SPECTRAL STRUCTURE OF TRANSFER OPERATORS FOR EXPANDING CIRCLE MAPS

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Abstract. We explicitly determine the spectrum of transfer operators (acting on spaces of holomorphic functions) associated to analytic expanding circle maps arising from finite Blaschke products. This is achieved by deriving a convenient natural representation of the respective adjoint operators.

1. Introduction

One of the major strands of modern ergodic theory is to exploit the rich links between dynamical systems theory and functional analysis, making the powerful tools of the latter available for the benefit of understanding complex dynamical behaviour. In classical ergodic theory, composition operators occur naturally as basic objects for formulating concepts such as ergodicity or mixing [28]. These operators, known in this context as Koopman operators, are the formal adjoints of transfer operators, the spectral data of which provide insight into fine statistical properties of the underlying dynamical systems, such as rates of mixing (see, for example, [2, 7]). There exists a large body of literature on spectral estimates for transfer operators (and hence mixing properties) for various one- and higher-dimensional systems with different degree of expansivity or hyperbolicity (see [1] for an overview). However, to the best of our knowledge, no sufficiently rich class of dynamical systems together with a machinery allowing for the explicit analytic determination of the entire spectrum is known. This is the gap the current contribution aims to fill.

In the literature, the term 'composition operator' is mostly used to refer to compositions with analytic functions mapping a disk into itself, a setting in which operator-theoretic properties such as boundedness, compactness, and most importantly explicit spectral information are well-established (good references are [23] or the encyclopedia on the subject [9]). The purpose of the present article is to demonstrate that in a particular analytic setting, the spectra of transfer operators can be deduced directly from certain composition operators.

Let τ be a real analytic expanding map on the circle and $\{\phi_k \colon k = 1, \ldots, K\}$ the set of local inverse branches of τ. Then the associated transfer operator $\mathcal{L}$ (also referred to as Ruelle-Perron-Frobenius or simply Ruelle operator), defined by

$$(\mathcal{L}f)(z) = \pm \sum_{k=1}^{K} \phi_k'(z)(f \circ \phi_k)(z), \tag{1.1}$$

preserves and acts compactly on certain spaces of functions holomorphic on a neighbourhood of the unit circle [19, 21, 24]. Here, the negative sign is chosen if τ is orientation-reversing.

The spectrum of $\mathcal{L}$ can be understood by passing to its (Banach space) adjoint operator $\mathcal{L}'$. This strategy has been explored in the context of generalised Ruelle operators acting on spaces of functions locally analytic on the Julia set of a rational function (see [3, 12, 13, 27]); in particular, explicit expressions for Fredholm determinants of certain generalised Ruelle operators associated with polynomial maps have been derived. In our setting of analytic expanding circle maps, we adopt a similar approach, that is, we analyse the spectrum of $\mathcal{L}$ by deriving a natural explicit representation of $\mathcal{L}'$ (Proposition 4.6). As a by-product we obtain a more conceptual proof of results in [24], where the spectrum of $\mathcal{L}$ for a certain family of analytic circle maps was determined using a block-triangular matrix representation.

In the spirit of approaching problems in the world of real numbers by making recourse to complex numbers, we consider finite Blaschke products, a class of rational maps on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ preserving the unit circle. One of the striking features of Blaschke products is that they partition the Riemann sphere into simple dynamically invariant regions: the unit circle, the unit disk and the exterior disk in $\hat{\mathbb{C}}$. As a consequence, the spectrum of $\mathcal{L}'$ can be determined by studying the spectrum of composition operators on spaces of holomorphic functions on these dynamically invariant regions.

Our main result (Theorem 5.4) can be summarized as follows. Let B be a finite Blaschke product such that its restriction τ to the unit circle $\mathbb{T} = \{z : |z| = 1\}$ is expanding. Denote by $H^2(A)$ the Hardy-Hilbert space of functions which are holomorphic on some suitable annulus A (containing $\mathbb{T}$) and square integrable on its boundary ∂A (see Definition 2.1). Then the transfer operator $\mathcal{L}$ associated to τ is compact on $H^2(A)$, with spectrum

$$\sigma(\mathcal{L}) = \{1\} \cup \{\lambda(z_0)^n : n \in \mathbb{N}\} \cup \{\overline{\lambda(z_0)}^n : n \in \mathbb{N}\} \cup \{0\}$$

where $\lambda(z_0)$ is the multiplier of the unique attracting fixed point z_0 of B in the unit disk. This implies that for finite Blaschke products which give rise to analytic expanding circle maps, the derivative of the fixed point in the unit disk completely determines the spectrum of $\mathcal{L}$.

This result gives rise to several interesting applications. In particular, it provides counterexamples to a variant of a conjecture of Mayer due to Levin on the reality of spectra of transfer operators (see [24]). Furthermore, it can be used to obtain interval maps with arbitrarily fast exponential mixing (corresponding to arbitrarily small nonzero subleading eigenvalues) but bounded Lyapunov exponents, see [25].

The paper is organized as follows. In Section 2, we review basic definitions and facts about Hardy-Hilbert spaces on annuli. The following Section 3 is devoted to analytic expanding circle maps and their corresponding transfer operators. In Section 4, we explicitly derive the structure of the corresponding adjoint operators after having established a suitable representation of the dual space. This structure is then used in Section 5 to obtain the spectrum of transfer operators associated to analytic expanding circle maps arising from finite Blaschke products, thus proving our main result.

2. Hardy-Hilbert spaces

Throughout this article $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the one point compactification of $\mathbb{C}$. For $U \subset \hat{\mathbb{C}}$, we write $\mathrm{cl}(U)$ to denote the closure of U in $\hat{\mathbb{C}}$. For $r > 0$ we use

$$\begin{aligned} \mathbb{T}_r &= \{z \in \mathbb{C} : |z| = r\}, \\ \mathbb{T} &= \mathbb{T}_1 \end{aligned}$$

to denote circles centred at 0, and

$$D_r = \{z \in \mathbb{C} : |z| < r\},$$
$$D_r^\infty = \{z \in \hat{\mathbb{C}} : |z| > r\},$$
$$\mathbb{D} = D_1$$

to denote disks centred at 0 and ∞. Given $r < 1 < R$ the symbol

$$A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$$

will denote an open annulus containing the unit circle $\mathbb{T}$.

We write $L^p(\mathbb{T}_r) = L^p(\mathbb{T}_r, d\theta/2\pi)$ with $1 \leq p < \infty$ for the Banach space of p-integrable functions with respect to normalized one-dimensional Lebesgue measure on $\mathbb{T}_r$. Finally, for U an open subset of $\hat{\mathbb{C}}$ we use $\mathrm{Hol}(U)$ for the space of holomorphic functions on U.

Hardy-Hilbert spaces on disks and annuli will provide a convenient setting for our analysis. We briefly recall their properties in the following.

Definition 2.1 (Hardy-Hilbert spaces)**.** For $\rho > 0$ and $f : \mathbb{T}_\rho \to \mathbb{C}$ write

$$M_\rho(f) = \int_0^{2\pi} |f(\rho e^{i\theta})|^2 \,\frac{d\theta}{2\pi}.$$

Then the Hardy-Hilbert spaces on D_r and $A_{r,R}$ are given by

$$H^2(D_r) = \left\{ f \in \mathrm{Hol}(D_r) : \sup_{\rho \nearrow r} M_\rho(f) < \infty \right\},$$

and

$$H^2(A_{r,R}) = \left\{ f \in \mathrm{Hol}(A_{r,R}) : \sup_{\rho \nearrow R} M_\rho(f) + \sup_{\rho \searrow r} M_\rho(f) < \infty \right\}.$$

The Hardy-Hilbert space on the exterior disk D_R^∞ is defined accordingly, that is $f \in H^2(D_R^\infty)$ if $f \in \mathrm{Hol}(D_R^\infty)$ (or, equivalently, $f \circ \varsigma$ holomorphic on $D_{1/R}$ with $\varsigma(z) = 1/z$) and $\sup_{\rho \searrow R} M_\rho(f) < \infty$. Finally, $H_0^2(D_R^\infty) \subset H^2(D_R^\infty)$ denotes the subspace of functions vanishing at infinity.

A comprehensive account of Hardy spaces over general domains is given in the classic text [10]. A crisp treatment of Hardy spaces on the unit disk can be found in [18, Chapter 17]), while a good reference for Hardy spaces on annuli is [22]. We shall now collect a number of results which will be useful in what follows.

Any function in $H^2(U)$, where U is a disk or an annulus, can be extended to the boundary in the following sense. For $f \in H^2(D_r)$ there is an $f^* \in L^2(\mathbb{T}_r)$ such that

$$\lim_{\rho \nearrow r} f(\rho e^{i\theta}) = f^*(re^{i\theta}) \quad \text{for a.e. } \theta,$$

and analogously for $f \in H^2(D_R^\infty)$. Similarly, for $f \in H^2(A_{r,R})$ there are $f_1^* \in L^2(\mathbb{T}_r)$ and $f_2^* \in L^2(\mathbb{T}_R)$, with $\lim_{\rho \searrow r} f(\rho e^{i\theta}) = f_1^*(re^{i\theta})$ and $\lim_{\rho \nearrow R} f(\rho e^{i\theta}) = f_2^*(Re^{i\theta})$ for a.e. θ. It turns out that the spaces $H^2(U)$ are Hilbert spaces with inner products

$$(f,g)_{H^2(D_r)} = \int_0^{2\pi} f^*(re^{i\theta})\overline{g^*(re^{i\theta})} \,\frac{d\theta}{2\pi}$$

and

$$(f,g)_{H^2(A_{r,R})} = \int_0^{2\pi} f_1^*(Re^{i\theta})\overline{g_1^*(Re^{i\theta})}\,\frac{d\theta}{2\pi} + \int_0^{2\pi} f_2^*(re^{i\theta})\overline{g_2^*(re^{i\theta})}\,\frac{d\theta}{2\pi}.$$

Similarly, for $H^2(D_R^\infty)$.

Notation 2.2. In order to avoid cumbersome notation, we shall write $f(z)$ instead of $f^*(z)$ for z on the boundary of the domain.

Remark 2.3. It is not difficult to see that $\mathcal{E} = \{e_n : n \in \mathbb{Z}\}$ where

$$e_n(z) = \frac{z^n}{d_n}, \quad \text{with } d_n = \sqrt{r^{2n} + R^{2n}} \tag{2.1}$$

is an orthonormal basis for $H^2(A_{r,R})$. In particular, it follows that $f \in \mathrm{Hol}(A_{r,R})$ is in $H^2(A_{r,R})$ if and only if $f(z) = \sum_{n=-\infty}^{\infty} c_n e_n(z)$ with $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$, where the coefficients are given by $c_n = c_n(f) = (f, e_n)_{H^2(A_{r,R})}$. Note also that $\|f\|^2_{H^2(A_{r,R})} = \sum_{n=-\infty}^{\infty} |c_n|^2$.

3. Circle maps and transfer operators

The purpose of this section is to establish compactness of transfer operators on Hardy-Hilbert spaces for analytic expanding circle maps, defined as follows.

Definition 3.1. We say $\tau \colon \mathbb{T} \to \mathbb{T}$ is an *analytic expanding circle map* if

(i) τ is analytic on $\mathbb{T}$;
(ii) $|\tau'(z)| > 1$ for all $z \in \mathbb{T}$.

In particular, τ is a K-fold covering for some $K > 1$. With a slight abuse of notation we continue to write τ for its holomorphic extension to an open annulus $A_{r,R}$ for $r < 1 < R$ and let

$$\mathcal{A} = \{A_{r,R} : \tau \text{ and } 1/\tau \text{ holomorphic on } A_{r,R}\}\,.$$

Before proceeding to the definition of transfer operator associated to τ we require some more notation. Given two open subsets U and V of $\mathbb{C}$ we write

$$U \Subset V$$

if $\mathrm{cl}(U)$ is a compact subset of V.

By the expansivity of τ and [24, Lemma 2.2], we can choose A_0, A' and A in $\mathcal{A}$ with

$$A_0 \Subset A' \Subset A \text{ and } \tau(\partial A_0) \cap \mathrm{cl}(A) = \emptyset. \tag{3.1}$$

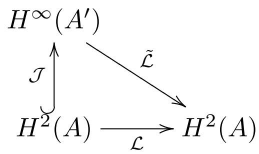
Given an analytic expanding circle map τ, we associate with it a transfer operator $\mathcal{L}$ by setting

$$(\mathcal{L}f)(z) = \sum_{k=1}^{K} \phi_k'(z)(f \circ \phi_k)(z)\,, \tag{3.2}$$

where ϕ_k denotes the k-th local inverse of τ.

As we shall see presently, the above choices of the annuli guarantee that $\mathcal{L}$ is a well-defined linear operator which maps $H^2(A)$ compactly to itself. In order to show this, we shall employ a factorization argument, similar to the ones used in [6, 24]. Let $H^\infty(A')$ be the Banach space of bounded holomorphic functions on A' equipped with the supremum norm. We can write $\mathcal{L} = \tilde{\mathcal{L}}\mathcal{J}$, where $\tilde{\mathcal{L}} \colon H^\infty(A') \to$

$H^2(A)$ is a lifted transfer operator given by the same functional expression (3.2) and $\mathcal{J}\colon H^2(A) \to H^\infty(A')$ is the canonical embedding



We use $H^\infty(A')$ instead of $H^2(A')$ as this choice allows for an easy proof of continuity of $\tilde{\mathcal{L}}$ in Lemma 3.2.

Let R, R' denote the radii of the circles forming the 'exterior' boundaries, and r, r' the radii of the circles forming the 'interior' boundaries of A and A', respectively, that is, $A = A_{r,R}$ and $A' = A_{r',R'}$.

Lemma 3.2. *The transfer operator $\tilde{\mathcal{L}}$ given by* (3.2) *maps $H^\infty(A')$ continuously to $H^2(A)$.*

Proof. We can factorize $\tilde{\mathcal{L}}$ as $\tilde{\mathcal{L}} = \hat{\mathcal{J}}\hat{\mathcal{L}}$, where $\hat{\mathcal{L}}\colon H^\infty(A') \to H^\infty(A)$, given by the functional expression (3.2), is continuous by [24, Lemma 2.5], and $\hat{\mathcal{J}}\colon H^\infty(A) \hookrightarrow H^2(A)$ is the canonical embedding. □

Next, we establish compactness of $\mathcal{J}\colon H^2(A) \hookrightarrow H^\infty(A')$ given by

$$(\mathcal{J}f)(z) = f(z) \quad \text{for } z \in A'.$$

Let $\{e_n : n \in \mathbb{Z}\}$ be the orthonormal basis for $H^2(A)$ given by (2.1), then any $f \in H^2(A)$ can be uniquely expressed as $f = \sum_{n\in\mathbb{Z}} c_n(f)e_n$. For $N \in \mathbb{N}$ define the finite rank operator $\mathcal{J}_N\colon H^2(A) \to H^\infty(A')$ by

$$(\mathcal{J}_N f)(z) = \sum_{n=-N+1}^{N-1} c_n(f)e_n(z) \quad \text{for } z \in A'.$$

Lemma 3.3. *Let $\mathcal{J}$ and $\mathcal{J}_N$ be as above. Then*

$$\lim_{N\to\infty} \|\mathcal{J} - \mathcal{J}_N\|_{H^2(A)\to H^\infty(A')} = 0.$$

In particular, the embedding $\mathcal{J}$ is compact.

Proof. For $z \in A'$, it follows by the Cauchy-Schwarz inequality that

$$\begin{aligned}
|(\mathcal{J}f)(z) - (\mathcal{J}_N f)(z)| &\leq \left(\sum_{|n|\geq N} |c_n(f)|^2\right)^{1/2} \left(\sum_{|n|\geq N} |e_n(z)|^2\right)^{1/2} \\
&\leq \|f\|_{H^2(A)} \left(\sum_{|n|\geq N} \frac{|z^n|^2}{r^{2n} + R^{2n}}\right)^{1/2} \\
&\leq \|f\|_{H^2(A)} \left(\sum_{n\geq N} \left|\frac{z}{R}\right|^{2n} + \sum_{n\geq N} \left|\frac{r}{z}\right|^{2n}\right)^{1/2}.
\end{aligned}$$

Thus,

$$\|\mathcal{J}f - \mathcal{J}_N f\|_{H^\infty(A')} \leq \|f\|_{H^2(A)} \left(\left(\frac{R'}{R}\right)^{2N} \frac{1}{1-(\frac{R'}{R})^2} + \left(\frac{r}{r'}\right)^{2N} \frac{1}{1-(\frac{r}{r'})^2}\right)^{1/2},$$

and the assertion follows. □

The factorization $\mathcal{L} = \tilde{\mathcal{L}}\mathcal{J}$ together with Lemmas 3.2 and 3.3 now imply the following result.

Proposition 3.4. *The transfer operator* $\mathcal{L}\colon H^2(A) \to H^2(A)$ *in* (3.2) *is compact.*

Remark 3.5. Closer inspection of Lemma 3.3 reveals that the singular values of $\mathcal{J}$ decay at an exponential rate. Thus $\mathcal{J}$ and hence $\mathcal{L}$ are trace-class. In fact, using results from [4] it is possible to show that both the singular values and the eigenvalues of $\mathcal{L}$ decay at an exponential rate, a property that $\mathcal{L}$ shares with other transfer operators arising from analytic maps (see, for example, [5, 11]).

4. Adjoint operator

A central step in showing our main result is to find an appropriate representation of the dual space on which the adjoint of the transfer operator has a simple structure.

For the remainder of this section we set $A = A_{r,R}$ and denote by $H^2(A)'$ the strong dual of $H^2(A)$, that is, the space of continuous linear functionals on $H^2(A)$ equipped with the topology of uniform convergence on the unit ball. We will show that $H^2(A)'$ is isomorphic to the topological direct sum $H^2(D_r) \oplus H^2_0(D_R^\infty)$, equipped with the norm $\|(h_1,h_2)\|^2 = \|h_1\|^2_{H^2(D_r)} + \|h_2\|^2_{H^2_0(D_R^\infty)}$. Similar representations of the duals of Hardy spaces for multiply connected regions can be found in [17, Proposition 3]. The present set-up is sufficiently simple to allow for a short proof of the representation.

Proposition 4.1. *The dual space* $H^2(A)'$ *is isomorphic to* $H^2(D_r) \oplus H^2_0(D_R^\infty)$ *with the isomorphism given by*

$$\begin{aligned} J\colon H^2(D_r) \oplus H^2_0(D_R^\infty) &\to H^2(A)' \\ (h_1,h_2) &\mapsto l, \end{aligned}$$

where

$$l(f) = \frac{1}{2\pi i}\int_{\mathbb{T}_r} f(z)h_1(z)\,dz + \frac{1}{2\pi i}\int_{\mathbb{T}_R} f(z)h_2(z)\,dz \quad (f \in H^2(A)). \tag{4.1}$$

Proof. We will first show that (4.1) defines a continuous functional $l \in H^2(A)'$ and that J is a bounded linear operator. In order to see this note that for any $(h_1,h_2) \in H^2(D_r) \oplus H^2_0(D_R^\infty)$ the linear functional $l = J(h_1,h_2)$ is bounded, since for any $f \in H^2(A)$ with $\|f\|_{H^2(A)} \leq 1$

$$|l(f)| \leq \left(r\,\|h_1\|_{H^2(D_r)} + R\,\|h_2\|_{H^2_0(D_R^\infty)} \right).$$

It follows that

$$\|J(h_1,h_2)\|_{H^2(A)'} \leq \sqrt{r^2+R^2}\sqrt{\|h_1\|^2_{H^2(D_r)} + \|h_2\|^2_{H^2_0(D_R^\infty)}}$$

and $\|J\|_{H^2(D_r)\oplus H^2_0(D_R^\infty)\to H^2(A)'} \leq \sqrt{r^2+R^2}$. Hence, J is well defined and bounded.

For injectivity, we suppose that $l = J(h_1,h_2) = 0$ and show that $h_1 = 0$ and $h_2 = 0$. In order to see this note that any $(h_1,h_2) \in H^2(D_r) \oplus H^2_0(D_R^\infty)$ can be written $h_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $h_2(z) = \sum_{n=1}^{\infty} a_{-n} z^{-n}$ with suitable coefficients $a_n \in \mathbb{C}$. Now let

$$\mathcal{E} = \{e_n : n \in \mathbb{Z}\} \quad \text{with} \quad e_n(z) = \frac{z^n}{d_n}$$

denote the orthonormal basis of $H^2(A)$ given in Remark 2.3. A short calculation using Lebesgue dominated convergence shows that

$$0 = (J(h_1, h_2))(e_n) = \frac{a_{-n-1}}{d_n} \quad \text{for all } n \in \mathbb{Z}, \tag{4.2}$$

which implies $h_1 = 0$ and $h_2 = 0$. Thus J is injective.

Finally, in order to show that J is surjective, fix $l \in H^2(A)'$. We will construct $(h_1, h_2) \in H^2(D_r) \oplus H_0^2(D_R^\infty)$ such that $J(h_1, h_2) = l$.

By the Riesz representation theorem there is a unique $g \in H^2(A)$ such that $l(f) = (f, g)_{H^2(A)}$ for all $f \in H^2(A)$. Moreover, g can be uniquely expressed as $g = \sum_{n \in \mathbb{Z}} c_n(g) e_n$. Now define

$$\begin{aligned} h_1(z) &= \sum_{n=0}^{\infty} \overline{c_{-n-1}(g)} d_{-n-1} z^n \quad \text{for } z \in D_r, \\ h_2(z) &= \sum_{n=1}^{\infty} \overline{c_{n-1}(g)} d_{n-1} z^{-n} \quad \text{for } z \in D_R^\infty. \end{aligned} \tag{4.3}$$

Using $\|g\|_{H^2(A)}^2 = \sum_{n \in \mathbb{Z}} |c_n(g)|^2 < \infty$, it follows that $h_1 \in H^2(D_r)$ and $h_2 \in H_0^2(D_R^\infty)$. Combining (4.2) and (4.3) we obtain

$$(J(h_1, h_2))(e_n) = \frac{a_{-n-1}}{d_n} = \frac{\overline{c_n(g)} d_n}{d_n} = \overline{c_n(g)} = (e_n, g)_{H^2(A)}$$

for every $n \in \mathbb{Z}$. Since the above equality also holds for all finite linear combinations of elements in $\mathcal{E}$ the continuity of J implies

$$(J(h_1, h_2))(f) = (f, g)_{H^2(A)} = l(f)$$

for all $f \in H^2(A)$. Thus J is surjective. □

Remark 4.2. The inverse J^{-1} of J can be obtained using the kernel $K_z \in H^2(A)$ defined by $K_z(w) = 1/(z - w)$ for $z \in \hat{\mathbb{C}} \setminus \mathrm{cl}(A)$. More precisely, J^{-1} is given by $l \mapsto (h_1, h_2)$, where $h_1(z) = l(-K_z)$ for $z \in D_r$ and $h_2(z) = l(K_z)$ for $z \in D_R^\infty$.

Returning to the setting of Section 3, let τ be an analytic expanding circle map and $A = A_{r,R} \in \mathcal{A}$ an annulus satisfying (3.1) such that the associated transfer operator $\mathcal{L}\colon H^2(A) \to H^2(A)$ is well defined and compact. We shall first assume that τ is orientation-preserving and comment on the orientation-reversing case at the end of this section. Using the representation of the dual space $H^2(A)'$ obtained in the previous lemma, we shall shortly derive an explicit form for the adjoint operator of $\mathcal{L}$.

Before doing so we require some more notation. Define $C^{(r)}\colon H^2(D_r) \to L^2(\mathbb{T}_r)$ by

$$(C^{(r)} h)(z) = h(\tau(z)) \quad \text{for } z \in \mathbb{T}_r\,, \tag{4.4}$$

and $C^{(R)}\colon H_0^2(D_R^\infty) \to L^2(\mathbb{T}_R)$ by

$$(C^{(R)} h)(z) = h(\tau(z)) \quad \text{for } z \in \mathbb{T}_R\,. \tag{4.5}$$

It turns out that $C^{(r)}$ and $C^{(R)}$ are compact, the proof of which relies on the following fact.

Lemma 4.3. *Let K be a compact subset of a disk D in $\mathbb{C}$. Then there exists a constant c_K depending on K only such that for any $f \in H^2(D)$*

$$\sup_{z\in K} |f(z)| \leq c_K \|f\|_{H^2(D)} .$$

Proof. This follows, for example, from [6, Lem. 2.9], or by a calculation using the Cauchy-Schwarz inequality similar to the proof of Lemma 3.3. □

We now have the following.

Lemma 4.4. *The operators $C^{(r)}$ and $C^{(R)}$ are compact.*

Proof. The choice of $A = A_{r,R}$ in (3.1) implies that $r_0 = \sup_{z\in\mathbb{T}_r} |\tau(z)| < r$, and we can choose a disk $D_{r'}$ with $D_{r_0} \subset\subset D_{r'} \subset\subset D_r$.

Let $\tilde{C}^{(r)} : H^2(D_{r'}) \to L^2(\mathbb{T}_r)$ be defined by the functional expression as in (4.4), but now considered on $H^2(D_{r'})$. The operator is continuous since

$$\left\|\tilde{C}^{(r)}h\right\|_{L^2(\mathbb{T}_r)} \leq \sup_{z\in\tau(\mathbb{T}_r)} |h(z)| \leq \sup_{z\in\mathrm{cl}(D_{r_0})} |h(z)| \leq c_K \|h\|_{H^2(D_{r'})} ,$$

where we have used Lemma 4.3 with $K = \mathrm{cl}(D_{r_0})$. The lemma follows since we can write $C^{(r)} = \tilde{C}^{(r)}\tilde{\mathcal{J}}$ with $\tilde{\mathcal{J}} \colon H^2(D_r) \hookrightarrow H^2(D_{r'})$ denoting the canonical embedding, which is compact (see, for example, [6, Lemma 2.9]). The proof for $C^{(R)}$ is similar. □

Remark 4.5. An argument similar to the proof of Lemma 3.3 shows that the embedding $\tilde{\mathcal{J}} \colon H^2(D_r) \hookrightarrow H^2(D_{r'})$ is trace class; in fact, its singular values decay at an exponential rate (see Remark 3.5). Thus $C^{(r)}$ is trace class — and so is $C^{(R)}$, by a similar argument.

Next, we need to define certain projection operators on $L^2(\mathbb{T}_\rho)$. For any $g \in L^2(\mathbb{T}_\rho)$ we can write $g(z) = \sum_{n\in\mathbb{Z}} g_n z^n$, so that $g = g_+ + g_-$ with $g_+(z) = \sum_{n=0}^{\infty} g_n z^n$ and $g_-(z) = \sum_{n=1}^{\infty} g_{-n} z^{-n}$. Since $\|g\|^2_{L^2(\mathbb{T}_\rho)} = \sum_{n=-\infty}^{\infty} |g_n|^2 \rho^{2n} < \infty$, the functions g_+ and g_- can be viewed as functions in $H^2(D_\rho)$ and $H^2_0(D^\infty_\rho)$, respectively. Then we define the bounded projection operators $\Pi^{(\rho)}_+ \colon L^2(\mathbb{T}_\rho) \to H^2(D_\rho)$ and $\Pi^{(\rho)}_- \colon L^2(\mathbb{T}_\rho) \to H^2_0(D^\infty_\rho)$ by

$$\Pi^{(\rho)}_+(g) = g_+ \quad \text{and} \quad \Pi^{(\rho)}_-(g) = g_-. \tag{4.6}$$

Finally, let $\mathcal{L}' \colon H^2(A)' \to H^2(A)'$ denote the adjoint operator of $\mathcal{L}$ in the Banach space sense, that is, $(\mathcal{L}'l)(f) = l(\mathcal{L}f)$ for all $l \in H^2(A)'$ and $f \in H^2(A)$. The following proposition provides an explicit representation $\mathcal{L}^\dagger$ of $\mathcal{L}'$ via

$$\mathcal{L}^\dagger = J^{-1}\mathcal{L}'J,$$

as an operator on $H^2(D_r) \oplus H^2_0(D^\infty_R)$ given by compositions of $C^{(\rho)}$, $\Pi^{(\rho)}_-$ and $\Pi^{(\rho)}_+$ for $\rho = r, R$.

Proposition 4.6. *Let $\mathcal{L} \colon H^2(A) \to H^2(A)$ be the transfer operator associated to an analytic orientation-preserving expanding circle map τ, with $A \in \mathcal{A}$ as in (3.1). Then the isomorphism J conjugates the adjoint $\mathcal{L}'$ of $\mathcal{L}$ to*

$$\mathcal{L}^\dagger \colon H^2(D_r) \oplus H^2_0(D^\infty_R) \to H^2(D_r) \oplus H^2_0(D^\infty_R)$$

given[1] *by*

$$\mathcal{L}^\dagger = \begin{pmatrix} \Pi_+^{(r)} C^{(r)} & \Pi_+^{(R)} C^{(R)} \\ \Pi_-^{(r)} C^{(r)} & \Pi_-^{(R)} C^{(R)} \end{pmatrix}, \tag{4.7}$$

that is $\mathcal{L}^\dagger = J^{-1}\mathcal{L}'J$.

Proof. We want to show that $\mathcal{L}'J = J\mathcal{L}^\dagger$, that is,

$$(\mathcal{L}'J(h_1,h_2))(f) = (J\mathcal{L}^\dagger(h_1,h_2))(f) \tag{4.8}$$

for all $(h_1,h_2) \in H^2(D_r) \oplus H_0^2(D_R^\infty)$ and $f \in H^2(A)$. For any such (h_1,h_2) and f, the adjoint property yields

$$\begin{aligned}(\mathcal{L}'J(h_1,h_2))(f) &= (J(h_1,h_2))(\mathcal{L}f) \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}_r} (\mathcal{L}f)(z)h_1(z)\,dz + \frac{1}{2\pi i}\int_{\mathbb{T}_R} (\mathcal{L}f)(z)h_2(z)\,dz\ .\end{aligned}$$

Now let a basis for $H^2(D_r) \oplus H_0^2(D_R^\infty)$ be given by $\mathcal{P} = \{(p_n,0) : n \in \mathbb{N}_0\} \cup \{(0,p_{-n}) : n \in \mathbb{N}\}$ with $p_n(z) = z^n$, where $p_n \in H^2(D_r)$ if $n \geq 0$ and $p_n \in H_0^2(D_R^\infty)$ if $n < 0$.

Take $f \in \mathcal{E}$, where $\mathcal{E}$ is the basis for $H^2(A)$ given by (2.1). For $n \in \mathbb{N}_0$ and $(h_1,h_2) = (p_n,0) \in \mathcal{P}$ we get

$$\begin{aligned}(\mathcal{L}'J(h_1,0))(f) &= \frac{1}{2\pi i}\int_{\mathbb{T}_r} (\mathcal{L}f)(z)h_1(z)\,dz \\ &\overset{(a)}{=} \frac{1}{2\pi i}\int_{\mathbb{T}} (\mathcal{L}f)(z)h_1(z)\,dz \\ &= \frac{1}{2\pi i}\sum_{k=1}^{K}\int_{\mathbb{T}} \phi_k'(z)(f\circ\phi_k)(z)h_1(z)\,dz \\ &\overset{(b)}{=} \frac{1}{2\pi i}\sum_{k=1}^{K}\int_{\phi_k(\mathbb{T})} f(w)(h_1\circ\tau)(w)\,dw \\ &\overset{(c)}{=} \frac{1}{2\pi i}\int_{\mathbb{T}} f(w)(h_1\circ\tau)(w)\,dw \\ &\overset{(d)}{=} \frac{1}{2\pi i}\int_{\mathbb{T}_r} f(w)(h_1\circ\tau)(w)\,dw\,.\end{aligned}$$

Here, equalities (a) and (d) follow since the integrands are analytic on A, equality (b) follows by change of variables with $w = \phi_k(z)$ and $\tau(w) = z$, and equality (c) is a consequence of the fact that $\bigcup_k \phi_k(\mathbb{T}) = \mathbb{T}$ up to measure zero. Then, by the definition of $\Pi_+^{(r)}$ and $\Pi_-^{(r)}$,

$$\begin{aligned}&(\mathcal{L}'J(h_1,0))(f) \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}_r} f(w)(\Pi_+^{(r)}(h_1\circ\tau))(w)\,dw + \frac{1}{2\pi i}\int_{\mathbb{T}_r} f(w)(\Pi_-^{(r)}(h_1\circ\tau))(w)\,dw \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}_r} f(w)(\Pi_+^{(r)}(h_1\circ\tau))(w)\,dw + \frac{1}{2\pi i}\int_{\mathbb{T}_R} f(w)(\Pi_-^{(r)}(h_1\circ\tau))(w)\,dw \\ &= (J\mathcal{L}^\dagger(h_1,0))(f).\end{aligned}$$

[1]Note that $H^2(D_R)$ can be viewed as a subspace of $H^2(D_r)$, and similarly $H_0^2(D_r^\infty)$ as a subspace of $H_0^2(D_R^\infty)$. Thus the off-diagonal elements of $\mathcal{L}^\dagger$ are well defined.

The penultimate equality follows from the fact that $\Pi_-^{(r)}(h_1 \circ \tau) \in H_0^2(D_r^\infty)$.

Analogously, for $n \in \mathbb{N}$ and $(h_1, h_2) = (0, p_{-n}) \in \mathcal{P}$, the same argument shows

$$(\mathcal{L}' J(0, h_2))(f) = (J\mathcal{L}^\dagger(0, h_2))(f).$$

Hence, for $f \in \mathcal{E}$, by linearity (4.8) holds for all finite linear combinations of basis elements (h_1, h_2) in $\mathcal{P}$. Since these form a dense subspace of $H^2(D_r) \oplus H_0^2(D_R^\infty)$, and $\mathcal{L}'$, $\mathcal{L}^\dagger$ and J are continuous operators, equality (4.8) holds for all $(h_1, h_2) \in H^2(D_r) \oplus H_0^2(D_R^\infty)$ and $f \in \mathcal{E}$. By continuity, this extends to all $f \in H^2(A)$, which completes the proof. □

Remark 4.7. Lemma 4.4 and continuity of the projection operators in (4.6) imply that $\mathcal{L}^\dagger$ is compact. Note, however, that this also follows from compactness of $\mathcal{L}$ guaranteed by the choice of A in (3.1).

Remark 4.8. The above proposition requires only minor modifications if τ is assumed to be orientation-reversing. The operators $C^{(r)}$ and $C^{(R)}$ in (4.4) and (4.5) are replaced with $\hat{C}^{(r)} \colon H^2(D_r) \to L^2(\mathbb{T}_R)$ and $\hat{C}^{(R)} \colon H_0^2(D_R^\infty) \to L^2(\mathbb{T}_r)$, defined by

$$(\hat{C}^{(r)} h)(z) = h(\tau(z)) \quad \text{for } z \in \mathbb{T}_R$$

and

$$(\hat{C}^{(R)} h)(z) = h(\tau(z)) \quad \text{for } z \in \mathbb{T}_r\,,$$

which are compact by the same argument as in Lemma 4.4. The adjoint operator in the proposition is then represented by

$$\mathcal{L}^\dagger = -\begin{pmatrix} \Pi_+^{(R)} \hat{C}^{(r)} & \Pi_+^{(r)} \hat{C}^{(R)} \\ \Pi_-^{(R)} \hat{C}^{(r)} & \Pi_-^{(r)} \hat{C}^{(R)} \end{pmatrix}.$$

5. Spectrum for Blaschke products

Having discussed transfer operators $\mathcal{L}$ associated with analytic expanding circle maps and a convenient representation of the corresponding adjoint operators (Proposition 4.6), we shall now use this representation to obtain the full spectrum of $\mathcal{L}$ for finite Blaschke products, a class of circle maps defined as follows.

Definition 5.1. For $n \geq 2$, let $\{a_1, \ldots, a_n\}$ be a finite set of complex numbers in the open unit disk $\mathbb{D}$. A *finite Blaschke product* is a map of the form

$$B(z) = C \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i} z},$$

where $|C| = 1$.

It follows from the definition that

(i) B is a meromorphic function on $\hat{\mathbb{C}}$ with zeros a_i and poles $1/\overline{a}_i$;
(ii) B is holomorphic on a neighbourhood of $\overline{\mathbb{D}}$ with $B(\mathbb{D}) = \mathbb{D}$ and $B(\mathbb{T}) = \mathbb{T}$.

Note also that a function f is holomorphic on an open neighbourhood of $\overline{\mathbb{D}}$ with $f(\mathbb{T}) = \mathbb{T}$ if and only if f is a finite Blaschke product (see, for example, [8, Exercise 6.12]).

Let $\tau \colon \mathbb{T} \to \mathbb{T}$ denote the restriction of a finite Blaschke product B to $\mathbb{T}$. A short calculation shows that τ is expanding if $\sum_{i=1}^{n}(1 - |a_i|)/(1 + |a_i|) > 1$ (see

[14, Corollary to Prop. 1] for details). Expansiveness of τ can also be expressed in terms of the nature of the fixed points of B, as the following result shows.

Proposition 5.2. *Let B and τ be as above and $|\tau'(z)| > 1$ for all $z \in \mathbb{T}$. Then B has exactly $n-1$ fixed points on $\mathbb{T}$, which are repelling, and two fixed points $z_0 \in \mathbb{D}$ and $\hat{z}_0 = 1/\overline{z}_0 \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, which are attracting.*

Proof. See [16, Prop. 2.1] and [26]. □

Crucial for the proof of our main theorem is the notion of a composition operator, which we briefly recall.

Definition 5.3. Let U be an open region in $\hat{\mathbb{C}}$. If $\psi\colon U \to U$ is holomorphic, then $C_\psi\colon \mathrm{Hol}(U) \to \mathrm{Hol}(U)$ defined by $C_\psi f = f \circ \psi$ is called a *composition operator (with symbol ψ)*.

Note that in the literature the term 'composition operator' is mostly used in the context of holomorphic functions. The operators in (4.4) considered on $L^2(\mathbb{T}_r)$ do not formally fall into this category, but will turn out to be composition operators for Blaschke product symbols.

We are now able to state our main result.

Theorem 5.4. *Let B be a finite Blaschke product such that $\tau = B|_{\mathbb{T}}$ is an analytic expanding circle map. Then*

(a) the transfer operator $\mathcal{L}\colon H^2(A) \to H^2(A)$ associated with τ is well defined and compact for some annulus $A \in \mathcal{A}$, and

(b) the spectrum of $\mathcal{L}\colon H^2(A) \to H^2(A)$ is given by

$$\sigma(\mathcal{L}) = \{1\} \cup \{\lambda(z_0)^n : n \in \mathbb{N}\} \cup \{\lambda(\hat{z}_0)^n : n \in \mathbb{N}\} \cup \{0\}\,, \tag{5.1}$$

where $\lambda(z_0)$ and $\lambda(\hat{z}_0) = \overline{\lambda(z_0)}$ are the multipliers[2] of the unique fixed points z_0 and $\hat{z}_0$ of B in $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, respectively.

Moreover, the algebraic multiplicity of the leading eigenvalue is 1, while for each other nonzero eigenvalue the algebraic (and geometric) multiplicity is equal to its number of occurrences in the list (5.1), meaning it is 1 if $\lambda(z_0)^n \neq \lambda(\hat{z}_0)^n$ and 2 otherwise.

Proof. The first assertion is obvious, as τ is an analytic expanding circle map and we can choose $A = A_{r,R} \in \mathcal{A}$ as in (3.1) such that $\mathcal{L}$ is well defined and compact by the results in Section 3.

For the second claim, we will use the fact that the spectrum of $\mathcal{L}$ coincides with that of its adjoint $\mathcal{L}'$, which together with the structure of the representation $\mathcal{L}^\dagger$ of $\mathcal{L}'$ will allow us to deduce (5.1).

We start by observing that for the chosen A we have $B(\partial A) \cap \mathrm{cl}(A) = \emptyset$, as well as $B(D_r) \Subset D_r$ and $B(D_R^\infty) \Subset D_R^\infty$. It follows that $f \circ B \in H^2(D_r)$ for any $f \in H^2(D_r)$, and $f \circ B \in H^2(D_R^\infty)$ for any $f \in H^2(D_R^\infty)$, so that $C_B^{(r)} f = f \circ B$ and $C_B^{(R)} f = f \circ B$ define composition operators on $H^2(D_r)$ and $H^2(D_R^\infty)$, respectively. It is a standard fact that $B(D_r) \Subset D_r$ guarantees compactness of $C_B^{(r)}$ (see, for example, [9, pp. 128-129]). Similarly for $C_B^{(R)}$. It is also well known (see [15,

[2]Recall that the multiplier $\lambda(z^*)$ of a fixed point z^* of a rational map R is given by $R'(z^*)$ if $z^* \in \mathbb{C}$ and $1/R'(z^*)$ if $z^* = \infty$. For Blaschke products the equality $\lambda(\hat{z}_0) = \overline{\lambda(z_0)}$ follows from a straightforward calculation.

Lem. 7.10] or [9, Thm. 7.20]) that all eigenvalues of a compact composition operator C_ψ are simple and are given by the non-negative integer powers of the multiplier of the unique attracting fixed point of ψ. Hence,

$$\sigma(C_B^{(r)}) = \{\lambda(z_0)^n : n \in \mathbb{N}_0\} \cup \{0\}$$

and

$$\sigma(C_B^{(R)}) = \{\lambda(\hat{z_0})^n : n \in \mathbb{N}_0\} \cup \{0\},$$

where z_0 and $\hat{z_0}$ are the unique attracting fixed points of B in D_r and D_R^∞, respectively (see Proposition 5.2).

We now explain how to use these observations to determine the spectrum of $\mathcal{L}^\dagger$ given in (4.7). Note that $\Pi_+^{(r)} C_B^{(r)} = C_B^{(r)}$ and $\Pi_-^{(r)} C_B^{(r)} = 0$, where $\Pi_+^{(r)}$ and $\Pi_-^{(r)}$ are the projection operators in (4.6). Thus the operator $\mathcal{L}^\dagger$ is given by

$$\mathcal{L}^\dagger = \begin{pmatrix} C_B^{(r)} & \Pi_+^{(R)} C_B^{(R)} \\ 0 & \Pi_-^{(R)} C_B^{(R)} \end{pmatrix}. \tag{5.2}$$

In particular, $\mathcal{L}^\dagger$ leaves $H^2(D_r) \oplus \{0\}$ invariant. The operator $\Pi_-^{(R)} C_B^{(R)}$ is not a composition operator on $H_0^2(D_R^\infty)$, but we can relate its spectrum to the spectrum of $C_B^{(R)}$ on $H^2(D_R^\infty)$. More precisely,

$$\sigma(\Pi_-^{(R)} C_B^{(R)}) = \sigma(C_B^{(R)}) \setminus \{1\}, \tag{5.3}$$

as we shall see below. Then, using (5.3) the assertion of the theorem follows, since

$$\begin{aligned} \sigma(\mathcal{L}^\dagger) &= \sigma(C_B^{(r)}) \cup \sigma(\Pi_-^{(R)} C_B^{(R)}) \\ &= \{1\} \cup \{\lambda(z_0)^n : n \in \mathbb{N}\} \cup \{\lambda(\hat{z_0})^n : n \in \mathbb{N}\} \cup \{0\}, \end{aligned}$$

and $\sigma(\mathcal{L}) = \sigma(\mathcal{L}') = \sigma(\mathcal{L}^\dagger)$.

It remains to prove (5.3). For brevity, we drop the superscript (R) from $\Pi_-^{(R)}$, $\Pi_+^{(R)}$ and $C_B^{(R)}$ since we only consider functions in $H^2(D_R^\infty)$ in what follows. Observe that for $f \in H^2(D_R^\infty)$, we have $(\Pi_+ f)(z) = f(\infty)$, which implies

$$C_B \Pi_+ = \Pi_+ \quad \text{and} \quad \Pi_- C_B = \Pi_- C_B \Pi_- . \tag{5.4}$$

Note that 1 is an eigenvalue of C_B if and only if the corresponding eigenfunction is constant. Take $\mu \in \sigma(C_B)$ with $\mu(1-\mu) \neq 0$. Since C_B is compact, there is a non-zero $f \in H^2(D_R^\infty)$ with $C_B f = \mu f$. The second equality in (5.4) now implies $\Pi_- C_B \Pi_- f = \mu \Pi_- f$. But since $\mu \neq 1$ the eigenvector f is non-constant, so we have $0 \neq \Pi_- f \in H_0^2(D_R^\infty)$ and thus $\mu \in \sigma(\Pi_- C_B)$.

To show the converse inclusion, take $\mu \in \sigma(\Pi_- C_B)$ with $\mu \neq 0$. Since $\Pi_- C_B$ is compact, there is a non-zero $f \in H_0^2(D_R^\infty)$ with $\Pi_- C_B f = \mu f$. First observe that[3] $\mu \neq 1$. Next we note that if $\mu(\mu - 1) \neq 0$, then $(1-\mu) f - \Pi_+ C_B f \neq 0$ (for otherwise f would be zero). Finally, we use (5.4) to show that $(1-\mu) f - \Pi_+ C_B f$ is an eigenfunction of C_B with eigenvalue μ:

$$\begin{aligned} C_B\left((1-\mu) f - \Pi_+ C_B f\right) &= (1-\mu)(C_B f + (\mu f - \Pi_- C_B f)) - C_B \Pi_+ C_B f \\ &= \mu(1-\mu) f + (1-\mu)(I - \Pi_-) C_B f - \Pi_+ C_B f \\ &= \mu\left((1-\mu) f - \Pi_+ C_B f\right). \end{aligned}$$

[3] In order to see this, note that otherwise $\Pi_- C_B f = f$, which implies $f \circ B - f = \text{const}$. However $f(B(\hat{z_0})) - f(\hat{z_0}) = 0$, which implies $f = f \circ B$. Thus $f = \text{const}$, so $\Pi_- C_B f = 0$, contradicting the fact that $\mu \neq 0$.

Thus $\sigma(\Pi_- C_B) = \sigma(C_B) \setminus \{1\}$ as claimed. □

Remark 5.5. Since the entries of $\mathcal{L}^\dagger$ in (5.2) are trace class operators (see Remark 4.5), it is also possible to determine the spectrum of $\mathcal{L}^\dagger$ from the zeros of the corresponding Fredholm determinant, which, in this case, is explicitly computable. The route taken above, however, is considerably more direct and conceptually simpler.

Remark 5.6. The main tool in the above proof is the representation of the adjoint operator in terms of compressions of composition operators, which in turn is based on the identification of the dual space with a space of functions holomorphic on the complement of a neighbourhood of the unit circle. This idea of investigating the spectral properties of transfer operators by passing to their adjoints using suitable dual space representations is not new. It has proved fruitful in studying transfer operators associated with analytic hyperbolic maps (see [20]) and in characterising spectral properties of generalised transfer operators associated to certain polynomial maps or rational maps possessing a unique attracting fixed point in $\hat{\mathbb{C}}$ (see [12, 13]).

The novelty of our result is the representation of adjoint operators for expanding circle maps, which in turn yields completely explicit expressions for the spectra of transfer operators arising from Blaschke maps.

The following examples illustrate our main result.

Example 5.7. The map $B(z) = z^n$ for $n \geq 2$ has two attracting fixed points $z_0 = 0$ and $\hat{z_0} = \infty$ with $\lambda(z_0) = \lambda(\hat{z}_0) = 0$. Thus $\sigma(\mathcal{L}) = \{1, 0\}$.

Curiously enough, these are not the only examples for which $\sigma(\mathcal{L}) = \{1, 0\}$, as the next example shows.

Example 5.8. Let $B(z) = z^m(z - b)/(1 - bz)$ for $b \in (-1, 1)$ and fixed integer $m \geq 2$. Then $B|_{\mathbb{T}}$ is an expanding $(m+1)$-to-1 circle map. As the multipliers of the attracting fixed points of B are vanishing, just as in Example 5.7, we get $\sigma(\mathcal{L}) = \{1, 0\}$.

Example 5.9. For the family of maps $B(z) = z(\mu - z)/(1 - \overline{\mu}z)$ considered in [24], the restriction $B|_{\mathbb{T}}$ is an expanding circle map for any $\mu \in \mathbb{D}$. The attracting fixed points are $z_0 = 0$ and $\hat{z_0} = \infty$ with $\lambda(z_0) = \mu$ and $\lambda(\hat{z}_0) = \overline{\mu}$. Thus

$$\sigma(\mathcal{L}) = \{1\} \cup \{\mu^n : n \in \mathbb{N}\} \cup \{\overline{\mu}^n : n \in \mathbb{N}\} \cup \{0\} \ .$$

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