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Gauge Invariants and Correlators in Flavoured Quiver Gauge Theories

Paolo Mattioli [1] **and Sanjaye Ramgoolam** [2]

Centre for Research in String Theory, School of Physics and Astronomy,
Queen Mary University of London,
Mile End Road, London E1 4NS, UK

ABSTRACT

In this paper we study the construction of holomorphic gauge invariant operators for general quiver gauge theories with flavour symmetries. Using a characterisation of the gauge invariants in terms of equivalence classes generated by permutation actions, along with representation theory results in symmetric groups and unitary groups, we give a diagonal basis for the 2-point functions of holomorphic and anti-holomorphic operators. This involves a generalisation of the previously constructed Quiver Restricted Schur operators to the flavoured case. The 3-point functions are derived and shown to be given in terms of networks of symmetric group branching coefficients. The networks are constructed through cutting and gluing operations on the quivers.

[1] p.mattioli@qmul.ac.uk
[2] s.ramgoolam@qmul.ac.uk

Contents

1 Introduction

Finite N aspects of AdS/CFT [1–3], such as giant gravitons [4], the stringy exclusion principle [5] and LLM geometries [6], have motivated the study of multi-matrix sectors of $\mathcal{N} = 4$ SYM, associated with different BPS sectors of the theory. These multi-matrix systems are also of interest purely from the point of view of supersymmetric gauge theory and their moduli spaces (*e.g.* [7]).

In this paper we study correlation functions of holomorphic and anti-holomorphic gauge invariant operators in quiver gauge theories with flavour symmetries, in the zero coupling limit. This builds on the results in our previous paper focused on enumeration of gauge invariant operators [8] and proceeds to explicit construction of the operators and consideration of free field two and three point functions. These have non-trivial dependences on the structure of the operators and on the ranks of the gauge and flavour symmetries. Our results are exact in the ranks, and their expansions contain information about the planar limit as well as all order expansions. The techniques we use build on earlier work exploiting representation theory techniques in the context of $\mathcal{N} = 4$ SYM [9–19]. The zero coupling results contain information about a singular limit from the point of view of the dual AdS. For special BPS sectors, where non-renormalization theorems are available, the representation theory methods have made contact with branes and geometries in the semiclassical AdS background. These representation theoretic studies were extended beyond $\mathcal{N} = 4$ SYM to ABJM [20] and conifolds [21–24]. The case of general quivers was studied in [25] and related work on quivers has since appeared in [26–29].

In the context of AdS/CFT, adding matter to $\mathcal{N} = 4$ SYM introduces flavour symmetries [30–34]. Typically, the added matter transforms in fundamental and anti-fundamental representations of these flavour symmetries. Matrix invariants in flavoured gauge theories do not need to be invariant under the flavour group: on the contrary, they have free indices living in the representation carried by their constituent fields. In this paper, we consider a general class of flavoured free gauge theories parametrised by a quiver. A quiver is a directed graph comprising of round nodes (gauge groups) and square nodes (flavour groups). The directed edges which join the round nodes corresponds to fields transforming in the bi-fundamental representation of the gauge group, as illustrated in subsection 1.1. Edges stretching between a round and a square node correspond to fields carrying a fundamental or antifundamental representation of the flavour group, depending on their orientation. We will call them simply quarks and antiquarks.

It was shown in [25] that the quiver, besides being a compact way to encode all the gauge groups and the matter content of the theory, is a powerful computational tool for correlators of gauge invariants. In that paper a generalisation of permutation group characters, called *quiver characters*, was introduced, involving branching coefficients of permutation groups in a non-trivial way. Obtaining the quiver character from the quiver diagram involves splitting each gauge node into *two* nodes, called positive and negative nodes. The first one collects all the fields coming into the original node, while the second one collects all the fields outgoing from the original node. A new line is added to join the positive and the negative node of the split-node diagram. Each edge in this modified quiver is decorated with appropriate representation theory data, as will be explained in the following sections. The properties of these characters, which have natural pictorial representations, allowed the derivation of counting formulae for the gauge invariants and expressions for the correlation functions.

In this paper, we will be concerned with the construction of a basis for the Hilbert space of holomorphic matrix invariants for the class of quiver gauge theories with $\prod_a U(N_a)$ gauge group and $\prod_a SU(F_a)\times SU(\bar{F}_a)\times U(1)$ flavour group. This basis is obtained in terms of *Quiver Restricted Schur Polynomials* $\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})$, that we define in Section 3. These are a generalisation of the restricted Schur operators introduced in [12–14,35,36]. In [25], the non-flavoured versions of these objects were called Generalised Restricted Schur operators, constructed in terms of quiver characters $\chi_{\mathcal{Q}}(\boldsymbol{L})$ where $\boldsymbol{L}$ is a collection of representation theory labels. In this flavoured case, we will find generalisations of these quiver characters, where the representation labels will include flavour states organised according to irreducible representations of the flavour groups. The advantages of using this approach is twofold. On the one hand, the Quiver Restricted Schur polynomials are orthogonal in the free field metric, as we will show, even for flavoured gauge theories. This leads to the simple expression for the two point function in eq. (4.1):

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\, \mathcal{O}^{\dagger}_{\mathcal{Q}}(\boldsymbol{L'}) \right\rangle = \delta_{\boldsymbol{L},\boldsymbol{L'}}\; c_{\vec{n}} \prod_a f_{N_a}(R_a) \tag{1.1}$$

In this equation $f_{N_a}(R_a)$ represents the product of weights of the $U(N_a)$ representation R_a, where a runs over the gauge nodes of the quiver. $c_{\vec{n}}$ is a constant depending on the matter content of the matrix invariant $\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})$, given in (3.27). On the other hand, the Quiver Restricted Schur polynomial formalism offers a simple way to capture the finite N constraints of matrix invariants. This can be seen directly from (1.1): each $f_{N_a}(R_a)$ vanishes if the length of the first column of the R_a Young diagram exceeds N_a.

In subsection 4.2 we give an N-exact expression for the three point function of matrix invariants in the free limit. This computation is performed using the Quiver Restricted Schur polynomial basis. Specifically, we will derive the $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$ coefficients in

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(1)})\, \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(2)})\, \mathcal{O}^{\dagger}_{\mathcal{Q}}(\boldsymbol{L}^{(3)}) \right\rangle = c_{\vec{n}^{(3)}}\; G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}} \prod_a f_{N_a}\left(R_a^{(3)}\right) \tag{1.2}$$

The analytical expression for $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$ looks rather complicated, but it can be easily understood in terms of diagrams. Although the identities we need appear somewhat complex, they all have a simple diagrammatic interpretation. Diagrammatics therefore play a central role in this paper: all the quantities we define and the calculational steps we perform can be visualised in terms of networks involving *symmetric group branching coefficients* and *Clebsch-*

Gordan coefficients. Both these quantities are defined in Section 3. The quantity $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$ is actually found to be a product over the gauge groups: for each gauge group there is a network of symmetric group branching coefficients and a single Clebsch-Gordan coefficient.

The organisation of this paper is as follows. In the next section we establish the notation we will use throughout the paper, and we specify the class of quiver theories we will focus on.

In Section 2 we describe a permutation based approach to label matrix invariants of the flavoured gauge theories under study. A matrix invariant will be constructed using a set of permutations (schematically σ) associated with gauge nodes of the quiver, and by a collection of fundamental and antifundamental states (schematically $s, \bar{s}$) of the flavour group, associated with external flavour nodes. In this section we highlight how the simplicity of apparently complex formulae can be understood via diagrammatic techniques. We describe equivalence relations, generated by the action of permutations associated with edges of the quiver (schematically η), acting on the gauge node permutations and flavour states. Equivalent data label the same matrix invariant. The equivalence is explained further and illustrated in Appendix A. The equivalences η can be viewed as "permutation gauge symmetries", while the $(\sigma, s, \bar{s})$ can be viewed as "matter fields" for these permutation gauge symmetries.

In Section 3 we give a basis of the matrix invariants using representation theory data, $\boldsymbol{L}$. This can be viewed as a *dual basis* where representation theory is used to perform a Fourier transformation on the equivalence classes of the permutation description. We refer to these gauge invariants, polynomial in the bi-fundamental and fundamental matter fields, as Quiver Restricted Schur polynomials. In this section we introduce the two main mathematical ingredients needed in this formalism. These are the symmetric group branching coefficients and the Clebsch-Gordan coefficients. Their definition will be accompanied by a corresponding diagram.

In Section 4 we derive the results for the free field two and three point function of gauge invariants. In subsection 4.1 we show that the two point function which couples holomorphic and anti-holomorphic matrix invariants is diagonal in the basis of Quiver Restricted Schur polynomials. In subsection 4.2 we give a diagrammatic description of the structure constants of the ring of Holomorphic Gauge Invariant Operators (GIOs). In particular, we present a step by step procedure to obtain such a diagram for the example of an $\mathcal{N} = 2$ SQCD, starting from its split-node diagram. Using these formulae, we identify four selection rules, all expressed in terms of symmetric group representation theory data. The analytical calculations are reported in Appendices C and D, and rely on the Quiver Restricted Schur polynomial technology introduced in the previous section.

Finally, in Section 5, we give some examples of the matrix invariants we can build using our method, for the case of an $\mathcal{N} = 2$ SQCD.

1.1 Definitions and framework

In this paper we consider free quiver gauge theories with gauge group $\prod_{a=1}^{n} U(N_a)$ and flavour symmetry of the general schematic form $\prod_{a=1}^{n} U(F_a) \times U(\bar{F}_a)$. Specifically, to work in the most general configuration, we choose to focus our attention to the subgroup $\prod_{a=1}^{n}[\times_\beta U(F_{a,\beta}) \times_\gamma U(\bar{F}_{a,\gamma})]$ of the flavour symmetry where $F_a = \sum_\beta F_{a,\beta}$ and $\bar{F}_a = \sum_\gamma \bar{F}_{a,\gamma}$. This more general flavour symmetry, where the $U(F_a) \times U(\bar{F}_a)$ is broken to a product of unitary groups for the quarks and anti-quarks, is likely to be useful when interactions are turned on. Our calculations work without any significant modification for this case of product global symmetry, hence we will work in this generality.

To recover the results for the global symmetry $U(F_a) \times U(\bar{F}_a)$ it is enough to drop the β, γ labels from all the equations that we are going to write. The constraint $F_a = \bar{F}_a$ solves chiral gauge anomaly conditions. As a last remark, notice that strictly speaking the global symmetry of the free theory contains only the determinant one part $S(U(F_{a,1}) \times U(F_{a,2}) \times \cdots U(F_{a,M_a}) \times U(\bar{F}_{a,1}) \times \cdots \times U(\bar{F}_{a,\bar{M}_a}))$. This means that, although for simplicity we write $\prod_{a=1}^{n} \left[\times_\beta U(F_{a,\beta}) \times_\gamma U(\bar{F}_{a,\gamma})\right]$ as the global symmetry, all the states we will write are neutral under the $U(1)$ which acts with a phase on all of the chiral fields and with the opposite phase on all of the anti-chiral fields. This $U(1)$ is part of the $U(N_a)$ gauge symmetry.

We now introduce the diagrammatic notation for the quivers. We follow the usual convention according to which round nodes in the quiver correspond to gauge groups, whereas square nodes correspond to global symmetries. Fields leaving gauge node a and arriving at gauge node b are be denoted by $\Phi_{ab,\alpha}$, and transform in the antifundamental representation $\bar{V}_{N_a}$ of $U(N_a)$ and the fundamental representation V_{N_b} of $U(N_b)$. The third label α takes values in $\{1, ..., M_{ab}\}$, and is used to distinguishes between M_{ab} different fields that transform in the same way under the gauge group. We can think of each $\Phi_{ab,\alpha}$ as a map

$$\Phi_{ab,\alpha} : \; V_{N_a} \to V_{N_b} \tag{1.3}$$

At every gauge node a we allow M_a different families of quarks $\{Q_{a,\beta}, \beta = 1, ..., M_a\}$ transforming in the antifundamental of $U(N_a)$ and $\bar{M}_a$ different families of antiquarks $\{\bar{Q}_{a,\gamma}, \gamma = 1, ..., \bar{M}_a\}$, transforming in the fundamental of $U(N_a)$. As for the field Φ, the greek letters β and γ distinguish the multiplicities of the quarks and antiquarks respectively. $U(F_{a,\beta})$ and $U(F_{a,\gamma})$ represent the flavour group of the quark $Q_{a,\beta}$ and of the antiquark $\bar{Q}_{a,\gamma}$ respectively. Figure 1 explicitly show this field configuration for one node a of the quiver. Table 1 summarises instead all the gauge and flavour group representations carried by every field in the quiver.

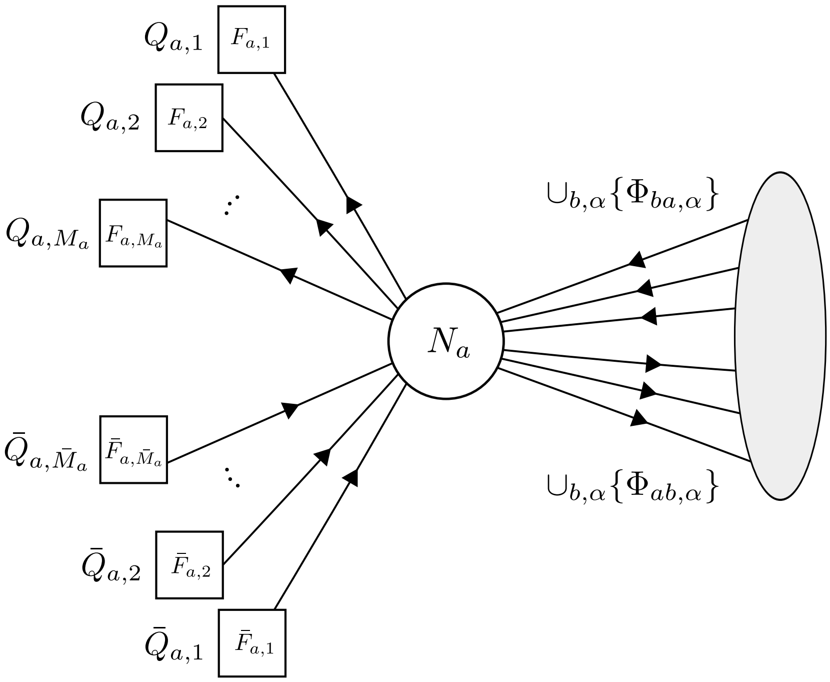

Figure 1: Pictorial representation of the fundamental fields (oriented edges), flavour group (square nodes) for a single gauge node labelled a.

	$U(N_a)$	$U(N_b)$	$U(F_{a,\beta})$	$U(\bar{F}_{a,\gamma})$
$\Phi_{ab,\alpha}$	$\bar{\square}$	$\square$	$\mathbf{1}$	$\mathbf{1}$
$\Phi_{aa,\alpha}$	Adj	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$Q_{a,\beta}$	$\bar{\square}$	$\mathbf{1}$	$\square$	$\mathbf{1}$
$\bar{Q}_{a,\gamma}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\bar{\square}$

Table 1: Gauge and flavour group representations carried by $\Phi_{ab,\alpha}$, $Q_{a,\beta}$ and $\bar{Q}_{a,\gamma}$. $\square$, $\bar{\square}$ and $\mathbf{1}$ are respectively the fundamental, antifundamental and trivial representations of the corresponding group.

2 Gauge Invariant Operators and Permutations

In this section we will present a systematic approach to list and label every holomorphic matrix invariant in quiver gauge theories of the type discussed above. We also allow for a flavour symmetry of the type discussed in Section 1.1. The operators we consider are polynomial in the Φ, Q and $\bar{Q}$ type fields that are invariant under gauge transformations. Therefore, all colour indices are contracted to produce traces and products of traces of these fields. For example

$$
\begin{aligned}
&\cdot \left(\Phi_{ab}\Phi_{bc}\cdots\Phi_{ta}\right) \quad &&\cdot \left(\Phi_{ab}\Phi_{bc}\right)\left(\Phi_{tt}\right)\\
&\cdot \left(\bar{Q}^k_a Q_{la}\right) &&\cdot \left(\bar{Q}^k_a \Phi_{ab}\Phi_{bc}\cdots\Phi_{qt}Q_{lq}\right)
\end{aligned}
\tag{2.1}
$$

and products thereof are suitable matrix invariants. In these examples round brackets denote contraction of gauge indices (*i.e.* traces), while k, l are flavour indices. The last two examples belong to the class of GIOs that in the literature has been called 'generalised mesons' (see *e.g.* [37]). In order to label these matrix polynomials, the first ingredient we need to specify is the number of fundamental fields that they contain. Let $n_{ab,\alpha}$ be the number of copies of $\Phi_{ab,\alpha}$ fields that are used to build the GIO. Similarly, let $n_{a,\beta}$ ($\bar{n}_{a,\gamma}$) be the number of copies of $Q_{a,\beta}$ quarks ($\bar{Q}_{a,\gamma}$ antiquarks) used in the GIO. In other words, the polynomial is characterised by degrees $\vec{n}$ given by

$$
\vec{n} = \cup_a \left\{ \cup_{b,\alpha}\, n_{ab,\alpha}; \cup_\beta\, n_{a,\beta}; \cup_\gamma\, \bar{n}_{a,\gamma} \right\}
\tag{2.2}
$$

For fixed degrees there is a large number of gauge invariant polynomials, differing in how the gauge indices are contracted. To guarantee gauge invariance we have to impose that the GIO does not have any free gauge indices. This condition implies the constraint on $\vec{n}$

$$
\begin{cases}
n_a = \sum_{b,\alpha} n_{ab,\alpha} + \sum_\beta n_{a,\beta} = \sum_{b,\alpha} n_{ba,\alpha} + \sum_\gamma \bar{n}_{a,\gamma} \qquad \forall a\\[2ex]
n_\alpha = \sum_a \sum_\beta n_{a,\beta} = \sum_a \sum_\gamma \bar{n}_{a,\gamma}
\end{cases}
\tag{2.3}
$$

We now introduce a second vector-like quantity, $\vec{s}$. It will store the information about the states of the quarks and antiquarks in the matrix invariant. To do so, let us first define the states

$$
|\boldsymbol{s}_{a,\beta}\rangle \in V_{F_{a,\beta}}^{\otimes n_{a,\beta}}\,, \qquad \langle \bar{\boldsymbol{s}}_{\boldsymbol{a},\boldsymbol{\gamma}}| \in \bar{V}_{\bar{F}_{a,\gamma}}^{\otimes \bar{n}_{a,\gamma}}
\tag{2.4}
$$

Here $V_{F_{a,\beta}}$ is the fundamental representation of $U(F_{a,\beta})$ and $\bar{V}_{\bar{F}_{a,\gamma}}$ is the antifundamental representation of $U(\bar{F}_{a,\gamma})$. Therefore, $|\boldsymbol{s}_{a,\beta}\rangle$ is the tensor product of all the $U(F_{a,\beta})$ fundamental representation states of the $n_{a,\beta}$ quarks $Q_{a,\beta}$. Similarly, $\langle \boldsymbol{s}_{a,\gamma}|$ is the tensor product of all the $U(\bar{F}_{a,\gamma})$ antifundamental representation states of the $\bar{n}_{a,\gamma}$ quarks $\bar{Q}_{a,\gamma}$. We define the vector $\vec{s}$ as the collection of these state labels:

$$
\vec{s} = \cup_a \left\{ \cup_\beta\, \boldsymbol{s}_{a,\beta}; \cup_\gamma\, \bar{\boldsymbol{s}}_{a,\gamma} \right\}
\tag{2.5}
$$

In the framework that we are going to introduce in this section, the building blocks of any matrix invariant are the tensor products of the fundamental fields $\Phi_{ab,\alpha}^{\otimes n_{ab,\alpha}}$, $Q_{a,\beta}^{\otimes n_{a,\beta}}$ and $\bar{Q}_{a,\gamma}^{\otimes \bar{n}_{a,\gamma}}$.

Let us then introduce the states

$$|I_{ab,\alpha}\rangle = |i_1, ..., i_{n_{ab,\alpha}}\rangle \in V_{N_a}^{\otimes n_{ab,\alpha}} \,, \qquad |I_{a,\beta}\rangle = |i_1, ..., i_{n_{a,\beta}}\rangle \in V_{N_a}^{\otimes n_{a,\beta}}$$

$$|J_{ab,\alpha}\rangle = |j_1, ..., j_{n_{ab,\alpha}}\rangle \in V_{N_a}^{\otimes n_{ab,\alpha}} \,, \qquad |\bar{J}_{a,\gamma}\rangle = |\bar{j}_1, ..., \bar{j}_{\bar{n}_{a,\gamma}}\rangle \in V_{N_a}^{\otimes \bar{n}_{a,\gamma}}$$

Using these definitions, together with eq. (2.4), we can write the matrix elements of every $\Phi_{ab,\alpha}^{\otimes n_{ab,\alpha}}$ tensor product as

$$\left(\Phi_{ab,\alpha}^{\otimes n_{ab,\alpha}}\right)^{I_{ab,\alpha}}_{J_{ab,\alpha}} = \left\langle I_{ab,\alpha} \middle| \Phi_{ab,\alpha}^{\otimes n_{ab,\alpha}} \middle| J_{ab,\alpha} \right\rangle \tag{2.6}$$

and similarly for $Q_{a,\beta}^{\otimes n_{a,\beta}}$ and $\bar{Q}_{a,\gamma}^{\otimes \bar{n}_{a,\gamma}}$:

$$\left(Q_{a,\beta}^{\otimes n_{a,\beta}}\right)^{I_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} = \left\langle I_{a,\beta} \middle| Q_{a,\beta}^{\otimes n_{a,\beta}} \middle| \boldsymbol{s}_{a,\beta} \right\rangle \,, \qquad \left(\bar{Q}_{a,\gamma}^{\otimes \bar{n}_{a,\gamma}}\right)^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{J}_{a,\gamma}} = \left\langle \bar{\boldsymbol{s}}_{a,\gamma} \middle| \bar{Q}_{a,\gamma}^{\otimes \bar{n}_{a,\gamma}} \middle| \bar{J}_{a,\gamma} \right\rangle \tag{2.7}$$

We will now present the first of the many diagrammatic techniques that we will use throughout this paper. We draw the matrix components of fundamental fields $(\Phi_{ab,\alpha})^i_j$, $(Q_{a,\beta})^i_s$ and $\left(\bar{Q}_{a,\gamma}\right)^{\bar{s}}_j$ as in Fig. 2.

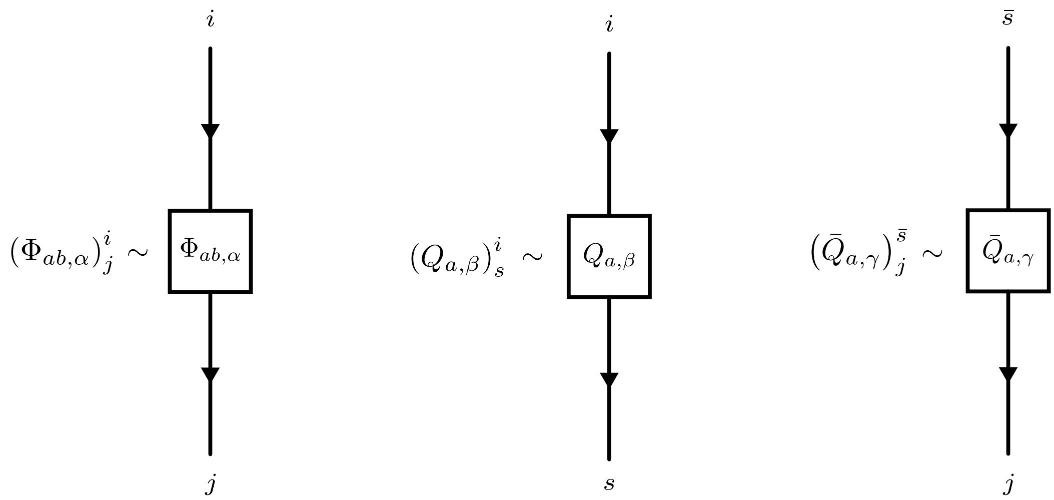

Figure 2: Diagrammatic description of the matrix elements of the fundamental fields Φ, Q and $\bar{Q}$.

This diagrammatic notation is then naturally extended to the tensor products $\left(\Phi_{ab,\alpha}^{\otimes n_{ab,\alpha}}\right)^{I_{ab,\alpha}}_{J_{ab,\alpha}}$, $\left(Q_{a,\beta}^{\otimes n_{a,\beta}}\right)^{I_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}$ and $\left(\bar{Q}_{a,\gamma}^{\otimes \bar{n}_{a,\gamma}}\right)^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{J}_{a,\gamma}}$, defined in eqs. 2.6 and 2.7, as in Fig. 3.

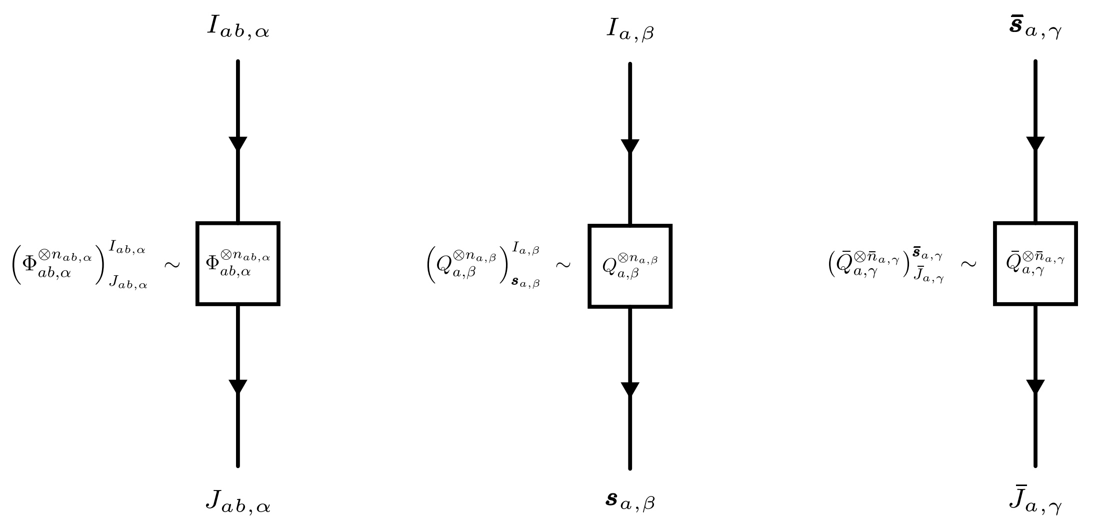

Figure 3: Diagrammatic description of the matrix elements of the tensor products of the fundamental fields Φ, Q and $\bar{Q}$.

Permutations act on a tensor product of states by rearranging the order in which the states are tensored together. For example, given a permutation $\sigma \in S_k$ and a tensor product of k states $|i_a\rangle$ $(1 \leq a \leq k)$ belonging to some vector space V, we have

$$\sigma|i_1, i_2, ..., i_k\rangle = |i_{\sigma(1)}, i_{\sigma(2)}, ..., i_{\sigma(k)}\rangle \tag{2.8}$$

Therefore, there is a natural permutation action on the states (2.4) and (2.6).

The gauge invariant polynomial is constructed by contracting the upper n_a indices of all the fields incident at the node a with their lower n_a indices. We describe these gauge invariants as follows. First we choose an ordering for all the fields with an upper $U(N_a)$ index. Then we fix a set of labelled upper indices: this means that we have picked an embedding of subsets into the set $[n_a] \equiv \{1, \cdots, n_a\}$, *i.e.*

$$[n_{a1,\alpha=1}] \sqcup [n_{a1,\alpha=2}] \sqcup \cdots \sqcup [n_{a2,\alpha=1}] \sqcup [n_{a2,\alpha=2}] \sqcup \cdots \sqcup [n_{a,\beta=1}] \sqcup [n_{a,\beta=2}] \sqcup \cdots \to [n_a] \tag{2.9}$$

which gives a set-partition of $[n_a]$. Similarly, there is an embedding into $[n_a]$ corresponding to the ordering of the lower $U(N_a)$ indices, namely

$$[n_{1a,\alpha=1}] \sqcup [n_{1a,\alpha=2}] \sqcup \cdots \sqcup [n_{2a,\alpha=1}] \sqcup [n_{2a,\alpha=2}] \sqcup \cdots \sqcup [\bar{n}_{a,\gamma=1}] \sqcup [\bar{n}_{a,\gamma=2}] \sqcup \cdots \to [n_a] \tag{2.10}$$

Now we contract the upper indices of these fields with their lower indices, *after* a permutation $\sigma_a \in S_{n_a}$ of their labels. We will therefore be considering permutations $\sigma_a \in S_{n_a}$, where $n_a = \sum_{b,\alpha} n_{ab,\alpha} + \sum_\beta n_{a,\beta} = \sum_{b,\alpha} n_{ba,\alpha} + \sum_\gamma n_{a,\gamma}$. Along the lines of eqs. (2.6) and (2.7) we can define the matrix elements of σ_a as

$$(\sigma_a)^{\times_{b,\alpha} J_{ba,\alpha} \times_\gamma \bar{J}_{a,\gamma}}_{\times_{b,\alpha} I_{ab,\alpha} \times_\beta I_{a,\beta}} = \left(\otimes_{b,\alpha}\langle J_{ba,\alpha}| \otimes_\gamma \langle \bar{J}_{a,\gamma}|\right) \sigma_a \left(\otimes_{b,\alpha}|I_{ab,\alpha}\rangle \otimes_\beta |I_{a,\beta}\rangle\right) \tag{2.11}$$

where the product symbols appearing in the upper and lower indices of σ_a are ordered as in (2.9) and (2.10). We depict these matrix elements as in Fig. 4.

$$(\sigma_a)^{\times_{b,\alpha} J_{ba,\alpha} \times_\gamma \bar{J}_{a,\gamma}}_{\times_{b,\alpha} I_{ab,\alpha} \times_\beta I_{a,\beta}} \sim$$

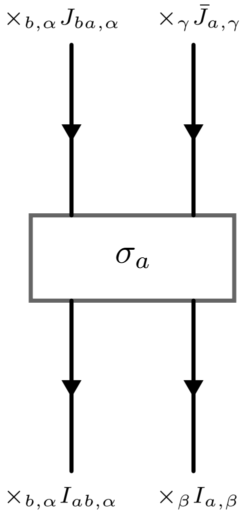

Figure 4: Diagrammatic description of the matrix elements of the permutation σ.

Following the approach of [25], we can write any GIO $\mathcal{O}_{\mathcal{Q}}$ of a quiver gauge theory $\mathcal{Q}$ with flavour symmetry as

$$\mathcal{O}_{\mathcal{Q}}(\vec{n};\, \vec{s};\, \vec{\sigma}) = \prod_a \left[\prod_{b,\alpha} \left(\Phi_{ab,\alpha}^{\otimes n_{ab,\alpha}} \right)^{I_{ab,\alpha}}_{J_{ab,\alpha}} \right] \otimes \left[\prod_\beta \left(Q_{a,\beta}^{\otimes n_{a,\beta}} \right)^{I_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} \right] \otimes \left[\prod_\gamma \left(\bar{Q}_{a,\gamma}^{\otimes \bar{n}_{a,\gamma}} \right)^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{J}_{a,\gamma}} \right]$$

$$\times \prod_a (\sigma_a)^{\times_{b,\alpha} J_{ba,\alpha} \times_\gamma \bar{J}_{a,\gamma}}_{\times_{b,\alpha} I_{ab,\alpha} \times_\beta I_{a,\beta}} \tag{2.12}$$

Here $\vec{\sigma} = \cup_a \{\sigma_a\}$ is a collection of permutations $\sigma_a \in S_{n_a}$, where $n_a = \sum_{b,\alpha} n_{ab,\alpha} + \sum_\beta n_{a,\beta}$. The purpose of $\vec{\sigma}$ is to contract all the gauge indices of the Φ, Q and $\bar{Q}$ fields to make a proper GIO. This formula looks rather complicated. However, it can be nicely interpreted in a diagrammatic way. We will now give an example of such a diagrammatic approach. Consider an $\mathcal{N} = 2$ SCQD theory. The $\mathcal{N} = 1$ quiver for this model is illustrated in Fig. 5.

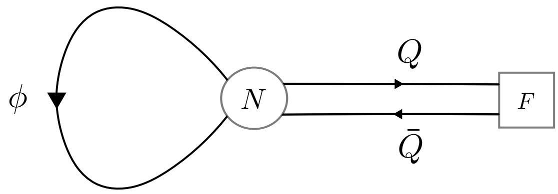

Figure 5: The $\mathcal{N} = 1$ quiver for an $\mathcal{N} = 2$ SQCD model.

We labelled the fields of this quiver by ϕ, Q and $\bar{Q}$, simplifying the notation of given in table 1. Consider now the GIO $(\bar{Q}\phi Q)^{\bar{s}_1}_{s_1} (\bar{Q}Q)^{\bar{s}_2}_{s_2}$. Here s_1, s_2 and $\bar{s}_1$, $\bar{s}_2$ are states of the fundamental and antifundamental representation of $SU(F)$ respectively, and the round brackets denotes $U(N)$ indices contraction. Figure 6 shows the diagrammatic interpretation of this GIO.

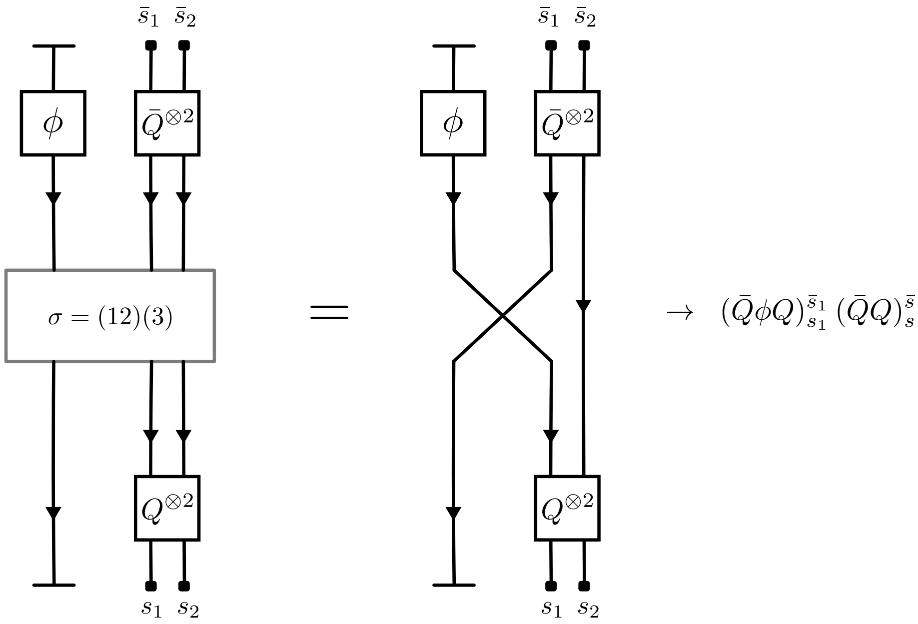

Figure 6: Diagrammatic description of the GIO $(\bar{Q}\phi Q)^{\bar{s}_1}_{s_1}\,(\bar{Q}Q)^{\bar{s}_2}_{s_2}$ in an $\mathcal{N}=2$ SQCD. The horizontal bars are to be identified.

For fixed $\vec{n}$, the data $\vec{\sigma}, \vec{s}$ determines a gauge invariant. However changing $\vec{\sigma}, \vec{s}$ can produce the same invariant. This fact can be described in terms of an equivalence relation generated by the action of permutations, associated with edges of the quiver, on the data $\vec{\sigma}, \vec{s}$. This has been discussed for the case without flavour symmetry in [25] and we will extend the discussion to flavours here. Continuing the example of the $\mathcal{N}=2$ SQCD introduced above, let us consider a matrix invariant built with n adjoint fields ϕ and n_q quarks and antiquarks Q and $\bar{Q}$. We label the tensor product of all the n_q quark states $|s_i\rangle \in V_{SU(F)}$ with the shorthand notation $|\boldsymbol{s}\rangle = \otimes_{i=1}^{n_q}|s_i\rangle$. Here $V_{SU(F)}$ is the fundamental representation of $SU(F)$. Similarly, $\langle\bar{\boldsymbol{s}}| = \otimes_{i=1}^{n_q}\langle\bar{s}_i|$ will be the tensor product of all the antiquarks states $\langle\bar{s}_i| \in \bar{V}_{SU(F)}$, where $\bar{V}_{SU(F)}$ is the antifundamental representation of $SU(F)$. In this model, a matrix invariant can be labelled by the triplet $(\sigma, \boldsymbol{s}, \bar{\boldsymbol{s}})$. The redundancy discussed above is captured by the identification

$$(\sigma,\ \boldsymbol{s},\ \bar{\boldsymbol{s}})\ \sim\ \big((\eta\times\bar{\rho})\sigma(\eta^{-1}\times\rho^{-1}),\ \rho(\boldsymbol{s}),\ \bar{\rho}(\bar{\boldsymbol{s}})\big) \tag{2.13}$$

where $\eta \in S_n$, $\rho, \bar{\rho} \in S_{n_q}$ and $\rho(\boldsymbol{s}) = (s_{\rho(1)}, s_{\rho(2)}, ..., s_{\rho(n_q)})$, $\bar{\rho}(\bar{\boldsymbol{s}}) = (\bar{s}_{\bar{\rho}(1)}, \bar{s}_{\bar{\rho}(2)}, ..., \bar{s}_{\bar{\rho}(n_q)})$. The last two equations are to be interpreted as the action of ρ and $\bar{\rho}^{-}1$ on the states $|\boldsymbol{s}\rangle$ and $\langle\bar{\boldsymbol{s}}|$:

$$\rho|\boldsymbol{s}\rangle = |s_{\rho(1)}, s_{\rho(2)}, ..., s_{\rho(n_q)}\rangle\,, \qquad \langle\bar{\boldsymbol{s}}|\bar{\rho}^{-1} = \langle\bar{s}_{\bar{\rho}(1)}, \bar{s}_{\bar{\rho}(2)}, ..., \bar{s}_{\bar{\rho}(n_q)}| \tag{2.14}$$

We refer to Appendix A for a diagrammatic interpretation of this equivalence.

For the general case of a gauge theory with flavour symmetry, the degeneracy is described by the identity

$$\mathcal{O}_{\mathcal{Q}}(\vec{n};\, \vec{s};\, \vec{\sigma}) = \mathcal{O}_{\mathcal{Q}}(\vec{n};\, \vec{\rho}\,(\vec{s});\, \mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})) \tag{2.15}$$

Here we introduced the permutations

$$\vec{\eta} = \cup_{a,b,\alpha}\{\eta_{ab,\alpha}\} \quad , \quad \eta_{ab,\alpha} \in S_{n_{ab,\alpha}} \tag{2.16a}$$

$$\vec{\rho} = \cup_a\{\cup_\beta\, \rho_{a,\beta};\, \cup_\gamma\, \bar{\rho}_{a,\gamma}\} \quad , \quad \rho_{a,\beta} \in S_{n_{a,\beta}} \quad , \quad \bar{\rho}_{a,\gamma} \in S_{\bar{n}_{a,\gamma}} \tag{2.16b}$$

and we defined

$$\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma}) = \cup_a\{(\times_{b,\alpha}\eta_{ba,\alpha} \times_\gamma \bar{\rho}_{a,\gamma})\sigma_a(\times_{b,\alpha}\eta^{-1}_{ab,\alpha} \times_\beta \rho^{-1}_{a,\beta})\} \ , \tag{2.17}$$

$$\vec{\rho}\,(\vec{s}) = \cup_a\{\cup_\beta\, \rho_{a,\beta}(\boldsymbol{s}_{a,\beta});\, \cup_\gamma\, \bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})\} \tag{2.18}$$

In Appendix A we will derive the constraint (2.15). This is essentially a set of equivalences of the type (2.13), iterated over all the nodes and edges of the quiver. The permutations $\eta_{ab,\alpha}, \rho_{a,\beta}, \bar{\rho}_{a,\gamma}$ can be viewed as "permutation gauge symmetries", associated with the edges of the quiver. The permutations $\vec{\sigma}$ and state labels $\vec{s}$ can be viewed as "matter fields" for the permutation gauge symmetries, associated with the nodes of the quiver. It is very intriguing that, in terms of the original Lie group gauge symmetry, the round nodes were associated with gauge groups $U(N_a)$, while the edges were matter. In this world of permutations, these roles are reversed, with the edges being associated with gauge symmetries and the nodes with matter.

So far we have used a permutation basis approach to characterise the quiver matrix invariants. This has offered a nice diagrammatic interpretation, but on the other hand it is subject to the complicated constraint in eq. (2.15). In the following section we are going to introduce a Fourier Transformation (FT) from this permutation description to its dual space, which is described in terms of representation theory quantities. In other words, we are going to change the way we label the matrix invariants: instead of using permutation data, we are going to use representation theory data. The upshot of doing so is twofold. On one hand the new basis will not be subject to any equivalence relation such as the one in (2.15). On the other hand, as a consequence of the Schur-Weyl duality (see *e.g.* [38]), it offers a simple way to capture the finite N constraints of the GIOs. Schematically, using this FT we trade the set of labels $\{\vec{n};\, \vec{s};\, \vec{\sigma}\}$ of any GIO for the new set $\{R_a, r_{ab,\alpha}, r_{a,\beta}, S_{a,\beta}, \bar{r}_{a,\gamma}, \bar{S}_{a,\gamma}, \nu_a^+, \nu_a^-\}$, that we denote with the shorthand notation $\boldsymbol{L}$:

$$\mathrm{FT} : \{\vec{n};\, \vec{s};\, \vec{\sigma}\} \;\; \rightarrow \;\; \boldsymbol{L} = \{R_a, r_{ab,\alpha}, r_{a,\beta}, S_{a,\beta}, \bar{r}_{a,\gamma}, \bar{S}_{a,\gamma}, \nu_a^+, \nu_a^-\} \tag{2.19}$$

Each R_a is a representation of the symmetric group S_{n_a}, where n_a has been defined in (2.3). $r_{ab,\alpha}, r_{a,\beta}, \bar{r}_{a,\gamma}$ are partitions of $n_{ab,\alpha}, n_{a,\beta}, \bar{n}_{a,\gamma}$ respectively. $S_{a,\beta}$ and $\bar{S}_{a,\gamma}$ are $U(F_{a,\beta})$ and $U(\bar{F}_{a,\gamma})$ states in the representation specified by the partitions $r_{a,\beta}$ and $\bar{r}_{a,\gamma}$ respectively. The integers $\nu_a^{\pm}$ are symmetric group multiplicity labels, a pair for each node in the quiver. Their meaning will be explained in the next section. Graphically, at each node a of the quiver we change the description of any matrix invariant as in Fig. 7. The diagram on the right in this figure is also called a *split-node* [25].

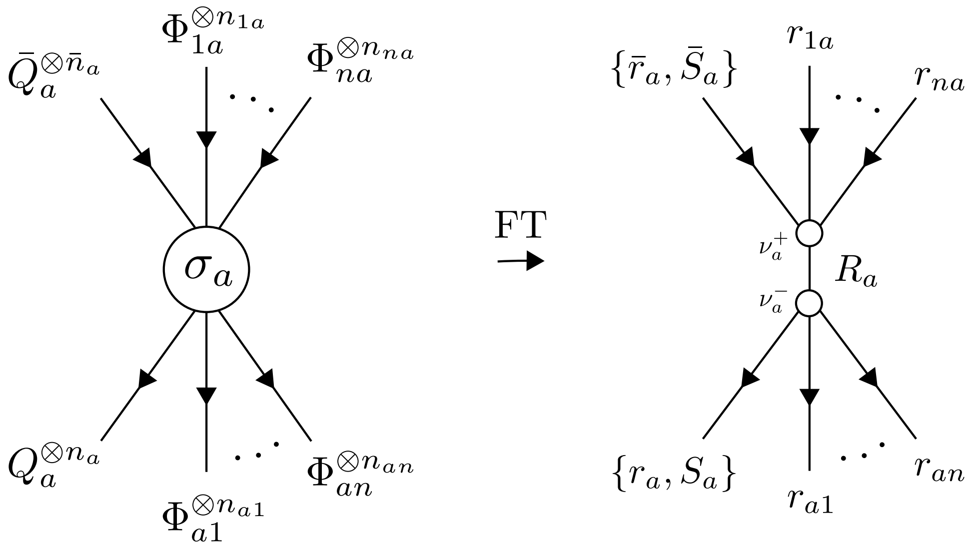

Figure 7: Pictorial representation of the Fourier transform discussed in the text. The multiplicity labels of the fields are not displayed.

We call the Fourier transformed operators *Quiver Restricted Schur polynomials*, or quiver Schurs for short. These are a generalisation of the Restricted Schur polynomials that first appeared in the literature in [12–14, 35, 36]. In section 4.1 we will show how the quiver Schurs form a basis for the Hilbert space of holomorphic operators.

3 The Quiver Restricted Schur Polynomials

In this section we describe the FT introduced above. In other words, we will explicitly construct the map

$$\mathrm{FT}: \ \mathcal{O}_{\mathcal{Q}}(\vec{n}; \vec{s}; \vec{\sigma}) \ \rightarrow \ \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}) \tag{3.1}$$

In order to do so, we need to introduce two main mathematical ingredients. These are the symmetric group branching coefficients and the Clebsch-Gordan coefficients. For each of these quantities we give both an analytic and a diagrammatic description: the latter will aid to make notationally heavy formulae easier to understand.

We begin by focusing on the symmetric group branching coefficients. Consider the symmetric group restriction

$$\times_{i=1}^{k} S_{n_i} \to S_n\,, \qquad \sum_{i=1}^{k} n_i = n \tag{3.2}$$

For each representation $V_R^{S_n}$ of S_n, this restriction induces the representation branching

$$V_R^{S_n} \simeq \bigoplus_{\substack{r_1\vdash n_1\\ r_2\vdash n_2\\ \dots\\ r_k\vdash n_k}} \left(\bigotimes_{i=1}^{k} V_{r_i}^{S_{n_i}}\right) \otimes V_R^{\vec{r}}\,, \qquad \vec{r} = (r_1, r_2, ..., r_k) \tag{3.3}$$

$V_R^{\vec{r}}$ is the *multiplicity* vector space, in case the representation $\otimes_i V_{r_i}$ appears more than once in the decomposition (3.3). The dimension of this space is $\dim(V_R^{\vec{r}}) = g\left(\cup_{i=1}^{k} r_i; R\right)$, where $g\left(\cup_{i=1}^{k} r_i; R\right) = g\left(r_1, r_2, ..., r_k; R\right)$ are Littlewood-Richardson coefficients [38].

In the following, the vectors belonging to any vector space V will be denoted using a bra-ket notation. The symbol $\langle\cdot|\cdot\rangle$ will indicate the inner product in V. Let then the set of vectors $\{\otimes_{i=1}^{k} |r_i, l_i, \nu\rangle\}$ be an orthonormal basis for $\bigoplus_{\vec{r}}\left(\bigotimes_{i=1}^{k} V_{r_i}^{S_{n_i}}\right) \otimes V_R^{\vec{r}}$. Here l_i is a state in $V_{r_i}^{S_{n_i}}$ and $\nu = 1, ..., g\left(\cup_{i=1}^{k} r_i; R\right)$ is a multiplicity label. We adopt the convention that $\otimes_{i=1}^{k} |r_i, l_i, \nu\rangle \equiv | \cup_i r_i, \cup_i l_i, \nu\rangle$. Similarly, let the set of vectors $\{|R, j\rangle\,,\, j = 1, ..., \dim(V_R^{S_n})\}$ be an orthonormal basis for $V_R^{S_n}$. The branching coefficients $B_{j\to\cup_i l_i}^{R\to\cup_i r_i;\,\nu}$ are the matrix entries of the linear invertible operator B, mapping

$$B: \quad V_R^{S_n} \longrightarrow \bigoplus_{\vec{r}}\left(\bigotimes_{i=1}^{k} V_{r_i}^{S_{n_i}}\right) \otimes V_R^{\vec{r}} \tag{3.4}$$

so that

$$B_{j\to\cup_k l_k}^{R\to\cup_k r_k;\,\nu}\; |R, j\rangle = | \cup_i r_i, \cup_i l_i, \nu\rangle \tag{3.5}$$

The sum over repeated indices is understood. By acting with $\langle S, i|$ on the left of both sides of (3.5) we then have

$$B_{i\to\cup_k l_k}^{S\to\cup_k r_k;\,\nu} = \langle S, i| \cup_i r_i, \cup_i l_i, \nu\rangle \tag{3.6}$$

Since B is an automorphism that maps an orthonormal basis to an orthonormal basis, it follows that B is an unitary operator, $B^\dagger = B^{-1}$. We can then write

$$\sum_j B_{j\to\cup_i l_i}^{R\to\cup_i r_i;\,\nu}\; (B^\dagger)_{\cup_i q_i\to j}^{\cup_i s_i;\,\mu\to R} = \left(\prod_i \delta^{s_i, r_i}\, \delta_{q_i, l_i}\right) \delta^{\mu,\nu} \tag{3.7}$$

However, since all the irreducible representations of any symmetric group can be chosen to be real [39], there exists a convention in which the branching coefficients (3.6) are also real. Therefore $B^\dagger = B^T$, where B^T is the transpose of the map (3.5). Using this last fact we can write the chain of equalities

$$\langle S, i| \cup_i r_i, \cup_i l_i, \nu\rangle = B^{S\to\cup_k r_k;\,\nu}_{i\to\cup_k l_k} = (B^T)^{\cup_k r_k;\,\nu\to S}_{\cup_k l_k\to i} = (B^{-1})^{\cup_k r_k;\,\nu\to S}_{\cup_k l_k\to i} = \langle \cup_i r_i, \cup_i l_i, \nu|S, i\rangle \quad (3.8)$$

We draw the branching coefficients (3.6) as in Fig. 8. The orientation of the arrows can be reversed because of the identities in (3.8).

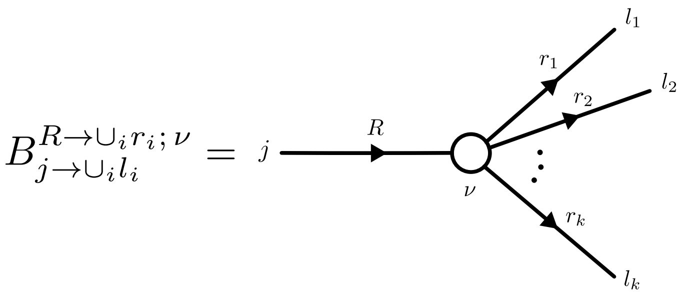

Figure 8: Pictorial description of the symmetric group branching coefficients.

Consider now taking k irreducible representations $V^{U(N)}_{r_i}$ of the unitary group $U(N)$, $i = 1, 2, ..., k$. For each $V^{U(N)}_{r_i}$, r_i is a partition of some integer n_i. This partition is associated with a Young diagram which is used to label the representation. If we tensor together all the $V^{U(N)}_{r_i}$'s, we generally end up with a reducible representation, and we have the isomorphism (see *e.g.* [39])

$$\bigotimes_{i=1}^{k} V^{U(N)}_{r_i} \simeq \bigoplus_{\substack{R\vdash n \\ c_1(R)\le N}} V^{U(N)}_R \otimes V^{\vec r}_R\,, \qquad n = \sum_{i=1}^{k} n_i\,, \quad (3.9)$$

Here R is a partition of $n = \sum_i n_i$. The direct sum on the RHS above is restricted to the Young diagrams R whose first column length $c_1(R)$ does not exceed the rank N of the gauge group. $V^{\vec r}_R$, with $\vec r = (r_1, r_2, ..., r_k)$, is the multiplicity vector space, satisfying $\dim(V^{\vec r}_R) = g\left(\cup_{i=1}^k r_i; R\right)$. The $g\left(\cup_{i=1}^k r_i; R\right)$ coefficients that appear in this formula are the same Littlewood-Richardson coefficients that we used in the above description of the symmetric group branching coefficients. Now let the set of vectors $\{|r_i, K_j\rangle\}$ be an orthonormal basis for $V^{U(N)}_{r_i}$, for $i = 1, 2, ..., k$. Here K_j is a state in $V^{U(N)}_{r_i}$. Also let $\{|R, M; \nu\rangle\}$ be an orthonormal basis for $\bigoplus_{R\vdash n} V^{U(N)}_R \otimes V^{\vec r}_R$. Here M is a state in the $U(N)$ representation $V^{U(N)}_R$ and ν is a multiplicity index. The Clebsch-Gordan coefficients $C^{R;\nu\to\cup_i r_i}_{M\to\cup_i K_i}$ are the matrix entries of the linear invertible operator C, mapping

$$C: \quad \bigotimes_{i=1}^k V^{U(N)}_{r_i} \longrightarrow \bigoplus_{R\vdash n} V^{U(N)}_R \otimes V^{\vec r}_R \quad (3.10)$$

so that

$$C^{R;\nu\to\cup_i r_i}_{M\to\cup_i K_i}\ |\cup_i r_i, \cup_i K_i\rangle = |R, M;\nu\rangle \tag{3.11}$$

The sum over repeated indices is understood. By acting on the left of both sides of (3.11) with $\langle\cup_i s_i, \cup_i P_i|$, where P_i are states of the $U(N)$ representations $V^{U(N)}_{s_i}$, we get

$$C^{R;\nu\to\cup_i s_i}_{M\to\cup_i P_i} = \langle\cup_i s_i, \cup_i P_i|R, M;\nu\rangle \tag{3.12}$$

From (3.11), we see that the automorphism C maps an orthonormal basis to an orthonormal basis. This makes C an unitary operator, $C^\dagger = C^{-1}$, and we can therefore write

$$\sum_{\vec{r},\vec{K}} C^{R;\nu\to\cup_i r_i}_{M\to\cup_i K_i}\ (C^\dagger)^{\cup_i r_i\to S;\mu}_{\cup_i K_i\to P} = \delta^{S,R}\,\delta_{P,M}\,\delta^{\mu,\nu} \tag{3.13}$$

As with the branching coefficients, it is always possible to choose a consistent convention in which all the $U(N)$ Clebsch-Gordan coefficients (3.12) are real. If we choose to work with such a convention, C becomes an orthogonal operator: $C^T = C^{-1}$. We then have, in the same fashion of (3.8)

$$\langle\cup_i s_i, \cup_i P_i|R, M;\nu\rangle = C^{R;\nu\to\cup_i s_i}_{M\to\cup_i P_i} = (C^T)^{\cup_i s_i\to R;\nu}_{\cup_i P_i\to M} = (C^{-1})^{\cup_i s_i\to R;\nu}_{\cup_i P_i\to M} = \langle R, M;\nu|\cup_i s_i, \cup_i P_i\rangle \tag{3.14}$$

We draw the Clebsch-Gordan coefficients as in Fig. 9. Again, the orientation of the arrows can be reversed, due to (3.14).

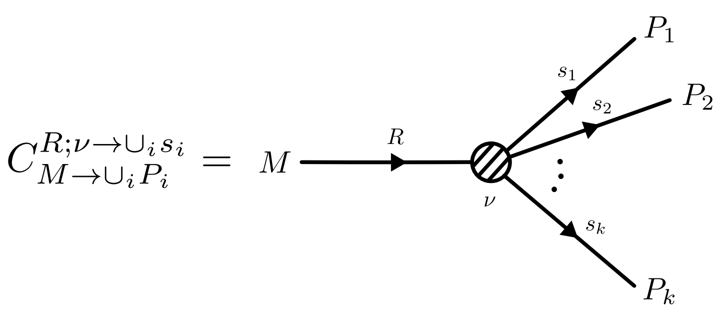

Figure 9: Pictorial representation of the $U(N)$ Clebsch-Gordan coefficient in eq. (3.12).

Consider now the particular case of (3.9) in which every representation $V^{U(N)}_{r_i}$ tensored on the LHS coincides with the $U(N)$ fundamental[1] representation, that for simplicity we just call V for the remainder of this section. This configuration allows us to use the Schur-Weyl

[1]We can get similar results by replacing the fundamental with the antifundamental representation of $U(N)$. The quantities we define here get modified accordingly.

duality to write

$$\overbrace{V \otimes \cdots \otimes V}^{k \text{ times}} = V^{\otimes k} \simeq \bigoplus_{\substack{R \vdash k \\ c_1(R) \leq N}} V_R^{U(N)} \otimes V_R^{S_k} \tag{3.15}$$

where $V_R^{U(N)}$ and $V_R^{S_k}$ are irreducible representations of $U(N)$ and S_k respectively. They correspond to the Young diagrams specified by the partition R of k. By comparing (3.15) with (3.9), we see that the representation $V_R^{S_k}$ has now taken the place of the generic multiplicity vector space $V_R^{\vec{r}}$. Since the Schur-Weyl decomposition will play a major role in this paper, we are now going to introduce a more compact notation for its Clebsch-Gordan coefficients. Let us consider the states

$$|\boldsymbol{s}\rangle = \otimes_{j=1}^{k} |s_j\rangle \in V^{\otimes k}\,, \;\; |s_j\rangle \in V\,, \qquad |R; M, i\rangle = |R, M\rangle \otimes |R, i\rangle \in V_R^{U(N)} \otimes V_R^{S_k} \tag{3.16}$$

where $\{|R, M\rangle\,,\, M = 1, ..., \dim(V_R^{U(N)})\}$ and $\{|R, i\rangle\,,\, i = 1, ..., \dim(V_R^{S_k})\}$ are orthonormal bases of $V_R^{U(N)}$ and $V_R^{S_k}$ respectively. The equations (3.11) and (3.14) imply

$$C_{\boldsymbol{s}}^{R,M,i}\; |\boldsymbol{s}\rangle = |R, M, i\rangle \tag{3.17}$$

and

$$C_{\boldsymbol{t}}^{R,M,i} = \langle \boldsymbol{t} | R, M, i\rangle = \langle R, M, i | \boldsymbol{t}\rangle = C^{\boldsymbol{t}}_{R,M,i} \tag{3.18}$$

respectively. We draw these quantities as in Fig. 10.

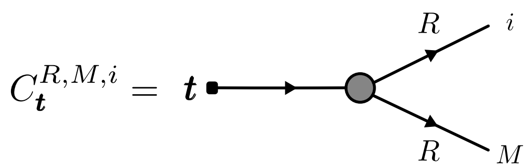

Figure 10: Pictorial representation of the $U(N)$ Clebsch-Gordan coefficients (3.18) for the Schur-Weyl duality (3.15).

3.1 The quiver characters

We now have all the tools necessary to introduce a key quantity, the *quiver characters* $\chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma})$. Here $\boldsymbol{L}$ is the set of representation theory labels defined in (2.19). The quiver characters are the expansion coefficients of the FT (3.1):

$$\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}) = \sum_{\vec{s}} \sum_{\vec{\sigma}} \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma})\, \mathcal{O}_{\mathcal{Q}}(\vec{n}, \vec{s}, \vec{\sigma}) \tag{3.19}$$

We define them as

$$\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma}) = c_{\boldsymbol{L}} \sum_{\substack{\{l_{ab,\alpha}\}\\ \{l_{a,\beta}\},\{\bar{l}_{a,\gamma}\}}} \prod_a \sum_{i_a,j_a} D^{R_a}_{i_a,j_a}(\sigma_a)\; B^{R_a\to\cup_{b,\alpha} r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{j_a\to\cup_{b,\alpha} l_{ab,\alpha}\cup_\beta l_{a,\beta}} \prod_\beta C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}$$

$$\times\, B^{R_a\to\cup_{b,\alpha} r_{ba,\alpha}\cup_\gamma \bar{r}_{a,\gamma};\nu_a^+}_{i_a\to\cup_{b,\alpha} l_{ba,\alpha}\cup_\gamma \bar{l}_{a,\gamma}} \prod_\gamma C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}} \tag{3.20}$$

where the coefficient $c_{\boldsymbol{L}}$ is the normalisation constant

$$c_{\boldsymbol{L}} = \prod_a \left(\frac{d(R_a)}{n_a!}\right)^{\frac{1}{2}} \left(\prod_{b,\alpha}\frac{1}{d(r_{ab,\alpha})}\right)^{\frac{1}{2}} \left(\prod_{\beta}\frac{1}{d(r_{a,\beta})}\right)^{\frac{1}{2}} \left(\prod_{\gamma}\frac{1}{d(\bar{r}_{a,\gamma})}\right)^{\frac{1}{2}} \tag{3.21}$$

Since we chose to work in the convention in which all symmetric group representations and Clebsch-Gordan coefficients are real, then the quiver characters are real quantities as well:

$$\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma}) = \chi^*_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma}) \tag{3.22}$$

This convention will be convenient when we compute the 2-point functions of holomorphic and anti-holomorphic matrix invariants in section 4.1.

These quantities have a pictorial interpretation. We have already introduced a diagrammatic notation for the branching and Clebsch-Gordan coefficients B and C in Fig. 8 and in Fig. 10 respectively. The pictorial notation for the i,j matrix element of the permutation σ in the irreducible representation R, $D^R_{i,j}(\sigma)$, is displayed in Fig. 11. All the edges of these diagrams are to be contracted together as per instructions of formula (3.20).

$$D^R_{i,j}(\sigma) = i \longrightarrow \overset{R}{\boxed{\sigma}} \longrightarrow j$$

Figure 11: Pictorial description of the matrix element $D^R_{i,j}(\sigma)$ of the S_n symmetric group representation R.

Let us give an example of the diagrammatic of the quiver character of a well-known flavoured gauge theory. Consider the $\mathcal{N}=1$ quiver for the flavoured conifold [33, 34, 40, 41] in Fig. 12.

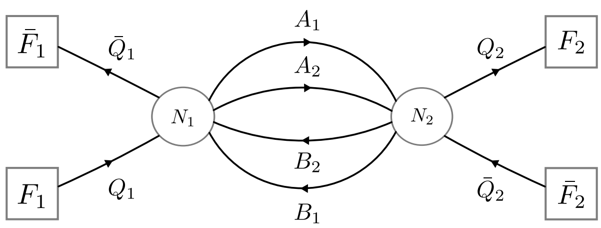

Figure 12: $\mathcal{N}=1$ quiver for the flavoured conifold gauge theory.

The quiver character for this model is depicted in Fig. 13. This figure explicitly shows how all the symmetric group matrix elements, the branching coefficients and the Clebsch-Gordan coefficients are contracted together.

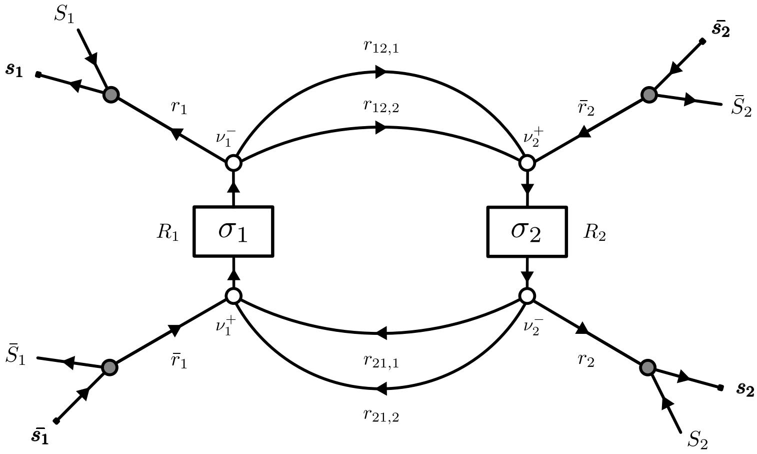

Figure 13: The quiver character diagram for the flavoured conifold gauge theory.

For completeness we also give a diagram for the the most generic quiver character $\chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma})$. This is done in Fig. 14. In this picture, we factored the quiver character into a product over the gauge nodes a of the quiver. All the internal edges (that is, the ones that are not connected to a Clebsch-Gordan coefficient) are contracted following the prescription of (3.20).

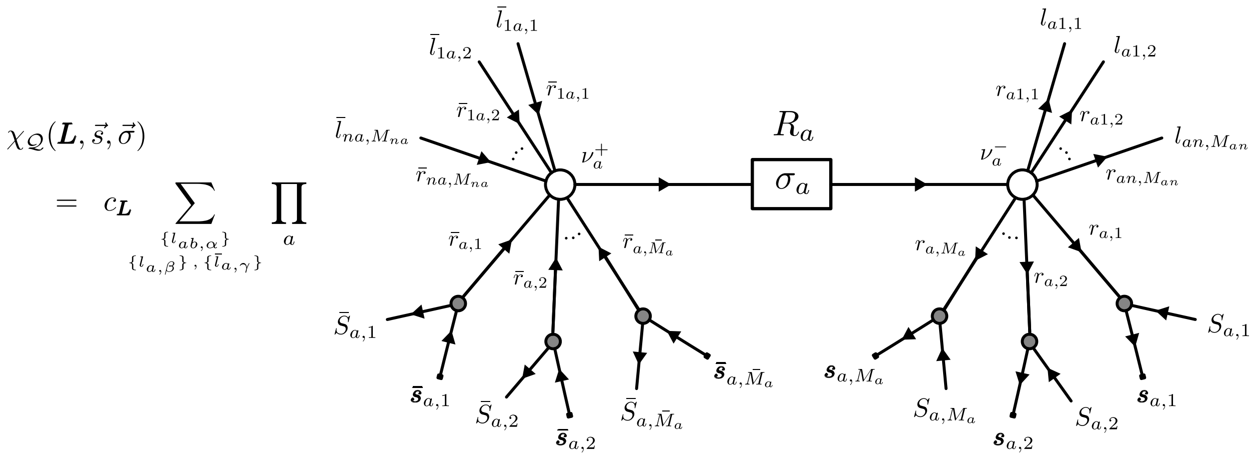

Figure 14: Pictorial description of the quiver characters $\chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma})$.

The quiver characters (3.20) satisfy the invariance relation

$$\chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma}) = \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{\rho}\,(\vec{s}\,), \mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})) \tag{3.23}$$

where $\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})$ has been defined in (2.17):

$$\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma}) = \cup_a \{(\times_{b,\alpha}\eta_{ba,\alpha} \times_\gamma \bar{\rho}_{a,\gamma})\sigma_a(\times_{b,\alpha}\eta^{-1}_{ab,\alpha} \times_\beta \rho^{-1}_{a,\beta})\} \tag{3.24}$$

They also satisfy the two orthogonality relations

$$\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}},\vec{s},\vec{\sigma})=\delta_{\boldsymbol{L},\tilde{\boldsymbol{L}}} \tag{3.25}$$

and

$$\sum_{\boldsymbol{L}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{t},\vec{\tau})=\frac{1}{c_{\vec{n}}}\sum_{\vec{\eta}\times\vec{\rho}}\delta\left(\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})\,\vec{\tau}^{-1}\right)\delta_{\vec{\rho}(\vec{s}),\vec{t}} \tag{3.26}$$

where we introduced the normalisation constant

$$c_{\vec{n}}=\prod_{a}\left(\prod_{b,\alpha}n_{ab,\alpha}!\right)\left(\prod_{\beta}n_{a,\beta}!\right)\left(\prod_{\gamma}n_{a,\gamma}!\right) \tag{3.27}$$

It is worthwhile to note that this quantity can be interpreted as the order of the permutation gauge symmetry group. All of these equations are derived in Appendix B.

The set of operators (3.19) form the Quiver Restricted Schur polynomial basis. Using (3.23) we can immediately check that such operators are invariant under the constraint (2.15). We have

$$\begin{aligned}
\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})&=\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\mathcal{O}_{\mathcal{Q}}(\vec{n},\vec{s},\vec{\sigma})\\
&=\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\mathcal{O}_{\mathcal{Q}}(\vec{n},\,\vec{\rho}\,(\vec{s}),\,\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma}))\\
&=\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{\rho}\,(\vec{s}),\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma}))\,\mathcal{O}_{\mathcal{Q}}(\vec{n},\,\vec{\rho}\,(\vec{s}),\,\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma}))\\
&=\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\mathcal{O}_{\mathcal{Q}}(\vec{n},\,\vec{s},\,\vec{\sigma})=\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})
\end{aligned} \tag{3.28}$$

were in the second line we used the constraint (2.15), in the third one the invariance of the quiver characters (3.23) and in the fourth one we relabelled the dummy variables of the double sum.

Finally, the FT (3.19) can be easily inverted. Starting from

$$\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})=\sum_{\vec{t}}\sum_{\vec{\tau}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{t},\vec{\tau})\,\mathcal{O}_{\mathcal{Q}}(\vec{n},\vec{t},\vec{\tau}) \tag{3.29}$$

we multiply both sides by $\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})$ and we take the sum over the set of labels in $\boldsymbol{L}$ to get

$$\sum_{\boldsymbol{L}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})=\sum_{\vec{t}}\sum_{\vec{\tau}}\left(\sum_{\boldsymbol{L}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{t},\vec{\tau})\right)\mathcal{O}_{\mathcal{Q}}(\vec{n},\vec{t},\vec{\tau}) \tag{3.30}$$

Using the orthogonality relation (3.26), the above equation becomes

$$\sum_{\boldsymbol{L}} \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma})\, \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}) = \sum_{\vec{t}} \sum_{\vec{\tau}} \left(\frac{1}{c_{\vec{n}}} \sum_{\vec{\eta}\times\vec{\rho}} \delta\left(\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})\, \vec{\tau}^{\,-1}\right) \delta_{\vec{\rho}(\vec{s}),\vec{t}} \right) \mathcal{O}_{\mathcal{Q}}(\vec{n}, \vec{t}, \vec{\tau})$$

$$= \frac{1}{c_{\vec{n}}} \sum_{\vec{\eta}\times\vec{\rho}} \mathcal{O}_{\mathcal{Q}}\left(\vec{n}, \vec{\rho}(\vec{s}), \mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})\right) = \frac{1}{c_{\vec{n}}} \sum_{\vec{\eta}\times\vec{\rho}} \mathcal{O}_{\mathcal{Q}}(\vec{n}, \vec{s}, \vec{\sigma}) \tag{3.31}$$

where in the last line we used the constraint (2.15). Now the sum over the permutations $\vec{\eta}$, $\vec{\rho}$ is trivial, and it just gives a factor of $c_{\vec{n}}$. We then have that the inverse of the map (3.19) is simply

$$\mathcal{O}_{\mathcal{Q}}(\vec{n}, \vec{s}, \vec{\sigma}) = \sum_{\boldsymbol{L}} \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma})\, \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}) \tag{3.32}$$

4 Two and Three Point Functions

In this section we will derive an expression for the two and three point function of matrix invariants, using the free field metric. All the computations are done using the Quiver Restricted Schur polynomials. The result for the two point function is rather compact, and offers a nice way to describe the Hilbert space of holomorphic GIOs. On the other hand, the expression for the three point function is still quite involved. We give a diagrammatic description of the answer in section 4.2, leaving the analytical expression and its derivation in Appendix D.

4.1 Hilbert space of holomorphic gauge invariant operators

In the free field metric, the Quiver Restricted Schur polynomials (3.19) form an orthogonal basis for the 2-point functions of holomorphic and anti-holomorphic matrix invariants. Explicitly, in Appendix C we derive the equation

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\, \mathcal{O}^{\dagger}_{\mathcal{Q}}(\boldsymbol{L'}) \right\rangle = \delta_{\boldsymbol{L},\boldsymbol{L'}}\; c_{\vec{n}} \prod_a f_{N_a}(R_a) \tag{4.1}$$

where $c_{\vec{n}}$ is given in (3.27). The quantity $f_{N_a}(R_a)$ is the product of weights of the $U(N_a)$ representation R_a, and it is defined as

$$f_{N_a}(R_a) = \prod_{i,j} (N_a - i + j) \tag{4.2}$$

Here i and j label the row and column of the Young diagram R_a. At finite N_a, this quantity vanishes if the length of the first column of its Young diagram exceeds N_a, that is if $c_1(R_a) > N_a$. This means that for a generic quiver $\mathcal{Q}$ the Hilbert space $\mathcal{H}_{\mathcal{Q}}$ of holomorphic GIOs can be described by

$$\mathcal{H}_{\mathcal{Q}} = \text{Span}\left\{\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})|\ \boldsymbol{L}\ s.t.\ c_1(R_a) \leq N_a,\, \forall a\right\} \tag{4.3}$$

We can see how the finite N_a constraints of any matrix invariant are captured by the simple rule $c_1(R_a) \leq N_a$. We leave the formal proof of (4.1) in Appendix C, and we present here only the main steps. In the free field metric, the only non-zero correlators are the ones that couple fields of the same kind ($e.g.$ $\Phi_{ab,\alpha}$ with $\Phi^\dagger_{ab,\alpha}$):

$$\left\langle (\Phi_{ab,\alpha})^i_j\, (\Phi^\dagger_{ab,\alpha})^k_l \right\rangle = \delta^i_l \delta^k_j\,, \quad \left\langle (Q_{a,\beta})^i_s\, (Q^\dagger_{a,\beta})^p_l \right\rangle = \delta^i_l \delta^p_s\,, \quad \left\langle (\bar{Q}_{a,\gamma})^{\bar{s}}_j\, (\bar{Q}^\dagger_{a,\gamma})^k_{\bar{p}} \right\rangle = \delta^k_j \delta^{\bar{s}}_{\bar{p}} \tag{4.4}$$

Consequently, we can use Wick contractions to find the 2-point function of matrix invariants in the permutation basis of eq. (2.12):

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\vec{n},\vec{s},\vec{\sigma})\mathcal{O}^\dagger_{\mathcal{Q}}(\vec{n},\vec{s}\,',\vec{\sigma}\,') \right\rangle = \sum_{\vec{\eta},\,\vec{\rho}} \delta_{\vec{s}\,',\vec{\rho}(\vec{s})} \prod_a \text{Tr}_{V_{N_a}^{\otimes n_a}} \left[\text{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\,(\sigma'_a)^{-1}\right] \tag{4.5}$$

The details of the computations are shown in Appendix C. Here the trace is taken over the product space $V_{N_a}^{\otimes n_a}$, V_{N_a} being the fundamental representation of $U(N_a)$ and $n_a = \sum_{b,\alpha} n_{ab,\alpha} + \sum_\beta n_{a,\beta}$. The trace in the product is also equal to N_a raised to the power of the number of cycles in the permutation appearing as an argument. Using the definition (3.19) we get

$$\begin{aligned}
\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\mathcal{O}^\dagger_{\mathcal{Q}}(\boldsymbol{L'}) \right\rangle &= \sum_{\vec{s},\vec{s}\,'} \sum_{\vec{\sigma},\vec{\sigma}\,'} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\chi^*_{\mathcal{Q}}(\boldsymbol{L'},\vec{s}\,',\vec{\sigma}\,') \left\langle \mathcal{O}_{\mathcal{Q}}(\vec{n},\vec{s},\vec{\sigma})\mathcal{O}^\dagger_{\mathcal{Q}}(\vec{n},\vec{s}\,',\vec{\sigma}\,') \right\rangle \\
&= \sum_{\vec{s}} \sum_{\vec{\sigma},\vec{\sigma}\,'} \sum_{\vec{\eta}\,,\vec{\rho}} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\chi_{\mathcal{Q}}(\boldsymbol{L'},\vec{\rho}(\vec{s}),\vec{\sigma}\,') \prod_a \text{Tr}_{V_{N_a}^{\otimes n_a}} \left[\text{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\,(\sigma'_a)^{-1}\right]
\end{aligned} \tag{4.6}$$

where we also exploited the reality of the quiver characters. Using the invariance and orthogonality properties (3.23) and (3.25) we get

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\mathcal{O}^\dagger_{\mathcal{Q}}(\boldsymbol{L'}) \right\rangle = \sum_{\vec{s}} \sum_{\vec{\sigma},\vec{\sigma}\,'} \sum_{\vec{\eta},\,\vec{\rho}} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\chi_{\mathcal{Q}}(\boldsymbol{L'},\vec{s},\vec{\sigma}\,') \prod_a \text{Tr}_{V_{N_a}^{\otimes n_a}} \left[\sigma_a\,(\sigma'_a)^{-1}\right] \tag{4.7}$$

Now the sum over the permutations $\vec{\eta}$ and $\vec{\rho}$ is trivial, and can be computed to give the factor $c_{\vec{n}}$ that appears in (4.1). Using the substitution $\sigma_a \to \tau_a \cdot \sigma'_a$, the identity (B.15), and computing explicitly the trace in (4.7) allows us to obtain the result (4.1).

4.2 Chiral ring structure constants and three point functions

In Appendix D we derive an equation for the holomorphic GIO ring structure constants $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$, defined as the coefficients of the operator product expansion

$$\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(1)})\,\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(2)}) = \sum_{\boldsymbol{L}^{(3)}} G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}\,\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(3)}) \tag{4.8}$$

Because of the orthogonality of the two point function (4.1), we also obtain an equation for the three point function:

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(1)})\,\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(2)})\,\mathcal{O}^{\dagger}_{\mathcal{Q}}(\boldsymbol{L}^{(3)})\right\rangle = c_{\vec{n}^{(3)}}\;G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}\prod_a f_{N_a}\left(R_a^{(3)}\right) \tag{4.9}$$

We only give here a pictorial interpretation of the equation we derived for $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$, leaving the technicalities in Appendix D. In particular, eq. (D.45) gives the analytical formula for the $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$ coefficients.

Let us begin by considering an example. We will show how to draw the diagram for the chiral ring structure constants for an $\mathcal{N}=2$ SCQD, through a step-by-step procedure. The quiver for this theory is shown in Fig. 5. As we discussed in the previous section, for any given model, a basis of GIOs is labelled by $\boldsymbol{L}=\{R_a, r_{ab,\alpha}, r_{a,\beta}, S_{a,\beta}, \bar{r}_{a,\gamma}, \bar{S}_{a,\gamma}, \nu_a^+, \nu_a^-\}$. However, for an $\mathcal{N}=2$ SQCD theory, many of these $a, b, \alpha, \beta, \gamma$ indices are redundant: for this reason we can simplify $\boldsymbol{L}$ as

$$\boldsymbol{L} = \{R, r,\, r_q, S,\, \bar{r}_q, \bar{S},\, \nu^+, \nu^-\} \tag{4.10}$$

Here r is the representation associated with the adjoint field ϕ; S denotes a state in the $SU(F)$ representation r_q and $\bar{S}$ denotes a state in the $SU(F)$ representation $\bar{r}_q$. R is the representation associated with the gauge group, $U(N)$. We therefore want to compute the three point function (4.9), where all the $\boldsymbol{L}^{(i)}$, $i=1,2,3$, are of the form given in (4.10). We split this process into five steps, that we now describe.

i) **Create the split node quiver diagram.** The first step is to create the split-node quiver diagram from the $\mathcal{N}=2$ SCQD quiver of Fig. 5. This involves separating the gauge node into two components, one that collects all the incoming edges and one from which all the edges exit. The former is called a positive node of the split-node quiver, the latter is called a negative node. These two are then joined by an edge, called a gauge edge, directed from the positive to the negative node. We then decorate all the edges in the split-node quiver with symmetric group representation labels. The positive and negative nodes in the split-node diagram are points where the edges meet. Since the edges now carry a symmetric group representation, we interpret them as representation branching points, to which we associate a branching coefficient (3.4). To the positive node we associate the branching multiplicity ν^+, to the negative node we associate the branching multiplicity ν^-. Finally, we label the open endpoints of the quark and antiquark edges with $U(F)$ fundamental and antifundamental representation state labels, S and $\bar{S}$. The resulting diagram is shown on the left of Fig. 15. Notice that such a diagram contains all the labels in $\boldsymbol{L} = \{R, r, r_q, S, \bar{r}_q, \bar{S}, \nu^+, \nu^-\}$.

ii) **Cut the edges in the split-node quiver.** In this step we will cut all the edges in the split-node diagram, as shown in the middle picture of Fig. 15. After all the cuts have been performed, we are left with two trivalent vertices and two edges corresponding to the quark and the antiquark fields. As previously stated, the trivalent vertices will be interpreted as branching coefficients (see Fig. 8). We group these four object into two pairs, depending whether their edges are connected to the positive or negative node of the split-node diagram. This is shown in the rightmost picture of Fig. 15.

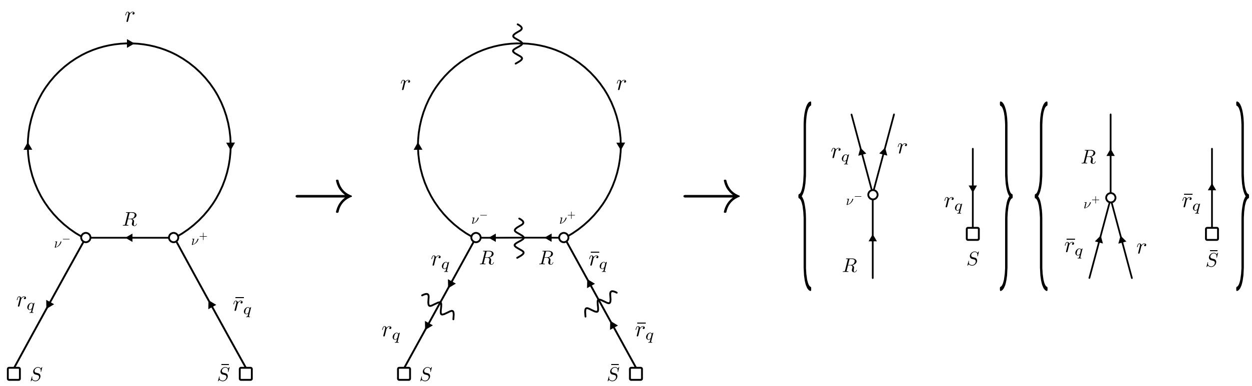

Figure 15: From left to right: the split-node quiver for the $\mathcal{N} = 2$ SQCD, the same diagram with the cut edges, and the two components of the negative and positive node of the split-node quiver.

iii) **Merge the edges connected to the negative node.** We consider the set of edges connected to the negative node of the split-node quiver. In order to compute the three point function (4.9), we need three copies of these sets, one for each field $\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(1)})$, $\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(2)})$, $\mathcal{O}^\dagger_{\mathcal{Q}}(\boldsymbol{L}^{(3)})$. These sets are shown in Fig. 16. The orientation of the edges in the last pair is reversed: this is because the third field on the LHS of (4.9) is hermitian conjugate.

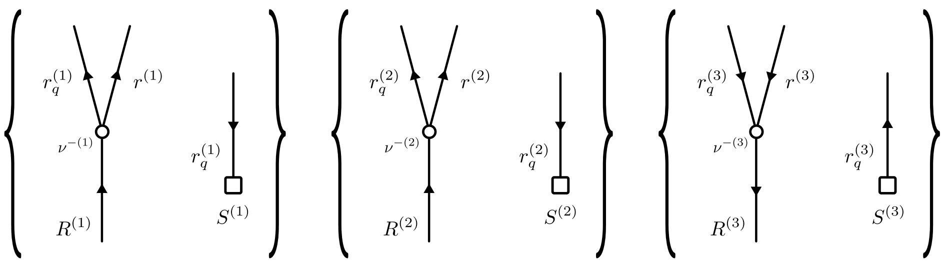

Figure 16: The three sets of trivalent vertices and edges needed to construct part of the $\mathcal{N} = 2$ SQCD three point function diagram.

We will now suitably merge the three trivalent vertices (branching coefficients) in Fig. 16, and join the three edges corresponding to the quark fields. The outcome of this fusing process is shown in Fig. 17. We introduced three new trivalent vertices, which as usual we interpret as branching coefficients: the labels μ, ν_r and ν_q denote their multiplicity. The fusing of the three quark edges has been achieved by introducing a Clebsch-Gordan coefficient, see Fig. 9. We further impose that the label for the multiplicity of the representation branching $r_q^{(1)} \otimes r_q^{(2)} \to r_q^{(3)}$ is the same in both the Clebsch-Gordan coefficient and the branching coefficient that appear in Fig. 16. In the figure we also inserted a permutation λ_- in the edge carrying the representation $R^{(3)}$. The purpose of this permutation is to rearrange tensor factors given the two different factorisation of $R^{(3)}$, that is from $(r^{(1)} \otimes r^{(1)}) \otimes (r_q^{(2)} \otimes r_q^{(2)}) \to r^{(3)} \otimes r_q^{(3)} \to R^{(3)}$ to $R^{(3)} \to R^{(1)} \otimes R^{(2)} \to (r^{(1)} \otimes r_q^{(1)}) \otimes (r^{(2)} \otimes r_q^{(2)})$.

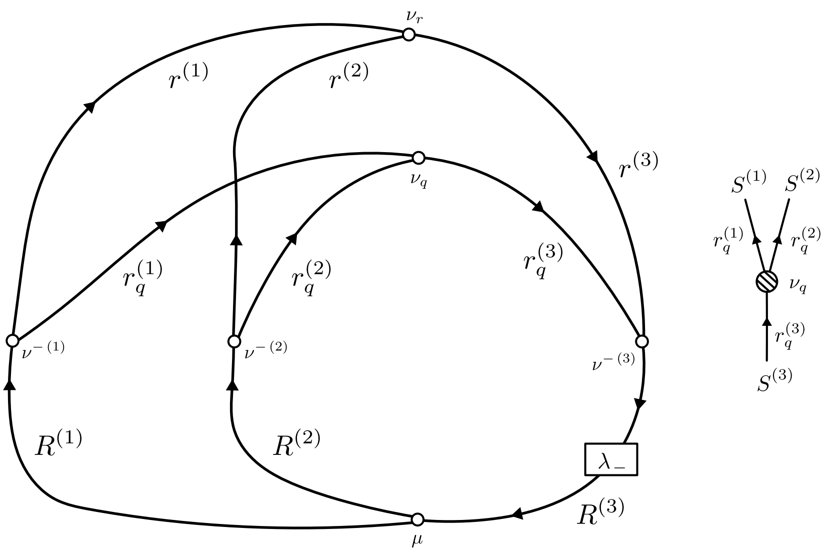

Figure 17: Merging of branching coefficients and quarks labels for the three sets in Fig. 16.

We thus obtained a closed network of branching coefficients, together with a single $SU(F)$ Clebsch-Gordan coefficient. All the edges involved into this process were the ones connected to the negative node of the split-node diagram they belonged to.

iv) **Merge the edges connected to the positive node.** By repeating the fusing process presented in point iii) for all the edges connected to the positive node of the split-node quiver, we obtain a diagram very similar to the one in Fig. 17. The only rule that we impose is that the multiplicity labels for representation branchings which appear in both these diagrams have to be the same. In our example, the branching of $R^{(3)}$ into $R^{(1)}$ and $R^{(2)}$ will appear in both diagrams. This is because the edge carrying the representation label R is connected to both the positive and negative node of the split-node quiver, as it can be seen from Fig. 15. Therefore these two branching coefficients will share the same multiplicity label, μ. Similarly, the branching of $r^{(1)}$ and $r^{(2)}$ into $r^{(3)}$ will be present in both diagrams too. Following the same rule, these two branching coefficients will then have the same multiplicity label, ν_r.

v) **Combine the diagrams and sum over multiplicities.** To obtain the final expression for the three point function, we just need put together the two diagram we obtained in the steps iv) and v) and sum over the multiplicities μ, ν_r, ν_q and $\bar{\nu}_q$. This final diagram is shown in Fig. 18.

$$G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}} \propto \sum_{\mu}\sum_{\nu_r}\sum_{\nu_q}\sum_{\bar{\nu}_q}$$

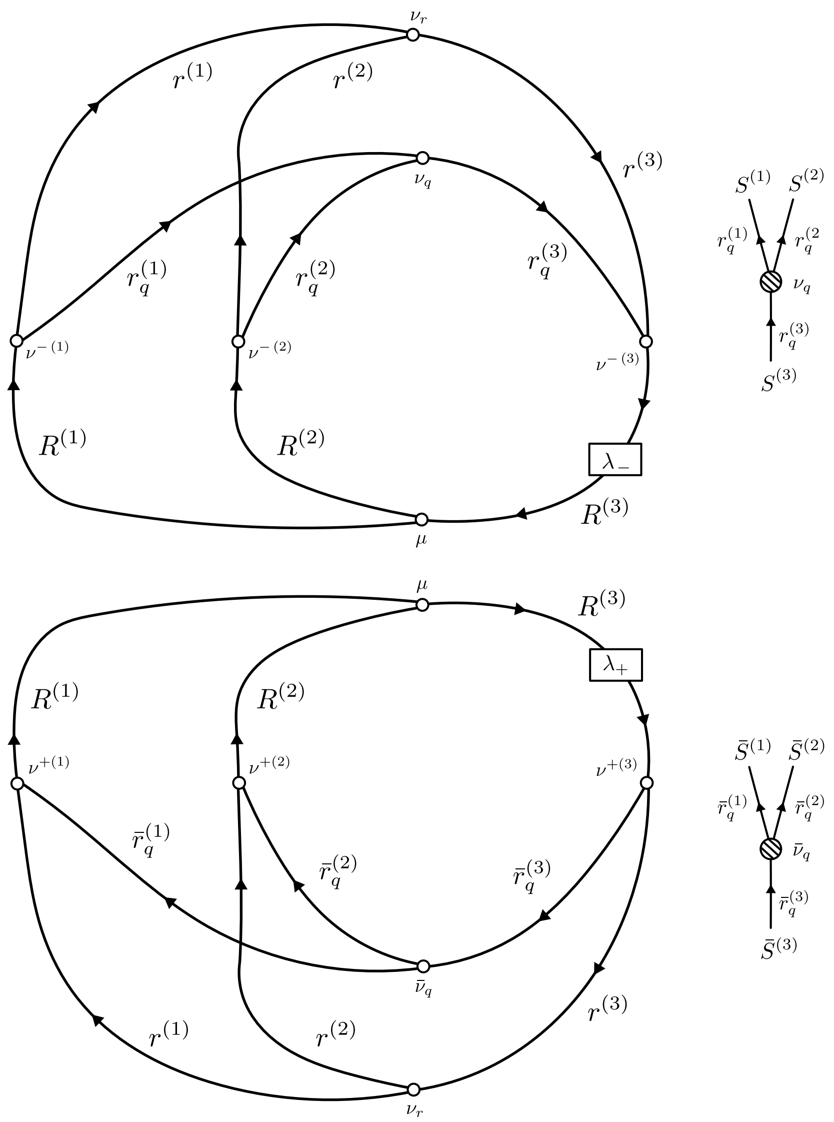

Figure 18: The diagram of the three point function (4.9) for the $\mathcal{N} = 2$ SQCD.

In Appendix D.1 we give a purely diagrammatic derivation of this result. We can see how the answer for the three point function factorises into two components: the former features only edges connected to the negative node of the split-node diagram, the latter only involves edges connected to its positive node. The same behaviour can be observed in the answer for the three point function of matrix invariants of generic quivers. We are now going to present this general result. The diagram for the three point function (4.9) is shown in Fig. 19.

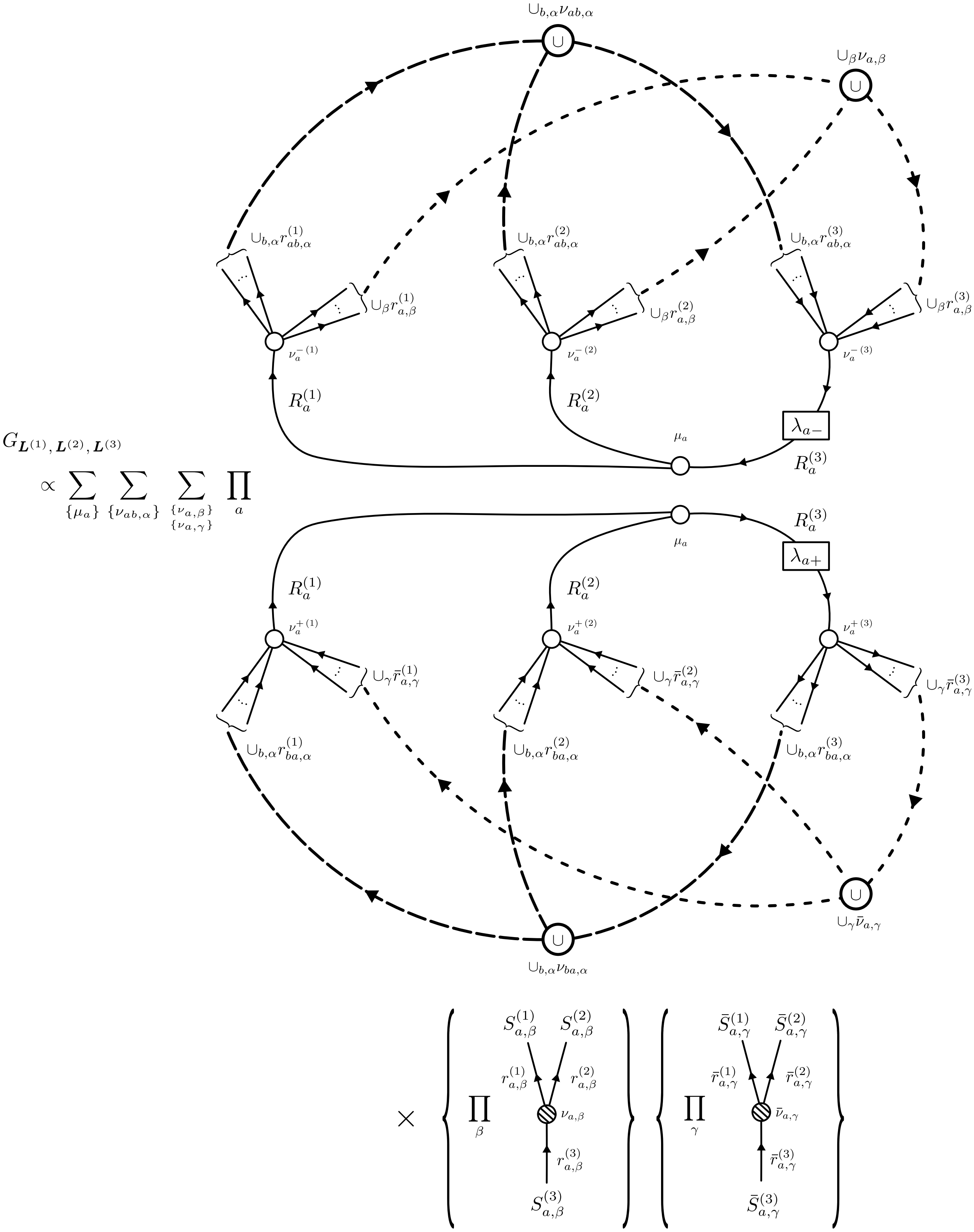

Figure 19: Pictorial description of the expression for the holomorphic GIO ring structure constants $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$, corresponding to eq. (D.45).

In drawing this picture we used the diagrammatic shorthand notation displayed in Fig. 20.

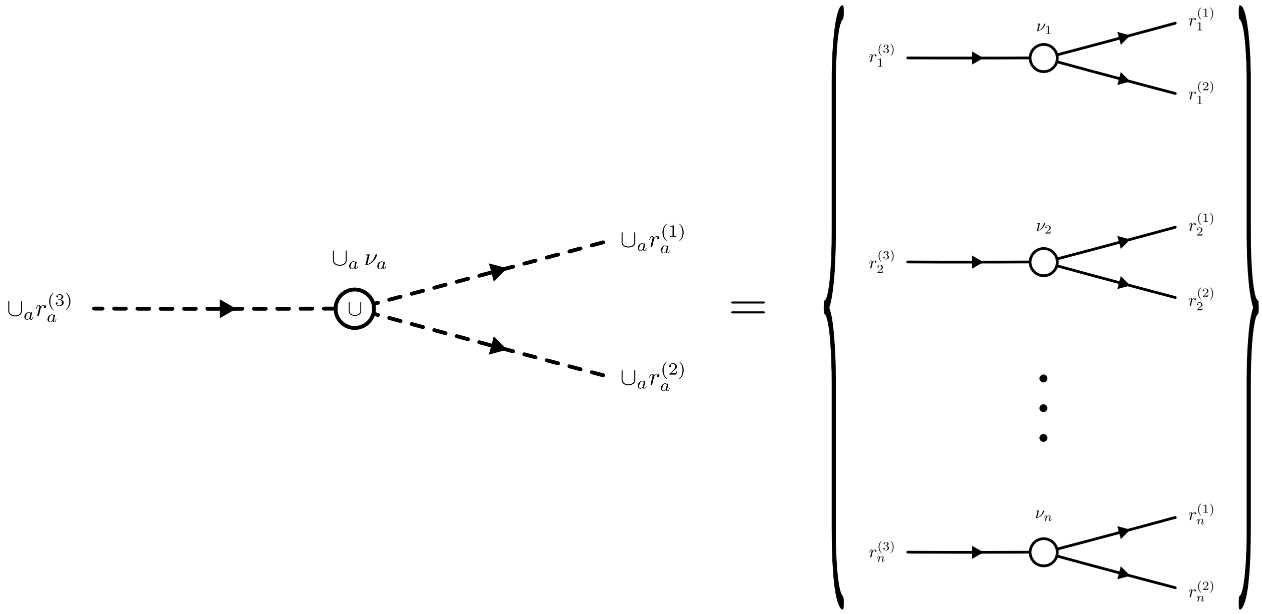

Figure 20: A shorthand notation for a collection of branching coefficients.

The λ_{a-} and λ_{a+} in Fig. 19 are permutations of $n_a^{(3)}$ elements, defined by the equations (D.2) and (D.3). Figure 19 shows that the holomorphic GIO ring structure constants factorise into a product over all the gauge nodes a of the quiver. Each one of these terms, whose diagrammatic interpretation is drawn in the figure, further factorises into a product of two components. They correspond to the positive and negative nodes of the split node a, with $a = 1, 2, ..., n$ (see also Fig. 7). Notice that the multiplicity labels μ_a, $\nu_{ab,\alpha}$, $\nu_{a,\beta}$ and $\bar{\nu}_{a,\gamma}$ always appear in pairs. For example, μ_a appears both in the upper and lower (disconnected) parts of the split-node a diagram. In the same diagram, $\nu_{a,\beta}$ appears in both a symmetric group branching coefficient and in a Clebsch-Gordan coefficient.

By inspecting Fig. 19 we can write four selection rules for the holomorphic GIO ring structure constants:

i) upon the restriction $S_{n_a^{(3)}}\Big|_{H_a}$, where $H_a = S_{n_a^{(1)}} \times S_{n_a^{(2)}}$, the $S_{n_a^{(3)}}$ representation $R_a^{(3)}$ becomes reducible. This reduction must contain the tensor product representation $R_a^{(1)} \otimes R_a^{(2)}$, $\forall\, a$. This implies the constraint $g(R_a^{(1)}, R_a^{(2)}; R_a^{(3)}) \neq 0$, $\forall\, a$.

ii) upon the restriction $S_{n_{ab,\alpha}^{(3)}}\Big|_{H_{ab,\alpha}}$, where $H_{ab,\alpha} = S_{n_{ab,\alpha}^{(1)}} \times S_{n_{ab,\alpha}^{(2)}}$, the $S_{n_{ab,\alpha}^{(3)}}$ representation $r_{ab,\alpha}^{(3)}$ becomes reducible. This reduction must contain the tensor product representation $r_{ab,\alpha}^{(1)} \otimes r_{ab,\alpha}^{(2)}$, $\forall\, a,\, b,\, \alpha$. This implies the constraint $g(r_{ab,\alpha}^{(1)}, r_{ab,\alpha}^{(2)}; r_{ab,\alpha}^{(3)}) \neq 0$, $\forall\, a,\, b,\, \alpha$.

iii) upon the restriction $S_{n_{a,\beta}^{(3)}}\Big|_{H_{a,\beta}}$, where $H_{a,\beta} = S_{n_{a,\beta}^{(1)}} \times S_{n_{a,\beta}^{(2)}}$, the $S_{n_{a,\beta}^{(3)}}$ representation $r_{a,\beta}^{(3)}$ becomes reducible. This reduction must contain the tensor product representation $r_{a,\beta}^{(1)} \otimes r_{a,\beta}^{(2)}$, $\forall\, a,\, \beta$. This implies the constraint $g(r_{a,\beta}^{(1)}, r_{a,\beta}^{(2)}; r_{a,\beta}^{(3)}) \neq 0$, $\forall\, a,\, \beta$.

iv) upon the restriction $S_{\bar{n}^{(3)}_{a,\gamma}}\Big|_{H_{a,\gamma}}$, where $H_{a,\gamma} = S_{\bar{n}^{(1)}_{a,\gamma}} \times S_{\bar{n}^{(2)}_{a,\gamma}}$, the $S_{\bar{n}^{(3)}_{a,\gamma}}$ representation $\bar{r}^{(3)}_{a,\gamma}$ becomes reducible. This reduction must contain the tensor product representation $\bar{r}^{(1)}_{a,\gamma} \otimes \bar{r}^{(2)}_{a,\gamma}$, $\forall\, a,\, \gamma$. This implies the constraint $g(\bar{r}^{(1)}_{a,\gamma}, \bar{r}^{(2)}_{a,\gamma}; \bar{r}^{(3)}_{a,\gamma}) \neq 0$, $\forall\, a,\, \gamma$.

All these rules are enforced by the branching coefficients networks in Fig. 19. Given two matrix invariants labelled by $\boldsymbol{L}^{(1)}$ and $\boldsymbol{L}^{(2)}$ respectively, we conclude that $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}} \neq 0$ if and only if $\boldsymbol{L}^{(3)}$ satisfies the selection rules i) - iv) above.

5 An Example: Quiver Restricted Schur Polynomials for an $\mathcal{N} = 2$ SQCD

We will now present some explicit examples of quiver Schurs for an $\mathcal{N} = 2$ SQCD, whose $\mathcal{N} = 1$ quiver is depicted in Fig. 5. We will begin by listing all the matrix invariants in the permutation basis (2.12) that it is possible to build using a fixed amount $\vec{n}$ of fundamental fields. We will then Fourier transform these operators to the quiver Schurs basis using (3.19). The set of representation theory labels needed to identify any matrix invariant in an $\mathcal{N} = 2$ SQCD has been explicitly given in (4.10). In the following we will continue to use such a convention.

The permutation basis is generated by

$$\mathcal{O}(\vec{n},\, \vec{s},\, \sigma) = \left(\phi^{\otimes\, n}\right)^{I}_{J} \otimes \left(Q^{\otimes\, n_Q}\right)^{I_Q}_{\boldsymbol{s}} \otimes \left(\bar{Q}^{\otimes\, \bar{n}_Q}\right)^{\bar{\boldsymbol{s}}}_{J_Q}\ \left(\sigma\right)^{J\times J_Q}_{I\times I_Q} \tag{5.1}$$

where $\vec{n} = \{n,\, n_Q,\, \bar{n}_Q\}$ specifies the field content of the operator $\mathcal{O}$, and $\vec{s} = (\boldsymbol{s}, \bar{\boldsymbol{s}})$. As we previously stated, we construct the quiver Schurs $\mathcal{O}(\boldsymbol{L})$ by using the Fourier transform (3.19):

$$\mathcal{O}(\boldsymbol{L}) = \sum_{\sigma,\, \vec{s}} \chi(\boldsymbol{L},\, \vec{s},\, \sigma)\, \mathcal{O}(\vec{n},\, \vec{s},\, \sigma) \tag{5.2}$$

where $\boldsymbol{L} = \{R, r,\, r_q, S,\, \bar{r}_q, \bar{S},\, \nu^+, \nu^-\}$ has been defined in eq. (4.10). In this formula $\chi(\boldsymbol{L},\, \vec{s},\, \sigma)$ is the $\mathcal{N} = 2$ SQCD quiver character, which reads

$$\chi(\boldsymbol{L},\, \vec{s},\, \sigma) = c_{\boldsymbol{L}}\, D^{R}_{i,j}(\sigma)\, \left\{B^{R\to r,\, r_q\,;\nu^-}_{j\to l,\, p}\ C^{\,r_q,\, S,\, p}_{\boldsymbol{s}}\right\} \left\{B^{R\to r,\, \bar{r}_q\,;\nu^+}_{i\to l,\, t}\ C^{\,\bar{r}_q,\, \bar{S},\, t}_{\bar{\boldsymbol{s}}}\right\} \tag{5.3}$$

Figure 21 shows the diagram for this quantity.

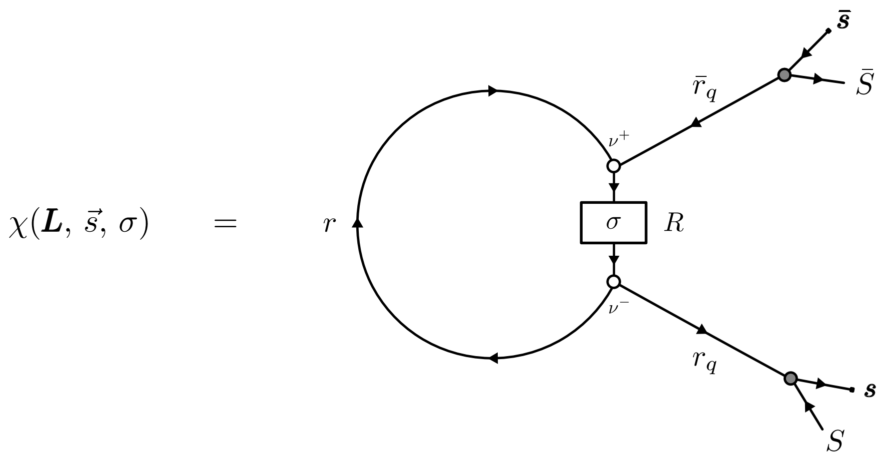

Figure 21: Diagram for the $\mathcal{N}=2$ SQCD quiver character, corresponding to eq. (5.3).

We now focus on some fixed values of $\vec{n}$.

- $\vec{n} = (2,1,1)$ field content

We start by listing the Fourier transformed holomorphic GIOs (3.19) that we can build with the set of fields $\{\phi, \phi, Q, \bar{Q}\}$, that is with the choice $\vec{n} = (2,1,1)$. In the permutation basis, these operators read

$$\begin{aligned}
&\mathcal{O}(\vec{n}\,, s, \bar{s}\,, (1)) = (\phi)(\phi)(\bar{Q}Q)^{\bar{s}}_{s}\,, \qquad && \mathcal{O}(\vec{n}\,, s, \bar{s}\,, (12)) = (\phi\phi)(\bar{Q}Q)^{\bar{s}}_{s}\,, \\
&\mathcal{O}(\vec{n}\,, s, \bar{s}\,, (13)) = (\phi)(\bar{Q}\phi Q)^{\bar{s}}_{s}\,, \qquad && \mathcal{O}(\vec{n}\,, s, \bar{s}\,, (23)) = (\phi)(\bar{Q}\phi Q)^{\bar{s}}_{s}\,, \\
&\mathcal{O}(\vec{n}\,, s, \bar{s}\,, (123)) = (\bar{Q}\phi\phi Q)^{\bar{s}}_{s}\,, \qquad && \mathcal{O}(\vec{n}\,, s, \bar{s}\,, (132)) = (\bar{Q}\phi\phi Q)^{\bar{s}}_{s}
\end{aligned} \tag{5.4}$$

where the round brackets denote $U(N)$ indices contraction. Notice that in this case $\vec{s} = (s, \bar{s})$. We will now construct the Fourier transformed operators. For this field content we do not have any branching multiplicity ν^+, ν^-: we can drop them from the set of labels $\boldsymbol{L}$, which now reads $\boldsymbol{L} = \{R, r, r_q, S, \bar{r}_q, \bar{S}\}$. We then look for the operators $\mathcal{O}(\boldsymbol{L}_i)$, $i = 1,2,3,4$, where

$$\boldsymbol{L}_1 = \{\yng(3), \yng(2), \square\,, S\,, \bar{\square}\,, \bar{S}\}\,, \qquad \boldsymbol{L}_2 = \left\{\yng(1,1,1), \yng(1,1), \square\,, S\,, \bar{\square}\,, \bar{S}\right\}\,, \tag{5.5}$$

$$\boldsymbol{L}_3 = \left\{\yng(2,1), \yng(2), \square\,, S\,, \bar{\square}\,, \bar{S}\right\}\,, \qquad \boldsymbol{L}_4 = \left\{\yng(2,1), \yng(1,1), \square\,, S\,, \bar{\square}\,, \bar{S}\right\}$$

We left the states S, $\bar{S}$ of the fundamental and antifundamental representation of $SU(F)$ implicit.

We first notice that, having one quark-antiquark pair only, the Clebsch-Gordan coefficients simplify as

$$C_{\boldsymbol{s}}^{r_q,\,S,\,p} = C_{s}^{\ydiagram{1},\,S,\,p} \equiv \delta_{s}^{S}\,, \qquad C_{\bar{\boldsymbol{s}}}^{\bar{r}_q,\,\bar{S},\,t} = C_{\bar{s}}^{\bar{\ydiagram{1}},\,\bar{S},\,t} \equiv \delta_{\bar{s}}^{\bar{S}} \tag{5.6}$$

We can then easily compute $\chi(\boldsymbol{L}_1)$ and $\chi(\boldsymbol{L}_2)$. Both the symmetric group representation branching $\ydiagram{3} \to \ydiagram{2} \otimes \ydiagram{1}$ and $\ydiagram{1,1,1} \to \ydiagram{1,1} \otimes \ydiagram{1}$ describe the branching of a 1-dimensional space into itself: as such their associate branching coefficients equal 1 identically. On the other hand, $D^{\ydiagram{3}}(\sigma) = 1 \;\forall\; \sigma$ and $D^{\ydiagram{1,1,1}}(\sigma) = \mathrm{sign}(\sigma)$. We then have

$$\chi(\boldsymbol{L}_1,\, s,\, \bar{s},\, \sigma) = \frac{1}{\sqrt{3!}}\; \delta_s^S\, \delta_{\bar{s}}^{\bar{S}}\,, \qquad \chi(\boldsymbol{L}_1,\, s,\, \bar{s},\, \sigma) = \frac{1}{\sqrt{3!}}\; \mathrm{sign}(\sigma)\, \delta_s^S\, \delta_{\bar{s}}^{\bar{S}} \tag{5.7}$$

The S_3 irrep $\ydiagram{2,1}$ is two dimensional, and we work in an orthonormal basis $\{e_1,\, e_2\}$ in which it reads[2]

$$D^{\ydiagram{2,1}}((1)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D^{\ydiagram{2,1}}((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D^{\ydiagram{2,1}}((13)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$
$$D^{\ydiagram{2,1}}((23)) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad D^{\ydiagram{2,1}}((123)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad D^{\ydiagram{2,1}}((132)) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{5.8}$$

If we restrict S_3 to $S_2 \times S_1$, the $\ydiagram{2,1}$ reduces as

$$\ydiagram{2,1}\,\Big|_{S_2 \times S_1} = \ydiagram{2} \otimes \ydiagram{1} \;\;\oplus\;\; \ydiagram{1,1} \otimes \ydiagram{1} \tag{5.9}$$

The restricted group $S_3|_{S_2 \times S_1}$ only contains two elements: $S_3|_{S_2 \times S_1} = \{(1),\, (12)\}$. The branching coefficients for this restriction are the matrix elements of the orthogonal operator B such that

$$B^{-1} D^{\ydiagram{2,1}}((12)) B = D^{\ydiagram{2}}((12)) \otimes D^{\ydiagram{1}}((1)) \;\oplus\; D^{\ydiagram{1,1}}((12)) \otimes D^{\ydiagram{1}}((1)) = \mathrm{diag}(1, -1) \tag{5.10}$$

With our basis choice for $\ydiagram{2,1}$ such a decomposition is already manifest, as it is clear from the matrix expression of the identity element and the (12) transposition in (5.8). Therefore, for this particular configuration, B is just the two dimensional identity matrix: $B = 1_2$. If we label f_1 the only state in the $\ydiagram{2}$ of S_2 and f_2 the only state in the $\ydiagram{1,1}$ of S_2, the branching

[2]Note that this is *not* the convention used in the SageMath software.

coefficients read

$$B_{j\to 1,1}^{\tiny\yng(2,1)\to\yng(2),\yng(1)} = (e_j,\, f_1) = \delta_{j,1}\,, \qquad\qquad B_{j\to 1,1}^{\tiny\yng(2,1)\to\yng(1,1),\yng(1)} = (e_j,\, f_2) = \delta_{j,2} \tag{5.11}$$

Inserting this result in (5.3) we obtain an expression for $\chi(\boldsymbol{L}_3)$ and $\chi(\boldsymbol{L}_4)$:

$$\begin{aligned}
\chi(\boldsymbol{L}_3,\, s,\, \bar{s},\, \sigma) &= \frac{1}{\sqrt{3}}\,\mathrm{Tr}\left[D^{\tiny\yng(2,1)}(\sigma)\, P^{\tiny\yng(2,1)\to\yng(2),\yng(1)}\right]\ \delta_s^S\, \delta_{\bar{s}}^{\bar{S}}\,,\\
\chi(\boldsymbol{L}_4,\, s,\, \bar{s},\, \sigma) &= \frac{1}{\sqrt{3}}\,\mathrm{Tr}\left[D^{\tiny\yng(2,1)}(\sigma)\, P^{\tiny\yng(2,1)\to\yng(1,1),\yng(1)}\right]\ \delta_s^S\, \delta_{\bar{s}}^{\bar{S}}
\end{aligned} \tag{5.12}$$

Here $P^{\tiny\yng(2,1)\to\yng(2),\yng(1)}$ and $P^{\tiny\yng(2,1)\to\yng(1,1),\yng(1)}$ are the projection operators of the $\yng(2,1)$ of S_3 on the $\yng(2)\otimes\yng(1)$ of $S_2\times S_1$ and the $\yng(2,1)$ of S_3 on the $\yng(1,1)\otimes\yng(1)$ of $S_2\times S_1$:

$$P^{\tiny\yng(2,1)\to\yng(2),\yng(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \qquad\qquad P^{\tiny\yng(2,1)\to\yng(1,1),\yng(1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.13}$$

We are now ready to write down the Fourier transformed operators. Using the definition (5.2) and the results (5.7) and (5.12), we find that

$$\begin{aligned}
\mathcal{O}(\boldsymbol{L}_1) &= \frac{1}{\sqrt{3!}}\left((\phi)(\phi)(\bar{Q}Q)_S^{\bar{S}} + (\phi\phi)(\bar{Q}Q)_S^{\bar{S}} + 2(\phi)(\bar{Q}\phi Q)_S^{\bar{S}} + 2(\bar{Q}\phi\phi Q)_S^{\bar{S}}\right)\,,\\
\mathcal{O}(\boldsymbol{L}_2) &= \frac{1}{\sqrt{3!}}\left((\phi)(\phi)(\bar{Q}Q)_S^{\bar{S}} - (\phi\phi)(\bar{Q}Q)_S^{\bar{S}} - 2(\phi)(\bar{Q}\phi Q)_S^{\bar{S}} + 2(\bar{Q}\phi\phi Q)_S^{\bar{S}}\right)\,,\\
\mathcal{O}(\boldsymbol{L}_3) &= \frac{1}{\sqrt{3}}\left((\phi)(\phi)(\bar{Q}Q)_S^{\bar{S}} + (\phi\phi)(\bar{Q}Q)_S^{\bar{S}} - (\phi)(\bar{Q}\phi Q)_S^{\bar{S}} - (\bar{Q}\phi\phi Q)_S^{\bar{S}}\right)\,,\\
\mathcal{O}(\boldsymbol{L}_4) &= \frac{1}{\sqrt{3}}\left((\phi)(\phi)(\bar{Q}Q)_S^{\bar{S}} - (\phi\phi)(\bar{Q}Q)_S^{\bar{S}} + (\phi)(\bar{Q}\phi Q)_S^{\bar{S}} - (\bar{Q}\phi\phi Q)_S^{\bar{S}}\right)
\end{aligned} \tag{5.14}$$

We can now perform some checks on this result. First of all, we expect to see the finite N constraints to manifest themselves if the gauge group of the theory is either $N = 1$ or $N = 2$. In the former case, only $\mathcal{O}(\boldsymbol{L}_1)$ should remain, and it is in fact easy to see that for $N = 1$ all the other operators are identically zero. For the latter case, we expect $\mathcal{O}(\boldsymbol{L}_2)$ to vanish, as $l(\yng(1,1,1)) > 2$, and as such it violates the finite N constraints. Indeed, using the identity $\phi^2 = (\phi)\phi - \det(\phi)1_2$, which follows from the Cayley-Hamilton theorem, one can verify that $\mathcal{O}(\boldsymbol{L}_2) = 0$ for a $U(2)$ gauge group.

We also expect these operators to be orthogonal in the free field metric. According to eq.

(4.5), the two point function in the permutation basis is simply

$$\left\langle \mathcal{O}(\vec{n}\,, s, \bar{s}\,, \sigma)\, \mathcal{O}^{\dagger}(\vec{n}\,, t, \bar{t}\,, \tau) \right\rangle = \delta_{s,t}\, \delta_{\bar{s},\bar{t}} \sum_{\eta \in S_2} N^{C\left[(\eta\times 1)\,\sigma\,(\eta\times 1)^{-1}\,\tau^{-1}\right]}\,, \qquad \vec{n} = (2,1,1) \tag{5.15}$$

were $C\,[\sigma]$ is the number of cycles in the permutation σ. With this equation we can check that all the states in (5.14) are orthogonal, and that

$$\begin{aligned}
&\left\langle \mathcal{O}(\boldsymbol{L}_1)\, \mathcal{O}^{\dagger}(\boldsymbol{L}_1) \right\rangle = 2\, N(N+1)(N+2)\,, && \left\langle \mathcal{O}(\boldsymbol{L}_2)\, \mathcal{O}^{\dagger}(\boldsymbol{L}_2) \right\rangle = 2\, N(N-1)(N-2)\,, \\
&\left\langle \mathcal{O}(\boldsymbol{L}_3)\, \mathcal{O}^{\dagger}(\boldsymbol{L}_3) \right\rangle = 2\, N(N^2-1)\,, && \left\langle \mathcal{O}(\boldsymbol{L}_4)\, \mathcal{O}^{\dagger}(\boldsymbol{L}_4) \right\rangle = 2\, N(N^2-1)
\end{aligned} \tag{5.16}$$

in agreement with (4.1).

- $\vec{n} = (1,2,2)$ field content

We now consider a different field content, that is $\{\phi, Q, Q, \bar{Q}, \bar{Q}\}$. This choice corresponds to $\vec{n} = (1,2,2)$. In the permutation basis, the GIOs that we can form with these fields are

$$\begin{aligned}
&\mathcal{O}(\vec{n}\,, \vec{s}, (1)) = (\phi)\, (\bar{Q}Q)^{\bar{s}_1}_{s_1}\, (\bar{Q}Q)^{\bar{s}_2}_{s_2}\,, && \mathcal{O}(\vec{n}\,, \vec{s}, (12)) = (\bar{Q}\phi Q)^{\bar{s}_1}_{s_1}\, (\bar{Q}Q)^{\bar{s}_2}_{s_2}\,, \\
&\mathcal{O}(\vec{n}\,, \vec{s}, (13)) = (\bar{Q}\phi Q)^{\bar{s}_2}_{s_2}\, (\bar{Q}Q)^{\bar{s}_1}_{s_1}\,, && \mathcal{O}(\vec{n}\,, \vec{s}, (23)) = (\phi)\, (\bar{Q}Q)^{\bar{s}_1}_{s_2}\, (\bar{Q}Q)^{\bar{s}_2}_{s_1}\,, \\
&\mathcal{O}(\vec{n}\,, \vec{s}, (123)) = (\bar{Q}\phi Q)^{\bar{s}_2}_{s_1}\, (\bar{Q}Q)^{\bar{s}_1}_{s_2}\,, && \mathcal{O}(\vec{n}\,, \vec{s}, (132)) = (\bar{Q}\phi Q)^{\bar{s}_1}_{s_2}\, (\bar{Q}Q)^{\bar{s}_2}_{s_1}
\end{aligned} \tag{5.17}$$

Here $\vec{s} = (s_1,\, s_2\,, \bar{s}_1,\, \bar{s}_2)$, and the round brackets denote $U(N)$ indices contraction.

Let us now construct the Fourier transformed operators. As in the previous example, for this fields content we do not have any branching multiplicity ν^+, ν^-, so that we will drop them from the set of labels in $\boldsymbol{L}$. We will now write the expression for the six operators $\mathcal{O}(\boldsymbol{L}_i)$, $i = 1, 2, ..., 6$, with

$$\begin{aligned}
&\boldsymbol{L}_1 = \left\{\yng(3), \yng(1), \yng(2), S\,, \overline{\yng(2)}, \bar{S}\right\}, && \boldsymbol{L}_2 = \left\{\yng(1,1,1), \yng(1), \yng(1,1), S\,, \overline{\yng(1,1)}, \bar{S}\right\}, \\
&\boldsymbol{L}_3 = \left\{\yng(2,1), \yng(1), \yng(2), S\,, \overline{\yng(2)}, \bar{S}\right\}, && \boldsymbol{L}_4 = \left\{\yng(2,1), \yng(1), \yng(1,1), S\,, \overline{\yng(1,1)}, \bar{S}\right\}, \\
&\boldsymbol{L}_5 = \left\{\yng(2,1), \yng(1), \yng(2), S\,, \overline{\yng(1,1)}, \bar{S}\right\}, && \boldsymbol{L}_6 = \left\{\yng(2,1), \yng(1), \yng(1,1), S\,, \overline{\yng(2)}, \bar{S}\right\}
\end{aligned} \tag{5.18}$$

As in the previous example, we leave the $SU(F)$ states S, $\bar{S}$ implicit.

The symmetric branching group coefficients are similar to the ones already introduced in the previous example. Both the branchings $\ydiagram{3} \to \ydiagram{1} \oplus \ydiagram{2}$ and $\ydiagram{1,1,1} \to \ydiagram{1} \oplus \ydiagram{1,1}$ are trivial, as they correspond to a branching of a 1-dimensional space into itself. These branching coefficients are therefore equal to 1 identically:

$$B^{\ydiagram{3} \to \ydiagram{1}, \ydiagram{2}}_{1\to 1,1} \equiv 1\,, \qquad\qquad B^{\ydiagram{1,1,1} \to \ydiagram{1}, \ydiagram{1,1}}_{1\to 1,1} \equiv 1 \tag{5.19}$$

We now turn to the reduction

$$\ydiagram{2,1}\,\Bigg|_{S_1\times S_2} = \ydiagram{1} \otimes \ydiagram{2} \;\oplus\; \ydiagram{1} \otimes \ydiagram{1,1} \tag{5.20}$$

As in the previous example, the group $S_3|_{S_1\times S_2}$ only contains two elements, but this time they are $S_3|_{S_1\times S_2} = \{(1),\,(23)\}$. This is because the $(1)\times(12) \in S_1 \times S_2$ has to be embedded into S_3, where it corresponds to the transposition (23). The branching coefficients for the reduction in (5.20) will be the matrix elements of the orthogonal operator B such that

$$B^{-1} D^{\ydiagram{2,1}}((23)) B = D^{\ydiagram{1}}((1)) \otimes D^{\ydiagram{2}}((12)) \;\oplus\; D^{\ydiagram{1}}((1)) \otimes D^{\ydiagram{1,1}}((12)) = \mathrm{diag}(1,-1) \tag{5.21}$$

We equip the $\ydiagram{2,1}$ of S_3 with a basis $\{e_1,\,e_2\}$, in which the representation takes the explicit form (5.8). We then choose f_1 and f_2 to be the basis vectors of the $\ydiagram{2}$ and the $\ydiagram{1,1}$ of S_2 respectively. In this basis the orthogonal matrix B must then take the form

$$B = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \tag{5.22}$$

We then have, by construction, $Be_1 = f_1$ and $Be_2 = f_2$. The branching coefficients for the reduction (5.20) then read

$$\begin{aligned} B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{2}}_{1\to 1,1} &= (e_1,\ f_1) = \tfrac{1}{2}\,, & B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{1,1}}_{1\to 1,1} &= (e_1,\ f_2) = -\tfrac{\sqrt{3}}{2}\,, \\ B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{2}}_{2\to 1,1} &= (e_2,\ f_1) = \tfrac{\sqrt{3}}{2}\,, & B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{1,1}}_{2\to 1,1} &= (e_2,\ f_2) = \tfrac{1}{2} \end{aligned} \tag{5.23}$$

It is useful to define the orthogonal projectors

$$P^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{2}}_{i,j} = B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{2}}_{i\to 1,1} B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{2}}_{j\to 1,1}\,, \qquad P^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{1,1}}_{i,j} = B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{1,1}}_{i\to 1,1} B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{1,1}}_{j\to 1,1} \tag{5.24}$$

projecting the $\ydiagram{2,1}$ of S_3 on the $\ydiagram{1} \otimes \ydiagram{2}$ and on the $\ydiagram{1} \otimes \ydiagram{1,1}$ of $S_1 \times S_2$ respectively. We also define the linear operator T through its matrix elements as

$$T_{i,j} = B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{2}}_{i\to 1,1} B^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{1,1}}_{j\to 1,1} \tag{5.25}$$

Explicitly, these matrices read

$$
P^{\ydiagram{2,1}\to\ydiagram{1},\ydiagram{2}} = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}\,, \qquad P^{\ydiagram{2,1}\to\ydiagram{1},\ydiagram{1,1}} = \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\,, \qquad T = \frac{1}{4}\begin{pmatrix} -\sqrt{3} & 1 \\ -3 & \sqrt{3} \end{pmatrix} \tag{5.26}
$$

We will use these quantities to compactly write the quiver characters.

We now turn to the Clebsch-Gordan coefficients, $C^{r_q,\,S,\,p}_{s_1,\,s_2}$ and $C^{\bar{r}_q,\,\bar{S},\,t}_{\bar{s}_1,\,\bar{s}_2}$, where r_q and $\bar{r}_q$ are both either $\ydiagram{2}$ or $\ydiagram{1,1}$. First of all notice that we can drop the symmetric group state labels p and t, because all the irreducible representation of S_2 are 1-dimensional. Let us call V_F the the fundamental representation of $SU(F)$, and let us choose an orthonormal basis e_i, $i = 1, 2, ..., F$. Consider now the $V_F \otimes V_F$ vector space, equipped with the induced basis $\{e_{i,j} = e_i \otimes e_j\}_{ij}$. The $\ydiagram{2}$ of $SU(F)$ is spanned by every symmetric permutation of the $e_{i,j} = e_i \otimes e_j$ basis vectors of $V_F \otimes V_F$. We can label an orthonormal basis for this representation with the notation $\ytableaushort{ij}$, where

$$
\ytableaushort{ii} = e_i \otimes e_i\,, \tag{5.27a}
$$

$$
\ytableaushort{ij} = \frac{1}{\sqrt{2}}(e_i \otimes e_j + e_j \otimes e_i)\,, \qquad i \neq j \tag{5.27b}
$$

On the other hand, the $\ydiagram{1,1}$ of $SU(F)$ is spanned by every antisymmetric permutation of the $e_{i,j} = e_i \otimes e_j$ basis vectors of $V_F \otimes V_F$. We can label an orthonormal basis for this representation with the notation $\ytableaushort{i,j}$, where

$$
\ytableaushort{i,j} = \frac{1}{\sqrt{2}}(e_i \otimes e_j - e_j \otimes e_i) \tag{5.28}
$$

We can therefore easily compute the Clebsch-Gordan coefficients (3.18). To optimise the notation, we use the Young tableaux $\ytableaushort{ij}$ and $\ytableaushort{i,j}$ to label both the $SU(F)$ representations and their states. The Clebsch-Gordan coefficients then read

$$
\begin{aligned}
C^{\ytableaushort{ii}}_{k,l} &= (e_{k,l},\ \ytableaushort{ii}) = (e_k \otimes e_l,\, e_i \otimes e_i) = \delta_{k,i}\,\delta_{l,i}\,, \\
C^{\ytableaushort{ij}}_{k,l} &= (e_{k,l},\ \ytableaushort{ij}) = \frac{1}{\sqrt{2}}\,(e_k \otimes e_l,\, e_i \otimes e_j + e_j \otimes e_i) = \frac{1}{\sqrt{2}}\,(\delta_{k,i}\,\delta_{l,j} + \delta_{k,j}\,\delta_{l,i})\,, \quad i \neq j\,, \\
C^{\ytableaushort{i,j}}_{k,l} &= (e_{k,l},\ \ytableaushort{i,j}) = \frac{1}{\sqrt{2}}\,(e_k \otimes e_l,\, e_i \otimes e_j - e_j \otimes e_i) = \frac{1}{\sqrt{2}}\,(\delta_{k,i}\,\delta_{l,j} - \delta_{k,j}\,\delta_{l,i})
\end{aligned} \tag{5.29}
$$

A similar approach can be used to derive the Clebsch-Gordan coefficients for the decomposition of the $\bar{V}_F \otimes \bar{V}_F$ representation of $SU(F)$, which gives similar results to the ones in (5.29).

We can now write the quiver characters for the six states (5.18). Denoting the generic flavour state $|S\rangle \in V_{r_q}^{SU(F)}$ as in (5.27) for $r_q = \ydiagram{2}$ and as in (5.28) for $r_q = \ydiagram{1,1}$ (and similarly for $|\bar{S}\rangle \in V_{\bar{r}_q}^{SU(F)}$), the labels in (5.18) read now

$$
\begin{aligned}
\boldsymbol{L}_1 &= \left\{ \ydiagram{3}, \ydiagram{1}, \ytableaushort{ij}, \overline{\ytableaushort{pq}} \right\}, & \boldsymbol{L}_2 &= \left\{ \ydiagram{1,1,1}, \ydiagram{1}, \ytableaushort{i,j}, \overline{\ytableaushort{p,q}} \right\}, \\
\boldsymbol{L}_3 &= \left\{ \ydiagram{2,1}, \ydiagram{1}, \ytableaushort{ij}, \overline{\ytableaushort{pq}} \right\}, & \boldsymbol{L}_4 &= \left\{ \ydiagram{2,1}, \ydiagram{1}, \ytableaushort{i,j}, \overline{\ytableaushort{p,q}} \right\}, \\
\boldsymbol{L}_5 &= \left\{ \ydiagram{2,1}, \ydiagram{1}, \ytableaushort{ij}, \overline{\ytableaushort{p,q}} \right\}, & \boldsymbol{L}_6 &= \left\{ \ydiagram{2,1}, \ydiagram{1}, \ytableaushort{i,j}, \overline{\ytableaushort{pq}} \right\}
\end{aligned}
\tag{5.30}
$$

The quiver characters are

$$
\begin{aligned}
\chi(\boldsymbol{L}_1, \vec{s}, \sigma) &= \frac{1}{\sqrt{3!}}\, C^{\ytableaushort{ij}}_{s_1,s_2}\, C^{\overline{\ytableaushort{pq}}}_{\bar{s}_1,\bar{s}_2}, \\
\chi(\boldsymbol{L}_2, \vec{s}, \sigma) &= \frac{1}{\sqrt{3!}}\, \mathrm{sign}(\sigma)\, C^{\ytableaushort{i,j}}_{s_1,s_2}\, C^{\overline{\ytableaushort{p,q}}}_{\bar{s}_1,\bar{s}_2}, \\
\chi(\boldsymbol{L}_3, \vec{s}, \sigma) &= \frac{1}{\sqrt{3}}\, \mathrm{Tr}\left[D^{\ydiagram{2,1}}(\sigma)\, P^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{2}}\right]\, C^{\ytableaushort{ij}}_{s_1,s_2}\, C^{\overline{\ytableaushort{pq}}}_{\bar{s}_1,\bar{s}_2}, \\
\chi(\boldsymbol{L}_4, \vec{s}, \sigma) &= \frac{1}{\sqrt{3}}\, \mathrm{Tr}\left[D^{\ydiagram{2,1}}(\sigma)\, P^{\ydiagram{2,1} \to \ydiagram{1}, \ydiagram{1,1}}\right]\, C^{\ytableaushort{i,j}}_{s_1,s_2}\, C^{\overline{\ytableaushort{p,q}}}_{\bar{s}_1,\bar{s}_2}, \\
\chi(\boldsymbol{L}_5, \vec{s}, \sigma) &= \frac{1}{\sqrt{3}}\, \mathrm{Tr}\left[D^{\ydiagram{2,1}}(\sigma)\, T\right]\, C^{\ytableaushort{ij}}_{s_1,s_2}\, C^{\overline{\ytableaushort{p,q}}}_{\bar{s}_1,\bar{s}_2}, \\
\chi(\boldsymbol{L}_6, \vec{s}, \sigma) &= \frac{1}{\sqrt{3}}\, \mathrm{Tr}\left[D^{\ydiagram{2,1}}(\sigma)\, T^{\mathrm{t}}\right]\, C^{\ytableaushort{i,j}}_{s_1,s_2}\, C^{\overline{\ytableaushort{pq}}}_{\bar{s}_1,\bar{s}_2}
\end{aligned}
\tag{5.31}
$$

where T^{t} denotes the transpose of the matrix T, defined in (5.26).

Defining the normalisation constants

$$
f_{i,j} = \begin{cases} 1 & \text{if} \quad i \neq j \\ \frac{1}{\sqrt{2}} & \text{if} \quad i = j \end{cases}
\tag{5.32}
$$

which keeps track of the different normalisation of the Clebsch-Gordan coefficients (5.27a) and (5.27b), the Fourier transformed operators take the explicit form

$$
\begin{aligned}
\mathcal{O}(\boldsymbol{L}_1) &= \frac{f_{i,j}\, f_{\bar{p},\bar{q}}}{\sqrt{3!}} \left((\phi)\, (\bar{Q}Q)^{(\bar{p}}_{(i}\, (\bar{Q}Q)^{\bar{q})}_{j)} + 2(\bar{Q}\phi Q)^{(\bar{p}}_{(i}\, (\bar{Q}Q)^{\bar{q})}_{j)} \right), \\
\mathcal{O}(\boldsymbol{L}_2) &= \frac{1}{\sqrt{3!}} \left((\phi)\, (\bar{Q}Q)^{[\bar{p}}_{[i}\, (\bar{Q}Q)^{\bar{q}]}_{j]} - 2(\bar{Q}\phi Q)^{[\bar{p}}_{[i}\, (\bar{Q}Q)^{\bar{q}]}_{j]} \right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}(\boldsymbol{L}_3) &= \frac{f_{i,j}\, f_{\bar{p},\bar{q}}}{\sqrt{3}} \left((\phi)\, (\bar{Q}Q)^{(\bar{p}}_{(i}\, (\bar{Q}Q)^{\bar{q})}_{j)} - (\bar{Q}\phi Q)^{(\bar{p}}_{(i}\, (\bar{Q}Q)^{\bar{q})}_{j)} \right) , \\
\mathcal{O}(\boldsymbol{L}_4) &= \frac{1}{\sqrt{3}} \left((\phi)\, (\bar{Q}Q)^{[\bar{p}}_{[i}\, (\bar{Q}Q)^{\bar{q}]}_{j]} + (\bar{Q}\phi Q)^{[\bar{p}}_{[i}\, (\bar{Q}Q)^{\bar{q}]}_{j]} \right) , \\
\mathcal{O}(\boldsymbol{L}_5) &= -f_{i,j}\, (\bar{Q}\phi Q)^{[\bar{p}}_{(i}\, (\bar{Q}Q)^{\bar{q}]}_{j)} , \\
\mathcal{O}(\boldsymbol{L}_6) &= -f_{\bar{p},\bar{q}}\, (\bar{Q}\phi Q)^{(\bar{p}}_{[i}\, (\bar{Q}Q)^{\bar{q})}_{j]}
\end{aligned}
\tag{5.33}
$$

Round brackets around the flavour indices denotes their symmetrisation, square brackets around them denotes their antisymmetrisation.

As in the previous case, we now run some tests on this result. It is easily seen that if the rank of the gauge group is $N = 1$, then among these six operators only $\mathcal{O}(\boldsymbol{L}_1)$ is non-zero, in agreement with our finite N constraints (4.3). Moreover, when $N = 2$, by explicitly writing all the components of $\mathcal{O}(\boldsymbol{L}_2)$ it is possible to check that $\mathcal{O}(\boldsymbol{L}_2) = 0$. This is a nontrivial result, once again predicted by the finite N constraints. Let us now check the orthogonality of these operators, in the free field metric. For this field content the two point function in the permutation basis, eq. (4.5), reads

$$
\left\langle \mathcal{O}(\vec{n}\,, \vec{s}\,, \sigma)\, \mathcal{O}^\dagger(\vec{n}\,, \vec{t}\,, \tau) \right\rangle = \sum_{\rho_1,\, \rho_2 \in S_2} \delta_{\rho_1(\boldsymbol{s}),\boldsymbol{t}}\, \delta_{\rho_2(\bar{\boldsymbol{s}}),\bar{\boldsymbol{t}}}\, N^{C\left[(1\times\rho_2)\,\sigma\,(1\times\rho_1)^{-1}\,\tau^{-1}\right]} , \qquad \vec{n} = (1, 2, 2)
\tag{5.34}
$$

As in the previous example, $C\,[\sigma]$ is the number of cycles in the permutation σ. Using this equation we can verify that the states in (5.33) are indeed orthogonal. Similarly, their squared norm are

$$
\begin{aligned}
&\left\langle \mathcal{O}(\boldsymbol{L}_1)\, \mathcal{O}^\dagger(\boldsymbol{L}_1) \right\rangle = 4\, N(N+1)(N+2) , && \left\langle \mathcal{O}(\boldsymbol{L}_2)\, \mathcal{O}^\dagger(\boldsymbol{L}_2) \right\rangle = 4\, N(N-1)(N-2) , \\
&\left\langle \mathcal{O}(\boldsymbol{L}_3)\, \mathcal{O}^\dagger(\boldsymbol{L}_3) \right\rangle = 4\, N(N^2-1) , && \left\langle \mathcal{O}(\boldsymbol{L}_4)\, \mathcal{O}^\dagger(\boldsymbol{L}_4) \right\rangle = 4\, N(N^2-1) , \\
&\left\langle \mathcal{O}(\boldsymbol{L}_5)\, \mathcal{O}^\dagger(\boldsymbol{L}_5) \right\rangle = 4\, N(N^2-1) , && \left\langle \mathcal{O}(\boldsymbol{L}_6)\, \mathcal{O}^\dagger(\boldsymbol{L}_6) \right\rangle = 4\, N(N^2-1)
\end{aligned}
\tag{5.35}
$$

in agreement with our prediction (4.1).

- $\vec{n} = (2, 2, 2)$ field content

Consider now the field content $\{\phi, \phi, Q, Q, \bar{Q}, \bar{Q}\}$, that is $\vec{n} = (2, 2, 2)$. Using the same notation of the previous examples, the quiver Schurs for this subspace can be labelled by the fourteen sets

$$
\begin{aligned}
&\boldsymbol{L}_1=\left\{\ydiagram{4}\,,\ydiagram{2}\,,\ytableaushort{ij}\,,\overline{\ytableaushort{pq}}\right\}, &&\boldsymbol{L}_2=\left\{\ydiagram{1,1,1,1}\,,\ydiagram{1,1}\,,\ytableaushort{i,j}\,,\overline{\ytableaushort{p,q}}\right\},\\
&\boldsymbol{L}_3=\left\{\ydiagram{3,1}\,,\ydiagram{2}\,,\ytableaushort{ij}\,,\overline{\ytableaushort{pq}}\right\}, &&\boldsymbol{L}_4=\left\{\ydiagram{3,1}\,,\ydiagram{1,1}\,,\ytableaushort{ij}\,,\overline{\ytableaushort{pq}}\right\},\\
&\boldsymbol{L}_5=\left\{\ydiagram{3,1}\,,\ydiagram{2}\,,\ytableaushort{i,j}\,,\overline{\ytableaushort{p,q}}\right\}, &&\boldsymbol{L}_6=\left\{\ydiagram{3,1}\,,\ydiagram{2}\,,\ytableaushort{ij}\,,\overline{\ytableaushort{p,q}}\right\},\\
&\boldsymbol{L}_7=\left\{\ydiagram{3,1}\,,\ydiagram{2}\,,\ytableaushort{i,j}\,,\overline{\ytableaushort{pq}}\right\}, &&\boldsymbol{L}_8=\left\{\ydiagram{2,1,1}\,,\ydiagram{1,1}\,,\ytableaushort{i,j}\,,\overline{\ytableaushort{p,q}}\right\},\\
&\boldsymbol{L}_9=\left\{\ydiagram{2,1,1}\,,\ydiagram{1,1}\,,\ytableaushort{ij}\,,\overline{\ytableaushort{pq}}\right\}, &&\boldsymbol{L}_{10}=\left\{\ydiagram{2,1,1}\,,\ydiagram{2}\,,\ytableaushort{i,j}\,,\overline{\ytableaushort{p,q}}\right\},\\
&\boldsymbol{L}_{11}=\left\{\ydiagram{2,1,1}\,,\ydiagram{1,1}\,,\ytableaushort{ij}\,,\overline{\ytableaushort{p,q}}\right\}, &&\boldsymbol{L}_{12}=\left\{\ydiagram{2,1,1}\,,\ydiagram{1,1}\,,\ytableaushort{i,j}\,,\overline{\ytableaushort{pq}}\right\},\\
&\boldsymbol{L}_{13}=\left\{\ydiagram{2,2}\,,\ydiagram{2}\,,\ytableaushort{ij}\,,\overline{\ytableaushort{pq}}\right\}, &&\boldsymbol{L}_{14}=\left\{\ydiagram{2,2}\,,\ydiagram{1,1}\,,\ytableaushort{i,j}\,,\overline{\ytableaushort{p,q}}\right\}
\end{aligned}
\tag{5.36}
$$

As usual, we left the states $\ytableaushort{ij}$ and $\ytableaushort{i,j}$ (with $i,j=1,2,...,F$) of the symmetric and antisymmetric representation of $SU(F)$ unspecified.

The quiver Schurs explicitly read

$$
\begin{aligned}
\mathcal{O}(\boldsymbol{L}_1)=\frac{f_{i,j}\,f_{\bar{p},\bar{q}}}{\sqrt{3!}}\,\Big(&2(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi\phi Q)^{\bar{q})}_{j)}+(\bar{Q}\phi Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi Q)^{\bar{q})}_{j)}+\frac{1}{2}(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi)^2+\\
&+\frac{1}{2}(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi\phi)+2(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi Q)^{\bar{q})}_{j)}\,(\phi)\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}(\boldsymbol{L}_2)=\frac{1}{\sqrt{3!}}\,\Big(&2(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi\phi Q)^{\bar{q}]}_{j]}+(\bar{Q}\phi Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi Q)^{\bar{q}]}_{j]}+\frac{1}{2}(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi)^2+\\
&-\frac{1}{2}(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi\phi)-2(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi Q)^{\bar{q}]}_{j]}\,(\phi)\Big)\,,
\end{aligned}
$$

$$
\mathcal{O}(\boldsymbol{L}_3)=\frac{f_{i,j}\,f_{\bar{p},\bar{q}}}{2\sqrt{2}}\,\Big(-2(\bar{Q}\phi Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi Q)^{\bar{q})}_{j)}+(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi)^2+(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi\phi)\Big)\,,
$$

$$
\begin{aligned}
\mathcal{O}(\boldsymbol{L}_4)=\frac{f_{i,j}\,f_{\bar{p},\bar{q}}}{2\sqrt{2}}\,\Big(&-2(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi\phi Q)^{\bar{q})}_{j)}+(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi)^2+\\
&-(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi\phi)+2(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi Q)^{\bar{q})}_{j)}\,(\phi)\Big)\,,
\end{aligned}
$$

$$\mathcal{O}(\boldsymbol{L}_5) = \frac{1}{2\sqrt{2}} \Big(2(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi\phi Q)^{\bar{q}]}_{j]} + (\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi)^2 +$$

$$+(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi\phi) + 2(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi Q)^{\bar{q}]}_{j]}\,(\phi)\Big)\,, \qquad (5.37)$$

$$\mathcal{O}(\boldsymbol{L}_6) = -f_{i,j}\,\Big((\bar{Q}Q)^{[\bar{p}}_{(i}\,(\bar{Q}\phi\phi Q)^{\bar{q}]}_{j)} + (\bar{Q}Q)^{[\bar{p}}_{(i}\,(\bar{Q}\phi Q)^{\bar{q}]}_{j)}\,(\phi)\Big)\,,$$

$$\mathcal{O}(\boldsymbol{L}_7) = -f_{\bar{p},\bar{q}}\,\Big((\bar{Q}Q)^{(\bar{p}}_{[i}\,(\bar{Q}\phi\phi Q)^{\bar{q})}_{j]} + (\bar{Q}Q)^{(\bar{p}}_{[i}\,(\bar{Q}\phi Q)^{\bar{q})}_{j]}\,(\phi)\Big)\,,$$

$$\mathcal{O}(\boldsymbol{L}_8) = \frac{1}{2\sqrt{2}}\,\Big(-2(\bar{Q}\phi Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi Q)^{\bar{q}]}_{j]} + (\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi)^2 - (\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi\phi)\Big)\,,$$

$$\mathcal{O}(\boldsymbol{L}_9) = \frac{f_{i,j}\,f_{\bar{p},\bar{q}}}{2\sqrt{2}}\,\Big(2(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi\phi Q)^{\bar{q})}_{j)} + (\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi)^2 +$$

$$-(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi\phi) - 2(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi Q)^{\bar{q})}_{j)}\,(\phi)\Big)\,,$$

$$\mathcal{O}(\boldsymbol{L}_{10}) = \frac{1}{2\sqrt{2}}\,\Big(-2(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi\phi Q)^{\bar{q}]}_{j]} + (\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi)^2 +$$

$$+(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi\phi) - 2(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi Q)^{\bar{q}]}_{j]}\,(\phi)\Big)\,,$$

$$\mathcal{O}(\boldsymbol{L}_{11}) = -f_{i,j}\,\Big((\bar{Q}Q)^{[\bar{p}}_{(i}\,(\bar{Q}\phi\phi Q)^{\bar{q}]}_{j)} - (\bar{Q}Q)^{[\bar{p}}_{(i}\,(\bar{Q}\phi Q)^{\bar{q}]}_{j)}\,(\phi)\Big)\,,$$

$$\mathcal{O}(\boldsymbol{L}_{12}) = -f_{\bar{p},\bar{q}}\,\Big((\bar{Q}Q)^{(\bar{p}}_{[i}\,(\bar{Q}\phi\phi Q)^{\bar{q})}_{j]} - (\bar{Q}Q)^{(\bar{p}}_{[i}\,(\bar{Q}\phi Q)^{\bar{q})}_{j]}\,(\phi)\Big)\,,$$

$$\mathcal{O}(\boldsymbol{L}_{13}) = \frac{f_{i,j}\,f_{\bar{p},\bar{q}}}{\sqrt{3}}\,\Bigg(-(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi\phi Q)^{\bar{q})}_{j)} + (\bar{Q}\phi Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi Q)^{\bar{q})}_{j)} + \frac{1}{2}(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi)^2 +$$

$$+\frac{1}{2}(\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}Q)^{\bar{q})}_{j)}\,(\phi\phi) - (\bar{Q}Q)^{(\bar{p}}_{(i}\,(\bar{Q}\phi Q)^{\bar{q})}_{j)}\,(\phi)\Bigg)\,,$$

$$\mathcal{O}(\boldsymbol{L}_{14}) = \frac{1}{\sqrt{3}}\,\Bigg(-(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi\phi Q)^{\bar{q}]}_{j]} + (\bar{Q}\phi Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi Q)^{\bar{q}]}_{j]} + \frac{1}{2}(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi)^2 +$$

$$-\frac{1}{2}(\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}Q)^{\bar{q}]}_{j]}\,(\phi\phi) + (\bar{Q}Q)^{[\bar{p}}_{[i}\,(\bar{Q}\phi Q)^{\bar{q}]}_{j]}\,(\phi)\Bigg)$$

The convention for round and square brackets around flavour indices is the same as the one used in the previous example. The computation that leads to this result is summarised in Appendix E. Using Mathematica, we checked that all these operators are orthogonal in the free field metric, that their norm satisfy (4.1), and that they obey the finite N constraints (4.3).

6 Conclusions and Outlook

In this paper we considered free quiver gauge theories with gauge group $\prod_{a=1}^{n} U(N_a)$ and flavour group $\prod_{a=1}^{n} U(F_a) \times U(\bar{F}_a)$. We found that the basis of Quiver Restricted Schur polynomials (3.19) diagonalises the two point function (4.1). Relying on diagrammatic methods, we also provided an analytical finite N expression for the three point function of holomorphic matrix invariants. The relevant diagram is shown in Fig. 19.

For quiver gauge theories with bi-fundamental matter (no fundamental matter), the counting and correlators of gauge invariant operators can be expressed in terms of defect observables in two dimensional topological field theories (TFT2). These theories are based on lattice gauge theory where permutation groups play the role of gauge groups [25]. The relevant two dimensional surfaces were obtained by a process of thickening the quiver. This leads us to expect that the counting and correlators for the present case can be expressed in terms of defect observables in TFT2 on Riemann surfaces with boundary. It will be very interesting to elaborate on this in the future. Another interesting future direction is the relation of gauge invariant correlators to the counting of branched covers. This has been discussed for the case of a single gauge group and one or more adjoint fields [42–47]. The equation (4.5) giving the formula for the 2-point function in the permutation basis would be a good starting point. By tracing the flavour indices, we expect to see that powers of the flavour rank are related to the counting of covering surfaces with boundaries (see for example [48]).

For the case of a single gauge group but multi-matrices (quiver with one node and multiple edges), a complete set of charges measuring the group theoretic labels of orthogonal bases for gauge invariant operators were given in [17]. They were constructed from Noether charges for enhanced symmetries in the zero coupling limit. A minimal set of charges can be characterised by using properties of Permutation Centralizer Algebras (PCAs) [49]. We expect similar applications of PCAs to gauge invariant operators in general quiver theories (without fundamental matter) to proceed in a fairly similar manner. For the case of quivers with fundamental matter, we may expect that appropriate PCAs along with modules over these algebras will play a role. There are in fact two ways one might associate a PCA to quiver with fundamentals. One is to excise the flavour legs of the quiver to be left with a quiver with bi-fundamentals only. Putting back the legs might correspond to going from algebra to a broader construction involving modules over the algebra. The other way is to tie all the incoming and outgoing legs to a single new node, preserving their orientation. This latter procedure was useful in consideration of the counting of gauge invariant operators [8].

Another interesting line of research would be to study the action of the one-loop dilatation operator on the basis of matrix invariants (3.19) for flavoured theories, possibly in some simple subsector. The action of the one-loop dilatation operator on the Schur basis for $\mathcal{N} = 4$ SYM has already been studied [50, 51]. For example, in the giant graviton sector of $\mathcal{N} = 4$ SYM, the explicit action of the one-loop dilatation operator corresponds to moving a single box in the

Young diagram that parametrises the giant graviton. It is an open problem to find analogous results in flavoured theories: an interesting starting point would be $\mathcal{N} = 2$ SQCD with gauge group $SU(N)$ and flavour symmetry $SU(2N)$, which is a conformal theory. An explicit basis for its matrix invariants is given in (5.2).

The broad summary of the results of the present paper and of a number of future directions is that the quiver, combined with associated permutation algebras and topological field theories, can be a powerful device in constructing correlators of gauge invariant observables and exposing hidden geometrical structures associated with these.

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A Operator Invariance

In this appendix we will derive the identity (2.15). Let us consider a matrix Φ in the bi-fundamental $(\square, \bar{\square})$ representation of $U(N_a) \times U(N_b)$, and a permutation $\eta \in S_n$. Eq. (2.15) arises from the equivalence

$$\eta^{-1}\left(\Phi^{\otimes n}\right)\eta = \Phi^{\otimes n} \quad \Rightarrow \quad \left[\Phi^{\otimes n}, \eta\right] = 0 \tag{A.1}$$

which follows from the identities

$$\begin{aligned}\langle e^{i_1}, e^{i_2}, \cdots, e^{i_n}|\Phi^{\otimes n}|e_{j_1}, e_{j_2}, \cdots, e_{j_n}\rangle &= (\Phi^{\otimes n})^{i_1, i_2, \dots i_n}_{j_1, j_2, \dots, j_n} = \Phi^{i_1}_{j_1}\Phi^{i_2}_{j_2}\cdots\Phi^{i_n}_{j_n} = \Phi^{i_{\eta(1)}}_{j_{\eta(1)}}\Phi^{i_{\eta(2)}}_{j_{\eta(2)}}\cdots\Phi^{i_{\eta(n)}}_{j_{\eta(n)}} \\ &= (\Phi^{\otimes n})^{i_{\eta(1)}, i_{\eta(2)}, \dots i_{\eta(n)}}_{j_{\eta(1)}, j_{\eta(2)}, \dots, j_{\eta(n)}} = \langle e^{i_{\eta(1)}}, e^{i_{\eta(2)}}, \cdots, e^{i_{\eta(n)}}|\Phi^{\otimes n}|e_{j_{\eta(1)}}, e_{j_{\eta(2)}}, \cdots, e_{j_{\eta(n)}}\rangle \\ &= \langle e^{i_1}, e^{i_2}, \cdots, e^{i_n}|\eta^{-1}\Phi^{\otimes n}\eta|e_{j_1}, e_{j_2}, \cdots, e_{j_n}\rangle, \qquad \eta \in S_n\end{aligned} \tag{A.2}$$

Here $|e_{j_1}, e_{j_2}, \cdots, e_{j_n}\rangle \in V_{N_a}^{\otimes n}$ and $\langle e^{j_1}, e^{j_2}, \cdots, e^{j_n}| \in \bar{V}_{N_b}^{\otimes n}$, V_{N_a} and $\bar{V}_{N_b}$ being the fundamental and antifundamental representations of $U(N_a)$ and $U(N_b)$ respectively. In the following, we will need the two identities

$$\begin{aligned}\left(Q^{\otimes n}\rho\right)^{I}_{\boldsymbol{s}} &= \langle e^{i_1}, e^{i_2}, \cdots, e^{i_n}|Q^{\otimes n}\rho|e_{s_1}, e_{s_2}, \cdots, e_{s_n}\rangle \\ &= \langle e^{i_1}, e^{i_2}, \cdots, e^{i_n}|Q^{\otimes n}|e_{s_{\rho(1)}}, e_{s_{\rho(2)}}, \cdots, e_{s_{\rho(n)}}\rangle = \left(Q^{\otimes n}\right)^{I}_{\rho(\boldsymbol{s})}\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}\left(\bar{\rho}^{-1}\bar{Q}^{\otimes n}\right)^{\bar{\boldsymbol{s}}}_{J} &= \langle e^{\bar{s}_1}, e^{\bar{s}_2}, \cdots, e^{\bar{s}_n}|\bar{\rho}^{-1}\bar{Q}^{\otimes n}|e_{j_1}, e_{j_2}, \cdots, e_{j_n}\rangle \\ &= \langle e^{\bar{s}_{\bar{\rho}(1)}}, e^{\bar{s}_{\bar{\rho}(2)}}, \cdots, e^{\bar{s}_{\bar{\rho}(n)}}|\bar{Q}^{\otimes n}|e_{j_1}, e_{j_2}, \cdots, e_{j_n}\rangle = \left(\bar{Q}^{\otimes n}\right)^{\bar{\rho}(\bar{\boldsymbol{s}})}_{J}\end{aligned} \tag{A.4}$$

Now let us consider a generic GIO $\mathcal{O}_{\mathcal{Q}}(\vec{n};\, \vec{s};\, \vec{\sigma})$, built with $n_{ab,\alpha}$ type $\Phi_{ab,\alpha}$ fields, $n_{a,\beta}$ type $Q_{a,\beta}$ fields and $\bar{n}_{a,\gamma}$ type $\bar{Q}_{a,\gamma}$ fields. We also introduce the permutations

$$\vec{\eta} = \cup_{a,b,\alpha}\{\eta_{ab,\alpha}\} \quad , \quad \eta_{ab,\alpha} \in S_{n_{ab,\alpha}} \tag{A.5a}$$

$$\vec{\rho} = \cup_{a}\{\cup_\beta\, \rho_{a,\beta};\, \cup_\gamma\, \bar{\rho}_{a,\gamma}\} \quad , \quad \rho_{a,\beta} \in S_{n_{a,\beta}} \quad , \quad \bar{\rho}_{a,\gamma} \in S_{\bar{n}_{a,\gamma}} \tag{A.5b}$$

From (A.1), we then have the equivalences

$$\eta^{-1}_{ab,\alpha}\left(\Phi^{\otimes n_{ab,\alpha}}_{ab,\alpha}\right)\eta_{ab,\alpha} = \Phi^{\otimes n_{ab,\alpha}}_{ab,\alpha} \;, \quad \rho^{-1}_{a,\beta}\left(Q^{\otimes n_{a,\beta}}_{a,\beta}\right)\rho_{a,\beta} = Q^{\otimes n_{a,\beta}}_{a,\beta} \;, \quad \bar{\rho}^{-1}_{a,\gamma}\left(\bar{Q}^{\otimes \bar{n}_{a,\gamma}}_{a,\gamma}\right)\bar{\rho}_{a,\gamma} = \bar{Q}^{\otimes \bar{n}_{a,\gamma}}_{a,\gamma} \tag{A.6}$$

for every a, b, α, β, γ. Inserting these identities in (2.12) gives

$$\begin{aligned}\mathcal{O}_{\mathcal{Q}}(\vec{n};\, \vec{s};\, \vec{\sigma}) = \prod_a &\left[\prod_{b,\alpha}\left(\eta^{-1}_{ab,\alpha}\left(\Phi^{\otimes n_{ab,\alpha}}_{ab,\alpha}\right)\eta_{ab,\alpha}\right)^{I_{ab,\alpha}}_{J_{ab,\alpha}}\right] \otimes \left[\prod_{\beta}\left(\rho^{-1}_{a,\beta}\left(Q^{\otimes n_{a,\beta}}_{a,\beta}\right)\rho_{a,\beta}\right)^{I_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}\right] \\ &\otimes \left[\prod_{\gamma}\left(\bar{\rho}^{-1}_{a,\gamma}\left(\bar{Q}^{\otimes \bar{n}_{a,\gamma}}_{a,\gamma}\right)\bar{\rho}_{a,\gamma}\right)^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{J}_{a,\gamma}}\right] (\sigma_a)^{\times_{b,\alpha}J_{ba,\alpha}\times_\gamma \bar{J}_{a,\gamma}}_{\times_{b,\alpha}I_{ab,\alpha}\times_\beta I_{a,\beta}} \\ = \prod_a &\left[\prod_{b,\alpha}\left(\Phi^{\otimes n_{ab,\alpha}}_{ab,\alpha}\right)^{K_{ab,\alpha}}_{L_{ab,\alpha}}\right] \otimes \left[\prod_{\beta}\left(Q^{\otimes n_{a,\beta}}_{a,\beta}\rho_{a,\beta}\right)^{K_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}\right] \otimes \left[\prod_{\gamma}\left(\bar{\rho}^{-1}_{a,\gamma}\bar{Q}^{\otimes \bar{n}_{a,\gamma}}_{a,\gamma}\right)^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{L}_{a,\gamma}}\right] \\ \times &\left[\prod_{b,\alpha}(\eta_{ab,\alpha})^{L_{ab,\alpha}}_{J_{ab,\alpha}}\right]\left[\prod_{\gamma}(\bar{\rho}_{a,\gamma})^{\bar{L}_{a,\gamma}}_{\bar{J}_{a,\gamma}}\right](\sigma_a)^{\times_{b,\alpha}J_{ba,\alpha}\times_\gamma \bar{J}_{a,\gamma}}_{\times_{b,\alpha}I_{ab,\alpha}\times_\beta I_{a,\beta}}\left[\prod_{b,\alpha}(\eta^{-1}_{ab,\alpha})^{I_{ab,\alpha}}_{K_{ab,\alpha}}\right]\left[\prod_{\beta}(\rho^{-1}_{a,\beta})^{I_{a,\beta}}_{K_{a,\beta}}\right]\end{aligned} \tag{A.7}$$

Now we use the equations (A.3) and (A.4) to obtain

$$
\begin{aligned}
\mathcal{O}_{\mathcal{Q}}(\vec{n};\,\vec{s};\,\vec{\sigma}) = \prod_a \left[\prod_{b,\alpha}\left(\Phi_{ab,\alpha}^{\otimes n_{ab,\alpha}}\right)_{L_{ab,\alpha}}^{K_{ab,\alpha}}\right] \otimes \left[\prod_{\beta}\left(Q_{a,\beta}^{\otimes n_{a,\beta}}\right)_{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta})}^{K_{a,\beta}}\right] \otimes \left[\prod_{\gamma}\left(\bar{Q}_{a,\gamma}^{\otimes \bar{n}_{a,\gamma}}\right)_{\bar{L}_{a,\gamma}}^{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}\right] \\
\times \left((\times_{b,\alpha}\eta_{ba,\alpha} \times_{\gamma} \bar{\rho}_{a,\gamma})\sigma_a(\times_{b,\alpha}\eta_{ab,\alpha}^{-1} \times_{\beta} \rho_{a,\beta}^{-1})\right)_{\times_{b,\alpha}K_{ab,\alpha}\times_{\beta}K_{a,\beta}}^{\times_{b,\alpha}L_{ba,\alpha}\times_{\gamma}\bar{L}_{a,\gamma}} \\
= \mathcal{O}_{\mathcal{Q}}(\vec{n};\,\vec{\rho}\,(\vec{s});\,\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})) \qquad \text{(A.8)}
\end{aligned}
$$

where we also used the definition of $\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})$, eq. (2.17):

$$
\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma}) = \cup_a\{(\times_{b,\alpha}\eta_{ba,\alpha} \times_{\gamma} \bar{\rho}_{a,\gamma})\sigma_a(\times_{b,\alpha}\eta_{ab,\alpha}^{-1} \times_{\beta} \rho_{a,\beta}^{-1})\} \qquad \text{(A.9)}
$$

We thus have explicitly shown the equivalence (2.15).

As it usually is the case when working in this framework, (2.15) has a pictorial interpretation. We now give an example of this diagrammatic interpretation, for the simple case of an $\mathcal{N}=2$ SQCD. The $\mathcal{N}=1$ quiver for this model is the one depicted in Fig. 5. Let us then consider an $\mathcal{N}=2$ SQCD GIO built with n adjoint fields ϕ and n_q quarks Q and antiquarks $\bar{Q}$. Each quark comes with a fixed state s_i state belonging to the fundamental representation of the flavour group $SU(F)$. We label the collection of these n_q states as $\boldsymbol{s} = (s_1, s_2, ..., s_{n_q})$. Similarly, $\bar{\boldsymbol{s}} = (\bar{s}_1, \bar{s}_2, ..., \bar{s}_{n_q})$ is the collection of the $SU(F)$ antifundamental states of the antiquarks $\bar{Q}$. The generic GIO $\mathcal{O}_{\mathcal{Q}}(n, n_q;\,\boldsymbol{s}, \bar{\boldsymbol{s}};\,\sigma)$ can be drawn as in Fig. 22.

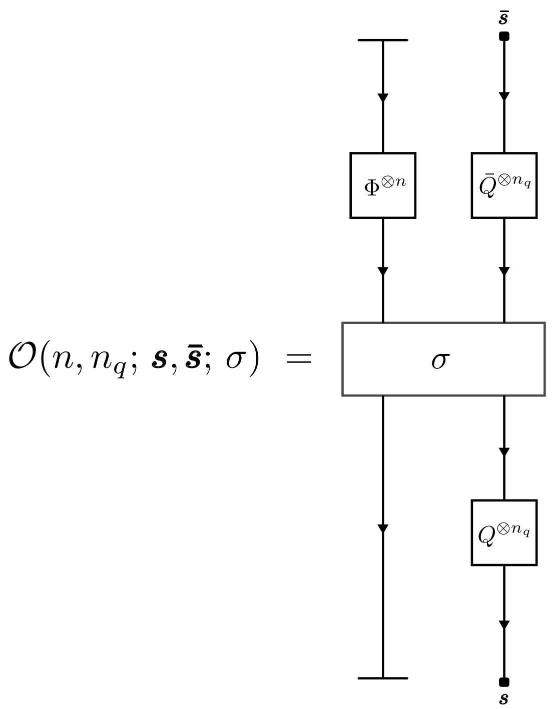

Figure 22: Diagram corresponding to a generic $\mathcal{N}=2$ SQCD GIO.

The horizontal bars denotes the identification of the indices. Specialising eq. (2.15) to this case, we have the identity

$$\mathcal{O}(n, n_q;\, \boldsymbol{s}, \bar{\boldsymbol{s}};\, \sigma) = \mathcal{O}(n, n_q;\, \rho(\boldsymbol{s}), \bar{\rho}(\bar{\boldsymbol{s}});\, \mathrm{Adj}_{\eta\times\rho}(\sigma)) \tag{A.10}$$

for $\sigma \in S_{n+n_q}$, $\eta \in S_n$ and $\rho\ \bar{\rho} \in S_{n_q}$. This equivalence is described in diagrammatic terms in Fig. 23.

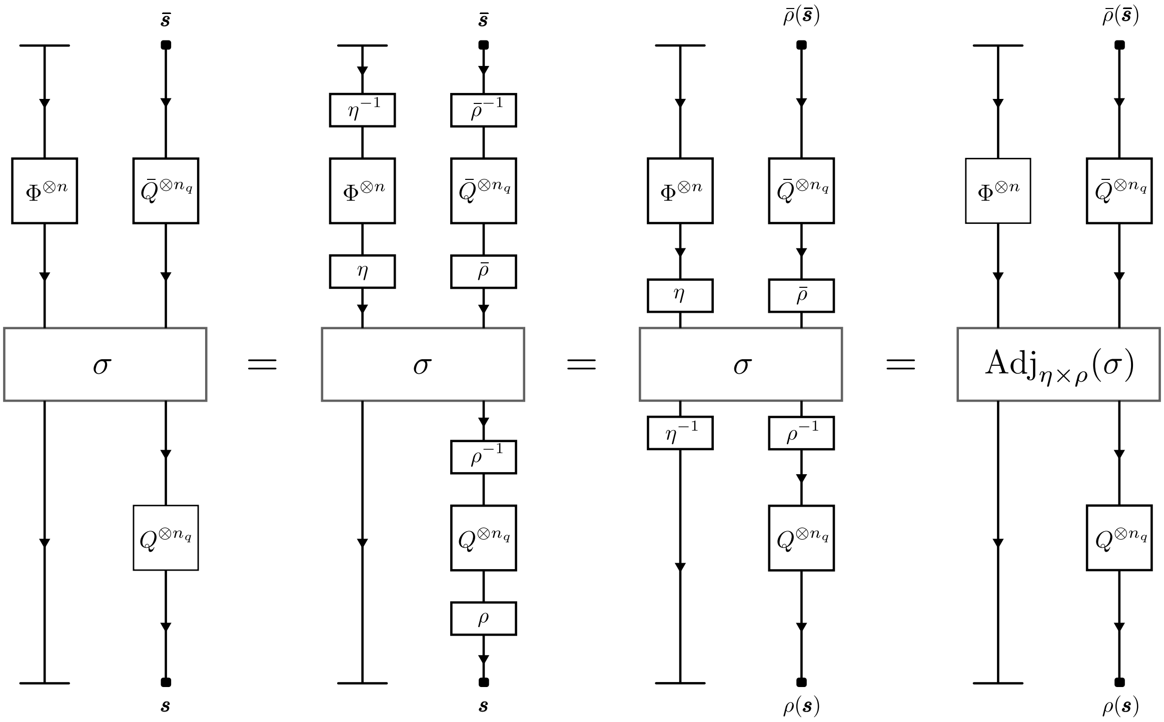

Figure 23: Diagrammatic interpretation of the identity (A.10).

B Quiver Character Identities

In this appendix we will derive equations (3.23), (3.25) and (3.26). Many of the symmetric group identities that we will use in this appendix were already introduced and discussed in Appendix A of [25].

B.1 Invariance Relation

In this section we will prove formula (3.23):

$$\chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma}) = \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{\rho}\,(\vec{s}), \mathrm{Adj}_{\vec{\rho}\times\vec{\eta}}(\vec{\sigma})) \tag{B.1}$$

Using the definition of $\mathrm{Adj}_{\vec{\rho}\times\vec{\eta}}(\vec{\sigma})$ given in (2.17)

$$\mathrm{Adj}_{\vec{\rho}\times\vec{\eta}}(\vec{\sigma}) = \cup_a\{(\times_{b,\alpha}\eta_{ba,\alpha}\times_\gamma\bar{\rho}_{a,\gamma})\sigma_a(\times_{b,\alpha}\eta^{-1}_{ab,\alpha}\times_\beta\rho^{-1}_{a,\beta})\} \tag{B.2}$$

we start by writing

$$\begin{aligned}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\mathrm{Adj}_{\vec{\rho}\times\vec{\eta}}(\vec{\sigma})) = c_{\boldsymbol{L}}\prod_a\sum_{i_a,j_a}\sum_{\substack{l_{ab,\alpha}\\ l_{a,\beta},\bar{l}_{a,\gamma}}} & D^{R_a}_{i_a,j_a}((\times_{b,\alpha}\eta_{ba,\alpha}\times_\gamma\bar{\rho}_{a,\gamma})\sigma_a(\times_{b,\alpha}\eta^{-1}_{ab,\alpha}\times_\beta\rho^{-1}_{a,\beta}))\\ &\times B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{j_a\to\cup_{b,\alpha}l_{ab,\alpha}\cup_\beta l_{a,\beta}}\left(\prod_\beta C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}\right)B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu_a^+}_{i_a\to\cup_{b,\alpha}l_{ba,\alpha}\cup_\gamma\bar{l}_{a,\gamma}}\left(\prod_\gamma C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}}\right)\\ =c_{\boldsymbol{L}}\prod_a\sum_{i_a,j_a}\sum_{\substack{l_{ab,\alpha}\\ l_{a,\beta},\bar{l}_{a,\gamma}}} & D^{R_a}_{i_a,i'_a}(\times_{b,\alpha}\eta_{ba,\alpha}\times_\gamma\bar{\rho}_{a,\gamma})D^{R_a}_{i'_a,j'_a}(\sigma_a)D^{R_a}_{j'_a,j_a}(\times_{b,\alpha}\eta^{-1}_{ab,\alpha}\times_\beta\rho^{-1}_{a,\beta})\\ &\times B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{j_a\to\cup_{b,\alpha}l_{ab,\alpha}\cup_\beta l_{a,\beta}}\left(\prod_\beta C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}\right)B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu_a^+}_{i_a\to\cup_{b,\alpha}l_{ba,\alpha}\cup_\gamma\bar{l}_{a,\gamma}}\left(\prod_\gamma C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}}\right)\end{aligned} \tag{B.3}$$

To ease the notation, for the remainder of this section we will drop the summation symbol in our equations. The sum over repeated symmetric group state indices will therefore be implicit. Notice however that there is no summation over the repeated representation labels $r_{ab,\alpha}$, $r_{a,\beta}$, $\bar{r}_{a,\gamma}$. Using the equivariance property of the branching coefficients [39]

$$D^R_{k,j}(\times_a\gamma_a)\,B^{R\to\cup_a r_a;\nu_a}_{j\to\cup_a l_a} = \left(\prod_a D^{r_a}_{l'_a,l_a}(\gamma_a)\right)B^{R\to\cup_a r_a;\nu_a}_{k\to\cup_a l'_a} \tag{B.4}$$

we can write

$$\begin{aligned}&D^{R_a}_{j'_a,j_a}(\times_{b,\alpha}\eta^{-1}_{ab,\alpha}\times_\beta\rho^{-1}_{a,\beta})\,B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{j_a\to\cup_{b,\alpha}l_{ab,\alpha}\cup_\beta l_{a,\beta}}\\ &\qquad=\left(\prod_{b,\alpha}D^{r_{ab,\alpha}}_{l'_{ab,\alpha},l_{ab,\alpha}}(\eta^{-1}_{ab,\alpha})\prod_\beta D^{r_{a,\beta}}_{l'_{a,\beta},l_{a,\beta}}(\rho^{-1}_{a,\beta})\right)B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta},\nu_a^-}_{j'_a\to\cup_{b,\alpha}l'_{ab,\alpha}\cup_\beta l'_{a,\beta}}\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}&D^{R_a}_{i_a,i'_a}(\times_{b,\alpha}\eta_{ba,\alpha}\times_\gamma\bar{\rho}_{a,\gamma})\,B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu_a^+}_{i_a\to\cup_{b,\alpha}l_{ba,\alpha}\cup_\gamma\bar{l}_{a,\gamma}}\\ &\qquad=\left(\prod_{b,\alpha}D^{r_{ba,\alpha}}_{l_{ba,\alpha},l_{ba,\alpha}''}(\eta_{ba,\alpha})\prod_\gamma D^{\bar{r}_{a,\gamma}}_{\bar{l}_{a,\gamma},\bar{l}_{a,\gamma}''}(\bar{\rho}_{a,\gamma})\right)B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu_a^+}_{i'_a\to\cup_{b,\alpha}l_{ba,\alpha}''\cup_\gamma l_{a,\gamma}''}\end{aligned} \tag{B.6}$$

Inserting the last two equations in (B.3) gives

$$
\begin{aligned}
\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\mathrm{Adj}_{\vec{\rho}\times\vec{\eta}}(\vec{\sigma})) =& c_{\boldsymbol{L}} \prod_a D^{R_a}_{i'_a,j'_a}(\sigma_a) \left\{ \prod_{b,\alpha} D^{r_{ab,\alpha}}_{l'_{ab,\alpha},l_{ab,\alpha}}(\eta^{-1}_{ab,\alpha}) D^{r_{ba,\alpha}}_{l_{ba,\alpha},l_{ba,\alpha}{}''}(\eta_{ba,\alpha}) \right\} \\
& \times B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta},\nu_a^-}_{j'_a\to\cup_{b,\alpha}l'_{ab,\alpha}\cup_\beta l'_{a,\beta}} \left\{ \prod_\beta D^{r_{a,\beta}}_{l_{a,\beta},l'_{a,\beta}}(\rho_{a,\beta})\, C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} \right\} \\
& \times B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu_a^+}_{i'_a\to\cup_{b,\alpha}l_{ba,\alpha}{}''\cup_\gamma l_{a,\gamma}{}''} \left\{ \prod_\gamma D^{\bar{r}_{a,\gamma}}_{\bar{l}_{a,\gamma},\bar{l}_{a,\gamma}{}''}(\bar{\rho}_{a,\gamma})\, C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}} \right\}
\end{aligned} \tag{B.7}
$$

A first simplification comes from noticing that

$$
\prod_{a,b,\alpha} D^{r_{ab,\alpha}}_{l'_{ab,\alpha},l_{ab,\alpha}}(\eta^{-1}_{ab,\alpha}) D^{r_{ba,\alpha}}_{l_{ba,\alpha},l_{ba,\alpha}{}''}(\eta_{ba,\alpha}) = \prod_{a,b,\alpha} \delta_{l'_{ab,\alpha},l_{ab,\alpha}{}''} \tag{B.8}
$$

We now focus on the Clebsch-Gordan coefficients. Let us first consider the chain of equalities

$$
\begin{aligned}
D^R_{i,i'}(\sigma) C^{R,M,i}_{\boldsymbol{s}} &= D^R_{i,i'}(\sigma)\langle \boldsymbol{s}|R,M,i\rangle = \langle \boldsymbol{s}|D(\sigma)|R,M,i'\rangle \\
&= \langle D(\sigma)^{-1}\boldsymbol{s}|R,M,i'\rangle = \langle \sigma^{-1}(\boldsymbol{s})|R,M,i'\rangle = C^{R,M,i'}_{\sigma^{-1}(\boldsymbol{s})}
\end{aligned} \tag{B.9}
$$

We can use this identity to write

$$
D^{r_{a,\beta}}_{l_{a,\beta},l'_{a,\beta}}(\rho_{a,\beta})\, C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} = C^{r_{a,\beta},S_{a,\beta},l'_{a,\beta}}_{\rho^{-1}_{a,\beta}(\boldsymbol{s}_{a,\beta})} \tag{B.10}
$$

and

$$
D^{\bar{r}_{a,\gamma}}_{\bar{l}_{a,\gamma},\bar{l}_{a,\gamma}{}''}(\bar{\rho}_{a,\gamma})\, C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}} = C^{\bar{\rho}^{-1}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}{}''} \tag{B.11}
$$

Using these results in (B.7) we then get

$$
\begin{aligned}
\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\mathrm{Adj}_{\vec{\rho}\times\vec{\eta}}(\vec{\sigma})) =& c_{\boldsymbol{L}} \prod_a D^{R_a}_{i'_a,j'_a}(\sigma_a) \left(\prod_{b,\alpha} \delta_{l'_{ab,\alpha},l_{ab,\alpha}{}''} \right) \\
& \times B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta},\nu_a^-}_{j'_a\to\cup_{b,\alpha}l'_{ab,\alpha}\cup_\beta l'_{a,\beta}} \left\{ \prod_\beta C^{r_{a,\beta},S_{a,\beta},l'_{a,\beta}}_{\rho^{-1}_{a,\beta}(\boldsymbol{s}_{a,\beta})} \right\} \\
& \times B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu_a^+}_{i'_a\to\cup_{b,\alpha}l_{ba,\alpha}{}''\cup_\gamma l_{a,\gamma}{}''} \left\{ \prod_\gamma C^{\bar{\rho}^{-1}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}{}''} \right\}
\end{aligned} \tag{B.12}
$$

$$
\begin{aligned}
=& c_{\boldsymbol{L}} \prod_a D^{R_a}_{i'_a, j'_a}(\sigma_a)\, B^{R_a \to \cup_{b,\alpha} r_{ab,\alpha} \cup_\beta r_{a,\beta}, \nu_a^-}_{j'_a \to \cup_{b,\alpha} l'_{ab,\alpha} \cup_\beta l'_{a,\beta}} \prod_\beta C^{r_{a,\beta}, S_{a,\beta}, l'_{a,\beta}}_{\rho^{-1}_{a,\beta}(\boldsymbol{s}_{a,\beta})} \\
& \times B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}; \nu_a^+}_{i'_a \to \cup_{b,\alpha} l'_{ba,\alpha} \cup_\gamma l_{a,\gamma}''} \prod_\gamma C^{\bar{\rho}^{-1}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar{r}_{a,\gamma}, \bar{S}_{a,\gamma}, \bar{l}_{a,\gamma}''} = \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{\rho}^{\,-1}(\vec{s}), \vec{\sigma})
\end{aligned}
$$

Substituting $\vec{s} \to \vec{\rho}\,(\vec{s})$, we finally get

$$
\chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma}) = \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{\rho}\,(\vec{s}), \mathrm{Adj}_{\vec{\rho} \times \vec{\eta}}(\vec{\sigma})) \tag{B.13}
$$

Our proposition is thus proven.

B.2 Orthogonality Relations

In this section we will prove the quiver character orthogonality equations (3.25) and (3.26).

B.2.1 Orthogonality in $\boldsymbol{L}$

Let us start with eq. (3.25):

$$
\sum_{\vec{s}} \sum_{\vec{\sigma}} \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma})\, \chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}}, \vec{s}, \vec{\sigma}) = \delta_{\boldsymbol{L}, \tilde{\boldsymbol{L}}} \tag{B.14}
$$

This formula is actually a particular case of the more general identity

$$
\begin{aligned}
& \sum_{\vec{s}} \sum_{\vec{\sigma}} \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma'} \cdot \vec{\sigma})\, \chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}}, \vec{s}, \vec{\sigma}) \qquad \text{(B.15)} \\
& \quad = c_{\boldsymbol{L}}\, c_{\tilde{\boldsymbol{L}}} \prod_a \frac{n_a!}{d(R_a)} \operatorname{Tr}\left(D^{R_a}(\sigma'_a) P^{\nu_a^+, \tilde{\nu}_a^+}_{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}} \right) \delta_{R_a, \tilde{R}_a} \\
& \quad \times \left(\prod_{b,\alpha} \delta_{r_{ab,\alpha}, \tilde{r}_{ab,\alpha}} \right) \left(\prod_\beta d(r_{a,\beta}) \delta_{r_{a,\beta}, \tilde{r}_{a,\beta}} \delta_{S_{a,\beta}, \tilde{S}_{a,\beta}} \right) \left(\prod_\gamma \delta_{\bar{r}_{a,\gamma}, \tilde{\bar{r}}_{a,\gamma}} \delta_{\bar{S}_{a,\gamma}, \tilde{\bar{S}}_{a,\gamma}} \right) \delta_{\nu_a^-, \tilde{\nu}_a^-}
\end{aligned}
$$

Here $P^{\nu_a^+, \tilde{\nu}_a^+}_{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}}$ is a linear operator whose matrix elements are

$$
P^{\nu_a^+, \tilde{\nu}_a^+}_{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}} \Big|_{\tilde{i}_a, i_a} = B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}; \tilde{\nu}_a^+}_{\tilde{i}_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar{l}_{a,\gamma}} B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}; \nu_a^+}_{i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar{l}_{a,\gamma}} \tag{B.16}
$$

Let us prove eq. (B.15). As a first step we expanding its LHS to get

$$\begin{aligned}
&\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma'}\cdot\vec{\sigma})\,\chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}},\vec{s},\vec{\sigma})\\
&\qquad= c_{\boldsymbol{L}}\,c_{\tilde{\boldsymbol{L}}}\sum_{\vec{s}}\sum_{\vec{\sigma}}\prod_{a} D^{R_a}_{i_a,j_a}(\sigma'_a\cdot\sigma_a)D^{\tilde{R}_a}_{\tilde{i}_a,\tilde{j}_a}(\sigma_a)\\
&\qquad\quad\times B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{j_a\to\cup_{b,\alpha}l_{ab,\alpha}\cup_\beta l_{a,\beta}}\,B^{\tilde{R}_a\to\cup_{b,\alpha}\tilde{r}_{ab,\alpha}\cup_\beta\tilde{r}_{a,\beta};\tilde{\nu}_a^-}_{\tilde{j}_a\to\cup_{b,\alpha}\tilde{l}_{ab,\alpha}\cup_\beta\tilde{l}_{a,\beta}}\prod_\beta C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}C^{\tilde{r}_{a,\beta},\tilde{S}_{a,\beta},\tilde{l}_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}\\
&\qquad\quad\times B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu_a^+}_{i_a\to\cup_{b,\alpha}l_{ba,\alpha}\cup_\gamma\bar{l}_{a,\gamma}}\,B^{\tilde{R}_a\to\cup_{b,\alpha}\tilde{r}_{ba,\alpha}\cup_\gamma\tilde{\bar{r}}_{a,\gamma};\tilde{\nu}_a^+}_{\tilde{i}_a\to\cup_{b,\alpha}\tilde{l}_{ba,\alpha}\cup_\gamma\tilde{\bar{l}}_{a,\gamma}}\prod_\gamma C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}}C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\tilde{\bar{r}}_{a,\gamma},\tilde{\bar{S}}_{a,\gamma},\tilde{\bar{l}}_{a,\gamma}}
\end{aligned}\tag{B.17}$$

The next step is to rewrite the known relation

$$\sum_{\sigma\in S_n} D^R_{i,j}(\sigma)D^{R'}_{p,q}(\sigma)=\frac{n!}{d(R)}\delta_{R,R'}\delta_{i,p}\delta_{j,q}\tag{B.18}$$

into the form

$$\begin{aligned}
\sum_{\sigma_a} D^{R_a}_{i_a,j_a}(\sigma'_a\cdot\sigma_a)D^{\tilde{R}_a}_{\tilde{i}_a,\tilde{j}_a}(\sigma_a) &= \sum_{k_a} D^{R_a}_{i_a,k_a}(\sigma'_a)\frac{n_a!}{d(R_a)}\delta_{R_a,\tilde{R}_a}\delta_{k_a,\tilde{i}_a}\delta_{j_a,\tilde{j}_a}\\
&= D^{R_a}_{i_a,\tilde{i}_a}(\sigma'_a)\frac{n_a!}{d(R_a)}\delta_{R_a,\tilde{R}_a}\delta_{j_a,\tilde{j}_a}
\end{aligned}\tag{B.19}$$

This identity can be inserted into eq. (B.17) to get

$$\begin{aligned}
&\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma'}\cdot\vec{\sigma})\,\chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}},\vec{s},\vec{\sigma})\\
&\quad= c_{\boldsymbol{L}}\,c_{\tilde{\boldsymbol{L}}}\sum_{\vec{s}}\prod_{a}\frac{n_a!}{d(R_a)}\delta_{R_a,\tilde{R}_a}\,D^{R_a}_{i_a,\tilde{i}_a}(\sigma'_a)\\
&\quad\quad\times\delta_{j_a,\tilde{j}_a}B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{j_a\to\cup_{b,\alpha}l_{ab,\alpha}\cup_\beta l_{a,\beta}}\,B^{R_a\to\cup_{b,\alpha}\tilde{r}_{ab,\alpha}\cup_\beta\tilde{r}_{a,\beta};\tilde{\nu}_a^-}_{\tilde{j}_a\to\cup_{b,\alpha}\tilde{l}_{ab,\alpha}\cup_\beta\tilde{l}_{a,\beta}}\prod_\beta C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}C^{\tilde{r}_{a,\beta},\tilde{S}_{a,\beta},\tilde{l}_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}\\
&\quad\quad\times B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu_a^+}_{i_a\to\cup_{b,\alpha}l_{ba,\alpha}\cup_\gamma\bar{l}_{a,\gamma}}\,B^{R_a\to\cup_{b,\alpha}\tilde{r}_{ba,\alpha}\cup_\gamma\tilde{\bar{r}}_{a,\gamma};\tilde{\nu}_a^+}_{\tilde{i}_a\to\cup_{b,\alpha}\tilde{l}_{ba,\alpha}\cup_\gamma\tilde{\bar{l}}_{a,\gamma}}\prod_\gamma C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}}C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\tilde{\bar{r}}_{a,\gamma},\tilde{\bar{S}}_{a,\gamma},\tilde{\bar{l}}_{a,\gamma}}
\end{aligned}\tag{B.20}$$

Now using the orthogonality relation (3.7) in eq. (B.20), we further obtain

$$\begin{aligned}
&\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma'}\cdot\vec{\sigma})\,\chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}},\vec{s},\vec{\sigma})\\
&\qquad= c_{\boldsymbol{L}}\,c_{\tilde{\boldsymbol{L}}}\sum_{\vec{s}}\prod_{a}\frac{n_a!}{d(R_a)}\delta_{R_a,\tilde{R}_a}\left(\prod_{b,\alpha}\delta_{r_{ab,\alpha},\tilde{r}_{ab,\alpha}}\delta_{l_{ab,\alpha},\tilde{l}_{ab,\alpha}}\right)\delta_{\nu_a^-,\tilde{\nu}_a^-}\,D^{R_a}_{i_a,\tilde{i}_a}(\sigma'_a)
\end{aligned}$$

$$
\times \left(\prod_\beta \delta_{r_{a,\beta},\tilde r_{a,\beta}} \delta_{l_{a,\beta},\tilde l_{a,\beta}} C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} C^{\tilde r_{a,\beta},\tilde S_{a,\beta},\tilde l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} \right)
$$

$$
\times B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar r_{a,\gamma};\nu_a^+}_{i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar l_{a,\gamma}} B^{R_a \to \cup_{b,\alpha} \tilde r_{ba,\alpha} \cup_\gamma \tilde{\bar r}_{a,\gamma};\tilde\nu_a^+}_{\tilde i_a \to \cup_{b,\alpha} \tilde l_{ba,\alpha} \cup_\gamma \tilde{\bar l}_{a,\gamma}} \prod_\gamma C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar r_{a,\gamma},\bar S_{a,\gamma},\bar l_{a,\gamma}} C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\tilde{\bar r}_{a,\gamma},\tilde{\bar S}_{a,\gamma},\tilde{\bar l}_{a,\gamma}}
$$

$$
= c_{\boldsymbol{L}}\, c_{\tilde{\boldsymbol{L}}} \sum_{\vec s} \prod_a \frac{n_a!}{d(R_a)} \delta_{R_a,\tilde R_a} \left(\prod_{b,\alpha} \delta_{r_{ab,\alpha},\tilde r_{ab,\alpha}} \right) \left(\prod_\beta \delta_{r_{a,\beta},\tilde r_{a,\beta}} \right) \delta_{\nu_a^-,\tilde\nu_a^-}\, D^{R_a}_{i_a,\tilde i_a}(\sigma'_a)
$$

$$
\times \prod_\beta C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} C^{r_{a,\beta},\tilde S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}
$$

$$
\times B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar r_{a,\gamma};\nu_a^+}_{i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar l_{a,\gamma}} B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \tilde{\bar r}_{a,\gamma};\tilde\nu_a^+}_{\tilde i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \tilde{\bar l}_{a,\gamma}} \prod_\gamma C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar r_{a,\gamma},\bar S_{a,\gamma},\bar l_{a,\gamma}} C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\tilde{\bar r}_{a,\gamma},\tilde{\bar S}_{a,\gamma},\tilde{\bar l}_{a,\gamma}} \tag{B.21}
$$

Let us focus on the pair of Clebsch-Gordan coefficients $C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} C^{r_{a,\beta},\tilde S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}$ in this formula. It is immediate to verify that, for a $U(F)$ Clebsch-Gordan coefficient $C^{r,S,i}_{\boldsymbol{s}}$

$$
\sum_{\boldsymbol{s}} C^{r,S,i}_{\boldsymbol{s}}\, C^{r',S',i'}_{\boldsymbol{s}} = \sum_{\boldsymbol{s}} \langle r,S,i|\boldsymbol{s}\rangle\ \langle r',S',i'|\boldsymbol{s}\rangle = \langle r,S,i| \left(\sum_{\boldsymbol{s}} |\boldsymbol{s}\rangle\, \langle \boldsymbol{s}| \right) |r',S',i'\rangle
$$

$$
= \langle r,S,i|1|r',S',i'\rangle = \delta_{r,r'}\, \delta_{S,S'}\, \delta_{i,i'} \tag{B.22}
$$

Therefore we can write

$$
\sum_{l_{a,\beta}} \sum_{\boldsymbol{s}_{a,\beta}} C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} C^{r_{a,\beta},\tilde S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} = \sum_{l_{a,\beta}} \delta_{S_{a,\beta},\tilde S_{a,\beta}} = d(r_{a,\beta})\, \delta_{S_{a,\beta},\tilde S_{a,\beta}} \tag{B.23}
$$

Inserting this in (B.21) we obtain

$$
\sum_{\vec s} \sum_{\vec\sigma} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec s,\vec{\sigma'}\cdot\vec\sigma)\, \chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}},\vec s,\vec\sigma)
$$

$$
= c_{\boldsymbol{L}}\, c_{\tilde{\boldsymbol{L}}} \prod_a \frac{n_a!}{d(R_a)} \delta_{R_a,\tilde R_a} \left(\prod_{b,\alpha} \delta_{r_{ab,\alpha},\tilde r_{ab,\alpha}} \right) \left(\prod_\beta d(r_{a,\beta}) \delta_{r_{a,\beta},\tilde r_{a,\beta}} \delta_{S_{a,\beta},\tilde S_{a,\beta}} \right) \delta_{\nu_a^-,\tilde\nu_a^-}\, D^{R_a}_{i_a,\tilde i_a}(\sigma'_a)
$$

$$
\times B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar r_{a,\gamma};\nu_a^+}_{i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar l_{a,\gamma}} B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \tilde{\bar r}_{a,\gamma};\tilde\nu_a^+}_{\tilde i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \tilde{\bar l}_{a,\gamma}} \prod_\gamma \sum_{\bar{\boldsymbol{s}}_{a,\gamma}} C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar r_{a,\gamma},\bar S_{a,\gamma},\bar l_{a,\gamma}} C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\tilde{\bar r}_{a,\gamma},\tilde{\bar S}_{a,\gamma},\tilde{\bar l}_{a,\gamma}}
$$

$$
= c_{\boldsymbol{L}}\, c_{\tilde{\boldsymbol{L}}} \prod_a \frac{n_a!}{d(R_a)} \delta_{R_a,\tilde R_a} \left(\prod_{b,\alpha} \delta_{r_{ab,\alpha},\tilde r_{ab,\alpha}} \right) \left(\prod_\beta d(r_{a,\beta}) \delta_{r_{a,\beta},\tilde r_{a,\beta}} \delta_{S_{a,\beta},\tilde S_{a,\beta}} \right) \delta_{\nu_a^-,\tilde\nu_a^-}\, D^{R_a}_{i_a,\tilde i_a}(\sigma'_a)
$$

$$
\times B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar r_{a,\gamma};\nu_a^+}_{i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar l_{a,\gamma}} B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \tilde{\bar r}_{a,\gamma};\tilde\nu_a^+}_{\tilde i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \tilde{\bar l}_{a,\gamma}} \left(\prod_\gamma \delta_{\bar r_{a,\gamma},\tilde{\bar r}_{a,\gamma}} \delta_{\bar S_{a,\gamma},\tilde{\bar S}_{a,\gamma}} \delta_{\bar l_{a,\gamma},\tilde{\bar l}_{a,\gamma}} \right)
$$

$$
\begin{aligned}
&= c_{\boldsymbol{L}}\, c_{\tilde{\boldsymbol{L}}} \prod_a \frac{n_a!}{d(R_a)} \delta_{R_a,\tilde{R}_a} \left(\prod_{b,\alpha} \delta_{r_{ab,\alpha},\tilde{r}_{ab,\alpha}}\right) \left(\prod_\beta d(r_{a,\beta}) \delta_{r_{a,\beta},\tilde{r}_{a,\beta}} \delta_{S_{a,\beta},\tilde{S}_{a,\beta}}\right) \\
&\quad \times \left(\prod_\gamma \delta_{\bar{r}_{a,\gamma},\tilde{\bar{r}}_{a,\gamma}} \delta_{\bar{S}_{a,\gamma},\tilde{\bar{S}}_{a,\gamma}}\right) \delta_{\nu_a^-,\tilde{\nu}_a^-}\; D^{R_a}_{i_a,\tilde{i}_a}(\sigma'_a) B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma};\nu_a^+}_{i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar{l}_{a,\gamma}} B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma};\tilde{\nu}_a^+}_{\tilde{i}_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar{l}_{a,\gamma}}
\end{aligned}
\tag{B.24}
$$

In the second equality above we again used (B.22):

$$
\sum_{\bar{\boldsymbol{s}}_{a,\gamma}} C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}} C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\tilde{\bar{r}}_{a,\gamma},\tilde{\bar{S}}_{a,\gamma},\tilde{\bar{l}}_{a,\gamma}} = \delta_{\bar{r}_{a,\gamma},\tilde{\bar{r}}_{a,\gamma}} \delta_{\bar{S}_{a,\gamma},\tilde{\bar{S}}_{a,\gamma}} \delta_{\bar{l}_{a,\gamma},\tilde{\bar{l}}_{a,\gamma}} \tag{B.25}
$$

We now define the projector-like operator $P^{\nu_a^+,\tilde{\nu}_a^+}_{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}}$, whose matrix elements are

$$
P^{\nu_a^+,\tilde{\nu}_a^+}_{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}}\Big|_{\tilde{i}_a,i_a} = B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma};\tilde{\nu}_a^+}_{\tilde{i}_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar{l}_{a,\gamma}} B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma};\nu_a^+}_{i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar{l}_{a,\gamma}} \tag{B.26}
$$

For $\nu_a^+ = \tilde{\nu}_a^+$ the operator $P^{\nu_a^+,\tilde{\nu}_a^+}_{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}}$ is the projector on the $(\cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}, \nu_a^+)$ subspace of R_a, but when $\nu_a^+ \neq \tilde{\nu}_a^+$ it is rather an intertwining operator mapping the copies ν_a^+ and $\tilde{\nu}_a^+$ of the same subspace $\cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma} \subset R_a$ one to another. With this definition, we can finally write

$$
\begin{aligned}
&\sum_{\vec{s}} \sum_{\vec{\sigma}} \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma'} \cdot \vec{\sigma})\, \chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}}, \vec{s}, \vec{\sigma}) \\
&\qquad = c_{\boldsymbol{L}}\, c_{\tilde{\boldsymbol{L}}} \prod_a \frac{n_a!}{d(R_a)} \operatorname{Tr}\left(D^{R_a}(\sigma'_a) P^{\nu_a^+,\tilde{\nu}_a^+}_{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}}\right) \delta_{R_a,\tilde{R}_a} \\
&\qquad\quad \times \left(\prod_{b,\alpha} \delta_{r_{ab,\alpha},\tilde{r}_{ab,\alpha}}\right) \left(\prod_\beta d(r_{a,\beta}) \delta_{r_{a,\beta},\tilde{r}_{a,\beta}} \delta_{S_{a,\beta},\tilde{S}_{a,\beta}}\right) \left(\prod_\gamma \delta_{\bar{r}_{a,\gamma},\tilde{\bar{r}}_{a,\gamma}} \delta_{\bar{S}_{a,\gamma},\tilde{\bar{S}}_{a,\gamma}}\right) \delta_{\nu_a^-,\tilde{\nu}_a^-}
\end{aligned}
\tag{B.27}
$$

which is eq. (B.15). Consider now the case in which $\vec{\sigma}' = \vec{1}$. Then

$$
\begin{aligned}
&\operatorname{Tr}\left(D^{R_a}(1) P^{\nu_a^+,\tilde{\nu}_a^+}_{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}}\right) = \operatorname{Tr}\left(P^{\nu_a^+,\tilde{\nu}_a^+}_{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}}\right) \\
&\qquad = \sum_{\substack{l_{ba,\alpha} \\ \bar{l}_{a,\gamma}}} \left(\sum_{i_a} B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma};\tilde{\nu}_a^+}_{i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar{l}_{a,\gamma}} B^{R_a \to \cup_{b,\alpha} r_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma};\nu_a^+}_{i_a \to \cup_{b,\alpha} l_{ba,\alpha} \cup_\gamma \bar{l}_{a,\gamma}}\right) \\
&\qquad = \delta_{\nu_a^+,\tilde{\nu}_a^+} \sum_{\substack{l_{ba,\alpha} \\ \bar{l}_{a,\gamma}}} \left(\prod_{b,\alpha} \delta_{l_{ba,\alpha},l_{ba,\alpha}}\right) \left(\prod_\gamma \delta_{\bar{l}_{a,\gamma},\bar{l}_{a,\gamma}}\right) = \delta_{\nu_a^+,\tilde{\nu}_a^+} \left(\prod_{b,\alpha} d(r_{ba,\alpha})\right) \left(\prod_\gamma d(\bar{r}_{a,\gamma})\right)
\end{aligned}
\tag{B.28}
$$

where the third equality follows from the orthogonality relation (3.7). Using this identity in (B.15) we get

$$\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}},\vec{s},\vec{\sigma})$$

$$= c_{\boldsymbol{L}}\, c_{\tilde{\boldsymbol{L}}} \prod_a \frac{n_a!}{d(R_a)}\,\delta_{R_a,\tilde{R}_a}\delta_{\nu_a^-,\tilde{\nu}_a^-}\delta_{\nu_a^+,\tilde{\nu}_a^+}\left(\prod_{b,\alpha} d(r_{ab,\alpha})\delta_{r_{ab,\alpha},\tilde{r}_{ab,\alpha}}\right)$$

$$\times\left(\prod_\beta d(r_{a,\beta})\delta_{r_{a,\beta},\tilde{r}_{a,\beta}}\delta_{S_{a,\beta},\tilde{S}_{a,\beta}}\right)\left(\prod_\gamma d(\bar{r}_{a,\gamma})\delta_{\bar{r}_{a,\gamma},\tilde{\bar{r}}_{a,\gamma}}\delta_{\bar{S}_{a,\gamma},\tilde{\bar{S}}_{a,\gamma}}\right) \tag{B.29}$$

Recalling the definition of the set of labels $\boldsymbol{L} = \{R_a, r_{ab,\alpha}, r_{a,\beta}, S_{a,\beta}, \bar{r}_{a,\gamma}, \bar{S}_{a,\gamma}, \nu_a^+, \nu_a^-\}$, we can thus write

$$\sum_{\vec{s}}\sum_{\vec{\sigma}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\tilde{\boldsymbol{L}},\vec{s},\vec{\sigma})$$

$$= \delta_{\boldsymbol{L},\tilde{\boldsymbol{L}}}\, c_{\boldsymbol{L}}\, c_{\tilde{\boldsymbol{L}}} \prod_a \frac{n_a!}{d(R_a)}\left(\prod_{b,\alpha} d(r_{ab,\alpha})\right)\left(\prod_\beta d(r_{a,\beta})\right)\left(\prod_\gamma d(\bar{r}_{a,\gamma})\right) = \delta_{\boldsymbol{L},\tilde{\boldsymbol{L}}} \tag{B.30}$$

The identity (B.14) is proven.

B.2.2 Orthogonality in $\vec{s}$, $\vec{\sigma}$

In this section we are going to prove (3.26):

$$\sum_{\boldsymbol{L}}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{t},\vec{\tau}) = \frac{1}{c_{\vec{n}}}\sum_{\vec{\eta}\times\vec{\rho}}\delta\left(\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})\,\vec{\tau}^{\,-1}\right)\delta_{\vec{\rho}(\vec{s}),\vec{t}} \tag{B.31}$$

We start by writing two useful identities, which will allow us to connect state indices appearing in the first quiver character with state indices appearing in the second quiver character. Consider contracting both sides of the equation [39]

$$\sum_{\sigma\in S_n} D^r_{i,j}(\sigma)\, D^{r'}_{k,l}(\sigma) = \frac{n!}{d(r)}\delta_{r,r'}\delta_{i,k}\delta_{j,l} \tag{B.32}$$

with $B^{R\to r,\cdots;\nu^-}_{I\to i,\cdots}\; B^{R\to r,\cdots;\nu^+}_{J\to j,\cdots}\; B^{R\to r',\cdots;\nu^+}_{K\to k,\cdots}\; B^{R\to r',\cdots;\nu^-}_{L\to l,\cdots}$ and then summing over the representation $r'\vdash n$. By doing so, we get the identity

$$B^{R\to r,\cdots;\nu^-}_{I\to i,\cdots}\; B^{R\to r,\cdots;\nu^+}_{K\to i,\cdots}\; B^{R\to r,\cdots;\nu^-}_{L\to l,\cdots}\; B^{R\to r,\cdots;\nu^+}_{J\to l,\cdots} \tag{B.33}$$

$$= \frac{d(r)}{n!}\sum_{\sigma\in S_n}\sum_{r'\vdash n} B^{R\to r,\cdots;\nu^-}_{I\to i,\cdots}\; B^{R\to r,\cdots;\nu^+}_{J\to j,\cdots}\; B^{R\to r',\cdots;\nu^+}_{K\to k,\cdots}\; B^{R\to r',\cdots;\nu^-}_{L\to l,\cdots}\, D^r_{i,j}(\sigma)\, D^{r'}_{k,l}(\sigma)$$

Alternatively, contracting both sides of (B.32) with $C^{r',S,k}_{\boldsymbol{s}} C^{r',S,l}_{\boldsymbol{t}}$ and summing over the representations $r' \vdash n$, we obtain

$$C^{r,S,i}_{\boldsymbol{s}} C^{r,S,j}_{\boldsymbol{t}} = \frac{d(r)}{n!} \sum_{\sigma\in S_n} D^r_{i,j}(\sigma) \left(\sum_{r'\vdash n} D^{r'}_{k,l}(\sigma)\, C^{r',S,k}_{\boldsymbol{s}} C^{r',S,l}_{\boldsymbol{t}} \right) \tag{B.34}$$

This is the second identity we are going to need.

Let us then consider the product

$$\begin{aligned}
\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{t},\vec{\tau}) &= c^2_{\boldsymbol{L}} \prod_a D^{R_a}_{i_a,j_a}(\sigma_a)\, D^{R_a}_{i'_a,j'_a}(\tau_a) \\
&\times B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu^-_a}_{j_a\to\cup_{b,\alpha}l_{ab,\alpha}\cup_\beta l_{a,\beta}} \left(\prod_\beta C^{r_{a,\beta},S_{a,\beta},l_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}\right) B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu^+_a}_{i_a\to\cup_{b,\alpha}l_{ba,\alpha}\cup_\gamma\bar{l}_{a,\gamma}} \left(\prod_\gamma C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}_{a,\gamma}}\right) \\
&\times B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu^-_a}_{j'_a\to\cup_{b,\alpha}l'_{ab,\alpha}\cup_\beta l'_{a,\beta}} \left(\prod_\beta C^{r_{a,\beta},S_{a,\beta},l'_{a,\beta}}_{\boldsymbol{t}_{a,\beta}}\right) B^{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu^+_a}_{i'_a\to\cup_{b,\alpha}l'_{ba,\alpha}\cup_\gamma\bar{l}'_{a,\gamma}} \left(\prod_\gamma C^{\bar{\boldsymbol{t}}_{a,\gamma}}_{\bar{r}_{a,\gamma},\bar{S}_{a,\gamma},\bar{l}'_{a,\gamma}}\right)
\end{aligned} \tag{B.35}$$

Using (B.33) and (B.34) in (B.35) we find

$$\begin{aligned}
\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{t},\vec{\tau}) &= c^2_{\boldsymbol{L}} \sum_{\vec{\eta},\vec{\rho}} \sum_{\{r'_{ab,\alpha}\}} \sum_{\{r'_{a,\beta}\}} \sum_{\{\bar{r}'_{a,\gamma}\}} \\
&\times \prod_a \left(\prod_{b,\alpha} \frac{d(r_{ab,\alpha})}{n_{ab,\alpha}!}\right) \left(\prod_\beta \frac{d(r_{a,\beta})}{n_{a,\beta}!}\right) \left(\prod_\gamma \frac{d(\bar{r}_{a,\gamma})}{\bar{n}_{a,\gamma}!}\right) D^{R_a}_{i_a,j_a}(\sigma_a)\, D^{R_a}_{i'_a,j'_a}(\tau_a) \\
&\times \left[\left(\prod_{b,\alpha} D^{r_{ab,\alpha}}_{p_{ab,\alpha},p'_{ab,\alpha}}(\eta_{ab,\alpha})\right) \left(\prod_\beta D^{r_{a,\beta}}_{p_{a,\beta},p'_{a,\beta}}(\rho_{a,\beta})\right) B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu^-_a}_{j_a\to\cup_{b,\alpha}p_{ab,\alpha}\cup_\beta p_{a,\beta}} \right] \\
&\times \left(\prod_\beta D^{r'_{a,\beta}}_{q_{a,\beta},q'_{a,\beta}}(\rho_{a,\beta})\, C^{r'_{a,\beta},S_{a,\beta},q_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}\, C^{r'_{a,\beta},S_{a,\beta},q'_{a,\beta}}_{\boldsymbol{t}_{a,\beta}}\right) \\
&\times \left[\left(\prod_{b,\alpha} D^{r'_{ba,\alpha}}_{q_{ba,\alpha},q'_{ba,\alpha}}(\eta_{ba,\alpha})\right) \left(\prod_\gamma D^{\bar{r}_{a,\gamma}}_{\bar{p}_{a,\gamma},\bar{p}'_{a,\gamma}}(\bar{\rho}_{a,\gamma})\right) B^{R_a\to\cup_{b,\alpha}r'_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu^+_a}_{i_a\to\cup_{b,\alpha}q_{ba,\alpha}\cup_\gamma\bar{p}_{a,\gamma}} \right] \\
&\times \left(\prod_\gamma D^{\bar{r}'_{a,\gamma}}_{\bar{q}_{a,\gamma},\bar{q}'_{a,\gamma}}(\bar{\rho}_{a,\gamma})\, C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{r}'_{a,\gamma},\bar{S}_{a,\gamma},\bar{q}_{a,\gamma}}\, C^{\bar{\boldsymbol{t}}_{a,\gamma}}_{\bar{r}'_{a,\gamma},\bar{S}_{a,\gamma},\bar{q}'_{a,\gamma}}\right) \\
&\times B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu^-_a}_{j'_a\to\cup_{b,\alpha}p'_{ab,\alpha}\cup_\beta p'_{a,\beta}}\, B^{R_a\to\cup_{b,\alpha}r'_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma};\nu^+_a}_{i'_a\to\cup_{b,\alpha}q'_{ba,\alpha}\cup_\gamma\bar{p}'_{a,\gamma}}
\end{aligned} \tag{B.36}$$

where $\{r'_{ab,\alpha}\}$, $\{r'_{a,\beta}\}$ and $\{\bar{r}'_{a,\gamma}\}$ are shorthands for $\cup_{a,b,\alpha}\{r'_{ab,\alpha}\}$, $\cup_{a,\beta}\{r'_{a,\beta}\}$ and $\cup_{a,\gamma}\{\bar{r}'_{a,\gamma}\}$

respectively. We now use the equivariance property of the branching coefficients (eq. (B.4)) to rewrite the terms in the square brackets above as

$$\left(\prod_{b,\alpha} D^{r_{ab,\alpha}}_{p_{ab,\alpha},p'_{ab,\alpha}}(\eta_{ab,\alpha})\right)\left(\prod_{\beta} D^{r_{a,\beta}}_{p_{a,\beta},p'_{a,\beta}}(\rho_{a,\beta})\right) B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{j_a\to\cup_{b,\alpha}p_{ab,\alpha}\cup_\beta p_{a,\beta}}$$
$$= D^{R_a}_{j_a,l_a}\left(\times_{b,\alpha}\eta_{ab,\alpha}\times_\beta\rho_{a,\beta}\right) B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{l_a\to\cup_{b,\alpha}p'_{ab,\alpha}\cup_\beta p'_{a,\beta}} \tag{B.37}$$

and

$$\left(\prod_{b,\alpha} D^{r'_{ba,\alpha}}_{q_{ba,\alpha},q'_{ba,\alpha}}(\eta_{ba,\alpha})\right)\left(\prod_{\gamma} D^{\bar r_{a,\gamma}}_{\bar p_{a,\gamma},\bar p'_{a,\gamma}}(\bar\rho_{a,\gamma})\right) B^{R_a\to\cup_{b,\alpha}r'_{ba,\alpha}\cup_\gamma \bar r_{a,\gamma};\nu_a^+}_{i_a\to\cup_{b,\alpha}q_{ba,\alpha}\cup_\gamma \bar p_{a,\gamma}}$$
$$= D^{R_a}_{i_a,l'_a}\left(\times_{b,\alpha}\eta_{ba,\alpha}\times_\gamma\rho_{a,\gamma}\right) B^{R_a\to\cup_{b,\alpha}r'_{ba,\alpha}\cup_\gamma \bar r_{a,\gamma};\nu_a^+}_{l'_a\to\cup_{b,\alpha}q'_{ba,\alpha}\cup_\gamma \bar p'_{a,\gamma}} \tag{B.38}$$

On the other hand, we can use eqs. (B.10) and (B.11) to write the Clebsch-Gordan coefficient terms as

$$\prod_\beta D^{r'_{a,\beta}}_{q_{a,\beta},q'_{a,\beta}}(\rho_{a,\beta})\, C^{r'_{a,\beta},S_{a,\beta},q_{a,\beta}}_{\boldsymbol{s}_{a,\beta}}\; C^{r'_{a,\beta},S_{a,\beta},q'_{a,\beta}}_{\boldsymbol{t}_{a,\beta}} = \prod_\beta C^{r'_{a,\beta},S_{a,\beta},q'_{a,\beta}}_{\rho^{-1}_{a,\beta}(\boldsymbol{s}_{a,\beta})}\; C^{r'_{a,\beta},S_{a,\beta},q'_{a,\beta}}_{\boldsymbol{t}_{a,\beta}} \tag{B.39}$$

(there is no sum over the $r_{a,\beta}$ and $S_{a,\beta}$ labels) and

$$\prod_\gamma D^{\bar r'_{a,\gamma}}_{\bar q_{a,\gamma},\bar q'_{a,\gamma}}(\bar\rho_{a,\gamma})\, C^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar r'_{a,\gamma},\bar S_{a,\gamma},\bar q_{a,\gamma}}\; C^{\bar{\boldsymbol{t}}_{a,\gamma}}_{\bar r'_{a,\gamma},\bar S_{a,\gamma},\bar q'_{a,\gamma}} = \prod_\gamma C^{\bar\rho^{-1}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar r'_{a,\gamma},\bar S_{a,\gamma},\bar q'_{a,\gamma}}\; C^{\bar{\boldsymbol{t}}_{a,\gamma}}_{\bar r'_{a,\gamma},\bar S_{a,\gamma},\bar q'_{a,\gamma}} \tag{B.40}$$

(again no sum over the $\bar r_{a,\gamma}$ and $\bar S_{a,\gamma}$ labels).

Inserting the last four equations in (B.36), taking the transpose of the matrix element on the RHS of (B.38) and relabelling the dummy permutation variables as $\vec\eta\to\vec\eta^{\,-1}, \vec\rho\to\vec\rho^{\,-1}$ gives

$$\chi_{\mathcal Q}(\boldsymbol{L},\vec s,\vec\sigma)\,\chi_{\mathcal Q}(\boldsymbol{L},\vec t,\vec\tau) = \frac{1}{c_{\vec n}}\sum_{\vec\eta,\vec\rho}\;\sum_{\{s_{ab,\alpha}\}}\;\sum_{\{s_{a,\beta}\}}\;\sum_{\{\bar s_{a,\gamma}\}}\;\prod_a\frac{d(R_a)}{n_a!}$$
$$\times D^{R_a}_{l'_a,i_a}\left(\times_{b,\alpha}\eta_{ba,\alpha}\times_\gamma\rho_{a,\gamma}\right) D^{R_a}_{i_a,j_a}(\sigma_a) D^{R_a}_{j_a,l_a}\left(\times_{b,\alpha}\eta^{-1}_{ab,\alpha}\times_\beta\rho^{-1}_{a,\beta}\right) D^{R_a}_{i'_a,j'_a}(\tau_a)$$
$$\times\left[B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{l_a\to\cup_{b,\alpha}p'_{ab,\alpha}\cup_\beta p'_{a,\beta}}\; B^{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta};\nu_a^-}_{j'_a\to\cup_{b,\alpha}p'_{ab,\alpha}\cup_\beta p'_{a,\beta}}\right]$$
$$\times\left[B^{R_a\to\cup_{b,\alpha}s_{ba,\alpha}\cup_\gamma \bar r_{a,\gamma};\nu_a^+}_{l'_a\to\cup_{b,\alpha}q'_{ba,\alpha}\cup_\gamma \bar p'_{a,\gamma}}\; B^{R_a\to\cup_{b,\alpha}s_{ba,\alpha}\cup_\gamma \bar r_{a,\gamma};\nu_a^+}_{i'_a\to\cup_{b,\alpha}q'_{ba,\alpha}\cup_\gamma \bar p'_{a,\gamma}}\right]$$
$$\times\left(\prod_\beta C^{s_{a,\beta},S_{a,\beta},q_{a,\beta}}_{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta})}\; C^{s_{a,\beta},S_{a,\beta},q_{a,\beta}}_{\boldsymbol{t}_{a,\beta}}\right)\left(\prod_\gamma C^{\bar\rho_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar s_{a,\gamma},\bar S_{a,\gamma},\bar q_{a,\gamma}}\; C^{\bar{\boldsymbol{t}}_{a,\gamma}}_{\bar s_{a,\gamma},\bar S_{a,\gamma},\bar q_{a,\gamma}}\right) \tag{B.41}$$

where we also used the definitions of $c_{\boldsymbol{L}}$ and $c_{\vec{n}}$ given in (3.21) and (3.27). Now, from eq. (B.16) we have

$$\left[B^{R_a \to \cup_{b,\alpha} r_{ab,\alpha} \cup_\beta r_{a,\beta};\nu_a^-}_{l_a \to \cup_{b,\alpha} p'_{ab,\alpha} \cup_\beta p'_{a,\beta}} \; B^{R_a \to \cup_{b,\alpha} r_{ab,\alpha} \cup_\beta r_{a,\beta};\nu_a^-}_{j'_a \to \cup_{b,\alpha} p'_{ab,\alpha} \cup_\beta p'_{a,\beta}} \right] = P^{\nu_a^-,\nu_a^-}_{R_a \to \cup_{b,\alpha} r_{ab,\alpha} \cup_\beta r_{a,\beta}} \Big|_{l_a, j'_a} \tag{B.42}$$

$$\left[B^{R_a \to \cup_{b,\alpha} r'_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma};\nu_a^+}_{l'_a \to \cup_{b,\alpha} q'_{ba,\alpha} \cup_\gamma \bar{p}'_{a,\gamma}} \; B^{R_a \to \cup_{b,\alpha} r'_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma};\nu_a^+}_{i'_a \to \cup_{b,\alpha} q'_{ba,\alpha} \cup_\gamma \bar{p}'_{a,\gamma}} \right] = P^{\nu_a^+,\nu_a^+}_{R_a \to \cup_{b,\alpha} r'_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}} \Big|_{l'_a, i'_a} \tag{B.43}$$

so that we can write

$$\begin{aligned}
\chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{s}, \vec{\sigma})\, \chi_{\mathcal{Q}}(\boldsymbol{L}, \vec{t}, \vec{\tau}) &= \frac{1}{c_{\vec{n}}} \sum_{\vec{\eta},\vec{\rho}} \sum_{\{r'_{ab,\alpha}\}} \prod_a \frac{d(R_a)}{n_a!} D^{R_a}_{l'_a, l_a} \left(\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\right) D^{R_a}_{i'_a, j'_a}(\tau_a) \\
&\times P^{\nu_a^-,\nu_a^-}_{R_a \to \cup_{b,\alpha} r_{ab,\alpha} \cup_\beta r_{a,\beta}} \Big|_{l_a, j'_a} P^{\nu_a^+,\nu_a^+}_{R_a \to \cup_{b,\alpha} r'_{ba,\alpha} \cup_\gamma \bar{r}_{a,\gamma}} \Big|_{l'_a, i'_a} \\
&\times \left(\sum_{\{r'_{a,\beta}\}} \prod_\beta C^{r'_{a,\beta}, S_{a,\beta}, q_{a,\beta}}_{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta})} C^{r'_{a,\beta}, S_{a,\beta}, q_{a,\beta}}_{\boldsymbol{t}_{a,\beta}} \right) \left(\sum_{\{\bar{r}'_{a,\gamma}\}} \prod_\gamma C^{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar{r}'_{a,\gamma}, \bar{S}_{a,\gamma}, \bar{q}_{a,\gamma}} C^{\bar{\boldsymbol{t}}_{a,\gamma}}_{\bar{r}'_{a,\gamma}, \bar{S}_{a,\gamma}, \bar{q}_{a,\gamma}} \right)
\end{aligned} \tag{B.44}$$

where we defined

$$\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a) = (\times_{b,\alpha} \eta_{ba,\alpha} \times_\gamma \rho_{a,\gamma})(\sigma_a)(\times_{b,\alpha} \eta^{-1}_{ab,\alpha} \times_\beta \rho^{-1}_{a,\beta}) \tag{B.45}$$

Now we can proceed to sum over $\boldsymbol{L} = \{R_a, r_{ab,\alpha}, r_{a,\beta}, S_{a,\beta}, \bar{r}_{a,\gamma}, \bar{S}_{a,\gamma}, \nu_a^+, \nu_a^-\}$. This introduces, among others, a summation over the flavour states $S_{a,\beta}$ and $\bar{S}_{a,\gamma}$. Consider then a pair of Clebsch-Gordan coefficients like the ones appearing in the last line of eq. (B.44). It is easy to write the relation

$$\sum_{r,S,i} C^{r,S,i}_{\rho(\boldsymbol{s})} C^{r,S,i}_{\boldsymbol{t}} = \langle \rho(\boldsymbol{s}) | \left(\sum_{r,S,i} |r,S,i\rangle\langle r,S,i| \right) |\boldsymbol{t}\rangle = \langle \rho(\boldsymbol{s}) | 1 | \boldsymbol{t} \rangle = \delta_{\rho(\boldsymbol{s}),\boldsymbol{t}} \tag{B.46}$$

We then have the identity

$$\sum_{r'_{a,\beta},\, S_{a,\beta},\, q_{a,\beta}} C^{r'_{a,\beta}, S_{a,\beta}, q_{a,\beta}}_{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta})} C^{r'_{a,\beta}, S_{a,\beta}, q_{a,\beta}}_{\boldsymbol{t}_{a,\beta}} = \delta_{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta}), \boldsymbol{t}_{a,\beta}} \tag{B.47}$$

and similarly

$$\sum_{\bar{r}'_{a,\gamma},\, \bar{S}_{a,\gamma},\, \bar{q}_{a,\gamma}} C^{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar{r}'_{a,\gamma}, \bar{S}_{a,\gamma}, \bar{q}_{a,\gamma}} C^{\bar{\boldsymbol{t}}_{a,\gamma}}_{\bar{r}'_{a,\gamma}, \bar{S}_{a,\gamma}, \bar{q}_{a,\gamma}} = \delta_{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma}), \bar{\boldsymbol{t}}_{a,\gamma}} \tag{B.48}$$

Inserting this result in eq. (B.44) we obtain

$$\begin{aligned}
&\sum_{\boldsymbol{L}} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{t},\vec{\tau})\\
&=\frac{1}{c_{\vec{n}}}\sum_{\vec{\eta},\vec{\rho}}\prod_{a}\sum_{R_a}\frac{d(R_a)}{n_a!}D^{R_a}_{l'_a,l_a}\left(\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\right)\,D^{R_a}_{i'_a,j'_a}(\tau_a)\\
&\quad\times\sum_{\cup_{b,\alpha}\{r_{ab,\alpha}\}}\ \sum_{\substack{\cup_\beta\{r_{a,\beta}\}\\ \nu_a^-}} P^{\nu_a^-,\nu_a^-}_{R_a\to\cup_{b,\alpha}r_{ab,\alpha}\cup_\beta r_{a,\beta}}\Big|_{l_a,j'_a}\ \sum_{\cup_{b,\alpha}\{r'_{ba,\alpha}\}}\ \sum_{\substack{\cup_\gamma\{\bar{r}_{a,\gamma}\}\\ \nu_a^+}} P^{\nu_a^+,\nu_a^+}_{R_a\to\cup_{b,\alpha}r'_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma}}\Big|_{l'_a,i'_a}\\
&\qquad\qquad\times\left(\prod_\beta\delta_{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta}),\boldsymbol{t}_{a,\beta}}\right)\left(\prod_\gamma\delta_{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma}),\bar{\boldsymbol{t}}_{a,\gamma}}\right)
\end{aligned}\tag{B.49}$$

Now using the projector identity

$$\sum_{\cup_i\{r_i\},\,\nu} P^{\nu,\nu}_{R\to\cup_i r_i}\Big|_{k,l}=\delta_{k,l}\tag{B.50}$$

we further get

$$\begin{aligned}
\sum_{\boldsymbol{L}} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{t},\vec{\tau}) &= \frac{1}{c_{\vec{n}}}\sum_{\vec{\eta},\vec{\rho}}\prod_a\sum_{R_a}\frac{d(R_a)}{n_a!}\chi_{R_a}\left(\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\,\tau_a^{-1}\right)\\
&\qquad\times\left(\prod_\beta\delta_{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta}),\boldsymbol{t}_{a,\beta}}\right)\left(\prod_\gamma\delta_{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma}),\bar{\boldsymbol{t}}_{a,\gamma}}\right)
\end{aligned}\tag{B.51}$$

Finally, through the identity

$$\sum_{R\vdash n}\frac{d(R)}{n!}\chi_R(\sigma)=\delta(\sigma)\tag{B.52}$$

we can rewrite (B.51) as

$$\begin{aligned}
&\sum_{\boldsymbol{L}} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\,\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{t},\vec{\tau}) =\\
&\qquad=\frac{1}{c_{\vec{n}}}\sum_{\vec{\eta},\vec{\rho}}\prod_a\delta\left(\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\,\tau_a^{-1}\right)\left(\prod_\beta\delta_{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta}),\boldsymbol{t}_{a,\beta}}\right)\left(\prod_\gamma\delta_{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma}),\bar{\boldsymbol{t}}_{a,\gamma}}\right)\\
&\qquad=\frac{1}{c_{\vec{n}}}\sum_{\vec{\eta},\vec{\rho}}\delta\left(\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma})\,\vec{\tau}^{\,-1}\right)\,\delta_{\vec{\rho}(\vec{s}),\vec{t}}
\end{aligned}\tag{B.53}$$

This last equation is exactly (3.26).

C Two Point Function: Proof of Orthogonality

In this section we will prove the orthogonality formula (4.1):

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\, \mathcal{O}^{\dagger}_{\mathcal{Q}}(\boldsymbol{L'}) \right\rangle = \delta_{\boldsymbol{L},\boldsymbol{L'}}\ c_{\vec{n}} \prod_{a} f_{N_a}(R_a) \tag{C.1}$$

The first ingredient we need is the Hermitean conjugated version of the operator defined in (2.12), which is simply

$$\begin{aligned}\mathcal{O}^{\dagger}_{\mathcal{Q}}(\vec{n};\,\vec{s};\,\vec{\sigma}) = \prod_{a}\left[\prod_{b,\alpha}\left(\Phi^{\dagger\,\otimes n_{ab,\alpha}}_{ab,\alpha}\right)^{J_{ab,\alpha}}_{I_{ab,\alpha}}\right]\otimes\left[\prod_{\beta}\left(Q^{\dagger\,\otimes n_{a,\beta}}_{a,\beta}\right)^{\boldsymbol{s}_{a,\beta}}_{I_{a,\beta}}\right]\otimes\left[\prod_{\gamma}\left(\bar{Q}^{\dagger\,\otimes \bar{n}_{a,\gamma}}_{a,\gamma}\right)^{\bar{J}_{a,\gamma}}_{\bar{\boldsymbol{s}}_{a,\gamma}}\right]\\ \times\prod_{c}\left(\sigma_c^{-1}\right)^{\cup_{b,\alpha} I_{cb,\alpha}\cup_{\beta} I_{a,\beta}}_{\cup_{b,\alpha} J_{bc,\alpha}\cup_{\gamma}\bar{J}_{a,\gamma}}\end{aligned} \tag{C.2}$$

Here we used $(\sigma)^{j}_{i} = (\sigma^{-1})^{i}_{j}$. Using the free field metric

$$\left\langle (\Phi_{ab,\alpha})^{i}_{j}\,(\Phi^{\dagger}_{ab,\alpha})^{k}_{l}\right\rangle = \delta^{i}_{l}\delta^{k}_{j}\,,\quad \left\langle (Q_{a,\beta})^{i}_{s}\,(Q^{\dagger}_{a,\beta})^{p}_{l}\right\rangle = \delta^{i}_{l}\delta^{p}_{s}\,,\quad \left\langle (\bar{Q}_{a,\gamma})^{\bar{s}}_{j}\,(\bar{Q}^{\dagger}_{a,\gamma})^{k}_{\bar{p}}\right\rangle = \delta^{k}_{j}\delta^{\bar{s}}_{\bar{p}} \tag{C.3}$$

(the remaining correlators are zero) we get

$$\left\langle \left(\Phi^{\otimes n_{ab,\alpha}}_{ab,\alpha}\right)^{I_{ab,\alpha}}_{J_{ab,\alpha}} \left(\Phi^{\dagger\,\otimes n_{ab,\alpha}}_{ab,\alpha}\right)^{J'_{ab,\alpha}}_{I'_{ab,\alpha}} \right\rangle = \sum_{\eta\in S_{n_{ab,\alpha}}} \delta^{\eta(I_{ab,\alpha})}_{I'_{ab,\alpha}}\,\delta^{J'_{ab,\alpha}}_{\eta(J_{ab,\alpha})} \tag{C.4}$$

In this formula the sum over permutations represents all possible Wick contractions of the labels $I_{ab,\alpha} = \{i_1, ..., i_{n_{ab,\alpha}}\}$, $J_{ab,\alpha} = \{j_1, ..., j_{n_{ab,\alpha}}\}$. Denoting the states belonging to the fundamental and the antifundamental representation of $U(N)$ by $|e_j\rangle$ and $\langle e^j|$respectively, we have the identities

$$\delta^{J'_{ab,\alpha}}_{\eta(J_{ab,\alpha})} = \langle e^{j'_1}, ..., e^{j'_{n_{ab,\alpha}}}|e_{j_{\eta(1)}}, ..., e_{j_{\eta(n_{ab,\alpha})}}\rangle = \langle e^{j'_1}, ..., e^{j'_{n_{ab,\alpha}}}|\eta|e_{j_1}, ..., e_{j_{n_{ab,\alpha}}}\rangle = (\eta)^{J'_{ab,\alpha}}_{J_{ab,\alpha}} \tag{C.5}$$

and

$$\delta^{I'_{ab,\alpha}}_{\eta(I_{ab,\alpha})} = (\eta)^{I'_{ab,\alpha}}_{I_{ab,\alpha}} = \left(\eta^{-1}\right)^{I_{ab,\alpha}}_{I'_{ab,\alpha}} = \delta^{I_{ab,\alpha}}_{\eta^{-1}(I'_{ab,\alpha})} \tag{C.6}$$

Performing similar steps on the correlators of quarks and antiquarks, we can then write

$$\left\langle \left(\Phi^{\otimes n_{ab,\alpha}}_{ab,\alpha}\right)^{I_{ab,\alpha}}_{J_{ab,\alpha}} \left(\Phi^{\dagger\,\otimes n_{ab,\alpha}}_{ab,\alpha}\right)^{J'_{ab,\alpha}}_{I'_{ab,\alpha}} \right\rangle = \sum_{\eta\in S_{n_{ab,\alpha}}} \left(\eta^{-1}\right)^{I_{ab,\alpha}}_{I'_{ab,\alpha}}\,(\eta)^{J'_{ab,\alpha}}_{J_{ab,\alpha}} \tag{C.7a}$$

$$\left\langle \left(Q^{\otimes n_{a,\beta}}_{a,\beta}\right)^{I_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} \left(Q^{\dagger\,\otimes n_{a,\beta}}_{a,\beta}\right)^{\boldsymbol{s'}_{a,\beta}}_{I'_{a,\beta}} \right\rangle = \sum_{\rho\in S_{n_{a,\beta}}} \left(\rho^{-1}\right)^{I_{a,\beta}}_{I'_{a,\beta}}\,(\rho)^{\boldsymbol{s'}_{a,\beta}}_{\boldsymbol{s}_{a,\beta}} = \sum_{\rho\in S_{n_{a,\beta}}} \left(\rho^{-1}\right)^{I_{a,\beta}}_{I'_{a,\beta}}\,\delta^{\boldsymbol{s'}_{a,\beta}}_{\rho(\boldsymbol{s}_{a,\beta})} \tag{C.7b}$$

$$\left\langle \left(\bar{Q}_{a,\gamma}^{\otimes \bar{n}_{a,\gamma}}\right)^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{J}_{a,\gamma}} \left(\bar{Q}_{a,\gamma}^{\dagger\,\otimes \bar{n}_{a,\gamma}}\right)^{\bar{J}'_{a,\gamma}}_{\bar{\boldsymbol{s}}'_{a,\gamma}} \right\rangle = \sum_{\bar{\rho}\in S_{\bar{n}_{a,\gamma}}} (\bar{\rho}^{-1})^{\bar{\boldsymbol{s}}_{a,\gamma}}_{\bar{\boldsymbol{s}}'_{a,\gamma}} \left(\bar{\rho}\right)^{\bar{J}'_{a,\gamma}}_{\bar{J}_{a,\gamma}} = \sum_{\bar{\rho}\in S_{\bar{n}_{a,\gamma}}} \left(\bar{\rho}\right)^{\bar{J}'_{a,\gamma}}_{\bar{J}_{a,\gamma}} \delta^{\bar{\rho}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar{\boldsymbol{s}}'_{a,\gamma}} \tag{C.7c}$$

We therefore get

$$\begin{aligned}
\left\langle \mathcal{O}_{\mathcal{Q}}(\vec{n},\vec{s},\vec{\sigma})\mathcal{O}^{\dagger}_{\mathcal{Q}}(\vec{n},\vec{s}\,',\vec{\sigma}\,') \right\rangle &= \sum_{\vec{\eta},\vec{\rho}} \prod_a \left[\prod_{b,\alpha} \left(\eta^{-1}_{ab,\alpha}\right)^{I_{ab,\alpha}}_{I'_{ab,\alpha}} \left(\eta_{ab,\alpha}\right)^{J'_{ab,\alpha}}_{J_{ab,\alpha}}\right] \left[\prod_\beta \left(\rho^{-1}_{a,\beta}\right)^{I_{a,\beta}}_{I'_{a,\beta}} \delta^{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta})}_{\boldsymbol{s}'_{a,\beta}}\right] \\
&\times \left[\prod_\gamma \left(\bar{\rho}_{a,\gamma}\right)^{\bar{J}'_{a,\gamma}}_{\bar{J}_{a,\gamma}} \delta^{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar{\boldsymbol{s}}'_{a,\gamma}}\right] \left(\sigma_a\right)^{\cup_{b,\alpha} J_{ba,\alpha}\cup_\gamma \bar{J}_{a,\gamma}}_{\cup_{b,\alpha} I_{ab,\alpha}\cup_\beta I_{a,\beta}} \left((\sigma'_a)^{-1}\right)^{\cup_{b,\alpha} I'_{ab,\alpha}\cup_\beta I'_{a,\beta}}_{\cup_{b,\alpha} J'_{ba,\alpha}\cup_\gamma \bar{J}'_{a,\gamma}} \\
&= \sum_{\vec{\eta},\vec{\rho}} \prod_a \mathrm{Tr}_{V_{N_a}^{\otimes n_a}} \left[\left(\times_{b,\alpha}\eta_{ba,\alpha} \times_\gamma \bar{\rho}_{a,\gamma}\right)\sigma_a \left(\times_{b,\alpha}\eta^{-1}_{ab,\alpha} \times_\beta \rho^{-1}_{a,\beta}\right) (\sigma'_a)^{-1}\right] \\
&\times \left[\prod_\beta \delta^{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta})}_{\boldsymbol{s}'_{a,\beta}}\right] \left[\prod_\gamma \delta^{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar{\boldsymbol{s}}'_{a,\gamma}}\right]
\end{aligned} \tag{C.8}$$

where, as we defined in (2.16),

$$\vec{\eta} = \cup_{a,b,\alpha}\{\eta_{ab,\alpha}\} \;\;,\quad \eta_{ab,\alpha} \in S_{n_{ab,\alpha}} \tag{C.9a}$$

$$\vec{\rho} = \cup_a\{\cup_\beta\, \rho_{a,\beta};\, \cup_\gamma\, \bar{\rho}_{a,\gamma}\} \;\;,\quad \rho_{a,\beta} \in S_{n_{a,\beta}} \;\;,\quad \bar{\rho}_{a,\gamma} \in S_{\bar{n}_{a,\gamma}} \tag{C.9b}$$

The trace is taken over the product space $V_{N_a}^{\otimes n_a}$, V_{N_a} being the fundamental representation of $U(N_a)$ and $n_a = \sum_{b,\alpha} n_{ab,\alpha} + \sum_\beta n_{a,\beta}$. Recalling (2.17),

$$\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma}) = \cup_a\{(\times_{b,\alpha}\eta_{ba,\alpha} \times_\gamma \bar{\rho}_{a,\gamma})\sigma_a(\times_{b,\alpha}\eta^{-1}_{ab,\alpha} \times_\beta \rho^{-1}_{a,\beta})\} \tag{C.10}$$

we can finally write

$$\begin{aligned}
\left\langle \mathcal{O}_{\mathcal{Q}}(\vec{n},\vec{s},\vec{\sigma})\mathcal{O}^{\dagger}_{\mathcal{Q}}(\vec{n},\vec{s}\,',\vec{\sigma}\,') \right\rangle &= \sum_{\vec{\eta},\vec{\rho}} \prod_a \mathrm{Tr}_{V_{N_a}^{\otimes n_a}} \left[\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\,(\sigma'_a)^{-1}\right] \\
&\times \left[\prod_\beta \delta^{\rho_{a,\beta}(\boldsymbol{s}_{a,\beta})}_{\boldsymbol{s}'_{a,\beta}}\right] \left[\prod_\gamma \delta^{\bar{\rho}_{a,\gamma}(\bar{\boldsymbol{s}}_{a,\gamma})}_{\bar{\boldsymbol{s}}'_{a,\gamma}}\right]
\end{aligned} \tag{C.11}$$

which is eq. (4.5).

Using the definition of the Fourier transformed operator (3.19) we then get

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\mathcal{O}^{\dagger}_{\mathcal{Q}}(\boldsymbol{L'})\right\rangle = \sum_{\vec{s},\vec{s}\,'}\sum_{\vec{\sigma},\vec{\sigma}\,'} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\chi^{\dagger}_{\mathcal{Q}}(\boldsymbol{L'},\vec{s}\,',\vec{\sigma}\,')\left\langle \mathcal{O}_{\mathcal{Q}}(\vec{n},\vec{s},\vec{\sigma})\mathcal{O}^{\dagger}_{\mathcal{Q}}(\vec{n},\vec{s}\,',\vec{\sigma}\,')\right\rangle \tag{C.12}$$

$$= \sum_{\vec{s}}\sum_{\vec{\sigma},\vec{\sigma}\,'}\sum_{\vec{\eta},\vec{\rho}} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\chi_{\mathcal{Q}}(\boldsymbol{L'},\vec{\rho}(\vec{s}),\vec{\sigma}\,')\prod_a \mathrm{Tr}_{V_{N_a}^{\otimes n_a}}\left[\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\,(\sigma'_a)^{-1}\right]$$

where to get the second equality we summed over $\vec{s}\,'$, used the Kronecker delta functions and used the reality of the quiver characters. Redefining the dummy variable $\vec{s}\to\vec{\rho}^{\,-1}(\vec{s})$ in (C.12) we obtain

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\mathcal{O}^{\dagger}_{\mathcal{Q}}(\boldsymbol{L'})\right\rangle$$

$$= \sum_{\vec{s}}\sum_{\vec{\sigma},\vec{\sigma}\,'}\sum_{\vec{\eta},\vec{\rho}} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{\rho}^{\,-1}(\vec{s}),\vec{\sigma})\chi_{\mathcal{Q}}(\boldsymbol{L'},\vec{s},\vec{\sigma}\,')\prod_a \mathrm{Tr}_{V_{N_a}^{\otimes n_a}}\left[\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\,(\sigma'_a)^{-1}\right]$$

$$= \sum_{\vec{s}}\sum_{\vec{\sigma},\vec{\sigma}\,'}\sum_{\vec{\eta},\vec{\rho}} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\vec{\sigma}))\chi_{\mathcal{Q}}(\boldsymbol{L'},\vec{s},\vec{\sigma}\,')\prod_a \mathrm{Tr}_{V_{N_a}^{\otimes n_a}}\left[\mathrm{Adj}_{\vec{\eta}\times\vec{\rho}}(\sigma_a)\,(\sigma'_a)^{-1}\right]$$

$$= \sum_{\vec{s}}\sum_{\vec{\sigma},\vec{\sigma}\,'}\sum_{\vec{\eta},\vec{\rho}} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\chi_{\mathcal{Q}}(\boldsymbol{L'},\vec{s},\vec{\sigma}\,')\prod_a \mathrm{Tr}_{V_{N_a}^{\otimes n_a}}\left[\sigma_a\,(\sigma'_a)^{-1}\right] \tag{C.13}$$

To get the second equality we used the invariance relation (3.23), and in the third we relabelled the dummy variable $\vec{\sigma}\to\mathrm{Adj}_{\vec{\rho}\times\vec{\eta}}(\vec{\sigma})$. We then see that the dependence on the permutations $\vec{\eta}$ and $\vec{\rho}$ drops out, so that their sums can be trivially computed to obtain

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\mathcal{O}^{\dagger}_{\mathcal{Q}}(\boldsymbol{L'})\right\rangle = \sum_{\vec{s}}\sum_{\vec{\sigma},\vec{\sigma}\,'} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\sigma})\chi_{\mathcal{Q}}(\boldsymbol{L'},\vec{s},\vec{\sigma}\,')$$

$$\times\prod_a\left(\prod_{b,\alpha} n_{ab,\alpha}!\right)\left(\prod_{\beta} n_{a,\beta}!\right)\left(\prod_{\gamma}\bar{n}_{a,\gamma}!\right)\mathrm{Tr}_{V_{N_a}^{\otimes n_a}}\left[\sigma_a\,(\sigma'_a)^{-1}\right] \tag{C.14}$$

Now let us further relabel $\sigma_a\to\tau_a\cdot\sigma'_a$ and use the definition of $c_{\vec{n}}$ given in (3.27) to get

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\mathcal{O}^{\dagger}_{\mathcal{Q}}(\boldsymbol{L'})\right\rangle = c_{\vec{n}}\sum_{\vec{s}}\sum_{\vec{\tau},\vec{\sigma}\,'} \chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\tau}\cdot\vec{\sigma}\,')\chi_{\mathcal{Q}}(\boldsymbol{L'},\vec{s},\vec{\sigma}\,')\prod_a \mathrm{Tr}_{V_{N_a}^{\otimes n_a}}\left[\tau_a\right] \tag{C.15}$$

The only dependence on $\vec{\sigma}'$ and $\vec{s}$ is now inside the two quiver characters, so that we can use

(B.15) and write

$$
\begin{aligned}
&\sum_{\vec{s}}\sum_{\vec{\sigma}\,'}\chi_{\mathcal{Q}}(\boldsymbol{L},\vec{s},\vec{\tau}\cdot\vec{\sigma}\,')\,\chi_{\mathcal{Q}}(\boldsymbol{L'},\vec{s},\vec{\sigma}\,')\\
&\qquad= c_{\boldsymbol{L}}c_{\boldsymbol{L'}}\prod_a \frac{n_a!}{d(R_a)}\,\mathrm{Tr}\left(D^{R_a}(\tau_a)P^{\nu_a^+,\nu_a^{+\prime}}_{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma}}\right)\,\delta_{R_a,R'_a}\\
&\qquad\quad\times\left(\prod_{b,\alpha}\delta_{r_{ab,\alpha},r'_{ab,\alpha}}\right)\left(\prod_\beta d(r_{a,\beta})\delta_{r_{a,\beta},r'_{a,\beta}}\delta_{S_{a,\beta},S'_{a,\beta}}\right)\left(\prod_\gamma\delta_{\bar{r}_{a,\gamma},\bar{r}'_{a,\gamma}}\delta_{\bar{S}_{a,\gamma},\bar{S}'_{a,\gamma}}\right)\delta_{\nu_a^-,\nu_a^{-\prime}}
\end{aligned}
\tag{C.16}
$$

Inserting this equation into (C.15) and recalling the identity

$$
\mathrm{Tr}_{V_N^{\otimes n}}(\sigma)=N^{C[\sigma]} \tag{C.17}
$$

we get

$$
\begin{aligned}
\left\langle\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\mathcal{O}^\dagger_{\mathcal{Q}}(\boldsymbol{L'})\right\rangle &= c_{\vec{n}}\sum_{\vec{\tau}}c_{\boldsymbol{L}}c_{\boldsymbol{L'}}\prod_a\frac{n_a!}{d(R_a)}\delta_{R_a,R'_a}\delta_{\nu_a^-,\nu_a^{-\prime}}\left(\prod_{b,\alpha}\delta_{r_{ab,\alpha},r'_{ab,\alpha}}\right)\\
&\quad\times\left(\prod_\beta d(r_{a,\beta})\delta_{r_{a,\beta},r'_{a,\beta}}\delta_{S_{a,\beta},S'_{a,\beta}}\right)\left(\prod_\gamma\delta_{\bar{r}_{a,\gamma},\bar{r}'_{a,\gamma}}\delta_{\bar{S}_{a,\gamma},\bar{S}'_{a,\gamma}}\right)\\
&\quad\times\mathrm{Tr}\left(D^{R_a}(\tau_a)P^{\nu_a^+,\nu_a^{+\prime}}_{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma}}\right)N_a^{c[\tau_a]}
\end{aligned}
\tag{C.18}
$$

The last piece we need to obtain eq. (C.1) is the identity

$$
\sum_{\tau_a}\mathrm{Tr}\left(D^{R_a}(\tau_a)P^{\nu_a^+,\nu_a^{+\prime}}_{R_a\to\cup_{b,\alpha}r_{ba,\alpha}\cup_\gamma\bar{r}_{a,\gamma}}\right)N_a^{c[\tau_a]}=\delta_{\nu_a^+,\nu_a^{+\prime}}\left(\prod_{b,\alpha}d(r_{ba,\alpha})\right)\left(\prod_\gamma d(\bar{r}_{a,\gamma})\right)f_{N_a}(R_a)
\tag{C.19}
$$

a proof of which can be found in *e.g.* [39]. Inserting it in (C.18) we finally get

$$
\begin{aligned}
\left\langle\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\mathcal{O}^\dagger_{\mathcal{Q}}(\boldsymbol{L'})\right\rangle &= c_{\vec{n}}\,c_{\boldsymbol{L}}c_{\boldsymbol{L'}}\prod_a\frac{n_a!}{d(R_a)}\delta_{R_a,R'_a}\delta_{\nu_a^-,\nu_a^{-\prime}}\delta_{\nu_a^+,\nu_a^{+\prime}}\left(\prod_{b,\alpha}d(r_{ab,\alpha})\delta_{r_{ab,\alpha},r'_{ab,\alpha}}\right)\\
&\quad\times\left(\prod_\beta d(r_{a,\beta})\delta_{r_{a,\beta},r'_{a,\beta}}\delta_{S_{a,\beta},S'_{a,\beta}}\right)\left(\prod_\gamma \bar{d}(\bar{r}_{a,\gamma})\delta_{\bar{r}_{a,\gamma},\bar{r}'_{a,\gamma}}\delta_{\bar{S}_{a,\gamma},\bar{S}'_{a,\gamma}}\right)f_{N_a}(R_a)
\end{aligned}
$$

$$= \delta_{\boldsymbol{L},\boldsymbol{L}'}\, c_{\vec{n}}\, c_{\boldsymbol{L}}^2 \prod_a \frac{n_a!}{d(R_a)} \left(\prod_{b,\alpha} d(r_{ab,\alpha})\right) \left(\prod_\beta d(r_{a,\beta})\right) \left(\prod_\gamma \bar{d}(\bar{r}_{a,\gamma})\right) f_{N_a}(R_a) \tag{C.20}$$

which, using the normalisation constant $c_{\boldsymbol{L}}$ defined in (3.21), reduces to eq. (4.1):

$$\left\langle \mathcal{O}_{\mathcal{Q}}(\boldsymbol{L})\, \mathcal{O}^\dagger_{\mathcal{Q}}(\boldsymbol{L}') \right\rangle = \delta_{\boldsymbol{L},\boldsymbol{L}'}\; c_{\vec{n}} \prod_a f_{N_a}(R_a) \tag{C.21}$$

The orthogonality of the Fourier transformed operators is thus proven.

D Deriving the Holomorphic Gauge Invariant Operator Ring Structure Constants

In this appendix we will derive the analytical expression for the holomorphic GIO ring structure constants $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$, corresponding to the diagram given in Fig. 19. We will divide the computation into five main steps, for improved clarity. In the following subsection D.1 we will explicitly derive the chiral ring structure constants for an $\mathcal{N}=2$ SQCD, by using diagrammatic techniques alone.

1) The permutation basis product

In this first step we are going to rewrite the product of two operators in the permutation basis, $\mathcal{O}_{\mathcal{Q}}(\vec{n}_1, \vec{s}^{(1)}, \vec{\sigma}^{(1)})\, \mathcal{O}_{\mathcal{Q}}(\vec{n}_2, \vec{s}^{(2)}, \vec{\sigma}^{(2)})$, as a single operator $\mathcal{O}_{\mathcal{Q}}(\vec{n}_3, \vec{s}^{(3)}, \vec{\sigma}^{(3)})$, specified by appropriate labels $\vec{n}_3$, $\vec{s}^{(3)}$ and $\vec{\sigma}^{(3)}$. We use the defining equation (2.12) for $\mathcal{O}_{\mathcal{Q}}(\vec{n}, \vec{s}, \vec{\sigma})$ to write this product as

$$
\begin{aligned}
&\mathcal{O}_{\mathcal{Q}}(\vec{n}_1, \vec{s}^{(1)}, \vec{\sigma}^{(1)})\, \mathcal{O}_{\mathcal{Q}}(\vec{n}_2, \vec{s}^{(2)}, \vec{\sigma}^{(2)}) \\
&= \prod_a \left[\prod_{b,\alpha} \left(\Phi_{ab,\alpha}^{\otimes n^{(1)}_{ab,\alpha}}\right)^{I^{(1)}_{ab,\alpha}}_{J^{(1)}_{ab,\alpha}}\right] \otimes \left[\prod_\beta \left(Q_{a,\beta}^{\otimes n^{(1)}_{a,\beta}}\right)^{I^{(1)}_{a,\beta}}_{\boldsymbol{s}^{(1)}_{a,\beta}}\right] \otimes \left[\prod_\gamma \left(\bar{Q}_{a,\gamma}^{\otimes \bar{n}^{(1)}_{a,\gamma}}\right)^{\bar{\boldsymbol{s}}^{(1)}_{a,\gamma}}_{\bar{J}^{(1)}_{a,\gamma}}\right] \left(\sigma^{(1)}_a\right)^{\times_{b,\alpha} J^{(1)}_{ba,\alpha} \times_\gamma \bar{J}^{(1)}_{a,\gamma}}_{\times_{b,\alpha} I^{(1)}_{ab,\alpha} \times_\beta I^{(1)}_{a,\beta}} \\
&\times \prod_a \left[\prod_{b,\alpha} \left(\Phi_{ab,\alpha}^{\otimes n^{(2)}_{ab,\alpha}}\right)^{I^{(2)}_{ab,\alpha}}_{J^{(2)}_{ab,\alpha}}\right] \otimes \left[\prod_\beta \left(Q_{a,\beta}^{\otimes n^{(2)}_{a,\beta}}\right)^{I^{(2)}_{a,\beta}}_{\boldsymbol{s}^{(2)}_{a,\beta}}\right] \otimes \left[\prod_\gamma \left(\bar{Q}_{a,\gamma}^{\otimes \bar{n}^{(2)}_{a,\gamma}}\right)^{\bar{\boldsymbol{s}}^{(2)}_{a,\gamma}}_{\bar{J}^{(2)}_{a,\gamma}}\right] \left(\sigma^{(2)}_a\right)^{\times_{b,\alpha} J^{(2)}_{ba,\alpha} \times_\gamma \bar{J}^{(2)}_{a,\gamma}}_{\times_{b,\alpha} I^{(2)}_{ab,\alpha} \times_\beta I^{(2)}_{a,\beta}} \\
&= \prod_a \left[\prod_{b,\alpha} \left(\Phi_{ab,\alpha}^{\otimes\left(n^{(1)}_{ab,\alpha}+n^{(2)}_{ab,\alpha}\right)}\right)^{I^{(1)}_{ab,\alpha}\times I^{(2)}_{ab,\alpha}}_{J^{(1)}_{ab,\alpha}\times J^{(2)}_{ab,\alpha}}\right] \left[\prod_\beta \left(Q_{a,\beta}^{\otimes\left(n^{(1)}_{a,\beta}+n^{(2)}_{a,\beta}\right)}\right)^{I^{(1)}_{a,\beta}\times I^{(2)}_{a,\beta}}_{\boldsymbol{s}^{(1)}_{a,\beta}\times \boldsymbol{s}^{(2)}_{a,\beta}}\right] \\
&\qquad \otimes \left[\prod_\gamma \left(\bar{Q}_{a,\gamma}^{\otimes\left(\bar{n}^{(1)}_{a,\gamma}+\bar{n}^{(1)}_{a,\gamma}\right)}\right)^{\bar{\boldsymbol{s}}^{(1)}_{a,\gamma}\times \bar{\boldsymbol{s}}^{(2)}_{a,\gamma}}_{\bar{J}^{(1)}_{a,\gamma}\times \bar{J}^{(2)}_{a,\gamma}}\right] \left(\sigma^{(1)}_a \times \sigma^{(2)}_a\right)^{\times_{b,\alpha} J^{(1)}_{ba,\alpha} \times_\gamma \bar{J}^{(1)}_{a,\gamma} \times_{b,\alpha} J^{(2)}_{ba,\alpha} \times_\gamma \bar{J}^{(2)}_{a,\gamma}}_{\times_{b,\alpha} I^{(1)}_{ab,\alpha} \times_\beta I^{(1)}_{a,\beta} \times_{b,\alpha} I^{(2)}_{ab,\alpha} \times_\beta I^{(2)}_{a,\beta}}
\end{aligned}
\tag{D.1}
$$

In the following we will continue to use the shorthand notation

$$|e_{i_1}, e_{i_2}, ..., e_{i_n}\rangle = |I\rangle\,, \qquad I = (i_1, i_2, ..., i_n)\,,$$

$$\langle e^{j_1}, e^{j_2}, ..., e^{j_n}| = \langle J|\,, \qquad J = (j_1, j_2, ..., j_n)$$

which was already introduced in the previous sections. For each gauge node a, let us now define the λ_{a-} and λ_{a+} permutations such that

$$\lambda_{a-}\left|\times_{b,\alpha} I^{(1)}_{ab,\alpha} \times_\beta I^{(1)}_{a,\beta} \times_{b,\alpha} I^{(2)}_{ab,\alpha} \times_\beta I^{(2)}_{a,\beta}\right\rangle = \left|\times_{b,\alpha}\left(I^{(1)}_{ab,\alpha} \times I^{(2)}_{ab,\alpha}\right) \times_\beta \left(I^{(1)}_{a,\beta} \times I^{(2)}_{a,\beta}\right)\right\rangle \tag{D.2}$$

and

$$\lambda^{-1}_{a+}\left|\times_{b,\alpha} J^{(1)}_{ba,\alpha} \times_\gamma \bar{J}^{(1)}_{a,\gamma} \times_{b,\alpha} J^{(2)}_{ba,\alpha} \times_\gamma \bar{J}^{(2)}_{a,\gamma}\right\rangle = \left|\times_{b,\alpha}\left(J^{(1)}_{ba,\alpha} \times J^{(2)}_{ba,\alpha}\right) \times_\gamma \left(\bar{J}^{(1)}_{a,\gamma} \times \bar{J}^{(2)}_{a,\gamma}\right)\right\rangle \tag{D.3}$$

These permutations have been chosen such that, when suitably acting on the $\sigma^{(1)}_a \times \sigma^{(2)}_a$ component of (D.1), the resulting term has the right index structure to match the index structure of the associated field component,

$$\left[\prod_{b,\alpha} \Phi^{\otimes\left(n^{(1)}_{ab,\alpha}+n^{(2)}_{ab,\alpha}\right)}_{ab,\alpha}\right]\left[\prod_{\beta} Q^{\otimes\left(n^{(1)}_{a,\beta}+n^{(2)}_{a,\beta}\right)}_{a,\beta}\right]\left[\prod_{\gamma} \bar{Q}^{\otimes\left(\bar{n}^{(1)}_{a,\gamma}+\bar{n}^{(1)}_{a,\gamma}\right)}_{a,\gamma}\right]^{\times_{b,\alpha}\left(I^{(1)}_{ab,\alpha}\times I^{(2)}_{ab,\alpha}\right)\times_\beta\left(I^{(1)}_{a,\beta}\times I^{(2)}_{a,\beta}\right)}_{\times_{b,\alpha}\left(J^{(1)}_{ba,\alpha}\times J^{(2)}_{ba,\alpha}\right)\times_\gamma\left(\bar{J}^{(1)}_{a,\gamma}\times \bar{J}^{(2)}_{a,\gamma}\right)} \tag{D.4}$$

We have in fact

$$\left(\sigma^{(1)}_a \times \sigma^{(2)}_a\right)^{\times_{b,\alpha} J^{(1)}_{ba,\alpha}\times_\gamma \bar{J}^{(1)}_{a,\gamma}\times_{b,\alpha} J^{(2)}_{ba,\alpha}\times_\gamma \bar{J}^{(2)}_{a,\gamma}}_{\times_{b,\alpha} I^{(1)}_{ab,\alpha}\times_\beta I^{(1)}_{a,\beta}\times_{b,\alpha} I^{(2)}_{ab,\alpha}\times_\beta I^{(2)}_{a,\beta}} = \left(\lambda^{-1}_{a+}\left(\sigma^{(1)}_a \times \sigma^{(2)}_a\right)\lambda^{-1}_{a-}\right)^{\times_{b,\alpha}\left(J^{(1)}_{ba,\alpha}\times J^{(2)}_{ba,\alpha}\right)\times_\gamma\left(\bar{J}^{(1)}_{a,\gamma}\times \bar{J}^{(2)}_{a,\gamma}\right)}_{\times_{b,\alpha}\left(I^{(1)}_{ab,\alpha}\times I^{(2)}_{ab,\alpha}\right)\times_\beta\left(I^{(1)}_{a,\beta}\times I^{(2)}_{a,\beta}\right)} \tag{D.5}$$

The purpose of λ_{a-} and λ_{a+} is therefore to change the embedding into $[n_a]$ corresponding to the ordering of the upper (lower) $U(N_a)$ indices of the fields coming into (departing from) node a, eq. (2.9) (eq. (2.10)). It can be seen that the index structure of the RHS of (D.5) now matches the one in (D.4). Inserting (D.5) into (D.1), we then obtain

$$\mathcal{O}_{\mathcal{Q}}(\vec{n}_1, \vec{s}^{(1)}, \vec{\sigma}^{(1)})\,\mathcal{O}_{\mathcal{Q}}(\vec{n}_2, \vec{s}^{(2)}, \vec{\sigma}^{(2)}) = \mathcal{O}_{\mathcal{Q}}(\vec{n}_{1+2}, \vec{s}^{(1)} \cup \vec{s}^{(2)}, \vec{\lambda}^{-1}_+\left(\vec{\sigma}^{(1)} \times \vec{\sigma}^{(2)}\right)\vec{\lambda}^{-1}_-) \tag{D.6}$$

where

$$\vec{n}_{1+2} = \cup_a\left\{\cup_{b,\alpha}\{n^{(1)}_{ab,\alpha}, n^{(2)}_{ab,\alpha}\}; \cup_\beta\{n^{(1)}_{a,\beta}, n^{(2)}_{a,\beta}\}; \cup_\gamma\{\bar{n}^{(1)}_{a,\gamma}, \bar{n}^{(2)}_{a,\gamma}\}\right\}\,,$$

$$\vec{s}^{(1)} \cup \vec{s}^{(2)} = \cup_a\{\cup_\beta\{\boldsymbol{s}^{(1)}_{a,\beta},\, \boldsymbol{s}^{(2)}_{a,\beta}\}\,;\, \cup_\gamma\{\bar{\boldsymbol{s}}^{(1)}_{a,\gamma},\, \bar{\boldsymbol{s}}^{(2)}_{a,\gamma}\}\}\,, \tag{D.7}$$

$$\vec{\lambda}_+^{-1}\left(\vec{\sigma}^{(1)}\times\vec{\sigma}^{(2)}\right)\vec{\lambda}_-^{-1}=\cup_a\{\lambda_{a+}^{-1}\left(\sigma_a^{(1)}\times\sigma_a^{(2)}\right)\lambda_{a-}^{-1}\}$$

2) Using the inversion formula

In this step we are going to use eq. (D.6) to write a first expression for the $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$ coefficients. Let us start form the product $\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(1)})\,\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(2)})$, that we expand as

$$\begin{aligned}&\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(1)})\,\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(2)})\\&\quad=\sum_{\vec{s}^{(1)},\vec{s}^{(2)}}\sum_{\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}}\chi_Q(\boldsymbol{L}^{(1)},\vec{s}^{(1)},\vec{\sigma}^{(1)})\,\chi_Q(\boldsymbol{L}^{(2)},\vec{s}^{(2)},\vec{\sigma}^{(2)})\,\mathcal{O}_{\mathcal{Q}}(\vec{n}_1,\vec{s}^{(1)},\vec{\sigma}^{(1)})\,\mathcal{O}_{\mathcal{Q}}(\vec{n}_2,\vec{s}^{(2)},\vec{\sigma}^{(2)})\end{aligned}\tag{D.8}$$

Plugging eq. (D.6) into this equation we get

$$\begin{aligned}\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(1)})\,\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(2)})=&\sum_{\vec{s}^{(1)},\vec{s}^{(2)}}\sum_{\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}}\chi_Q(\boldsymbol{L}^{(1)},\vec{s}^{(1)},\vec{\sigma}^{(1)})\,\chi_Q(\boldsymbol{L}^{(2)},\vec{s}^{(2)},\vec{\sigma}^{(2)})\\&\times\mathcal{O}_{\mathcal{Q}}(\vec{n}_{1+2},\vec{s}^{(1)}\cup\vec{s}^{(2)},\vec{\lambda}_+^{-1}\left(\vec{\sigma}^{(1)}\times\vec{\sigma}^{(2)}\right)\vec{\lambda}_-^{-1})\end{aligned}\tag{D.9}$$

We now use the inversion formula (3.32) to get

$$\begin{aligned}\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(1)})\,\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(2)})=\sum_{\boldsymbol{L}^{(3)}}\Bigg\{&\sum_{\vec{s}^{(1)},\vec{s}^{(2)}}\sum_{\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}}\chi_Q(\boldsymbol{L}^{(1)},\vec{s}^{(1)},\vec{\sigma}^{(1)})\chi_Q(\boldsymbol{L}^{(2)},\vec{s}^{(2)},\vec{\sigma}^{(2)})\\&\times\chi_{\mathcal{Q}}(\boldsymbol{L}^{(3)},\vec{s}^{(1)}\cup\vec{s}^{(2)},\vec{\lambda}_+^{-1}\left(\vec{\sigma}^{(1)}\times\vec{\sigma}^{(2)}\right)\vec{\lambda}_-^{-1})\Bigg\}\,\mathcal{O}_{\mathcal{Q}}(\boldsymbol{L}^{(3)})\end{aligned}\tag{D.10}$$

from which we obtain an expression for $G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}$:

$$\begin{aligned}&G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}}\\&=\sum_{\vec{s}^{(1)},\vec{s}^{(2)}}\sum_{\vec{\sigma}^{\,(1)},\vec{\sigma}^{\,(2)}}\chi_Q(\boldsymbol{L}^{(1)},\vec{s}^{(1)},\vec{\sigma}^{\,(1)})\chi_Q(\boldsymbol{L}^{(2)},\vec{s}^{(2)},\vec{\sigma}^{\,(2)})\chi_{\mathcal{Q}}(\boldsymbol{L}^{(3)},\vec{s}^{(1)}\cup\vec{s}^{(2)},\vec{\lambda}_+^{-1}\left(\vec{\sigma}^{\,(1)}\times\vec{\sigma}^{\,(2)}\right)\vec{\lambda}_-^{-1})\\&=c_{\boldsymbol{L}^{(1)}}\,c_{\boldsymbol{L}^{(2)}}\,c_{\boldsymbol{L}^{(3)}}\sum_{\vec{s}^{(1)},\vec{s}^{(2)}}\sum_{\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}}\prod_a D^{R_a^{(1)}}_{i_a^{(1)},j_a^{(1)}}(\sigma_a^{(1)})D^{R_a^{(2)}}_{i_a^{(2)},j_a^{(2)}}(\sigma_a^{(2)})D^{R_a^{(3)}}_{i_a^{(3)},j_a^{(3)}}(\lambda_{a+}^{-1}\left(\sigma_a^{(1)}\times\sigma_a^{(2)}\right)\lambda_{a-}^{-1})\\&\quad\times\left(\prod_{p=1}^{3}B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{j_a^{(p)}\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\;B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ba,\alpha}\cup_\gamma\bar{r}^{(p)}_{a,\gamma};\nu_a^{+(p)}}_{i_a^{(p)}\to\cup_{b,\alpha}l^{(p)}_{ba,\alpha}\cup_\gamma\bar{l}^{(p)}_{a,\gamma}}\right)\\&\quad\times\left(\prod_\beta C^{r^{(1)}_{a,\beta},S^{(1)}_{a,\beta},l^{(1)}_{a,\beta}}_{\boldsymbol{s}^{(1)}_{a,\beta}}\,C^{r^{(2)}_{a,\beta},S^{(2)}_{a,\beta},l^{(2)}_{a,\beta}}_{\boldsymbol{s}^{(2)}_{a,\beta}}\,C^{r^{(3)}_{a,\beta},S^{(3)}_{a,\beta},l^{(3)}_{a,\beta}}_{\boldsymbol{s}^{(1)}_{a,\beta}\cup\boldsymbol{s}^{(2)}_{a,\beta}}\right)\left(\prod_\gamma C^{\bar{\boldsymbol{s}}^{(1)}_{a,\gamma}}_{\bar{r}^{(1)}_{a,\gamma},\bar{S}^{(1)}_{a,\gamma},\bar{l}^{(1)}_{a,\gamma}}\,C^{\bar{\boldsymbol{s}}^{(2)}_{a,\gamma}}_{\bar{r}^{(2)}_{a,\gamma},\bar{S}^{(2)}_{a,\gamma},\bar{l}^{(2)}_{a,\gamma}}\,C^{\bar{\boldsymbol{s}}^{(1)}_{a,\gamma}\cup\bar{\boldsymbol{s}}^{(2)}_{a,\gamma}}_{\bar{r}^{(3)}_{a,\gamma},\bar{S}^{(3)}_{a,\gamma},\bar{l}^{(3)}_{a,\gamma}}\right)\end{aligned}\tag{D.11}$$

3) Fusing of gauge edges

At this stage the chiral ring structure constants are given as a product of three definite quantities, *i.e.* three quiver characters. We now proceed to fuse together their gauge edges, by using standard representation theory identities. Let us then focus on the permutation dependent piece of eq. (D.11), namely

$$
\begin{aligned}
&\sum_{\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}} \prod_a D^{R_a^{(1)}}_{i_a^{(1)},j_a^{(1)}}(\sigma_a^{(1)}) D^{R_a^{(2)}}_{i_a^{(2)},j_a^{(2)}}(\sigma_a^{(2)}) D^{R_a^{(3)}}_{i_a^{(3)},j_a^{(3)}}(\lambda_{a+}^{-1}\left(\sigma_a^{(1)}\times\sigma_a^{(2)}\right)\lambda_{a-}^{-1}) \\
&\quad = \sum_{\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}} \prod_a D^{R_a^{(1)}}_{i_a^{(1)},j_a^{(1)}}(\sigma_a^{(1)}) D^{R_a^{(2)}}_{i_a^{(2)},j_a^{(2)}}(\sigma_a^{(2)}) D^{R_a^{(3)}}_{i_a^{(3)},h_a^{(3)}}(\lambda_{a+}^{-1}) D^{R_a^{(3)}}_{h_a^{(3)},g_a^{(3)}}(\sigma_a^{(1)}\times\sigma_a^{(2)}) D^{R_a^{(3)}}_{g_a^{(3)},j_a^{(3)}}(\lambda_{a-}^{-1})
\end{aligned}
\tag{D.12}
$$

Using the identity

$$
D^R_{ij}(\sigma^{(1)}\times\sigma^{(2)}) = \sum_{r_1,r_2,\,\mu} B^{R\to r_1,r_2;\mu}_{i\to l_1,l_2}\, B^{R\to r_1,r_2;\mu}_{j\to k_1,k_2} D^{r_1}_{l_1,k_1}(\sigma^{(1)})\, D^{r_2}_{l_2,k_2}(\sigma^{(2)}) \tag{D.13}
$$

we can write

$$
\begin{aligned}
&\sum_{\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}} \prod_a D^{R_a^{(1)}}_{i_a^{(1)},j_a^{(1)}}(\sigma_a^{(1)}) D^{R_a^{(2)}}_{i_a^{(2)},j_a^{(2)}}(\sigma_a^{(2)}) D^{R_a^{(3)}}_{i_a^{(3)},h_a^{(3)}}(\lambda_{a+}^{-1}) D^{R_a^{(3)}}_{h_a^{(3)},g_a^{(3)}}(\sigma_a^{(1)}\times\sigma_a^{(2)}) D^{R_a^{(3)}}_{g_a^{(3)},j_a^{(3)}}(\lambda_{a-}^{-1}) \\
&\quad = \sum_{\vec{\sigma}^{(1)},\vec{\sigma}^{(2)}} \prod_a D^{R_a^{(1)}}_{i_a^{(1)},j_a^{(1)}}(\sigma_a^{(1)}) D^{R_a^{(2)}}_{i_a^{(2)},j_a^{(2)}}(\sigma_a^{(2)})\, D^{R_a^{(3)}}_{i_a^{(3)},h_a^{(3)}}(\lambda_{a+}^{-1}) D^{R_a^{(3)}}_{g_a^{(3)},j_a^{(3)}}(\lambda_{a-}^{-1}) \\
&\quad \times \left(\sum_{S_a^{(1)},S_a^{(2)},\,\mu_a} B^{R_a^{(3)}\to S_a^{(1)},S_a^{(2)};\mu_a}_{h_a^{(3)}\to l_a^{(1)},l_a^{(2)}}\, B^{R_a^{(3)}\to S_a^{(1)},S_a^{(2)};\mu_a}_{g_a^{(3)}\to k_a^{(1)},k_a^{(2)}}\;\; D^{S_a^{(1)}}_{l_a^{(1)},k_a^{(1)}}(\sigma_a^{(1)}) D^{S_a^{(2)}}_{l_a^{(2)},k_a^{(2)}}(\sigma_a^{(2)}) \right) \\
&\quad = \prod_a \frac{n_a^{(1)}!\, n_a^{(2)}!}{d(R_a^{(1)})\, d(R_a^{(2)})}\, D^{R_a^{(3)}}_{i_a^{(3)},h_a^{(3)}}(\lambda_{a+}^{-1}) D^{R_a^{(3)}}_{g_a^{(3)},j_a^{(3)}}(\lambda_{a-}^{-1}) \\
&\qquad \times \sum_{S_a^{(1)},S_a^{(2)},\,\mu_a} B^{R_a^{(3)}\to S_a^{(1)},S_a^{(2)};\mu_a}_{h_a^{(3)}\to l_a^{(1)},l_a^{(2)}}\, B^{R_a^{(3)}\to S_a^{(1)},S_a^{(2)};\mu_a}_{g_a^{(3)}\to k_a^{(1)},k_a^{(2)}} \prod_{q=1}^{2} \delta_{R_a^{(q)},S_a^{(q)}}\, \delta_{i_a^{(q)},l_a^{(q)}}\, \delta_{j_a^{(q)},k_a^{(q)}} \\
&\quad = \prod_a \frac{n_a^{(1)}!\, n_a^{(2)}!}{d(R_a^{(1)})\, d(R_a^{(2)})} \\
&\qquad\quad \times \sum_{\mu_a} \left(D^{R_a^{(3)}}_{i_a^{(3)},h_a^{(3)}}(\lambda_{a+}^{-1})\, B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{h_a^{(3)}\to i_a^{(1)},i_a^{(2)}} \right) \left(D^{R_a^{(3)}}_{j_a^{(3)},g_a^{(3)}}(\lambda_{a-}) B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{g_a^{(3)}\to j_a^{(1)},j_a^{(2)}} \right)
\end{aligned}
\tag{D.14}
$$

where in the second equality we used

$$\sum_{\sigma\in S_n} D^R_{ij}(\sigma)D^S_{kl}(\sigma) = \frac{n!}{d(R)}\delta_{R,S}\,\delta_{i,k}\,\delta_{j,l} \tag{D.15}$$

It is important to stress that all the steps that we will be describing in this appendix can be also interpreted diagrammatically. For example, (D.14) can be understood trough the diagram in Fig. 24.

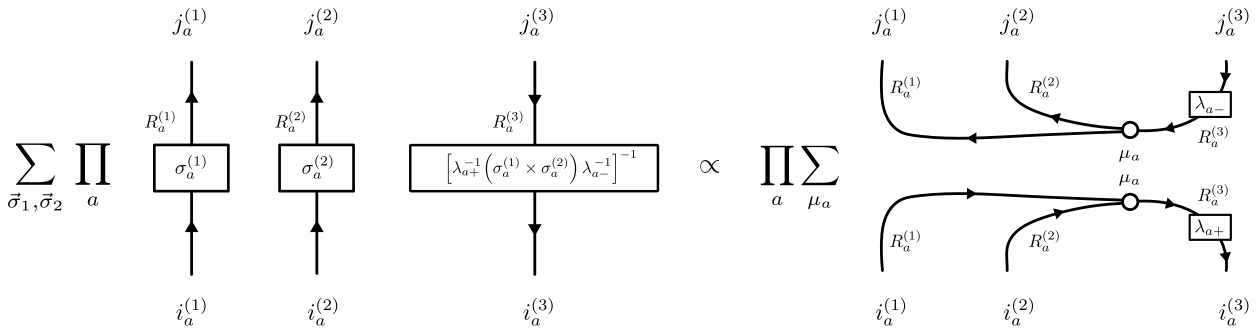

Figure 24: Diagrammatic interpretation of eq. (D.14).

Similar pictures can be drawn for all the following steps. In equation (D.14) (or equivalently, in Fig. 24) we see the emergence of the first of the selection rules already anticipated in section 4.2. This selection rule is expressed by the terms

$$B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{h_a^{(3)}\to i_a^{(1)},i_a^{(2)}}\,,\qquad B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{g_a^{(3)}\to j_a^{(1)},j_a^{(2)}} \tag{D.16}$$

These coefficients are non-zero only if the restriction of the $S_{n_1+n_2}$ representation $R_a^{(3)}$ to $S_{n_1}\times S_{n_2}$ contains the representation $R_a^{(1)}\otimes R_a^{(2)}$, $\forall\, a$.

4) Fusing of the quark/antiquark edges

In this step we will perform the fusing of the edges corresponding to the fundamental/anti-fundamental matter fields. This involves summing over the quark/antiquark states $\boldsymbol{s}^{(1,2)}_{a,\beta}$ and $\bar{\boldsymbol{s}}^{(1,2)}_{a,\gamma}$. Let us then turn to the Clebsch-Gordan parts of equation (D.11), that is

$$\sum_{\boldsymbol{s}^{(1)}_{a,\beta},\boldsymbol{s}^{(2)}_{a,\beta}} C^{r^{(1)}_{a,\beta},S^{(1)}_{a,\beta},l^{(1)}_{a,\beta}}_{\boldsymbol{s}^{(1)}_{a,\beta}}\, C^{r^{(2)}_{a,\beta},S^{(2)}_{a,\beta},l^{(2)}_{a,\beta}}_{\boldsymbol{s}^{(2)}_{a,\beta}}\, C^{r^{(3)}_{a,\beta},S^{(3)}_{a,\beta},l^{(3)}_{a,\beta}}_{\boldsymbol{s}^{(1)}_{a,\beta}\cup\boldsymbol{s}^{(2)}_{a,\beta}} \tag{D.17}$$

and

$$\sum_{\bar{\boldsymbol{s}}^{(1)}_{a,\gamma},\bar{\boldsymbol{s}}^{(2)}_{a,\gamma}} C^{\bar{\boldsymbol{s}}^{(1)}_{a,\gamma}}_{\bar{r}^{(1)}_{a,\gamma},\bar{S}^{(1)}_{a,\gamma},\bar{l}^{(1)}_{a,\gamma}}\, C^{\bar{\boldsymbol{s}}^{(2)}_{a,\gamma}}_{\bar{r}^{(2)}_{a,\gamma},\bar{S}^{(2)}_{a,\gamma},\bar{l}^{(2)}_{a,\gamma}}\, C^{\bar{\boldsymbol{s}}^{(1)}_{a,\gamma}\cup\bar{\boldsymbol{s}}^{(2)}_{a,\gamma}}_{\bar{r}^{(3)}_{a,\gamma},\bar{S}^{(3)}_{a,\gamma},\bar{l}^{(3)}_{a,\gamma}} \tag{D.18}$$

Consider for example the former. Aiming at simplifying notation, we rewrite it here dropping the a, β labels:

$$\sum_{\boldsymbol{s}^{(1)},\,\boldsymbol{s}^{(2)}} C^{r^{(1)},S^{(1)},l^{(1)}}_{\boldsymbol{s}^{(1)}}\, C^{r^{(2)},S^{(2)},l^{(2)}}_{\boldsymbol{s}^{(2)}}\, C^{r^{(3)},S^{(3)},l^{(3)}}_{\boldsymbol{s}^{(1)}\cup\boldsymbol{s}^{(2)}} \tag{D.19}$$

We can expand this quantity as

$$\begin{aligned}
&\sum_{\boldsymbol{s}^{(1)},\,\boldsymbol{s}^{(2)}} C^{r^{(1)},S^{(1)},l^{(1)}}_{\boldsymbol{s}^{(1)}}\, C^{r^{(2)},S^{(2)},l^{(2)}}_{\boldsymbol{s}^{(2)}}\, C^{r^{(3)},S^{(3)},l^{(3)}}_{\boldsymbol{s}^{(1)}\cup\boldsymbol{s}^{(2)}} \\
&\qquad = \sum_{\boldsymbol{s}^{(1)},\,\boldsymbol{s}^{(2)}} \langle r^{(1)}, S^{(1)}, l^{(1)}|\boldsymbol{s}^{(1)}\rangle\, \langle r^{(2)}, S^{(2)}, l^{(2)}|\boldsymbol{s}^{(2)}\rangle\, \langle r^{(3)}, S^{(3)}, l^{(3)}|\boldsymbol{s}^{(1)}\cup\boldsymbol{s}^{(2)}\rangle \\
&\qquad = \sum_{\boldsymbol{s}^{(1)},\,\boldsymbol{s}^{(2)}} \big(\langle r^{(1)}, S^{(1)}, l^{(1)}|\otimes\langle r^{(2)}, S^{(2)}, l^{(2)}|\big)\, \big(|\boldsymbol{s}^{(1)}\rangle\otimes|\boldsymbol{s}^{(2)}\rangle\big)\, \langle r^{(3)}, S^{(3)}, l^{(3)}|\boldsymbol{s}^{(1)}\cup\boldsymbol{s}^{(2)}\rangle \\
&\qquad = \langle r^{(1)}, S^{(1)}, l^{(1)}|\otimes\langle r^{(2)}, S^{(2)}, l^{(2)}| \left(\sum_{\boldsymbol{s}^{(1)},\,\boldsymbol{s}^{(2)}} |\boldsymbol{s}^{(1)}\rangle\otimes|\boldsymbol{s}^{(2)}\rangle\, \langle\boldsymbol{s}^{(1)}|\otimes\langle\boldsymbol{s}^{(2)}|\right) |r^{(3)}, S^{(3)}, l^{(3)}\rangle \\
&\qquad = \big(\langle r^{(1)}, S^{(1)}, l^{(1)}|\otimes\langle r^{(2)}, S^{(2)}, l^{(2)}|\big)\, |r^{(3)}, S^{(3)}, l^{(3)}\rangle \\
&\qquad = \langle\{r^{(1)}, r^{(2)}\},\, \{S^{(1)}, S^{(2)}\},\, \{l^{(1)}, l^{(2)}\}|r^{(3)}, S^{(3)}, l^{(3)}\rangle
\end{aligned} \tag{D.20}$$

Since the generic state $|r,S,l\rangle \in V^{S_n}_r \otimes V^{U(F)}_r$ is by definition the tensor product $|r,S,l\rangle = |r,S\rangle\otimes|r,l\rangle$, we may separately decompose the two states $|r^{(3)}, S^{(3)}, l^{(3)}\rangle$ and $|\{r^{(1)}, r^{(2)}\}$, $\{S^{(1)}, S^{(2)}\}$, $\{l^{(1)}, l^{(2)}\}\rangle$ as follows. We factorise the former according to the decomposition (3.3), which in this case reads

$$V^{S_{n^{(3)}}}_{r^{(3)}} = \bigoplus_{u^{(1)}\vdash n^{(1)}} \bigoplus_{u^{(2)}\vdash n^{(2)}} \left(V^{S_{n^{(1)}}}_{u^{(1)}} \otimes V^{S_{n^{(2)}}}_{u^{(2)}}\right) \otimes V^{u^{(1)},u^{(2)}}_{r^{(3)}} \tag{D.21}$$

We then write

$$\begin{aligned}
\big|r^{(3)}, S^{(3)}, l^{(3)}\big\rangle &= \big|r^{(3)}, S^{(3)}\big\rangle\otimes\big|r^{(3)}, l^{(3)}\big\rangle \\
&= \sum_{u^{(1)},\,u^{(2)}}\ \sum_{p^{(1)},\,p^{(2)}}\ \sum_{\nu} B^{r^{(3)}\to u^{(1)},u^{(2)};\nu}_{l^{(3)}\to p^{(1)},p^{(2)}}\, \big|r^{(3)}, S^{(3)}\big\rangle\otimes\big|\{u^{(1)}, u^{(2)}\}, \{p^{(1)}, p^{(2)}\}; \nu\big\rangle \\
&= \sum_{u^{(1)},\,u^{(2)}}\ \sum_{p^{(1)},\,p^{(2)}}\ \sum_{\nu} B^{r^{(3)}\to u^{(1)},u^{(2)};\nu}_{l^{(3)}\to p^{(1)},p^{(2)}}\, \big|\{u^{(1)}, u^{(2)}, r^{(3)}\},\, \{p^{(1)}, p^{(2)}\},\, S^{(3)}\,; \nu\big\rangle
\end{aligned} \tag{D.22}$$

For the latter we use instead the the unitary group decomposition (3.9), which in this case takes the explicit form

$$V^{U(F)}_{r^{(1)}} \otimes V^{U(F)}_{r^{(2)}} = \bigoplus_{u^{(3)} \vdash n^{(3)}} V^{U(F)}_{u^{(3)}} \otimes V^{r^{(1)},r^{(2)}}_{u^{(3)}}\,, \qquad\qquad n^{(3)} = n^{(1)} + n^{(2)} \tag{D.23}$$

We therefore have

$$\begin{aligned}
&\big|\{r^{(1)}, r^{(2)}\},\, \{S^{(1)}, S^{(2)}\},\, \{l^{(1)}, l^{(2)}\}\big\rangle = \big|\{r^{(1)}, r^{(2)}\},\, \{S^{(1)}, S^{(2)}\}\big\rangle \otimes \big|\{r^{(1)}, r^{(2)}\},\, \{l^{(1)}, l^{(2)}\}\big\rangle \\
&\quad = \sum_{u^{(3)}} \sum_{P^{(3)}} \sum_{\tilde{\nu}} C^{u^{(3)};\tilde{\nu}\to r^{(1)},r^{(2)}}_{P^{(3)}\to S^{(1)},S^{(2)}} \,\big|u^{(3)},\, P^{(3)}\,;\tilde{\nu}\big\rangle \otimes \big|\{r^{(1)}, r^{(2)}\},\, \{l^{(1)}, l^{(2)}\}\big\rangle \\
&\quad = \sum_{u^{(3)}} \sum_{P^{(3)}} \sum_{\tilde{\nu}} C^{u^{(3)};\tilde{\nu}\to r^{(1)},r^{(2)}}_{P^{(3)}\to S^{(1)},S^{(2)}} \,\big|\{r^{(1)}, r^{(2)}, u^{(3)}\},\, \{l^{(1)}, l^{(2)}\},\, P^{(3)}\,;\tilde{\nu}\big\rangle
\end{aligned} \tag{D.24}$$

The vector spaces $V^{u^{(1)},u^{(2)}}_{r^{(3)}}$ in (D.21) and $V^{r^{(1)},r^{(2)}}_{u^{(3)}}$ in (D.23) are both multiplicity vector spaces. We recall that $\dim(V^{r^{(1)},r^{(2)}}_{r^{(3)}}) = g(r^{(1)}, r^{(2)}; r^{(3)})$, where g is the Littlewood-Richardson coefficient. Notice that both the states on the far RHSs of (D.22) and (D.24) live in the tensor space $\mathcal{W}$, where

$$\mathcal{W} = V^{S_{n^{(1)}}}_{r^{(1)}} \otimes V^{S_{n^{(2)}}}_{r^{(2)}} \otimes V^{U(F)}_{r^{(3)}} \otimes V^{r^{(1)},r^{(2)}}_{r^{(3)}} \tag{D.25}$$

Taking the scalar product of (D.22) and (D.23) then gives

$$\begin{aligned}
&\langle \{r^{(1)}, r^{(2)}\}, \{S^{(1)}, S^{(2)}\},\, \{l^{(1)}, l^{(2)}\} | r^{(3)}, S^{(3)}, l^{(3)}\rangle \\
&\qquad = \left(\prod_{k=1}^{3} \sum_{u^{(k)}}\right) \left(\prod_{q=1}^{2} \sum_{p^{(q)}}\right) \sum_{P^{(3)}} \sum_{\nu,\,\tilde{\nu}} \; B^{r^{(3)}\to u^{(1)},u^{(2)};\nu}_{l^{(3)}\to p^{(1)},p^{(2)}} \; C^{u^{(3)};\tilde{\nu}\to r^{(1)},r^{(2)}}_{P^{(3)}\to S^{(1)},S^{(2)}} \\
&\qquad\qquad \times \left(\prod_{k=1}^{3} \delta_{r^{(k)},\,u^{(k)}}\right) \left(\prod_{q=1}^{2} \delta_{l^{(q)},\,p^{(q)}}\right) \delta_{S^{(3)},\,P^{(3)}}\, \delta_{\nu,\,\tilde{\nu}} \\
&\qquad = \sum_{\nu} \; B^{r^{(3)}\to r^{(1)},r^{(2)};\nu}_{l^{(3)}\to l^{(1)},l^{(2)}} \; C^{r^{(3)};\nu\to r^{(1)},r^{(2)}}_{S^{(3)}\to S^{(1)},S^{(2)}}
\end{aligned} \tag{D.26}$$

We conclude that

$$\sum_{\boldsymbol{s}^{(1)},\,\boldsymbol{s}^{(2)}} C^{r^{(1)},S^{(1)},l^{(1)}}_{\boldsymbol{s}^{(1)}} \; C^{r^{(2)},S^{(2)},l^{(2)}}_{\boldsymbol{s}^{(2)}} \; C^{r^{(3)},S^{(3)},l^{(3)}}_{\boldsymbol{s}^{(1)}\cup\boldsymbol{s}^{(2)}} = \sum_{\nu} \; B^{r^{(3)}\to r^{(1)},r^{(2)};\nu}_{l^{(3)}\to l^{(1)},l^{(2)}} \; C^{r^{(3)};\nu\to r^{(1)},r^{(2)}}_{S^{(3)}\to S^{(1)},S^{(2)}} \tag{D.27}$$

The diagrammatic interpretation of eq. (D.27) is drawn in Fig. 25.

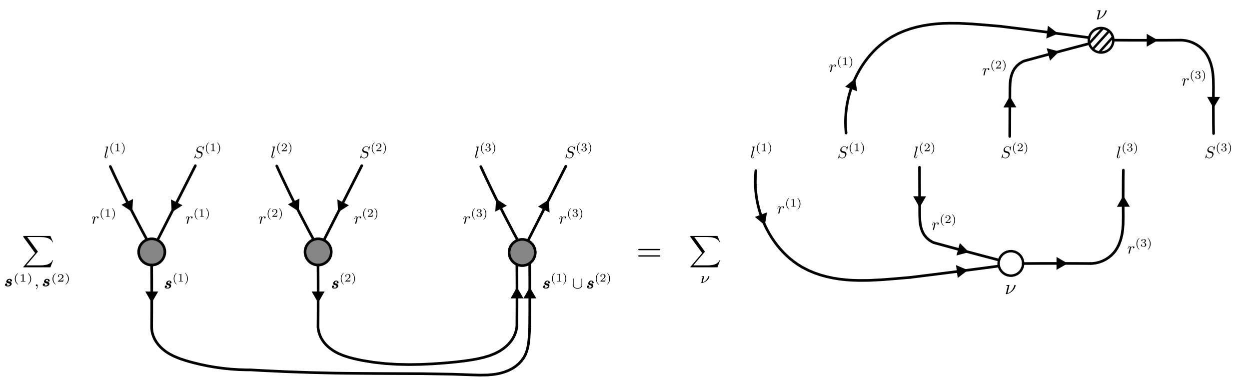

Figure 25: Diagrammatic interpretation of eq. (D.27).

Reintroducing the a,β notation, we then obtain

$$\sum_{\mathbf{s}^{(1)}_{a,\beta},\,\mathbf{s}^{(2)}_{a,\beta}} C^{r^{(1)}_{a,\beta},S^{(1)}_{a,\beta},l^{(1)}_{a,\beta}}_{\mathbf{s}^{(1)}_{a,\beta}}\, C^{r^{(2)}_{a,\beta},S^{(2)}_{a,\beta},l^{(2)}_{a,\beta}}_{\mathbf{s}^{(2)}_{a,\beta}}\, C^{r^{(3)}_{a,\beta},S^{(3)}_{a,\beta},l^{(3)}_{a,\beta}}_{\mathbf{s}^{(1)}_{a,\beta}\cup\mathbf{s}^{(2)}_{a,\beta}} = \sum_{\nu_{a,\beta}} B^{r^{(3)}_{a,\beta}\to r^{(1)}_{a,\beta},r^{(2)}_{a,\beta};\nu_{a,\beta}}_{l^{(3)}_{a,\beta}\to l^{(1)}_{a,\beta},l^{(2)}_{a,\beta}}\, C^{r^{(3)}_{a,\beta};\nu_{a,\beta}\to r^{(1)}_{a,\beta},r^{(2)}_{a,\beta}}_{S^{(3)}_{a,\beta}\to S^{(1)}_{a,\beta},S^{(2)}_{a,\beta}} \tag{D.28}$$

Similarly, we can show that for (D.18)

$$\sum_{\bar{\mathbf{s}}^{(1)}_{a,\gamma},\,\bar{\mathbf{s}}^{(2)}_{a,\gamma}} C^{\bar{\mathbf{s}}^{(1)}_{a,\gamma}}_{\bar r^{(1)}_{a,\gamma},\bar S^{(1)}_{a,\gamma},\bar l^{(1)}_{a,\gamma}}\, C^{\bar{\mathbf{s}}^{(2)}_{a,\gamma}}_{\bar r^{(2)}_{a,\gamma},\bar S^{(2)}_{a,\gamma},\bar l^{(2)}_{a,\gamma}}\, C^{\bar{\mathbf{s}}^{(1)}_{a,\gamma}\cup\bar{\mathbf{s}}^{(2)}_{a,\gamma}}_{\bar r^{(3)}_{a,\gamma},\bar S^{(3)}_{a,\gamma},\bar l^{(3)}_{a,\gamma}} = \sum_{\bar\nu_{a,\gamma}} B^{\bar r^{(3)}_{a,\gamma}\to\bar r^{(1)}_{a,\gamma},\bar r^{(2)}_{a,\gamma};\bar\nu_{a,\gamma}}_{\bar l^{(3)}_{a,\gamma}\to\bar l^{(1)}_{a,\gamma},\bar l^{(2)}_{a,\gamma}}\, C^{\bar r^{(3)}_{a,\gamma};\bar\nu_{a,\gamma}\to\bar r^{(1)}_{a,\gamma},\bar r^{(2)}_{a,\gamma}}_{\bar S^{(3)}_{a,\gamma}\to\bar S^{(1)}_{a,\gamma},\bar S^{(2)}_{a,\gamma}} \tag{D.29}$$

From eq. (D.28) and (D.29) (or equivalently by considering Fig. 25) one can see the manifestation of another selection rule for the holomorphic GIO ring structure constants. In particular, the coefficients $B^{r^{(3)}_{a,\beta}\to r^{(1)}_{a,\beta},r^{(2)}_{a,\beta};\nu_{a,\beta}}_{l^{(3)}_{a,\beta}\to l^{(1)}_{a,\beta},l^{(2)}_{a,\beta}}$ are identically zero if the restriction of the $S_{n^{(1)}_{a,\beta}+n^{(2)}_{a,\beta}}$ representation $r^{(3)}_{a,\beta}$ to $S_{n^{(1)}_{a,\beta}}\times S_{n^{(2)}_{a,\beta}}$ does not contain the representation $r^{(1)}_{a,\beta}\otimes r^{(2)}_{a,\beta}$. A similar condition holds for the coefficients $B^{\bar r^{(3)}_{a,\gamma}\to\bar r^{(1)}_{a,\gamma},\bar r^{(2)}_{a,\gamma};\bar\nu_{a,\gamma}}_{\bar l^{(3)}_{a,\gamma}\to\bar l^{(1)}_{a,\gamma},\bar l^{(2)}_{a,\gamma}}$.

Inserting eqs. (D.14), (D.28) and (D.29) into (D.11) we then get

$$\begin{aligned} G_{\mathbf{L}^{(1)},\mathbf{L}^{(2)},\mathbf{L}^{(3)}} = c_{\mathbf{L}^{(1)}}\, c_{\mathbf{L}^{(2)}}\, c_{\mathbf{L}^{(3)}} \prod_a \frac{n^{(1)}_a!\, n^{(2)}_a!}{d(R^{(1)}_a)\, d(R^{(2)}_a)} \\ \times \sum_{\mu_a} \left(D^{R^{(3)}_a}_{i^{(3)}_a,h^{(3)}_a}(\lambda^{-1}_{a+})\, B^{R^{(3)}_a\to R^{(1)}_a,R^{(2)}_a;\mu_a}_{h^{(3)}_a\to i^{(1)}_a,i^{(2)}_a}\right) \left(D^{R^{(3)}_a}_{j^{(3)}_a,g^{(3)}_a}(\lambda_{a-}) B^{R^{(3)}_a\to R^{(1)}_a,R^{(2)}_a;\mu_a}_{g^{(3)}_a\to j^{(1)}_a,j^{(2)}_a}\right) \\ \times \left(\prod_{p=1}^{3} B^{R^{(p)}_a\to\cup_{b,\alpha} r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu^{-(p)}_a}_{j^{(p)}_a\to\cup_{b,\alpha} l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\; B^{R^{(p)}_a\to\cup_{b,\alpha} r^{(p)}_{ba,\alpha}\cup_\gamma \bar r^{(p)}_{a,\gamma};\nu^{+(p)}_a}_{i^{(p)}_a\to\cup_{b,\alpha} l^{(p)}_{ba,\alpha}\cup_\gamma \bar l^{(p)}_{a,\gamma}} \right) \\ \times \left(\prod_\beta \sum_{\nu_{a,\beta}} B^{r^{(3)}_{a,\beta}\to r^{(1)}_{a,\beta},r^{(2)}_{a,\beta};\nu_{a,\beta}}_{l^{(3)}_{a,\beta}\to l^{(1)}_{a,\beta},l^{(2)}_{a,\beta}}\; C^{r^{(3)}_{a,\beta};\nu_{a,\beta}\to r^{(1)}_{a,\beta},r^{(2)}_{a,\beta}}_{S^{(3)}_{a,\beta}\to S^{(1)}_{a,\beta},S^{(2)}_{a,\beta}} \right) \end{aligned}$$

$$\times\left(\prod_{\gamma}\sum_{\bar{\nu}_{a,\gamma}} B^{\bar{r}^{(3)}_{a,\gamma}\to\bar{r}^{(1)}_{a,\gamma},\bar{r}^{(2)}_{a,\gamma};\bar{\nu}_{a,\gamma}}_{\bar{l}^{(3)}_{a,\gamma}\to\bar{l}^{(1)}_{a,\gamma},\bar{l}^{(2)}_{a,\gamma}}\; C^{\bar{r}^{(3)}_{a,\gamma};\bar{\nu}_{a,\gamma}\to\bar{r}^{(1)}_{a,\gamma},\bar{r}^{(2)}_{a,\gamma}}_{\bar{S}^{(3)}_{a,\gamma}\to\bar{S}^{(1)}_{a,\gamma},\bar{S}^{(2)}_{a,\gamma}}\right) \tag{D.30}$$

5) Fusing the bi-fundamental edges and factorising the $\pm$ nodes

The two tasks of this last step are to fuse the edges corresponding to the bi-fundamental fields and to factorise the positive and negative node of the split-node quiver. We start by considering the product

$$D^{R^{(3)}_a}_{j^{(3)}_a,g^{(3)}_a}(\lambda_{a-})\, B^{R^{(3)}_a\to R^{(1)}_a,R^{(2)}_a;\mu_a}_{g^{(3)}_a\to j^{(1)}_a,j^{(2)}_a}\left(\prod_{p=1}^{3} B^{R^{(p)}_a\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu^{-(p)}_a}_{j^{(p)}_a\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\right) \tag{D.31}$$

which appears in eq. (D.30). We want to decompose this term into a product of branching coefficients of the form $B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha}}_{l^{(3)}_{ab,\alpha}\to l^{(1)}_{ab,\alpha},l^{(2)}_{ab,\alpha}}$.

First we notice that the equivariance property of the branching coefficients

$$D^{R}_{k,j}(\times_a\gamma_a)\, B^{R\to\cup_a r_a;\nu_a}_{j\to\cup_a l_a} = \left(\prod_a D^{r_a}_{l'_a,l_a}(\gamma_a)\right) B^{R\to\cup_a r_a;\nu_a}_{k\to\cup_a l'_a} \tag{D.32}$$

also implies

$$B^{R\to\cup_a r_a;\nu_a}_{i\to\cup_a l_a} = D^{R}_{i,k}(\times_a\gamma_a)\left(\prod_a D^{r_a}_{l_a,l'_a}(\gamma_a)\right) B^{R\to\cup_a r_a;\nu_a}_{k\to\cup_a l'_a} \tag{D.33}$$

for a collection of permutations $\cup_a\{\gamma_a\in S_{n_a}\}$, where each r_a is a partition of the integer n_a. We can use this identity to write (D.31) as

$$\begin{aligned}
&D^{R^{(3)}_a}_{j^{(3)}_a,g^{(3)}_a}(\lambda_{a-})\, B^{R^{(3)}_a\to R^{(1)}_a,R^{(2)}_a;\mu_a}_{g^{(3)}_a\to j^{(1)}_a,j^{(2)}_a}\left(\prod_{p=1}^{3} B^{R^{(p)}_a\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu^{-(p)}_a}_{j^{(p)}_a\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\right)\\
&\quad=\Big(D^{R^{(1)}_a}_{j^{(1)}_a,k^{(1)}_a}(\times_{b,\alpha}\eta^{(1)}_{ab,\alpha}\times 1)\, D^{R^{(2)}_a}_{j^{(2)}_a,k^{(2)}_a}(\times_{b,\alpha}\eta^{(2)}_{ab,\alpha}\times 1)\\
&\qquad\times D^{R^{(3)}_a}_{j^{(3)}_a,k^{(3)}_a}\Big(\lambda_{a-}\Big(\times_{b,\alpha}\eta^{(1)}_{ab,\alpha}\times 1\times_{b,\alpha}\eta^{(2)}_{ab,\alpha}\times 1\Big)\Big)\; B^{R^{(3)}_a\to R^{(1)}_a,R^{(2)}_a;\mu_a}_{k^{(3)}_a\to k^{(1)}_a,k^{(2)}_a}\Big)\\
&\qquad\times\left(\prod_{p=1}^{3} B^{R^{(p)}_a\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu^{-(p)}_a}_{j^{(p)}_a\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\right)
\end{aligned} \tag{D.34}$$

where $(\times_{b,\alpha}\eta^{(p)}_{ab,\alpha}\times 1)\in S_{n^{(p)}_a}$ and $\eta^{(p)}_{ab,\alpha}\in S_{n^{(p)}_{ab,\alpha}}$, for $p=1,2$.

Let us now go back to the equation defining the λ_{a-} permutations, (D.2). It is easy to see that

$$\lambda_{a-}\left(\times_{b,\alpha}\eta^{(1)}_{ab,\alpha}\times_\beta \rho^{(1)}_{a,\beta}\times_{b,\alpha}\eta^{(2)}_{ab,\alpha}\times_\beta \rho^{(2)}_{a,\beta}\right)=\left[\times_{b,\alpha}\left(\eta^{(1)}_{ab,\alpha}\times\eta^{(2)}_{ab,\alpha}\right)\times_\beta\left(\rho^{(1)}_{a,\beta}\times\rho^{(2)}_{a,\beta}\right)\right]\lambda_{a-} \tag{D.35}$$

We can use this identity in (D.34) to get

$$\begin{aligned}
&D^{R^{(3)}_a}_{j^{(3)}_a,g^{(3)}_a}(\lambda_{a-})\,B^{R^{(3)}_a\to R^{(1)}_a,R^{(2)}_a;\mu_a}_{g^{(3)}_a\to j^{(1)}_a,j^{(2)}_a}\left(\prod_{p=1}^{3}B^{R^{(p)}_a\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu^{-(p)}_a}_{j^{(p)}_a\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\right)\\
&\quad=\Big(D^{R^{(1)}_a}_{j^{(1)}_a,k^{(1)}_a}(\times_{b,\alpha}\eta^{(1)}_{ab,\alpha}\times 1)\,D^{R^{(2)}_a}_{j^{(2)}_a,k^{(2)}_a}(\times_{b,\alpha}\eta^{(2)}_{ab,\alpha}\times 1)\\
&\qquad\times D^{R^{(3)}_a}_{j^{(3)}_a,k^{(3)}_a}\left(\left(\times_{b,\alpha}\left(\eta^{(1)}_{ab,\alpha}\times\eta^{(2)}_{ab,\alpha}\right)\times 1\right)\lambda_{a-}\right)\;B^{R^{(3)}_a\to R^{(1)}_a,R^{(2)}_a;\mu_a}_{k^{(3)}_a\to k^{(1)}_a,k^{(2)}_a}\Big)\\
&\qquad\times\left(\prod_{p=1}^{3}B^{R^{(p)}_a\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu^{-(p)}_a}_{j^{(p)}_a\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\right)
\end{aligned}\tag{D.36}$$

Next we use the identity (D.32) in eq. (D.36) as follows, for $p=1,2$:

$$\begin{aligned}
&D^{R^{(p)}_a}_{j^{(p)}_a,k^{(p)}_a}(\times_{b,\alpha}\eta^{(p)}_{ab,\alpha}\times 1)\,B^{R^{(p)}_a\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu^{-(p)}_a}_{j^{(p)}_a\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\\
&\quad=\left(\prod_{b,\alpha}D^{r^{(p)}_{ab,\alpha}}_{l^{(p)}_{ab,\alpha},q^{(p)}_{ab,\alpha}}\left(\eta^{(p)}_{ab,\alpha}\right)\right)\left(\prod_\beta\delta_{l^{(p)}_{a,\beta},q^{(p)}_{a,\beta}}\right)B^{R^{(p)}_a\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu^{-(p)}_a}_{k^{(p)}_a\to\cup_{b,\alpha}q^{(p)}_{ab,\alpha}\cup_\beta q^{(p)}_{a,\beta}}\\
&\quad=\left(\prod_{b,\alpha}D^{r^{(p)}_{ab,\alpha}}_{l^{(p)}_{ab,\alpha},q^{(p)}_{ab,\alpha}}\left(\eta^{(p)}_{ab,\alpha}\right)\right)B^{R^{(p)}_a\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu^{-(p)}_a}_{k^{(p)}_a\to\cup_{b,\alpha}q^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}
\end{aligned}\tag{D.37}$$

Similarly, we use (D.32) also for the term

$$\begin{aligned}
&D^{R^{(3)}_a}_{j^{(3)}_a,k^{(3)}_a}\left(\left(\times_{b,\alpha}\left(\eta^{(1)}_{ab,\alpha}\times\eta^{(2)}_{ab,\alpha}\right)\times 1\right)\lambda_{a-}\right)\,B^{R^{(3)}_a\to\cup_{b,\alpha}r^{(3)}_{ab,\alpha}\cup_\beta r^{(3)}_{a,\beta};\nu^{-(3)}_a}_{j^{(3)}_a\to\cup_{b,\alpha}l^{(3)}_{ab,\alpha}\cup_\beta l^{(3)}_{a,\beta}}\\
&=D^{R^{(3)}_a}_{g^{(3)}_a,k^{(3)}_a}(\lambda_{a-})\,D^{R^{(3)}_a}_{j^{(3)}_a,g^{(3)}_a}\left(\times_{b,\alpha}\left(\eta^{(1)}_{ab,\alpha}\times\eta^{(2)}_{ab,\alpha}\right)\times 1\right)\,B^{R^{(3)}_a\to\cup_{b,\alpha}r^{(3)}_{ab,\alpha}\cup_\beta r^{(3)}_{a,\beta};\nu^{-(3)}_a}_{j^{(3)}_a\to\cup_{b,\alpha}l^{(3)}_{ab,\alpha}\cup_\beta l^{(3)}_{a,\beta}}\\
&=D^{R^{(3)}_a}_{g^{(3)}_a,k^{(3)}_a}(\lambda_{a-})\left(\prod_{b,\alpha}D^{r^{(3)}_{ab,\alpha}}_{l^{(3)}_{ab,\alpha},q^{(3)}_{ab,\alpha}}\left(\eta^{(1)}_{ab,\alpha}\times\eta^{(2)}_{ab,\alpha}\right)\right)B^{R^{(3)}_a\to\cup_{b,\alpha}r^{(3)}_{ab,\alpha}\cup_\beta r^{(3)}_{a,\beta};\nu^{-(3)}_a}_{g^{(3)}_a\to\cup_{b,\alpha}q^{(3)}_{ab,\alpha}\cup_\beta l^{(3)}_{a,\beta}}
\end{aligned}\tag{D.38}$$

Putting these last equations together, we get to

$$
\begin{aligned}
& D^{R_a^{(3)}}_{j_a^{(3)},g_a^{(3)}}(\lambda_{a-})\, B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{g_a^{(3)}\to j_a^{(1)},j_a^{(2)}} \left(\prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{j_a^{(p)}\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\right) \\
&= \left(\prod_{b,\alpha} D^{r^{(1)}_{ab,\alpha}}_{l^{(1)}_{ab,\alpha},q^{(1)}_{ab,\alpha}}\left(\eta^{(1)}_{ab,\alpha}\right) D^{r^{(2)}_{ab,\alpha}}_{l^{(2)}_{ab,\alpha},q^{(2)}_{ab,\alpha}}\left(\eta^{(2)}_{ab,\alpha}\right) D^{r^{(3)}_{ab,\alpha}}_{l^{(3)}_{ab,\alpha},q^{(3)}_{ab,\alpha}}\left(\eta^{(1)}_{ab,\alpha}\times\eta^{(2)}_{ab,\alpha}\right)\right) B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{k_a^{(3)}\to k_a^{(1)},k_a^{(2)}} \\
&\quad\times\left[D^{R_a^{(3)}}_{g_a^{(3)},k_a^{(3)}}(\lambda_{a-})\, B^{R_a^{(3)}\to\cup_{b,\alpha}r^{(3)}_{ab,\alpha}\cup_\beta r^{(3)}_{a,\beta};\nu_a^{-(3)}}_{g_a^{(3)}\to\cup_{b,\alpha}q^{(3)}_{ab,\alpha}\cup_\beta l^{(3)}_{a,\beta}}\right]\prod_{p=1}^{2} B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{k_a^{(p)}\to\cup_{b,\alpha}q^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}
\end{aligned}
\tag{D.39}
$$

Notice that the quantity on the LHS above is independent of the permutations η. We can then sum over all possible permutations η on the RHS, provided we divide by the number of permutations themselves: we thus obtain

$$
\begin{aligned}
& D^{R_a^{(3)}}_{j_a^{(3)},g_a^{(3)}}(\lambda_{a-})\, B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{g_a^{(3)}\to j_a^{(1)},j_a^{(2)}} \left(\prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{j_a^{(p)}\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\right) \\
&= \frac{1}{\prod_{b,\alpha} n^{(1)}_{ab,\alpha}!\, n^{(2)}_{ab,\alpha}!}\left\{\sum_{\cup_{b,\alpha}\{\eta^{(1)}_{b,\alpha},\eta^{(2)}_{ab,\alpha}\}}\prod_{b,\alpha} D^{r^{(1)}_{ab,\alpha}}_{l^{(1)}_{ab,\alpha},q^{(1)}_{ab,\alpha}}\left(\eta^{(1)}_{ab,\alpha}\right) D^{r^{(2)}_{ab,\alpha}}_{l^{(2)}_{ab,\alpha},q^{(2)}_{ab,\alpha}}\left(\eta^{(2)}_{ab,\alpha}\right)\right. \\
&\qquad\left.\times\, D^{r^{(3)}_{ab,\alpha}}_{l^{(3)}_{ab,\alpha},q^{(3)}_{ab,\alpha}}\left(\eta^{(1)}_{ab,\alpha}\times\eta^{(2)}_{ab,\alpha}\right)\right\} B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{k_a^{(3)}\to k_a^{(1)},k_a^{(2)}} \\
&\quad\times\left[D^{R_a^{(3)}}_{g_a^{(3)},k_a^{(3)}}(\lambda_{a-})\, B^{R_a^{(3)}\to\cup_{b,\alpha}r^{(3)}_{ab,\alpha}\cup_\beta r^{(3)}_{a,\beta};\nu_a^{-(3)}}_{g_a^{(3)}\to\cup_{b,\alpha}q^{(3)}_{ab,\alpha}\cup_\beta l^{(3)}_{a,\beta}}\right]\prod_{p=1}^{2} B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{k_a^{(p)}\to\cup_{b,\alpha}q^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}
\end{aligned}
\tag{D.40}
$$

The quantity inside the curvy brackets above has the same structure of the far LHS of eq. (D.14). Performing similar steps to the ones presented in that equation we obtain, dropping the a, b, α notation for improved clarity

$$
\begin{aligned}
& \sum_{\eta^{(1)},\,\eta^{(2)}} D^{r^{(1)}}_{l^{(1)},q^{(1)}}\left(\eta^{(1)}\right) D^{r^{(2)}}_{l^{(2)},q^{(2)}}\left(\eta^{(2)}\right) D^{r^{(3)}}_{l^{(3)},q^{(3)}}\left(\eta^{(1)}\times\eta^{(2)}\right) \\
&\qquad = \frac{n^{(1)}!\, n^{(2)}!}{d(r^{(1)})\, d(r^{(2)})}\sum_{\nu} B^{r^{(3)}\to r^{(1)},r^{(2)};\nu}_{l^{(3)}\to l^{(1)},l^{(2)}}\; B^{r^{(3)}\to r^{(1)},r^{(2)};\nu}_{q^{(3)}\to q^{(1)},q^{(2)}}
\end{aligned}
\tag{D.41}
$$

Inserting this identity in (D.40) we get

$$D^{R_a^{(3)}}_{j_a^{(3)},g_a^{(3)}}(\lambda_{a-})\,B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{g_a^{(3)}\to j_a^{(1)},j_a^{(2)}}\left(\prod_{p=1}^{3}B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{j_a^{(p)}\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\right)$$

$$=\sum_{\cup_{b,\alpha}\{\nu_{ab,\alpha}\}}\left\{\prod_{b,\alpha}\frac{1}{d(r^{(1)}_{ab,\alpha})\,d(r^{(2)}_{ab,\alpha})}\;B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{l^{(3)}_{ab,\alpha}\to l^{(1)}_{ab,\alpha},l^{(2)}_{ab,\alpha}}\;B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{q^{(3)}_{ab,\alpha}\to q^{(1)}_{ab,\alpha},q^{(2)}_{ab,\alpha}}\right\}\;B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{k_a^{(3)}\to k_a^{(1)},k_a^{(2)}}$$

$$\times\left[D^{R_a^{(3)}}_{g_a^{(3)},k_a^{(3)}}(\lambda_{a-})\,B^{R_a^{(3)}\to\cup_{b,\alpha}r^{(3)}_{ab,\alpha}\cup_\beta r^{(3)}_{a,\beta};\nu_a^{-(3)}}_{g_a^{(3)}\to\cup_{b,\alpha}q^{(3)}_{ab,\alpha}\cup_\beta l^{(3)}_{a,\beta}}\right]\prod_{p=1}^{2}B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{k_a^{(p)}\to\cup_{b,\alpha}q^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\tag{D.42}$$

Using the substitutions $k_a^{(3)}\to t_a^{(3)}$ and $g_a^{(3)}\to k_a^{(3)}$ we can then write

$$D^{R_a^{(3)}}_{j_a^{(3)},g_a^{(3)}}(\lambda_{a-})\,B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{g_a^{(3)}\to j_a^{(1)},j_a^{(2)}}\left(\prod_{p=1}^{3}B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{j_a^{(p)}\to\cup_{b,\alpha}l^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\right)$$

$$=\sum_{\cup_{b,\alpha}\{\nu_{ab,\alpha}\}}\left\{\prod_{b,\alpha}\frac{1}{d(r^{(1)}_{ab,\alpha})\,d(r^{(2)}_{ab,\alpha})}\;B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{l^{(3)}_{ab,\alpha}\to l^{(1)}_{ab,\alpha},l^{(2)}_{ab,\alpha}}\;B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{q^{(3)}_{ab,\alpha}\to q^{(1)}_{ab,\alpha},q^{(2)}_{ab,\alpha}}\right\}$$

$$\times B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{t_a^{(3)}\to k_a^{(1)},k_a^{(2)}}\left[D^{R_a^{(3)}}_{k_a^{(3)},t_a^{(3)}}(\lambda_{a-})\,B^{R_a^{(3)}\to\cup_{b,\alpha}r^{(3)}_{ab,\alpha}\cup_\beta r^{(3)}_{a,\beta};\nu_a^{-(3)}}_{k_a^{(3)}\to\cup_{b,\alpha}q^{(3)}_{ab,\alpha}\cup_\beta l^{(3)}_{a,\beta}}\right]$$

$$\times\prod_{p=1}^{2}B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{k_a^{(p)}\to\cup_{b,\alpha}q^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}$$

$$=\sum_{\cup_{b,\alpha}\{\nu_{ab,\alpha}\}}\left\{\prod_{b,\alpha}\frac{1}{d(r^{(1)}_{ab,\alpha})\,d(r^{(2)}_{ab,\alpha})}\;B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{l^{(3)}_{ab,\alpha}\to l^{(1)}_{ab,\alpha},l^{(2)}_{ab,\alpha}}\;B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{q^{(3)}_{ab,\alpha}\to q^{(1)}_{ab,\alpha},q^{(2)}_{ab,\alpha}}\right\}$$

$$\times D^{R_a^{(3)}}_{k_a^{(3)},t_a^{(3)}}(\lambda_{a-})\;B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{t_a^{(3)}\to k_a^{(1)},k_a^{(2)}}\prod_{p=1}^{3}B^{R_a^{(p)}\to\cup_{b,\alpha}r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{k_a^{(p)}\to\cup_{b,\alpha}q^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}}\tag{D.43}$$

We see here the manifestation of the last selection rule, enforced by the branching coefficients $B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{l^{(3)}_{ab,\alpha}\to l^{(1)}_{ab,\alpha},l^{(2)}_{ab,\alpha}}$. These quantities are non zero only if the restriction of the $S_{n^{(1)}_{ab,\alpha}+n^{(2)}_{ab,\alpha}}$ representation $r^{(3)}_{ab,\alpha}$ to $S_{n^{(1)}_{ab,\alpha}}\times S_{n^{(2)}_{ab,\alpha}}$ contains the representation $r^{(1)}_{ab,\alpha}\otimes r^{(2)}_{ab,\alpha}$.

With the identity (D.43) we have achieved a factorisation of the branching coefficients over all the nodes of the quiver. Moreover, the positive and negative node of every split-node a are now disentangled. There are no symmetric group states $q^{(i)}_{ab,\alpha}$ $(i=1,2,3)$, associated with the negative node of the split-node a, that mix with symmetric group states $l^{(i)}_{ab,\alpha}$ $(i=1,2,3)$, associated with its positive node.

Plugging eq. (D.43) into (D.30), we get

$$
\begin{aligned}
G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}} = c_{\boldsymbol{L}^{(1)}}\, c_{\boldsymbol{L}^{(2)}}\, c_{\boldsymbol{L}^{(3)}} \prod_a \frac{n_a^{(1)}!\, n_a^{(2)}!}{d(R_a^{(1)})\, d(R_a^{(2)})} \frac{1}{\prod_{b,\alpha} d(r^{(1)}_{ab,\alpha})\, d(r^{(2)}_{ab,\alpha})} \sum_{\mu_a} & \\
\times \left(D^{R_a^{(3)}}_{k_a^{(3)},t_a^{(3)}}\left(\lambda_{a-}\right)\ B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{t_a^{(3)}\to k_a^{(1)},k_a^{(2)}} \prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha} r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{k_a^{(p)}\to\cup_{b,\alpha} q^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}} \right) & \\
\times \left(D^{R_a^{(3)}}_{i_a^{(3)},h_a^{(3)}}\left(\lambda^{-1}_{a+}\right)\ B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{h_a^{(3)}\to i_a^{(1)},i_a^{(2)}} \prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha} r^{(p)}_{ba,\alpha}\cup_\gamma \bar{r}^{(p)}_{a,\gamma};\nu_a^{+(p)}}_{i_a^{(p)}\to\cup_{b,\alpha} l^{(p)}_{ba,\alpha}\cup_\gamma \bar{l}^{(p)}_{a,\gamma}} \right) & \\
\times \left(\prod_{b,\alpha} \sum_{\nu_{ab,\alpha}} B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{l^{(3)}_{ab,\alpha}\to l^{(1)}_{ab,\alpha},l^{(2)}_{ab,\alpha}}\ B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{q^{(3)}_{ab,\alpha}\to q^{(1)}_{ab,\alpha},q^{(2)}_{ab,\alpha}} \right) & \\
\times \left(\prod_{\beta} \sum_{\nu_{a,\beta}}\ B^{r^{(3)}_{a,\beta}\to r^{(1)}_{a,\beta},r^{(2)}_{a,\beta};\nu_{a,\beta}}_{l^{(3)}_{a,\beta}\to l^{(1)}_{a,\beta},l^{(2)}_{a,\beta}}\ C^{r^{(3)}_{a,\beta};\nu_{a,\beta}\to r^{(1)}_{a,\beta},r^{(2)}_{a,\beta}}_{S^{(3)}_{a,\beta}\to S^{(1)}_{a,\beta},S^{(2)}_{a,\beta}} \right) & \\
\times \left(\prod_{\gamma} \sum_{\bar{\nu}_{a,\gamma}}\ B^{\bar{r}^{(3)}_{a,\gamma}\to \bar{r}^{(1)}_{a,\gamma},\bar{r}^{(2)}_{a,\gamma};\bar{\nu}_{a,\gamma}}_{\bar{l}^{(3)}_{a,\gamma}\to \bar{l}^{(1)}_{a,\gamma},\bar{l}^{(2)}_{a,\gamma}}\ C^{\bar{r}^{(3)}_{a,\gamma};\bar{\nu}_{a,\gamma}\to \bar{r}^{(1)}_{a,\gamma},\bar{r}^{(2)}_{a,\gamma}}_{\bar{S}^{(3)}_{a,\gamma}\to \bar{S}^{(1)}_{a,\gamma},\bar{S}^{(2)}_{a,\gamma}} \right) &
\end{aligned}
\tag{D.44}
$$

The latter equation can be finally rewritten as

$$
\begin{aligned}
G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}} = c_{\boldsymbol{L}^{(1)}}\, c_{\boldsymbol{L}^{(2)}}\, c_{\boldsymbol{L}^{(3)}} \prod_a \frac{n_a^{(1)}!\, n_a^{(2)}!}{d(R_a^{(1)})\, d(R_a^{(2)})} \frac{1}{\prod_{b,\alpha} d(r^{(1)}_{ab,\alpha})\, d(r^{(2)}_{ab,\alpha})} \sum_{\mu_a} \left(\prod_{b,\alpha} \sum_{\nu_{ab,\alpha}} \right) & \\
\times \Bigg[D^{R_a^{(3)}}_{k_a^{(3)},t_a^{(3)}}\left(\lambda_{a-}\right)\ B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{t_a^{(3)}\to k_a^{(1)},k_a^{(2)}} \left(\prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha} r^{(p)}_{ab,\alpha}\cup_\beta r^{(p)}_{a,\beta};\nu_a^{-(p)}}_{k_a^{(p)}\to\cup_{b,\alpha} q^{(p)}_{ab,\alpha}\cup_\beta l^{(p)}_{a,\beta}} \right) & \\
\times \left(\prod_{b,\alpha} B^{r^{(3)}_{ab,\alpha}\to r^{(1)}_{ab,\alpha},r^{(2)}_{ab,\alpha};\nu_{ab,\alpha}}_{q^{(3)}_{ab,\alpha}\to q^{(1)}_{ab,\alpha},q^{(2)}_{ab,\alpha}} \right) \left(\prod_{\beta} \sum_{\nu_{a,\beta}}\ B^{r^{(3)}_{a,\beta}\to r^{(1)}_{a,\beta},r^{(2)}_{a,\beta};\nu_{a,\beta}}_{l^{(3)}_{a,\beta}\to l^{(1)}_{a,\beta},l^{(2)}_{a,\beta}}\ C^{r^{(3)}_{a,\beta};\nu_{a,\beta}\to r^{(1)}_{a,\beta},r^{(2)}_{a,\beta}}_{S^{(3)}_{a,\beta}\to S^{(1)}_{a,\beta},S^{(2)}_{a,\beta}} \right) \Bigg] & \\
\times \Bigg[D^{R_a^{(3)}}_{i_a^{(3)},h_a^{(3)}}\left(\lambda^{-1}_{a+}\right)\ B^{R_a^{(3)}\to R_a^{(1)},R_a^{(2)};\mu_a}_{h_a^{(3)}\to i_a^{(1)},i_a^{(2)}} \left(\prod_{p=1}^{3} B^{R_a^{(p)}\to\cup_{b,\alpha} r^{(p)}_{ba,\alpha}\cup_\gamma \bar{r}^{(p)}_{a,\gamma};\nu_a^{+(p)}}_{i_a^{(p)}\to\cup_{b,\alpha} l^{(p)}_{ba,\alpha}\cup_\gamma \bar{l}^{(p)}_{a,\gamma}} \right) & \\
\times \left(\prod_{b,\alpha} B^{r^{(3)}_{ba,\alpha}\to r^{(1)}_{ba,\alpha},r^{(2)}_{ba,\alpha};\nu_{ba,\alpha}}_{l^{(3)}_{ba,\alpha}\to l^{(1)}_{ba,\alpha},l^{(2)}_{ba,\alpha}} \right) \left(\prod_{\gamma} \sum_{\bar{\nu}_{a,\gamma}}\ B^{\bar{r}^{(3)}_{a,\gamma}\to \bar{r}^{(1)}_{a,\gamma},\bar{r}^{(2)}_{a,\gamma};\bar{\nu}_{a,\gamma}}_{\bar{l}^{(3)}_{a,\gamma}\to \bar{l}^{(1)}_{a,\gamma},\bar{l}^{(2)}_{a,\gamma}}\ C^{\bar{r}^{(3)}_{a,\gamma};\bar{\nu}_{a,\gamma}\to \bar{r}^{(1)}_{a,\gamma},\bar{r}^{(2)}_{a,\gamma}}_{\bar{S}^{(3)}_{a,\gamma}\to \bar{S}^{(1)}_{a,\gamma},\bar{S}^{(2)}_{a,\gamma}} \right) \Bigg] &
\end{aligned}
\tag{D.45}
$$

The last equation shows that, at each node a in the quiver, the holomorphic GIO ring structure constant factorises into two components, one associated with the positive node and one associated with the negative node of the corresponding split node a. Figure 19 shows a pictorial interpretation of this formula.

D.1 Diagrammatic derivation for an $\mathcal{N} = 2$ SQCD

We are now going to present a diagrammatic recap of this derivation, for the example of an $\mathcal{N} = 2$ SQCD already discussed in section 4.2. Our starting point is (D.11), where each $\boldsymbol{L}^{(i)}$ has been simplified as in eq. (4.10). We can depict this quantity as in Fig. 26.

$$G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}} = \sum_{\substack{\boldsymbol{s}^{(1)},\,\boldsymbol{s}^{(2)} \\ \bar{\boldsymbol{s}}^{(1)},\,\bar{\boldsymbol{s}}^{(2)}}} \sum_{\sigma^{(1)},\,\sigma^{(2)}}$$

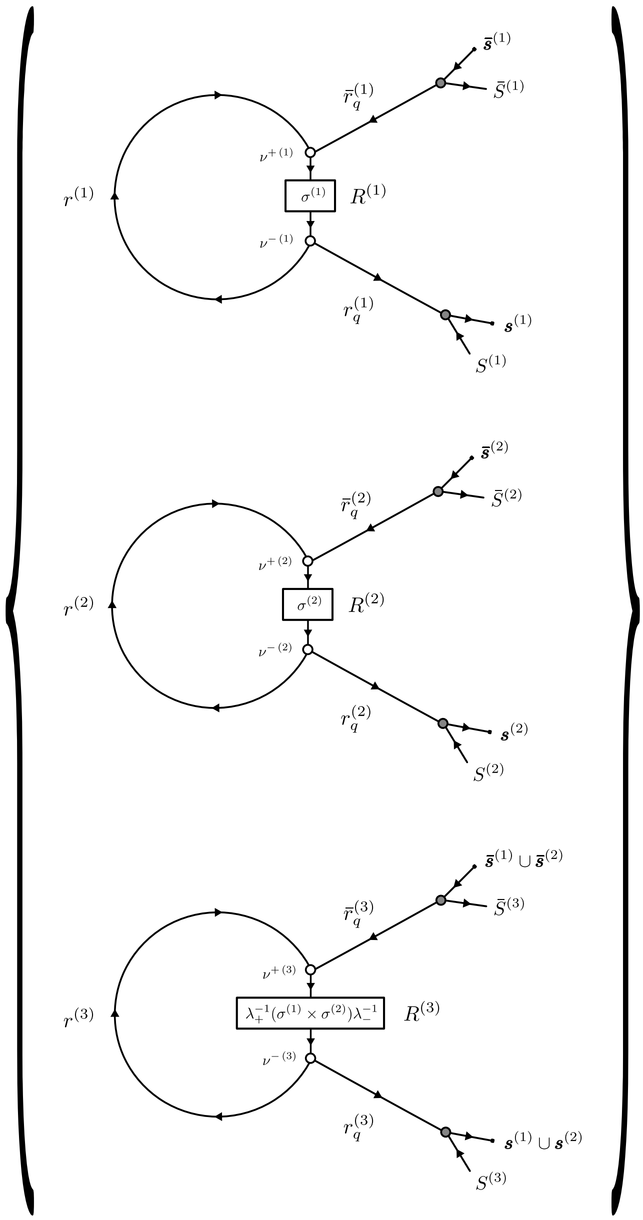

Figure 26: Diagrammatic representation of the chiral ring structure constants for an $\mathcal{N} = 2$ SQCD, corresponding to eq. (D.11).

After using identity (D.14), which is represented in Fig. 24, the diagram is transformed to the one in Fig. 27. We see that now the three disjoint diagrams of the previous Fig. 26 are now joined into a single connected component.

$$G_{\boldsymbol{L}^{(1)},\boldsymbol{L}^{(2)},\boldsymbol{L}^{(3)}} \propto \sum_{\boldsymbol{s}^{(1)},\boldsymbol{s}^{(2)}} \sum_{\bar{\boldsymbol{s}}^{(1)},\bar{\boldsymbol{s}}^{(2)}} \sum_{\mu}$$

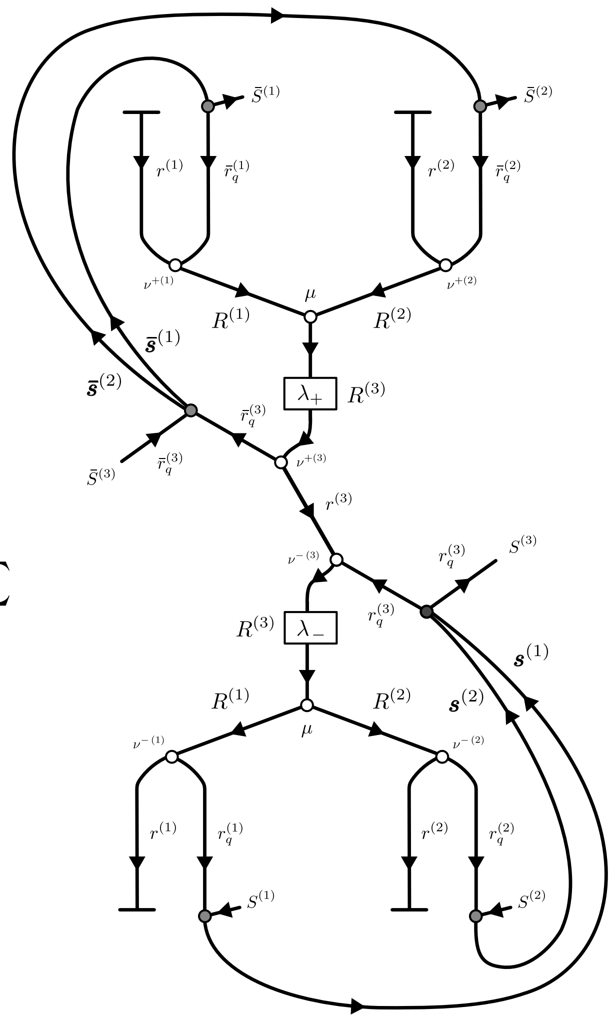

Figure 27: The diagram for the chiral ring structure constants after using the identity (D.14). The horizontal bars are to be identified.

Here we can see the relevance of the permutations λ_- and λ_+, which were previously obtained in the explicit derivation. They allow the fusing of all the state indices of the three disjoint pieces of Fig. 26. This can be understood by looking at Fig. 27. Let us follow the flow at the top of the diagram from $r^{(1)} \otimes \bar{r}_q^{(1)} \otimes r^{(2)} \otimes \bar{r}_q^{(2)}$ to $R^{(3)}$. This corresponds to the embeddings

$$S_{n^{(1)}} \times S_{n_q^{(1)}} \times S_{n^{(2)}} \times S_{n_q^{(2)}} \to S_{n^{(1)}+n_q^{(1)}} \times S_{n^{(2)}+n_q^{(2)}} \to S_{n^{(1)}+n^{(2)}+n_q^{(1)}+n_q^{(2)}} \tag{D.46}$$

and

$$[n^{(1)}] \sqcup [n_q^{(1)}] \sqcup [n^{(2)}] \sqcup [n_q^{(2)}] \to [n^{(1)} + n_q^{(1)}] \sqcup [n^{(2)} + n_q^{(2)}] \to [n^{(1)} + n_q^{(1)} + n^{(2)} + n_q^{(2)}] \tag{D.47}$$

The second embedding corresponds to the branching coefficient labelled by μ. In the branching *after* the λ_+ permutation, $R^{(3)}$ splits into $r^{(3)}$ and $r_q^{(3)}$. The relevant embedding is now

$$[n^{(1)} + n^{(2)}] \sqcup [n_q^{(1)} + n_q^{(2)}] \to [n^{(1)} + n_q^{(1)} + n^{(2)} + n_q^{(2)}] \tag{D.48}$$

which comes naturally from the construction of $\mathcal{O}(\boldsymbol{L}_3)$. The purpose of λ_+ is to allow the transition from (D.47) to (D.48). A similar (but reversed) role is played by the permutation λ_-.

Now we use the relation in Fig. 25 to separate the edges corresponding to the quark (and antiquark) fields from the rest of the diagram. We thus obtain Fig. 28.

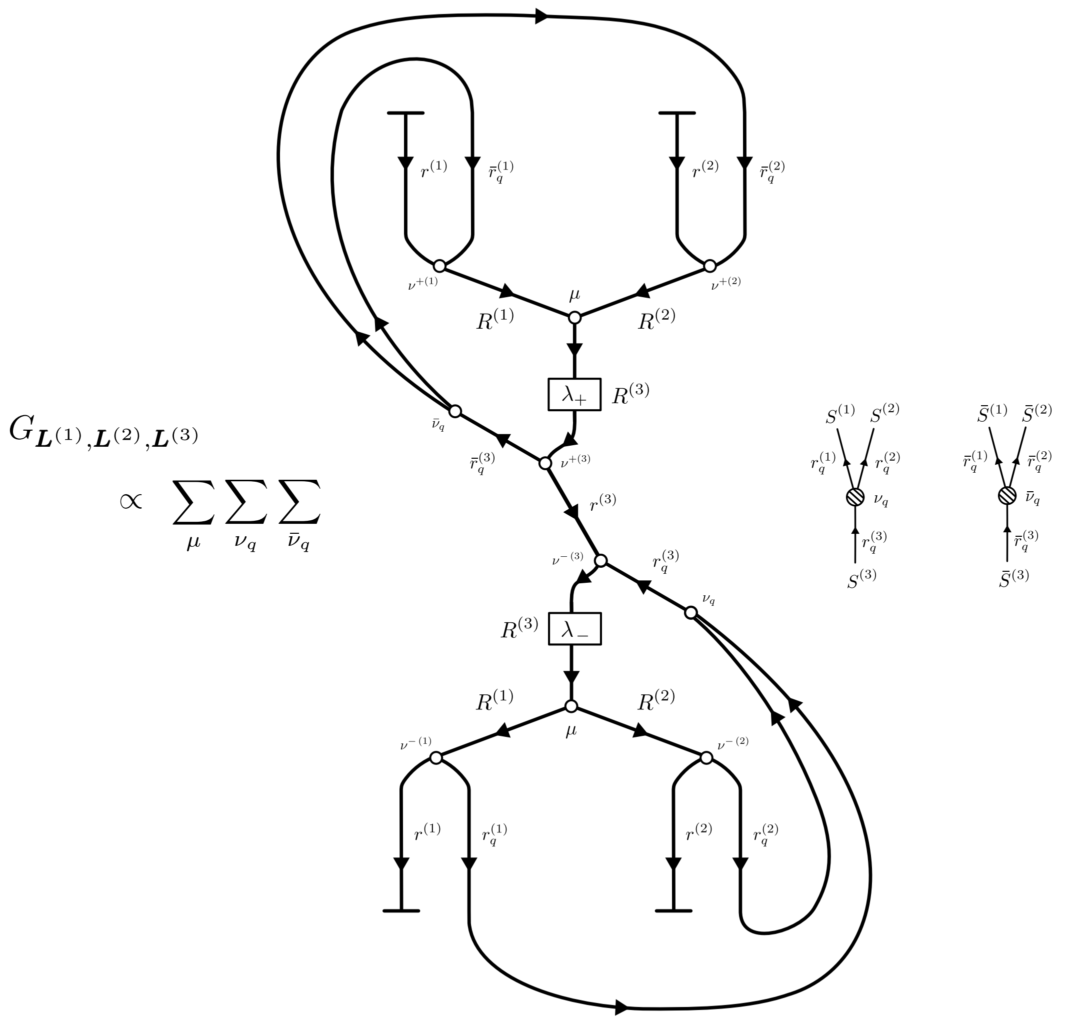

Figure 28: The outcome of inserting the identity described by Fig. 25 into Fig. 27. The horizontal bars are to be identified.

The last step is to separate all the edges connected to the negative node of the split-node from all the edges connected to its positive node. As explained in the derivation above, this operation is achieved through the identity (D.43), which in this example takes the form depicted in Fig. 29.

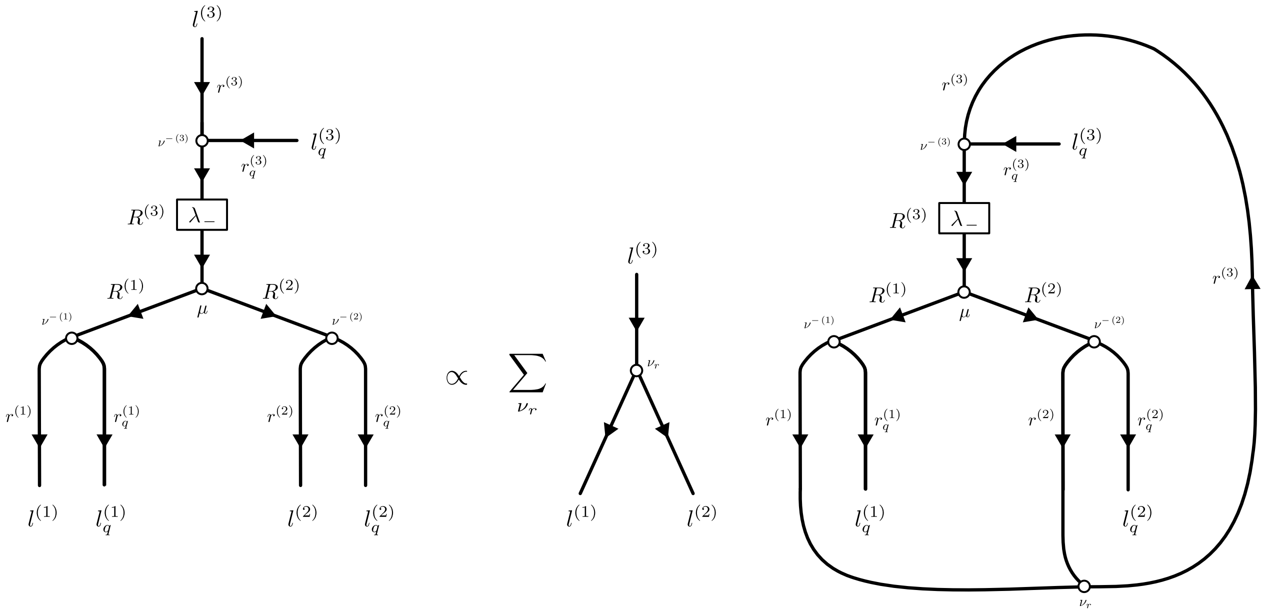

Figure 29: Diagrammatic description of eq. (D.43) for the $\mathcal{N}=2$ SQCD example.

Once this diagrammatic relation has been inserted into Fig. 28, we straightforwardly obtain the final diagram for the chiral ring structure constants for an $\mathcal{N}=2$ SQCD, depicted in Fig. 18.

E Quiver Restricted Schur Polynomials for an $\mathcal{N}=2$ SQCD: $\vec{n}=(2,2,2)$ Field Content

In this appendix we will summarise the main steps which led to the expression of the operators in (5.37). In particular we will derive all the fourteen different quiver characters, corresponding to the set of labels $\boldsymbol{L}_i$ described in (5.36), $i=1,2,...,14$. The operators (5.37) are then readily obtained by using the definition (3.19).

We start from $\mathcal{O}(\boldsymbol{L}_1)$ and $\mathcal{O}(\boldsymbol{L}_2)$. Their quiver characters can be immediately computed to be respectively

$$\chi(\boldsymbol{L}_1,\vec{s},\sigma)=\frac{1}{\sqrt{4!}}\,C^{\boxed{i}\boxed{j}}_{s_1,s_2}\,C^{\overline{\boxed{p}\boxed{q}}}_{\bar{s}_1,\bar{s}_2}\,,\qquad\qquad \chi(\boldsymbol{L}_2,\vec{s},\sigma)=\frac{1}{\sqrt{4!}}\,\mathrm{sign}(\sigma)\,C^{\begin{smallmatrix}\boxed{i}\\\boxed{j}\end{smallmatrix}}_{s_1,s_2}\,C^{\overline{\begin{smallmatrix}\boxed{p}\\\boxed{q}\end{smallmatrix}}}_{\bar{s}_1,\bar{s}_2} \tag{E.1}$$

Here we used the Clebsch-Gordan coefficients already derived in (5.29). We will keep using this notation for the rest of this appendix.

Let us now turn to the three dimensional representation $\yng(3,1)$ of S_4. We choose a basis $\{e_1,e_2,e_3\}$ in which the three Jucys-Murphy elements (12), $(13)+(23)$, $(14)+(24)+(34)$ of S_4 have the eigenvalues in table 2.

	(12)	(13) + (23)	(14) + (24) + (34)
e_1	1	-1	2
e_2	-1	1	2
e_3	1	2	-1

Table 2: Eigenvalues of the Jucys-Murphy elements (12), (13) + (23), (14) + (24) + (34) on our chosen basis $\{e_1, e_2, e_3\}$ for the standard representation of S_4.

Alternatively, we can specify our basis choice with the standard Young tableaux

$$e_1 \sim \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}\,, \qquad e_2 \sim \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}\,, \qquad e_3 \sim \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} \tag{E.2}$$

We now consider the group restriction $S_4|_{S_2 \times S_2} = \{(1), (12), (34), (12)(34)\}$. Under this restriction, the $\ydiagram{3,1}$ decomposes as

$$\ydiagram{3,1}\Big|_{S_2 \times S_2} = \ydiagram{2} \otimes \ydiagram{2} \;\oplus\; \ydiagram{1,1} \otimes \ydiagram{2} \;\oplus\; \ydiagram{2} \otimes \ydiagram{1,1} \tag{E.3}$$

The branching coefficients for this group reduction will then be the matrix elements of the orthogonal operator B such that

$$B^{-1} D^{\ydiagram{3,1}}(\,(1)\,) B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B^{-1} D^{\ydiagram{3,1}}(\,(12)\,) B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$B^{-1} D^{\ydiagram{3,1}}(\,(34)\,) B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B^{-1} D^{\ydiagram{3,1}}(\,(12)(34)\,) B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{E.4}$$

In our basis choice (E.2) the matrix B reads

$$B = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 0 & -1 \\ 0 & \sqrt{3} & 0 \\ 1 & 0 & \sqrt{2} \end{pmatrix} \tag{E.5}$$

The branching coefficient for (E.3) are then

$$
\begin{aligned}
&B_{1\to1,1}^{\yng(3,1)\to\yng(2),\yng(2)} = \sqrt{\tfrac{2}{3}}\,, && B_{1\to1,1}^{\yng(3,1)\to\yng(1,1),\yng(2)} = 0\,, && B_{1\to1,1}^{\yng(3,1)\to\yng(2),\yng(1,1)} = -\tfrac{1}{\sqrt{3}}\,, \\
&B_{2\to1,1}^{\yng(3,1)\to\yng(2),\yng(2)} = 0\,, && B_{2\to1,1}^{\yng(3,1)\to\yng(1,1),\yng(2)} = 1\,, && B_{2\to1,1}^{\yng(3,1)\to\yng(2),\yng(1,1)} = 0\,, \\
&B_{3\to1,1}^{\yng(3,1)\to\yng(2),\yng(2)} = 1\,, && B_{3\to1,1}^{\yng(3,1)\to\yng(1,1),\yng(2)} = 0\,, && B_{3\to1,1}^{\yng(3,1)\to\yng(2),\yng(1,1)} = \sqrt{\tfrac{2}{3}}
\end{aligned}
\tag{E.6}
$$

We now define the orthogonal projectors

$$
\begin{gathered}
P_{i,j}^{\yng(3,1)\to\yng(2),\yng(2)} = B_{i\to1,1}^{\yng(3,1)\to\yng(2),\yng(2)}\, B_{j\to1,1}^{\yng(3,1)\to\yng(2),\yng(2)}\,, \qquad P_{i,j}^{\yng(3,1)\to\yng(1,1),\yng(2)} = B_{i\to1,1}^{\yng(3,1)\to\yng(1,1),\yng(2)}\, B_{j\to1,1}^{\yng(3,1)\to\yng(1,1),\yng(2)}\,, \\
P_{i,j}^{\yng(3,1)\to\yng(2),\yng(1,1)} = B_{i\to1,1}^{\yng(3,1)\to\yng(2),\yng(1,1)}\, B_{j\to1,1}^{\yng(3,1)\to\yng(2),\yng(1,1)}
\end{gathered}
\tag{E.7}
$$

which project the $\yng(3,1)$ of S_4 on the $\yng(2)\otimes\yng(2)$, on the $\yng(1,1)\otimes\yng(2)$ and on the $\yng(2)\otimes\yng(1,1)$ of $S_2\times S_2$ respectively. We also define a fourth operator, that we label T, as

$$
T_{i,j} = B_{i\to1,1}^{\yng(3,1)\to\yng(2),\yng(2)}\, B_{j\to1,1}^{\yng(3,1)\to\yng(2),\yng(1,1)}
\tag{E.8}
$$

These matrices explicitly read

$$
\begin{aligned}
&P^{\yng(3,1)\to\yng(2),\yng(2)} = \tfrac{1}{3}\begin{pmatrix} 2 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 1 \end{pmatrix}, && P^{\yng(3,1)\to\yng(1,1),\yng(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&P^{\yng(3,1)\to\yng(2),\yng(1,1)} = \tfrac{1}{3}\begin{pmatrix} 1 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 2 \end{pmatrix}, && T = \tfrac{1}{3}\begin{pmatrix} -\sqrt{2} & 0 & 2 \\ 0 & 0 & 0 \\ -1 & 0 & \sqrt{2} \end{pmatrix}
\end{aligned}
\tag{E.9}
$$

The quiver character for $\mathcal{O}(\boldsymbol{L}_3)$, $\mathcal{O}(\boldsymbol{L}_4)$, $\mathcal{O}(\boldsymbol{L}_5)$, $\mathcal{O}(\boldsymbol{L}_6)$, $\mathcal{O}(\boldsymbol{L}_7)$ are then

$$
\begin{aligned}
\chi(\boldsymbol{L}_3,\vec{s},\sigma) &= \frac{1}{2\sqrt{2}}\ \mathrm{Tr}\left[D^{\yng(3,1)}(\sigma)\, P^{\yng(3,1)\to\yng(2),\yng(2)}\right]\ C_{s_1,s_2}^{\young(ij)}\ C_{\bar{s}_1,\bar{s}_2}^{\overline{\young(pq)}}\,, \\
\chi(\boldsymbol{L}_4,\vec{s},\sigma) &= \frac{1}{2\sqrt{2}}\ \mathrm{Tr}\left[D^{\yng(3,1)}(\sigma)\, P^{\yng(3,1)\to\yng(1,1),\yng(2)}\right]\ C_{s_1,s_2}^{\young(ij)}\ C_{\bar{s}_1,\bar{s}_2}^{\overline{\young(pq)}}\,, \\
\chi(\boldsymbol{L}_5,\vec{s},\sigma) &= \frac{1}{2\sqrt{2}}\ \mathrm{Tr}\left[D^{\yng(3,1)}(\sigma)\, P^{\yng(3,1)\to\yng(2),\yng(1,1)}\right]\ C_{s_1,s_2}^{\young(i,j)}\ C_{\bar{s}_1,\bar{s}_2}^{\overline{\young(p,q)}}\,,
\end{aligned}
\tag{E.10}
$$

$$\chi(\boldsymbol{L}_6, \vec{s}, \sigma) = \frac{1}{2\sqrt{2}}\ \mathrm{Tr}\left[D^{\ydiagram{3,1}}(\sigma)\, T\right]\ C^{\young(ij)}_{s_1,s_2}\ C^{\young(p,q)}_{\bar{s}_1,\bar{s}_2}\,,$$

$$\chi(\boldsymbol{L}_7, \vec{s}, \sigma) = \frac{1}{2\sqrt{2}}\ \mathrm{Tr}\left[D^{\ydiagram{3,1}}(\sigma)\, T^t\right]\ C^{\young(i,j)}_{s_1,s_2}\ C^{\young(pq)}_{\bar{s}_1,\bar{s}_2}$$

Here T^t is the transpose of the matrix T in (E.8).

We now focus on the $\ydiagram{2,1,1}$ representation of S_4. This representation can be obtained by tensoring together the standard and the sign representation of S_4:

$$\ydiagram{2,1,1} = \ydiagram{3,1} \otimes \ydiagram{1,1,1,1} \tag{E.11}$$

In the following, we will continue to use (E.2) as our basis choice for the standard representation $\ydiagram{3,1}$. Under the group restriction $S_4|_{S_2\times S_2} = \{(1),(12),(34),(12)(34)\}$, the $\ydiagram{2,1,1}$ decomposes as

$$\ydiagram{2,1,1}\,\Bigg|_{S_2\times S_2} = \ydiagram{2} \otimes \ydiagram{1,1} \ \oplus\ \ydiagram{1,1} \otimes \ydiagram{2} \ \oplus\ \ydiagram{1,1} \otimes \ydiagram{1,1} \tag{E.12}$$

As in the previous instance, the branching coefficients for this group reduction are the matrix elements of the orthogonal operator B, such that

$$B^{-1}D^{\ydiagram{2,1,1}}(\,(1)\,)B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B^{-1}D^{\ydiagram{2,1,1}}(\,(12)\,)B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$B^{-1}D^{\ydiagram{2,1,1}}(\,(34)\,)B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B^{-1}D^{\ydiagram{2,1,1}}(\,(12)(34)\,)B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E.13}$$

In our basis choice, the matrix B reads

$$B = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 & \sqrt{2} \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 1 \end{pmatrix} \tag{E.14}$$

The branching coefficient for (E.12) are thus

$$
\begin{aligned}
&B_{1\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{2},\ydiagram{1,1}}=0\,, &&B_{1\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{2}}=-\tfrac{1}{\sqrt{3}}\,, &&B_{1\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{1,1}}=\sqrt{\tfrac{2}{3}}\,,\\
&B_{2\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{2},\ydiagram{1,1}}=1\,, &&B_{2\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{2}}=0\,, &&B_{2\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{1,1}}=0\,,\\
&B_{3\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{2},\ydiagram{1,1}}=0\,, &&B_{3\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{2}}=\sqrt{\tfrac{2}{3}}\,, &&B_{3\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{1,1}}=\tfrac{1}{\sqrt{3}}
\end{aligned}
\tag{E.15}
$$

Closely following the procedure of the previous paragraph, we define the orthogonal projectors

$$
\begin{gathered}
P_{i,j}^{\ydiagram{2,1,1}\to\ydiagram{2},\ydiagram{1,1}}=B_{i\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{2},\ydiagram{1,1}}\,B_{j\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{2},\ydiagram{1,1}}\,,\qquad P_{i,j}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{2}}=B_{i\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{2}}\,B_{j\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{2}}\,,\\
P_{i,j}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{1,1}}=B_{i\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{1,1}}\,B_{j\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{1,1}}
\end{gathered}
\tag{E.16}
$$

These operators project the $\ydiagram{2,1,1}$ of S_4 on the $\ydiagram{2}\otimes\ydiagram{1,1}$, on the $\ydiagram{1,1}\otimes\ydiagram{2}$ and on the $\ydiagram{1,1}\otimes\ydiagram{1,1}$ of $S_2\times S_2$ respectively. We also introduce the operator V:

$$
V_{i,j}=B_{i\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{2}}\,B_{j\to1,1}^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{1,1}}
\tag{E.17}
$$

These matrices explicitly read

$$
\begin{aligned}
&P^{\ydiagram{2,1,1}\to\ydiagram{2},\ydiagram{1,1}}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, &&P^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{2}}=\tfrac{1}{3}\begin{pmatrix}1&0&-\sqrt{2}\\0&0&0\\-\sqrt{2}&0&2\end{pmatrix},\\
&P^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{1,1}}=\tfrac{1}{3}\begin{pmatrix}2&0&\sqrt{2}\\0&0&0\\-\sqrt{2}&0&1\end{pmatrix}, &&V=\tfrac{1}{3}\begin{pmatrix}-\sqrt{2}&0&-1\\0&0&0\\2&0&\sqrt{2}\end{pmatrix}
\end{aligned}
\tag{E.18}
$$

Notice that $V=T^t$, where T is the matrix defined in (E.9). The quiver character for $\mathcal{O}(\boldsymbol{L}_8)$, $\mathcal{O}(\boldsymbol{L}_9)$, $\mathcal{O}(\boldsymbol{L}_{10})$, $\mathcal{O}(\boldsymbol{L}_{11})$, $\mathcal{O}(\boldsymbol{L}_{12})$ are therefore

$$
\begin{aligned}
\chi(\boldsymbol{L}_8,\vec{s},\sigma)&=\frac{1}{2\sqrt{2}}\,\mathrm{Tr}\left[D^{\ydiagram{2,1,1}}(\sigma)\,P^{\ydiagram{2,1,1}\to\ydiagram{2},\ydiagram{1,1}}\right]\,C_{s_1,s_2}^{\ytableaushort{i,j}}\,C_{\bar{s}_1,\bar{s}_2}^{\overline{\ytableaushort{p,q}}}\,,\\
\chi(\boldsymbol{L}_9,\vec{s},\sigma)&=\frac{1}{2\sqrt{2}}\,\mathrm{Tr}\left[D^{\ydiagram{2,1,1}}(\sigma)\,P^{\ydiagram{2,1,1}\to\ydiagram{1,1},\ydiagram{2}}\right]\,C_{s_1,s_2}^{\ytableaushort{ij}}\,C_{\bar{s}_1,\bar{s}_2}^{\overline{\ytableaushort{pq}}}\,,
\end{aligned}
\tag{E.19}
$$

$$\chi(\boldsymbol{L}_{10}, \vec{s}, \sigma) = \frac{1}{2\sqrt{2}} \operatorname{Tr}\left[D^{\ydiagram{2,1,1}}(\sigma)\, P^{\ydiagram{2,1,1} \to \ydiagram{1,1}, \ydiagram{1,1}} \right] \; C_{s_1,s_2}^{\begin{ytableau} i \\ j \end{ytableau}} \; C_{\bar{s}_1,\bar{s}_2}^{\overline{\begin{ytableau} p \\ q \end{ytableau}}} \,,$$

$$\chi(\boldsymbol{L}_{11}, \vec{s}, \sigma) = \frac{1}{2\sqrt{2}} \operatorname{Tr}\left[D^{\ydiagram{2,1,1}}(\sigma)\, V \right] \; C_{s_1,s_2}^{\begin{ytableau} i & j \end{ytableau}} \; C_{\bar{s}_1,\bar{s}_2}^{\overline{\begin{ytableau} p \\ q \end{ytableau}}} \,,$$

$$\chi(\boldsymbol{L}_{12}, \vec{s}, \sigma) = \frac{1}{2\sqrt{2}} \operatorname{Tr}\left[D^{\ydiagram{2,1,1}}(\sigma)\, V^t \right] \; C_{s_1,s_2}^{\begin{ytableau} i \\ j \end{ytableau}} \; C_{\bar{s}_1,\bar{s}_2}^{\overline{\begin{ytableau} p & q \end{ytableau}}}$$

Two operators still remain. They can be obtained by considering the S_4 $\ydiagram{2,2}$ representation branching

$$\ydiagram{2,2}\,\Big|_{S_2\times S_2} = \ydiagram{2} \otimes \ydiagram{2} \;\oplus\; \ydiagram{1,1} \otimes \ydiagram{1,1} \tag{E.20}$$

The $\ydiagram{2,2}$ representation of S_4 is really a representation of the quotient group $S_4/\{(1), (12)(34), (13)(24), (14)(23)\}$, which in turn is isomorphic to S_3. This representation is thus just the standard representation of S_3 pulled back to S_4 via this quotient [38]. We choose a basis $\{e_1, e_2\}$ in which the Jucys-Murphy elements (12), (13) + (23), (14) + (24) + (34) of S_4 have the eigenvalues in table 3.

	(12)	(13) + (23)	(14) + (24) + (34)
e_1	1	-1	0
e_2	-1	1	0

Table 3: Eigenvalues of the Jucys-Murphy elements (12), (13) + (23), (14) + (24) + (34) on our chosen basis $\{e_1, e_2\}$ for the two-dimensional representation of S_4.

The standard Young tableaux labelling of this basis is

$$e_1 \sim \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}\,, \qquad\qquad e_2 \sim \begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau} \tag{E.21}$$

An explicit representation of $\yng(2,2)$ is therefore obtained by considering the set of matrices

$$
\begin{aligned}
&D^{\yng(2,2)}(\,(1)\,) = D^{\yng(2,2)}(\,(12)(34)\,) = D^{\yng(2,2)}(\,(13)(24)\,) = D^{\yng(2,2)}(\,(14)(23)\,) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,\\
&D^{\yng(2,2)}(\,(12)\,) = D^{\yng(2,2)}(\,(34)\,) = D^{\yng(2,2)}(\,(1324)\,) = D^{\yng(2,2)}(\,(1423)\,) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,\\
&D^{\yng(2,2)}(\,(13)\,) = D^{\yng(2,2)}(\,(24)\,) = D^{\yng(2,2)}(\,(1234)\,) = D^{\yng(2,2)}(\,(1432)\,) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} ,\\
&D^{\yng(2,2)}(\,(23)\,) = D^{\yng(2,2)}(\,(14)\,) = D^{\yng(2,2)}(\,(1342)\,) = D^{\yng(2,2)}(\,(1243)\,) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} ,\\
&D^{\yng(2,2)}(\,(123)\,) = D^{\yng(2,2)}(\,(243)\,) = D^{\yng(2,2)}(\,(142)\,) = D^{\yng(2,2)}(\,(134)\,) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} ,\\
&D^{\yng(2,2)}(\,(132)\,) = D^{\yng(2,2)}(\,(143)\,) = D^{\yng(2,2)}(\,(234)\,) = D^{\yng(2,2)}(\,(124)\,) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}
\end{aligned}
\tag{E.22}
$$

With this basis choice, under the group restriction $S_4|_{S_2\times S_2} = \{(1),(12),(34),(12)(34)\}$, we have

$$
\begin{aligned}
&D^{\yng(2,2)}(\,(1)\,) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad && D^{\yng(2,2)}(\,(12)\,) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,\\
&D^{\yng(2,2)}(\,(34)\,) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad && D^{\yng(2,2)}(\,(12)(34)\,) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
\tag{E.23}
$$

The decomposition (E.20) is already manifest. The branching coefficients for this reduction are then

$$
B^{\yng(2,2)\to\yng(2),\yng(2)}_{j\to 1,1} = \delta_{j,1}\,, \qquad B^{\yng(2,2)\to\yng(1,1),\yng(1,1)}_{j\to 1,1} = \delta_{j,2}\,, \qquad j = 1,2 \tag{E.24}
$$

We can now write the orthogonal projectors

$$
\begin{aligned}
P_{i,j}^{\ydiagram{2,2}\to\ydiagram{2},\ydiagram{2}} &= B_{i\to 1,1}^{\ydiagram{2,2}\to\ydiagram{2},\ydiagram{2}}\, B_{j\to 1,1}^{\ydiagram{2,2}\to\ydiagram{2},\ydiagram{2}} \quad\longrightarrow\quad P^{\ydiagram{2,2}\to\ydiagram{2},\ydiagram{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
P_{i,j}^{\ydiagram{2,2}\to\ydiagram{1,1},\ydiagram{1,1}} &= B_{i\to 1,1}^{\ydiagram{2,2}\to\ydiagram{1,1},\ydiagram{1,1}}\, B_{j\to 1,1}^{\ydiagram{2,2}\to\ydiagram{1,1},\ydiagram{1,1}} \quad\longrightarrow\quad P^{\ydiagram{2,2}\to\ydiagram{1,1},\ydiagram{1,1}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
\tag{E.25}
$$

projecting the $\ydiagram{2,2}$ of S_4 on the $\ydiagram{2}\otimes\ydiagram{2}$ and on the $\ydiagram{1,1}\otimes\ydiagram{1,1}$ of $S_2\times S_2$ respectively. The quiver characters for the remaining two operators, $\mathcal{O}(\boldsymbol{L}_{13})$ and $\mathcal{O}(\boldsymbol{L}_{14})$, are then

$$
\begin{aligned}
\chi(\boldsymbol{L}_{13},\vec{s},\sigma) &= \frac{1}{2\sqrt{3}}\,\mathrm{Tr}\left[D^{\ydiagram{2,2}}(\sigma)\,P^{\ydiagram{2,2}\to\ydiagram{2},\ydiagram{2}}\right]\; C_{s_1,s_2}^{\ytableaushort{ij}}\; C_{\bar{s}_1,\bar{s}_2}^{\overline{\ytableaushort{pq}}}, \\
\chi(\boldsymbol{L}_{14},\vec{s},\sigma) &= \frac{1}{2\sqrt{3}}\,\mathrm{Tr}\left[D^{\ydiagram{2,2}}(\sigma)\,P^{\ydiagram{2,2}\to\ydiagram{1,1},\ydiagram{1,1}}\right]\; C_{s_1,s_2}^{\ytableaushort{i,j}}\; C_{\bar{s}_1,\bar{s}_2}^{\overline{\ytableaushort{p,q}}}
\end{aligned}
\tag{E.26}
$$

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