k(G)$. By kM we mean the vector space on M with its product extended linearly and a group-like coproduct on basis elements. By $k(G)$ we mean the functions on G with usual pointwise multiplication and standard coproduct. From the 'quasi-action' $\triangleright$ in our matched pair data we have a right action of kM on $k(G)$ and make a cross product algebra. From the action $\triangleleft$ in our data we have a left coaction of $k(G)$ on kM and make a cross coproduct coalgebra. This gives $kM{\triangleright\!\!\blacktriangleleft}k(G)$ built on $kM\otimes k(G)$, with exactly the same formulae as for Hopf algebra bicrossproducts. Following the conventions of [28] we take a basis $\{s\otimes\delta_u\,|\,s\in M, u\in G\}$ and the algebra and coalgebra structure are explicitly

$$(s\otimes\delta_u)(t\otimes\delta_v) = s\cdot t\otimes\delta_v\,\delta_{u,t\triangleright v},$$

$$1 = \sum_u e\otimes\delta_u,$$

$$\Delta(s\otimes\delta_u) = \sum_{ab=u}(s\otimes\delta_a)\otimes(s\triangleleft a\otimes\delta_b),$$

$$\varepsilon(s\otimes\delta_u) = \delta_{u,e}.$$

Notice that since $\triangleleft$ is an actual group action it induces an actual coaction of $k(G)$ and Δ is therefore coassociative as a standard cross coproduct on the coalgebra of kM. The proof is identical to the proof [28] for ordinary bicrossproducts; it is not affected by M being nonassociative. Likewise when checking the homomorphism property $\Delta((s\otimes\delta_u)\cdot(t\otimes\delta_v)) = \Delta(s\otimes\delta_u)\cdot\Delta(t\otimes\delta_v)$ we only ever encounter in each tensor factor one product in M and one application of $\triangleleft$, so we never use any of the identities in Proposition 4.1.1 involving τ. All the others are identical to those in the matched pair conditions for ordinary bicrossproducts, so the proof is again line by line identical. Hence the only issue is the antipode.

Lemma 4.2.1. *Suppose M in Proposition 4.1.1 has right inverses and G is finite. The antipode of the bicrossproduct $kM{\triangleright\!\!\blacktriangleleft}k(G)$, if it is a Hopf quasigroup, necessarily takes the form*

$$S(s\otimes\delta_u) = (s\triangleleft u)^{-L}\otimes\delta_{(s\triangleright u)^{-1}},$$

for all $s\in M$ and $u\in G$.

Proof. Consider $S(s\otimes\delta_u) = \sum_t t\otimes f_t^{s,u}$ for some functions with $f_t^{s,u}\in k(G)$. Then considering the usual antipode property alone (set $g=1$ in the explicit formulae (3.3.1) in the definition of a Hopf quasigroup) applied to $h = s \otimes \delta_v$ we have

$$\begin{aligned} e \otimes 1\, \delta_{v,e} &= (S(s \otimes \delta_v)_{(1)})(s \otimes \delta_v)_{(2)} \\ &= \sum_a S(s \otimes \delta_a)(s{\triangleleft}a \otimes \delta_{a^{-1}v}) \\ &= \sum_{a,t} (t \otimes f_t^{s,a})(s{\triangleleft}a \otimes \delta_{a^{-1}v}) \\ &= \sum_{a,t} t \cdot (s{\triangleleft}a) \otimes \delta_{a^{-1}v}\, f_t^{s,a}((s{\triangleleft}a){\triangleright}(a^{-1}v)) \\ &= \sum_{a,t} t \cdot (s{\triangleleft}a) \otimes \delta_{a^{-1}v}\, f_t^{s,a}((s{\triangleright}a)^{-1}(s{\triangleright}v)), \end{aligned}$$

using the definitions, and (4.1.7) in the last step. We multiply both sides from the right by $(e \otimes \delta_{u^{-1}v})$ which picks out $a = u$ in the sum. Comparing results, we see that

$$\delta_{v,e}\, e = \sum_t t \cdot (s{\triangleleft}u)\, f_t^{s,u}((s{\triangleright}u)^{-1}(s{\triangleright}v)).$$

By right cancellation in M the basis elements appearing on the right are all distinct.

We look first at $t = (s{\triangleleft}u)^{-L}$ which gives e. Writing $f = f_t^{s,u}$ for brevity and $L_{s\triangleright u}(f) = f((s{\triangleright}u)^{-1}(\ ))$, we have,

$$L_{s\triangleright u}(f)(s{\triangleright}v) = \delta_e(v) = \delta_e(s{\triangleright}v),$$

where the last equality is because $v = e \Leftrightarrow s{\triangleright}v = e$. To see this, if $v = e$ then $s{\triangleright}v = e$ by (4.1.8). Conversely, if $s{\triangleright}v = e$ then from (4.1.15) we have $v = \tau^{-1}(s^{-L}, s)(s^{-L}{\triangleright}e)\tau(s^{-L}{\triangleleft}e, (s^{-L}{\triangleleft}e)^{-R}) = \tau^{-1}(s^{-L}, s)\tau(s^{-L}, s) = e$. Returning to our displayed equation, we see that $L_{s\triangleright u}(f) = \delta_e$ on $s{\triangleright}G$. But $s{\triangleright}G = G$ by Proposition 4.1.5 hence $f = \delta_{(s\triangleright u)^{-1}}$.

Looking now at $t \neq (s{\triangleleft}u)^{-L}$, each element of M in our sum occurs just once on the right and not at all on the left. Hence for these $f = f_t^{s,u}$ we have

$$L_{s\triangleright u}(f)(s{\triangleright}(\ )) = 0,$$

and as $s{\triangleright}G = G$ we conclude that $L_{s\triangleright u}(f) = 0$ and hence $f = 0$. $\square$

The following applies to the form of S in Lemma 4.2.1 which one can also arrive at from other considerations, for example by requiring that S is basis preserving. We continue to state it under the given assumption.

Theorem 4.2.2. *Suppose M in Proposition 4.1.1 has right-inverses and G is finite. The bicrossproduct $kM{\triangleright\!\!\blacktriangleleft}k(G)$ is a Hopf quasigroup iff M is a (two sided inverse property) quasigroup, and*

$$\tau(s{\triangleleft}(t{\triangleright}u), t{\triangleleft}u) = (s{\triangleright}(t{\triangleright}u))^{-1}\tau(s,t)(s{\triangleright}(t{\triangleright}u)), \tag{4.2.1}$$

holds for all $s, t \in M$ and $u \in G$. In this case $S^2 = \mathrm{id}$.

Proof. We suppose $kM{\triangleright\!\!\blacktriangleleft}k(G)$ is a Hopf quasigroup so S has the form found in Lemma 4.2.1. From one of the Hopf quasigroup identities we know that for all $s, t \in M$,

$$\begin{aligned} t\otimes\delta_e &= S((s\otimes\delta_e)_{(1)})((s\otimes\delta_e)_{(2)}(t\otimes\delta_e)) \\ &= \sum_a S(s\otimes\delta_a)((s{\triangleleft}a\otimes\delta_{a^{-1}})(t\otimes\delta_e)) \\ &= \sum_a((s{\triangleleft}a)^{-L}\otimes\delta_{(s{\triangleright}a)^{-1}})((s{\triangleleft}a)\cdot t\otimes\delta_e)\,\delta_{a^{-1},t{\triangleright}e} \\ &= (s^{-L}\otimes\delta_e)(s\cdot t\otimes\delta_e) \\ &= s^{-L}\cdot(s\cdot t)\otimes\delta_e, \end{aligned}$$

where only a such that $a^{-1} = t{\triangleright}e = e$, *i.e.* $a = e$ contributes in the sum. So we find that for all $s, t \in M$, $s^{-L}\cdot(s\cdot t) = t$. Similarly on the other side, we have

$$\begin{aligned} t\otimes\delta_e &= ((t\otimes\delta_e)(s\otimes\delta_e)_{(1)})S((s\otimes\delta_e)_{(2)}) \\ &= \sum_a (t\cdot s)\cdot s^{-L}\otimes\delta_{s{\triangleright}a}\,\delta_{e,s{\triangleright}a}\,\delta_{a,s^{-L}{\triangleright}(s{\triangleright}a)} \\ &= (t\cdot s)\cdot s^{-L}\otimes\delta_e \end{aligned}$$

where we find the only a that contributes is $a = e$ in view of the delta function $\delta_{e,s{\triangleright}a}$. So we see that $(t\cdot s)\cdot s^{-L} = t$ for all $s, t \in M$. Hence M is an (inverse property) quasigroup.

As this is necessary, we will now suppose that M is a quasigroup for the rest of the proof and show that we have a Hopf quasigroup iff the remaining condition (4.2.1) holds. Let us see first that it is necessary. We look at one of the Hopf quasigroup identities

$$\begin{aligned} t\otimes\delta_v &= (s\otimes\delta_e)_{(1)}(S((s\otimes\delta_e)_{(2)})(t\otimes\delta_v)) \\ &= \sum_a (s\otimes\delta_a)(S(s\triangleleft a\otimes\delta_{a^{-1}})(t\otimes\delta_v)) \\ &= \sum_a (s\otimes\delta_a)((s^{-L}\otimes\delta_{s\triangleright a})(t\otimes\delta_v)) \\ &= s\cdot(s^{-L}\cdot t)\otimes\delta_v \sum_a \delta_{s\triangleright a,t\triangleright v}\,\delta_{a,(s^{-L}\cdot t)\triangleright v} \\ &= s\cdot(s^{-L}\cdot t)\otimes\delta_v\,\delta_{s\triangleright((s^{-L}\cdot t)\triangleright v),t\triangleright v} \end{aligned}$$

where only $a=(s^{-L}\cdot t)\triangleright v$ contributes in the sum. We have already dealt with the first tensor factors (M is a quasigroup) and we conclude further that $s\triangleright((s^{-L}\cdot t)\triangleright v)=t\triangleright v$ for all $s,t\in M$ and $v\in G$. By changing variables we conclude

$$(s\cdot t)\triangleright u = s\triangleright(t\triangleright u), \tag{4.2.2}$$

for all $s,t\in M$ and $u\in G$. Comparing with (4.1.6) this is equivalent to the condition stated. It is straightforward to see that it is also equivalent to

$$\tau(s\triangleleft(t\triangleright u),t\triangleleft u) = ((s\cdot t)\triangleright u)^{-1}\tau(s,t)((s\cdot t)\triangleright u), \tag{4.2.3}$$

for all $s,t\in M$ and $u\in G$.

Now suppose that this condition holds and that M is a quasigroup. From a special case of (4.2.3) we see that the condition in Lemma 4.1.8 applies for all elements. It remains to verify all of the Hopf quasigroup identities. Thus,

$$\begin{aligned} &((t\otimes\delta_v)(s\otimes\delta_u)_{(1)})S((s\otimes\delta_u)_{(2)}) \\ &\quad= \sum_a ((t\otimes\delta_v)(s\otimes\delta_a))S(s\triangleleft a\otimes\delta_{a^{-1}u}) \\ &\quad= \sum_a (t\cdot s\otimes\delta_a)((s\triangleleft u)^{-L}\otimes\delta_{((s\triangleleft a)\triangleright(a^{-1}u))^{-1}})\,\delta_{v,s\triangleright a} \\ &\quad= \sum_a (t\cdot s\otimes\delta_a)((s\triangleleft u)^{-L}\otimes\delta_{(s\triangleright u)^{-1}(s\triangleright a)})\,\delta_{v,s\triangleright a} \\ &\quad= \sum_a (t\cdot s)\cdot(s\triangleleft u)^{-L}\otimes\delta_{(s\triangleright u)^{-1}(s\triangleright a)}\,\delta_{v,s\triangleright a}\,\delta_{a,(s\triangleleft u)^{-L}\triangleright((s\triangleright u)^{-1}(s\triangleright a))} \\ &\quad= \sum_a (t\cdot s)\cdot(s\triangleleft u)^{-L}\otimes\delta_{(s\triangleright u)^{-1}v}\,\delta_{v,s\triangleright a}\,\delta_{a,(s\triangleleft u)^{-L}\triangleright((s\triangleright u)^{-1}v)} \end{aligned}$$

$$= \sum_a (t \cdot s) \cdot (s \triangleleft u)^{-L} \otimes \delta_{(s \triangleright u)^{-1} v}\, \delta_{v, s \triangleright a}\, \delta_{a, ((s \triangleleft u)^{-L} \triangleright (s \triangleright u)^{-1})(((s \triangleleft u)^{-L} \triangleleft (s \triangleright u)^{-1}) \triangleright v)}$$
$$= \sum_a (t \cdot s) \cdot (s \triangleleft u)^{-L} \otimes \delta_{(s \triangleright u)^{-1} v}\, \delta_{v, s \triangleright a}\, \delta_{a, u^{-1}(s^{-L} \triangleright v)}$$
$$= (t \cdot s) \cdot (s \triangleleft u)^{-L} \otimes \delta_{(s \triangleright u)^{-1} v}\, \delta_{v, s \triangleright (u^{-1}(s^{-L} \triangleright v))},$$

where we used the definitions and (4.1.7), (4.1.13) to arrive at the third equality. We then compute the remaining algebra product and replace $s \triangleright a$ by v in some of the expressions in view of $\delta_{v,s \triangleright a}$ to arrive at the 5th equality. Next, we use Lemma 4.1.8 in view of our assumption, and Lemma 4.1.9 and simplify to arrive at the 7th equality. We now see that only one value of a contributes in the sum to arrive at the final expression. From (4.2.1) or rather the equivalent statement (4.2.2), we see that

$$v = s \triangleright (u^{-1}(s^{-L} \triangleright v)) \Leftrightarrow\ s^{-L} \triangleright v = s^{-L} \triangleright (s \triangleright (u^{-1}(s^{-L} \triangleright v))) = u^{-1}(s^{-L} \triangleright v) \Leftrightarrow u = e.$$

We know here that $s^{-L} \triangleright (\ )$ is bijective by Remark 4.1.6, hence we obtain that our original expression

$$= (t \cdot s) \cdot (s \triangleleft u)^{-L} \otimes \delta_{(s \triangleright u)^{-1} v}\, \delta_{u,e} = (t \cdot s) \cdot s^{-L} \otimes \delta_v\, \delta_{u,e} = t \otimes \delta_v\, \delta_{u,e},$$

as required.

Similarly,

$$(s \otimes \delta_u)_{(1)}(S((s \otimes \delta_u)_{(2)})(t \otimes \delta_v))$$
$$= \sum_a (s \otimes \delta_a)(S(s \triangleleft a \otimes \delta_{a^{-1}u})(t \otimes \delta_v))$$
$$= \sum_a (s \otimes \delta_a)(((s \triangleleft u)^{-L} \otimes \delta_{(s \triangleright u)^{-1}(s \triangleright a)})(t \otimes \delta_v))$$
$$= \sum_a (s \otimes \delta_a)((s \triangleleft u)^{-L} \cdot t \otimes \delta_v)\, \delta_{(s \triangleright u)^{-1}(s \triangleright a), t \triangleright v}$$
$$= \sum_a s \cdot ((s \triangleleft u)^{-L} \cdot t) \otimes \delta_v\, \delta_{(s \triangleright u)^{-1}(s \triangleright a), t \triangleright v}\, \delta_{a, ((s \triangleleft u)^{-L} \cdot t) \triangleright v}$$
$$= s \cdot ((s \triangleleft u)^{-L} \cdot t) \otimes \delta_v\, \delta_{s \triangleright (((s \triangleleft u)^{-L} \cdot t) \triangleright v), (s \triangleright u)(t \triangleright v)},$$

by similar computations for the antipode and multiplying out the products. Here only $a = ((s \triangleleft u)^{-L} \cdot t) \triangleright v$ contributes in the sum and we rearrange the final delta-function accordingly. But

$$
\begin{aligned}
&(s\triangleright(((s\triangleleft u)^{-L}\cdot t)\triangleright v)) = (s\triangleright u)(t\triangleright v)\\
&\qquad\Leftrightarrow ((s\triangleleft u)^{-L}\cdot t)\triangleright v = s^{-L}\triangleright((s\triangleright u)(t\triangleright v))\\
&\qquad\Leftrightarrow ((s\triangleleft u)^{-L}\cdot t)\triangleright v = (s^{-L}\triangleright(s\triangleright u))(((s^{-L}\triangleleft(s\triangleright u))\triangleright(t\triangleright v))\quad \text{by (4.1.7)},\\
&\qquad\Leftrightarrow ((s\triangleleft u)^{-L}\cdot t)\triangleright v = u(((s\triangleleft u)^{-L}\cdot t)\triangleright v)\quad\Leftrightarrow\quad u=e,
\end{aligned}
$$

using bijectivity of $s^{-L}\triangleright(\ )$, the identity (4.2.2) equivalent to our assumed condition, and (4.1.14). Hence our original expression

$$
= s\cdot((s\triangleleft u)^{-L}\cdot t)\otimes\delta_v\,\delta_{u,e} = s\cdot(s^{-L}\cdot t)\otimes\delta_v\,\delta_{u,e} = t\otimes\delta_v\,\delta_{u,e},
$$

as required.

We similarly compute

$$
\begin{aligned}
&S((s\otimes\delta_u)_{(1)})((s\otimes\delta_u)_{(2)}(t\otimes\delta_v))\\
&\qquad\qquad = \sum_a S(s\otimes\delta_a)((s\triangleleft a\otimes\delta_{a^{-1}u})(t\otimes\delta_v))\\
&\qquad\qquad = \sum_a ((s\triangleleft a)^{-L}\otimes\delta_{(s\triangleright a)^{-1}})((s\triangleleft a)\cdot t\otimes\delta_v)\,\delta_{a^{-1}u,t\triangleright v}\\
&\qquad\qquad = \sum_a (s\triangleleft a)^{-L}\cdot((s\triangleleft a)\cdot t)\otimes\delta_v\,\delta_{a^{-1}u,t\triangleright v}\,\delta_{(s\triangleright a)^{-1},((s\triangleleft a)\cdot t)\triangleright v}\\
&\qquad\qquad = \sum_a t\otimes\delta_v\,\delta_{a^{-1}u,t\triangleright v}\,\delta_{(s\triangleright a)^{-1},((s\triangleleft a)\cdot t)\triangleright v}.
\end{aligned}
$$

The final delta function gives terms when $(s\triangleleft a)^{-1} = ((s\triangleleft a)\cdot t)\triangleright v$, which by (4.1.13) and our assumption (4.2.2) is equivalent to $a^{-1} = t\triangleright v$. Hence there is only one value of a in the sum, and our original expression

$$
= t\otimes\delta_v\,\delta_{(t\triangleright v)u,t\triangleright v} = t\otimes\delta_v\,\delta_{u,e},
$$

as required. Finally,

$$
\begin{aligned}
&((t\otimes\delta_v)S((s\otimes\delta_u)_{(1)}))(s\otimes\delta_u)_{(2)}\\
&\qquad = \sum_a ((t\otimes\delta_v)S(s\otimes\delta_a))(s\triangleleft a\otimes\delta_{a^{-1}u})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_a ((t \otimes \delta_v)((s{\triangleleft}a)^{-L} \otimes \delta_{(s{\triangleright}a)^{-1}}))(s{\triangleleft}a \otimes \delta_{a^{-1}u}) \\
&= \sum_a (t \cdot (s{\triangleleft}a)^{-L} \otimes \delta_{(s{\triangleright}a)^{-1}})(s{\triangleleft}a \otimes \delta_{a^{-1}u})\, \delta_{v,(s{\triangleleft}a)^{-L}{\triangleright}(s{\triangleright}a)^{-1}} \\
&= \sum_a (t \cdot (s{\triangleleft}a)^{-L}) \cdot (s{\triangleleft}a) \otimes \delta_{a^{-1}u}\, \delta_{v,(s{\triangleleft}a)^{-L}{\triangleright}(s{\triangleright}a)^{-1}}\, \delta_{(s{\triangleright}a)^{-1},(s{\triangleleft}a){\triangleright}(a^{-1}u)} \\
&= \sum_a t \otimes \delta_{a^{-1}u}\, \delta_{v,a^{-1}}\, \delta_{(s{\triangleright}a)^{-1},(s{\triangleleft}a){\triangleright}(a^{-1}u)} \\
&= t \otimes \delta_{vu}\, \delta_{s{\triangleright}v^{-1},(s{\triangleright}u)^{-1}(s{\triangleright}v^{-1})} \;=\; t \otimes \delta_{vu}\, \delta_{s{\triangleright}u,e} \;=\; t \otimes \delta_v\, \delta_{u,e},
\end{aligned}
$$

where only $a = v^{-1}$ contributes and we used again that $s{\triangleright}(\ )$ is bijective.

For the computation of S^2 we use Lemma 4.1.8, Lemma 4.1.9 and Proposition 4.1.7 to get

$$
\begin{aligned}
S^2(s \otimes \delta_u) &= S((s{\triangleleft}u)^{-L} \otimes \delta_{(s{\triangleright}u)^{-1}}) \\
&= ((s{\triangleleft}u)^{-L}{\triangleleft}(s{\triangleright}u)^{-1})^{-L} \otimes \delta_{((s{\triangleleft}u)^{-L}{\triangleright}(s{\triangleright}u)^{-1})^{-1}} = (s^{-L})^{-L} \otimes \delta_u = s \otimes \delta_u.
\end{aligned}
$$

□

Remark 4.2.3. We remark that in [3, 39] the authors constructed a monoidal category of M-graded G-modules from the data in Proposition 4.1.1 and the existence of right inverses. In this case, where there is an obvious multiplicative functor to vectors spaces, one knows by Tannaka-Krein reconstruction that there is a Drinfeld quasi-Hopf algebra [13] generating this category as its modules. This was given in [39] and a close inspection shows that it has a bicrossproduct form. In our dual formulation this category is that of comodules of the bicrossproduct $kM{\triangleright\!\!\blacktriangleleft}k(G)$ with S as in Lemma 4.2.1 and a Drinfeld Hopf 3-cocycle

$$\phi(s \otimes \delta_u \otimes t \otimes \delta_v \otimes r \otimes \delta_w) = \delta_{u,\tau^{-1}(t,r)}\delta_{v,e}\delta_{w,e},$$

$$\phi^{-1}(s \otimes \delta_u \otimes t \otimes \delta_v \otimes r \otimes \delta_w) = \delta_{u,\tau(t,r)}\delta_{v,e}\delta_{w,e},$$

making it into a coquasi-Hopf algebra. Here the algebra product is associative up to conjugation by ϕ in a convolution sense:

$$\sum \phi(h_{(1)}, g_{(1)}, f_{(1)})(h_{(2)}g_{(2)})f_{(2)} = \sum h_{(1)}(g_{(1)}f_{(1)})\phi(h_{(2)}, g_{(2)}, f_{(2)}),$$

for all h, g, f and ϕ is invertible in the same convolution sense. The coquasi-Hopf structure in our case is easily verified by direct computation and does not require the two further conditions in Theorem 4.2.2, *i.e.* the bicrossproduct being a Hopf quasigroup is strictly stronger.

Incidentally, this remark means that there is a coquasi-Hopf algebra and monoidal category associated to any any finite left quasigroup M with right inverses as every such inverse property quasigroup can be expressed as a coset construction [35]. This will be discussed in more detail in Chapter 6.

It is not clear if our additional requirements for a Hopf quasigroup can ever be satisfied, starting now from M an inverse property quasigroup. One can show that the following special case can.

Corollary 4.2.4. *We obtain a bicrossproduct Hopf quasigroup in the context of Theorem 4.2.2 if M is an (inverse property) quasigroup and*

$$\tau(s, t{\triangleleft}u) = \tau(s, t), \quad \tau(s{\triangleleft}(s^{-L}{\triangleright}u), t) = u^{-1}\tau(s, t)u,$$

for all $s, t \in M$ and $u \in G$.

Proof. If we suppose that the first of these conditions holds then the requirement (4.2.1) on τ becomes, on a unique change of variables $t{\triangleright}u$ to v, which is possible by Remark 4.1.6,

$$\tau(s{\triangleleft}v, t) = (s{\triangleright}v)^{-1}\tau(s, t)(s{\triangleleft}v),$$

which is equivalent to the second condition under a further change of variables $s{\triangleright}v$ to u. □

The first condition in Corollary 4.2.4 says that the second argument of τ is constant on orbits of $\triangleleft$. The second condition says that the function in its first argument essentially intertwines $\triangleleft$ with the adjoint action (cf. a crossed module $\tau : M \to G$) except that it is twisted by $\triangleright$. This suggests to further simplify our search by focusing on the special case where $\triangleright$ is trivial. We denote by $[t]$ the orbit label or equivalence class of t under the remaining action $\triangleleft$. Note that if $\triangleright$ is trivial then the condition (4.2.1) on τ in Theorem 4.2.2 is already included as (4.1.6) in Proposition 4.1.1, *i.e.* there is no additional constraint in this case other than M

an inverse property quasigroup. However, we still have to solve for this data and we will do so in the special case of the corollary.

Corollary 4.2.5. *Suppose that $\triangleright$ is trivial. Then the data for constructing a bicrossproduct Hopf quasigroup in the special case of Corollary 4.2.4 become G finite and $\triangleleft, \tau$ such that*

(1) $\tau(s,t) = \tau(s,[t])$ *(depends only on $[t]$),*

(2) $u^{-1}\tau(s,[t])u = \tau(s\triangleleft u,[t])$ *(covariance condition),*

(3) $\tau(s,[t])\tau(s\cdot t,[r]) = \tau(s,[t\cdot r])\tau(t,[r])$ *(2-cocycle condition).*

Here we require that G acts on M by an action $\triangleleft$ respecting its structure as in (4.1.3)-(4.1.4), (4.1.8) *and that M is an inverse property quasigroup and quasi-associative in the sense* (4.1.5).

Proof. The first condition on τ in Corollary 4.2.4 is (1). Assuming (1) holds and $\triangleright$ is trivial, the second condition in Corollary 4.2.4 coincides with (4.1.6) and (2). Identity (4.1.7) is empty, while (4.1.2) simplifies to (3) using (1).

We still require (4.1.3)-(4.1.4), (4.1.8) as stated, however we note that (4.1.4) now simplifies to $(s\cdot t)\triangleleft u = (s\triangleleft u)\cdot(t\triangleleft u)$.

Clearly $s\triangleright(\ )$ is bijective as it is the identity so we have right inverses by Proposition 4.1.5. □

In this case X has a semidirect product form with cross relations $su = u(s\triangleleft u)$ and relates to the product of M by $st = \tau(s,t)s\cdot t$ as before. Its structure can be recovered given the stated data from

$$(u,s)(v,t) = (u\tau(s,[t])v, [(s\triangleleft v)\cdot t])$$

as a special case of Proposition 4.1.4. Also, if G is Abelian then the first two conditions of Corollary 4.2.5 say that $\tau(s,t) = \tau([s],[t])$ *i.e.* τ depends for both of its arguments only on the orbits in M under G.

For an example we let $M = \mathcal{G}_{\mathbb{O}}$ the octonion quasigroup taking the form discussed in Section 3.2, *i.e.* the group here consists of elements $\{\pm e_a\}$, where $a\in\mathbb{Z}_2^3$, sitting inside the octonion algebra with product $e_a\cdot e_b = F(a,b)e_{a+b}$ in terms of component-wise addition. The signs here are given in the following table; $F(a,b)$ takes values [2]

	000	001	010	011	100	101	110	111
000	1	1	1	1	1	1	1	1
001	1	−1	1	−1	1	−1	1	−1
010	1	−1	−1	1	1	−1	−1	1
011	1	1	−1	−1	−1	−1	1	1
100	1	−1	−1	1	−1	1	1	−1
101	1	1	1	1	−1	−1	−1	−1
110	1	−1	1	−1	−1	1	−1	1
111	1	1	−1	−1	1	1	−1	−1

The quasigroup in this form, like the octonion algebra, is quasi-associative in the sense [2]

$$(e_a \cdot e_b) \cdot e_c = \phi(a,b,c) e_a \cdot (e_b \cdot e_c), \quad \phi(a,b,c) = (-1)^{|abc|},$$

extended to signs, where we use the determinant of the matrix formed by the three vectors (in other words ϕ is -1 if and only if the three vectors are linearly independent in $\mathbb{Z}_2^3$ as a vector space over $\mathbb{Z}_2$ as shown in Lemma 3.2.1).

Example 4.2.6. Let X be the order 128 non-Abelian group with generators $\pm e_i, g_i$, $i = 1,2,3$ and relations

$$e_i e_j = \begin{cases} -e_j e_i & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}, \quad e_i g_j = \begin{cases} g_j e_i & \text{if } i \neq j \\ -g_j e_i & \text{if } i = j \end{cases}, \quad g_i g_j = \begin{cases} g_j g_i & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases},$$

and let $G = \mathbb{Z}_2^3$ as the subgroup of X generated by $\{g_i\}$ and Cl_3 be the subgroup generated by $\{e_i\}$. Then $X = \mathbb{Z}_2^3 \ltimes Cl_3$ and the transversal $M \subset X$ labeled by $\pm e_a$ and consisting of

$$\begin{aligned} e_{000} &= 1, & e_{001} &= e_3, \\ e_{010} &= e_2, & e_{011} &= -g_1 e_2 e_3, \\ e_{100} &= e_1, & e_{101} &= -g_2 e_1 e_3, \\ e_{110} &= -g_3 e_1 e_2, & e_{111} &= g_1 g_2 g_3 e_1 e_2 e_3, \end{aligned}$$

extended to signs, acquires the structure of the octonion quasigroup $\mathcal{G}_{\mathbb{O}}$. Moreover, the conditions of Theorem 4.2.2 hold and we have a Hopf quasigroup $k\mathcal{G}_{\mathbb{O}}{\blacktriangleright\!\!\!<} k[\mathbb{Z}_2^3]$.

Proof. This is constructed using Corollary 4.2.5 above. We know as in Example 3.3.16 that $\mathcal{G}_{\mathbb{O}}$ has an action $\triangleleft$ of the group $\mathbb{Z}_2^3$ given by $e_a{\triangleleft}g^b = (-1)^{a\cdot b}e_a$ for $a, b \in \mathbb{Z}_2^3$, where $g^b = g_1^{b_1}g_2^{b_2}g_3^{b_3}$ is the group written multiplicatively. We used the vector space dot product over $\mathbb{Z}_2$. Explicitly, $e_a{\triangleleft}g_i = (-1)^{a_i}e_a$. We know that this action respects the product. Next, we define

$$\tau(\pm e_a, \pm e_b) = g^{a\times b} \equiv g_1^{a_2b_3-a_3b_2}g_2^{a_3b_1-a_1b_3}g_3^{a_1b_2-a_2b_1},$$

where the vector space cross product in $\mathbb{Z}_2^3$ is viewed multiplicatively via g as shown explicitly. Clearly τ obeys condition (1) in Corollary 4.2.5 and also the 2-cocycle condition (3) due to linearity over $\mathbb{Z}_2$ of the cross product. We also require that the quasigroup obeys

$$(e_a \cdot e_b) \cdot e_c = (e_a{\triangleleft}\tau(e_b, e_c)) \cdot (e_b \cdot e_c),$$

which holds because $a \cdot (b \times c) = |abc|$. Similarly when there are $\pm$ signs. Finally, our special conditions for Theorem 4.2.2 hold because τ does not depend on the sign of its arguments and the orbit of e_a under $\triangleleft$ is $\{\pm e_a\}$ (the group G being Abelian, we require that both arguments depend only on the orbits). Hence, by Proposition 4.1.4, we have all the data for a quasigroup double cross product to yield a group $X = \mathbb{Z}_2^3 \ltimes \mathcal{G}_{\mathbb{O}}$. It remains to determine what this group is. It contains $G = \mathbb{Z}_2^3$ as a subgroup, and cross relations $e_ig_j = g_j(e_i{\triangleleft}g_j)$ where $e_1 = e_{100}$, $e_2 = e_{010}$ and $e_3 = e_{001}$ are (at this stage) elements of the quasigroup $\mathcal{G}_{\mathbb{O}}$. This gives the cross-relations stated. We also have $e_ae_b = \tau(e_a, e_b)e_a \cdot e_b = g^{a\times b}F(a, b)e_{a+b}$ for the product in X in terms of that in $G_{\mathbb{O}}$. Thus

$$e_1e_2 = g_3F(100, 010)e_{110} = -g_3e_{110} = -e_2e_1,$$

$$e_1e_3 = g_2F(100, 001)e_{101} = -g_2e_{101} = -e_3e_1,$$

$$e_2e_3 = g_1F(010, 001)e_{011} = -g_1e_{011} = -e_3e_1, \quad e_i^2 = -1,$$

since $F(a, b) = -F(b, a)$ when $a, b, a + b \neq 0$. This gives the relations of X in terms of the $\{e_i\}$ regarded now as generators of X. These relations and those of

$\mathbb{Z}_2^3$ provide for a basis $\{\pm e_1^{a_1} e_2^{a_2} e_3^{a_3} g_1^{b_1} g_2^{b_2} g_3^{b_3}\}$ which has order 128, hence these are all the relations. We also see in these calculations how the products are related to elements of $\mathcal{G}_{\mathbb{O}}$ and rearrange them to obtain the image of most of them in X. We similarly compute

$$e_1 e_2 e_3 = -g_3 e_{110} e_{001} = -g_3 g_1 g_2 F(110, 001) e_{111} = g_1 g_2 g_3 e_{111},$$

to obtain the last element e_{111} of the transversal. □

We find that the group X here is a semidirect product by $\mathbb{Z}_2^3$ of the 'Clifford group' Cl_3 generated by the $\pm e_i$. This is the set of signed monomials of these generators in the Clifford algebra in three dimensions (generalizing the way in which the quaternion group is defined from the quaternion algebra) and easily seen to form a group. The additional information provided by the transversal provides the quasigroup structure on the left coset space $G\backslash X$ according to our results above. For example,

$$e_{110} e_{001} = -g_3 e_1 e_2 e_3 = -g_1 g_2 e_{111},$$

induces $Ge_{110} \cdot Ge_{001} = G(-e_{111})$ at the level of cosets. In this way one can verify all the signs in the table of F for the $G_{\mathbb{O}}$ product as a useful check of all of our theory. The additional signs beyond those from the group Cl_3 come from moving all the $\{g_i\}$ to the far left where it is absorbed by G in the coset. We also obtain, of course, a new Hopf quasigroup. Its dual is a Hopf coquasigroup $k[G_{\mathbb{O}}] \rtimes k\mathbb{Z}_2^3$ with structure is similar to that of $k[S^7] \rtimes k\mathbb{Z}_2^3$ in Example 3.4.12, and could be obtained in a similar way as there. However, we have provided now a bicrossproduct point of view on it. By Remark 4.3 we also have the bicrossproduct as a coquasi-Hopf algebra and a monoidal category associated to $G_{\mathbb{O}}$ in this way

Chapter 5

Integral theory for Hopf (co)quasigroups

In this chapter we define integrals on Hopf (co)quasigroups to be linear forms on the Hopf (co)quasigroup with invariant properties. We show the existence and uniqueness of non-zero integrals on any finite dimensional Hopf quasigroup, by using a theorem of Brzezinski [5], and investigate their properties.

As the definition of a Hopf quasigroup is not self-dual, we do not define an integral on a Hopf coquasigroup in the way one may expect; we now define an integral as an element invariant under an induced coaction. We prove the existence and uniqueness of non-zero integrals on any finite dimensional Hopf coquasigroup by first proving a version of the Hopf module lemma for Hopf coquasigroups.

This leads us to define Fourier transformations on Hopf (co)quasigroups given in terms of integrals by the same formula as for Hopf algebras as a linear map from the object to its dual. We give its inverse and investigate the properties of this map.

Since the algebra structure of a Hopf coquasigroup is associative, the usual notions of Frobenius and separable algebras apply to these objects. We apply integrals to a version of Maschke's theorem for Hopf coquasigroups, following from that for Hopf algebras due to Larson and Sweedler [22]. It is described in [30] as follows. The classical result states for a finite group G, kG is semisimple *iff* $|G|^{-1} \in k$. Let $\Lambda = \sum_{g\in G} g$, then Λ is an integral in the Hopf algebra kG and $\varepsilon(\Lambda) = |G|$.

Thus, $|G|^{-1} \in k$ *iff* $\varepsilon(\Lambda) \neq 0$. This leads to the statement of Larson and Sweedler that a Hopf algebra with non-zero integral Λ is semisimple *iff* $\varepsilon(\Lambda) \neq 0$. We will demonstrate that this statement also holds for Hopf coquasigroups and briefly discuss the dual results for cosemisimple Hopf quasigroups.

The results in this chapter, unless otherwise stated, are taken from [19].

5.1 Integrals on Hopf quasigroups

Actions and coactions of Hopf (co)quasigroups are considered in Chapter 3, however, Brzezinski [5] extends the theory to defining Hopf modules over Hopf (co)quasigroups and proves a version of the Hopf module lemma adapted to Hopf quasigroups:

Definition 5.1.1. [5] Let H be a Hopf quasigroup. A vector space M is called a *right H-Hopf module* if there are maps $\alpha : M \otimes H \to M$, $m \otimes h \mapsto m \triangleleft h$, and $\rho : M \to M \otimes H$, $m \mapsto m^{(1)} \otimes m^{(2)}$, such that ρ is a coassociative counital right coaction and

$$(m \triangleleft h_{(1)}) \triangleleft Sh_{(2)} = \varepsilon(h)m = (m \triangleleft Sh_{(1)}) \triangleleft h_{(2)}, \qquad m \triangleleft 1 = m, \tag{5.1.1}$$

$$(m \triangleleft h)^{(1)} \otimes (m \triangleleft h)^{(2)} = m^{(1)} \triangleleft h_{(1)} \otimes m^{(2)} h_{(2)}, \tag{5.1.2}$$

for all $h \in H$ and $m \in M$.

In [5] it was shown that if M is a right H-Hopf module with action $\alpha(m \otimes h) = m \triangleleft h$, then M is a right H-Hopf module with the same coaction and the induced action

$$\widehat{\alpha}(m \otimes h) = (m^{(1)(1)} \triangleleft S m^{(1)(2)}) \triangleleft (m^{(2)} h).$$

A map $f : M \to N$ between right H-Hopf modules is an *H-Hopf module morphism* [5] if it commutes with the coaction and the induced action.

Left H-Hopf modules are similarly defined. Let H be a Hopf quasigroup and M be a right H-Hopf module with coaction ρ. Denote the set of *H-coinvariants* by M^{coH}, that is

$$M^{coH} = \{m \in M \mid \rho(m) = m \otimes 1\}.$$

Theorem 5.1.2. *[5] Let H be a Hopf quasigroup and M be a right H-Hopf module, then $M \cong M^{coH} \otimes H$ as right H-Hopf modules, where $M^{coH} \otimes H$ is a right H-Hopf module with action and coaction given by*

$$(m \otimes h) \triangleleft g = m \otimes hg, \quad m \otimes h \mapsto m \otimes h_{(1)} \otimes h_{(2)},$$

for all $m \in M$ and $h, g \in H$.

Proof. [5] The proof involves constructing an isomorphism $\sigma : M^{coH} \otimes H \to M$ defined by $m \otimes h \mapsto m \triangleleft h$, with inverse $\sigma^{-1} : M \to M^{coH} \otimes H$ sending $m \mapsto m^{(1)(1)} \triangleleft S m^{(1)(2)} \otimes m^{(2)}$. □

Let H be a finite dimensional Hopf quasigroup and H^* be the dual space with the natural Hopf coquasigroup structure.

Lemma 5.1.3. *H^* is a right H-Hopf module with action and coaction given by*

$$\varphi \leftharpoondown h = \varphi_{(1)} \langle \varphi_{(2)}, Sh \rangle, \tag{5.1.3}$$

$$\varphi^{(1)} \otimes \varphi^{(2)} = \sum_i f^i \varphi \otimes e_i, \tag{5.1.4}$$

for all $h \in H$ and $\varphi \in H^*$, where $\{e_i\}$ is a basis of H and $\{f^i\}$ is a dual basis.

Proof. Let $h, x \in H$ and $\varphi \in H^*$ then,

$$\begin{aligned} \langle (\varphi \leftharpoondown h_{(1)}) \leftharpoondown Sh_{(2)}, x \rangle &= \langle \varphi \leftharpoondown h_{(1)}, x S^2 h_{(2)} \rangle \\ &= \langle \varphi, (x S^2 h_{(2)}) S h_{(1)} \rangle \\ &= \langle \varphi, (x S(Sh)_{(1)})(Sh)_{(2)} \rangle \\ &= \langle \varepsilon(Sh)\varphi, x \rangle \quad = \quad \langle \varepsilon(h)\varphi, x \rangle. \end{aligned}$$

We used that S is anti-comulitplicative for the third equality and the Hopf quasigroup identity (3.3.2) for the fourth. Hence $(\varphi \leftharpoondown h_{(1)}) \leftharpoondown Sh_{(2)} = \varepsilon(h)\varphi$, and similarly, $(\varphi \leftharpoondown Sh_{(1)}) \leftharpoondown h_{(2)} = \varepsilon(h)\varphi$ for all $h \in H$ and $\varphi \in H^*$. In order to check that (5.1.4) is a coassociative coaction, we require

$$f^i \otimes e_{i(1)} \otimes e_{i(2)} = f^i f^j \otimes e_i \otimes e_j, \tag{5.1.5}$$

which is easily verified by evaluating against general elements. We recall that if $\{e_i\}$ is a basis for H and $\{f^i\}$ is a dual basis then

$$\langle f^i, h\rangle e_i = h \quad \text{and} \quad \langle \varphi, e_i\rangle f^i = \varphi. \tag{5.1.6}$$

Using these identities we find, for $\varphi \in H^*$,

$$\begin{aligned}
\varphi^{(1)} \otimes \varphi^{(2)}{}_{(1)} \otimes \varphi^{(2)}{}_{(2)} &= f^i\varphi \otimes e_{i(1)} \otimes e_{i(2)} \\
&= (f^i f^j)\varphi \otimes e_i \otimes e_j, \quad \text{by (5.1.5)}, \\
&= f^i(f^j\varphi) \otimes e_i \otimes e_j, \quad \text{since } H^* \text{ is associative}, \\
&= f^i\varphi^{(1)} \otimes e_i \otimes \varphi^{(2)}, \quad \text{by (5.1.4)}, \\
&= \varphi^{(1)(1)} \otimes \varphi^{(1)(2)} \otimes \varphi^{(2)}.
\end{aligned}$$

It is clear that $\varphi^{(1)}\varepsilon(\varphi^{(2)}) = \varphi$, hence (5.1.4) defines an H-coaction on H^*. Finally, we check the compatibility condition, for which we require another identity on the dual bases,

$$f^i{}_{(1)} \otimes f^i{}_{(2)} \otimes e_i = f^i \otimes f^j \otimes e_i e_j. \tag{5.1.7}$$

For all $\varphi \in H^*$ and $h \in H$,

$$\begin{aligned}
\varphi^{(1)} \leftharpoondown h_{(1)} \otimes \varphi^{(2)} h_{(2)} &= (f^i\varphi) \leftharpoondown h_{(1)} \otimes e_i h_{(2)} \\
&= f^i{}_{(1)}\varphi_{(1)} \otimes e_i h_{(2)} \langle f^i{}_{(2)}\varphi_{(2)}, Sh_{(1)}\rangle \\
&= f^i{}_{(1)}\varphi_{(1)} \otimes e_i h_{(2)} \langle f^i{}_{(2)}, Sh_{(1)(2)}\rangle \langle \varphi_{(2)}, Sh_{(1)(1)}\rangle \\
&= f^i\varphi_{(1)} \otimes (e_i e_j) h_{(2)(2)} \langle f^j, Sh_{(2)(1)}\rangle \langle \varphi_{(2)}, Sh_{(1)}\rangle \\
&= f^i\varphi_{(1)} \otimes (e_i Sh_{(2)(1)}) h_{(2)(2)} \langle \varphi_{(2)}, Sh_{(1)}\rangle \\
&= f^i\varphi_{(1)} \otimes e_i \langle \varphi_{(2)}, Sh\rangle \\
&= f^i(\varphi \leftharpoondown h) \otimes e_i \\
&= (\varphi \leftharpoondown h)^{(1)} \otimes (\varphi \leftharpoondown h)^{(2)}.
\end{aligned}$$

We have used the definitions of the coaction and action for the first and second equalities, anti-comultiplicity of the antipode for the third and (5.1.7) for the fourth. The fifth uses properties of the dual bases, the sixth uses the Hopf quasigroup axiom, the seventh uses (5.1.3) and we use (5.1.4) for the final equality. $\square$

Definition 5.1.4. The H-coinvariants $\lambda \in H^{*coH}$ are called *left integrals on H*.

The left integrals are precisely those elements $\lambda \in H^*$ satisfying

$$h_{(1)}\lambda(h_{(2)}) = \lambda(h) \cdot 1 \quad \forall\, h \in H. \tag{5.1.8}$$

A left integral λ is *normalized* if $\lambda(1) = 1$. We denote the space of left integrals on H by $\mathcal{I}_l^H$.

It is well-known [36, 1] that the space of integrals on a finite dimensional Hopf algebra is 1-dimensional. The proof uses the Hopf module lemma of Larson and Sweedler [22]. This statement is also true for Hopf quasigroups which we prove using Theorem 5.1.2 and Lemma 5.1.3.

Theorem 5.1.5. *If H is a finite dimensional Hopf quasigroup then* $\dim(\mathcal{I}_l^H) = 1$ *i.e. a left integral exists and is unique up to scalar multiples.*

Proof. Since H^* is a right H-Hopf module as shown in Lemma 5.1.3, by Theorem 5.1.2, we deduce that $H^* \cong H^{*coH} \otimes H$ as right H-Hopf modules, and since, $\dim H = \dim H^*$, we conclude that $\dim \mathcal{I}_l^H = 1$. It is straightforward to check that the integrals satisfy the property given in the definition:

$$\begin{aligned} H^{*coH} &= \{\lambda \in H^* \,|\, \langle \lambda^{(1)}, h\rangle \lambda^{(2)} = \langle \lambda, h\rangle \cdot 1 \,\forall\, h \in H\} \\ &= \{\lambda \in H^* \,|\, \langle \lambda, h_{(2)}\rangle h_{(1)} = \langle \lambda, 1\rangle \cdot 1 \,\forall\, h \in H\}. \end{aligned}$$

□

Corollary 5.1.6. *If H is a finite dimensional Hopf quasigroup then the antipode S is bijective.*

Proof. This follows the proof for Hopf algebras [36]. Let $h \in \ker S$ and λ be a non-zero left integral on H. Then under the isomorphism σ in Theorem 5.1.2 we have

$$\sigma(\lambda \otimes h) = \lambda \leftharpoondown h = \lambda_{(1)}\langle \lambda_{(2)}, Sh\rangle = \lambda_{(1)}\langle \lambda_{(2)}, 0\rangle = 0.$$

Since σ is an isomorphism, $\lambda \otimes h = 0$, but λ is non-zero, so $h = 0$. Thus $\ker S = \emptyset$ and S is injective. Since H is finite dimensional, it is also bijective. □

Right integrals on H are defined similarly. We find that an integral on H satisfies many of the same identities as integrals on Hopf algebras, for example, they are invariant under left multiplication:

Lemma 5.1.7. *Let H be a Hopf quasigroup, then $\lambda \in H^*$ is a left integral on H iff*

$$\varphi\lambda = \varepsilon(\varphi)\lambda, \quad \forall\, \varphi \in H^*.$$

Proof. Let $h \in H$ and $\varphi \in H^*$ then,

$$\langle \varphi\lambda, h\rangle = \langle \varphi, h_{(1)}\rangle\langle \lambda, h_{(2)}\rangle = \langle \varphi, h_{(1)}\lambda(h_{(2)})\rangle,$$

and,

$$\langle \varepsilon(\varphi)\lambda, h\rangle = \langle \varphi, 1\rangle\langle \lambda, h\rangle = \langle \varphi, \lambda(h)\cdot 1\rangle.$$

Hence, $h_{(1)}\lambda(h_{(2)}) = \lambda(h)\cdot 1 \Leftrightarrow \varphi\lambda = \varepsilon(\varphi)\lambda$. □

Following the terminology in classical Hopf algebra theory, elements $x \in H$ such that $hx = \varepsilon(h)x$ for all $h \in H$ are called left integrals *in* H. So we can restate Lemma 5.1.7 as λ is a left integral *on* H iff λ is a left integral *in* H^*.

Lemma 5.1.8. *Let λ be a left integral on a Hopf quasigroup H then,*

(1) *$S\lambda$ is a right integral on H,*

(2) *if $\varepsilon(\lambda) = 1$, then $\lambda = S\lambda$.*

Proof. For any $h \in H$ we have, by properties of the dual pairing,

$$\begin{aligned}
\langle S\lambda, h_{(1)}\rangle h_{(2)} &= \langle \lambda, Sh_{(1)}\rangle h_{(2)} \\
&= \langle \lambda, (Sh_{(1)})_{(2)}\rangle (Sh_{(1)})_{(1)} h_{(2)}, \quad \text{by (5.1.8)}, \\
&= \langle \lambda, Sh_{(1)(1)}\rangle (Sh_{(1)(2)}) h_{(2)} \\
&= \langle \lambda, Sh_{(1)}\rangle (Sh_{(2)(1)}) h_{(2)(2)}, \quad \text{by coassociativity}, \\
&= \langle \lambda, Sh\rangle \cdot 1 \quad = \quad \langle S\lambda, h\rangle \cdot 1.
\end{aligned}$$

Hence $S\lambda$ is a right integral, which proves (1). For (2), under our assumption, $\varepsilon(\lambda)=1$ hence also $\varepsilon(S\lambda)=1$. Therefore,

$$\lambda=\varepsilon(S\lambda)\lambda=(S\lambda)\lambda,$$

by Lemma 5.1.7. Now, since $S\lambda$ is a right integral, it satisfies $(S\lambda)\varphi=\varepsilon(\varphi)S\lambda$ for all $\varphi\in H^*$ by a right integral version of Lemma 5.1.7, and in particular, $(S\lambda)\lambda=\varepsilon(\lambda)S\lambda$. So, by our assumption, our displayed equation becomes,

$$(S\lambda)\varepsilon(\lambda)=S\lambda,$$

as required. □

Lemma 5.1.9. *Let H be a Hopf quasigroup and λ be a left integral on H, then for all $h,g\in H$,*

1. $h_{(1)}\lambda(h_{(2)}Sg)=g_{(2)}\lambda(hSg_{(1)})$,

2. $h_{(1)}\lambda(gh_{(2)})=(Sg_{(1)})\lambda(g_{(2)}h)$.

Proof. To prove (1), let $h,g\in H$; by the axioms of a Hopf quasigroup,

$$h_{(1)}\lambda(h_{(2)}Sg)=h_{(1)}\varepsilon(g_{(2)})\lambda(h_{(2)}Sg_{(1)})=(h_{(1)}Sg_{(2)(1)})g_{(2)(2)}\lambda(h_{(2)}Sg_{(1)}),$$

then in view of coassociativity and anti-multiplicity of the antipode, this is equal to

$$(h_{(1)}(Sg_{(1)})_{(1)})g_{(2)}\lambda(h_{(2)}(Sg_{(1)})_{(2)})=(hSg_{(1)})_{(1)}g_{(2)}\lambda(hSg_{(1)})_{(2)}=g_{(2)}\lambda(hSg_{(1)}),$$

where we use (5.1.8) for the final equality. Similarly, for $h,g\in H$,

$$\begin{aligned}h_{(1)}\lambda(gh_{(2)})&=\varepsilon(g_{(1)})h_{(1)}\lambda(g_{(2)}h_{(2)})\\&=(Sg_{(1)(1)})(g_{(1)(2)}h_{(1)})\lambda(g_{(2)}h_{(2)})\\&=(Sg_{(1)})(g_{(1)}h)_{(1)}\lambda(g_{(2)}h)_{(2)}\\&=(Sg_{(1)})\lambda(g_{(2)}h).\end{aligned}$$

□

There are analogous results for right integrals. We state them here without proof as we will refer to them later.

Definition 5.1.10. Let H be a Hopf quasigroup, $\lambda_R \in H^*$ is a *right integral on* H if for all $h \in H$,

$$\lambda_R(h_{(1)})h_{(2)} = \lambda_R(h) \cdot 1.$$

The space of right integrals on H is denoted $\mathcal{I}_r^H$.

Proposition 5.1.11. *Let H be a Hopf quasigroup and λ_R be a right integral on H then,*

(1) *λ_R is a right integral on H iff for all $\varphi \in H^*$, $\lambda_R \varphi = \varepsilon(\varphi)\lambda_R$,*

(2) *$\lambda_R((Sg)h_{(1)})h_{(2)} = \lambda_R((Sg_{(2)})h)g_{(1)}$ for all $h, g \in H$,*

(3) *$\lambda_R(g_{(1)}h)g_{(2)} = \lambda_R(gh_{(1)})Sh_{(2)}$ for all $h, g \in H$.*

Theorem 5.1.12. *If H is a finite dimensional Hopf quasigroup then* $\dim(\mathcal{I}_r^H) = 1$ *i.e. a right integral exists and is unique up to scalar multiples.*

5.2 Integrals on Hopf coquasigroups

As the definition of a Hopf quasigroup is not self-dual, we do not have the notion of a comodule or a Hopf module over a Hopf coquasigroup. An integral cannot be defined in the way one may expect since the proof requires a version of the Hopf module lemma. However, Brzezinski [5] introduced the following definition.

Definition 5.2.1. [5] Let A be a Hopf coquasigroup. A vector space M is a *right A-Hopf module* if there are maps $\alpha : M \otimes A \to M$, $m \otimes a \mapsto m \triangleleft a$ and $\rho : M \to M \otimes A$, $m \mapsto m^{(1)} \otimes m^{(2)}$, such that α is an associative unital right action on M and

$$m^{(1)(1)} \otimes (Sm^{(1)(2)})m^{(2)} = m \otimes 1 = m^{(1)(1)} \otimes m^{(1)(2)}Sm^{(2)}, \tag{5.2.1}$$

$$m^{(1)}\varepsilon(m^{(2)}) = m,$$

$$(m\triangleleft a)^{(1)} \otimes (m\triangleleft a)^{(2)} = m^{(1)}\triangleleft a_{(1)} \otimes m^{(2)}a_{(2)}, \tag{5.2.2}$$

for all $a \in A, m \in M$.

In [5] it was shown that if M is a right A-Hopf module with coaction $\rho(m) = m^{(1)} \otimes m^{(2)}$, then M is a right A-Hopf module with the same action and the induced coaction

$$\widehat{\rho}(m) = m^{(1)(1)}\triangleleft((Sm^{(1)(2)})m^{(2)}{}_{(1)}) \otimes m^{(2)}{}_{(2)}.$$

A map $f : M \to N$ between right A-Hopf modules is an *A-Hopf module morphism* [5] if it commutes with the action and the induced coaction.

For Hopf algebras and Hopf quasigroups, the space of coinvariants of a Hopf module consists of elements invariant under the coaction, however for a Hopf coquasigroup we require invariance under this induced coaction.

Definition 5.2.2. The space of *A-coinvariants* of a right A-Hopf module M with coaction ρ is defined to be

$$M^{coA} = \{m \in M \,|\, \widehat{\rho}(m) = m \otimes 1\}.$$

Using these definitions, we are able to prove a Hopf module lemma for Hopf coquasigroups.

Theorem 5.2.3. *If A is a Hopf coquasigroup and M is a right A-Hopf module, then $M \cong M^{coA} \otimes A$ as right A-Hopf modules, where $M^{coA} \otimes A$ is a right A-Hopf module by*

$$(m \otimes a)\triangleleft b = m \otimes ab, \qquad (m \otimes a)^{(1)} \otimes (m \otimes a)^{(2)} = m \otimes a_{(1)} \otimes a_{(2)},$$

for all $a, b \in A, m \in M$. The isomorphism is given explicitly by

$$\sigma : M^{coA} \otimes A \to M; \qquad m \otimes a \mapsto m\triangleleft a,$$

$$\sigma^{-1} : M \to M^{coA} \otimes A; \qquad m \mapsto m^{(1)(1)}\triangleleft Sm^{(1)(2)} \otimes m^{(2)}.$$

Proof. It is straightforward to check that σ as stated gives the required isomorphism, however here we will make use of Theorem 3.13 in [5], which gives an isomorphism $M \cong M^A \otimes A$, where M^A is the set of A-invariants defined by the coequalizer

$$M \otimes A \underset{\mathrm{id}\otimes\varepsilon}{\overset{\alpha}{\rightrightarrows}} M \xrightarrow{\pi_M} M^A$$

so that $\pi_M \circ \alpha = \pi_M(\mathrm{id} \otimes \varepsilon)$. As noted in [5], for Hopf algebras, there exists an isomorphism $M^A \cong M^{coA}$. We now show that the same maps also provide an isomorphism when A is a Hopf coquasigroup; define $\omega : M^{coA} \to M^A$ by $m \mapsto \pi_M(m)$ for $m \in M^{coA}$, and $\omega^{-1} : M^A \to M^{coA}$ by $\pi_M(m) \mapsto m^{(1)}{\triangleleft}Sm^{(2)}$ for all $m \in M$.

First we have to check that ω^{-1} is well-defined,

$$\begin{aligned}
&\widehat{\rho}(\omega^{-1}(\pi_M(m)))\\
&\quad= \widehat{\rho}(m^{(1)}{\triangleleft}Sm^{(2)})\\
&\quad= (m^{(1)}{\triangleleft}Sm^{(2)})^{(1)(1)}{\triangleleft}((S(m^{(1)}{\triangleleft}Sm^{(2)})^{(1)(2)})(m^{(1)}{\triangleleft}Sm^{(2)})^{(2)}{}_{(1)})\\
&\qquad\quad \otimes (m^{(1)}{\triangleleft}Sm^{(2)})^{(2)}{}_{(2)}\\
&\quad= (m^{(1)(1)(1)}{\triangleleft}(Sm^{(2)})_{(1)(1)}){\triangleleft}(S(m^{(1)(1)(2)}(Sm^{(2)})_{(1)(2)})(m^{(1)(2)}(Sm^{(2)})_{(2)})_{(1)})\\
&\qquad \otimes (m^{(1)(2)}(Sm^{(2)})_{(2)})_{(2)}\\
&\quad= m^{(1)(1)(1)}{\triangleleft}((Sm^{(2)})_{(1)(1)}S((Sm^{(2)})_{(1)(2)})(Sm^{(1)(1)(2)})(m^{(1)(2)}(Sm^{(2)})_{(2)})_{(1)})\\
&\qquad \otimes (m^{(1)(2)}(Sm^{(2)})_{(2)})_{(2)}\\
&\quad= m^{(1)(1)(1)}{\triangleleft}((Sm^{(1)(1)(2)})(m^{(1)(2)}Sm^{(2)})_{(1)}) \otimes (m^{(1)(2)}Sm^{(2)})_{(2)}\\
&\quad= m^{(1)}{\triangleleft}Sm^{(2)} \otimes 1\\
&\quad= \omega^{-1}(\pi_M(m)) \otimes 1,
\end{aligned}$$

where we used the definition of the induced coaction for the second equality, the compatibility condition (5.2.2) for the third, and associativity of the right action for the fourth. The fifth equality requires the antipode axiom (Proposition 3.4.3 (1)) on $(Sm^{(2)})_{(1)}$ and the sixth uses (5.2.1).

Finally, we check that these maps are mutually inverse; for all $m \in M$,

$$\begin{aligned}
\omega(\omega^{-1}(\pi_M(m))) &= \omega(m^{(1)}{\triangleleft}Sm^{(2)})\\
&= \pi_M(m^{(1)}{\triangleleft}Sm^{(2)})
\end{aligned}$$

$$
\begin{aligned}
&= \pi_M(m^{(1)}\varepsilon(Sm^{(2)})) \\
&= \pi_M(m), \quad \text{by (5.2.1)},
\end{aligned}
$$

where we used the defining property of the coequalizer for the third equality. Finally, for all $m \in M^{coA}$,

$$
\begin{aligned}
\omega^{-1}(\omega(m)) &= \omega^{-1}(\pi_M(m)) \\
&= m^{(1)} \triangleleft Sm^{(2)} \\
&= m^{(1)(1)} \triangleleft Sm^{(1)(2)}\varepsilon(m^{(2)}), \quad \text{since the coaction is counital,} \\
&= m^{(1)(1)} \triangleleft ((Sm^{(1)(2)})m^{(2)}{}_{(1)}Sm^{(2)}{}_{(2)}), \quad \text{by the antipode axiom on } m^{(2)}, \\
&= (m^{(1)(1)} \triangleleft ((Sm^{(1)(2)})m^{(2)}{}_{(1)})) \triangleleft Sm^{(2)}{}_{(2)}, \quad \text{by associativity of the action,} \\
&= m \triangleleft S1, \quad \text{since } m \in M^{coA}, \\
&= m.
\end{aligned}
$$

□

The following lemma proves some useful identities for A-Hopf modules.

Lemma 5.2.4. *Let A be a Hopf coquasigroup and M be a right A-Hopf module, then for all $a \in A$ and $m \in M$,*

(1) $(m \triangleleft a)^{(1)} \triangleleft S(m \triangleleft a)^{(2)} = m^{(1)} \triangleleft Sm^{(2)}\varepsilon(a)$,

(2) $(m^{(1)} \triangleleft Sm^{(2)})^{(1)(1)} \triangleleft S(m^{(1)} \triangleleft Sm^{(2)})^{(1)(2)} \otimes (m^{(1)} \triangleleft Sm^{(2)})^{(2)}$
$= m^{(1)} \triangleleft Sm^{(2)} \otimes 1$,

(3) *if* $m^{(1)} \triangleleft Sm^{(2)} = m$, *then,* $m^{(1)(1)} \triangleleft Sm^{(1)(2)} \otimes m^{(2)} = m \otimes 1$.

Proof. All properties follow from the properties of an A-Hopf module given in Definition 5.2.1 and (1) is from [5]. We will give a detailed proof as they are important for the next result. (1) follows from the compatibility condition (5.2.2) and associativity of the action,

$$
\begin{aligned}
(m \triangleleft a)^{(1)} \triangleleft S(m \triangleleft a)^{(2)} &= (m^{(1)} \triangleleft a_{(1)}) \triangleleft S(m^{(2)}a_{(2)}) \\
&= m^{(1)} \triangleleft (a_{(1)}(Sa_{(2)})Sm^{(2)}) \\
&= m^{(1)} \triangleleft Sm^{(2)}\varepsilon(a).
\end{aligned}
$$

Property (2) uses (5.2.2), (1) and (5.2.1):

$$\begin{aligned}
&(m^{(1)}\triangleleft Sm^{(2)})^{(1)(1)}\triangleleft S(m^{(1)}\triangleleft Sm^{(2)})^{(1)(2)} \otimes (m^{(1)}\triangleleft Sm^{(2)})^{(2)} \\
&\quad = (m^{(1)(1)}\triangleleft (Sm^{(2)})_{(1)})^{(1)}\triangleleft S(m^{(1)(1)}\triangleleft (Sm^{(2)})_{(1)})^{(2)} \otimes m^{(1)(2)}(Sm^{(2)})_{(2)} \\
&\quad = m^{(1)(1)(1)}\triangleleft Sm^{(1)(1)(2)} \otimes m^{(1)(2)}(Sm^{(2)})_{(2)}\varepsilon((Sm^{(2)})_{(1)}), \quad \text{by (1)}, \\
&\quad = m^{(1)(1)(1)}\triangleleft Sm^{(1)(1)(2)} \otimes m^{(1)(2)}Sm^{(2)} \\
&\quad = m^{(1)}\triangleleft Sm^{(2)} \otimes 1, \quad \text{by (5.2.1)}.
\end{aligned}$$

Finally, if $m^{(1)}\triangleleft Sm^{(2)} = m$ then

$$\begin{aligned}
m^{(1)(1)}\triangleleft Sm^{(1)(2)} \otimes m^{(2)} &= (m^{(1)}\triangleleft Sm^{(2)})^{(1)(1)}\triangleleft S(m^{(1)}\triangleleft Sm^{(2)})^{(1)(2)} \otimes (m^{(1)}\triangleleft Sm^{(2)})^{(2)} \\
&= m^{(1)}\triangleleft Sm^{(2)} \otimes 1 = m \otimes 1,
\end{aligned}$$

where we used our assumption for the first and second equality, (2) for the third and our assumption again for the fourth. □

Using these identities, we can prove two equivalent notions of the set of A-coinvariants.

Lemma 5.2.5. *For any right A-Hopf module M,*

$$M^{coA} = \{m \in M \,|\, m^{(1)}\triangleleft Sm^{(2)} = m\}.$$

Proof. Equality of M^{coA} and $\{m \in M \,|\, m^{(1)}\triangleleft Sm^{(2)} = m\}$ is due to part (2) of Lemma 5.2.4. First suppose that $m \in M^{coA}$ then

$$(m^{(1)(1)}\triangleleft ((Sm^{(1)(2)})m^{(2)}{}_{(1)}))\triangleleft Sm^{(2)}{}_{(2)} = m\triangleleft S1 = m.$$

But, by associativity of the action and the algebra, the LHS also equals

$$m^{(1)(1)}\triangleleft ((Sm^{(1)(2)})m^{(2)}{}_{(1)}Sm^{(2)}{}_{(2)}) = m^{(1)(1)}\triangleleft Sm^{(2)(1)}\varepsilon(m^{(2)}) = m^{(1)}\triangleleft Sm^{(2)},$$

hence $M^{coA} \subseteq \{m \in M \,|\, m^{(1)}\triangleleft Sm^{(2)} = m\}$.

Conversely, suppose $m \in M$ satisfies $m^{(1)}\triangleleft Sm^{(2)} = m$, then by associativity of the action and Lemma 5.2.4 (3),

$$\widehat{\rho}(m) = (m^{(1)(1)} \triangleleft S m^{(1)(2)}) \triangleleft m^{(2)}{}_{(1)} \otimes m^{(2)}{}_{(2)} = m \triangleleft 1 \otimes 1 = m \otimes 1,$$

and $\{m \in M \,|\, m^{(1)} \triangleleft S m^{(2)} = m\} \subseteq M^{coA}$. □

Lemma 5.2.6. *For any right A-Hopf module M,*

$$M^{coA} = \{m \in M \,|\, m^{(1)(1)} \triangleleft S m^{(1)(2)} \otimes m^{(2)} = m \otimes 1\}.$$

Proof. From Lemma 5.2.5, if $m \in M^{coA}$ then $m^{(1)} \triangleleft S m^{(2)} = m$, hence,

$$\begin{aligned} m \otimes 1 &= m^{(1)} \triangleleft S m^{(2)} \otimes 1 \\ &= (m^{(1)} \triangleleft S m^{(2)})^{(1)(1)} \triangleleft S (m^{(1)} \triangleleft S m^{(2)})^{(1)(2)} \otimes (m^{(1)} \triangleleft S m^{(2)})^{(2)}, \\ &\qquad \text{by Lemma 5.2.4 (2),} \\ &= m^{(1)(1)} \triangleleft S m^{(1)(2)} \otimes m^{(2)}, \quad \text{since } m^{(1)} \triangleleft S m^{(2)} = m. \end{aligned}$$

Conversely, if $m^{(1)(1)} \triangleleft S m^{(1)(2)} \otimes m^{(2)} = m \otimes 1$ then

$$m^{(1)} \triangleleft S m^{(2)} = m^{(1)(1)} \triangleleft S m^{(1)(2)} \varepsilon(m^{(2)}) = m\varepsilon(1) = m.$$

□

Lemma 5.2.7. *Let A be a finite dimensional Hopf coquasigroup, then A^* is a right A-Hopf module with action and coaction given by*

$$\varphi \triangleleft a = \varphi \leftharpoondown a = \varphi_{(1)} \langle \varphi_{(2)}, Sa \rangle, \tag{5.2.3}$$

$$\varphi^{(1)} \otimes \varphi^{(2)} = \sum_i f^i \varphi \otimes e_i, \tag{5.2.4}$$

for all $a \in A, \varphi \in A^*$, where $\{e_i\}$ is a basis for H and $\{f^i\}$ is a dual basis.

Proof. Since A is finite dimensional, A^* is a Hopf quasigroup and hence coassociative. To show that the action is associative, let $\varphi \in A^*$ and $a, b \in A$ then

$$(\varphi \leftharpoondown a) \leftharpoondown b = \varphi_{(1)} \leftharpoondown b \langle \varphi_{(2)}, Sa \rangle$$

$$
\begin{aligned}
&= \varphi_{(1)(1)}\langle\varphi_{(1)(2)}, Sb\rangle\langle\varphi_{(2)}, Sa\rangle\\
&= \varphi_{(1)}\langle\varphi_{(2)(1)}, Sb\rangle\langle\varphi_{(2)(2)}, Sa\rangle\\
&= \varphi_{(1)}\langle\varphi_{(2)}, S(ab)\rangle\\
&= \varphi \leftharpoondown (ab).
\end{aligned}
$$

We used (5.2.3) for the first and second equalities, coassociativity of A^* for the third, properties of the dual pairing for the fourth, and finally (5.2.3). Clearly, $\varphi \leftharpoondown 1 = \varphi$, so (5.2.3) is an associative unital right A-action. For the coaction we evaluate against a general element $a \in A$ in the first factor and find,

$$
\begin{aligned}
\langle\varphi^{(1)(1)}, a\rangle(S\varphi^{(1)(2)})\varphi^{(2)} &= \langle(f^i\varphi)^{(1)}, a\rangle(S(f^i\varphi)^{(2)})e_i, \quad \text{by (5.2.4)},\\
&= \langle f^j(f^i\varphi), a\rangle(Se_j)e_i\\
&= \langle f^j, a_{(1)}\rangle\langle f^i, a_{(2)(1)}\rangle\langle\varphi, a_{(2)(2)}\rangle(Se_j)e_i\\
&= \langle\varphi, a_{(2)(2)}\rangle(Sa_{(1)})a_{(2)(1)}, \quad \text{by (5.1.6)},\\
&= \langle\varphi, a\rangle \cdot 1.
\end{aligned}
$$

The final equality uses the Hopf coquasigroup axiom (3.4.2). This holds for all $a \in A$, hence, $\varphi^{(1)(1)} \otimes (S\varphi^{(1)(2)})\varphi^{(2)} = \varphi \otimes 1$. Similarly, $\varphi^{(1)(1)} \otimes \varphi^{(1)(2)}S\varphi^{(2)} = \varphi \otimes 1$. Finally, we have to check the compatibility condition holds; let $a, b \in A$ and $\varphi \in A^*$ then,

$$
\begin{aligned}
\langle\varphi^{(1)} \leftharpoondown a_{(1)}, b\rangle\varphi^{(2)}a_{(2)} &= \langle(f^i\varphi) \leftharpoondown a_{(1)}, b\rangle e_i a_{(2)}\\
&= \langle f^i{}_{(1)}\varphi_{(1)}, b\rangle\langle f^i{}_{(2)}\varphi_{(2)}, Sa_{(1)}\rangle e_i a_{(2)}\\
&= \langle f^i\varphi_{(1)}, b\rangle\langle f^j\varphi_{(2)}, Sa_{(1)}\rangle e_i e_j a_{(2)}, \quad \text{by (5.1.7)},\\
&= \langle f^i\varphi_{(1)}, b\rangle\langle f^j, Sa_{(1)(2)}\rangle\langle\varphi_{(2)}, Sa_{(1)(1)}\rangle e_i e_j a_{(2)}\\
&= \langle f^i\varphi_{(1)}, b\rangle\langle\varphi_{(2)}, Sa_{(1)(1)}\rangle e_i(Sa_{(1)(2)})a_{(2)}, \quad \text{by (5.1.6)},\\
&= \langle f^i\varphi_{(1)}, b\rangle\langle\varphi_{(2)}, Sa\rangle e_i,\\
&\qquad \text{by the Hopf coquasigroup axiom},\\
&= \langle f^i(\varphi \leftharpoondown a), b\rangle e_i\\
&= \langle(\varphi \leftharpoondown a)^{(1)}, b\rangle(\varphi \leftharpoondown a)^{(2)}.
\end{aligned}
$$

Hence, $(\varphi \leftharpoondown a)^{(1)} \otimes (\varphi \leftharpoondown a)^{(2)} = \varphi^{(1)} \leftharpoondown a_{(1)} \otimes \varphi^{(2)}a_{(2)}$ for all $a \in A, \varphi \in A^*$, and A^* is a right A-Hopf module. □

Definition 5.2.8. The coinvariants $\lambda \in A^{*coA}$ are called *left integrals* on A. We denote the space of left integrals on A by $\mathcal{I}_l^A$.

We note that left integrals on A are those elements $\lambda \in A^*$ satisfying

$$\langle \lambda, a \rangle = \langle f^j(f^i\lambda), a(Se_{i(1)})S^2 e_j \rangle e_{i(2)} = \langle f^i\lambda, aS^2 e_i \rangle = \langle f^j(f^i\lambda), aS^2 e_j \rangle e_i,$$

for all $a \in A$. The following theorem follows immediately from Theorem 5.2.3:

Theorem 5.2.9. *Let A be a finite dimensional Hopf coquasigroup, then a non-zero left integral exists on A and is unique up to scale.*

Proof. From Theorem 5.2.3 and Lemma 5.2.7, we have an isomorphism $\mathcal{I}_l^A \otimes A \cong A^*$. Since $\dim A = \dim A^*$, it follows that $\dim \mathcal{I}_l^A = 1$. □

Corollary 5.2.10. *If A is a finite dimensional Hopf coquasigroup then the antipode S is bijective.*

Proof. The proof is as in Corollary 5.2.10. □

As in the previous section, right integrals on Hopf coquasigroups are similarly defined with analogous properties.

5.3 Fourier transformations on Hopf quasigroups

For this section H will be a finite dimensional Hopf quasigroup, and H^* its dual space with the structure of a Hopf coquasigroup. We refer to [28] for the treatment of Fourier transformations on Hopf algebras.

Definition 5.3.1. Let λ be a left integral on H. A *Fourier transformation* on H is a linear map $F_\lambda : H \to H^*$ defined by

$$F_\lambda(h) = \lambda \leftharpoonup h, \tag{5.3.1}$$

for all $h \in H$, where the right action of H on H^* is given by (5.1.3).

Proposition 5.3.2. *A Fourier transformation on a Hopf quasigroup H is an H-Hopf module morphism, that is*

$$F_\lambda(hg) = \widehat{\alpha}(F_\lambda(h) \otimes g),$$

$$\rho(F_\lambda(h)) = (F_\lambda \otimes \mathrm{id})\Delta(h),$$

for all $h, g \in H$.

Proof. This is immediate from Theorem 5.1.2 since $F_\lambda(h) = \sigma(\lambda \otimes h)$ and σ is an H-Hopf module morphism, however we will give the full proof here. By the compatibility condition (5.1.2),

$$\rho(F_\lambda(h)) = \lambda^{(1)} \leftharpoondown h_{(1)} \otimes \lambda^{(2)} h_{(2)} = \lambda \leftharpoondown h_{(1)} \otimes h_{(2)} = (F_\lambda \otimes \mathrm{id})\Delta(h).$$

We use this identity for the induced action:

$$\begin{aligned}\widehat{\alpha}(F_\lambda(h) \otimes g) &= (F_\lambda(h)^{(1)(1)} \leftharpoondown SF_\lambda(h)^{(1)(2)}) \leftharpoondown (F_\lambda(h)^{(2)} g)\\ &= (F_\lambda(h_{(1)(1)}) \leftharpoondown Sh_{(1)(2)}) \leftharpoondown (h_{(2)} g)\\ &= ((\lambda \leftharpoondown h_{(1)(1)}) \leftharpoondown Sh_{(1)(2)}) \leftharpoondown (h_{(2)} g),\\ &= \lambda \leftharpoondown (hg), \quad \text{by (5.1.1) on } h_{(1)},\\ &= F_\lambda(hg).\end{aligned}$$

□

Lemma 5.3.3. *Let λ be a left integral on H and $F_\lambda : H \to H^*$ be a Fourier transform on H, then for all $h \in H$, $\varphi \in H^*$,*

(1) $F_\lambda(h_{(1)} Sh_{(2)}) = F_\lambda(h_{(1)}) \leftharpoondown Sh_{(2)} = \varepsilon(h)\lambda$,

(2) $F_\lambda((Sh_{(1)})h_{(2)}) = F_\lambda(Sh_{(1)}) \leftharpoondown h_{(2)} = \varepsilon(h)\lambda$,

(3) $F_\lambda(\varphi \rightharpoonup h) = \varphi F_\lambda(h)$,

(4) *if H is cocommutative and flexible then* $\langle F_\lambda(h_{(1)} g), h_{(2)} \rangle = \langle F_\lambda(h_{(1)}) \leftharpoondown g, h_{(2)} \rangle$,

where the right coaction is given by (5.1.4) and the left action of H^* on H is given by $\varphi \rightharpoonup h = h_{(1)} \langle \varphi, h_{(2)} \rangle$ for all $h \in H$ and $\varphi \in H^*$.

Proof. It is clear using the antipode axiom and the definition of the Fourier transformation on the LHS of (1) that $F_\lambda(h_{(1)}Sh_{(2)}) = F_\lambda(\varepsilon(h)) = \varepsilon(h)F_\lambda(1) = \varepsilon(h)\lambda$ for all $h \in H$. Also we know from Theorem 5.1.5 that H^* is a right H-Hopf module with this action and a given coaction. In particular $\leftharpoondown$ satisfies (5.1.1) hence,

$$F_\lambda(h_{(1)}) \leftharpoondown Sh_{(2)} = (\lambda \leftharpoondown h_{(1)}) \leftharpoondown Sh_{(2)} = \varepsilon(h)\lambda.$$

Thus $F_\lambda(h_{(1)}Sh_{(2)}) = F_\lambda(h_{(1)}) \leftharpoondown Sh_{(2)} = \varepsilon(h)\lambda$. (2) is proved similarly. To prove (3), let $x \in H$ then,

$$\begin{aligned}\langle F_\lambda(\varphi \rightharpoonup h), x\rangle &= \langle F_\lambda(h_{(1)}), x\rangle\langle\varphi, h_{(2)}\rangle\\ &= \langle\lambda, xSh_{(1)}\rangle\langle\varphi, h_{(2)}\rangle, \quad \text{by (5.3.1)},\\ &= \langle\lambda, x_{(2)}Sh\rangle\langle\varphi, x_{(1)}\rangle, \quad \text{by Lemma 5.1.9 (1)},\\ &= \langle F_\lambda(h), x_{(2)}\rangle\langle\varphi, x_{(1)}\rangle, \quad \text{by (5.3.1)},\\ &= \langle\varphi F_\lambda(h), x\rangle.\end{aligned}$$

This holds for all $x \in H$, hence $F_\lambda(\varphi \rightharpoonup h) = \varphi F_\lambda(h)$ for all $h \in H, \varphi \in H^*$. Finally, if H is cocommutative and flexible and $h, g \in H$ we find,

$$\begin{aligned}\langle F_\lambda(h_{(1)}) \leftharpoondown g, h_{(2)}\rangle &= \langle F_\lambda(h_{(1)}), h_{(2)}Sg\rangle\\ &= \langle\lambda, (h_{(2)}Sg)Sh_{(1)}\rangle\\ &= \langle\lambda, (h_{(1)}Sg)Sh_{(2)}\rangle, \quad \text{since } H \text{ is cocommutative},\\ &= \langle\lambda, h_{(1)}((Sg)Sh_{(2)})\rangle, \quad \text{by Lemma 3.3.6},\\ &= \langle\lambda, h_{(1)}S(h_{(2)}g)\rangle\\ &= \langle\lambda, h_{(2)}S(h_{(1)}g)\rangle\\ &= \langle F_\lambda(h_{(1)}g), h_{(2)}\rangle,\end{aligned}$$

as required. □

Lemma 5.3.4. *Let H be a Hopf quasigroup with Fourier transform $F_\lambda : H \to H^*$ defined by* (5.3.1)*, then for all $g, h \in H$,*

$$F_\lambda(g)F_\lambda(h) = F_\lambda(F_\lambda(g) \rightharpoonup h).$$

Proof. Let $x \in H$, then

$$\begin{aligned}\langle F_\lambda(g)F_\lambda(h), x\rangle &= \langle F_\lambda(g), x_{(1)}\rangle\langle F_\lambda(h), x_{(2)}\rangle \\ &= \langle F_\lambda(g), x_{(1)}\rangle\langle \lambda, x_{(2)}Sh\rangle, \quad \text{by (5.3.1)}, \\ &= \langle F_\lambda(g), h_{(2)}\rangle\langle \lambda, xSh_{(1)}\rangle, \quad \text{by Lemma 5.1.9 (1)}, \\ &= \langle \lambda, xS(F_\lambda(g) \rightharpoonup h)\rangle \\ &= \langle F_\lambda(F_\lambda(g) \rightharpoonup h), x\rangle.\end{aligned}$$

Since this holds for all $x \in H$ we are done. □

Let λ be a left integral on H and λ_R^* be the unique right integral on H^* such that $\langle \lambda, \lambda_R^*\rangle = 1$. Define the map ${F_\lambda}^{-1} : H^* \to H$ by

$${F_\lambda}^{-1}(\varphi) = \varphi \rightharpoonup \lambda_R^*, \tag{5.3.2}$$

for all $\varphi \in H^*$, where the left action $\rightharpoonup$ is given in Lemma 5.3.3. That ${F_\lambda}^{-1}$ and F_λ are mutually inverse is immediate from Theorem 5.1.2 since $F_\lambda(h) = \sigma(\lambda \otimes a)$ and ${F_\lambda}^{-1}(\varphi) = (\lambda_R^* \otimes \mathrm{id})\sigma^{-1}(\varphi)$.

For $g, h \in H$ and a left integral $\lambda \in H^*$, define the *convolution product* $g * h \in H$ by

$$g * h = h_{(1)}\langle \lambda h_{(2)} Sg\rangle \tag{5.3.3}$$

Proposition 5.3.5. *Let H be a Hopf quasigroup with non-zero left integral λ, then the Fourier transform $F_\lambda : H \to H^*$ maps the convolution product in H to the product in H^*, that is,*

$$F(g * h) = F(g)F(h) \quad \forall\, g, h \in H.$$

Proof. Let $g, h, x \in H$ then,

$$\begin{aligned}\langle F(g)F(h), x\rangle &= \langle F(F(g) \rightharpoonup h), x\rangle \quad \text{by Lemma 5.3.4}, \\ &= \langle F(h_{(1)}), x\rangle\langle F(g), h_{(2)}\rangle \\ &= \langle F(h_{(1)}), x\rangle\langle \lambda, h_{(2)}Sg\rangle \\ &= \langle F(h_{(1)})\langle \lambda h_{(1)}Sg\rangle, x\rangle \\ &= \langle F(g * h), x\rangle.\end{aligned}$$

This holds for all $x \in H$, hence, $F(g)F(h) = F(g * h)$ for all $g, h \in H$. □

Lemma 5.3.6. *If λ is a left integral on H such that $\lambda(hg) = \lambda(gS^2h)$ for all $h, g \in H$, then*

$$S(g * h) = S(h) * S(g),$$

for all $g, h \in H$.

Proof. Let $h, g \in H$ then,

$$\begin{aligned} S(g * h) &= Sh_{(1)}\langle\lambda, h_{(2)}Sg\rangle \\ &= Sh_{(1)}\langle\lambda, (Sg)S^2h_{(2)}\rangle \\ &= Sh_{(1)}\langle S\lambda, (Sh_{(2)})g\rangle \\ &= Sg_{(2)}\langle S\lambda, (Sh)g_{(1)}\rangle \\ &= Sg_{(2)}\langle\lambda, (Sg_{(1)})S^2h\rangle \\ &= (Sg)_{(1)}\langle\lambda, (Sg)_{(2)}S^2h\rangle \\ &= (Sh) * (Sg). \end{aligned}$$

We used the definition of the convolution product for the first equality, our assumption for the second, properties of the antipode and dual pairing for the third, and Proposition 5.1.11 (2) for the fourth. We apply properties of the antipode for the fifth, and (5.3.3) for the final equality. □

5.3.1 Fourier Transformations on Hopf coquasigroups

We take A to be a finite dimensional Hopf coquasigroup so that non-zero integrals necessarily exist and A^* is a Hopf quasigroup.

Definition 5.3.7. A *Fourier transformation* on a finite dimensional Hopf coquasigroup A with integral λ is a map $F_\lambda : A \to A^*$ defined by

$$F_\lambda(a) = \lambda \leftharpoondown a = \lambda_{(1)}\langle\lambda_{(2)}, Sa\rangle,$$

for $a \in A$.

Proposition 5.3.8. *A Fourier transformation on a Hopf coquasigroup A is an A-Hopf module morphism, that is*

$$F_\lambda(ab) = F_\lambda(a) \leftharpoondown b,$$

$$\widehat{\rho}(F_\lambda(a)) = (F_\lambda \otimes \mathrm{id})\Delta(a),$$

for all $a, b \in A$.

Proof. This is immediate from Theorem 5.2.3 since $F_\lambda(a) = \sigma(\lambda \otimes a)$ and σ is an A-Hopf module morphism. The explicit proof is dual to that of Proposition 5.3.2, but we include it here. Since the action of A^* on A is associative, for all $a, b \in A$,

$$F_\lambda(ab) = \lambda \leftharpoondown (ab) = (\lambda \leftharpoondown a) \leftharpoondown b = F_\lambda(a) \leftharpoondown b.$$

Now to see that F_λ commutes with the induced coaction, let $a \in A$ then,

$$\begin{aligned}
\widehat{\rho}(F_\lambda(a)) &= (\lambda \leftharpoondown a)^{(1)(1)} \leftharpoondown S(\lambda \leftharpoondown a)^{(1)(2)}(\lambda \leftharpoondown a)^{(2)}{}_{(1)} \otimes (\lambda \leftharpoondown a)^{(2)}{}_{(2)} \\
&= (\lambda^{(1)(1)} \leftharpoondown a_{(1)(1)}) \leftharpoondown S(\lambda^{(1)(2)}a_{(1)(2)})(\lambda^{(2)}{}_{(1)}a_{(2)(1)}) \otimes \lambda^{(2)}{}_{(2)}a_{(2)(2)} \\
&= ((\lambda^{(1)(1)} \leftharpoondown (a_{(1)(1)}Sa_{(1)(2)})) \leftharpoondown S\lambda^{(1)(2)}) \leftharpoondown \lambda^{(2)}{}_{(1)}a_{(2)(1)} \otimes \lambda^{(2)}{}_{(2)}a_{(2)(2)} \\
&= (\lambda^{(1)(1)} \leftharpoondown S\lambda^{(1)(2)}) \leftharpoondown \lambda^{(2)}{}_{(1)}a_{(1)} \otimes \lambda^{(2)}{}_{(2)}a_{(2)} \\
&= \lambda \leftharpoondown a_{(1)} \otimes a_{(2)} \quad \text{by Lemma 5.2.6,} \\
&= F_\lambda(a_{(1)}) \otimes a_{(2)} = (F_\lambda \otimes \mathrm{id})\Delta(a),
\end{aligned}$$

where we used the definition of $\widehat{\rho}$ for the first equality, the compatibility condition (5.2.2) for the second, associativity of the action for the third and the antipode identity on $a_{(1)}$ for the fourth. □

Proposition 5.3.9. *Let A be a finite dimensional Hopf coquasigroup and $\lambda \in A^*$ be a left integral on A, then*

$$F_\lambda^{-1}(\varphi) = (\mathrm{id} \otimes \varphi)\Delta(\lambda_R^*),$$

where $\lambda_R^* \in A$ is the unique right integral in A satisfying $\langle \lambda, \lambda_R^* \rangle = 1$.

Proof. Since $F_\lambda(a) = \sigma(\lambda \otimes a)$, from Theorem 5.2.3,

$$F_\lambda^{-1}(\varphi) = \langle \varphi^{(1)(1)} \leftharpoondown \varphi^{(1)(2)}, \lambda_R^* \rangle \varphi^{(2)}.$$

Then,

$$\begin{aligned} F_\lambda^{-1}(\varphi) &= \langle \varphi^{(1)(1)} \leftharpoondown \varphi^{(1)(2)}, \lambda_R^* \rangle \varphi^{(2)} \\ &= \langle \varphi^{(1)(1)}, \lambda_R^* S^2 \varphi^{(1)(2)} \rangle \varphi^{(2)} \\ &= \langle \varphi^{(1)(1)}, \lambda_R^* \varepsilon(S^2 \varphi^{(1)(2)}) \rangle \varphi^{(2)} \\ &= \langle \varphi^{(1)}, \lambda_R^* \rangle \varphi^{(2)} \\ &= \langle \varphi, \lambda_{R(2)}^* \rangle \lambda_{R(1)}^*. \end{aligned}$$

□

5.4 Frobenius and separable Hopf coquasigroups

We show that every non-zero integral on a Hopf coquasigroup A gives rise to an associative non-degenerate bilinear form on A, and thus every Hopf coquasigroup with a non-zero integral is Frobenius. Since the algebra structure of a Hopf coquasigroup is associative, we adopt the usual definitions of bilinear forms and Frobenius algebras:

Definition 5.4.1. Let A be a Hopf coquasigroup. A bilinear form $B : A \times A \to k$ on A is *associative* if

$$B(a, bc) = B(ab, c), \qquad \forall\, a, b, c \in A,$$

and it is *non-degenerate* if

$$B(a, b) = 0\ \forall\, b \in H \Rightarrow a = 0, \quad \text{and} \quad B(a, b) = 0\ \forall\, a \in H \Rightarrow b = 0.$$

Definition 5.4.2. A Hopf coquasigroup A is *Frobenius* if there exists an associative non-degenerate bilinear form on A.

Proposition 5.4.3. *Every finite dimensional Hopf coquasigroup is Frobenius.*

Proof. By Theorem 5.2.3 we have an isomorphism $A^* \cong \lambda \otimes A$. There is a left action of A on A^* defined by $a \rightharpoonup \varphi = \varphi_{(1)}\langle \varphi_{(2)}, a\rangle$ for $a \in A, \varphi \in A^*$, so $\varphi \leftharpoonup a = Sa \rightharpoonup \varphi$ for all $a \in A, \varphi \in A^*$, where the right action is defined in (5.2.3). Using this we have

$$A^* = \lambda \leftharpoonup A = SA \rightharpoonup \lambda = A \rightharpoonup \lambda,$$

the last equality coming from the bijectivity of the antipode from Corollary 5.2.10. By Theorem 5.2.9, a unique left integral $\lambda \in A^*$ on A exists. Define a form $B : A \times A \to k$ by $B(a, b) = \langle \lambda, ab\rangle$ for $a, b \in A$. Clearly B is associative since A is:

$$B(a, bc) = \langle \lambda, a(bc)\rangle = \langle \lambda, (ab)c\rangle = B(ab, c).$$

It remains to show that B is non-degenerate. Assume there exists $x \in A$ such that $B(x, A) = 0$, then,

$$0 = B(x, A) = \langle \lambda, xA\rangle = \langle A \rightharpoonup \lambda, x\rangle = \langle A^*, x\rangle.$$

By the non-degeneracy of the dual pairing, this implies that $x = 0$, and B is left non-degenerate. Since A is finite dimensional this is sufficient to prove that B is non-degenerate. □

Definition 5.4.4. A Hopf coquasigroup A is *separable* if there exists $\omega = \sum \omega^{(1)} \otimes \omega^{(2)} \in A \otimes A$, such that $\sum \omega^{(1)}\omega^{(2)} = 1$ and $\sum a\omega^{(1)} \otimes \omega^{(2)} = \sum \omega^{(1)} \otimes \omega^{(2)}a$ for all $a \in A$.

Recall from Section 5.1 that an integral in a Hopf coquasigroup A is equivalent to a left integral on a Hopf quasigroup.

Proposition 5.4.5. *Let A be a Hopf coquasigroup with left integral $\Lambda \in A$ in A such that $\varepsilon(\Lambda) \neq 0$, then A is separable.*

Proof. Let $\Lambda \in A$ be a left integral in A with $\varepsilon(\Lambda) \neq 0$, that is $a\Lambda = \varepsilon(a)\Lambda$ for all $a \in A$. By uniqueness of the integral up to scalar multiples, we can assume that $\varepsilon(\Lambda) = 1$ Define $\omega = \Lambda_{(1)} \otimes S\Lambda_{(2)}$. For all $a \in A$,

$$\Delta(\Lambda) \otimes a = \Delta(\varepsilon(a_{(1)})\Lambda) \otimes a_{(2)} = \Delta(a_{(1)}\Lambda) \otimes a_{(2)},$$

so,

$$\begin{aligned}\Lambda_{(1)} \otimes (S\Lambda_{(2)})a &= a_{(1)(1)}\Lambda_{(1)} \otimes S(a_{(1)(2)}\Lambda_{(2)})a_{(2)}\\ &= a_{(1)(1)}\Lambda_{(1)} \otimes ((S\Lambda_{(2)})Sa_{(1)(2)})a_{(2)}\\ &= a\Lambda_{(1)} \otimes S\Lambda_{(2)}, \quad \text{by (3.4.2).}\end{aligned}$$

Also, $\omega^{(1)}\omega^{(2)} = \Lambda_{(1)}S\Lambda_{(2)} = \varepsilon(\Lambda) = 1$, hence ω is the required separability element. □

Corollary 5.4.6. *Every finite dimensional Hopf coquasigroup with non-zero integral is separable.*

Proof. An integral *in* a finite dimensional Hopf coquasigroup exists by Theorem 5.1.5, and we can apply Proposition 5.4.5. □

5.5 Semisimplicity and cosemisimplicity

The algebra structure of a Hopf coquasigroup A is associative and unital, hence we adopt the usual definition of semisimplicity and the equivalent statements. The *radical* of an algebra A contains all nilpotent ideals in A; a finite dimensional algebra is *semisimple* if its radical contains only the zero ideal. Equivalently,

Definition 5.5.1. An algebra A is *semisimple* if every left A-module is completely reducible *i.e.* for any A-submodule N of an A-module M, there exists a projection $p : M \to N$ such that p is an A-module morphism.

Definition 5.5.2. A Hopf coquasigroup is *semisimple* if it is semisimple as an algebra.

Theorem 5.5.3. *Let A be a finite dimensional Hopf coquasigroup and Λ be a non-zero left integral in A. Then A is semisimple* iff $\varepsilon(\Lambda) \neq 0$.

Proof. The proof is as in [22], although we now require the Hopf coquasigroup axioms. Suppose $\varepsilon(\Lambda) = 0$. Then by Lemma 5.1.7, $\Lambda^2 = \varepsilon(\Lambda)\Lambda = 0$, so $k\Lambda$ is a nilpotent left ideal in A. Therefore, if A is semisimple then $\varepsilon(\Lambda) \neq 0$.

Conversely, suppose $\varepsilon(\Lambda) \neq 0$. By uniqueness of the integral up to scale, we can assume $\varepsilon(\Lambda) = 1$. To prove A is semisimple, let M be a left A-module (by an associative, unital action labeled $\triangleright$) and $N \subset M$ be an A-submodule; we show that N has a complement in M. Let $i : M \to N$ be a linear projection of M onto N and define $\pi : M \to N$ by

$$\pi(m) = \Lambda_{(1)} \triangleright i(S\Lambda_{(2)} \triangleright m),$$

for all $m \in M$. Then, for any $n \in N$,

$$\pi(n) = \Lambda_{(1)} \triangleright i(S\Lambda_{(2)} \triangleright n) = \Lambda_{(1)} \triangleright (S\Lambda_{(2)} \triangleright n) = \varepsilon(\Lambda)n = n.$$

It is easily seen that,

$$\Delta(\Lambda) \otimes a = a_{(1)(1)}\Lambda_{(1)} \otimes a_{(1)(2)}\Lambda_{(2)} \otimes a_{(2)},$$

by using the property $a\Lambda = \varepsilon(a)\Lambda$. So for all $m \in M, a \in A$,

$$\begin{aligned}
\pi(a \triangleright m) &= \Lambda_{(1)} \triangleright i(S\Lambda_{(2)} \triangleright (a \triangleright m)) \\
&= a_{(1)(1)}\Lambda_{(1)} \triangleright i(S(a_{(1)(2)}\Lambda_{(2)}) \triangleright (a_{(2)} \triangleright m)) \\
&= a_{(1)(1)}\Lambda_{(1)} \triangleright i((S\Lambda_{(2)})(Sa_{(1)(2)})a_{(2)} \triangleright m), \quad \text{since the action is associative,} \\
&= a\Lambda_{(1)} \triangleright i(S\Lambda_{(2)} \triangleright m), \quad \text{by the Hopf coquasigroup axiom,} \\
&= a \triangleright \pi(m).
\end{aligned}$$

Thus π is a projection and an A-module map, and $P = \ker\pi$ is a complement for N as a submodule of M. □

Since the algebra structure of a Hopf quasigroup is nonassociative, there is no notion of a semisimple Hopf quasigroup, however, the definition of semisimplicity and Maschke's theorem can of course be dualized. We end this chapter with a dual Maschke's theorem for Hopf quasigroups, beginning with the standard definition of a cosemisimple coalgebra.

Definition 5.5.4. A coalgebra C is *cosemisimple* if every left C-comodule is completely reducible, *i.e.* for any left C-subcomodule N of a left C-comodule M, there exists a C-colinear morphism $p : M \to N$ such that $p \circ i = \mathrm{id}_N$ where $i : M \to N$ is the canonical inclusion.

Definition 5.5.5. A Hopf quasigroup H is *cosemisimple* if it is cosemisimple as a coalgebra.

Theorem 5.5.6. *A Hopf quasigroup H is cosemisimple* iff *there exists a normalized left integral $\lambda : H \to k$ on H.*

Proof. The proof is dual to that of Theorem 5.5.3, but we give it here explicitly. Assume such a left integral λ exists on H and let M be a left H-comodule, $N \subset M$ be a left H-subcomodule and $i : M \to N$ be a linear projection of M onto N. We denote a left coaction of H on m by $\rho(m) = m^{(0)} \otimes m^{(1)} \in H \otimes M$. Define $\pi : M \to N$ by

$$\pi(m) = i(m^{(1)})^{(1)}\lambda(m^{(0)}Si(m^{(1)})^{(0)}).$$

Then π is itself a projection and

$$\begin{aligned}
(\mathrm{id} \otimes \pi)\rho(m) &= m^{(0)} \otimes \pi(m^{(1)}) \\
&= m^{(0)} \otimes i(m^{(1)(1)})^{(1)}\lambda(m^{(1)(0)}Si(m^{(1)(1)})^{(0)}) \\
&= m^{(0)}{}_{(1)} \otimes i(m^{(1)})^{(1)}\lambda(m^{(0)}{}_{(2)}Si(m^{(1)})^{(0)}), \\
&\qquad \text{since the coaction of } H \text{ is coassociative,} \\
&= i(m^{(1)})^{(0)}{}_{(2)} \otimes i(m^{(1)})^{(1)}\lambda(m^{(0)}Si(m^{(1)})^{(0)}{}_{(1)}), \\
&\qquad \text{by Lemma 5.1.9 (1),} \\
&= i(m^{(1)})^{(1)(0)} \otimes i(m^{(1)})^{(1)(1)}\lambda(m^{(0)}Si(m^{(1)})^{(0)}) \\
&= \rho(\pi(m)).
\end{aligned}$$

Conversely, suppose every left H-comodule is completely reducible. H and k are left H-comodules by Δ and the unit map η, respectively, and k is a subcomodule of H. It is immediate that η is an H-comodule morphism, and by assumption H is completely reducible. Thus there exists an H-comodule morphism $\varphi : H \to k$ such that φ is an H-comodule morphism and $\varphi(1) = 1$. The condition that φ is a comodule morphism is exactly that $h_{(1)}\varphi(h_{(2)}) = \varphi(h) \cdot 1$, hence φ is a normalized left integral on H. □

Chapter 6

From quasigroups to quasi-Hopf algebras

Quasi-Hopf algebras originally appeared in [13]. The main difference with Hopf algebras is that the coproduct is no longer coassociative, instead it is only coassociative up to conjugation by some invertible element $\phi \in H^3$. Drinfeld further defined quasitriangular quasi-Hopf algebras. We give the definitions here following [28] and show how the category of left modules of a quasi-Hopf algebra is a rigid braided monoidal category.

We recall the definition of the quantum double associated to a finite group given explicitly in [34], as an example of the categorical construction given in [29], generalizing the original construction of the quantum double on Hopf algebras due to Drinfeld [12]. The double provides nontrivial examples of noncocommutative examples of quasi-Hopf algebras.

Next we discuss the theory of transmutation; this process takes a noncocommutative object and provides a commutative braided group in a braided category. This reconstruction theorem was first introduced for Hopf algebras by Majid [25], and the structure given explicitly for quasi-Hopf algebras by Bulacu and Nauwelaerts [9]; we recall the structure of this braided group and extend the computations.

There is a dual process to transmutation called bosonization; given a braided group in the category of modules of a quasitriangular Hopf algebra, one can construct an equivalent Hopf algebra. This was introduced by Majid in [27] and the structure for quasi-Hopf algebras was given explicitly by Bulacu and Nauwelaerts in [9]. They obtain the bosonization structure as a case of Radford's biproduct, we derive this same structure using the method in [28].

Finally, we discuss the well-known result [35] that given any quasigroup one can express it as a coset construction and obtain the data detailed in Chapter 4. We then demonstrate for the octonion quasigroup how one can associate a quasitriangular quasi-Hopf algebra.

6.1 Introduction to quasi-Hopf algebras

Definition 6.1.1. A *quasi-bialgebra* [13] over k is $(H, \Delta, \varepsilon, \phi)$ where H is a unital associative algebra over k, $\Delta : H \to H \otimes H$, $\varepsilon : H \to k$ are algebra morphisms satisfying

$$(\mathrm{id} \otimes \Delta)\Delta(h) = \phi(\Delta \otimes \mathrm{id})\Delta(h)\phi^{-1}, \tag{6.1.1}$$

$$(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id})\Delta,$$

for all $h \in H$, and the *Drinfeld associator* $\phi \in H^3$ is an invertible 3-cocycle *i.e.*

$$(1 \otimes \phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi)(\phi \otimes 1) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi), \tag{6.1.2}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\phi) = 1 \otimes 1 \otimes 1.$$

Further, H is a *quasi-Hopf algebra* if there exists an anti-homomorphism $S : H \to H$ called the antipode and elements $\alpha, \beta \in H$ such that

$$\sum (Sh_{(1)})\alpha h_{(2)} = \varepsilon(h)\alpha, \qquad \sum h_{(1)}\beta Sh_{(2)} = \varepsilon(h)\beta, \tag{6.1.3}$$

$$X^1\beta(SX^2)\alpha X^3 = 1, \qquad (Sx^1)\alpha x^2\beta Sx^3 = 1$$

for all $h \in H$, where $\phi = X^1 \otimes X^2 \otimes X^3$ is written in capital letters, and $\phi^{-1} = x^1 \otimes x^2 \otimes x^3$ is written in lower case letters.

The antipode is uniquely determined up to a transformation $\alpha \mapsto U\alpha, \beta \mapsto \beta U^{-1}, Sh \mapsto U(Sh)U^{-1}$, for any invertible $U \in H$, hence, without loss of generality, we assume $\varepsilon(\alpha) = \varepsilon(\beta) = 1$.

The antipode of a quasi-Hopf algebra is not a coalgebra anti-homomorphism as it is for a Hopf algebra or a Hopf (co)quasigroup, however up to a twist it is anti-comultiplicative *i.e.* there exists $f \in H \otimes H$ such that

$$f\Delta(Sh)f^{-1} = S(\Delta^{op}(h)), \tag{6.1.4}$$

for all $h \in H$. Following [13], define $\gamma, \partial \in H \otimes H$ by

$$\gamma = S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4, \tag{6.1.5}$$

$$\partial = B^1\beta S(B^4) \otimes B^2 \otimes S(B^3),$$

where

$$A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\phi \otimes 1)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi^{-1}),$$

$$B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi)(\phi^{-1} \otimes 1).$$

The twist f and its inverse, f^{-1} are given by

$$f = (S \otimes S)(\Delta^{op}(x^1))\gamma\Delta(x^2\beta Sx^3),$$

$$f^{-1} = \Delta((Sx^1)\alpha x^2)\partial(S \otimes S)(\Delta^{op}(x^3)).$$

Further, the following relations hold,

$$f\Delta(\alpha) = \gamma, \qquad \Delta(\beta)f^{-1} = \partial, \tag{6.1.6}$$

$$\Delta(X^1)\partial(S \otimes S)(\Delta^{op}(X^2))\gamma\Delta(X^3) = 1,$$

$$(S \otimes S)(\Delta^{op}(x^1))\gamma\Delta(x^2)\partial(S \otimes S)(\Delta^{op}(x^3)) = 1.$$

It is useful to define elements

$$q = q^1 \otimes q^2 = \sum X^1 \otimes S^{-1}(\alpha X^3)X^2, \tag{6.1.7}$$

$$p = p^1 \otimes p^2 = \sum x^1 \otimes x^2\beta S(x^3), \tag{6.1.8}$$

in $H \otimes H$. Then, for all $h \in H$,

$$\Delta(h_{(1)})p(1 \otimes S(h_{(2)})) = p(h \otimes 1),$$

$$(1 \otimes S^{-1}(h_{(2)}))q\Delta(h_{(1)}) = (h \otimes 1)q,$$

$$\Delta(q^1)p(1 \otimes S(q^2)) = 1 \otimes 1,$$

$$(1 \otimes S^{-1}(p^2))q\Delta(p^1) = 1 \otimes 1.$$

Definition 6.1.2. A quasi-Hopf algebra $(H, \Delta, \varepsilon, S, \alpha, \beta, \phi)$ is *quasitriangular* [13] if there is an invertible element $\mathcal{R} \in H \otimes H$ such that,

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \phi_{312}\mathcal{R}_{13}\phi_{132}^{-1}\mathcal{R}_{23}\phi, \tag{6.1.9}$$

$$(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \phi_{231}^{-1}\mathcal{R}_{13}\phi_{213}\mathcal{R}_{12}\phi^{-1}, \tag{6.1.10}$$

$$\Delta^{op}(h) = \mathcal{R}\Delta(h)\mathcal{R}^{-1}, \tag{6.1.11}$$

for all $h \in H$.

Writing $\phi = \sum X^1 \otimes X^2 \otimes X^3$, $\phi_{ijk} \in H \otimes H \otimes H$ has X^1 in the i-th position, X^2 in the j-th position and X^3 in the k-th position, for example, $\phi_{312} = X^2 \otimes X^3 \otimes X^1$. Similarly for $\phi = \sum x^1 \otimes x^2 \otimes x^3$. The inverse [10] is given by

$$\mathcal{R}^{-1} = X^1\beta S(Y^2\mathcal{R}^{(1)}x^1X^2)\alpha Y^3x^3{X^3}_{(2)} \otimes Y^1\mathcal{R}^{(2)}x^2{X^3}_{(1)}.$$

As for quasitriangular Hopf algebras, $(\varepsilon \otimes \mathrm{id})\mathcal{R} = (\mathrm{id} \otimes \varepsilon)\mathcal{R} = 1 \otimes 1$.

Example 6.1.3. The group function algebra $k(G)$ of a group G can be viewed as a quasi-Hopf algebra if there is an invertible element $\varphi \in k(G)^3$ such that

$$\varphi(s,t,u)\varphi(r,st,u)\varphi(r,s,t) = \varphi(r,s,tu)\varphi(rs,t,u),$$

and $\varphi(r,e,t)$. Then with $\alpha = 1$, $\beta = \sum_{t \in G} \delta_t\ \varphi(t^{-1},t,t^{-1})$ and the associator given by

$$\phi = \sum_{r,s,t \in G} \delta_r \otimes \delta_s \otimes \delta_t\ \varphi(r,s,t),$$

$k_\phi(G)$ is a quasi-Hopf algebra. Further, if there is an invertible element $r \in k(G)^2$ such that

$$r(st,u) = r(s,u)r(t,u)\varphi(u,s,t)\varphi(s,t,u)\varphi^{-1}(s,u,t),$$

$$r(u,st) = r(u,s)r(u,t)\varphi(s,u,t)\varphi^{-1}(u,s,t)\varphi^{-1}(s,t,u),$$

and $r(s,e) = r(e,s) = 1$, then $k_\phi(G)$ is quasitriangular with $\mathcal{R} = \sum_{s,t \in G} \delta_s \otimes$

$\delta_t\ r(s,t)$.

Given a quasitriangular quasi-Hopf algebra H, the set of left H-modules forms a rigid, braided monoidal category:

Example 6.1.4. [13] Let H be a unital algebra, then the category ${}_H\mathcal{M}$ of left H-modules consists of objects, the vector spaces V on which H acts, and morphisms, the linear maps f which commute with the action of H, *i.e.* $f(h\triangleright v) = h\triangleright f(v)$ for all $v \in V$ and $h \in H$. If H is a quasi-bialgebra, then $\otimes$, defined by $h\triangleright(v \otimes w) = h_{(1)}\triangleright v \otimes h_{(2)}\triangleright w$, and

$$\Phi_{U,V,W}((u \otimes v) \otimes w) = \phi^{(1)}\triangleright u \otimes (\phi^{(2)}\triangleright v \otimes \phi^{(3)}\triangleright w),$$

for all $u \in U$, $v \in V$ and $w \in W$ where U, V, W are objects in ${}_H\mathcal{M}$, makes ${}_H\mathcal{M}$ into a monoidal category. If H is a quasi-Hopf algebra, then ${}_H\mathcal{M}$ is a rigid monoidal category with $(h\triangleright f)(v) = f(S(h)\triangleright v)$ for all $v \in V$, $f \in V^*$ and $h \in H$, and

$$\mathrm{ev}(f \otimes v) = f(\alpha\triangleright v),$$

$$\mathrm{coev} = \sum_a \beta\triangleright e_a \otimes f^a,$$

where $\{e_a\}$ is a basis for V and $\{f^a\}$ a dual basis. Finally, if H is a quasi-triangular quasi-Hopf algebra, then ${}_H\mathcal{M}$ is a braided monoidal category with the braiding defined by

$$\Psi_{U,V}(u \otimes v) = \mathcal{R}^{(2)}\triangleright v \otimes \mathcal{R}^{(1)}\triangleright u,$$

for all $u \in U, v \in V$.

The dual of a quasi-Hopf algebra is something that is associative only up to conjugation in a convolution product sense by a 3-cocycle.

Definition 6.1.5. A *coquasi-Hopf algebra* is a counital coassociative coalgebra A with a product satisfying

$$a_{(1)}(b_{(1)}c_{(1)})\phi(a_{(2)}, b_{(2)}, c_{(2)}) = \phi(a_{(1)}, b_{(1)}, c_{(1)})(a_{(2)}b_{(2)})c_{(2)},$$

for all $a, b, c \in A$. Here $\phi : A \otimes A \otimes A \to k$ is a unital 3-cocycle *on* A so it satisfies

$$\phi(b_{(1)}, c_{(1)}, d_{(1)})\phi(a_{(1)}, b_{(2)}c_{(2)}, d_{(2)})\phi(a_{(2)}, b_{(3)}, c_{(3)})$$

$$= \phi(a_{(1)}, b_{(1)}, c_{(1)}d_{(1)})\phi(a_{(2)}b_{(2)}, c_{(2)}, d_{(2)}),$$

and $\phi(a, 1, b) = 1$ for all $a, b, c, d \in A$. Finally, there are morphisms $S : A \to A$ and $\alpha, \beta : A \to k$ such that S is anti-comultiplicative and

$$(Sa_{(1)})a_{(3)}\alpha(a_{(2)}) = 1 \cdot \alpha(a), \quad a_{(1)}Sa_{(3)}\beta(a_{(2)}) = 1 \cdot \beta(a),$$

$$\phi(a_{(1)}, Sa_{(3)}, a_{(5)})\beta(a_{(2)})\alpha(a_{(4)}) = \varepsilon(a),$$

$$\phi^{-1}(Sa_{(1)}, a_{(3)}, Sa_{(5)})\alpha(a_{(2)})\beta(a_{(4)}) = \varepsilon(a).$$

6.1.1 The quantum double

Given any quasi-Hopf algebra H with bijective antipode, one can associate a quasitriangular quasi-Hopf algebra, the quantum double [12] $D(H)$ of this finite dimensional quasi-Hopf algebra H. This was described in [34] and we recall the definition here adopting the notation of [8].

Let H be a finite dimensional quasi-Hopf algebra with basis $\{e_i\}$ and corresponding dual basis $\{e^i\}$ of H^*. Consider $\Omega \in H^5$ given by

$$\Omega = \sum X^1{}_{(1)(1)}y^1x^1 \otimes X^1{}_{(1)(2)}y^2x^2{}_{(1)} \otimes X^1{}_{(2)}y^3x^2{}_{(2)} \otimes S^{-1}(f^1X^2x^3) \otimes S^{-1}(f^2X^3).$$

The quantum double $D(H) = H^* \bowtie H$ is defined on $H^* \otimes H$ by

$$(\varphi \bowtie h)(\psi \bowtie g) = \sum(\Omega^1 \rightharpoonup \varphi \leftharpoonup \Omega^5)(\Omega^2 h_{(1)(1)} \rightharpoonup \psi \leftharpoonup S^{-1}(h_{(2)})\Omega^4) \bowtie \Omega^3 h_{(1)(2)}g,$$

$$\Delta(\varphi \bowtie h) = \sum(\varepsilon \bowtie X^1Y^1)(p^1{}_{(1)}x^1 \rightharpoonup \varphi_{(2)} \leftharpoonup Y^2S^{-1}(p^2) \bowtie p^1{}_{(2)}x^2h_{(1)}) \otimes(X^2{}_{(1)} \rightharpoonup \varphi_{(1)} \leftharpoonup S^{-1}(X^3) \bowtie X^2{}_{(2)}Y^3x^3h_{(2)}),$$

$$\varepsilon(\varphi \bowtie h) = \varepsilon(h)\varphi(S^{-1}\alpha),$$

where we write an element of the double as $\varphi \bowtie h$ for $\varphi \in H^*$ and $h \in H$. The antipode is given by

$$S(\varphi \bowtie h) = \sum (\varepsilon \bowtie (Sh)f^1)(p^1{}_{(1)}U^1 \rightharpoonup S^{-1}\varphi \leftharpoonup f^2 S^{-1}p^2 \bowtie p^1{}_{(2)}U^2),$$

where $U = \sum g^1 Sq^2 \otimes g^2 Sq^1$ and the actions of H on H^* are given as in Chapter 5 by

$$h \rightharpoonup \varphi = \varphi_{(1)}\langle \varphi_{(2)}, h\rangle, \quad \varphi \leftharpoonup h = \varphi_{(2)}\langle \varphi_{(1)}, h\rangle.$$

Finally, $D(H)$ is quasitriangular with structure

$$\mathcal{R} = \sum_i (\varepsilon \bowtie S(p^2)e_i p^1{}_{(1)}) \otimes (e^i \bowtie p^1{}_{(2)}).$$

6.2 Transmutation of quasi-Hopf algebras

A bialgebra in a monoidal category $\mathcal{C}$ is $(B, \underline{m}, \underline{\eta}, \underline{\Delta}, \underline{\varepsilon})$, where B is an object in $\mathcal{C}$, and the morphism $\underline{m} : B \otimes B \to B$ is associative in the category up to the associator Φ, *i.e.*

$$\underline{m}(\underline{m} \otimes \mathrm{id}) = \underline{m}(\mathrm{id} \otimes \underline{m})\Phi_{B,B,B}.$$

The identity is $\underline{\eta} : \underline{1} \to B$, on the unit object $\underline{1} \in \mathcal{C}$, such that $\underline{m}(\underline{\eta} \otimes \mathrm{id}) = \mathrm{id} = \underline{m}(\mathrm{id} \otimes \mathrm{id})$. Similarly, the morphism $\underline{\Delta} : B \to B \otimes B$ is coassociative up to Φ, and $\underline{\varepsilon} : B \to \underline{1}$ satisfies the usual counit axiom in $\mathcal{C}$. Both $\underline{\Delta}$ and $\underline{\varepsilon}$ are algebra morphisms in $\mathcal{C}$ up to the isomorphisms Φ, Ψ, for example, $\underline{\Delta}$ satisfies

$$\underline{\Delta} \circ \underline{m} = (\underline{m} \otimes \underline{m})\Phi^{-1}_{B,B,B\otimes B}(\mathrm{id} \otimes \Phi)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\mathrm{id} \otimes \Phi^{-1})\Phi_{B,B,B\otimes B}(\underline{\Delta} \otimes \underline{\Delta}).$$

If there is a morphism $\underline{S} : B \to B$ satisfying the usual antipode axioms but now in $\mathcal{C}$, then B is called a *braided group* [24].

Following [26], consider monoidal categories $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{D}$ braided, and functors $F, V \otimes F : \mathcal{C} \to \mathcal{D}$. Suppose there is an object $B \in \mathcal{D}$ such that for all $V \in \mathcal{D}$, $\mathrm{Mor}(V, B) \cong \mathrm{Nat}(V \otimes F, F)$ by functorial isomorphisms θ_V, where $\mathrm{Nat}(V \otimes F, F)$ is the set of natural transformations between the functors $V \otimes F$ and F introduced in Section 2.2. Let

$$\alpha = \{\alpha_M : B \otimes F(M) \to F(M) | M \in \mathcal{C}\},$$

be the natural transformation corresponding to the identity morphism id_B in $\mathrm{Mor}(V,B)$. Then, using α, and the braiding we get induced maps

$$\theta_V^n : \mathrm{Mor}(V, B^{\otimes n}) \to \mathrm{Nat}(V \otimes F^n, F^n),$$

and we assume these are bijections. This is called the *representability assumption for modules.* Then, using these bijections, we can define a multiplication, a unit, a coproduct, a counit and an antipode for B.

For example, note that $\alpha_M(\mathrm{id}\otimes\alpha_M)\Phi_{B,B,F(M)} : (B\otimes B)\otimes F(M) \to F(M)$ is a natural transformation in $\mathrm{Nat}(B\otimes B\otimes F, F)$, and hence corresponds to a unique map $B\otimes B \to B$ under $\theta^{-1}_{B\otimes B}$, which must be the multiplication on B.

Theorem 6.2.1. *[26] Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories with $\mathcal{D}$ braided, and $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor satisfying the representability assumption for modules. Then B, as above, is a bialgebra in $\mathcal{D}$. Moreover, if $\mathcal{D}$ is rigid, then B is a Hopf algebra in $\mathcal{D}$.*

In [9] the authors applied this theory to H a quasitriangular quasi-Hopf algebra.

Theorem 6.2.2. *[9] Let H be a quasitriangular quasi-Hopf algebra. Then H gives a braided cocommutative Hopf algebra B in ${}_H\mathcal{M}$ with structure*

$$\underline{m}(b\otimes b') = q^1(x^1{\triangleright}b)S(q^2)x^2b'S(x^3),$$

$$\underline{\eta}(1) = \beta,$$

$$\underline{\Delta}(b) = x^1X^1b_{(1)}g^1S(x^2\mathcal{R}^{(2)}y^3{X^3}_{(2)}) \otimes (x^3\mathcal{R}^{(1)}){\triangleright}(y^1X^2b_{(2)}g^2S(y^2{X^3}_{(1)})),$$

$$\underline{\varepsilon}(b) = \varepsilon(b),$$

$$\underline{S}(b) = X^1\mathcal{R}^{(2)}x^2\beta S(q^1(X^2\mathcal{R}^{(1)}x^1{\triangleright}b)S(q^2)X^3x^3),$$

where the left action $\triangleright$ of H on B is the left adjoint action and q is as given in (6.1.8)*. Moreover, the antipode is bijective.*

The proof of the theorem is given in [9], however the proof of the coproduct was omitted. The proof involves only the application of the identities in Section 6.1. I have included the proof in Appendix A.

6.3 Bosonization of braided groups

Let H be a quasitriangular quasi-Hopf algebra. Given a braided group in ${}_H\mathcal{M} = \mathcal{C}$ we can reconstruct an ordinary quasi-Hopf algebra, or more specifically a quasi-Hopf algebra in the category of vector spaces. We follow the strategy in [27]. If B is a braided Hopf algebra in $\mathcal{C}$, then a left braided B-module is an object V in $\mathcal{C}$ and an action $\alpha_V^B : B \otimes V \to V$ in $\mathcal{C}$. Since α_V^B is a morphism in the category it intertwines the action of H, that is $\alpha_V^B(h\triangleright(b\otimes v)) = h\triangleright\alpha_V^B(b\otimes v)$, for all $h \in H, b \in B, v \in V$; equivalently,

$$h\triangleright(b\triangleright v) = (h_{(1)}\triangleright b)\triangleright(h_{(2)}\triangleright v), \tag{6.3.1}$$

where the notation for the actions of H on B, H on V and B on V are understood. The category ${}_B\mathcal{C}$ of left braided B-modules in $\mathcal{C}$ is a braided monoidal category with the braiding of $\mathcal{C}$.

Theorem 6.3.1. *Let H be a quasitriangular quasi-Hopf algebra, and $B \in {}_H\mathcal{M}$ be a braided group. Then there is an ordinary quasi-Hopf algebra $B \rtimes H$ built on the vector space $B \otimes H$ with structure*

$$(b\otimes h)(c\otimes g) = (x^1\triangleright b)\underline{\cdot}((x^2h_{(1)})\triangleright c)\otimes x^3h_{(2)}g,$$

$$\eta(1) = \underline{1}\otimes 1,$$

$$\begin{aligned}\Delta(b\otimes h) = (y^1X^1)\triangleright b_{\underline{(1)}} &\otimes y^2Y^1\mathcal{R}^{(2)}x^2{X^3}_{(1)}h_{(1)}\\ &\otimes({y^3}_{(1)}Y^2\mathcal{R}^{(1)}x^1X^2)\triangleright b_{\underline{(2)}}\otimes {y^3}_{(2)}Y^3x^3{X^3}_{(2)}h_{(2)},\end{aligned}$$

$$\varepsilon(b\otimes h) = \underline{\varepsilon}(b)\varepsilon(h),$$

$$\begin{aligned}S(b\otimes h) = ((S(X^1{x^1}_{(1)}\mathcal{R}^{(2)}h)\alpha)_{(1)}X^2{x^1}_{(2)}\mathcal{R}^{(1)})\triangleright\underline{S}(b)\\ \otimes(S(X^1{x^1}_{(1)}\mathcal{R}^{(2)}h)\alpha)_{(2)}X^3x^2\beta S(x^3),\end{aligned}$$

$$\alpha_{B\rtimes H} = \underline{1}\otimes\alpha,$$

$$\beta_{B\rtimes H} = \underline{1}\otimes\beta,$$

$$\phi_{B\rtimes H} = \underline{1}\otimes X^1\otimes\underline{1}\otimes X^2\otimes\underline{1}\otimes X^3.$$

Proof. Given a braided B-module V in $\mathcal{C}$, we have an action of B on V and an action of H on V. The action of $B \rtimes H$ on V is

$$(b \otimes h) \triangleright v = b \triangleright (h \triangleright v), \tag{6.3.2}$$

for all $v \in V, b \in B$ and $h \in H$. Note, that since the action of B on V is a morphism in $\mathcal{C}$ and hence associative up to Φ, it satisfies

$$b \triangleright (c \triangleright v) = ((x^1 \triangleright b) \underline{\cdot} (x^2 \triangleright c)) \triangleright (x^3 \triangleright v), \tag{6.3.3}$$

for $b, c \in B, v \in V$, where we use $\underline{\cdot}$ to denote the multiplication in B. Since $B \rtimes H$ is a quasi-Hopf algebra in the category of vector spaces, its action on V is associative. These properties are enough to determine the multiplication in $B \rtimes H$; let $b, c \in B$ and $h, g \in H$ then,

$$\begin{aligned}
((b \otimes h)(c \otimes g)) \triangleright v &= (b \otimes h) \triangleright ((c \otimes g) \triangleright v) \\
&= b \triangleright (h \triangleright ((c \otimes g) \triangleright v)), \quad \text{by (6.3.2)}, \\
&= b \triangleright (h \triangleright (c \triangleright (g \triangleright v))) \\
&= b \triangleright ((h_{(1)} \triangleright c) \triangleright (h_{(2)} \triangleright (g \triangleright v))), \quad \text{by(6.3.1)}, \\
&= b \triangleright ((h_{(1)} \triangleright c) \triangleright ((h_{(2)} g) \triangleright v)), \quad \text{by associativity of the action of } H, \\
&= ((x^1 \triangleright b) \underline{\cdot} (x^2 \triangleright (h_{(1)} \triangleright c))) \triangleright (x^3 \triangleright ((h_{(2)} g) \triangleright v)), \quad \text{by (6.3.3)}, \\
&= ((x^1 \triangleright b) \underline{\cdot} ((x^2 h_{(1)}) \triangleright c)) \triangleright ((x^3 h_{(2)} g) \triangleright v).
\end{aligned}$$

Thus,

$$(b \otimes h)(c \otimes g) = (x^1 \triangleright b) \underline{\cdot} (x^2 h_{(1)} \triangleright c) \otimes x^3 h_{(2)} g.$$

For the coproduct we note that as for Hopf algebras, a quasi-Hopf algebra acts on a tensor product piecewise, but also, by (6.3.2), $(b \otimes h) \triangleright (v \otimes w) = b \triangleright (h \triangleright (v \otimes w)) = b \triangleright (h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w)$. Thus the coproduct of $B \rtimes H$ is characterized by

$$(b \otimes h)_{(1)} \triangleright v \otimes (b \otimes h)_{(2)} \triangleright w = b \triangleright (h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w). \tag{6.3.4}$$

The braided group B also acts on the tensor product $V \otimes W$ piecewise, but now in the category, *i.e.* as

$$\alpha^B_{V \otimes W} = (\alpha^B_V \otimes \alpha^B_W) \Phi^{-1}_{B,V,B \otimes W} (\mathrm{id} \otimes \Phi_{V,B,W})(\mathrm{id} \otimes \Psi_{B,V} \otimes \mathrm{id})$$

$$(\mathrm{id}\otimes\Phi^{-1}_{B,V,W})\Phi_{B,B,V\otimes W}(\underline{\Delta}\otimes\mathrm{id}\otimes\mathrm{id}),$$

that is,

$$\begin{aligned}b\triangleright(v\otimes w)=&((y^1X^1)\triangleright b_{\underline{(1)}})\triangleright((y^2Y^1\mathcal{R}^{(2)}x^2X^3{}_{(1)})\triangleright v)\\&\otimes((y^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1X^2)\triangleright b_{\underline{(2)}})\triangleright((y^3{}_{(2)}Y^3x^3X^3{}_{(2)})\triangleright w).\end{aligned}$$

So, by associativity of the action of H,

$$\begin{aligned}b\triangleright(h_{(1)}\triangleright v\otimes h_{(2)}\triangleright w)=&((y^1X^1)\triangleright b_{\underline{(1)}})\triangleright((y^2Y^1\mathcal{R}^{(2)}x^2X^3{}_{(1)}h_{(1)})\triangleright v)\\&\otimes((y^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1X^2)\triangleright b_{\underline{(2)}})\triangleright((y^3{}_{(2)}Y^3x^3X^3{}_{(2)}h_{(2)})\triangleright w).\end{aligned}$$

Comparing the above equation with (6.3.4), we obtain

$$\begin{aligned}\Delta(b\otimes h)=&(y^1X^1)\triangleright b_{\underline{(1)}}\otimes y^2Y^1\mathcal{R}^{(2)}x^2X^3{}_{(1)}h_{(1)}\\&\otimes(y^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1X^2)\triangleright b_{\underline{(2)}}\otimes y^3{}_{(2)}Y^3x^3X^3{}_{(2)}h_{(2)}.\end{aligned}$$

as required.

For the antipode, given $V\in{}_B\mathcal{C}$ we have to consider how H and B act on the dual object V^*. Since $B\rtimes H$ is a quasi-Hopf algebra, it is known that it acts on the dual space V^* as $((b\otimes h)\triangleright v^*)(v)=v^*(S(b\otimes h)\triangleright v)$ for all $v^*\in V^*, v\in V, b\in B, h\in H$. By (6.3.2), we also have $((b\otimes h)\triangleright v^*)(v)=(b\triangleright(h\triangleright v^*))(v)$, and so the antipode is determined by

$$v^*(S(b\otimes h)\triangleright v)=(b\triangleright(h\triangleright v^*))(v). \tag{6.3.5}$$

It remains to find how B acts on the dual space; if V is a left B-module, then V^* is a right B-module by $\alpha^*:V^*\otimes B\to V^*$ defined as

$$(\alpha^*(v^*\otimes b))(v)=v^*((S(X^1x^1{}_{(1)})\alpha)\triangleright\{((X^2x^1{}_{(2)})\triangleright b)\triangleright((X^3x^2\beta S(x^3))\triangleright v)\}).$$

Then, V^* becomes a left B-module by

$$\begin{aligned}\alpha^B_{V^*}(b\otimes v^*)(v)&=\alpha^*(\mathrm{id}\otimes\underline{S})\Psi_{B,V^*}(b\otimes v^*)(v)\\&=v^*((S(X^1x^1{}_{(1)}\mathcal{R}^{(2)})\alpha)\triangleright(((X^2x^1{}_{(2)}\mathcal{R}^{(1)})\triangleright\underline{S}(b))\triangleright((X^3x^2\beta S(x^3))\triangleright v))).\end{aligned} \tag{6.3.6}$$

Using (6.3.5) we find

$$\begin{aligned}
&v^*(S(b\otimes h)\triangleright v) = (b\triangleright(h\triangleright v^*))(v)\\
&= (h\triangleright v^*)((S(X^1{x^1}_{(1)}\mathcal{R}^{(2)})\alpha)\triangleright\{((X^2{x^1}_{(2)}\mathcal{R}^{(1)})\triangleright\underline{S}(b))\triangleright((X^3x^2\beta S(x^3))\triangleright v)\})\\
&\qquad\text{by the action of } B \text{ on the dual space (6.3.6)},\\
&= v^*((S(X^1{x^1}_{(1)}\mathcal{R}^{(2)}h)\alpha)\triangleright\{((X^2{x^1}_{(2)}\mathcal{R}^{(1)})\triangleright\underline{S}(b))\triangleright((X^3x^2\beta S(x^3))\triangleright v)\})\\
&= v^*(\{(S(X^1{x^1}_{(1)}\mathcal{R}^{(2)}h)\alpha)_{(1)}X^2{x^1}_{(2)}\mathcal{R}^{(1)}\triangleright\underline{S}(b)\}\triangleright\\
&\qquad\qquad \{(S(X^1{x^1}_{(1)}\mathcal{R}^{(2)}h)\alpha)_{(2)}X^3x^2\beta S(x^3)\triangleright v\})\quad\text{by (6.3.1).}
\end{aligned}$$

Thus we find that the antipode has the form required. □

We note that the bosonization construction for quasi-Hopf algebras was given explicitly in [9] as a particular case of Radford's biproduct.

Corollary 6.3.2. *[11] The modules of B in the braided category ${}_H\mathcal{M}$ are in one to one correspondence to the ordinary modules of $B \rtimes H$.*

Proof. $B \rtimes H$ is a smash product when considered as an algebra; this structure was found in [11], and this correspondence is given as follows. Let V be a $B\rtimes H$-module with structure given by $(b\otimes h)\cdot v$. Define maps $j : H \to B\rtimes H$ and $i : B \to B\rtimes H$ by $j(h) = 1\otimes h$ and $i(b) = b\otimes 1$. Then V becomes a left H-module by $h\triangleright v = j(h)\cdot v$, and V becomes a braided B-module by $b\triangleright v = i(b)\cdot v$. Conversely, if V is a braided module in ${}_H\mathcal{M}$, define the action of $B \rtimes H$ on V by $(b\otimes h)\cdot v = b\triangleright(h\triangleright v)$. □

Example 6.3.3. The octonions $\mathbb{O} = \{\sum_{a\in\mathbb{Z}_2^3} u_a e_a \,|\, u_a \in \mathbb{R}\}$ live in the category of $\mathbb{Z}_2^3$-graded spaces; the grading is given by $|e_a| = a$. We can view this $\mathbb{Z}_2^3$-graded quasialgebra as a $k_\phi(\mathbb{Z}_2^3)$-module with action given by $\delta_b\triangleright e_a = \delta_b(|e_a|)\, e_a = e_a\,\delta_{a,b}$ on homogeneous elements. Thus we can construct the bosonization of the octonions $\mathbb{O} \rtimes k_\phi(\mathbb{Z}_2^3)$ as an algebra.

$$\begin{aligned}
(e_a\otimes\delta_s)(e_b\otimes\delta_t) &= (x^1\triangleright e_a)\cdot(x^2\delta_{s(1)}\triangleright e_b)\otimes x^3\delta_{s(2)}\delta_t\\
&= \sum_{x,y,z}\sum_{p+q=s}(\delta_x\triangleright e_a)\cdot(\delta_y\,\delta_p\triangleright e_b)\otimes\delta_z\,\delta_q\,\delta_t\,\phi(x,y,z)\\
&= \sum_{x,y,z}\sum_{p+q=s} e_a\cdot e_b\otimes\delta_t\;\delta_{x,a}\,\delta_{y,p}\,\delta_{p,b}\,\delta_{z,t}\,\delta_{q,t}\,\phi(x,y,z)\\
&= e_a\cdot e_b\otimes\delta_t\;\;\delta_{s,t+b}\,\phi(a,b,t)
\end{aligned}$$

$$= (-1)^{|abt|} e_a \cdot e_b \otimes \delta_t \; \delta_{s,t+b},$$

for all $a, b, s, t \in \mathbb{Z}_2^3$.

It is clear that $(1 \otimes \delta_s)(1 \otimes \delta_t) = 1 \otimes \delta_s \delta_t$, and so $k_\phi(\mathbb{Z}_2^3) \subset \mathbb{O} \rtimes k_\phi(\mathbb{Z}_2^3)$ as a subalgebra. It also contains an algebra with the following structure:

$$\begin{aligned}(e_a \otimes 1)(e_b \otimes 1) &= \sum_{s,t} (-1)^{|abt|} e_a \cdot e_b \otimes \delta_t \, \delta_{s,t+b} \\ &= \sum_{t} (-1)^{|abt|} e_a \cdot e_b \otimes \delta_t \\ &= e_a \cdot e_b \otimes \chi(a,b),\end{aligned}$$

where $\chi(a,b) = \sum_t (-1)^{|abt|} \delta_t$, and we note

$$\chi(a,b) = \begin{cases} 2(\delta_0 + \delta_a + \delta_b + \delta_{a+b}) - 1 & \text{if } a, b \text{ linearly independent over } \mathbb{Z}_2 \\ 1 & \text{otherwise} \end{cases}$$

Finally, the commutation relations are

$$\begin{aligned}(e_a \otimes 1)(1 \otimes \delta_t) &= \sum_s e_a \otimes \delta_{s,t} \delta_t \\ &= e_a \otimes \delta_t,\end{aligned}$$

$$\begin{aligned}(1 \otimes \delta_s)(e_b \otimes 1) &= \sum_t e_b \otimes \delta_{s+b,t} \delta_t \\ &= e_b \otimes \delta_{s+b}.\end{aligned}$$

So we find that $f e_a = e_a L_a(f)$ for all $f \in k_\phi(\mathbb{Z}_2^3)$ and $a \in \mathbb{Z}_2^3$, where $L_a(f)(s) = f(a+s)$.

6.4 Coset construction

As remarked in Section 4.2, given any inverse property quasigroup, we can find an associated quasi-Hopf algebra H. This is due to the fact that every such quasigroup can be expressed as a coset construction [35].

We adopt the notation of [35]; let $(Q, *)$ be a set with a binary multiplication. For an element $q \in Q$, the *right multiplication* $R_q : Q \to Q$ is a map defined by $R_q : x \mapsto x * q$. The *left multiplication* is a map $L_q : Q \to Q$ similarly defined.

The *right multiplication group* is a subgroup of the permutation group of Q generated by $\{R_q | q \in Q\}$.

Proposition 6.4.1. *[35] Let $(Q, *, e)$ be an inverse property quasigroup. Then there is a transversal M to a subgroup G of a group X such that $(Q, *, e)$ is isomorphic to $(M, \cdot, 1)$, where M acquires the structure detailed in Chapter 4.*

Proof. The proof in [35] gives the construction, which we outline here. Let X be the right multiplication group of Q, and G be the stabilizer of the element e in X. Then, with M as the set of generators of X, *i.e.* $M = \{R_x | x \in Q\}$, we obtain a transversal to G in X. The required isomorphism is $R : Q \to M$ taking $x \mapsto R_x$. □

The proof of this proposition gives the construction from the initial quasigroup of the group, subgroup and transversal. As noted in Remark 4.2.3, from this construction we obtain a coquasi-Hopf algebra, although not necessarily a Hopf quasigroup.

Example 6.4.2. Consider $M = \mathcal{G}_{\mathbb{O}} = \{\pm e_a | a \in \mathbb{Z}_2^3\}$ the octonion quasigroup. Following the proof of the proposition above, we construct the right multiplication group X of order 128 with 16 generators and obtain a subgroup G of order 8 with 3 generators as the stabilizer of $e_0 \in X$. These groups are exactly those found in Example 4.2.6. By Theorem 4.2.2, we can construct $k\mathcal{G}_{\mathbb{O}} \ltimes k[\mathbb{Z}_2^3]$, and, since G is finite, its dual $k[\mathcal{G}_{\mathbb{O}}] \rtimes k\mathbb{Z}_2^3$.

Once again, we label the elements of $\mathbb{Z}_2^3$ by σ_a where $a \in \mathbb{Z}_2^3$ and we will write the generators of $k[\mathbb{Z}_2^3]$ as $\{\omega^u \,|\, u \in \mathbb{Z}_2^3\}$ defined on $\mathbb{Z}_2^3$ as $\omega^u(\sigma_v) = \delta_{u,v}$. Then $k\mathcal{G}_{\mathbb{O}} \ltimes k[\mathbb{Z}_2^3]$ is a Hopf quasigroup with unit $1 = \sum_u e_o \otimes \omega^u$, counit $\varepsilon(e_a \otimes \omega^u) = \delta_{u,0}$, and

$$(e_a \otimes \omega^u)(e_b \otimes \omega^v) = e_a \cdot e_b \otimes \omega^v \, \delta_{u,v},$$

$$\Delta(e_a \otimes \omega^u) = \sum_{v+w=u} (-1)^{a \cdot v} \, (e_a \otimes \omega^v) \otimes (e_a \otimes \omega^w),$$

$$S(e_a \otimes \omega^u) = (-1)^{a \cdot u} \, F(a, a) \, e_a \otimes \omega^u.$$

This object becomes a coquasi-Hopf algebra with $\alpha = \beta = \varepsilon$ and

$$\phi^{\pm 1}((e_a \otimes \omega^u) \otimes (e_b \otimes \omega^v) \otimes (e_c \otimes \omega^w)) = \delta_{u,b\times c}\, \delta_{v,0}\, \delta_{w,0}.$$

Dually, $k[\mathcal{G}_{\mathbb{O}}]$ is the commutative polynomial algebra $k[x_a \,|\, a \in \mathbb{Z}_2^3]$ with relations $\sum_a x_a^2 = 1$ and $x_a x_b = 0$ if $a \neq b$. Following the theory in Chapter 4, we find the quasi-Hopf algebra structure of $k[\mathcal{G}_{\mathbb{O}}] \rtimes k\mathbb{Z}_2^3$ has unit $\sum_a x_a^2 \otimes \sigma_0$, counit $\varepsilon(x_a \otimes \sigma_u) = \delta_{a,0}$ and

$$(x_a \otimes \sigma_u)(x_b \otimes \sigma_v) = (-1)^{a\cdot u}\, x_a^2 \otimes \sigma_u \sigma_v\, \delta_{a,b},$$

$$\Delta(x_a \otimes \sigma_u) = \sum_{b+c=a} (x_b \otimes \sigma_u) \otimes (x_c \otimes \sigma_u)\, F(b,c),$$

$$S(x_a \otimes \sigma_u) = (-1)^{a\cdot u}\, x_a \otimes \sigma_u\, F(a,a),$$

$$\phi = \sum_{a,b,c} (x_a^2 \otimes \sigma_{b\times c}) \otimes (x_b^2 \otimes \sigma_0) \otimes (x_c^2 \otimes \sigma_0),$$

and $\alpha = \beta = 1$.

Following the construction in Section 6.1.1, we obtain the structure of the double as

$$\begin{aligned} &((e_a \otimes \omega^t) \bowtie (x_b \otimes \sigma_u))((e_c \otimes \omega^v) \bowtie (x_d \otimes \sigma_w)) \\ &\qquad = (e_a \cdot e_c \otimes \omega^v) \bowtie (x_d^2 \otimes \sigma_{a\times c} \sigma_u \sigma_w)\; \delta_{b,d}\, \delta_{v,t}\, R(c,d) (-1)^{c\cdot u + c\cdot v + d\cdot u}, \end{aligned}$$

$$\begin{aligned} &\Delta((e_a \otimes \omega^u) \bowtie (x_b \otimes \sigma_v)) \\ &= \sum_{s+t=b} \sum_{j+k=u} (-1)^{a\cdot j}\, F(s,t)\, ((e_a \otimes \omega^{k+(s\times t)}) \bowtie (x_s \otimes \sigma_v)) \otimes ((e_a \otimes \omega^g) \otimes (x_t \otimes \sigma_v)), \end{aligned}$$

$$S((e_a \otimes \omega^u) \bowtie (x_b \otimes \sigma_v)) = (-1)^{a\cdot v + b\cdot v} R(a,b) F(a,a) F(b,b)\, (e_a \otimes \omega^u) \bowtie (x_b \otimes \sigma_v).$$

The antipode can be simplified to

$$S((e_a \otimes \omega^u) \bowtie (x_b \otimes \sigma_v)) = (-1)^{a\cdot v} F(a,a) R(a,b)\, (e_a \otimes \omega^u) \bowtie S(x_b \otimes \sigma_v).$$

It is immediate from the structure of the double in Section 6.1.1 that this quasi-Hopf algebra contains $k[\mathcal{G}_{\mathbb{O}}] \rtimes k\mathbb{Z}_2^3$ as a subalgebra. However, $k\mathcal{G}_{\mathbb{O}} \ltimes k[\mathbb{Z}_2^3]$ is not a subalgebra, and this is true in general [16]; for a double $D(H)$ built on $H^* \otimes H$, the dual space is not a subalgebra unless the coproduct on H is coassociative, i.e. H is a Hopf algebra.

The commutation relations can be found by calculating

$$((e_a \otimes \omega^t) \bowtie 1)(1 \bowtie (x_d \otimes \sigma_w)) = (e_a \otimes \omega^t) \bowtie (x_d \otimes \sigma_w),$$

$$(1 \bowtie (x_b \otimes \sigma_u))((e_c \otimes \omega^v) \bowtie 1) = (e_c \otimes \omega^v) \bowtie (x_b \otimes \sigma_u)\, R(c,b)\, (-1)^{c \cdot u + c \cdot v}.$$

So we can write the commutation relations as

$$(x_a \otimes \sigma_u)\varphi = \chi_{a,u}(\varphi)(x_a \otimes \sigma_u),$$

for all $\varphi \in k\mathcal{G}_{\mathbb{O}} \ltimes k[\mathbb{Z}_2^3]$ and $a, u \in \mathbb{Z}_2^3$, where

$$\chi_{a,u}(e_b \otimes \omega^v) = (-1)^{b \cdot u + b \cdot v} R(a,b) e_b \otimes \omega^v.$$

So, given any quasigroup M, we can construct a quasitriangular quasi-Hopf algebra H as the double of the bicrossproduct formed from M. Clearly then we can go further to obtain a braided group in the category of $D(H)$-modules using Theorem 6.2.2.

Appendix A

Proof of Theorem 6.2.2

In this appendix we give the explicit proof that the given form of the coproduct of the braided group in Theorem 6.2.2 satisfies the required identities.

Proposition. *The coproduct* $\underline{\Delta}$ *given in Theorem 6.2.2 satisfies*

$$\alpha_{M\otimes N} = (\alpha_M \otimes \alpha_N)\Phi^{-1}_{\underline{H},M,\underline{H}\otimes N}(\mathrm{id}\otimes\Phi_{M,\underline{H},N})(\mathrm{id}\otimes(\Psi_{\underline{H},M}\otimes\mathrm{id}))\circ$$
$$(\mathrm{id}\otimes\Phi^{-1}_{\underline{H},M,N})\Phi_{\underline{H},\underline{H},M\otimes N}(\underline{\Delta}\otimes\mathrm{id}\otimes\mathrm{id}).$$

Proof. From [26], the comultiplication $\underline{\Delta}$ of $\underline{H}$, is obtained as the unique morphism in ${}_H\mathcal{M}$ satisfying the identity in the statement of this proposition. That is, $\underline{\Delta}$ satisfies

$$\Delta(q^1 b S(q^2)) = q^1(y^1X^1\triangleright b_{\underline{(1)}})S(q^2)y^2Y^1R^{(2)}x^2X^3{}_{(1)}$$
$$\otimes Q^1(y^3{}_{(1)}Y^2R^{(1)}x^1X^2\triangleright b_{\underline{(2)}})S(Q^2)y^3{}_{(2)}Y^3x^3X^3{}_{(2)},$$

for all $b\in\underline{H}$.

We check that the comultiplication given in the statement of the Theorem 6.2.2 satisfies the identity above. Starting from the RHS,

$$q^1(y^1X^1\triangleright b_{\underline{(1)}})S(q^2)y^2Y^1\mathcal{R}^{(2)}x^2X^3{}_{(1)}$$
$$\otimes Q^1(y^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1X^2\triangleright b_{\underline{(2)}})S(Q^2)y^3{}_{(2)}Y^3x^3X^3{}_{(2)}$$
$$= \quad q^1(y^1X^1\triangleright w^1A^1b_{(1)}g^1S(w^2\mathcal{R}'^{(2)}z^3A^3{}_{(2)}))S(q^2)y^2Y^1\mathcal{R}^{(2)}x^2X^3{}_{(1)}$$

$$
\begin{aligned}
&\quad \otimes Q^1(y^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1X^2w^3\mathcal{R}'^{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))\\
&\qquad S(Q^2)y^3{}_{(2)}Y^3x^3X^3{}_{(2)}\\
=\;& W^1y^1{}_{(1)}X^1{}_{(1)}w^1A^1b_{(1)}g^1S(W^2y^1{}_{(2)}X^1{}_{(2)}w^2\mathcal{R}'^{(2)}z^3A^3{}_{(2)})\alpha\\
&\qquad W^3y^2\mathcal{R}^{(2)}x^2X^3{}_{(1)}\\
&\quad \otimes Q^1(y^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1X^2w^3\mathcal{R}'^{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))\\
&\qquad S(Q^2)y^3{}_{(2)}Y^3x^3X^3{}_{(2)}\\
\overset{(6.1.2)}{=}\;& W^1y^1{}_{(1)}w^1X^1A^1b_{(1)}g^1S(W^2y^1{}_{(2)}w^2T^1X^2{}_{(1)}\mathcal{R}'^{(2)}z^3A^3{}_{(2)})\alpha\\
&\qquad W^3y^2Y^1\mathcal{R}^{(2)}x^2w^3{}_{(2)(1)}T^3{}_{(1)}X^3{}_{(1)}\\
&\quad \otimes Q^1(y^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1w^3{}_{(1)}T^2X^2{}_{(2)}\mathcal{R}'^{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))\\
&\qquad S(Q^2)y^3{}_{(2)}Y^3x^3w^3{}_{(2)(2)}T^3{}_{(2)}X^3{}_{(2)}\\
\overset{(6.1.1),(6.1.11)}{=}\;& W^1y^1{}_{(1)}w^1X^1A^1b_{(1)}g^1S(W^2y^1{}_{(2)}w^2T^1X^2{}_{(1)}\mathcal{R}'^{(2)}z^3A^3{}_{(2)})\alpha\\
&\qquad W^3y^2w^3{}_{(1)}Y^1\mathcal{R}^{(2)}x^2T^3{}_{(1)}X^3{}_{(1)}\\
&\quad \otimes Q^1(y^3{}_{(1)}w^3{}_{(2)(1)}Y^2\mathcal{R}^{(1)}x^1T^2X^2{}_{(2)}\mathcal{R}'^{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))\\
&\qquad S(Q^2)y^3{}_{(2)}w^3{}_{(2)(2)}Y^3x^3T^3{}_{(2)}X^3{}_{(2)}\\
\overset{(6.1.2)}{=}\;& y^1X^1A^1b_{(1)}g^1S(y^2{}_{(1)}w^1T^1X^2{}_{(1)}\mathcal{R}'^{(2)}z^3A^3{}_{(2)})\alpha\\
&\qquad y^2{}_{(2)}w^2Y^1\mathcal{R}^{(2)}x^2T^3{}_{(1)}X^3{}_{(1)}\\
&\quad \otimes Q^1(y^3{}_{(1)}w^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1T^2X^2{}_{(2)}\mathcal{R}'^{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))\\
&\qquad S(Q^2)y^3{}_{(2)}w^3{}_{(2)}Y^3x^3T^3{}_{(2)}X^3{}_{(2)}\\
\overset{(6.1.3)}{=}\;& X^1A^1b_{(1)}g^1S(w^1T^1X^2{}_{(1)}\mathcal{R}'^{(2)}z^3A^3{}_{(2)})\alpha w^2Y^1\mathcal{R}^{(2)}x^2T^3{}_{(1)}X^3{}_{(1)}\\
&\quad \otimes Q^1(w^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1T^2X^2{}_{(2)}\mathcal{R}'^{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))\\
&\qquad S(Q^2)w^3{}_{(2)}Y^3x^3T^3{}_{(2)}X^3{}_{(2)}\\
\overset{(6.1.11)}{=}\;& X^1A^1b_{(1)}g^1S(w^1T^1\mathcal{R}'^{(2)}X^2{}_{(2)}z^3A^3{}_{(2)})\alpha w^2Y^1\mathcal{R}^{(2)}x^2T^3{}_{(1)}X^3{}_{(1)}\\
&\quad \otimes Q^1(w^3{}_{(1)}Y^2\mathcal{R}^{(1)}x^1T^2\mathcal{R}'^{(1)}X^2{}_{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))\\
&\qquad S(Q^2)w^3{}_{(2)}Y^3x^3T^3{}_{(2)}X^3{}_{(2)}\\
\overset{(6.1.2),(6.1.3)}{=}\;& X^1A^1b_{(1)}g^1S(w^1t^1T^1\mathcal{R}'^{(2)}X^2{}_{(2)}z^3A^3{}_{(2)})\alpha w^2t^2{}_{(1)}\mathcal{R}^{(2)}x^2T^3{}_{(1)}X^3{}_{(1)}\\
&\quad \otimes Q^1(w^3t^2{}_{(2)}\mathcal{R}^{(1)}x^1T^2\mathcal{R}'^{(1)}X^2{}_{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))\\
&\qquad S(Q^2)t^3x^3T^3{}_{(2)}X^3{}_{(2)}\\
\overset{(6.1.11)}{=}\;& X^1A^1b_{(1)}g^1S(w^1t^1T^1\mathcal{R}'^{(2)}X^2{}_{(2)}z^3A^3{}_{(2)})\alpha w^2\mathcal{R}^{(2)}t^2{}_{(2)}x^2T^3{}_{(1)}X^3{}_{(1)}\\
&\quad \otimes Q^1(w^3\mathcal{R}^{(1)}t^2{}_{(1)}x^1T^2\mathcal{R}'^{(1)}X^2{}_{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))
\end{aligned}
$$

$$
\begin{aligned}
&\qquad S(Q^2)t^3x^3T^3{}_{(2)}X^3{}_{(2)}\\
\overset{(6.1.2)}{=}\;& X^1A^1b_{(1)}g^1S(w^1T^1x^1{}_{(1)}\mathcal{R}'^{(2)}X^2{}_{(2)}z^3A^3{}_{(2)})\alpha w^2\mathcal{R}^{(2)}T^3x^2X^3{}_{(1)}\\
&\otimes Q^1(w^3\mathcal{R}^{(1)}T^2x^1{}_{(2)}\mathcal{R}'^{(1)}X^2{}_{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))S(Q^2)x^3X^3{}_{(2)}\\
\overset{(6.1.11)}{=}\;& X^1A^1b_{(1)}g^1S(w^1T^1\mathcal{R}'^{(2)}x^1{}_{(2)}X^2{}_{(2)}z^3A^3{}_{(2)})\alpha w^2\mathcal{R}^{(2)}T^3x^2X^3{}_{(1)}\\
&\otimes Q^1(w^3\mathcal{R}^{(1)}T^2\mathcal{R}'^{(1)}x^1{}_{(1)}X^2{}_{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))S(Q^2)x^3X^3{}_{(2)}\\
\overset{(6.1.10)}{=}\;& X^1A^1b_{(1)}g^1S(\mathcal{R}^{(2)}{}_{(1)}Y^2x^1{}_{(2)}X^2{}_{(2)}z^3A^3{}_{(2)})\alpha\mathcal{R}^{(2)}{}_{(2)}Y^3x^2X^3{}_{(1)}\\
&\otimes Q^1(\mathcal{R}^{(1)}Y^1x^1{}_{(1)}X^2{}_{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))S(Q^2)x^3X^3{}_{(2)}\\
\overset{(6.1.3)}{=}\;& X^1A^1b_{(1)}g^1S(Y^2x^1{}_{(2)}X^2{}_{(2)}z^3A^3{}_{(2)})\alpha Y^3x^2X^3{}_{(1)}\\
&\otimes Q^1(Y^1x^1{}_{(1)}X^2{}_{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))S(Q^2)x^3X^3{}_{(2)}\\
\overset{(6.1.7)}{=}\;& X^1A^1b_{(1)}g^1S(Y^2x^1{}_{(2)}X^2{}_{(2)}z^3A^3{}_{(2)})\alpha Y^3x^2X^3{}_{(1)}\\
&\otimes W^1Y^1{}_{(1)}x^1{}_{(1)(1)}X^2{}_{(1)(1)}z^1A^2b_{(2)}g^2S(W^2Y^1{}_{(2)}x^1{}_{(1)(2)}X^2{}_{(1)(2)}z^2A^3{}_{(1)})\\
&\qquad \alpha W^3x^3X^3{}_{(2)}\\
\overset{(6.1.1)}{=}\;& X^1A^1b_{(1)}g^1S(Y^2z^3x^1{}_{(2)(2)}X^2{}_{(2)(2)}A^3{}_{(2)})\alpha Y^3x^2X^3{}_{(1)}\\
&\otimes W^1Y^1{}_{(1)}z^1x^1{}_{(1)}X^2{}_{(1)}A^2b_{(2)}g^2S(W^2Y^1{}_{(2)}z^2x^1{}_{(2)(1)}X^2{}_{(2)(1)}A^3{}_{(1)})\\
&\qquad \alpha W^3x^3X^3{}_{(2)}\\
\overset{(6.1.2)}{=}\;& X^1A^1{}_{(1)}b_{(1)}g^1S(Y^2z^3x^1{}_{(2)(2)}y^2{}_{(2)}X^3{}_{(1)(2)}A^2{}_{(2)})\alpha Y^3x^2y^3{}_{(1)}X^3{}_{(2)(1)}A^3{}_{(1)}\\
&\otimes W^1Y^1{}_{(1)}z^1x^1{}_{(1)}y^1X^2A^1{}_{(2)}b_{(2)}g^2S(W^2Y^1{}_{(2)}z^2x^1{}_{(2)(1)}y^2{}_{(1)}X^3{}_{(1)(1)}A^2{}_{(1)})\\
&\qquad \alpha W^3x^3y^3{}_{(2)}X^3{}_{(2)(2)}A^3{}_{(2)}\\
\overset{(6.1.2),(6.1.3)}{=}\;& X^1A^1{}_{(1)}b_{(1)}g^1S(T^2Y^2{}_{(2)}x^1{}_{(2)(2)}y^2{}_{(2)}X^3{}_{(1)(2)}A^2{}_{(2)})\alpha\\
&\qquad T^3Y^3x^2y^3{}_{(1)}X^3{}_{(2)(1)}A^3{}_{(1)}\\
&\otimes W^1Y^1x^1{}_{(1)}y^1X^2A^1{}_{(2)}b_{(2)}g^2S(W^2T^1Y^2{}_{(1)}x^1{}_{(2)(1)}y^2{}_{(1)}X^3{}_{(1)(1)}A^2{}_{(1)})\\
&\qquad \alpha W^3x^3y^3{}_{(2)}X^3{}_{(2)(2)}A^3{}_{(2)}\\
\overset{(6.1.2)}{=}\;& X^1A^1{}_{(1)}b_{(1)}g^1S(T^2y^2{}_{(1)(2)}x^1{}_{(2)}X^3{}_{(1)(2)}A^2{}_{(2)})\alpha T^3y^2{}_{(2)}x^2X^3{}_{(2)(1)}A^3{}_{(1)}\\
&\otimes W^1y^1X^2A^1{}_{(2)}b_{(2)}g^2S(W^2T^1y^2{}_{(1)(1)}x^1{}_{(1)}X^3{}_{(1)(1)}A^2{}_{(1)})\\
&\qquad \alpha W^3y^3x^3X^3{}_{(2)(2)}A^3{}_{(2)}\\
\overset{(6.1.1)}{=}\;& X^1A^1{}_{(1)}b_{(1)}g^1S(T^2y^2{}_{(1)(2)}X^3{}_{(1)(1)(2)}x^1{}_{(2)}A^2{}_{(2)})\alpha T^3y^2{}_{(2)}X^3{}_{(1)(2)}x^2A^3{}_{(1)}\\
&\otimes W^1y^1X^2A^1{}_{(2)}b_{(2)}g^2S(W^2T^1y^2{}_{(1)(1)}X^3{}_{(1)(1)(1)}x^1{}_{(1)}A^2{}_{(1)})\\
&\qquad \alpha W^3y^3X^3{}_{(2)}x^3A^3{}_{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(6.1.1),(6.1.3)}{=} && X^1A^1{}_{(1)}b_{(1)}g^1S(T^2x^1{}_{(2)}A^2{}_{(2)})\alpha T^3x^2A^3{}_{(1)}\\
& && \otimes W^1y^1X^2A^1{}_{(2)}b_{(2)}g^2S(W^2y^2X^3{}_{(1)}T^1x^1{}_{(1)}A^2{}_{(1)})\alpha W^3y^3X^3{}_{(2)}x^3A^3{}_{(2)}\\
&= && X^1A^1{}_{(1)}b_{(1)}g^1S(T^2x^1{}_{(2)}A^2{}_{(2)})\alpha T^3x^2A^3{}_{(1)}\\
& && \otimes X^2A^1{}_{(2)}b_{(2)}g^2S(X^3{}_{(1)}T^1x^1{}_{(1)}A^2{}_{(1)})\alpha X^3{}_{(2)}x^3A^3{}_{(2)}\\
&\overset{(6.1.3)}{=} && A^1{}_{(1)}b_{(1)}g^1S(T^2x^1{}_{(2)}A^2{}_{(2)})\alpha T^3x^2A^3{}_{(1)}\\
& && \otimes A^1{}_{(2)}b_{(2)}g^2S(T^1x^1{}_{(1)}A^2{}_{(1)})\alpha x^3A^3{}_{(2)}\\
&= && X^1{}_{(1)}b_{(1)}g^1S(X^2{}_{(2)})S(Y^2x^1{}_{(2)})\alpha Y^3x^2X^3{}_{(1)}\\
& && \otimes X^1{}_{(2)}b_{(2)}g^2S(X^2{}_{(1)})S(Y^1x^1{}_{(1)})\alpha x^3X^3{}_{(2)}, \quad \text{relabeling,}\\
&\overset{(6.1.4),(6.1.5)}{=} && X^1{}_{(1)}b_{(1)}S(X^2)_{(1)}g^1\gamma^1X^3{}_{(1)} \otimes X^1{}_{(2)}b_{(2)}S(X^2)_{(2)}g^2\gamma^2X^3{}_{(2)}\\
&\overset{(6.1.6)}{=} && X^1{}_{(1)}b_{(1)}S(X^2)_{(1)}\alpha_{(1)}X^3{}_{(1)} \otimes X^1{}_{(2)}b_{(2)}S(X^2)_{(2)}\alpha_{(2)}X^3{}_{(2)}\\
&= && \Delta(X^1bS(X^2)\alpha X^3)\\
&\overset{(6.1.7)}{=} && \Delta(q^1bS(q^2)).
\end{aligned}
$$

$\square$

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